

Minimax Estimation for Periodically Correlated Stochastic Processes

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Abstract The problem of optimal linear estimation of functional depending on the unknown values of periodically correlated stochastic process from observations of this process for is considered. Formulas that determine the least favorable processes and the minimax estimation for functional are proposed for the given class of admissible processes.

Keywords Periodically correlated process, minimax estimation, mean square error, least favorable process, minimax-robust spectral characteristic

1 Introduction

Research methods of estimation problems of unknown values of stationary stochastic processes (extrapolation, interpolation and filtering problems) are developed in the works of Kolmogorov [1], Wiener [2], Yaglom [3, 4]. These methods are based on the assumption that spectral densities of processes are exactly known. In the case when complete information on the spectral densities is impossible, but a set of admissible densities is given, the minimax approach to estimation problem is used. That is we find the estimate that minimizes the mean square error for all spectral densities from a given class simultaneously. Ulf Grenander [5] was the first who applied the minimax estimation method to the extrapolation problem for stationary processes. Moklyachuk [6], Moklyachuk and Masyutka [7] studied the extrapolation, interpolation and filtering problems for stationary processes and sequences. In the paper by Gladyshev [8] the investigation of periodically correlated processes was started. The analysis of properties of correlation function and representation of periodically correlated processes is presented. The connection between periodically correlated processes and stationary processes is proposed by Makagon [9, 10].

2 Periodically correlated processes and generated vector stationary sequences

Definition 1. Mean square continuous stochastic process $\zeta : \mathbb{R} \rightarrow H = L_2(\Omega, \mathcal{F}, \mathbb{P})$, $E\zeta(t) = 0$, is called periodically correlated (PC) with period T , if its correlation function $K(t+u, u) = E\zeta(t+u)\overline{\zeta(u)}$ for all $t, u \in \mathbb{R}$ and some fixed T is such that

$$K(t+u, u) = K(t+u+T, u+T).$$

Consider the problem of optimal linear estimation of the functional

$$A\zeta = \int_0^\infty a(t)\zeta(t)dt$$

depending on the unknown values of PC stochastic process $\zeta(t)$ from the class \mathbf{Y} of mean square continuous PC processes $\zeta(t)$ such that $E\zeta(t) = 0$, $E|\zeta(t)|^2 \leq P/T$. The estimation is based on observations of the process $\zeta(t)$ for $t < 0$. The function $a(t)$, $t \in \mathbb{R}$ satisfies the condition $\int_0^\infty |a(t)|dt < \infty$.

The functional $A\zeta$ can be written as

$$A\zeta = \int_0^\infty a(t)\zeta(t)dt = \sum_{j=0}^\infty \int_0^T a_j(u)\zeta_j(u)du,$$

$$a(u+jT) = a_j(u), \zeta(u+jT) = \zeta_j(u), u \in [0, T).$$

$\{\zeta_j(u), u \in [0, T), j \in \mathbb{Z}\}$ is $L_2([0, T); H)$ -valued stationary stochastic sequence $\{\zeta_j, j \in \mathbb{Z}\}$ with the correlation function

$$B(l, j) = \langle \zeta_l, \zeta_j \rangle_H = \int_0^T K(u+(l-j)T, u)du = B(l-j).$$

Consider in $L_2([0, T); \mathbb{R})$ the orthonormal basis

$$\{\tilde{e}_k = T^{-1/2} e^{2\pi i \{(-1)^k \lceil k/2 \rceil\} u/T}, k = 1, 2, 3, \dots\},$$

$$\langle \tilde{e}_j, \tilde{e}_k \rangle = \delta_{kj}.$$

The stationary sequence $\{\zeta_j, j \in \mathbb{Z}\}$ can be represented in the form

$$\zeta_j = \sum_{k=1}^{\infty} \zeta_{kj} \tilde{e}_k,$$

$$\zeta_{kj} = \langle \zeta_j, \tilde{e}_k \rangle = T^{-1/2} \int_0^T \zeta_j(v) e^{-2\pi i \{(-1)^k [k/2]\} v/T} dv,$$

and the functional $A\zeta$ has the following form

$$A\zeta = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} a_{kj} \zeta_{kj} = \sum_{j=0}^{\infty} \vec{a}_j^\top \vec{\zeta}_j,$$

$$\vec{\zeta}_j = (\zeta_{kj}, k = 1, 2, \dots),$$

$$\vec{a}_j = (a_{kj}, k = 1, 2, \dots) =$$

$$= (a_{1j}, a_{2j}, a_{3j}, \dots, a_{2k+1,j}, a_{2k,j}, \dots)^\top, \quad a_{kj} = \langle a_j, \tilde{e}_k \rangle.$$

Components ζ_{kj} are such that [11]

$$E\zeta_{kj} = 0,$$

$$\|\zeta_j\|_H^2 = \sum_{k=1}^{\infty} E|\zeta_{kj}|^2 \leq P,$$

$$E\zeta_{kl} \overline{\zeta_{nj}} = \langle R(l-j)e_k, e_n \rangle.$$

The correlation function $R(n)$ of stationary sequence $\{\zeta_j, j \in \mathbb{Z}\}$ is a correlation operator function in ℓ_2 .

The regular stationary sequence $\{\zeta_j, j \in \mathbb{Z}\}$ has the spectral density operator function $f(\lambda), \lambda \in [-\pi, \pi]$ in ℓ_2 and satisfy the equality

$$\langle R(l-j)e_k, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(l-j)\lambda} \langle f(\lambda)e_k, e_n \rangle d\lambda, \quad k, n \geq 1.$$

The spectral density $f(\lambda)$ a.e. on $[-\pi, \pi]$ is a kernel operator with integrable kernel norm

$$\sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle f(\lambda)e_k, e_k \rangle d\lambda = \|\zeta_j\|_H^2 \leq P. \quad (1)$$

Regular stationary sequence $\{\zeta_j, j \in \mathbb{Z}\}$ admits the canonical moving average representation of components [11]

$$\zeta_{kj} = \sum_{s=-\infty}^{\infty} \sum_{m=1}^{\infty} g_{km}(j-s) \varepsilon_m(s), \quad (2)$$

where $\varepsilon_m(s)$, $m = 1, \dots, M$, $s \in \mathbb{Z}$ are mutually orthogonal sequences in H with orthonormal values; M is the multiplicity of $\{\zeta_j\}$; $g_{km}(s)$, $k = 1, 2, \dots$, $m = 1, \dots, M$, $s = 0, 1, \dots$, are sequences such that $\sum_{s=0}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^M |g_{km}(s)|^2 \leq P$.

As the consequence of the representation (2) the optimal linear estimation of components of stationary sequence can be written in the form

$$\hat{\zeta}_{kj} = \sum_{s=-\infty}^{\infty} \sum_{m=1}^M g_{km}(j-s) \varepsilon_m(s). \quad (3)$$

The spectral density $f(\lambda)$ of regular stationary sequence $\{\zeta_j, j \in \mathbb{Z}\}$ admits canonical factorization

$$f_{kn}(\lambda) = \sum_{m=1}^M \hat{g}_{km}(e^{i\lambda}) \overline{\hat{g}_{nm}(e^{i\lambda})}, \quad k, n \geq 1, \quad (4)$$

$$\hat{g}_{km}(e^{i\lambda}) = \sum_{s=0}^{\infty} g_{km}(s) e^{-is\lambda}$$

. This means that

$$f(\lambda) = G(\lambda)G^*(\lambda), \quad G(\lambda) = \{\hat{g}_{km}(e^{i\lambda})\}_{k=1, \infty}^{m=1, M}.$$

3 Minimax estimation of linear functional

Let us assume that coefficients $\{\vec{a}_j, j \geq 0\}$ satisfy conditions

$$\sum_{j=0}^{\infty} \|\vec{a}_j\| < \infty, \quad \sum_{j=0}^{\infty} (j+1) \|\vec{a}_j\|^2 < \infty, \quad (5)$$

$$\|\vec{a}_j\|^2 = \sum_{k=1}^{\infty} |a_{kj}|^2.$$

Denote by Λ the set of all linear estimates of functional $A\zeta$ based on observation of process $\zeta(t)$ for $t < 0$. Let \mathbf{Y}_R denotes the class of all regular stationary sequences that satisfy the condition $\|\zeta_j\|_H^2 \leq P$.

We calculate the largest values of mean square errors $\Delta(\zeta, \hat{A}) = E|A\zeta - \hat{A}\zeta|^2$ of estimation $\hat{A}\zeta$ of the functional $A\zeta$ and $\Delta(\zeta, \hat{A}_N) = E|A_N\zeta - \hat{A}_N\zeta|^2$ of estimation $\hat{A}_N\zeta$ of the functional $A_N\zeta = \sum_{j=0}^{\infty} \vec{a}_j^\top \vec{\zeta}_j$.

Theorem 1. *Let the coefficients $\{\vec{a}_j, j = 0, \dots, N\}$ satisfy conditions*

$$\|\vec{a}_j\| < \infty, \quad j = 0, 1, \dots, N. \quad (6)$$

Then the function $\Delta(\zeta, \hat{A}_N)$ has a saddle point on the set $\mathbf{Y} \times \Lambda$:

$$\min_{\hat{A}_N \in \Lambda} \max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}_N) = \max_{\zeta \in \mathbf{Y}} \min_{\hat{A}_N \in \Lambda} \Delta(\zeta, \hat{A}_N) = P \cdot \nu_N^2,$$

where ν_N^2 is the greatest eigenvalue of the self-adjoint compact operator $Q_N = \{Q_N(p, q)\}_{p, q=0}^N$ in the space ℓ_2 determined by block-matrices $Q_N(p, q) = \{Q_{kn}^N(p, q)\}_{k, n=1}^{\infty}$ with elements

$$Q_{kn}^N(p, q) = \sum_{s=0}^{\min(N-p, N-q)} a_{k, s+p} \cdot \overline{a_{n, s+q}}, \quad (7)$$

$$k, n = 1, 2, 3, \dots, \quad p, q = 0, 1, \dots, N.$$

Proof: Upper bound. Let denote Λ_1 the class of all linear estimates of the functional $A_N\zeta$ of the form $\hat{A}_N\zeta = \sum_{j=-\infty}^{-1} \vec{c}_j^\top \vec{\zeta}_j$. Taking into account the spectral decomposition of stationary sequence and of its correlation function [12], the following relation holds true

$$\Delta(\zeta, \hat{A}_N) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (A_N(e^{i\lambda}) - C(e^{i\lambda}))^* f(\lambda) (A_N(e^{i\lambda}) - C(e^{i\lambda})) d\lambda,$$

$$A_N(e^{i\lambda}) = \sum_{j=0}^{\infty} \vec{a}_j e^{ij\lambda}, \quad C(e^{i\lambda}) = \sum_{j=-\infty}^{-1} \vec{c}_j e^{ij\lambda}.$$

Then

$$\max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}_N) \leq \max_{\zeta \in \mathbf{Y}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \|A_N(e^{i\lambda}) - C(e^{i\lambda})\|^2 \|f(\lambda)\| d\lambda \leq$$

$$\leq \max_{\lambda \in [-\pi, \pi]} \|A_N(e^{i\lambda}) - C(e^{i\lambda})\|^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(\lambda)\| d\lambda \leq$$

$$\leq P \cdot \max_{\lambda \in [-\pi, \pi]} \|A_N(e^{i\lambda}) - C(e^{i\lambda})\|^2,$$

what follows from the inequality $\|f(\lambda)\| = \left(\sum_{k, n=1}^{\infty} |f_{kn}(\lambda)|^2\right)^{1/2} \leq \text{Tr } f(\lambda)$ and integrability of kernel norm of spectral density (1).

To calculate $\max_{\lambda \in [-\pi, \pi]} \|A_N(e^{i\lambda}) - C(e^{i\lambda})\|^2$ we consider the class of all power series $\tilde{f}(z) = \sum_{j=0}^{\infty} \tilde{a}_j z^j$, which are regular for $|z| < 1$ and first $(N+1)$ summands $\sum_{j=0}^N \tilde{a}_j z^j$ are given. Denote by ω_N^2 the greatest eigenvalue of the matrix $D_N = \{D_N(p, q)\}_{p, q=0}^N$ constructed by block-matrices

$$D_N(p, q) = \sum_{s=0}^{\min(N-p, N-q)} \tilde{a}_{N-p+s} \cdot \tilde{a}_{N-q+s}^*,$$

$$p, q = 0, 1, \dots, N.$$

Then $\min_{\{\tilde{a}_j: j \geq N+1\}} \max_{|z|=1} \|\tilde{f}(z)\|^2 = \omega_N^2$ as it follows from [13]. Since $\Lambda_1 \subset \Lambda$, we have

$$\min_{\hat{A}_N \in \Lambda} \max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}_N) \leq \min_{\hat{A}_N \in \Lambda_1} \max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}_N) \leq P \cdot \omega_N^2. \quad (8)$$

Lower bound. Using the canonical decompositions of components of regular stationary sequence (2) and components of optimal linear estimation (3), we obtain

$$\min_{\hat{A} \in \Lambda} \Delta(\zeta, \hat{A}) = E \left| \sum_{k=1}^{\infty} \sum_{j=0}^N a_{kj} (\zeta_{kj} - \hat{\zeta}_{kj}) \right|^2 =$$

$$= \sum_{k, n=1}^{\infty} \sum_{m=1}^M \sum_{p, q=0}^N g_{km}(p) \overline{g_{nm}(q)} Q_{kn}^N(p, q), \quad (9)$$

where operator $Q_N = \{Q_N(p, q)\}_{p, q=0}^N$ in the space ℓ_2 determined by block-matrices $Q_N(p, q) = \{Q_{kn}^N(p, q)\}_{k, n=1}^{\infty}$ with elements (7). It can be represented in the form $Q_N = A_N A_N^*$, where the matrix operator $A_N = \{A_N(p, q)\}_{p, q=0}^{\infty}$ is determined by the vector columns $A_N(p, q) = \tilde{a}_{p+q}$, $p+q \leq N$, and $A_N(p, q) = \vec{0}$, $p+q > N$. The squared absolute norm of operator A_N equals $\mathcal{N}^2(A_N) = \sum_{p=0}^{\infty} (p+1) \|\tilde{a}_p\|^2 < \infty$, because of (6). Then operator A_N is continuous [14]. The operator Q_N is continuous and has real nonnegative eigenvalues.

Denote by $g_N = (g(0), g(1), \dots, g(N))^T$ the vector columns with matrix elements

$$g(p) = \{g_{km}(p)\}_{k=1, \infty}^{m=1, M}, p = 0, 1, \dots, N.$$

Operator Q_N acts in the vector g_N as follows

$$Q_N g_N = \left\{ \sum_{q=0}^N Q_N(p, q) g(q) \right\}_{p=0}^N.$$

The relation (9) has the form

$$\min_{\hat{A}_N \in \Lambda} \Delta(\zeta, \hat{A}_N) = \sum_{p=0}^N \sum_{q=0}^N (Q_N(p, q) \overline{g(q)}, \overline{g(p)}) =$$

$$= (Q_N \overline{g_N}, \overline{g_N}),$$

where (\cdot, \cdot) is the scalar product in ℓ_2 .

Let us denote $\tilde{g} = \overline{g_N} P^{-1/2}$. Then the extremal problem

$$|(Q_N \tilde{g}, \tilde{g})| \rightarrow \max, \|\tilde{g}\| = 1$$

has solutions [15]. This solution is the eigenvalue of the operator Q_N . The appropriate eigenvalue is $\nu_N^2 = \max_{\|\tilde{g}\| \leq 1} |(Q_N \tilde{g}, \tilde{g})| = \|Q_N\|$. Thus we have

$$\max_{\zeta \in \mathbf{Y}_R} \min_{\hat{A}_N \in \Lambda} \Delta(\zeta, \hat{A}_N) = P \cdot \max_{\|\tilde{g}\| \leq 1} |(Q_N \tilde{g}, \tilde{g})| =$$

$$= P \cdot \nu_N^2 \leq \max_{\zeta \in \mathbf{Y}} \min_{\hat{A}_N \in \Lambda} \Delta(\zeta, \hat{A}_N). \quad (10)$$

Let us denote $D_N(N-p, N-q) = Q_N(p, q)$. Then $\omega_N^2 = \nu_N^2$. Relations (8) and (10) give us the inequality

$$\min_{\hat{A}_N \in \Lambda} \max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}_N) \leq P \cdot \nu_N^2 \leq \max_{\zeta \in \mathbf{Y}} \min_{\hat{A}_N \in \Lambda} \Delta(\zeta, \hat{A}_N). \quad (11)$$

Since the opposite inequality always holds true, we have equality in (11).

Theorem 2. Let the coefficients $\{\tilde{a}_j, j = 0, 1, \dots\}$ satisfy conditions (5). Then the function $\Delta(\zeta, \hat{A})$ has a saddle point on the set $\mathbf{Y} \times \Lambda$ and the following equality holds true

$$\min_{\hat{A} \in \Lambda} \max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}) = \max_{\zeta \in \mathbf{Y}} \min_{\hat{A} \in \Lambda} \Delta(\zeta, \hat{A}) = P \cdot \nu^2, \quad (12)$$

where ν^2 is the greatest eigenvalue of the self-adjoint compact operator $Q = \{Q(p, q)\}_{p, q=0}^{\infty}$ in the space ℓ_2 determined by block-matrices $Q(p, q) = \{Q_{kn}(p, q)\}_{k, n=1}^{\infty}$ with elements

$$Q_{kn}(p, q) = \sum_{s=0}^{\infty} a_{k, s+p} \cdot \overline{a_{n, s+q}},$$

$$k, n = 1, 2, 3, \dots, \quad p, q = 0, 1, \dots$$

The least favorable stochastic sequence in the class \mathbf{Y} for the optimal estimate of the functional $A\zeta$ is a one-sided moving average sequence of the form

$$\vec{\zeta}_j = \sum_{s=-\infty}^j g(j-s) \vec{\varepsilon}(s),$$

where $g = (g(p))_{p=0}^{\infty}$ is the eigenvector, that corresponds to ν^2 , is constructed from matrices $g(s) = \{g_{km}(s)\}_{k=1, \infty}^{m=1, M}$ and is determined by the condition $\|g\|^2 = \sum_{p=0}^{\infty} \|g(p)\|^2 = P$, $\vec{\varepsilon}(s) = \{\varepsilon_m(s)\}_{m=1}^M$ is a vector stationary stochastic sequence with orthogonal values.

Proof: Upper bound. Let us consider the norm approximation of the self-adjoint continuous [14] operator $Q = AA^*$, $A = \{A(p, q)\}_{p, q=0}^{\infty} = \{\tilde{a}_{p+q}\}_{p, q=0}^{\infty}$ in ℓ_2 by the sequence of operators $Q_N = A_N A_N^*$ from Theorem 1. Then we have

$$\min_{\hat{A} \in \Lambda} \max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}) = \lim_{N \rightarrow \infty} (\min_{\hat{A}_N \in \Lambda} \max_{\zeta \in \mathbf{Y}} \Delta(\zeta, \hat{A}_N)) =$$

$$= P \lim_{N \rightarrow \infty} \nu_N^2 = P \cdot \nu^2. \quad (13)$$

Lower bound. Solution of the extremal problem

$$|(Q \tilde{g}, \tilde{g})| \rightarrow \max, \|\tilde{g}\| = 1$$

is an eigenvector, that corresponds to the greatest eigenvalue [15]

$$\nu^2 = \max_{\|\tilde{g}\| \leq 1} |(Q \tilde{g}, \tilde{g})| = \|Q\|.$$

Thus, the following inequalities hold true

$$\max_{\zeta \in \mathbf{Y}_R} \min_{\hat{A} \in \Lambda} \Delta(\zeta, \hat{A}) = P \cdot \max_{\|\tilde{g}\| \leq 1} |(Q \tilde{g}, \tilde{g})| =$$

$$= P \cdot \nu^2 \leq \max_{\zeta \in \mathbf{Y}} \min_{\hat{A} \in \Lambda} \Delta(\zeta, \hat{A}). \quad (14)$$

From relations (13) and (14) we have only possible equality (12). Theorem is proved.

4 Conclusions

Formulas for calculating the maximum values of the mean square errors of optimal linear estimation of functionals $A\zeta$ and $A_N\zeta$ that depend on the unknown values of periodically correlated stochastic process $\zeta(t)$ from the class \mathbf{Y} are proposed. The estimation is based on observations of the process $\zeta(t)$ for $t < 0$. It is shown that the maximum value of the error for $A\zeta$ gives the one-sided moving average sequence.

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