# On the Identifiability of Tensor Ranks via Prior Predictive Matching

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Selecting the latent dimensions (ranks) in tensor factorization is a central challenge that often relies on heuristic methods. This paper introduces a rigorous approach to determine rank identifiability in probabilistic tensor models, based on prior predictive moment matching. We transform a set of moment matching conditions into a log-linear system of equations in terms of marginal moments, prior hyperparameters, and ranks; establishing an equivalence between rank identifiability and the solvability of such system. We apply this framework to four foundational tensor-models, demonstrating that the linear structure of the PARAFAC/CP model, the chain structure of the Tensor Train model, and the closed-loop structure of the Tensor Ring model yield solvable systems, making their ranks identifiable. In contrast, we prove that the symmetric topology of the Tucker model leads to an underdetermined system, rendering the ranks unidentifiable by this method. For the identifiable models, we derive explicit closed-form rank estimators based on the moments of observed data only. We empirically validate these estimators and evaluate the robustness of the proposal.

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## I. INTRODUCTION

Tensor factorizations have become indispensable tools for analyzing multi-way data across diverse contexts and fields such as neuroscience [9, 17], bioinformatics and genomics [1], chemometrics [3], computational social science [22], network science [10], recommender systems [2, 12], supervised learning [23], and compression of large neural networks [18]. These methods decompose a high-dimensional tensor into a set of lower-dimensional factors, revealing latent structures and facilitating data interpretation and prediction [6, 7, 16].

A fundamental challenge in the design and aplication of tensor models is the selection of their latent dimensions, i.e., the **ranks**  $(r_p)$ , which dictate the model's complexity and its ability to capture the observed statistics of the data, without overfitting. In contrast to simple matrix factorizations, most tensor models have multiple ranks, i.e.,  $p \geq 1$ , depending on the specific form of the factorization. The determination of the tensor ranks is, in general, a NP-Complete problem for tensors over  $\mathbb{Q}$  [14], NP-Hard for tensors over  $\mathbb{R}$  [15] and hard to approximate [24].

In practice, Bayesian approaches to tensor factorization [6, 21, 27] provide a principled alternative to exhaustive search, enabling uncertainty quantification, prior incorporation, and even automatic rank inference via nonparametric methods [20] or automatic relevance determination [27]. Yet, parametric models often demand userspecified ranks or strong priors, complicating deployment in real-world scenarios where domain knowledge is scarce.

For matrix factorization models, the prior specification hurdle can be mitigated through prior predictive checks or empirical Bayes techniques [11, 26]. That is, we can determine the ideal rank by comparing observed data with data simulated from the model under alternative ranks. This established technique for matrix factorization inspires an analogous development for tensor factorization.

Given this context, the fundamental question we ask here is when can a prior predictive moment matching approach identify the ranks of tensor factorization models. To answer the question, we need to establish a range of tools for connecting the moments of the prior predictive distribution with the ranks of the tensor model and an approach to use them for characterizing the rank identifiability of the model.

In this work, we formulate a strategy for analyzing the identifiability of ranks in probabilistic tensor factorization models, by analyzing a log-linear system of equations derived from matching model-specific prior predictive moments with low-order moments of the observed tensor-data. We show that the algebraic structure of this system, which is determined by the chosen tensor decomposition, dictates whether the ranks can be uniquely identified by the method of moment-matching. We formalize this analysis by transforming the multiplicative moment equations into a log-linear system, where identifiability is equivalent to the solvability of the system with respect to the unknown ranks. Notably, we provide

a clear theoretical characterization and distinction between different tensor models, demonstrating that rank identifiability is a direct consequence of the interaction topology of their latent factors. Besides, we showcase how ranks can be empirically estimated from observed tensor-data according to the prior predictive matching framework.

Our specific contributions are:

- 1. We establish a general, prior predictive based framework for analyzing rank identifiability in probabilistic tensor models, by examining the rank of a log-linear system of moment equations.
- 2. We apply this framework to four foundational tensor decompositions: **Tucker** [25],

PARAFAC/CP [5, 13], Tensor Train [TT, 19], and Tensor Ring [TR, 28].

- 3. We prove that the ranks of the standard Tucker model are not identifiable with this method due to symmetries in its moment structure leading to a degenerate, unsolvable system of equations.
- 4. We prove that the ranks of the PARAFAC/CP, TT and TR models are identifiable from the first and second moments and derive explicit closed-form estimators for them.
- 5. We present and evaluate a complete and robust pipeline for the estimation of ranks from observed tensor-data in identifiable models.

# II. A GENERAL FRAMEWORK FOR RANK IDENTIFIABILITY

We hereby introduce the notation and assumptions to establish a general framework for rank identifiability in probabilistic tensor factorization models.

Notation. Let  $Y \in \mathbb{R}^{N_1 \times \cdots \times N_M}$  be an observed tensor with M modes. A mode of an order-M tensor is one of its M dimensions. Let  $\mathbf{i}_{\alpha} = (i_m)_{m \in [M]} \in \mathop{\textstyle \times}_{m=1}^M [N_m]$  be the multi-index from the observed tensor domain, where  $[N_m] = \{1, \ldots, N_m\}$ . Let  $\mathbf{r} = (r_p)_{p \in [R]}$  be the multi-rank vector of size  $R \leq M$ . Let  $\mathbf{i}_{\beta} = (i_1, \ldots, i_R) \in \mathop{\textstyle \times}_{p=1}^R [r_p]$  be the multi-index of the product of rank domains  $[r_p]$ . For multi-indices  $\mathbf{i}, \mathbf{j}$ , the shared-mode set is  $S(\mathbf{i}, \mathbf{j}) = \{k : i_k = j_k\}$ . We use bold sans-serif for tensors  $(\boldsymbol{\eta})$  and vectors/matrices  $(\boldsymbol{\theta}^{(p)})$ . All latent factor multi-indices  $\mathbf{i}_p(\alpha,\beta)$  are formed by combining some components of  $\mathbf{i}_{\alpha}$  and some components of  $\mathbf{i}_{\beta}$ , and each factor is probabilistically defined as  $\theta_{\mathbf{i}_p(\alpha,\beta)}^{(p)} \sim \pi_p(\mu_p,\sigma_p^2)$ , i.i.d. across indices with p indexing and grouping the factors.

**Tensor diagram.** We use compact tensor networks to visualize factorizations. Each node is a tensor (core/factor); solid legs are indices. A single dangling leg denotes a physical mode (observed dimension); edges connecting nodes are latent "bond" indices whose sizes are the ranks.

TABLE I: Summary of identifiability and closed-form estimators from second moments. All  $v_S$  are pure interaction terms obtained via exact-sharing covariances and Möbius inversion;  $E[Y] = \mathbb{E}[Y]$ .

Model	Identifiable?	Rank parameters	s Closed-form from observables			
Tucker	No (from 2nd moments)	, , , , ,	Symmetry induces $M$ identities $\log v_{\{p\}} - 2\log \mathbb{E}[Y] = \log v_{\{G,p\}} - \log v_{\{G\}}$ ; the reduced log-rank system is degenerate (non-identifiable).			
PARAFAC/CP	Yes	$r \ (r_1,\ldots,r_{M-1})$	For any $p \neq q$ : $r = \frac{v_{\{p,q\}} (\mathbb{E}[Y])^2}{v_{\{p\}} v_{\{q\}}}$ .			
Tensor Train (TT)	Yes	$(r_1,\ldots,r_{M-1})$				
			$\begin{cases} r_p = \frac{v_{\{p,p+1\}} \ v_{\{p-1,p+2\}}}{v_{\{p+1\}} \ v_{\{p-1,p,p+2\}}} & \text{Interior } 1  3 \end{cases}$			
Tensor Ring (TR)	Yes	$(r_1,\ldots,r_M)$	Define $\xi_p = \frac{(\mathbb{E}[Y])^2 v_{\{p,p+1\}}}{v_{\{p\}} v_{\{p+1\}}} = r_p \frac{r_{p-1}}{r_{p+1}}$ . Solve $x_p + x_{p-1} - x_{p+1} = \psi_p$ with $x_p = \log r_p$ , $\psi_p = \log \xi_p$ ; circulant system is invertible for $M \ge 3$ .			

Chains of nodes depict TT; a cycle depicts TR; a starlike spoke pattern with a central core depicts Tucker; CP is a sum of rank-1 outer products (often drawn as r parallel spoke motifs). Diagrammatic equalities correspond to index contractions; our counting arguments track how many independent latent sums (i.e., free edges) remain under each exact-sharing pattern, which become exponents of the ranks in the monomials.

**Definition II.1** (The Probabilistic Tensor-Model). We assume that elements in  $\mathbf{Y} \in \mathbb{R}^{N_1 \times \cdots \times N_M}$  are probabilistically generated from a rate tensor  $\boldsymbol{\eta} \in \mathbb{R}^{N_1 \times \cdots \times N_M}$ , whose structure is defined by a (model-dependent) tensor decomposition with latent factors of ranks  $\mathbf{r} = (r_1, \dots, r_R)$  of the following generic form:

$$\eta_{\mathbf{i}_{\alpha}} = \sum_{\mathbf{i}_{\beta}} \prod_{p} \theta_{\mathbf{i}_{p}(\alpha,\beta)}^{(p)}, \tag{1}$$

with latent indices defined by  $\mathbf{i}_p(\alpha, \beta)$  and parameters  $\theta_{\mathbf{i}_{\{\alpha,\beta\}}}^{(p)} \sim \pi_p(\mu_p, \sigma_p^2)$ , drawn independently from a factor p-specific location-scale prior distribution  $\pi_p(\cdot)$  with upto-second order moments  $(\mu_p, \sigma_p^2)$ .

Tensor observations  $Y_i$  are conditionally independent given the rate tensor  $\eta$ , with conditional moments  $\mathbb{E}[Y_i|\eta] = \eta_i$  and  $\operatorname{Var}(Y_i|\eta) = \phi(\eta_i)$  for some model-dependent positive function  $\phi(\cdot)$ .

This formulation allows for different observation models (e.g. Poisson with  $\phi(\eta) = \eta$ , or Gaussian with  $\phi(\eta) = \sigma_Y^2$  variance), and it generalizes existing matrix factorization frameworks [11], setting the stage for the prior predictive moment matching analysis. Commonly used factorization models such as Tucker, PARAFAC/CP, TR and TT, are all instances of this general formulation [29].

Observable data and moments of interest. The second moments of the observable tensor-data can be described by two key quantities: the total covariance and pure interaction terms, which form the key building blocks for the theoretical analysis that follows.

**Definition II.2** (Total Covariance and Pure Interaction Terms). Let  $\mathbf{i} = (i_1, \dots, i_M)$  and  $\mathbf{j} = (j_1, \dots, j_M)$  be two multi-indices from the tensor's domain. Let  $S(\mathbf{i}, \mathbf{j}) = \{k \mid i_k = j_k\}$  be the set of indices they share. The **total observable covariance**,  $C_S$ , is the theoretical covariance between two multi-index tensor entries  $\mathbf{i}$  and  $\mathbf{j}$ :

$$C_S = Cov(Y_i, Y_j)$$
.

Given a factorization structure, this value depends only on the set of shared indices  $S(\mathbf{i}, \mathbf{j})$ . For entries sharing no indices, we assume  $C_{\emptyset} = 0$ .

The pure interaction term,  $v_S$ , represents the variance arising uniquely from the interaction of factors in  $S(\mathbf{i}, \mathbf{j})$ , and induces the definition of the total covariance in terms of their additive contribution over the non-empty subsets of  $S(\mathbf{i}, \mathbf{j})$ :

$$C_S = \sum_{\emptyset \neq S' \subset S} v_{S'} .$$

These pure interaction terms can be also defined via the principle of inclusion-exclusion from the observable covariances:

$$v_S = \sum_{S' \subset S} (-1)^{|S| - |S'|} C_{S'}.$$

Connecting theorical and observable moments. For many tensor decomposition models, the theoretical moments of the rate tensor  $\eta$  can be expressed as polynomials: i.e., as products of the unknown parameters (ranks and prior moments).

**Proposition II.3** (Polynomial Structure of Rate Moments). For a probabilistic tensor model congruent with Definition II.1, every pure interaction term  $v_S$  is a monomial in the unknown model hyperparameters  $(r_p, \mu_p, \sigma_p^2)$ . Therefore, the marginal mean  $\mathbb{E}[\eta]$ , the variance  $\operatorname{Var}(\eta)$ , and the covariance terms  $\operatorname{Cov}(\eta_i, \eta_j)$  can be expressed as polynomials of  $\{r_p, \mu_p, \sigma_p^2\}$ .

*Proof.* Each rate element  $\eta_{\mathbf{i}_{\alpha}}$  is a sum-product of the latent factor elements  $\{\theta_{\mathbf{i}_{\{\alpha,\beta\}}}^{(p)}\}$ . Applying total expectation we obtain  $\mathbb{E}[\mathbf{Y}_{\mathbf{i}_{\alpha}}] = \sum_{i_{\beta}} \prod_{p:i_{p}(\alpha,\beta)} \mu_{p}$ , with the sum-

mation index depending on the rank dimensions, and resulting in the monomial given by products of ranks  $\{r_p\}$  and means  $\{\mu_p\}$ . Applying total covariance and conditional independence, the computation leads to moments  $\mathbb{E}[\eta_{\mathbf{i}_{\alpha}}\eta_{\mathbf{j}_{\alpha}}]$ , which resolve into a double sum-product over latent indexes, with  $\sigma_p^2$  appearing for factors with the same latent index,  $\mu_p^2$  for independent factors, and powers of  $r_p$  from interactions of repeated sums and multiplications, leading to a polynomial on prior parameters  $\{\mu_p, \sigma_p^2\}$  and ranks  $\{r_p\}$ .

# A. The Log-Linear System of Prior Predictive Moment Equations

The insight established by Proposition II.3 is that observable moments (like covariances  $C_S$ ) are polynomials of the model parameters. More precisely, the individual terms of these polynomials are the **pure interaction terms**  $v_S$ , which are **monomials**. This monomial (multiplicative) structure is ideal for linearization. Namely, by taking the logarithm of the equation for each pure interaction term, we transform the system of prior predictive moments into a linear one, amenable to standard algebraic analysis.

**Definition II.4** (The Log-Linear System). The log-linear system derived from the monomial equations of pure interaction terms is given by  $\mathbf{A}\mathbf{x} = \mathbf{b}$ , where  $\mathbf{x}$  is the vector of the logarithms of the unknown parameters, i.e.,  $\log(r_p), \log(\mu_p^2), \log(\sigma_p^2)$ ;  $\mathbf{b}$  is the vector of the logarithms of the (estimable) moment terms, i.e.,  $\log(E[Y]), \log(v_S)$ ; and  $\mathbf{A}$  is the integer-valued **design matrix**, where each entry  $A_{ij}$  is the exponent of the j-th unknown parameter in the monomial expression for the i-th moment term.

While the invertibility of the full matrix  $\mathbf{A}$  determines if all parameters are identifiable, our primary interest is in the ranks of the tensors. The identifiability of the ranks hinges on whether they can be isolated from the other unknown parameters. This leads to a more precise, and relevant, principle of rank identifiability:

**Proposition II.5** (Principle of Rank Identifiability). The ranks  $\{r_p\}$  of a tensor model are identifiable from the second moments of tensor-data if and only if the unknown prior-moment parameters can be algebraically eliminated

from the full log-linear system of moments  $\mathbf{A}\mathbf{x} = \mathbf{b}$ , to yield a non-trivial **reduced system** of the log-ranks  $\mathbf{x}_r$  alone: i.e.,  $\mathbf{A}_{red}\mathbf{x}_r = \mathbf{b}_{red}$ , where the reduced design matrix  $\mathbf{A}_{red}$  has full rank. A full-rank  $\mathbf{A}_{red}$  guarantees a unique solution for the ranks, meaning they occupy a parameter subspace that can be uniquely determined from the data's moments.

Propositions II.3 and II.5 establish a direct link between the model-dependent, prior predictive moments and the moments of the observed tensor-data. This connection is formalized through an identifiability analysis of a reduced, rank-specific log-linear system. However, the utility of this link for rank estimation is fully model-dependent.

We first present in Section III a cautionary example where the dependencies between prior hyperparameters and observed second moments are not identifiable for a popular tensor model. In contrast, we highlight in Section IV several models where identifiability holds, enabling the design and implementation of a novel method for estimating tensor ranks directly and exclusively from observed moments.

# III. A CAUTIONARY TALE: THE TUCKER MODEL

We now examine the standard Tucker decomposition and show that its symmetric "hub-and-spoke" [4] interaction topology precludes forming a solvable reduced system of equations for the Tucker model's ranks.

# A. Model Definition and Its Moments

**Definition III.1** (The Tucker model). For an order-M tensor, with ranks  $\mathbf{r} = (r_1, \ldots, r_M)$  the rate  $\boldsymbol{\eta}$  is defined by the Tucker decomposition as

$$\eta_{i_1...i_M} = \sum_{k_1=1}^{r_1} \cdots \sum_{k_M=1}^{r_M} G_{k_1...k_M} \prod_{p=1}^M \theta_{i_p,k_p}^{(p)}.$$

This model, introduced by Tucker [25], is widely used for its interpretability and versatibility in multi-way analysis.

The ranks to be estimated  $(r_1, \ldots, r_M)$  are the dimensions of the core tensor G. As per Definition II.1 the elements of G and each factor  $\theta^{(p)}$  are i.i.d. random variables with moments  $(\mu_G, \sigma_G^2)$  and  $(\mu_p, \sigma_p^2)$ , respectively.

**Lemma III.2** (Moment Structure of the Tucker Model). For the order-M Tucker model, the squared mean and the pure interaction terms involving the core  $(v_G)$ , a single

factor  $(v_p)$ , or both  $(v_{G,p})$  obey:

$$\begin{split} (E[Y])^2 &= \mu_G^2 \prod_{q=1}^M (r_q^2 \mu_q^2) \;, \\ v_G &= \sigma_G^2 \prod_{q=1}^M (r_q^2 \mu_q^2) \;, \\ v_p &= \mu_G^2 (r_p \sigma_p^2) \prod_{q \neq p} (r_q^2 \mu_q^2) \;, \\ v_{G,p} &= \sigma_G^2 (r_p \sigma_p^2) \prod_{q \neq p} (r_q^2 \mu_q^2) \;. \end{split}$$

## B. Rank Identifiability Analysis

We now prove that the symmetric topology of the Tucker model leads to a degeneracy in its moment equations by showing that the reduced design matrix for the ranks is the zero matrix. Hence, this structure makes the ranks fundamentally unrecoverable from observed tensordata.

**Lemma III.3** (An Identity Among Observables). In the log-linear system for the Tucker model moments, the following M independent relations hold for every mode p = 1, ..., M:

$$\log v_p - 2\log E[Y] = \log v_{G,p} - \log v_G. \tag{2}$$

*Proof.* We show that both sides of the equation reduce to the same expression. Using the monomials from Lemma III.2, the left-hand side is:

$$\begin{split} \log v_p - 2\log E[Y] &= \log \left[ \mu_G^2 r_p \sigma_p^2 \prod_{q \neq p} (r_q^2 \mu_q^2) \right] \\ &- 2\log \left[ \mu_G \prod_{q=1}^M (r_q \mu_q) \right] \\ &= \log \left( \frac{\mu_G^2 r_p \sigma_p^2 \prod_{q \neq p} (r_q^2 \mu_q^2)}{\mu_G^2 \prod_{q=1}^M (r_q^2 \mu_q^2)} \right) \\ &= \log \left( \frac{1}{r_p} \frac{\sigma_p^2}{\mu_p^2} \right) \; . \end{split}$$

An identical calculation shows that  $\log v_{G,p} - \log v_G$  also simplifies to  $\log \left(\frac{1}{r_p} \frac{\sigma_p^2}{\mu_p^2}\right)$ , demonstrating the identity in Equation 2.

**Theorem III.4** (Non-identifiability of Tucker ranks). The Tucker model ranks  $(r_1, ..., r_M)$  are **not identifiable** directly from the second moments of the tensor-data.

*Proof.* We prove that the reduced design matrix  $\mathbf{A}_{\text{red}}$  is the zero matrix. The identity established in Lemma III.3 provides M linear dependencies among the rows of the

full design matrix  $\mathbf{A}$ . Because these M identities express every standard-basis vector  $\mathbf{e}_p$  (the column that selects  $\log r_p$ ) as a linear combination of columns that are free of any log-rank variable, the M columns corresponding to the log-ranks are redundant. Consequently, the reduced design matrix  $\mathbf{A}_{\text{red}}$  of Proposition II.5 is the  $0 \times M$  zero matrix, and the ranks are unidentifiable.

# IV. EXEMPLARY TALES: IDENTIFIABLE RANKS WITH CLOSED-FORM EXPRESSION

In contrast to the previous section, we present in the following the general solutions for the ranks of the PARAFAC/CP, Tensor Train, and Tensor Ring models for an arbitrary order M tensor, showcasing how to establish identifiability according to the principles of Proposition II.5.

# A. PARAFAC/CP Model

The PARAFAC or Canonical Polyadic (CP) model's rank is identifiable due to a unique property of its second moments: the additivity of its variance components. Precisely, its total observable covariance decomposes into a clean sum of pure, monomial interaction terms, because the model is defined as a sum of r independent rank-one tensors. This structure provides the necessary constraints for a solvable rank system.

# 1. Model Definition and Its Moments

**Definition IV.1** (The PARAFAC/CP model). For an order-M tensor, the rate  $\eta$  of a PARAFAC/CP decomposition of rank r follows:

$$\eta_{\mathbf{i}} = \eta_{i_1...i_M} = \sum_{k=1}^r \prod_{p=1}^M \theta_{i_p,k}^{(p)}.$$

The PARAFAC/CP decomposition, originally proposed by Harshman et al. [13] and Carroll and Chang [5], assumes a sum of rank-1 tensors and is widely used in multiple fields given its simple structure and applicability. The variance calculation of the CP model results in a simple and regular form for its pure interaction terms, as formalized below, a fundamental property leading to a closed-form expression for the rank r.

**Lemma IV.2** (Monomial Moment structure of PARAFAC/CP). For an order-M PARAFAC/CP model with rank r and factor moments  $(\mu_k, \sigma_k^2)$ , the squared

mean and the pure-interaction terms obey

$$\begin{split} (E[Y])^2 &= r^2 \prod_{k=1}^M \mu_k^2 \;, \\ v_{\{p\}} &= r \sigma_p^2 \prod_{k \neq p} \mu_k^2 \;, \\ v_{\{p,q\}} &= r \sigma_p^2 \sigma_q^2 \prod_{k \neq p,q} \mu_k^2 \qquad (p \neq q) \;. \end{split}$$

*Proof.* The rate is  $\eta_{\mathbf{i}} = \sum_{s=1}^{r} \prod_{k=1}^{M} \theta_{i_k,s}^{(k)}$ . Because all factor entries across different components s are independent, the variance of the sum is the sum of the variances. The pure interaction term  $v_S$  is the component of the total variance arising from the variance of factors in the set S and the mean of factors not in S. For  $v_{\{p\}}$ , this is:

$$\begin{split} v_{\{p\}} &= \mathrm{Var}_{\mathrm{pure}}(\eta_{\mathbf{i}} \mid \mathrm{mode}\ p) \\ &= \sum_{s=1}^{r} \mathrm{Var}(\theta_{i_{p},s}^{(p)}) \prod_{k \neq p} \left( E[\theta_{i_{k},s}^{(k)}] \right)^{2} \\ &= r \sigma_{p}^{2} \prod_{k \neq p} \mu_{k}^{2} \ . \end{split}$$

The derivations for other pure terms follow the same principle of summing variances over r independent components.

# 2. Rank Identifiability Analysis

The simple monomial structure of the moments leads directly to a closed-form solution for the rank.

**Theorem IV.3** (Closed-Form Rank Estimator for PARAFAC/CP). For any two distinct modes  $p \neq q$ , the rank r is identifiable and given by:

$$r = \frac{v_{\{p,q\}} \cdot (E[Y])^2}{v_{\{p\}} \cdot v_{\{q\}}} \ . \tag{3}$$

Proof. We substitute the monomials from Lemma IV.2 into the expression:

$$\begin{split} \frac{v_{\{p,q\}}\cdot(E[Y])^2}{v_{\{p\}}\cdot v_{\{q\}}} &= \frac{\left[r\sigma_p^2\sigma_q^2\prod_{k\neq p,q}\mu_k^2\right]\cdot\left[r^2\prod_{k=1}^M\mu_k^2\right]}{\left[r\sigma_p^2\prod_{k\neq p}\mu_k^2\right]\cdot\left[r\sigma_q^2\prod_{k\neq q}\mu_k^2\right]} \\ &= \frac{r^3\left(\prod_{k\neq p,q}\mu_k^2\right)\left(\mu_p^2\mu_q^2\prod_{k\neq p,q}\mu_k^2\right)}{r^2\left(\mu_q^2\prod_{k\neq p,q}\mu_k^2\right)\cdot\left(\mu_p^2\prod_{k\neq p,q}\mu_k^2\right)} \\ &= r. \end{split}$$

Every nuisance parameter cancels, yielding the rank itself. Hence, r is identifiable and the formula is exact.  $\square$ 

## B. The Tensor Train Model

The Tensor Train (TT) model's topology is an open, asymmetric chain of interactions [19]. This sequential structure breaks the symmetries that cause non-identifiability in the Tucker model, providing a sufficient number of distinct algebraic constraints to solve for all of the model's internal ranks.

## 1. Model Definition and Its Moments

**Definition IV.4** (The Tensor-Train model). For an order-M tensor, the rate  $\eta$  of a Tensor Train decomposition with TT-ranks  $(r_1, \ldots, r_{M-1})$  follows

$$\eta_{i_1\dots i_M} = \sum_{k_1=1}^{r_1} \cdots \sum_{k_{M-1}=1}^{r_{M-1}} \theta_{1,i_1,k_1}^{(1)} \theta_{k_1,i_2,k_2}^{(2)} \cdots \theta_{k_{M-1},i_M,1}^{(M)}.$$

The ranks to be estimated are the internal bond dimensions  $(r_1, \ldots, r_{M-1})$ , which we refer to as *interior ranks*. A full derivation of the general TT moments involves a series of (repeted) algebraic calculations, which we present in Appendix A 3.

However, their algebraic structure contains the necessary asymmetry for rank identification. The key is that pure interaction terms involving non-contiguous modes (e.g.,  $\{p-1,p+2\}$ ) have a different dependence on the ranks than terms involving contiguous modes (e.g.,  $\{p,p+1\}$ ).

**Lemma IV.5** (Monomial Moment Structure of Tensor-Train). The pure-interaction monomials for the TT model are functions of the ranks and prior moments. Their exact form depends on the chosen modes in S, but they all exhibit a crucial asymmetry. For an interior rank  $r_p$ , the relevant monomials for its identification are:

$$\begin{split} v_{p+1} &= r_p r_{p+1} r_{p+2} \mu_p^2 \sigma_{p+1}^2 \mu_{p+2}^2 \prod_{k \neq p, p+1, p+2} \mu_k^2 \;, \\ v_{p,p+1} &= r_p r_{p+1} r_{p+2} \sigma_p^2 \sigma_{p+1}^2 \mu_{p+2}^2 \prod_{k \neq p, p+1, p+2} \mu_k^2 \;, \\ v_{p-1,p+2} &= r_{p-1} r_p r_{p+1} r_{p+2} \sigma_{p-1}^2 \mu_p^2 \sigma_{p+2}^2 \prod_{k \neq p-1, p, p+2} \mu_k^2 \;, \end{split}$$

$$v_{p-1,p,p+2} = r_{p-1}r_pr_{p+1}r_{p+2}\sigma_{p-1}^2\sigma_p^2\sigma_{p+2}^2\prod_{k\neq p-1,p,p+2}\mu_k^2$$

Notably, the following identities hold:

$$\frac{v_{\{p,p+1\}}}{v_{\{p+1\}}} = \frac{\sigma_p^2}{\mu_p^2} \,, \tag{4}$$

$$\frac{v_{\{p-1,p,p+2\}}}{v_{\{p-1,p+2\}}} = \frac{1}{r_p} \frac{\sigma_p^2}{\mu_p^2} \,. \tag{5}$$

Informal Proof. The TT rate is a sum over products of core tensor entries. Each pure-interaction term is obtained by calculating the covariance, which involves a double summation over all internal rank indices. The exponents of the ranks in the final monomial count how many of these summations remain independent after constraints from the interaction set are applied. The chain-like structure ensures that non-contiguous interactions constrain the summations differently than contiguous ones, producing the necessary asymmetry —see Appendix A 3 for detailed derivations.

## 2. Rank Identifiability Analysis

The asymmetry of the moment structure allows us to construct a solvable system for each interior rank.

**Theorem IV.6** (Closed-Form Interior-Rank Estimator for TT). For any interior rank  $1 , <math>r_p$  is identifiable and given by

$$r_p = \frac{v_{\{p,p+1\}} \cdot v_{\{p-1,p+2\}}}{v_{\{p+1\}} \cdot v_{\{p-1,p,p+2\}}}.$$
 (6)

*Proof.* The proof follows by dividing the two identities from Lemma IV.5. The unknown coefficient of variation term,  $\sigma_p^2/\mu_p^2$ , cancels, resulting in the rank equation:

$$r_p = \frac{v_{\{p,p+1\}}/v_{\{p+1\}}}{v_{\{p-1,p,p+2\}}/v_{\{p-1,p+2\}}} = \frac{v_{\{p,p+1\}} \cdot v_{\{p-1,p+2\}}}{v_{\{p+1\}} \cdot v_{\{p-1,p,p+2\}}}.$$

Since the rank can be expressed purely in terms of observable moments, it is identifiable.  $\Box$ 

## C. The Tensor Ring (TR) Model

The Tensor Ring (TR) decomposition generalizes the Tensor Train by imposing cyclic boundary conditions that connect the first and last latent dimensions, closing the chain into a ring. Formally, for an order-M tensor with TR ranks  $(r_1, \ldots, r_M)$  and indices modulo M, the mean rate tensor is defined as

$$\eta_{i_1\dots i_M} = \operatorname{Tr}\left(\Theta_{i_1}^{(1)}\Theta_{i_2}^{(2)}\cdots\Theta_{i_M}^{(M)}\right),\,$$

where each core matrix  $\Theta_{i_p}^{(p)} \in \mathbb{R}^{r_{p-1} \times r_p}$  and the boundary condition  $r_0 = r_M$  closes the loop. All entries of  $\Theta_{i_p}^{(p)}$  are i.i.d. with moments  $(\mu_p, \sigma_p^2)$ .

## 1. Moment structure

The TR model preserves the factorized moment structure of the TT and Tucker models while introducing new constraints from its cyclic topology. In particular, the first and second prior predictive moments yield the monomials

$$(E[Y])^{2} = \left(\prod_{k=1}^{M} r_{k} \mu_{k}\right)^{2} \tag{7}$$

$$v_{\{p,p+1\}} = \left(\prod_{k=1}^{M} r_k\right) \sigma_p^2 \sigma_{p+1}^2 \prod_{k \notin \{p,p+1\}} \mu_k^2 \tag{8}$$

$$v_{\{p\}} = \left(\prod_{k \notin \{p-1, p\}} r_k^2\right) r_{p-1} r_p^2 \sigma_p^2 \prod_{k \neq p} \mu_k^2, \qquad (9)$$

whose asymmetric exponents  $(r_{p-1}^1 \text{ vs. } r_p^2)$  reflect the directed nature of the incoming and outgoing links at each variance node. A full derivation with explicit double-counting verification appears in Appendix A 4 b.

## 2. Rank Identifiability Analysis

Combining these three observable moment families produces a simple, exactly-cancelling statistic:

$$\xi_p := \frac{(E[Y])^2 v_{\{p,p+1\}}}{v_{\{p\}} v_{\{p+1\}}}, \qquad \Rightarrow \qquad \xi_p = r_p \frac{r_{p-1}}{r_{p+1}}.$$
(10)

This relation—proved in Appendix A 4 b—encodes all rank dependencies implied by the cyclic topology. Taking logarithms,  $x_p = \log r_p$ ,  $\psi_p = \log \xi_p$ , yields the linear circulant system

$$x_p + x_{p-1} - x_{p+1} = \psi_p, \qquad p = 1, \dots, M.$$

Closed-form recovery and identifiability. The coefficient matrix of the above system is circulant with first row  $(1, -1, 0, \ldots, 0, 1)$ . Its discrete Fourier eigenvalues are  $\lambda_k = 1 + 2i\sin(2\pi k/M)$ , all nonzero for  $M \geq 3$ , implying full-rank invertibility and hence identifiability of the TR ranks. The solution has a simple closed form in the Fourier domain:

$$\widehat{x}_k = \widehat{\psi}_k / \lambda_k, \qquad x = DFT^{-1}(\widehat{x}), \qquad r_p = \exp(x_p).$$

Edge cases follow the expected pattern: M=2 reduces to a matrix/CP model, and M=3 remains identifiable (eigenvalues  $1, 1 \pm i\sqrt{3}$ ).

The TR model thus shares the clean identifiability property of the TT model, despite its cyclic symmetry. The key difference is that the ring topology leads to a first-order circulant system whose unique invertibility ensures all ranks  $(r_1, \ldots, r_M)$  are determined directly from first and second moments. All intermediate algebraic steps and the detailed counting argument are given in Appendix A 4 b.

# V. THE TENSOR-RANK ESTIMATION PIPELINE

We now present a complete, algorithmically specified, robust pipeline for applying the theoretical results of this work. It consists of estimating from tensor-data the pure-interaction terms  $\{\hat{v}_S\}$  and the estimated mean  $\hat{E}[Y]$ , to replace them in the specific closed-form expression derived in Section IV for the ranks of each identifiable model. Recall that all rank estimation expressions are written as a division, i.e., the rank  $r_p$  is always calculated via an equation of the  $\frac{\text{Num}_p}{\text{Den}_p}$  form, with  $\text{Num}_p$  the numerator and  $\text{Den}_p$  the denominator, with expressions adjusted for each specific model.

Our tensor-rank estimation procedure is based on (1) computing an unbiased, single-pass estimation of the aforementioned moments; (2) a bootstrap procedure to provide a distribution of empirical (regularized) rank estimates; and (3) a summarization step to yield robust point and uncertainty-aware rank estimates.

Inner loop: The Unbiased Moment Estimation. The fundamental step is to compute unbiased estimates of pure interaction terms  $\{\hat{v}_S\}$  from the observed tensordata  $\mathcal{Y}$ :

- 1. The Global Mean: Compute the empirical global mean of the tensor,  $\hat{E}[Y] = |\mathcal{Y}|^{-1} \sum_{\mathbf{i}} Y_{\mathbf{i}}$ .
- 2. Total Covariances: For every non-empty pattern of shared indices  $S \subseteq \{1, \ldots, M\}$ , compute the sample covariance  $\hat{C}_S$  by drawing a (large) number of pairs of entries  $(\mathbf{a}_j, \mathbf{b}_j)$  that match exactly on the index set S:

$$\hat{C}_S = (N_{\text{cov}} - 1)^{-1} \sum_{j=1}^{N_{\text{cov}}} (Y_{\mathbf{a}_j} - \hat{E}[Y])(Y_{\mathbf{b}_j} - \hat{E}[Y]) .$$

3. Pure Interactions: Calculate the pure interaction terms from the total covariances using the exact inclusion-exclusion principle:

$$\hat{v}_S = \sum_{S' \subseteq S} (-1)^{|S| - |S'|} \hat{C}_{S'} .$$

Outer loop: The Bootstrapped Distribution of Rank Estimates. To ensure robustness and quantify the estimation uncertainty, we repeat the moment estimation procedure above on bootstrap replicates of the observed tensor-data. For  $b = 1, \ldots, B$ :

- 1. A bootstrap sample  $Y^{(b)}$  is generated from the original tensor Y, using a block bootstrap on tensor slices to preserve potential dependencies.
- 2. The complete "Inner loop" estimation procedure is applied to  $\mathbf{Y}^{(b)}$  to obtain a set of bootstrap pure term estimates,  $\{\hat{v}_S\}^{(b)}$ .
- 3. These bootstrap moments are plugged into the appropriate closed-form formula described in Section IV to yield the numerator  $\operatorname{Num}_p^{(b)}$  and denominator  $\operatorname{Den}_p^{(b)}$  for each bootstrapped replicate of the identifiable rank  $r_p^{(b)}$ :
  - PARAFAC/CP (order M, single rank r): for  $p \neq q$ , use Equation 3.
  - TT (order M, ranks  $r_1, \ldots, r_{M-1}$ ): for interior ranks 1 , use Equation 6.

4. Additionally, each bootstrap estimate is regularized to prevent numerical instability. For each replicate b, we propose a regularized rank

$$\hat{r}_p^{(b)} = \operatorname{sign}(\operatorname{Den}_p^{(b)}) \cdot \frac{\operatorname{Num}_p^{(b)}}{|\operatorname{Den}_p^{(b)}| + \varepsilon_p}.$$

The shrinkage term  $\varepsilon_p = 1.96 \cdot \text{SE}(\{\text{Den}_p^{(j)}\}_{j=1}^B)$  is the data-driven standard error of the denominator, estimated from all bootstrap replicates.

The Final Rank Estimate. We provide point rank estimates with bootstraped confidence intervals based on the collection of regularized estimates  $\{\hat{r}_p^{(b)}\}_{b=1}^B$ :

- the collection of regularized estimates  $\{\hat{r}_p^{(b)}\}_{b=1}^B$ :

   Point Estimate: The median is used for its robustness to outliers,  $\hat{r}_p^{\rm med} = \mathrm{median}\{\hat{r}_p^{(b)}\}$ .

   Confidence Interval: The  $(1-\alpha)$  percentile interval in the state of the state of
- Confidence Interval: The  $(1 \alpha)$  percentile interval is computed from the bootstrap distribution:  $[\hat{r}_p^{\text{low}}, \hat{r}_p^{\text{up}}] = \text{percentile}(\{\hat{r}_p^{(b)}\}, [\alpha/2, 1 \alpha/2]).$

## VI. EMPIRICAL VALIDATION

We validate the theoretical findings for the identifiable models using simulation studies, confirming that the closed-form estimators accurately recover the true latent ranks when applied on finite data.

## A. Simulation Setup

Generative Process. We generated synthetic tensordata according to the PARAFAC/CP. The elements of the latent factors  $(\theta^{(p)})$  for both models were drawn independently from Gamma distributions. The shape and scale parameters of the Gamma priors were chosen to control the factor-specific means and variances  $(\mu_p, \sigma_p^2)$ while ensuring the resulting rate tensor  $\eta$  was strictly positive. The final observed tensor elements  $Y_i$  were then drawn from a Poisson distribution with the corresponding rate  $\eta_i$ :

$$Y_{\mathbf{i}} \sim \text{Poisson}(\eta_{\mathbf{i}})$$
.

**Experimental Design.** We study the precision of the constructed pipeline by generating data in a wide range of conditions:

• For PARAFAC/CP, varying the tensor order  $(M \in \{3,4\})$ , the tensor dimensions  $(N_p \in \{(100,100,100),(50,50,50,50)\})$ , and the ground-truth latent ranks  $(r_p \in \{5,\ldots,65\})$ .

For each case, we aggregate results over multiple independent runs, i.e., realizations of both the data and the latents using a set of values for the prior described in Appendix C. For each run, a median rank was estimated from a distribution of bootstrap replicates, and the plots show the mean of these median estimates.

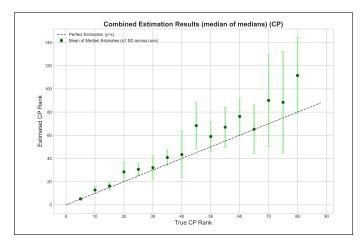


FIG. 1: Combined estimation results for the PARAFAC/CP model. The dashed line represents perfect estimation (y=x). Green dots show the mean of median estimates across multiple runs, with error bars indicating  $\pm 1$  standard deviation.

#### B. Results for Identifiable Models

PARAFAC/CP Model. The results for the PARAFAC/CP model provide strong empirical support for its theoretical identifiability. Figure 1 shows the estimated CP rank versus the true CP rank, aggregated across all simulation runs.

The mean of the estimated ranks (green dots) closely tracks the line of perfect estimation. For true ranks ranging from low values up to 90, the estimator correctly identifies the latent dimension and follows the linear trend. Best performance (both in closer mean and reduced estimation uncertainty) is achieved for reduced (< 25) rank values.

We observe a slight positive bias and an increase in estimation variance (error bars) as the true rank grows. This is expected, as higher-rank structures are more complex, and their corresponding moment-based signatures are more sensitive to the statistical noise inherent in finite data sampling.

Overall, the results confirm that the estimator is effective and that the CP rank is practically identifiable.

Further empirical studies for TT and TR are presented in the Appendix C.

## VII. DISCUSSION

Our prior predictive moment matching framework reveals that rank identifiability in probabilistic tensor models hinges on the latent factors' interaction topology. More precisely, we demonstrate that linear (additive) structures like PARAFAC/CP yield monomial moments with clean algebraic cancellations, enabling closed-form estimators. Chain-like topologies (TT) introduce asym-

metry via contiguous/non-contiguous interactions, providing independent equations per rank that also enable their empirical estimation. The cyclic structure in TR maintains solvability through invertible circulant matrices, despite its symmetry. In contrast, multiplicative "hub-and-spoke" topologies (e.g., Tucker) induce linear dependencies, rendering the reduced system degenerate — we especulate that higher moments or correlations structure in the core tensor could render the models' ranks identifiable.

This topology-based lens offers a principled tensor model analysis toolbox: for identifiable models, low-order moments suffice for exact rank recovery; for others, heuristics like matricization (see Appendix A 2) implicitly reduce to CP-like problems.

Limitations of the presented work include the i.i.d. prior assumption, and the focus on second-order moments only —we hypothesize extensions to higher-order moments or correlated priors could broaden its applicability. Empirically, our pipeline is robust to moderate noise, but sensitive to extreme sparsity which requires further numerical considerations.

A natural next-step is the application of the proposed framework to real-data sets (e.g., on fMRI tensors). More ambitious future work might consider integration of the prior predictive moment matching approach into variational inference for scalable tensor ML, or exploring non-Euclidean topologies (e.g., hyperbolic embeddings). A natural generalization of our framework is to consider generic tensor factorizations defined by arbitrary tensor networks [8]. In this case, the latent ranks would correspond to the bond dimensions along network edges, and identifiability would depend on the network's topology (e.g., tree-like vs. loopy structures).

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## Appendix A: Theoretical Details and Proofs

This appendix provides complete, self-contained derivations that justify all claims used in the main text while preserving the main text's numbering and statements. We (i) clarify the definitions of observable covariances and pure interaction terms; (ii) give a general counting lemma that turns index-sharing patterns into monomial exponents; and (iii) supply full proofs for Tucker, TT, and TR.

## 1. Preliminaries: observation model, covariances, and pure terms

**Notation.** Let  $\mathbf{Y} \in \mathbb{R}^{N_1 \times \cdots \times N_M}$  be an observed tensor with M modes. A mode of an order-M tensor is one of its M dimensions. Let  $\mathbf{i} = (i_m)_{m \in [M]} \in X_{m=1}^M[N_m]$  be the multi-index from the observed tensor domain, where  $[N_m] = \{1, \ldots, N_m\}$ . Let  $\mathbf{r} = (r_p)_{p \in [R]}$  be the multi-rank vector of size  $R \leq M$ . Let the complete multi-index be  $(i_1, \ldots, i_R) \in X_{p=1}^R[r_p]$  for the product of all R latent rank domains  $[r_p]$ , and let  $\mathbf{i}_\beta$  be a derived multi-index for latent domains  $\beta \in \mathcal{K} \subset [R]$  representing a subset  $\mathcal{K}$  of all possible latent dimensions. For multi-indices  $\mathbf{i}, \mathbf{j}$ , the shared-mode set is  $S(\mathbf{i}, \mathbf{j}) = \{k : i_k = j_k\}$ . We use bold sans-serif for tensors  $(\eta)$  and vectors/matrices  $(\boldsymbol{\theta}^{(p)})$ . All latent factor multi-indices  $\mathbf{i}_p(\mathbf{i}, \beta)$  are formed by combining some components of the observed tensor indices  $\mathbf{i}$  and some components latent domains  $\mathbf{i}_\beta$ , and each factor is probabilistically defined as  $\theta_{\mathbf{i}_p(\mathbf{i},\beta)}^{(p)} \sim \pi_p(\mu_p, \sigma_p^2)$ , i.i.d. across indices with  $p\mathcal{P}$  groups the factors/cores. The notation  $\mathbf{i}_p(\mathbf{i},\beta)$  is small change (a explicit dependence on observation tensor multi-index) on the main article notation  $\mathbf{i}_p(\mathbf{o},\beta)$  (section II) for convenience in some calculation in the Appendix. **Observation model.** Throughout, conditional on the rate tensor  $\eta$ , entries are conditionally independent:

$$\mathbb{E}[Y_{\mathbf{i}} \mid \boldsymbol{\eta}] = \eta_{\mathbf{i}}, \quad \operatorname{Var}(Y_{\mathbf{i}} \mid \boldsymbol{\eta}) = \phi(\eta_{\mathbf{i}}),$$

for some nonnegative function  $\phi(\cdot)$ . Hence, for distinct entries the observable cross-covariance equals the covariance of rates:

$$\mathrm{Cov}(Y_{\mathbf{i}},Y_{\mathbf{j}}) = \mathrm{Cov}(\eta_{\mathbf{i}},\eta_{\mathbf{j}}) \quad \text{if } \mathbf{i} \neq \mathbf{j}.$$

Therefore, all results below are agnostic to the choice of  $\phi$  for cross terms and only use  $\mathbb{E}[Y] = \mathbb{E}[\eta]$ .

**Exact-sharing covariances.** For two multi-indices  $\mathbf{i}, \mathbf{j} \in \prod_{m=1}^{M} [N_m]$  and a subset  $S \subseteq [M]$ , we say the pair shares exactly S if  $i_k = j_k$  for  $k \in S$  and  $(i_k, j_k)$  are free (not constrained to be equal) for  $k \notin S$ . The total observable covariance for pattern S is

$$C_S := \mathbb{E}[\text{Cov}(Y_{\mathbf{i}}, Y_{\mathbf{j}} | \mathbf{i}, \mathbf{j} \text{ share exactly } S)],$$

which depends only on S under the iid factor assumptions.

Pure interaction terms and inclusion–exclusion. Define  $v_S$  by the Moebius inversion (pure term) over the lattice of subsets:

$$C_S = \sum_{\emptyset \neq S' \subset S} v_{S'}, \qquad v_S = \sum_{S' \subseteq S} (-1)^{|S| - |S'|} C_{S'}.$$
 (A1)

In all derivations we work with exact-sharing  $C_S$  and pure terms  $v_S$  defined by equation A1.

Augmented interaction sets for Tucker. For the Tucker model we sometimes want to distinguish contributions that involve the core tensor G versus factor variances. For clarity, we allow augmented symbols such as  $v_{\{G\}}$  and  $v_{\{G,p\}}$  to denote the pure contribution that is proportional to  $\sigma_G^2$  (respectively to  $\sigma_G^2 \sigma_p^2$ ) when all physical modes are shared. This does not change the observable definition in equation A1; it is a bookkeeping device that uniquely decomposes  $C_{[M]}$  by variance sources.

a. A counting lemma for rank exponents

All models considered can be written as

$$\eta_{\mathbf{i}} = \sum_{\beta \in \mathcal{K}} \prod_{p \in \mathcal{P}} \theta_{\mathbf{i}_p(\mathbf{i}, \beta)}^{(p)},$$

where  $\beta$  collects all *latent* indices (their cardinalities are products of the ranks  $\{r\}$ ), and the product ranges over parameter groups p (factors/cores), each drawn iid with  $\mathbb{E}[\theta^{(p)}] = \mu_p$  and  $\operatorname{Var}(\theta^{(p)}) = \sigma_p^2$ .

**Lemma A.1** (Exponent counting). Fix a model topology (CP/TT/TR/Tucker), a sharing pattern  $S \subseteq [M]$ , and consider the second moment  $\mathbb{E}[\eta_i \eta_j]$  with  $(\mathbf{i}, \mathbf{j})$  constrained to share exactly S. Write it as a double sum over two latent index tuples  $(\beta, \beta') \in \mathcal{K} \times \mathcal{K}$ . Then:

- 1. For each parameter group p, the factor  $\theta^{(p)}$  contributes either a mean-square term  $\mu_p^2$  (if the two indices of  $\theta^{(p)}$  are independent across  $(\beta, \beta')$  or across  $(\mathbf{i}, \mathbf{j})$ ) or a variance term  $\sigma_p^2$  (if they coincide).
- 2. Each latent link (summation index) contributes an exponent of  $r^{\alpha}$  where  $\alpha \in \{0,1,2\}$  equals the number of independent surviving sums across  $(\beta, \beta')$  for that link after enforcing the constraints from S and from which factors are in variance mode.

Consequently, every pure term  $v_S$  is a monomial in  $\{r\} \cup \{\mu_p, \sigma_p^2\}$  for CP/TT/TR; for Tucker,  $v_{\{G\}}$ ,  $v_{\{p\}}$ , and  $v_{\{G,p\}}$  are monomials when all physical modes are shared.

*Proof.* Expand  $\mathbb{E}[\eta_i\eta_j]$  into a double sum over latent indices for the two entries. Independence across different latent tuples makes the expectation factor across parameter groups p. Whether the same random variable appears twice (giving  $\sigma_p^2$ ) or two independent copies (giving  $\mu_p^2$ ) is determined by index equalities induced by S and by whether the two latent index tuples coincide on the arguments of  $\theta^{(p)}$ . The number of independent sums that remain for each latent link determines the power of the corresponding rank. Subtracting lower-order  $C_T$  by inclusion–exclusion isolates the pure term  $v_S$ , which retains a single monomial.

# 2. Theoretical analysis of the Tucker model

We work with the standard order-M Tucker rate

$$\eta_{i_1...i_M} = \sum_{k_1=1}^{r_1} \cdots \sum_{k_M=1}^{r_M} G_{k_1...k_M} \prod_{p=1}^M \theta_{i_p,k_p}^{(p)},$$

with iid  $G_k$  having mean  $\mu_G$  and variance  $\sigma_G^2$ , and iid factor entries  $\theta^{(p)}$  having moments  $(\mu_p, \sigma_p^2)$ . **Global mean.** By independence,

$$\mathbb{E}[\eta_{i_1...i_M}] = \sum_{k_1,...,k_M} \mu_G \prod_{p=1}^M \mu_p = \left(\prod_{p=1}^M r_p\right) \mu_G \prod_{p=1}^M \mu_p.$$

Thus,

$$(E[Y])^{2} = \left(\prod_{p=1}^{M} r_{p}\right)^{2} \mu_{G}^{2} \prod_{p=1}^{M} \mu_{p}^{2}.$$
 (A2)

Pure terms when all modes are shared. Consider  $C_{[M]} = \text{Cov}(Y_i, Y_j)$  for pairs sharing all M physical indices (this is the only case in which core-variance and factor-variance mix coherently in Tucker). Decomposing by the variance sources we obtain four monomials:

**Lemma A.2** (Tucker monomials for shared-all-modes). With the above conventions, the following hold:

$$v_{\{G\}} = \sigma_G^2 \prod_{p=1}^M (r_p^2 \mu_p^2),$$
 (A3)

$$v_{\{p\}} = \mu_G^2 (r_p \sigma_p^2) \prod_{q \neq p} (r_q^2 \mu_q^2),$$
 (A4)

$$v_{\{G,p\}} = \sigma_G^2 \left( r_p \sigma_p^2 \right) \prod_{q \neq p} \left( r_q^2 \mu_q^2 \right), \tag{A5}$$

and equation A2 for  $(E[Y])^2$  remains unchanged.

Proof. Expand  $\mathbb{E}[\eta_{\mathbf{i}}\eta_{\mathbf{j}}]$  as a double sum over the core indices (k,k') and factor indices for two copies. For the pure term  $v_{\{G\}}$ , variance arises from G with k=k' (contributing  $\sigma_G^2$ ). All factor groups are in mean mode and appear in both copies, leaving two independent summations per factor index; each contributes  $r_q^2 \mu_q^2$ . Thus  $v_{\{G\}} = \sigma_G^2 \prod_q (r_q^2 \mu_q^2)$ .

For  $v_{\{p\}}$ , variance arises from the single factor at mode p (giving  $r_p \sigma_p^2$ ), while the core and all other factors contribute means in both copies: hence  $\mu_G^2$  and, for  $q \neq p$ ,  $r_q^2 \mu_q^2$ . This yields equation A4.

For  $v_{\{G,p\}}$ , combine the two variance sources: G in variance and factor p in variance; the remaining factors contribute mean-squared terms across the two copies, producing equation A5.

A Tucker identity and its consequence. Using equation A2-equation A5, we obtain

$$\log v_{\{p\}} - 2\log(E[Y]) = \log\left[\mu_G^2(r_p\sigma_p^2)\prod_{q\neq p}(r_q^2\mu_q^2)\right] - 2\log\left[\left(\prod_q r_q\right)\mu_G\prod_q \mu_q\right]$$

$$= \log\left(\frac{\sigma_p^2}{r_p\mu_p^2}\right), \tag{A6}$$

and similarly

$$\log v_{\{G,p\}} - \log v_{\{G\}} = \log \left[ \sigma_G^2(r_p \sigma_p^2) \prod_{q \neq p} (r_q^2 \mu_q^2) \right] - \log \left[ \sigma_G^2 \prod_q (r_q^2 \mu_q^2) \right]$$

$$= \log \left( \frac{\sigma_p^2}{r_p \mu_p^2} \right). \tag{A7}$$

Hence we have the exact identity

$$\log v_{\{p\}} - 2\log(E[Y]) = \log v_{\{G,p\}} - \log v_{\{G\}}$$
(A8)

which is the clarified version of Eq. (3.1) in the main text (Lemma III.2/Lemma III.3 there).

**Theorem A.3** (Non-identifiability of Tucker ranks: full proof). Let  $\mathbf{A}\mathbf{x} = \mathbf{b}$  be the log-linear system built from the monomials for  $(E[Y])^2$ ,  $v_{\{G\}}$ ,  $v_{\{p\}}$ ,  $v_{\{G,p\}}$  with  $p = 1, \ldots, M$ . Then the M columns corresponding to  $\log r_p$  lie in the span of the columns that do not involve  $\log r_p$ ; equivalently, the reduced design matrix for  $\mathbf{x}_r = (\log r_1, \ldots, \log r_M)$  is the zero matrix. Thus the ranks  $(r_1, \ldots, r_M)$  are not identifiable from second moments.

*Proof.* Each identity equation A8 eliminates  $\log r_p$  when forming the linear combination of rows

$$(\log v_{\{p\}}) - 2\log(E[Y]) - \left[(\log v_{\{G,p\}}) - \log v_{\{G\}}\right] = 0.$$

Hence the coordinate vector that picks  $\log r_p$  is a combination of columns that correspond only to nuisance parameters (log-means and log-variances). Doing this for each p shows the reduced matrix has zero columns.

# 3. Theoretical analysis of the Tensor Train model

This appendix provides complete, self-contained derivations for the TT model. We (i) formalize the monomial structure of pure terms  $v_S$  for arbitrary exact-sharing patterns S, (ii) derive the two key identities used to identify interior ranks, and (iii) provide boundary-mode analogues and closed-form estimators for boundary ranks.

**Model.** For an order-M TT, the rate is

$$\eta_{i_1...i_M} = \sum_{k_1=1}^{r_1} \cdots \sum_{k_{M-1}=1}^{r_{M-1}} \theta_{1,i_1,k_1}^{(1)} \, \theta_{k_1,i_2,k_2}^{(2)} \cdots \theta_{k_{M-1},i_M,1}^{(M)},$$

where entries of core p have moments  $(\mu_p, \sigma_p^2)$  and are iid across indices. Exact-sharing covariances  $C_S$  and pure terms  $v_S$  follow Appendix A 1.

a. A general monomial formula for TT pure terms

**Proposition A.4** (TT pure terms are monomials with explicit rank exponents). Let  $S \subseteq \{1, ..., M\}$  be an exact-sharing pattern. For each TT link  $\ell \in \{1, ..., M-1\}$ , define

$$\alpha_{\ell}(S) := 2 - \mathbf{1}\{\ell \in S \text{ or } \ell + 1 \in S\} \in \{1, 2\}.$$

Then the pure term associated with S is the single monomial

$$v_S = \left(\prod_{\ell=1}^{M-1} r_\ell^{\alpha_\ell(S)}\right) \left(\prod_{p \in S} \sigma_p^2\right) \left(\prod_{p \notin S} \mu_p^2\right). \tag{A9}$$

Proof. Expand  $\mathbb{E}[\eta_{\mathbf{i}}\eta_{\mathbf{j}}]$  as a double sum over latent tuples  $(k_1,\ldots,k_{M-1})$  and  $(k'_1,\ldots,k'_{M-1})$ . For a fixed S, modes in S contribute variance  $(\sigma_p^2)$ , modes not in S contribute squared means  $(\mu_p^2)$  after subtracting  $\mathbb{E}[\eta_{\mathbf{i}}] \mathbb{E}[\eta_{\mathbf{j}}]$ . A TT  $link \ \ell$  is the pair  $(k_\ell,k'_\ell)$ . If neither adjacent physical mode  $\ell$  nor  $\ell+1$  lies in S, the two copies keep independent sums over  $k_\ell$  and  $k'_\ell$  (yielding  $r_\ell^2$ ). If at least one of  $\ell$  or  $\ell+1$  lies in S, equality constraints collapse two sums into one (yielding  $r_\ell$ ). This gives the exponent  $\alpha_\ell(S)$ . Moebius inversion over subsets  $T \subseteq S$  cancels all mixed patterns, leaving the single monomial equation A9.

Consequences. Proposition A.4 immediately yields a catalogue of pure terms:

$$\text{Singleton } S = \{p\} \colon \quad v_{\{p\}} = \left(\prod_{\ell \neq p-1,p} r_\ell^2\right) r_{p-1} \, r_p \ \sigma_p^2 \prod_{k \neq p} \mu_k^2.$$
 
$$\text{Adjacent pair } S = \{p,p+1\} \colon \quad v_{\{p,p+1\}} = \left(\prod_{\ell \notin \{p-1,p,p+1\}} r_\ell^2\right) r_{p-1} \, r_p \, r_{p+1} \ \sigma_p^2 \sigma_{p+1}^2 \prod_{k \notin \{p,p+1\}} \mu_k^2.$$
 
$$\text{Noncontiguous pair } S = \{p-1,p+2\} \colon \quad v_{\{p-1,p+2\}} = \left(\prod_{\ell \notin \{p-2,p-1,p,p+1,p+2\}} r_\ell^2\right) r_{p-2} \, r_{p-1} \, r_p^2 \, r_{p+1} \, r_{p+2}$$
 
$$\times \ \sigma_{p-1}^2 \sigma_{p+2}^2 \prod_{k \notin \{p-1,p+2\}} \mu_k^2.$$
 
$$\text{Triple } S = \{p-1,p,p+2\} \colon \quad v_{\{p-1,p,p+2\}} = \left(\prod_{\ell \notin \{p-2,p-1,p,p+1,p+2\}} r_\ell^2\right) r_{p-2} \, r_{p-1} \, r_p^1 \, r_{p+1} \, r_{p+2}$$
 
$$\times \ \sigma_{p-1}^2 \sigma_p^2 \sigma_{p+2}^2 \prod_{k \notin \{p-1,p,p+2\}} \mu_k^2.$$

The highlighted exponents of  $r_p$  differ by one (2 vs. 1), which drives the  $1/r_p$  factor used below.

A.3.2 Interior-mode identities and estimator

**Lemma A.5** (Interior identities). For any interior mode p with 1 :

$$\frac{v_{\{p,p+1\}}}{v_{\{p+1\}}} = \frac{\sigma_p^2}{\mu_p^2},\tag{A10}$$

$$\frac{v_{\{p-1,p,p+2\}}}{v_{\{p-1,p+2\}}} = \frac{1}{r_p} \frac{\sigma_p^2}{\mu_p^2}.$$
(A11)

Proof. Using equation A9: In equation A10, the link exponents coincide for every  $\ell$  (both sets include p+1), so the rank parts cancel; the moment parts differ only at mode p, giving  $\sigma_p^2/\mu_p^2$ . In equation A11, the moment parts differ only at p, again giving  $\sigma_p^2/\mu_p^2$ , while the rank parts differ by exactly one power of  $r_p$  because  $\alpha_p(\{p-1,p+2\})=2$  (neither p nor p+1 is in the set) but  $\alpha_p(\{p-1,p,p+2\})=1$  (now  $p\in S$ ).

**Theorem A.6** (Closed-form estimator for interior TT ranks). For 1 ,

$$r_p = \frac{v_{\{p,p+1\}} \, v_{\{p-1,p+2\}}}{v_{\{p+1\}} \, v_{\{p-1,p,p+2\}}}. \tag{A12}$$

*Proof.* Divide equation A10 by equation A11.

b. Detailed TT moment calculations (interior and boundary cases)

This section expands Proposition A.4 for patterns used in practice and covers boundary modes p = 1 and p = M - 1. Interior patterns (full monomials) For 1 :

$$\begin{split} v_{\{p\}} &= \left(\prod_{\ell \notin \{p-1,p\}} r_\ell^2\right) r_{p-1} r_p \ \sigma_p^2 \prod_{k \neq p} \mu_k^2, \\ v_{\{p+1\}} &= \left(\prod_{\ell \notin \{p,p+1\}} r_\ell^2\right) r_p r_{p+1} \ \sigma_{p+1}^2 \prod_{k \neq p+1} \mu_k^2, \\ v_{\{p,p+1\}} &= \left(\prod_{\ell \notin \{p-1,p,p+1\}} r_\ell^2\right) r_{p-1} r_p r_{p+1} \ \sigma_p^2 \sigma_{p+1}^2 \prod_{k \notin \{p,p+1\}} \mu_k^2, \\ v_{\{p-1,p+2\}} &= \left(\prod_{\ell \notin \{p-2,p-1,p,p+1,p+2\}} r_\ell^2\right) r_{p-2} r_{p-1} r_p^2 r_{p+1} r_{p+2} \ \sigma_{p-1}^2 \sigma_{p+2}^2 \prod_{k \notin \{p-1,p+2\}} \mu_k^2, \\ v_{\{p-1,p,p+2\}} &= \left(\prod_{\ell \notin \{p-2,p-1,p,p+1,p+2\}} r_\ell^2\right) r_{p-2} r_{p-1} r_p^1 r_{p+1} r_{p+2} \ \sigma_{p-1}^2 \sigma_p^2 \sigma_{p+2}^2 \prod_{k \notin \{p-1,p,p+2\}} \mu_k^2. \end{split}$$

From these, Lemma A.5 follows by cancellation.

# Boundary patterns and identities

At the chain ends, the link  $r_1$  sits between modes 1 and 2; the link  $r_{M-1}$  sits between M-1 and M. We need two identities per boundary rank: one giving  $\sigma_p^2/\mu_p^2$ , and a second giving  $(\sigma_p^2/\mu_p^2)/r_p$ .

Left boundary (p=1). For  $M \geq 3$ ,

$$\frac{v_{\{1,2\}}}{v_{\{2\}}} = \frac{\sigma_1^2}{\mu_1^2},\tag{A13}$$

$$\frac{v_{\{1,3\}}}{v_{\{3\}}} = \frac{1}{r_1} \frac{\sigma_1^2}{\mu_1^2}.$$
 (A14)

*Proof.* The first ratio mirrors equation A10 with p=1. For the second, note that including mode 3 constrains link  $r_2$  in both numerator and denominator, while including mode 1 (only in the numerator) additionally constrains  $r_1$ , changing its exponent from 2 to 1. Moments differ only at mode 1. Hence the ratio equals  $(\sigma_1^2/\mu_1^2)/r_1$ .

Right boundary (p = M - 1). For  $M \ge 3$ ,

$$\frac{v_{\{M-1,M\}}}{v_{\{M\}}} = \frac{\sigma_{M-1}^2}{\mu_{M-1}^2},\tag{A15}$$

$$\frac{v_{\{M-3,M-1\}}}{v_{\{M-3\}}} = \frac{1}{r_{M-1}} \frac{\sigma_{M-1}^2}{\mu_{M-1}^2}.$$
(A16)

*Proof.* Symmetric to the left boundary with the appropriate shift (now link  $r_{M-1}$  is constrained iff mode M-1 or M is in S). The pair  $\{M-3\}$  vs.  $\{M-3, M-1\}$  plays the role of  $\{3\}$  vs.  $\{1,3\}$ .

**Theorem A.7** (Closed-form estimators for boundary TT ranks). For  $M \geq 3$ , the boundary ranks are identifiable via

$$r_1 = \frac{v_{\{1,2\}} \, v_{\{3\}}}{v_{\{2\}} \, v_{\{1,3\}}}, \qquad r_{M-1} = \frac{v_{\{M-1,M\}} \, v_{\{M-3\}}}{v_{\{M\}} \, v_{\{M-3,M-1\}}}. \tag{A17}$$

*Proof.* Divide equation A13 by equation A14 (left boundary) and equation A15 by equation A16 (right boundary).  $\Box$ 

**Edge-case notes.** (i) For M=3, the identities remain valid with the natural interpretation of sets (e.g.,  $\{3\}$  and  $\{1,3\}$  are defined). (ii) For M=2, TT reduces to a matrix factorization with a single rank and the CP formula applies.

Worked example (rank exponents table) For illustration, consider M=6 and p=3. The link exponents  $\alpha_{\ell}(S)$  from Proposition A.4 are:

S	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$
{3}	2	1	1	2	2
$\{4\}$	2	2	1	1	2
${3,4}$	2	1		1	2
$\{2, 5\}$	1	1	2	1	1
$\{2, 3, 5\}$	1	1	1	1	1

The highlighted entries show the drop  $2 \to 1$  at link 3 when moving from  $\{2,5\}$  to  $\{2,3,5\}$ , producing the  $1/r_3$  factor.

This completes the TT analysis with full monomial formulas, interior and boundary identities, and closed-form estimators for all TT ranks.

## 4. Theoretical analysis of the Tensor Ring model

We analyze the Tensor Ring (TR) model in full detail. Despite its cyclic symmetry, the first/second moments determine all ranks via a short linear system.

a. Model and notation

**Definition A.8** (Tensor Ring (TR)). For an order-M tensor with TR ranks  $(r_1, \ldots, r_M)$  (indices modulo M), the rate is

$$\eta_{i_1...i_M} = \operatorname{Tr}\left(\Theta_{i_1}^{(1)}\Theta_{i_2}^{(2)}\cdots\Theta_{i_M}^{(M)}\right),$$

where  $\Theta_{i_p}^{(p)} \in \mathbb{R}^{r_{p-1} \times r_p}$  has i.i.d. entries with moments  $(\mu_p, \sigma_p^2)$ , and  $r_0 \equiv r_M$  closes the ring. Conditional on  $\eta$ , the  $Y_i$  are independent with  $\mathbb{E}[Y_i \mid \eta] = \eta_i$ .

Let the TR latent indices be  $s_0, s_1, \ldots, s_{M-1}$  with  $s_0 = s_M$  and  $s_p \in [r_p]$ . Expanding the trace gives

$$\eta_{\mathbf{i}} = \sum_{s_0, \dots, s_{M-1}} \prod_{p=1}^{M} \theta_{s_{p-1}, i_p, s_p}^{(p)}, \qquad s_0 = s_M, \tag{A18}$$

where  $\theta^{(p)}$  denotes an entry of  $\Theta^{(p)}$ . As in Appendix A 1, for exact-sharing  $S \subseteq [M]$  we work with  $C_S = \text{Cov}(\eta_i, \eta_j)$  and the pure terms  $v_S$  via inclusion–exclusion.

b. TR prior predictive moments and rank-ratio identity

Throughout this section, indices are taken modulo M. Each TR core  $\Theta_{i_p}^{(p)} \in \mathbb{R}^{r_{p-1} \times r_p}$  has i.i.d. entries with

$$\mathbb{E}[\theta_{ab}^{(p)}] = \mu_p, \qquad \operatorname{Cov}(\theta_{ab}^{(p)}, \theta_{a'b'}^{(p)}) = \sigma_p^2 \, \delta_{aa'} \delta_{bb'}.$$

Conditional on  $\eta$ , the observations satisfy  $\mathbb{E}[Y_{\mathbf{i}} \mid \eta] = \eta_{\mathbf{i}}$ .

**Lemma A.9** (TR prior predictive moments under exact sharing). For any  $M \geq 3$ ,

$$(E[Y])^2 = \left(\prod_{k=1}^{M} r_k \mu_k\right)^2,$$
 (A19)

$$v_{\{p\}} = \left(\prod_{k \notin \{p-1, p\}} r_k^2\right) r_{p-1} r_p^2 \sigma_{k \neq p}^{2} \mu_k^2, \tag{A20}$$

$$v_{\{p,p+1\}} = \left(\prod_{k=1}^{M} r_k\right) \sigma_p^2 \sigma_{p+1}^2 \prod_{k \notin \{p,p+1\}} \mu_k^2. \tag{A21}$$

Proof. Mean. Expanding the trace,  $\mathbb{E}[\eta_{\mathbf{i}}] = \sum_{s_0,...,s_{M-1}} \prod_{p=1}^{M} \mathbb{E}[\theta_{s_{p-1},i_p,s_p}^{(p)}] = \prod_{p} (r_p \mu_p)$ , giving equation A19. Singleton  $v_{\{p\}}$ . Consider  $\text{Cov}(\eta_{\mathbf{i}},\eta_{\mathbf{j}})$  for the exact-sharing pattern  $S = \{p\}$  (only mode p shared, all others free).

Singleton  $v_{\{p\}}$ . Consider  $Cov(\eta_i, \eta_j)$  for the exact-sharing pattern  $S = \{p\}$  (only mode p shared, all others free). In  $\mathbb{E}[\eta_i \eta_j]$ , modes  $k \neq p$  contribute  $\mu_k^2$ , while mode p contributes  $\sigma_p^2$ . The rank exponents count independent latent sums: links  $k \notin \{p-1, p\}$  are free  $(r_k^2)$ ; the incoming link p-1 is jointly summed  $(r_{p-1})$ ; the outgoing link p remains decoupled  $(r_p^2)$ . This yields equation A20.

Adjacent pair  $v_{\{p,p+1\}}$ . For the exact-sharing pattern  $S=\{p,p+1\}$ , only cores p and p+1 are in variance, others in mean. Two consecutive variances glue the two copies into a single cyclic loop: all latent links participate in exactly one joint sum, hence each  $r_k$  appears once. Multiplying by  $\sigma_p^2 \sigma_{p+1}^2 \prod_{k \notin \{p,p+1\}} \mu_k^2$  gives equation A21.

**Remarks.** (i) Mean normalization of pure terms leaves  $\xi_p$  unchanged but improves numerical stability. (ii) Edge cases: M=2 reduces to a CP/matrix case; M=3 has eigenvalues  $1,1\pm i\sqrt{3}$  and remains identifiable.

**Proposition A.10** (Rank–ratio identity). For all p (indices modulo M),

$$\xi_p = r_p \frac{r_{p-1}}{r_{p+1}},$$
 (A22)

defining a circulant system  $\mathbf{C}\mathbf{x} = \boldsymbol{\psi}$  for  $x_p = \log r_p$  and  $\psi_p = \log \xi_p$ .

*Proof.* Substituting equation A19–equation A21, all  $\mu_k$  and  $\sigma_k$  cancel, leaving  $\xi_p = r_p \frac{r_{p-1}}{r_{p+1}}$ . Taking logs with  $x_p = \log r_p$  gives the circulant system

$$x_p + x_{p-1} - x_{p+1} = \psi_p, \quad p = 1, \dots, M.$$
 (A23)

i.e.,  $\mathbf{C}\mathbf{x} = \boldsymbol{\psi}$  with the  $M \times M$  circulant matrix whose first row is  $(1, -1, 0, \dots, 0, 1)$ :

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}. \tag{A24}$$

**Theorem A.11** (TR identifiability via DFT). For every integer  $M \ge 3$ , the matrix  $\mathbf{C}$  in equation A24 is invertible. Hence  $\mathbf{x} = \mathbf{C}^{-1} \boldsymbol{\psi}$  is uniquely determined from moments, and  $r_p = \exp(x_p)$  are identifiable for all p.

*Proof.* The coefficient matrix of equation A23 is circulant with first row  $(1, -1, 0, \dots, 0, 1)$ . Its DFT eigenvalues are

$$\lambda_k = 1 + 2i\sin(2\pi k/M), \quad k = 0, \dots, M - 1,$$

all nonzero for  $M \geq 3$ . Hence **C** is invertible and the ranks are uniquely recovered by  $\mathbf{x} = \mathbf{C}^{-1} \boldsymbol{\psi}$ ,  $r_p = \exp(x_p)$ .

**Computation.** Practically, solve  $\mathbf{C}\mathbf{x} = \boldsymbol{\psi}$  by FFTs: compute  $\widehat{\psi} = \mathrm{DFT}(\psi)$ , divide componentwise by  $\lambda_k$ , and invert the DFT to obtain x.

**Remarks.** (i) Mean normalization of pure terms leaves  $\xi_p$  unchanged but improves numerical stability. (ii) Edge cases: M=2 reduces to a CP/matrix case; M=3 has eigenvalues  $1,1\pm i\sqrt{3}$  and remains identifiable.

## Appendix B: Computational and Statistical Properties of the Estimators

## 1. Covariances, interaction sets, and pure terms

We estimate ranks in two steps. (i) Observable covariances. For a sharing set  $S \subseteq \{1, ..., M\}$ , the exact-sharing covariance  $\widehat{C}_S$  is computed from pairs  $(Y_i, Y_j)$  with indices equal on S and different on the complement. (ii) Pure terms. The pure interaction  $\widehat{v}_S$  is obtained from  $\{\widehat{C}_T : T \subseteq S\}$  via Möbius inversion:

$$\widehat{v}_S = \sum_{T \subseteq S} (-1)^{|S| - |T|} \widehat{C}_T.$$

These  $\hat{v}_S$  are the inputs to the closed-form estimators in CP/TT/TR.

**CP** (PARAFAC). With i.i.d. factor priors,  $v_S = R \prod_{p \in S} \sigma_p^2 \prod_{p \notin S} \mu_p^2$  for any nonempty S. Hence

$$\widehat{R}_S \ = \ \frac{\widehat{v}_S}{\prod_{p \in S} \sigma_p^2 \prod_{p \notin S} \mu_p^2}, \quad \text{aggregate over } S \in \{\{p\}\} \text{ and optionally } \{p,p+1\}.$$

Pure terms needed:

$$\widehat{v}_{\{p\}} = \widehat{C}_{\{p\}}, \qquad \widehat{v}_{\{i,j\}} = \widehat{C}_{\{i,j\}} - \widehat{C}_{\{i\}} - \widehat{C}_{\{j\}}.$$

**Tensor Train (TT).** Let  $r_1, \ldots, r_{M-1}$  be bond ranks (open chain). Using the monomial formula for pure terms (Appendix A 3), the closed forms are:

$$\begin{array}{ll} \text{Interior } (1$$

Special cases use triplets directly:

$$M=3: r_{1} = \frac{v_{\{1,2\}} v_{\{3\}}}{v_{\{2\}} v_{\{1,3\}}}, \quad r_{2} = \frac{v_{\{2,3\}} v_{\{1\}}}{v_{\{3\}} v_{\{1,2\}}}; \qquad M=4: \begin{cases} r_{1} = \frac{v_{\{1,2\}}}{v_{\{2\}}} \cdot \frac{v_{\{3,4\}}}{v_{\{1,3,4\}}}, \\ r_{2} = \frac{v_{\{2,3\}}}{v_{\{3\}}} \cdot \frac{v_{\{1,3,4\}}}{v_{\{1,2\}}}, \\ r_{3} = \frac{v_{\{3,4\}}}{v_{\{2\}}} \cdot \frac{v_{\{1,2\}}}{v_{\{1,3,4\}}}. \end{cases}$$

Required exact–sharing sets are: all singletons; adjacent pairs; the gap-two pair  $\{p-1, p+2\}$ ; the triplet  $\{p-1, p, p+2\}$ ; and the boundary helpers  $\{1, 3\}$ ,  $\{M-3\}$ ,  $\{M-3, M-1\}$ . For  $M \in \{3, 4\}$ , include  $\{1, 2, 3\}$  and  $\{1, 3, 4\}$ ,  $\{1, 2, 4\}$  as needed.

**Tensor Ring (TR).** Let  $\psi_p = \log(E[Y]^2) + \log v_{\{p,p+1\}} - \log v_{\{p\}} - \log v_{\{p+1\}}$  (indices mod M). Then the circulant system  $C \log r = \psi$  recovers all  $r_p$ . Only singletons and adjacent pairs are required.

Sampling kernel (all models). For each required set S, draw  $S_{cov}$  pairs per bootstrap with exact sharing pattern S, compute  $\widehat{C}_S$ , and obtain  $\widehat{v}_S$  by Möbius inversion over the (small) downward-closed family that contains S.

Complexity Analysis (Simplified). Let S be the number of distinct sharing sets used, B bootstraps, and  $S_{cov}$  pairs per set. Pair generation and access are O(1) amortized; Möbius is O(1) per set since  $|S| \leq 3$  here. The dominant cost is sampling:

Time 
$$\approx \Theta(SBS_{cov})$$
, Memory  $\approx \Theta(SB)$  (store-all) or  $\Theta(S)$  (stream).

Model-specific S (to estimate all ranks):

CP: S = M (singletons) or 2M (+ adjacent pairs).

TT: S = C(M-1), with small constant C (singletons, adjacents, gap-two, triplets).

TR: S = 2M (singletons + adjacent pairs) + one tiny linear solve  $O(M \log M)$ .

Thus all three scale *linearly* in M via S; TR has the cheapest set family, TT adds a constant-factor overhead for gap-two/triplets, and CP is minimal with singletons.

## 2. Variance and SNR of the Pure-Term Estimators

Among the family of moment-based estimators derived in this work, the *pure-term estimator* for the Tensor Train (TT) structure is the only one that relies *exclusively on second-order moments*— specifically, on covariances between entries sharing particular subsets of tensor modes. Unlike estimators that exploit higher-order cross-moments or structured mean terms, this construction avoids the need for explicit fourth-order moment tensors, making it computationally efficient and broadly applicable. However, this same property also makes it particularly sensitive to the *signal-to-noise ratio* (SNR) of the empirical covariances, because all rank information is inferred from ratios of covariances that may have small expected magnitude and high sampling variability.

In low-SNR regimes, random fluctuations in the empirical covariances can dominate the ratio computations, leading to unstable or biased rank estimates even when the theoretical ratio identities are exact. This motivates the detailed SNR analysis that follows: we study how the prior mean  $\mu$  and variance (via the coefficient of variation cv) of the latent factors jointly determine the effective SNR of the covariance-based estimator. The goal is to identify prior configurations that maximize the estimator's robustness without sacrificing the closed-form structure that makes it analytically tractable.

Let  $v_S$  be the TT pure interaction defined by exact-sharing Moebius inversion over covariances  $C_T$ . With per-mode prior mean  $\mu_m$ , variance  $\sigma_m^2$ , second raw moment  $m_{2,m} = \mu_m^2 + \sigma_m^2$  and fourth raw moment  $m_{4,m}$ , the TT population pure term is

$$v_S \ = \ \left(\prod_{\ell=1}^{M-1} r_\ell^{e_\ell(S)}\right) \left(\prod_{m \in S} \sigma_m^2\right) \left(\prod_{m \notin S} \mu_m^2\right), \quad e_\ell(S) = \mathbf{1}\{\ell \in S \text{ or } \ell+1 \in S\} + 1.$$

Let  $\widehat{C}_T$  be the exact-sharing covariance estimated from n paired samples and  $\widehat{v}_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} \widehat{C}_T$ . Under independent paired sampling per T and large n,

$$\operatorname{Var}(\widehat{C}_T) \approx \frac{1}{n} \left( \prod_{\ell} r_{\ell}^{h_{\ell}(T)} \right) \left( \prod_{m \in T} m_{4,m} \right) \left( \prod_{m \notin T} m_{2,m}^2 \right) - \frac{1}{n} \left( \prod_{\ell} r_{\ell}^{2e_{\ell}(T)} \right) \left( \prod_{m \in T} m_{2,m}^2 \right) \left( \prod_{m \notin T} \mu_m^4 \right),$$

where  $h_{\ell}(T) = 2e_{\ell}(T) \in \{2, 4\}$  are the fourth–moment exponents for TT. Consequently,

$$\operatorname{Var}(\widehat{v}_S) \ \approx \ \sum_{T \subset S} \frac{1}{n} (-1)^{2|S|-2|T|} \left[ \left( \prod_{\ell} r_\ell^{h_\ell(T)} \right) \left( \prod_{m \in T} m_{4,m} \right) \left( \prod_{m \notin T} m_{2,m}^2 \right) - \left( \prod_{\ell} r_\ell^{2e_\ell(T)} \right) \left( \prod_{m \in T} m_{2,m}^2 \right) \left( \prod_{m \notin T} \mu_m^4 \right) \right].$$

A convenient leading—order SNR follows by keeping the dominant term in the bracket:

$$\mathrm{SNR}(S) \equiv \frac{v_S}{\sqrt{\mathrm{Var}(\widehat{v}_S)}} \; \approx \; \sqrt{n} \; \prod_{m \in S} \frac{\sigma_m^2}{\sqrt{m_{4,m}}} \; \prod_{m \notin S} \frac{\mu_m^2}{m_{2,m}} \; \times \; \prod_{\ell} r_\ell^{e_\ell(S) - \frac{1}{2}h_\ell(S)}.$$

For TT,  $h_{\ell}(S) = 2e_{\ell}(S)$ , so the rank factor cancels and

SNR(S) 
$$\approx \sqrt{n} \prod_{m \in S} \frac{\sigma_m^2}{\sqrt{m_{4,m}}} \prod_{m \notin S} \frac{\mu_m^2}{\mu_m^2 + \sigma_m^2}$$
.

Gamma prior (equal across modes). If  $\theta^{(m)} \sim \text{Gamma}(\alpha, \theta)$  with mean  $\mu$  and  $\text{cv}^2 = 1/\alpha$ , then  $m_2 = \mu^2 (1 + \text{cv}^2)$  and  $m_4/\mu^4 = 1 + 6\text{cv}^2 + 3\text{cv}^4$ . Therefore

$$\mathrm{SNR}(S) \approx \sqrt{n} \; \mu^{|S|} \left( \frac{\mathrm{cv}^2}{\sqrt{1 + 6\mathrm{cv}^2 + 3\mathrm{cv}^4}} \right)^{|S|} \left( \frac{1}{1 + \mathrm{cv}^2} \right)^{M - |S|}.$$

This exhibits the empirical trade–off observed in practice: increasing cv boosts the shared–mode factor but diminishes the non–shared factor; an intermediate cv maximizes SNR for fixed |S| and M.

**Graphical analysis.** We can observe from Figure 2 that the SNR of the pure-interaction covariance estimator for |S| = 1 and |S| = 2 (different sizes of interaction sets) has a narrow band where it lead to joint high SNR. In most areas of the graph we have prior configurations that leads to low SNR, affecting the quality and variability of any estimator based on pure-interactin covariance values.

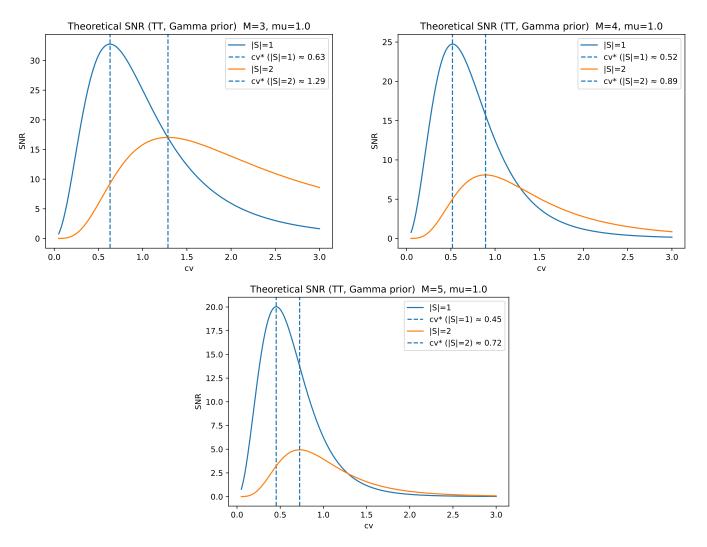


FIG. 2: Theoretical SNR (TT, Gamma prior) for  $M \in \{3,4,5\}$ ,  $\mu = 1.0$ . Vertical lines mark theoretical optimal region for cv\*.

## 3. Bias Analysis of the Regularized (and Mean-Normalized) Rank Estimators

All closed-form rank estimators in this work are ratios of estimated moments, and hence inherit the well-known sensitivity of ratio estimators when denominators are small or noisy. We analyze the regularized estimator used throughout and the effect of the optional mean normalization (dividing each  $\hat{v}_S$  by  $(\hat{E}[Y])^2$  before forming ratios). **Regularized ratio.** For any target rank written as  $r_p = \text{Num}_p/\text{Den}_p$ , we use

$$\hat{r}_p^{\text{reg}} = \text{sign}(\widehat{\text{Den}}_p) \frac{\widehat{\text{Num}}_p}{\left|\widehat{\text{Den}}_p\right| + \varepsilon_p}, \qquad \varepsilon_p = 1.96 \cdot \text{SE}_{\text{boot}}(\widehat{\text{Den}}_p),$$
(B1)

where  $SE_{boot}$  is the bootstrap standard error computed *across* bootstrap replicates. The small additive  $\varepsilon_p$  stabilizes divisions by noisy denominators. Using  $sign(\cdot)$  preserves the correct orientation when sampling variation flips the estimated sign (rare but possible for small-S pure terms).

**Mean normalization.** Define mean-normalized pure terms  $\tilde{v}_S := \hat{v}_S/((\hat{E}[Y])^2 + \varepsilon_E)$  with a tiny  $\varepsilon_E > 0$  (e.g.  $10^{-12}$ ). All CP/TT/TR rank identities are homogeneous in  $v_S$ , so replacing  $v_S$  by  $\tilde{v}_S$  does not change the population target. In finite samples it reduces dispersion because we scale by a high-SNR quantity,  $(\hat{E}[Y])^2$ . We therefore analyze both equation B1 with  $(\widehat{\text{Num}}, \widehat{\text{Den}})$  built from  $\hat{v}_S$  or from  $\tilde{v}_S$ .

**Delta-method bias.** Let  $R = \widehat{\text{Num}}/\widehat{\text{Den}}$  and  $f(x,y) = x/(y+\varepsilon)$  for a fixed  $\varepsilon \geq 0$ . A first-order delta-method

expansion around  $(\mu_X, \mu_Y) = (\mathbb{E}[\widehat{\text{Num}}], \mathbb{E}[\widehat{\text{Den}}])$  yields

$$\mathbb{E}[f(\widehat{\text{Num}}, \widehat{\text{Den}})] \approx \frac{\mu_X}{\mu_Y + \varepsilon} + \frac{1}{2} \Big( f_{xx} \text{Var}(\widehat{\text{Num}}) + 2 f_{xy} \text{Cov}(\widehat{\text{Num}}, \widehat{\text{Den}}) + f_{yy} \text{Var}(\widehat{\text{Den}}) \Big),$$

with  $f_x = 1/(\mu_Y + \varepsilon)$ ,  $f_y = -\mu_X/(\mu_Y + \varepsilon)^2$ ,  $f_{xx} = 0$ ,  $f_{xy} = -1/(\mu_Y + \varepsilon)^2$ ,  $f_{yy} = 2\mu_X/(\mu_Y + \varepsilon)^3$ . Ignoring the mixed term (or if it is small by design—e.g., disjoint pairs), the dominant bias contribution is

$$\mathrm{Bias} \; \approx \; \mathbb{E}[\hat{r}_p^{\mathrm{reg}}] - r_p \; \approx \; -r_p \cdot \frac{\varepsilon_p}{\mu_Y} \; + \; r_p \cdot \frac{\mathrm{Var}(\widehat{\mathrm{Den}})}{(\mu_Y + \varepsilon_p)^2} \; - \; \frac{\mathrm{Cov}(\widehat{\mathrm{Num}}, \widehat{\mathrm{Den}})}{(\mu_Y + \varepsilon_p)^2}. \tag{B2}$$

Thus the shrinkage term  $-r_p \varepsilon_p/\mu_Y$  is negative and of the same order as the stochastic error; the variance term is positive and of order  $1/N_{\text{cov}}$ ; the cross-covariance term often reduces in absolute value under mean normalization and when we estimate  $v_S$  via ridge-regularized Moebius regression.

## a. PARAFAC/CP

For CP,  $r = \frac{v_{\{p,q\}}(E[Y])^2}{v_{\{p\}}v_{\{q\}}}$  with two singletons in the denominator. By the delta method in the log domain, write

$$\log \hat{r} = \left(\log \hat{v}_{\{p,q\}} - \log \hat{v}_{\{p\}} - \log \hat{v}_{\{q\}}\right) + \left(2\log \widehat{E}[Y]\right),$$

so  $Var(\log \hat{r})$  is a sum of variances/covariances of these log-terms. Translating back, the relative bias of equation B1 satisfies

$$\frac{|\text{Bias}|}{r} = O\!\!\left(\text{CV}(\widehat{\text{Den}})\right) = O\!\!\left(\frac{1}{\sqrt{N_{\text{cov}}}}\right),$$

where  $CV(\widehat{Den}) = \sqrt{CV^2(\hat{v}_{\{p\}}) + CV^2(\hat{v}_{\{q\}}) + 2\rho CV(\hat{v}_{\{p\}})CV(\hat{v}_{\{q\}})}$  and  $\rho$  is the correlation between the singleton estimates (empirically small if pairs are sampled independently across S). Mean normalization divides all  $v_S$  by  $(\widehat{E}[Y])^2$ , shrinking both variance and covariance terms without changing the target.

## b. Tensor Train

For interior bonds,  $r_p = \frac{v_{\{p,p+1\}}v_{\{p-1,p+2\}}}{v_{\{p+1\}}v_{\{p-1,p,p+2\}}}$ . The denominator is a product; in the  $\log\ domain$  we have

$$\log \hat{r}_p = \left(\log \hat{v}_{\{p,p+1\}} + \log \hat{v}_{\{p-1,p+2\}}\right) - \left(\log \hat{v}_{\{p+1\}} + \log \hat{v}_{\{p-1,p,p+2\}}\right),$$

so  $Var(\log \hat{r}_p)$  is a sum of the four log-variances plus their cross-covariances. Exponentiating back implies

$$\frac{|\mathrm{Bias}|}{r_p} \ = \ O\!\!\left(\sqrt{\sum_{S \in \mathcal{S}_p} \mathrm{Var}\!\left(\log \hat{v}_S\right)}\right) \ = \ O\!\!\left(\frac{1}{\sqrt{N_{\mathrm{cov}}}}\right),$$

where  $S_p = \{\{p, p+1\}, \{p-1, p+2\}, \{p+1\}, \{p-1, p, p+2\}\}$ . Mean normalization replaces  $\hat{v}_S$  by  $\tilde{v}_S$ , reducing both  $\operatorname{Var}(\log \hat{v}_S)$  and cross-covariances in practice because  $(\widehat{E}[Y])^2$  is high-SNR. The regularization bias term in equation B2 remains  $O(1/\sqrt{N_{\text{cov}}})$  since  $\varepsilon_p \propto \operatorname{SE}(\widehat{\operatorname{Den}}_p)$ .

## c. Tensor Ring

For TR we estimate  $\xi_p = \frac{(E[Y])^2 v_{\{p,p+1\}}}{v_{\{p\}} v_{\{p+1\}}}$  and solve the linear system  $x_p + x_{p-1} - x_{p+1} = \psi_p$  with  $x_p = \log r_p$  and  $\psi_p = \log \hat{\xi}_p$ . Hence

$$\widehat{\mathbf{x}} = \mathbf{C}^{-1}\widehat{\boldsymbol{\psi}}, \qquad \operatorname{Var}(\widehat{\mathbf{x}}) = \mathbf{C}^{-1}\operatorname{Var}(\widehat{\boldsymbol{\psi}})(\mathbf{C}^{-1})^{\top}.$$

Each  $\psi_p$  is a log-ratio of estimated moments; its variance is reduced by mean normalization (replacing  $v_S$  by  $\tilde{v}_S$ ), and any regularization is applied componentwise to the underlying denominators that define  $\xi_p$  (as in equation B1). Since **C** is fixed and well-conditioned for  $M \geq 3$ , the bias order remains  $O(1/\sqrt{N_{\text{cov}}})$ .

Across CP/TT/TR, with or without mean normalization, and with denominator regularization equation B1, the induced bias satisfies

$$\frac{|\text{Bias}|}{r_p} \lesssim c \cdot \text{CV}(\widehat{\text{Den}}_p) = O\left(\frac{1}{\sqrt{N_{\text{cov}}}}\right),$$

for a constant c that depends smoothly on the moment mix and (weak) cross-covariances. In practice, the reduction in variance (and hence RMSE) from the stabilizer  $\varepsilon_p$  and from mean normalization outweighs the small  $O(1/\sqrt{N_{\rm cov}})$  bias. We therefore recommend (i) mean normalization of  $\hat{v}_S$  by  $(\hat{E}[Y])^2$ ; (ii) bootstrap-based  $\varepsilon_p$ ; and (iii) (optionally) least-squares Moebius inversion with a small ridge penalty when computing  $\hat{v}_S$ , which further reduces both variance and cross-covariances among the pure terms.

## Appendix C: Further Experiments

**Tensor Train (Figure 3).** Across settings, TT rank recovery tracks the y=x line for small and moderate true ranks, with dispersion increasing as ranks grow or the signal-to-noise ratio (SNR) drops (e.g., smaller Poisson means or more dispersed Gamma priors). This is expected for moment-based estimators: the pure terms  $v_S$  shrink in magnitude at low SNR, making inclusion-exclusion noisier and the denominators in the closed-form ratios closer to zero, which amplifies variance and occasionally induces slight upward bias. Interior bonds are typically better behaved than boundary bonds, which involve longer interaction sets. One practical mitigations help in our runsL mean normalization of all  $\hat{v}_S$  by  $(\widehat{E}[Y])^2$ , which reduces spread without changing the target. Across panels, estimation accuracy improves as the effective SNR increases: low-mean or high-variance priors yield underestimation and wide dispersion, while moderate  $(\mu, cv)$  combinations (as predicted by the SNR sweet-spot analysis in Appendix B2) yield nearly unbiased recovery. The results confirm that the covariance-only TT estimator remains consistent when the prior mean and variance are chosen to keep  $SNR(S) \gtrsim 5$  for the pure-term subsets involved in the ratio identities. **Tensor Ring (Figure 4).** For TR, the median estimates follow the y=x line across a wide range of ranks, with variance increasing at larger ranks and under lower SNR (smaller Poisson means / more dispersed Gamma priors). This behavior is consistent with our theory: each  $\xi_p$  is a log-ratio of estimated moments, so noise in singletons  $v_{\{p\}}$ and adjacent pairs  $v_{\{p,p+1\}}$  propagates through the circulant linear system for  $\{\log r_p\}$ . The cyclic coupling also means local errors can diffuse around the ring, slightly inflating uncertainty relative to TT at comparable ranks. In practice, mean normalization of  $\hat{v}_S$  by  $(\widehat{E}[Y])^2$  tighten the spread of  $\hat{\xi}_p$  and, consequently, the recovered ranks.

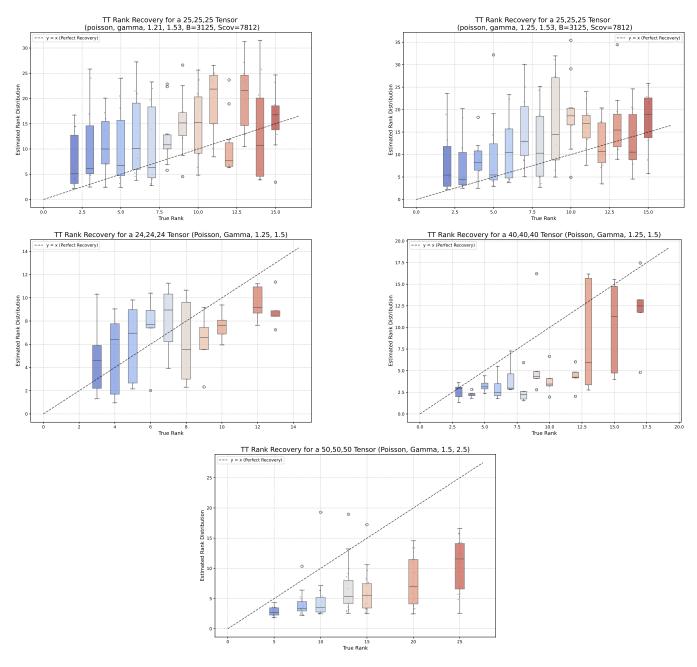
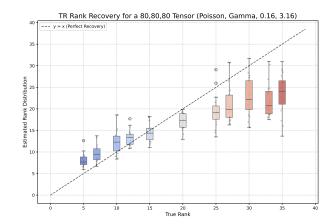


FIG. 3: Tensor Train (TT) rank estimation under varying dimensions and Gamma–Poisson generative settings. Each panel reports median rank estimates (over 20 independent runs) grouped by true rank, with boxplots showing the distribution of empirical estimates. The dashed diagonal (y=x) indicates perfect recovery. All models are trained using the covariance-only TT estimator. **Top row:** TT tensors of size (25, 25, 25) with Poisson likelihood and Gamma priors Gamma $(\alpha, \theta)$  set to (1.2, 1.5) (left) and (1.25, 1.5) (right). **Middle row:** TT tensors of sizes (24, 24, 24) and (40, 40, 40) with Gamma(1.25, 1.5) priors, showing improved stability with increasing tensor size. **Bottom:** TT tensor of size (50, 50, 50) with Gamma(1.5, 2.5) prior, corresponding to a higher mean and lower relative variance (lower cv).



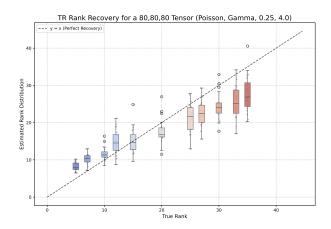


FIG. 4: Tensor Ring (TR) rank estimation on (80, 80, 80) tensors with Gamma–Poisson generative models. Each panel reports median rank estimates across 20 independent runs, grouped by true rank and visualized as boxplots. The dashed diagonal (y=x) denotes perfect recovery. Both experiments use a Poisson likelihood with Gamma priors  $Gamma(\alpha, \theta)$  given by (0.16, 3.16) (left) and (0.25, 4.0) (right). The shape and scale values correspond to different mean–variance tradeoffs, with the higher mean/lower variance prior (right) yielding improved accuracy and reduced dispersion, consistent with the expected gain in effective signal-to-noise ratio.