

Mean-Field Price Formation on Trees *

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Abstract

In this work, we combine the mean-field game theory with the classical idea of binomial tree framework, pioneered by Sharpe and Cox, Ross & Rubinstein, to solve the equilibrium price formation problem for the stock. For agents with exponential utilities and recursive utilities of exponential type, we prove the existence of a unique mean-field equilibrium and derive an explicit formula for equilibrium transition probabilities of the stock price by restricting its trajectories onto a binomial tree. The agents are subject to stochastic terminal liabilities and incremental endowments, both of which are dependent on unhedgeable common and idiosyncratic factors, in addition to the stock price path. Finally, we provide numerical examples to illustrate the qualitative effects of these components on the equilibrium price distribution.

Keywords: mean-field game theory, multi-population, equilibrium price formation, exponential utility, recursive utility, binomial tree

1 Introduction

The mean-field game (MFG) theory was pioneered independently by the seminal works of Lasry & Lions [33, 34, 35] and Huang et al. [27, 28, 29] in the mid- to late-2000s. These works are based on analytic methods using coupled Hamilton-Jacobi-Bellman and Kolmogorov equations. Later, the probabilistic approach to the MFG theory based on forward-backward stochastic differential equations (FBSDEs) of McKean-Vlasov type was established by Carmona & Delarue [9, 10]. The two volumes, [11] and [12], by the same authors, provide full details on the probabilistic approach and its applications.

The greatest advantage of the MFG theory is that it can convert a complex problem of stochastic differential games among many agents into a standard optimization problem and a fixed-point problem. There is a growing number of studies in the literature attempting to solve many-agent problems by the MFG technique. The MFG theory requires, in principle, symmetric interactions among the agents. We can find a particularly large number of applications of MFGs in financial and energy markets because symmetric interactions are standard in these settings. There are also many economic applications of mean field games, in particular, those focusing on general equilibrium models on growth, inequality and unemployment, dynamic demand response and persuasion problems, etc. See, for example, [1, 2, 3, 4, 5, 22] and the references therein.

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In recent years, there have also been major advances in MFG theory for applications in the equilibrium price-formation problem, where the asset price process is constructed endogenously to ensure that the asset's supply and demand always balance among the heterogeneous but exchangeable agents, rather than being exogenously given. Gomes & Saúde [24] present a deterministic model of electricity price. Its extension with random supply is given by Gomes et al. [23]. Ashrafyan et al. [6] propose a duality approach transforming these problems into variational ones that are numerically more tractable. Shrivats et al. [39] deal with a price formation problem for the solar renewable energy certificate (SREC) by solving FBSDEs of McKean-Vlasov type. Féron et al. [15] develop a tractable equilibrium model for intraday electricity markets. Sarto et al. [36] study cap-and-trade pollution regulation and derive the equilibrium price for the carbon emission. Regarding the price formation of securities, Fujii & Takahashi [16] show that the equilibrium price process can be characterized by FBSDEs of conditional McKean-Vlasov type. Its strong convergence to the mean-field limit from a finite-agent setting is proved in [17], and its extension to the presence of a major player is given in [18] by the same authors. Fujii [19] develops a model that allows the co-presence of cooperative and non-cooperative populations to learn how the price process is formed when the agents in one population act in a coordinated manner.

There remain two limitations in the above results: firstly, the relevant control of each agent is the trading rate and hence her asset position is constrained to follow an absolutely continuous process with respect to the Lebesgue measure dt ; secondly, the cost function of each agent consists of penalties on the trading speed and on the inventory size of the assets. Therefore, the above frameworks cannot deal with the general self-financing trading strategies nor the utility functions defined directly in terms of the wealth process of the portfolio. A major obstacle in dealing with a utility function of the wealth resides in the difficulty of guaranteeing the convexity of the Hamiltonian associated with the Pontryagin's maximum principle and in the difficulty of getting enough regularity to prove the well-posedness of the associated FBSDEs. These problems are solved by Fujii & Sekine [20, 21] for the agents with exponential utilities by applying the method of Hu, Imkeller & Müller [26] based on the martingale optimality principle. An extension to the setting with partial information is studied by Sekine [37]. They find that a novel quadratic-growth backward SDE (qg-BSDE) of conditional McKean-Vlasov type characterizes the equilibrium risk-premium process. Unfortunately, however, the existence of the solution of this mean-field qg-BSDE has been proved only under rather restrictive conditions. This is because the conditional McKean-Vlasov nature of the equation makes the traditional approach of Kobylanski [32] inapplicable anymore. Moreover, even if the well-posedness of the equation were to be completely solved, its numerical evaluation would continue to remain a highly challenging problem.

In this work, we study the price-formation problem for the risky stock. In order to understand how the equilibrium price process changes its behavior due to the market environment and the differences in the distribution of characteristics among the agents, it is necessary to get more explicit and more numerically tractable solutions than those in the existing literature. For proceeding toward this goal, in this paper, we propose an approach to combine the MFG theory with a classical idea of binomial trees, first proposed by Sharpe [38] and formalized by Cox, Ross & Rubinstein [13]. By restricting the stock price trajectories onto a binomial tree, we can search, in a rather straightforward manner, an appropriate set of transition probabilities so that the stock market clears. The agents, which are mainly intended to be financial and investment firms, are supposed to have standard exponential utilities or recursive utilities of exponential type. We allow the presence of terminal liability and incremental endowments, both of which are stochastic due

to unhedgeable common factors Y as well as idiosyncratic factors Z^i in addition to the stock price dependence. Since we are focusing on a trading desk of each firm, the incremental endowments represent non-tradable incomes originating from the firm's other business activities. We also study the impacts of external order flow from outside groups or possibly from a major financial player. Simple and explicit expressions for the equilibrium transition probabilities allow us to numerically evaluate the marginal equilibrium price distributions as well as conditional distributions with respect to the common factors Y and to study their qualitative behaviors with respect to the components just mentioned above. Moreover, our scheme can be readily extended to multi-population equilibrium problems. This is clear contrast to the continuous-time setting as in [20, 21], where the extension would require to solve a coupled system of complex mean-field qg-BSDEs, whose well-posedness would be much more challenging than a single population case.

We structure the rest of the paper as follows: Section 2 investigates the mean-field equilibrium among the agents with exponential utilities subject to terminal liabilities. Section 3 deals with an extension to recursive utilities and also introduces cash spending (i.e. nominal consumption) and incremental endowments. Section 4 provides several illustrative numerical examples and examines their implications. Section 5 summarizes our findings and discusses possible directions for future research.

2 Utility optimization for terminal wealth

We start our investigation into the mean-field price formation from a simple model where a countable number of agents, whose utilities are exponential type, are optimizing their terminal wealth by carrying out self-financing trading strategies on a deterministic money market account and a single risky stock. Each agent has to manage financial risk arising from the common market shocks as well as her own idiosyncratic shocks. Note that, in addition to the shocks from the stock price process, our model incorporates non-tradable macro-economic and/or environmental shocks that affect all agents.

2.1 The setup and notation

Let us start from explaining the relevant probability spaces. $(\Omega^0, \mathcal{F}^0, (\mathcal{F}_{t_n}^0)_{n=1}^N, \mathbb{P}^0)$ is a complete filtered probability space, where $0 = t_0 < t_1 < \dots < t_N = T$ is an equally spaced time sequence using a time step $\Delta := T/N$ where $T < \infty$ and $N \in \mathbb{N}$ are given constants. The filtration $(\mathcal{F}_{t_n}^0)_{n=0}^N$ is generated by two stochastic processes, one is a strictly positive process $(S_n := S(t_n))_{n=0}^N$ and the other is a d_Y -dimensional process $(Y_n = Y(t_n))_{n=0}^N$, i.e. $\mathcal{F}_{t_n}^0 := \sigma\{S_k, Y_k, 0 \leq k \leq n\}$. In the model below, we shall use S_n to denote the stock price at t_n and Y_n the common shocks affecting all the agents at t_n . $S_0 > 0$ and $Y_0 \in \mathbb{R}^{d_Y}$ are given constants and thus \mathcal{F}_0^0 is trivial.

In addition to the above space, we consider a countable set of complete filtered probability spaces $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_{t_n}^i)_{n=0}^N, \mathbb{P}^i)$, $i \in \mathbb{N}$. For each i , $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_{t_n}^i)_{n=0}^N, \mathbb{P}^i)$ is endowed with \mathcal{F}_0^i -measurable random variables (ξ^i, γ_i) as well as $(\mathcal{F}_{t_n}^i)_{n=0}^N$ -adapted stochastic process $(Z_n^i = Z^i(t_n))_{n=0}^N$. Here, ξ_i, γ_i are both \mathbb{R} -valued and ξ_i is used to represent the initial wealth and γ_i the size of absolute risk aversion of agent- i . The d_Z -dimensional process $(Z_n^i)_{n=0}^N$ is used to model idiosyncratic shocks to each agent. The fact that (ξ^i, γ_i) are \mathcal{F}_0^i -measurable means that the agent- i knows her initial wealth and the size of risk aversion at time zero.

By the standard procedures (see, for example, Klenke [31, Chapter 14]), the complete filtered

probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t_n})_{n=0}^N, \mathbb{P})$ is defined as the product of all the above spaces

$$(\Omega, \mathcal{F}, (\mathcal{F}_{t_n})_{n=0}^N, \mathbb{P}) := (\Omega^0, \mathcal{F}^0, (\mathcal{F}_{t_n}^0)_{n=0}^N, \mathbb{P}^0) \otimes_{i=1}^{\infty} (\Omega^i, \mathcal{F}^i, (\mathcal{F}_{t_n}^i)_{n=0}^N, \mathbb{P}^i)$$

which denotes the full probability space containing the entire environment of our model. Therefore, by construction, $((S_n), (Y_n))$ and $(\xi^i, \gamma_i, (Z_n^i))$, $i \in \mathbb{N}$ are mutually independent. On the other hand, the relevant probability space for each agent- i is the product probability space defined by

$$(\Omega^{0,i}, \mathcal{F}^{0,i}, (\mathcal{F}_{t_n}^{0,i})_{n=0}^N, \mathbb{P}^{0,i}) := (\Omega^0, \mathcal{F}^0, (\mathcal{F}_{t_n}^0)_{n=0}^N, \mathbb{P}^0) \otimes (\Omega^i, \mathcal{F}^i, (\mathcal{F}_{t_n}^i)_{n=0}^N, \mathbb{P}^i),$$

which reflects our assumption that common shocks are public knowledge, but the idiosyncratic shocks are private to each agent. We shall use the same symbols, such as $(S_n, Y_n, \gamma_i, \dots)$, if they are defined as trivial extensions on larger product probability spaces. Expectations with respect to \mathbb{P}^0 , $\mathbb{P}^{0,i}$ and \mathbb{P} are denoted by $\mathbb{E}^0[\cdot]$, $\mathbb{E}^{0,i}[\cdot]$ and $\mathbb{E}[\cdot]$, respectively. We also denote by $\bar{E}[\cdot]$ the expectation with respect to the product measure $\otimes_{i=1}^{\infty} \mathbb{P}^i$.

In this work, we restrict the trajectories of $(S_n)_{n=0}^N$ on a recombining binomial tree. For each $n = 1, \dots, N$, the random variable $\tilde{R}_n := S_n/S_{n-1}$ takes only the two possible values, either \tilde{u} or \tilde{d} . This means that the set of all possible values taken by S_n is given by $\mathcal{S}_n := \{S_0 \tilde{u}^k \tilde{d}^{n-k}, 0 \leq k \leq n\}$, which is a finite subset of $(0, \infty)$. Let $\mathcal{S}^n := \mathcal{S}_0 \times \mathcal{S}_1 \times \mathcal{S}_2 \times \dots \times \mathcal{S}_n$ be the set of all values taken by the stock price trajectory $(S_k)_{k=0}^n$. Moreover, in order to avoid technicalities regarding the conditional probabilities, we also assume that the process Y takes only a finite number of values at every t_n . We use \mathcal{Y}_n , which is a finite subset of \mathbb{R}^{d_Y} , to denote the set of all values taken by Y_n . We use, for each $0 \leq n \leq N$, $\mathcal{Y}^n := \mathcal{Y}_0 \times \mathcal{Y}_1 \times \dots \times \mathcal{Y}_n$ to represent the set of all values in $\mathbb{R}^{d_Y \times (n+1)}$ taken by the trajectory $(Y_k)_{k=0}^n$. In a similar manner, we denote the support of the random variable Z_n^i by \mathcal{Z}_n and define $\mathcal{Z}^n := \mathcal{Z}_0 \times \mathcal{Z}_1 \times \dots \times \mathcal{Z}_n$ as the support of $(Z_k^i)_{k=0}^n$. The time- t_n value of the risk-free money market account is given by $\exp(rn\Delta)$ where $r > 0$ is a positive constant denoting the risk-free (nominal) interest rate. We also use the symbol $\beta := \exp(r\Delta)$. For later use, let us also define

$$u := \tilde{u} - \exp(r\Delta), \quad d := \tilde{d} - \exp(r\Delta)$$

and

$$R_n := \tilde{R}_n - \exp(r\Delta), \quad 1 \leq n \leq N.$$

The random variable R_n takes the values either u or d .

In the following, to lighten the notational burden, we use $\mathbb{E}^{0,i}[\cdot | s, y, z^i, \gamma_i]$ to denote $\mathbb{E}^{0,i}[\cdot | S_{n-1} = s, Y_{n-1} = y, Z_{n-1}^i = z^i, \gamma_i = \gamma_i]$ for $(s, y, z^i) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1}$. With slight abuse of notation, we shall use the same symbols for the realizations of \mathcal{F}_0^i -measurable random variables (γ_i in the above example).

Assumption 2.1. (i): \tilde{u} and \tilde{d} are real constants satisfying

$$0 < \tilde{d} < \exp(r\Delta) < \tilde{u} < \infty.$$

(ii): Every $(\xi^i, \gamma_i, (Z_n^i)_{n=0}^N)$, $i = 1, 2, \dots$ has the same distribution.

(iii): There exist constants $\underline{\xi}, \bar{\xi}$ and $\underline{\gamma}, \bar{\gamma}$ so that for every $i \in \mathbb{N}$,

$$\xi_i \in [\underline{\xi}, \bar{\xi}] \subset \mathbb{R}, \quad \gamma_i \in \Gamma := [\underline{\gamma}, \bar{\gamma}] \subset (0, \infty).$$

(iv): For every $0 \leq n \leq N$, \mathcal{Z}_n is a bounded subset of \mathbb{R}^{dz} .

(v): For each i , $(Z_n^i)_{n=0}^N$ is a Markov process i.e. $\mathbb{E}^i[f(Z_n^i)|\mathcal{F}_{t_m}^i] = \mathbb{E}^i[f(Z_n^i)|Z_m^i]$ for every bounded measurable function f on \mathcal{Z}_n and $m \leq n$.

(vi): $(Y_n)_{n=0}^N$ is a Markov process i.e. $\mathbb{E}^0[f(Y_n)|\mathcal{F}_{t_m}^0] = \mathbb{E}^0[f(Y_n)|Y_m]$ for every bounded measurable function f on \mathcal{Y}_n and $m \leq n$.

(vii): The transition probabilities of $(S_n)_{n=0}^N$ satisfy, for every $0 \leq n \leq N-1$,

$$\mathbb{P}^0(S_{n+1} = \tilde{u}S_n | \mathcal{F}_{t_n}^0) = \mathbb{P}^0(S_{n+1} = \tilde{u}S_n | S_n, Y_n) = p_n(S_n, Y_n),$$

$$\mathbb{P}^0(S_{n+1} = \tilde{d}S_n | \mathcal{F}_{t_n}^0) = \mathbb{P}^0(S_{n+1} = \tilde{d}S_n | S_n, Y_n) = q_n(S_n, Y_n) := 1 - p_n(S_n, Y_n),$$

where $p_n, q_n : \mathcal{S}_n \times \mathcal{Y}_n \rightarrow \mathbb{R}$, $0 \leq n \leq N-1$ are bounded measurable functions satisfying

$$0 < p_n(s, y), q_n(s, y) < 1$$

for every $(s, y) \in \mathcal{S}_n \times \mathcal{Y}_n$.

Let us give some remarks on the above assumptions. Firstly, by the condition (i), we have $d < 0 < u$. It is well-known that the transition probabilities under the risk-neutral measure \mathbb{Q} for the classical binomial framework are given by $p^{\mathbb{Q}} := \frac{-d}{u-d}$ for the up-move and $q^{\mathbb{Q}} := 1 - p^{\mathbb{Q}}$ for the down-move. These probabilities are uniquely determined by the parameters (\tilde{u}, \tilde{d}) and the risk-free interest rate. In this paper, we fix the relative jump size (\tilde{u}, \tilde{d}) to be constant across all nodes; however this is done merely for simplicity. The entire discussion of our paper still holds even if (\tilde{u}, \tilde{d}) varies from node to node. In fact, the method works even for general non-recombining binomial trees. Moreover, thanks to the famous result by Derman & Kani (1994) [14], one can construct so-called “*implied binomial trees*” by adjusting the position (\tilde{u}, \tilde{d}) node by node to reproduce implied volatility surface in the market, while keeping the recombining property of binomial trees intact. Therefore, if necessary, our discussion below can be applied to a binomial tree whose risk-neutral distribution is consistent with the option market. The boundedness assumption in (iii) and (iv) is not crucial. One can relax it by adding appropriate integrability conditions.

Our goal in this section is to find a set of transition probabilities of the form $(p_n(s, y))_{n=0}^{N-1}$ so that the demand and supply of the stock are balanced among the agents at every node $(s, y) \in \mathcal{S}_n \times \mathcal{Y}_n$, $0 \leq n \leq N-1$. Note that we can assume, without any loss of generality, that the process $(Y_n)_{n=0}^N$ and $(Z_n^i)_{n=0}^N$ are Markov, since, if necessary, we can recover Markovian property by lifting Y, Z^i to higher dimensional processes. However, the condition (vii) is not trivial. In fact, in the next section, we shall study more general situations where the transition probability must be dependent on the past history of the stock price to achieve the market-clearing equilibrium. Under the current condition (vi) and (vii), (S_{n+1}, Y_{n+1}) satisfy the property:

$$\mathbb{E}^0[f(S_{n+1})g(Y_{n+1})|\mathcal{F}_{t_n}^0] = \mathbb{E}^0[f(S_{n+1})|S_n, Y_n]\mathbb{E}^0[g(Y_{n+1})|Y_n], \quad 0 \leq n \leq N-1, \quad (2.1)$$

for any bounded measurable functions f and g . We can interpret that the process $(Y_n)_{n=0}^N$ represents some standalone macro-economic and/or environmental factors which are not influenced by the agents’ trading activities. It may naturally serve as a state process in regime switching models.

Remark 2.1. Note that the bound for the transition probabilities in (vii) guarantees the equivalence of probability measures $\mathbb{P}^0 \circ S_n^{-1}$ and $\mathbb{Q} \circ S_n^{-1}$. Hence, our system is arbitrage free.

2.2 The individual optimization problem

We now explain the optimization problem for each agent. Agent- i , $i \in \mathbb{N}$, endowed with an initial wealth ξ_i , engages in self-financing trading involving the risk-free money market account and the risky stock. She adopts an $(\mathcal{F}_{t_n}^{0,i})_{n=0}^{N-1}$ -adapted trading strategy $(\phi_n^i)_{n=0}^{N-1}$ denoting the invested amount of cash in the stock at time $t = t_n$. The associated wealth process of agent- i is denoted by $(X_n^i = X^i(t_n))_{n=0}^N$ and follows the dynamics

$$\begin{aligned} X_{n+1}^i &= \exp(r\Delta)(X_n^i - \phi_n^i) + \phi_n^i \tilde{R}_{n+1} \\ &= \beta X_n^i + \phi_n^i R_{n+1}, \end{aligned}$$

where $X_0^i = \xi_i$ and $\beta = \exp(r\Delta)$. Recall that $R_{n+1} := \tilde{R}_{n+1} - \exp(r\Delta)$.

Each agent- i is supposed to solve the optimization problem:

$$\sup_{(\phi_n^i)_{n=0}^{N-1} \in \mathbb{A}^i} \mathbb{E}^{0,i} \left[-\exp \left(-\gamma_i (X_N^i - F(S_N, Y_N, Z_N^i)) \right) \middle| \mathcal{F}_0^{0,i} \right], \quad (2.2)$$

where

$$\mathbb{A}^i := \{ (\phi_n^i)_{n=0}^{N-1}; \phi_n^i \text{ is } \mathcal{F}_{t_n}^{0,i}\text{-measurable real random variable} \}$$

denotes the admissible control space. Here, we assume that agent- i has full knowledge of the common market information and her own private idiosyncratic information, but no knowledge of the private idiosyncratic information of the other agents.

Assumption 2.2. (i): $F : \mathcal{S}_N \times \mathcal{Y}_N \times \mathcal{Z}_N \rightarrow \mathbb{R}$ is a bounded measurable function.

(ii): Every agent is a price-taker in the sense that she considers that the stock price process (and hence its transition probability specified in Assumption 2.1 (vii)) to be exogenously given and unaffected by her trading strategies.

Here, the term $F(S_N, Y_N, Z_N^i)$ denotes the stochastic liability $((-1) \times \text{stochastic endowment})$ at the terminal time, which depends on S_N, Y_N and Z_N^i . Under the exponential utilities, the constant shift $F \rightarrow F + c$ does not change the optimization problem; only the dependence on (S_N, Y_N, Z_N^i) in the function F is relevant. The condition (ii) is a plausible assumption since every agent naturally knows her trading share is negligible in the market. This is also a key assumption for the standard MFG technique, in which we first solve a simple optimization problem with a given candidate for the equilibrium distribution, and then check its consistency with the obtained optimal solution.

Remark 2.2. Before going to solve the optimization problem, we give some economic motivations to include the stochastic terminal liability $F(S_N, Y_N, Z_N^i)$. Since we primarily want to model various financial firms by our agents, it is natural to suppose that they are subject to stochastic liabilities (such as portfolio of derivative contracts) dependent on the stock price. It is also very plausible that the size of liability varies from agent to agent by their idiosyncratic factors (Z_N^i) as well as the common macro-economic/environmental factors (Y_N) . This structure would naturally hold for non-financial firms, too.

We now characterize the optimal trading strategy for each agent. Applying the well-known scheme of backward induction for discrete-time models, we establish the following result. We shall see that the initial wealth ξ_i does not play any role, which is an important characteristic of exponential-type utilities.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 be in force. Then the problem (2.2) has a unique optimal solution $(\phi_{n-1}^{i,*})_{n=1}^N$, which is an a.s. bounded process defined by a measurable function $\phi_{n-1}^{i,*} : \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \rightarrow \mathbb{R}$ as $\phi_{n-1}^{i,*} := \phi_{n-1}^{i,*}(S_{n-1}, Y_{n-1}, Z_{n-1}^i, \gamma_i)$, where*

$$\begin{aligned} \phi_{n-1}^{i,*}(s, y, z^i, \gamma_i) &:= \frac{1}{\gamma_i(u-d)} \frac{\beta^n}{\beta^N} \left\{ \log \left(-\frac{p_{n-1}(s, y)u}{q_{n-1}(s, y)d} \right) + \log(f_{n-1}(s, y, z^i, \gamma_i)) \right\}, \\ f_{n-1}(s, y, z^i, \gamma_i) &:= \frac{\mathbb{E}^{0,i}[V_n(s\tilde{u}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i]}{\mathbb{E}^{0,i}[V_n(s\tilde{d}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i]}. \end{aligned} \quad (2.3)$$

Here, $f_{n-1} : \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \rightarrow \mathbb{R}$ and $V_n : \mathcal{S}_n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \rightarrow \mathbb{R}$, $1 \leq n \leq N$ are a.s. strictly positive bounded measurable functions defined recursively by

$$V_N(s, y, z^i, \gamma_i) := \exp(\gamma_i F(s, y, z^i)),$$

and

$$\begin{aligned} V_{n-1}(s, y, z^i, \gamma_i) &:= p_{n-1}(s, y) e^{-\gamma_i \frac{\beta^N}{\beta^n} \phi_{n-1}^{i,*}(s, y, z^i, \gamma_i) u} \mathbb{E}^{0,i}[V_n(s\tilde{u}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i] \\ &\quad + q_{n-1}(s, y) e^{-\gamma_i \frac{\beta^N}{\beta^n} \phi_{n-1}^{i,*}(s, y, z^i, \gamma_i) d} \mathbb{E}^{0,i}[V_n(s\tilde{d}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i]. \end{aligned} \quad (2.4)$$

Proof. With $V_N : \mathcal{S}_N \times \mathcal{Y}_N \times \mathcal{Z}_N \times \Gamma \rightarrow \mathbb{R}$ defined by $V_N(s, y, z^i, \gamma_i) = \exp(\gamma_i F(s, y, z^i))$, we temporarily suppose that the problem of agent- i at $t = t_{n-1}$, $1 \leq n \leq N$, is defined by

$$\sup_{\phi^i} \mathbb{E}^{0,i} \left[-\exp \left(-\gamma_i \frac{\beta^N}{\beta^n} X_n^i \right) V_n(S_n, Y_n, Z_n^i, \gamma_i) | \mathcal{F}_{t_{n-1}}^{0,i} \right], \quad (2.5)$$

with some a.s. strictly positive bounded measurable function $V_n : \mathcal{S}_n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \rightarrow \mathbb{R}$. Here, the supremum is taken over $\mathcal{F}_{n-1}^{0,i}$ -measurable real random variables. Let us solve the above problem on the set $\{(X_{n-1}^i, S_{n-1}, Y_{n-1}, Z_{n-1}^i, \gamma_i) = (x^i, s, y, z^i, \gamma_i)\}^1$. Then the above problem is equivalent to

$$\begin{aligned} &\inf_{\phi^i \in \mathbb{R}} \mathbb{E}^{0,i} \left[\exp \left(-\gamma_i \frac{\beta^N}{\beta^n} (\beta x^i + \phi^i R_n) \right) V_n(s\tilde{R}_n, Y_n, Z_n^i, \gamma_i) | x^i, y, z^i, \gamma_i \right] \\ &= \exp \left(-\gamma_i \frac{\beta^N}{\beta^{n-1}} x^i \right) \inf_{\phi^i \in \mathbb{R}} \left\{ p_{n-1}(s, y) e^{-\gamma_i \frac{\beta^N}{\beta^n} \phi^i u} \mathbb{E}^{0,i}[V_n(s\tilde{u}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i] \right. \\ &\quad \left. + q_{n-1}(s, y) e^{-\gamma_i \frac{\beta^N}{\beta^n} \phi^i d} \mathbb{E}^{0,i}[V_n(s\tilde{d}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i] \right\}, \end{aligned}$$

where we have used the property given in Assumption 2.1 (vii). Since $d < 0 < u$, the optimal

¹With slight abuse of notation, we use the same symbols for the realizations of \mathcal{F}_0^i -measurable random variables.

position $\phi^{i,*}$ is a.s. uniquely characterized by

$$\begin{aligned} p_{n-1}(s, y) u e^{-\gamma_i \frac{\beta^N}{\beta^n} \phi^{i,*} u} \mathbb{E}^{0,i}[V_n(s\tilde{u}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i] \\ + q_{n-1}(s, y) d e^{-\gamma_i \frac{\beta^N}{\beta^n} \phi^{i,*} d} \mathbb{E}^{0,i}[V_n(s\tilde{d}, Y_n, Z_n^i, \gamma_i)|y, z^i, \gamma_i] = 0. \end{aligned}$$

This gives the results given in (2.3), which is well-defined and bounded since V_n is a strictly positive and bounded, and $0 < p_{n-1}, q_{n-1} < 1$ by our assumption.

It follows that the function V_{n-1} defined by (2.4) becomes once again a.s. strictly positive and bounded on $\mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$. The value function at the next step t_{n-1} is now defined by

$$-\exp\left(-\gamma_i \frac{\beta^N}{\beta^{n-1}} X_{n-1}^i\right) V_{n-1}(S_{n-1}, Y_{n-1}, Z_{n-1}^i, \gamma_i)$$

and we have recovered the problem of the same form as in (2.5). Thus the repeat of the above procedures N times yields the desired conclusion. \square

2.3 Mean-field equilibrium under stochastic order flow

Our goal is to find a set of functions $p_n(s, y), (s, y) \in \mathcal{S}_n \times \mathcal{Y}_n, 0 \leq n \leq N-1$ that specify the transition probabilities at every node so that supply and demand match among the agents. To ensure generality, we also incorporate an external stochastic order flow, $L_{n-1}(S_{n-1}, Y_{n-1})$, which represents the stock supply per capita at each time $t = t_{n-1}$, for $1 \leq n \leq N$. The external order flow serves to model the net contribution from other populations or from a major financial institution, such as a central bank.

Assumption 2.3. *For every $1 \leq n \leq N$, $L_{n-1} : \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \rightarrow \mathbb{R}$ is a bounded measurable function.*

Definition 2.1. *We say that the system is in the mean-field equilibrium if*

$$\lim_{N_p \rightarrow \infty} \frac{1}{N_p} \sum_{i=1}^{N_p} \phi_{n-1}^{i,*}(S_{n-1}, Y_{n-1}, Z_{n-1}^i, \gamma_i) = L_{n-1}(S_{n-1}, Y_{n-1}),$$

\mathbb{P} -a.s. for every $1 \leq n \leq N$ with $\phi_{n-1}^{i,*}$ defined by (2.3).

Since $(Z^i, \gamma_i), i \in \mathbb{N}$ are independent and identically distributed and also independent of the process (S, Y) , the above condition for the mean-field equilibrium is equivalently expressed as

$$\mathbb{E}^1[\phi_{n-1}^{1,*}(s, y, Z_{n-1}^1, \gamma_1)] = L_{n-1}(s, y) \quad (2.6)$$

for every $(s, y) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1}, 1 \leq n \leq N$. Under the mean-field equilibrium, the excess demand/supply per capita converges to zero as the population size tends to infinity. Our first main result for this paper is then established as follows. One can observe conditional McKean-Vlasov nature as expected from the result in Fujii & Sekine [20].

Theorem 2.2. *Let Assumptions 2.1, 2.2 and 2.3 be in force. Then there exists a unique mean-field equilibrium and the associated transition probabilities of the stock are given by*

$$\begin{aligned} p_{n-1}(s, y) &:= \mathbb{P}^0(S_n = \tilde{u}S_{n-1} | (S_{n-1}, Y_{n-1}) = (s, y)) \\ &= (-d) / \left\{ u \exp\left(\frac{1}{\mathbb{E}^1[1/\gamma_1]} \left[\mathbb{E}^1\left(\frac{\log(f_{n-1}(s, y, Z_{n-1}^1, \gamma_1))}{\gamma_1}\right) - (u-d) \frac{\beta^N}{\beta^n} L_{n-1}(s, y) \right] \right) - d \right\}, \end{aligned} \quad (2.7)$$

for every $(s, y) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1}$, $1 \leq n \leq N$. Moreover, under the above transition probabilities, there exists some positive constant C_{n-1} such that

$$\mathbb{E} \left| \frac{1}{N_p} \sum_{i=1}^{N_p} \phi_{n-1}^{i,*}(S_{n-1}, Y_{n-1}, Z_{n-1}^i, \gamma_i) - L_{n-1}(S_{n-1}, Y_{n-1}) \right|^2 \leq \frac{C_{n-1}}{N_p} \quad (2.8)$$

for every $1 \leq n \leq N$, which gives the convergence rate in the large population limit.

Proof. The first claim (2.7) is a direct consequence of (2.3). The form of $(p_{n-1}(s, y))$ is uniquely determined by the condition (2.6). We only need to verify that it satisfies the condition (vii) in Assumption 2.1. At $t = N - 1$, f_{N-1} is an a.s. strictly positive and bounded function on $\mathcal{S}_{N-1} \times \mathcal{Y}_{N-1} \times \mathcal{Z}_{N-1} \times \Gamma$ by the corresponding assumption on F . Since $d < 0 < u$, it is easy to see $0 < p_{N-1}(s, y), q_{N-1}(s, y) < 1$ for every $(s, y) \in \mathcal{S}_{N-1} \times \mathcal{Y}_{N-1}$, and hence consistent with the condition. This then implies that $\phi_{N-1}^{i,*}$ is an a.s. bounded function on $\mathcal{S}_{N-1} \times \mathcal{Y}_{N-1} \times \mathcal{Z}_{N-1} \times \Gamma$, which results in a.s. strictly positive and bounded V_{N-1} given by (2.4) on $\mathcal{S}_{N-1} \times \mathcal{Y}_{N-1} \times \mathcal{Z}_{N-1} \times \Gamma$. This in turn ensures that f_{N-2} satisfies the desired properties, and so do $(p_{N-2}(s, y), q_{N-2}(s, y))$, $(s, y) \in \mathcal{S}_{N-2} \times \mathcal{Y}_{N-2}$. By simple induction, we get the desired consistency for every time step.

For the second claim, it suffices to show that there is some constant C_{n-1} such that an inequality

$$\mathbb{E} \left| \frac{1}{N_p} \sum_{i=1}^{N_p} \phi_{n-1}^{i,*}(s, y, Z_{n-1}^i, \gamma_i) - L_{n-1}(s, y) \right|^2 \leq \frac{C_{n-1}}{N_p}$$

holds for every $(s, y) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1}$, $1 \leq n \leq N$. With the equilibrium transition probabilities given by (2.7), one can show that the optimal position for agent- i is given by

$$\begin{aligned} \phi_{n-1}^{i,*}(s, y, Z_{n-1}^i, \gamma_i) &= \frac{1}{(u-d)\beta^N} \left\{ \frac{\log f_{n-1}(s, y, Z_{n-1}^i, \gamma_i)}{\gamma_i} - \frac{1/\gamma_i}{\mathbb{E}^1[1/\gamma_1]} \mathbb{E}^1 \left[\frac{\log f_{n-1}(s, y, Z_{n-1}^1, \gamma_1)}{\gamma_1} \right] \right\} \\ &\quad + \frac{1/\gamma_i}{\mathbb{E}^1[1/\gamma_1]} L_{n-1}(s, y). \end{aligned} \quad (2.9)$$

Thanks to the i.i.d. property of (Z_{n-1}^i, γ_i) and the boundedness of f_{n-1}, L_{n-1} and $1/\gamma_i$, it is easy to see that there exists some constant C_{n-1} such that

$$\begin{aligned} &\mathbb{E} \left| \frac{1}{N_p} \sum_{i=1}^{N_p} \phi_{n-1}^{i,*}(s, y, Z_{n-1}^i, \gamma_i) - L_{n-1}(s, y) \right|^2 \\ &\leq \frac{C_{n-1}}{N_p} \mathbb{E}^1 \left[\left| \frac{1}{\gamma_1} - \mathbb{E}^1 \left(\frac{1}{\gamma_1} \right) \right|^2 + \left| \frac{\log f_{n-1}(s, y, Z_{n-1}^1, \gamma_1)}{\gamma_1} - \mathbb{E}^1 \left[\frac{\log f_{n-1}(s, y, Z_{n-1}^1, \gamma_1)}{\gamma_1} \right] \right|^2 \right] \end{aligned}$$

which gives the desired result. Note that the variances in the left-hand side are finite uniformly in $(s, y) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1}$ thanks to the boundedness of functions f_{n-1} and $1/\gamma_i$. \square

2.4 Some implications

Let us consider some implications of the above findings. For simplicity, assume first that there is no external order flow $L \equiv 0$. Recalling that the risk-neutral probability of the up-move at each node is $p^\mathbb{Q} = \frac{(-d)}{u-d}$, one can see from (2.7) that $p_{n-1}(s, y) > p^\mathbb{Q}$ (i.e. positive excess return at this node) occurs if and only if $\mathbb{E}^1[\log(f_{n-1}(s, y, Z_{n-1}^1, \gamma_1))/\gamma_1] < 0$. This happens if V_n , which is determined by the function F , is a decreasing function of s . In Section 4, we shall see numerical examples confirming this point. The corresponding situation occurs when the agents' liability $((-1) \times \text{endowment})$ decreases when the stock price goes up. In this case, adding to the long position in the stock increases the risk, and hence the agents require higher risk premium. Therefore, for a liability whose size varies countercyclically, the more levered financial and investment firms are, the higher the risk premium is demanded. Suppose, on the other hand, that the agents' liability increases when the stock price goes up. For example, imagine that agents have net short position in call options on the stock. Then, the agents have a strong incentive to increase the long position in the same stock, and hence may accept even a negative risk premium.

As one can see from (2.7), there is no need to add idiosyncratic shocks to control the size of risk premium, which is mainly determined by the sensitivity of the liability to the stock price. However, the absence of idiosyncratic shocks gives rise to a very unrealistic market where there is no trade among the agents. From the expression of the optimal position in (2.9), we can observe that the trading volume per capita $\mathbb{E}[|\phi_n^{1,*}|^2]^{\frac{1}{2}}$ in the market is dictated by the variation of the idiosyncratic factors. Note that, by the definition of the mean-field equilibrium, $\mathbb{E}[\phi_n^{i,*}] = 0, \forall i \in \mathbb{N}$ when $L_n = 0$. Therefore, $\mathbb{E}[|\phi_n^{1,*}|^2]^{\frac{1}{2}}$ gives the standard deviation of the stock position among the agents at time $t = t_n$.

In addition to the condition $L \equiv 0$, let us now suppose that the function F is independent of the stock price. We then have $f_{N-1} = 1$ a.s. since V_N is S_N -independent and thus $\phi_{N-1}^{i,*} = 0$ a.s. by (2.9). This in turn makes V_{N-1} independent from S_{N-1} . In this way, a simple induction shows that $f_{n-1} = 1$ a.s. for every $1 \leq n \leq N$ and the equilibrium price distribution becomes equal to the one in the risk-neutral measure. In this case, there is no trade in the market although each agent has different risk aversion, which corresponds to the classical (but a bit uninteresting) example of the representative agent with CARA utility.

Finally, let us turn on the external order flow. It is clearly seen from (2.7), the positive inflow $L > 0$ to the stock market increases the equilibrium risk premium. This may sound slightly counter intuitive since we think a big sell-off in the stock should lead to a sharp decline in the stock price. In order to understand that there is no contradiction, it is important to recall that what we have found above is the transition probabilities so that there exists equilibrium. If there is positive supply of the stock, the agents must accept larger long position (and hence larger risk) in the stock to maintain the balance of demand and supply. Thus the agents require higher risk premium to compensate this additional risk. If the risk premium is not high enough, there would be no equilibrium and thus the stock market might crash.

3 Recursive utility optimization with path-dependent cash flows

In the previous section, we obtained mean-field equilibrium by choosing appropriate transition probabilities in the form of $p_n(s, y)$. Suppose now that the stochastic liability (or (-1) endowment) F is dependent not only on the terminal stock price S_N but also on the stock-price history $(S_n)_{n=0}^N$, which is just as plausible as the previous case. In this case, a quick inspection of the proofs for Theorems 2.1 and 2.2 shows that the transition probabilities of the simple form $p_n(s, y)$ can not clear the market anymore. It strongly suggests that we need path-dependence also in the transition probabilities. We also want to examine if we can include cash spending (i.e. nominal consumption) and to know its impact on the excess return. In this section, we shall thus adopt recursive utility that incorporates standard time-separable utility over nominal consumptions as its special case. We include a path-dependent terminal liability as well as path-dependent incremental endowments in the model.

3.1 The setup and notation

In this section, for each $i \in \mathbb{N}$, we assume that the probability space $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_{t_n}^i)_{n=0}^N, \mathbb{P}^i)$ is endowed with $(\xi_i, \gamma_i, \zeta_i, \psi_i, \delta_i)$ as \mathcal{F}_0^i -measurable random variables in addition to $(\mathcal{F}_{t_n}^i)_{n=0}^N$ -adapted stochastic process $(Z_n^i = Z^i(t_n))_{n=0}^N$. Here, ζ_i is the coefficient of absolute risk aversion for cash spending and the parameter ψ_i is used to control the importance of the continuation utility relative to the current spending. δ_i denotes the coefficient of time preference. We introduce an \mathcal{F}_0^i -measurable 4-tuple $\varrho_i := (\gamma_i, \zeta_i, \psi_i, \delta_i)$ for simpler notation. Let us also introduce the symbol $\mathbf{S}^n := (S_0, S_1, \dots, S_n)$ to denote a stock-price trajectory and $\mathbf{s}^n = (s_0, \dots, s_n) \in \mathcal{S}^n$ as its specific realization. For $\mathbf{s} \in \mathcal{S}^{n-1}$, we also use the symbols $(\mathbf{s}\tilde{u})^n := (\mathbf{s}^{n-1}, s_{n-1}\tilde{u}) \in \mathcal{S}^n$ and $(\mathbf{s}\tilde{d})^n := (\mathbf{s}^{n-1}, s_{n-1}\tilde{d}) \in \mathcal{S}^n$. As in the last section, we shall use the expressions such as $\mathbb{E}^{0,i}[\cdot | \mathbf{S}^{n-1} = \mathbf{s}, y, z^i, \varrho_i]$ for $(\mathbf{s}, y, z^i) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1}$ to denote the conditional expectation $\mathbb{E}^{0,i}[\cdot | \mathbf{S}^{n-1} = \mathbf{s}, Y_{n-1} = y, Z_{n-1}^i = z_i, \varrho_i = \varrho_i]$, where, with the slight abuse of notation, the same symbol is used for a realization of \mathcal{F}_0^i -measurable random variable ϱ_i . Except these points, we will use the same setup and notation given in Section 2.1. Now, let us update Assumption 2.1 for this section.

Assumption 3.1. (i): \tilde{u} and \tilde{d} are real constants satisfying

$$0 < \tilde{d} < \exp(r\Delta) < \tilde{u} < \infty.$$

(ii): Every $(\xi_i, \gamma_i, \zeta_i, \psi_i, \delta_i, (Z_n^i)_{n=0}^N), i = 1, 2, \dots$ has the same distribution.

(iii): There exist real constants $\underline{\xi}, \bar{\xi}, \underline{\gamma}, \bar{\gamma}, \underline{\zeta}, \bar{\zeta}, \underline{\psi}, \bar{\psi}$ and $\underline{\delta}, \bar{\delta}$ so that for every $i \in \mathbb{N}$,

$$\xi_i \in [\underline{\xi}, \bar{\xi}] \subset \mathbb{R},$$

$$\varrho_i := (\gamma_i, \zeta_i, \psi_i, \delta_i) \in \Gamma := [\underline{\gamma}, \bar{\gamma}] \times [\underline{\zeta}, \bar{\zeta}] \times [\underline{\psi}, \bar{\psi}] \times [\underline{\delta}, \bar{\delta}] \subset (0, \infty)^4.$$

(iv): For every $0 \leq n \leq N$, \mathcal{Z}_n is a bounded subset of \mathbb{R}^{d_z} .

(v): For each i , $(Z_n^i)_{n=0}^N$ is a Markov process i.e. $\mathbb{E}^i[f(Z_n^i) | \mathcal{F}_{t_m}^i] = \mathbb{E}^i[f(Z_n^i) | Z_m^i]$ for every bounded measurable function f on \mathcal{Z}_n and $m \leq n$.

(vi): $(Y_n)_{n=0}^N$ is a Markov process i.e. $\mathbb{E}^0[f(Y_n) | \mathcal{F}_{t_m}^0] = \mathbb{E}^0[f(Y_n) | Y_m]$ for every bounded measurable function f on \mathcal{Y}_n and $m \leq n$.

(vii): The transition probabilities of $(S_n)_{n=0}^N$ satisfy, for every $0 \leq n \leq N-1$,

$$\begin{aligned}\mathbb{P}^0(S_{n+1} = \tilde{u}S_n | \mathcal{F}_{t_n}^0) &= \mathbb{P}^0(S_{n+1} = \tilde{u}S_n | \mathbf{S}^n, Y_n) = p_n(\mathbf{S}^n, Y_n), \\ \mathbb{P}^0(S_{n+1} = \tilde{d}S_n | \mathcal{F}_{t_n}^0) &= \mathbb{P}^0(S_{n+1} = \tilde{d}S_n | \mathbf{S}^n, Y_n) = q_n(\mathbf{S}^n, Y_n) := 1 - p_n(\mathbf{S}^n, Y_n),\end{aligned}$$

where $p_n, q_n : \mathcal{S}^n \times \mathcal{Y}_n \rightarrow \mathbb{R}$, $0 \leq n \leq N-1$ are bounded measurable functions satisfying

$$0 < p_n(\mathbf{s}, y), q_n(\mathbf{s}, y) < 1$$

for every $(\mathbf{s}, y) \in \mathcal{S}^n \times \mathcal{Y}_n$.

Under the above assumptions, we have, instead of (2.1), the relation

$$\mathbb{E}^0[f(S_{n+1})g(Y_{n+1}) | \mathcal{F}_{t_n}^0] = \mathbb{E}^0[f(S_{n+1}) | \mathbf{S}^n, Y_n] \mathbb{E}^0[g(Y_{n+1}) | Y_n], \quad 0 \leq n \leq N-1, \quad (3.1)$$

for any bounded measurable functions f and g .

Remark 3.1. As in the last section, the condition $0 < p_n(\mathbf{s}, y), q_n(\mathbf{s}, y) < 1, \forall (\mathbf{s}, y) \in \mathcal{S}^n \times \mathcal{Y}_n, 0 \leq n \leq N-1$ guarantees the equivalence of $\mathbb{P}^0 \circ S_n^{-1}$ and $\mathbb{Q} \circ S_n^{-1}$. Hence the system is arbitrage free.

3.2 The individual optimization problem

In this section, as mentioned before, we assume that each agent- i not only engages in self-financing trading with the money-market account and the risky stock, but she is also allowed to spend some cash at the beginning of each period. Moreover, she receives a stochastic endowment at each time $t_n, 1 \leq n \leq N$. Thus the wealth of the agent- i ($X_n^i := X^i(t_n)$) $_{n=0}^N$ follows the dynamics

$$\begin{aligned}X_{n+1}^i &= \exp(r\Delta)(X_n^i - c_n^i \Delta - \phi_n^i) + \phi_n^i \tilde{R}_{n+1} + g_{n+1}(\mathbf{S}^{n+1}, Y_{n+1}, Z_{n+1}^i) \\ &= \beta(X_n^i - c_n^i \Delta) + \phi_n^i R_{n+1} + g_{n+1}(\mathbf{S}^{n+1}, Y_{n+1}, Z_{n+1}^i),\end{aligned}$$

where $X_0^i = \xi_i$. Here, $c_n^i, 0 \leq n \leq N-1$ denotes the cash spending at t_n , which is scaled to the period's rate. $g_n(\mathbf{S}^n, Y_n, Z_n^i), 1 \leq n \leq N$ is the stochastic endowment (i.e. income originating from the agent's other business lines) paid at t_n , which is dependent on the stock-price trajectory \mathbf{S}^n in addition to the common and the idiosyncratic shocks (Y_n, Z_n^i) . Recall that $R_n := \tilde{R}_n - \exp(r\Delta)$. As discussed in Remark 2.2, including such stochastic endowments appears to be almost unavoidable for a realistic model of financial firms.

We suppose that the $(\mathcal{F}_{t_n}^{0,i})_{n=0}^N$ -adapted process of utilities $(U_n^i)_{n=0}^N$ are defined recursively by

$$\begin{aligned}U_n^i &:= -\frac{1}{\zeta_i} \log \left\{ e^{-\zeta_i c_n^i \Delta} + \delta_i \exp \left(\frac{\psi_i}{\gamma_i} \log (\mathbb{E}^{0,i}[e^{-\gamma_i U_{n+1}^i} | \mathcal{F}_{t_n}^{0,i}]) \right) \right\}, \\ \Leftrightarrow e^{-\zeta_i U_n^i} &= e^{-\zeta_i c_n^i \Delta} + \delta_i \exp \left(\frac{\psi_i}{\gamma_i} \log (\mathbb{E}^{0,i}[e^{-\gamma_i U_{n+1}^i} | \mathcal{F}_{t_n}^{0,i}]) \right),\end{aligned} \quad (3.2)$$

with the terminal condition

$$U_N^i := X_N^i - F(\mathbf{S}^N, Y_N, Z_N^i).$$

Each agent- i is supposed to solve the optimization problem

$$\sup_{(\phi_n^i, c_n^i)_{n=0}^{N-1} \in \mathbb{A}^i} U_0^i, \quad (3.3)$$

over the admissible space defined by

$$\mathbb{A}^i := \{(\phi_n^i, c_n^i)_{n=0}^{N-1}; (\phi_n^i, c_n^i) \text{ is an } \mathcal{F}_n^{0,i}\text{-measurable } \mathbb{R}^2\text{-valued random variable}\}.$$

For simplicity, we do not restrict (c_n^i) to non-negative values. One may interpret negative spending as positive income from costly labor for the corresponding period.

Assumption 3.2. (i): *The function $F : \mathcal{S}^N \times \mathcal{Y}_N \times \mathcal{Z}_N \rightarrow \mathbb{R}$ is measurable and bounded.*
(ii): *For every $1 \leq n \leq N$, the function $g_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \rightarrow \mathbb{R}$ is measurable and bounded.*

Before going into the details, let us consider the special case: $\zeta_i = \gamma_i = \psi_i$. In this case, we have

$$\begin{aligned} e^{-\zeta_i U_n^i} &= e^{-\zeta_i c_n^i} \Delta + \delta_i \mathbb{E}^{0,i}[e^{-\zeta_i U_{n+1}^i} | \mathcal{F}_{t_n}^{0,i}] \\ &= e^{-\zeta_i c_n^i} \Delta + \mathbb{E}^{0,i}[\delta_i e^{-\zeta_i c_{n+1}^i} \Delta | \mathcal{F}_{t_n}^{0,i}] + \delta_i^2 \mathbb{E}^{0,i}[e^{-\zeta_i U_{n+2}^i} | \mathcal{F}_{t_n}^{0,i}] \\ &= \dots = e^{-\zeta_i c_n^i} \Delta + \mathbb{E}^{0,i}\left[\sum_{k=n+1}^{N-1} \delta_i^{k-n} e^{-\zeta_i c_k^i} \Delta + \delta_i^{N-n} e^{-\zeta_i U_N^i} | \mathcal{F}_{t_n}^{0,i}\right], \end{aligned}$$

which thus corresponds to the standard time-separable utility over nominal consumptions. One can see that the parameter ψ_i determines the relative importance of the continuation utility. We now derive the optimal strategy for each agent with respect to the above-defined recursive utility.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 be in force. Then the problem (3.3) has an unique optimal solution $(\phi_{n-1}^{i,*}, c_{n-1}^{i,*})_{n=1}^N$, where $(\phi_{n-1}^{i,*})_{n=1}^N$ and $(c_{n-1}^{i,*})_{n=1}^N$ are a.s. bounded processes defined by measurable functions $\phi_{n-1}^{i,*} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \rightarrow \mathbb{R}$ and $c_{n-1}^{i,*} : \mathbb{R} \times \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \rightarrow \mathbb{R}$ as $\phi_{n-1}^{i,*} := \phi_{n-1}^{i,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}, \varrho_i)$ and $c_{n-1}^{i,*} := c_{n-1}^{i,*}(X_{n-1}^i, \mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}, \varrho_i)$ respectively, where, for each $(x^i, \mathbf{s}, y, z^i, \varrho_i) \in \mathbb{R} \times \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$,*

$$\phi_{n-1}^{i,*}(\mathbf{s}, y, z^i, \varrho_i) := \frac{1}{\gamma_i \eta_n^i (u - d)} \left\{ \log \left(-\frac{p_{n-1}(\mathbf{s}, y) u}{q_{n-1}(\mathbf{s}, y) d} \right) + \log(f_{n-1}(\mathbf{s}, y, z^i, \varrho_i)) \right\}, \quad (3.4)$$

$$c_{n-1}^{i,*}(x^i, \mathbf{s}, y, z^i, \varrho_i) := \frac{\psi_i \eta_n^i \beta}{\zeta_i + \Delta \psi_i \eta_n^i \beta} x^i - \frac{1}{\zeta_i + \Delta \psi_i \eta_n^i \beta} \log \left\{ \frac{\delta_i \psi_i \eta_n^i \beta}{\zeta_i} \exp \left(\frac{\psi_i}{\gamma_i} \log \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \right) \right\}. \quad (3.5)$$

$f_{n-1} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \rightarrow \mathbb{R}$ is an a.s. strictly positive and bounded measurable function:

$$f_{n-1}(\mathbf{s}, y, z^i, \varrho_i) := \frac{\mathbb{E}^{0,i}[\exp(\gamma_i[V_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i]}{\mathbb{E}^{0,i}[\exp(\gamma_i[V_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i]}, \quad (3.6)$$

and $V_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \rightarrow \mathbb{R}$ is an a.s. bounded measurable function defined recursively by

$$V_{n-1}(\mathbf{s}, y, z^i, \varrho_i) := \frac{\eta_{n-1}^i}{\eta_n^i \gamma_i \beta} \log \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) + \frac{1}{\zeta_i + \Delta \psi_i \eta_n^i \beta} \log \left(\frac{\delta_i \psi_i \eta_n^i \beta}{\zeta_i} \right) - \frac{1}{\zeta_i} \log(\eta_{n-1}^i), \quad (3.7)$$

with

$$\begin{aligned} \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \\ := p_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi_{n-1}^{i,*} u} \mathbb{E}^{0,i} [\exp(\gamma_i [V_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i] \\ + q_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi_{n-1}^{i,*} d} \mathbb{E}^{0,i} [\exp(\gamma_i [V_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i], \end{aligned} \quad (3.8)$$

starting from the terminal condition $V_N(\mathbf{S}^N, Y_N, Z_N^i, \varrho_i) := F(\mathbf{S}^N, Y_N, Z_N^i)$. $(\eta_n^i)_{n=0}^N$ are strictly positive and bounded \mathcal{F}_0^i -measurable random variables given by the recursive relation:

$$\eta_{n-1}^i := \frac{\psi_i \eta_n^i \beta}{\zeta_i + \Delta \psi_i \eta_n^i \beta}, \quad \eta_N^i \equiv 1. \quad (3.9)$$

Proof. We first hypothesize that the utility U_n^i at $t = t_n$ is given by the following form:

$$U_n^i(X_n^i, \mathbf{S}^n, Y_n, Z_n^i, \varrho_i) = \eta_n^i X_n^i - V_n(\mathbf{S}^n, Y_n, Z_n^i, \varrho_i),$$

where $V_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \rightarrow \mathbb{R}$ is a measurable a.s. bounded function and η_n^i is \mathcal{F}_0^i -measurable strictly positive and bounded random variable. The hypothesis obviously holds at the terminal point with $V_N(\mathbf{S}^N, Y_N, Z_N^i, \varrho_i) = F(\mathbf{S}^N, Y_N, Z_N^i)$ and $\eta_N^i \equiv 1$. We shall show by induction that our hypothesis holds in every period. Under the hypothesis, the problem for the agent- i at t_{n-1} becomes to find $\mathcal{F}_{n-1}^{0,i}$ -measurable strategy (ϕ^i, c^i) that solves

$$\inf_{(\phi^i, c^i)} \left\{ e^{-\zeta_i c^i} \Delta + \delta_i \exp \left(\frac{\psi_i}{\gamma_i} \log \left(\mathbb{E}^{0,i} [e^{-\gamma_i U_n^i(X_n^i, \mathbf{S}^n, Y_n, Z_n^i, \varrho_i)} | \mathcal{F}_{n-1}^{0,i}] \right) \right) \right\}. \quad (3.10)$$

We consider the problem on the set $\{(X_{n-1}^i, \mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i) = (x^i, \mathbf{s}, y, z^i, \varrho_i)\}$. By Assumption 3.1 (vii), (3.1), and the above hypothesis, we have

$$\begin{aligned} \mathbb{E}^{0,i} [e^{-\gamma_i U_n^i(X_n^i, \mathbf{S}^n, Y_n, Z_n^i, \varrho_i)} | x^i, \mathbf{s}, y^i, z^i, \varrho_i] \\ = \mathbb{E}^{0,i} \left[\exp \left(-\gamma_i \eta_n^i (\beta(x^i - c^i \Delta) + \phi^i R_n + g_n(\mathbf{S}^n, Y_n, Z_n^i)) + \gamma_i V_n(\mathbf{S}^n, Y_n, Z_n^i, \varrho_i) \right) | \mathbf{s}, y, z^i, \varrho_i \right] \\ = e^{-\gamma_i \eta_n^i \beta(x^i - c^i \Delta)} \left\{ p_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i u} \mathbb{E}^{0,i} [e^{\gamma_i [V_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i)]} | y, z^i, \varrho_i] \right. \\ \left. + q_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i d} \mathbb{E}^{0,i} [e^{\gamma_i [V_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i)]} | y, z^i, \varrho_i] \right\}. \end{aligned}$$

Thus the problem (3.10) can be restated as

$$\begin{aligned} \inf_{(\phi^i, c^i)} \left\{ e^{-\zeta_i c^i} \Delta + \delta_i e^{-\psi_i \eta_n^i \beta(x^i - c^i \Delta)} \right. \\ \times \exp \left(\frac{\psi_i}{\gamma_i} \log \left\{ p_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i u} \mathbb{E}^{0,i} [e^{\gamma_i [V_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i)]} | y, z^i, \varrho_i] \right. \right. \\ \left. \left. + q_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i d} \mathbb{E}^{0,i} [e^{\gamma_i [V_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i)]} | y, z^i, \varrho_i] \right\} \right\}. \end{aligned}$$

The optimization over (ϕ^i, c^i) can now be solved separately. Since $d < 0 < u$, the optimal $\phi^{i,*}$

is characterized uniquely by

$$0 = p_{n-1}(\mathbf{s}, y) u e^{-\gamma_i \eta_n^i \phi^i u} \mathbb{E}^{0,i} \left[e^{\gamma_i [V_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{u})^n, Y_n, Z_n^i)]} | y, z^i, \varrho_i \right] \\ + q_{n-1}(\mathbf{s}, y) d e^{-\gamma_i \eta_n^i \phi^i d} \mathbb{E}^{0,i} \left[e^{\gamma_i [V_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\tilde{d})^n, Y_n, Z_n^i)]} | y, z^i, \varrho_i \right],$$

which gives the desired solution (3.4) for $\phi_{n-1}^{i,*}$ with f_{n-1} defined as in (3.6). Thanks to the boundedness of g_n and our hypothesis on V_n , f_{n-1} is proved to be an a.s. strictly positive and bounded function on $\mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$. Combined with the assumption on (p_{n-1}, q_{n-1}) and our hypothesis on η_n^i , $\phi_{n-1}^{i,*}$ is also a.s. bounded on $\mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$.

With \tilde{V}_{n-1} defined as in (3.8), the optimization with respect to c^i reduces to

$$\inf_{c^i} \left\{ e^{-\zeta_i c^i} \Delta + \delta_i e^{-\psi_i \eta_n^i \beta (x^i - c^i \Delta)} \exp \left(\frac{\psi_i}{\gamma_i} \log \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \right) \right\}.$$

Thus the optimal solution is characterized by the equation:

$$0 = -\zeta_i e^{-\zeta_i c^i} + \delta_i \psi_i \eta_n^i \beta e^{-\psi_i \eta_n^i \beta (x^i - c^i \Delta)} \exp \left(\frac{\psi_i}{\gamma_i} \log \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \right), \quad (3.11)$$

which gives the unique solution $c_{n-1}^{i,*}$ in (3.5) as desired. Note that it is a.s. bounded if the wealth x^i at t_{n-1} is bounded. Therefore, once our induction is complete, the spending process is shown to be a.s. bounded since ξ_i takes values in a bounded interval $[\xi, \bar{\xi}]$.

In order to complete the induction argument, we need to obtain the utility U_{n-1}^i for the next period. By (3.2), it satisfies

$$e^{-\zeta_i U_{n-1}^i} = e^{-\zeta_i c_{n-1}^{i,*}} \Delta + \delta_i e^{-\psi_i \eta_n^i \beta (x^i - c_{n-1}^{i,*} \Delta)} \exp \left(\frac{\psi_i}{\gamma_i} \log \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \right).$$

The right-hand side of the above equality can be evaluated by using (3.11) as

$$e^{-\zeta_i c_{n-1}^{i,*}} \Delta + \frac{1}{\psi_i \eta_n^i \beta} \zeta_i e^{-\zeta_i c_{n-1}^{i,*}} = \frac{\zeta_i + \Delta \psi_i \eta_n^i \beta}{\psi_i \eta_n^i \beta} e^{-\zeta_i c_{n-1}^{i,*}} \\ = \frac{\zeta_i + \Delta \psi_i \eta_n^i \beta}{\psi_i \eta_n^i \beta} \exp \left\{ -\frac{\zeta_i \psi_i \eta_n^i \beta}{\zeta_i + \Delta \psi_i \eta_n^i \beta} x^i + \frac{\zeta_i}{\zeta_i + \Delta \psi_i \eta_n^i \beta} \log \left[\frac{\delta_i \psi_i \eta_n^i \beta}{\zeta_i} \exp \left(\frac{\psi_i}{\gamma_i} \log \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \right) \right] \right\}.$$

It follows that the utility U_{n-1}^i is given by

$$U_{n-1}^i(x^i, \mathbf{s}, y, z^i, \varrho_i) = \frac{\psi_i \eta_n^i \beta}{\zeta_i + \Delta \psi_i \eta_n^i \beta} x^i - \frac{1}{\zeta_i + \Delta \psi_i \eta_n^i \beta} \log \left\{ \frac{\delta_i \psi_i \eta_n^i \beta}{\zeta_i} \exp \left(\frac{\psi_i}{\gamma_i} \log \tilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \right) \right\} \\ - \frac{1}{\zeta_i} \log \left(\frac{\zeta_i + \Delta \psi_i \eta_n^i \beta}{\psi_i \eta_n^i \beta} \right).$$

on the set $\{(X_{n-1}^i, \mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i) = (x^i, \mathbf{s}, y, z^i, \varrho_i)\}$. By setting the right-hand side equal to $\eta_{n-1}^i x^i - V_{n-1}(\mathbf{s}, y, z^i, \varrho_i)$, we obtained the desired recursive relation for η_n^i and V_n . It is now clear that $(\eta_n^i)_{n=1}^N$ are \mathcal{F}_0^i -measurable, strictly positive and bounded, and that V_n is a bounded function on $\mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \rightarrow \mathbb{R}$ for every $0 \leq n \leq N$. \square

3.3 Mean-field equilibrium among the agents with recursive utilities

Finally, as a main goal of this section, we shall derive a set of transition probabilities of the stock so that the mean-field equilibrium holds among the agents with recursive utilities. As before, we incorporate the existence of stochastic external order flow L_n at each t_n , but it is now allowed to be path-dependent on the stock price:

Assumption 3.3. *For every $1 \leq n \leq N$, $L_{n-1} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \rightarrow \mathbb{R}$ is a bounded measurable function.*

Definition 3.1. *We say that the system is in the mean-field equilibrium if*

$$\lim_{N_p \rightarrow \infty} \frac{1}{N_p} \sum_{i=1}^{N_p} \phi_{n-1}^{i,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i) = L_{n-1}(\mathbf{S}^{n-1}, Y_{n-1}),$$

\mathbb{P} -a.s. for every $1 \leq n \leq N$ with $\phi_{n-1}^{i,*}$ defined by (3.4).

Since $(Z^i, \varrho_i), i \in \mathbb{N}$ are independent and identically distributed and also independent of the process (S, Y) , the above condition for the mean-field equilibrium can be represented by

$$\mathbb{E}^1[\phi_{n-1}^{1,*}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1)] = L_{n-1}(\mathbf{s}, y)$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$, $1 \leq n \leq N$. It is now straightforward to derive the counterpart of Theorem 2.2.

Theorem 3.2. *Let Assumptions 3.1, 3.2 and 3.3 be in force. Then there exists a unique mean-field equilibrium and the associated transition probabilities of the stock are given by*

$$\begin{aligned} p_{n-1}(\mathbf{s}, y) &:= \mathbb{P}^0(S_n = \tilde{u}S_{n-1} | (\mathbf{S}^{n-1}, Y_{n-1}) = (\mathbf{s}, y)) \\ &= (-d) / \left\{ u \exp\left(\frac{1}{\mathbb{E}^1[1/(\gamma_1 \eta_n^1)]} \left[\mathbb{E}^1\left(\frac{\log(f_{n-1}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1))}{\gamma_1 \eta_n^1}\right) - (u-d)L_{n-1}(\mathbf{s}, y) \right] \right) - d \right\} \end{aligned}$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$, $1 \leq n \leq N$. Moreover, with the above transition probabilities, there exists some positive constant C_{n-1} such that

$$\mathbb{E} \left| \frac{1}{N_p} \sum_{i=1}^{N_p} \phi_{n-1}^{i,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i) - L_{n-1}(\mathbf{S}^{n-1}, Y_{n-1}) \right|^2 \leq \frac{C_{n-1}}{N_p}$$

for every $1 \leq n \leq N$, which gives the convergence rate in the large population limit.

Proof. The proof is analogous to that of Theorem 2.2. The expression for the transition probabilities $(p_{n-1}(\mathbf{s}, y))$ is a direct consequence of (3.4). Assumptions 3.2 and 3.3 guarantees that f_{N-1} is a.s. strictly positive and bounded and hence the condition (vii) in Assumption 3.1 is satisfied for (p_{N-1}, q_{N-1}) . It then makes $\phi_{N-1}^{i,*}$ a.s. bounded and thus \tilde{V}_{N-1} becomes a.s. strictly positive and bounded. It then follows that V_{N-1} and hence f_{N-2} are a.s. bounded functions, which shows that (p_{N-2}, q_{N-2}) satisfies the condition (vii). In this way, a simple induction shows that the transition probabilities satisfy Assumption 3.1 (vii) for every period. The second claim can be

proved by noticing that the optimal position $\phi_{n-1}^{i,*}$ in the equilibrium is given by for $(\mathbf{s}, y, z^i, \varrho_i) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$,

$$\begin{aligned} \phi_{n-1}^{i,*}(\mathbf{s}, y, z^i, \varrho_i) = & \frac{1}{(u-d)} \left\{ \frac{\log f_{n-1}(\mathbf{s}, y, z^i, \varrho_i)}{\gamma_i \eta_n^i} - \frac{1/(\gamma_i \eta_n^i)}{\mathbb{E}^1[1/(\gamma_1 \eta_n^1)]} \mathbb{E}^1 \left(\frac{\log f_{n-1}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1)}{\gamma_1 \eta_n^1} \right) \right\} \\ & + \frac{1/(\gamma_i \eta_n^i)}{\mathbb{E}^1[1/(\gamma_1 \eta_n^1)]} L_{n-1}(\mathbf{s}, y), \end{aligned} \quad (3.12)$$

and the fact that all the relevant components are bounded. \square

Remark 3.2. *It is easy to check that the required time-horizon of the path dependence in the transition probabilities is equal to that in the liability and the incremental endowments. In particular, if F and g_n depend only on S_N and S_n respectively as in the previous section, the transition probabilities of the form $(p_{n-1}(s, y), q_{n-1}(s, y))$ with $s \in \mathcal{S}_{n-1}$ are enough to achieve the equilibrium. Simple replacement of $\mathbf{s} \in \mathcal{S}^{n-1}$ by $s \in \mathcal{S}_{n-1}$, we obtain the corresponding results for Theorems 3.1 and 3.2.*

On the relation between the mean-field price distribution and the one in the risk-neutral measure, most of the discussions given in Section 2.4 still hold. In particular, the situation $p_{n-1}(\mathbf{s}, y) > p^Q$ (i.e. positive excess return) occurs if and only if $\mathbb{E}^1 \left[\frac{\log(f_{n-1}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1))}{\gamma_1 \eta_n^1} \right] < 0$, when there is no external order flow $L_{n-1} \equiv 0$. As one can see from (3.6), f_{n-1} has contributions from the liability V_n and the incremental endowment g_n . If the liability size decreases and the endowment size simultaneously increases as the stock price grows, both effects will amplify the deviations from the risk-neutral distribution toward a higher excess return.

Since the equilibrium price distribution of the stock is determined by the need for risk hedging, the relative importance of the continuation utility with respect to the current nominal consumption is a crucial factor in controlling the size of the risk premium. From the expressions in (3.7) and (3.8), one can expect that the ratio

$$\frac{\eta_{n-1}^i}{\eta_n^i}$$

is the key value. Using $\beta \simeq 1$ and $\Delta \ll 1$, we have $\eta_{n-1}^i/\eta_n^i \simeq \psi_i/\zeta_i$ from (3.9). Therefore, if $\psi_i < \zeta_i$ holds for the majority of agents, we expect that the relative importance of the continuation utility quickly decays and we would have only a small impact from it for earlier periods. In this case, significant deviations from the risk-neutral price distribution can be observed only in the later periods near the maturity. On the other hand, in the case of $\psi_i > \zeta_i$, we can expect to see significant deviations throughout the interval. We shall confirm this behavior in Section 4.

As for the expected trading volume $\mathbb{E}[|\phi_n^{1,*}|^2]^{\frac{1}{2}}$, which gives the standard deviation of the stock position among the agents at $t = t_n$, we have the consistent result as in Section 2.4. The expression for $\phi_{n-1}^{i,*}$ in (3.12) shows that its size is governed by the variation of idiosyncratic factors defined on the space $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$. It is also quite consistent with our intuition that the agents' heterogeneity in idiosyncratic factors is the origin of the trading activity in the market. Moreover, we can make use of the degrees of freedom in the process (Z_n^i) , in particular its volatility, to obtain the desired trading volume. We shall confirm this claim in Section 4.2.

Finally, let us comment on the fact that the constant shifts in F and g_n , i.e. $F \mapsto F + c$ and $g_n \mapsto g_n + c'$ with some constants $c, c' \in \mathbb{R}$ do not affect the equilibrium price distribution. This property can be checked by a simple induction as follows:

- By (3.6), the value of f_{N-1} remains unchanged and so are $p_{N-1}(\mathbf{s}, y)$ and $\phi_{N-1}^{i,*}$.
- The value of \tilde{V}_{N-1} is changed only by an \mathcal{F}_0^i -measurable multiplicative factor.
- The value of V_{N-1} is only shifted by an \mathcal{F}_0^i -measurable term.
- Thus f_{N-2} remains once again unchanged, and so are $p_{N-2}(\mathbf{s}, y)$ and $\phi_{N-2}^{i,*}$. \dots

Therefore, the signs of F and g_n can be altered without affecting the equilibrium price distribution. Note however that, the cash spending is affected by these shifts.

3.4 Mean-field equilibrium of multiple populations

Let us briefly mention the construction of the mean-field equilibrium for multiple populations. The population ratio among the groups must be kept constant when the large population limit is taken. Suppose, for example, that population ratio among the m groups are given by $w_p \in (0, 1)$, $p = 1, \dots, m$ satisfying $\sum_{p=1}^m w_p = 1$. We enlarge the product probability space by introducing $(\Omega^{i,p}, \mathcal{F}^{i,p}, (\mathcal{F}_{t_n}^{i,p})_{n=0}^N, \mathbb{P}^{i,p})$ for every $p = 1, \dots, m$ and $i \in \mathbb{N}$. If we denote the optimal trade position of the agent- i in the group p by $\phi_{n-1}^{i,p,*}$, the condition for the mean-field equilibrium is, with obvious notation,

$$\sum_{p=1}^m w_p \mathbb{E}^{1,p} [\phi_{n-1}^{1,p,*}(\mathbf{s}, y, Z_{n-1}^{1,p}, \varrho_{1,p})] = L_{n-1}(\mathbf{s}, y) \quad (3.13)$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$, $1 \leq n \leq N$. Since the factor $\log\left(-\frac{p_{n-1}(\mathbf{s}, y)u}{q_{n-1}(\mathbf{s}, y)d}\right)$ in the optimal position $\phi_{n-1}^{i,p,*}$ is the same across the populations, solving the equation (3.13) is straightforward. We can even mix the populations with standard exponential utilities and the recursive utilities just discussed.

As a simple example, consider the situation where the external order flow is absent $L \equiv 0$ and the agents discussed in Section 2 and those in Section 3 have the same population size in the market. Distinguishing the expectations and variables for the first group by tilde (such as $\tilde{\mathbb{E}}^1[\cdot]$ and $\tilde{\gamma}_1$) for simplicity, the equation (3.13) yields

$$\begin{aligned} 0 = & \log\left(-\frac{p_{n-1}(\mathbf{s}, y)u}{q_{n-1}(\mathbf{s}, y)d}\right) \left(\mathbb{E}^1\left[\frac{1}{\gamma_1 \eta_n^1}\right] + \frac{\beta^n}{\beta^N} \tilde{\mathbb{E}}^1\left[\frac{1}{\tilde{\gamma}_1}\right] \right) \\ & + \mathbb{E}^1\left[\frac{\log f_{n-1}(\mathbf{s}, y, Z_{n-1}^1, \varrho_i)}{\gamma_1 \eta_n^1}\right] + \frac{\beta^n}{\beta^N} \tilde{\mathbb{E}}^1\left[\frac{\log \tilde{f}_{n-1}(s, y, \tilde{Z}_{n-1}^1, \tilde{\gamma}_1)}{\tilde{\gamma}_1}\right] \end{aligned}$$

on the set $\{(\mathbf{S}^{n-1}, Y_{n-1}) = (\mathbf{s}, y)\}$, where $s := s_{n-1}$. It is easy to solve this equation to get the

equilibrium transition probability as

$$\begin{aligned} p_{n-1}(\mathbf{s}, y) &= \mathbb{P}^0(S_n = \tilde{u}S_{n-1} | (\mathbf{S}^{n-1}, Y_{n-1}) = (\mathbf{s}, y)) \\ &= (-d) / \left\{ u \exp \left(\frac{\mathbb{E}^1 \left[\frac{\log f_{n-1}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1)}{\gamma_1 \eta_n^1} \right] + \frac{\beta^n}{\beta^N} \tilde{\mathbb{E}}^1 \left[\frac{\log \tilde{f}_{n-1}(s, y, \tilde{Z}_{n-1}^1, \tilde{\gamma}_1)}{\tilde{\gamma}_1} \right]}{\mathbb{E}^1 \left[\frac{1}{\gamma_1 \eta_n^1} \right] + \frac{\beta^n}{\beta^N} \tilde{\mathbb{E}}^1 \left[\frac{1}{\tilde{\gamma}_1} \right]} \right) - d \right\}. \end{aligned}$$

One can prove its well-posedness across the whole time interval in the same way as in Theorems 2.2 and 3.2. Moreover, if we turn off the path-dependence in F and g_n at least, the numerical evaluation of the equilibrium distribution can be carried out in a simple manner as in the next section.

Note that these results are quite remarkable when compared with the situation in the continuous-time setting. In fact, if we try to solve the corresponding problem in the formulation of Fujii & Sekine [20], we would obtain a coupled system of mean-field qg-BSDEs, whose well-posedness would be much harder than the single population case, not to mention its numerical evaluation.

4 Numerical examples

In this section, we provide some numerical examples for the models introduced in Sections 2 and 3 but without path dependence. Since the models are too flexible for thorough analysis, we focus only on a few simple setups to grasp characteristic behaviors of the mean-field price distributions. In order to reduce numerical cost, we assume that Y and Z^i are also one-dimensional discrete processes taking values on binomial trees, and that all the \mathcal{F}_0^i -measurable random variables are uniformly distributed over finite sets.

4.1 Utility for terminal wealth

Let us first consider the model discussed in Section 2. γ_i is assumed to have a uniform distribution over $(N_\gamma + 1)$ discrete values given by

$$\gamma_i(k_\gamma) := \underline{\gamma} + (\bar{\gamma} - \underline{\gamma}) \frac{k_\gamma}{N_\gamma}, \quad k_\gamma = 0, \dots, N_\gamma.$$

The process $(Z_n^i)_{n=0}^N$ is supposed to follow a one-dimensional binomial process modeled by

$$Z_{n+1}^i = Z_n^i R_{n+1}^i,$$

where (R_n^i) is an $(\mathcal{F}_{t_n}^i)$ -adapted process taking values either u_z or d_z , where $R_{n+1}^i = u_z$ occurs with probability p_z and $R_{n+1}^i = d_z$ with $q_z := 1 - p_z$. We take $u_z = (d_z)^{-1} = \exp(\sigma_z \sqrt{\Delta})$. We also assume $Z_0^i = z_0 \in (0, \infty)$ common for all the agents in order to reduce numerical costs. We model the process $(Y_n)_{n=0}^N$ similarly but it is assumed to follow an approximate Gaussian process:

$$Y_{n+1} = Y_n + R_{n+1}^y,$$

where (R_n^y) is an $(\mathcal{F}_{t_n}^0)$ -adapted process taking values either u_y or d_y , where $R_{n+1}^y = u_y$ with probability p_y and $R_{n+1}^y = d_y$ with $q_y := 1 - p_y$. We take $u_y = (-d_y) = \sigma_y \sqrt{\Delta}$. Finally, for the stock-price process (S_n) , we set $\tilde{u} = (\tilde{d})^{-1} = \exp(\sigma \sqrt{\Delta})$.

The parameter values to be used throughout this subsection are summarized in Table 1 below. Recall that the initial wealth ξ_i is irrelevant for our analysis.

parameter	$\underline{\gamma}$	$\bar{\gamma}$	N_γ	z_0	σ_z	p_z	Y_0	σ_y	p_y	S_0	σ	r	T	N
value	0.5	1.5	4	1.0	12%	0.5	1.0	12%	0.5	1.0	15%	3.3%	3yr	48

Table 1: parameter values

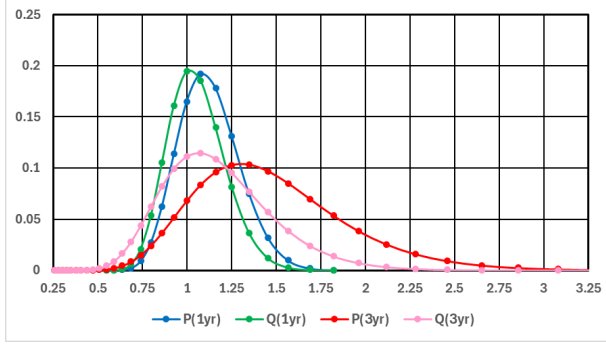


Figure 1: Comparison of the marginal price distributions under the equilibrium measure (P) and the risk-neutral measures (Q) at 1-year and 3-year points for (4.1).

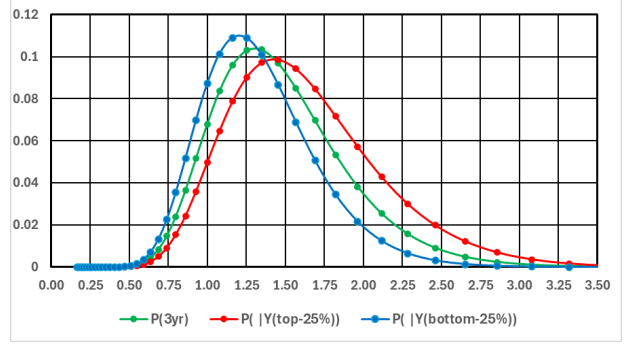


Figure 2: Comparison of the marginal price distribution P and the conditional price distributions $P(\cdot|Y^{\text{top}-25\%})$ and $P(\cdot|Y^{\text{bottom}-25\%})$ at 3-year point for (4.1).

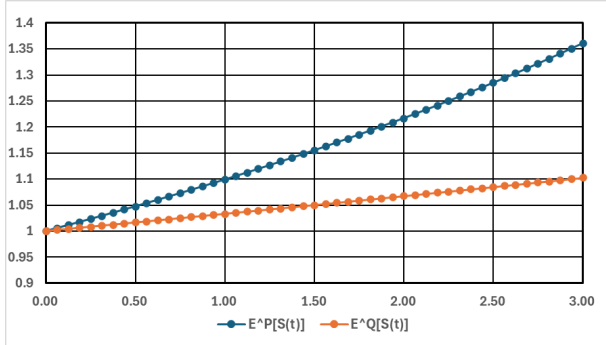


Figure 3: Comparison of the expected values of $S(t_n)$ under the equilibrium and the risk-neutral measures for (4.1).

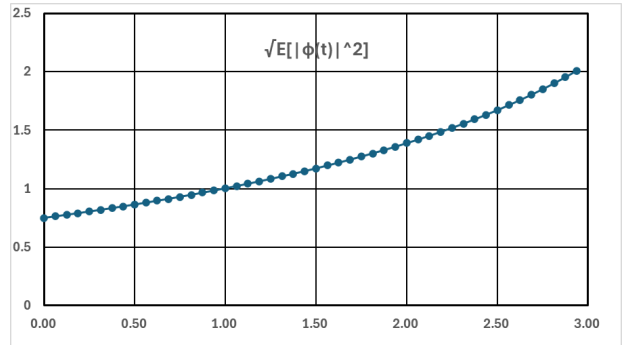


Figure 4: The time evolution of the expected trading volume $\mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$ of the optimal stock position for (4.1).

Let us first assume that there is no external order flow $L_n \equiv 0$, and that the terminal liability F is given by the function

$$F(S_N, Y_N, Z_N^i) := C - 3S_N Y_N Z_N^i, \quad (4.1)$$

where $C \in \mathbb{R}$ is an arbitrary real constant. Since the result is unchanged by the constant shift, one can adjust, if necessary, the constant C to make the liability positive. From the discussion in Section 2.4, we expect that the equilibrium price distribution for this liability has positive excess return. We can check if this is actually the case by observing Figure 1, which presents the comparison of the marginal price distributions under the equilibrium measure (P) and the

risk-neutral measure (Q) at 1-year and 3-year points.

We can also provide the conditional price distribution $\mathbb{P}(S_n \in A | Y_n = y)$, $\forall A \subset \mathcal{S}_n$ for each $y \in \mathcal{Y}_n$. At 3-year point, the value of Y that marks the top 25% ($Y^{\text{top-25}}$) (resp. bottom 25% ($Y^{\text{bottom-25}}$)) is equivalent to a total of 36 up moves (resp. 12 up moves). Comparison of the conditional distributions $\{\mathbb{P}(S_n = s | Y^{\text{top-25}}), \mathbb{P}(S_n = s | Y^{\text{bottom-25}}), \forall s \in \mathcal{S}_n\}$ and the marginal distribution at 3-year point ($n = 48$) is given in Figure 2. As expected from the functional form in (4.1), the deviations from the risk-neutral distribution are positive and become larger for the larger value of Y . In Figure 3, we provide the time-evolution of $(\mathbb{E}^{\mathbb{P}}[S(t)], \mathbb{E}^Q[S(t)])$, the expected value of the stock price under the equilibrium measure and the risk-neutral measure, respectively. Figure 4 gives the time-evolution of the expected trading volume $\sqrt{\mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2]}$.

In Figures 5, 6, 7 and 8, we present the corresponding results for the different liability function

$$F(S_N, Y_N, Z_N^i) := C + 3S_N Y_N Z_N^i \quad (4.2)$$

which exhibits the opposite sign of sensitivity to the stock price. We see that the excess return becomes negative in this case.

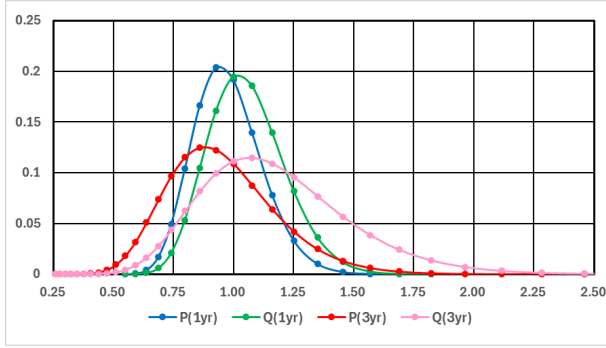


Figure 5: Comparison of the marginal price distributions under the equilibrium measure (P) and the risk-neutral measures (Q) at 1-year and 3-year points for (4.2).

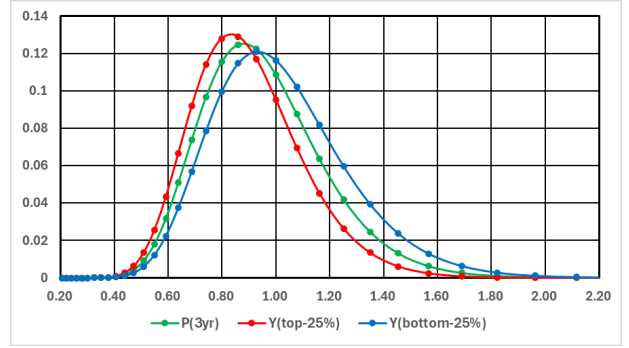


Figure 6: Comparison of the marginal price distribution P and the conditional price distributions $P(\cdot | Y^{\text{top-25}})$ and $P(\cdot | Y^{\text{bottom-25}})$ at 3-year point for (4.2).

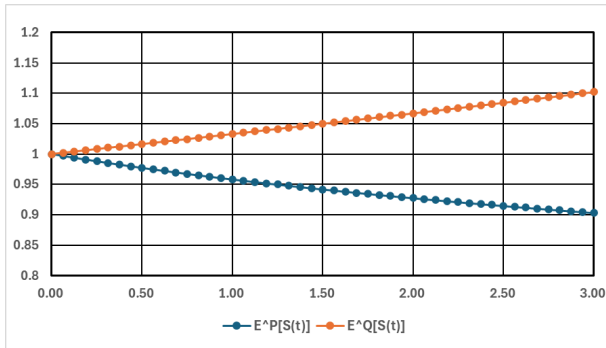


Figure 7: Comparison of the expected values of $S(t_n)$ under the equilibrium and the risk-neutral measures for (4.2).

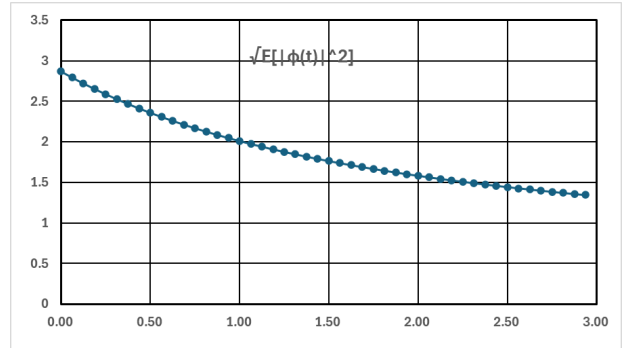


Figure 8: The time evolution of the expected trading volume $\mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$ of the optimal stock position for (4.2).

Now, let us turn on the external order flow. We continue to use the same parameter values in Table 1 and the liability function defined by (4.1), but now with external order flow (without y -dependence):

$$L_n(s) := a \max(s - c, 0). \quad (4.3)$$

We set $c = 1.75$ and consider two scenarios, one is $a = 7$ and the other is $a = -7$. In the former case, there is positive supply of the stock when the stock price is very large $s > 1.75$, and in the latter case the positive demand (i.e. negative supply) for $s > 1.75$. In Figure 9, we compare the marginal as well as conditional price distribution as in Figure 2 with the positive external order flow in the left panel and the negative one in the right panel. We can clearly observe that the positive supply generates a heavy right tail in the equilibrium distributions and that the positive demand causes the opposite. In order to absorb positive supply, higher excess return is required by the agents.

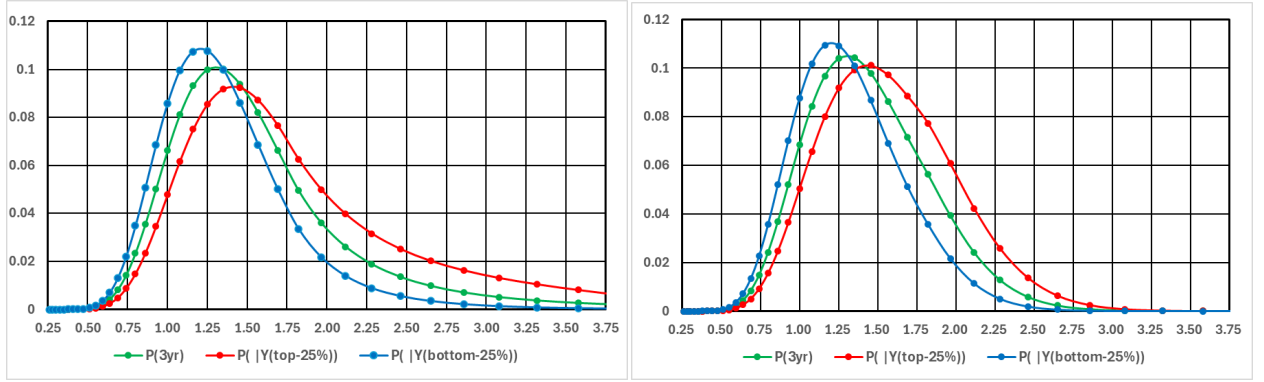


Figure 9: Comparison of the marginal price distribution P and the conditional price distributions $P(\cdot|Y^{\text{top-25\%}})$ and $P(\cdot|Y^{\text{bottom-25\%}})$ at 3-year point. F is given by (4.1) and the external order flow is equal to $L_n(s) = 7 \max(s - 1.75, 0)$ (i.e. positive supply) in the left panel and $L_n(s) = -7 \max(s - 1.75, 0)$ (i.e. positive demand) in the right panel.

4.2 Recursive utility

We now consider the recursive utility model discussed in Section 3. For numerical ease, we assume no path-dependence. The liability F is then assumed to depend solely on the terminal stock price S_N , while the incremental endowments g_n depend solely on the current price S_n . See Remark 3.2 for the corresponding results. We use the same models for $(S_n, Y_n, Z_n^i)_{n=0}^N$ and γ_i as in Section 4.1. ψ_i is assumed to have a uniform distribution over $(N_\psi + 1)$ discrete values given by

$$\psi_i(k_\psi) := \underline{\psi} + (\bar{\psi} - \underline{\psi}) \frac{k_\psi}{N_\psi}, \quad k_\psi = 0, \dots, N_\psi.$$

For simplicity, we assume that $\delta_i := \exp(-\rho\Delta)$ has a common value across the agents. Moreover, we assume that ζ_i and ψ_i are related by

$$\psi_i/\zeta_i = a_\zeta,$$

where a_ζ is a positive constant common across the agents. We shall use the parameter a_ζ to control the ratio η_{n-1}^i/η_n^i . The parameter values to be used throughout this subsection (except the last example) are summarized in Table 2 below.

parameter	$\underline{\gamma}$	$\bar{\gamma}$	N_γ	$\underline{\psi}$	$\bar{\psi}$	N_ψ	ρ	z_0	σ_z	p_z	Y_0	σ_y	p_y	S_0	σ	r	T	N
value	0.4	1.6	3	0.5	1.5	2	5.0%	1.0	12%	0.5	1.0	12%	0.5	1.0	15%	3.3%	3yr	48

Table 2: parameter values

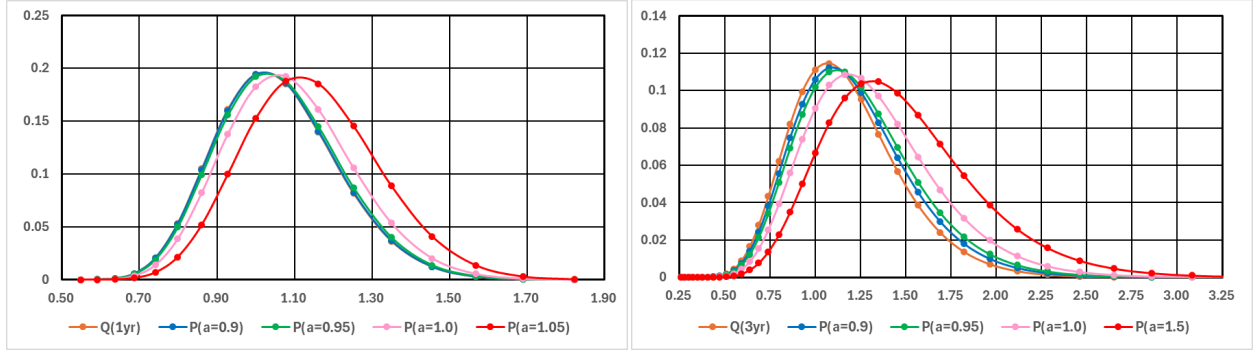


Figure 10: Comparison of the risk-neutral as well as the equilibrium marginal price distributions with $a_\zeta = 0.9, 0.95, 1.0, 1.05$ at 1-year (left panel) and 3-year (right panel) points.

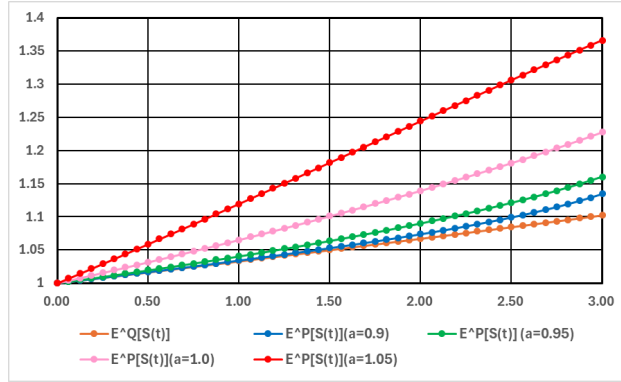


Figure 11: Comparison of the expected values of $S(t_n)$ under the risk-neutral as well as the equilibrium price distributions with $a_\zeta = 0.9, 0.95, 1.0, 1.05$.

Since the effects of the stochastic liability and incremental endowments on the equilibrium price distributions are as expected from the results in the previous subsection, let us first concentrate on the effects of the factor $a_\zeta = \psi_i/\zeta_i$. We set $L_n \equiv 0, \forall n$ and define the liability function and the incremental endowments in the following way:

$$F(S_N, Y_N, Z_N^i) := C - 2S_N Y_N Z_N^i, \quad (4.4)$$

$$g_n(S_n, Y_n, Z_n^i) := C' + 1.5\Delta S_n Y_n Z_n^i, \quad 1 \leq n \leq N. \quad (4.5)$$

Here, C, C' are arbitrary constants irrelevant for the equilibrium distributions. In Figure 10, we plot the risk-neutral as well as the equilibrium marginal price distributions with 4 different values of the ratio $a_\zeta = 0.9, 0.95, 1.0, 1.05$ at 1-year (left panel) and 3-year (right panel) points. Figure 11 provides the time evolution of the expected value of the stock price $S(t_n)$ for each case. As inferred from the discussion in the last part of Section 3.3, the deviations from the risk-neutral distribution become smaller as a_ζ decreases, and this effect (smaller deviations) is more profound in earlier periods. We can observe that the value of $a_\zeta = \psi_i/\zeta_i$ can efficiently control the level of the risk-premium without changing the other parameters.

Next, let us turn off the incremental endowments $g_n \equiv 0$ while keeping the liability function the same as in (4.4). We are now going to study the effects of an external order flow defined by

$$L(S_n) = 8 \max(S_n - 1.6, 0) - 8 \max(1.1 - S_n, 0), \quad 0 \leq n \leq N - 1. \quad (4.6)$$

to demonstrate how flexibly the shape of equilibrium distributions can change. (4.6) means that there is positive supply (i.e. sell orders from other groups) of the stock when the stock price is high and positive demand (i.e. buy orders from other groups) when the stock price is low. In Figure 12, with $a_\zeta = 1.07$, we compare the equilibrium price distribution at 3-year point with L_n given by (4.6) and that with $L_n \equiv 0$. It shows that the existence of the external order flow (4.6) makes the equilibrium distribution fat-tailed in both directions, which is as expected by the analysis made in the last paragraph of Section 2.4. Corresponding modifications in the terminal liability F and/or the incremental endowments g_n would yield similar results.

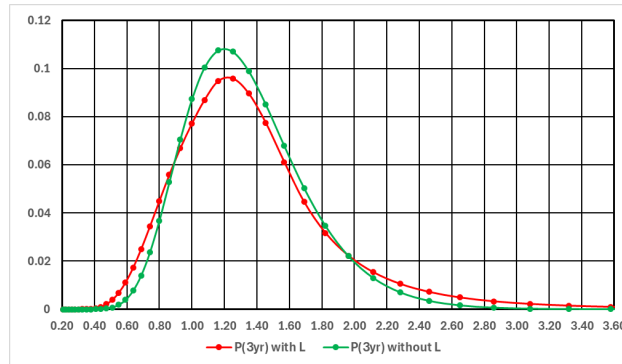


Figure 12: Comparison of the stock price distributions at 3-year point with L_n given by (4.6) and $L_n \equiv 0$.

In the last numerical example, we examine the effect of σ_z , the volatility of the process (Z_n^i) , on trading volume. This volume is quantified by the standard deviation of the stock position among the agents $\mathbb{E}^1[|\phi_t^{i,*}|^2]^{\frac{1}{2}}$, as discussed in Section 2.4. To highlight the effect of σ_z , we reduce the variation in $(\gamma_i, \psi_i, \zeta_i)$. The parameter values we use are summarized in Table 3 below:

parameter	$\underline{\gamma}$	$\bar{\gamma}$	N_γ	$\underline{\psi}$	$\bar{\psi}$	N_ψ	ρ	z_0	a_ζ	p_z	Y_0	σ_y	p_y	S_0	σ	r	T	N
value	0.95	1.05	2	0.95	1.05	2	5.0%	1.0	1.02	0.5	1.0	12%	0.5	1.0	15%	3.3%	3yr	48

Table 3: value of parameters for Figure 13.

We use the terminal liability and the incremental endowments defined by (4.4) and (4.5). In the right panel of Figure 13, we have plotted the evolution of the trading volume $\mathbb{E}^\mathbb{P}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$ for 5 different volatilities of the process (Z_n^i): $\sigma_z = 0\%, 5\%, 10\%, 15\%, 20\%$. We observe that the trading volume increases with the volatility σ_z . The non-zero trading volume, even when $\sigma_z = 0$, stems from the non-zero variation in the risk-aversion coefficients. The near-identical trading volume in the earliest period is a consequence of our assumption that the agents have the common initial value $Z_0^i \equiv z_0 = 1, \forall i \in \mathbb{N}$, an assumption made solely for numerical convenience.

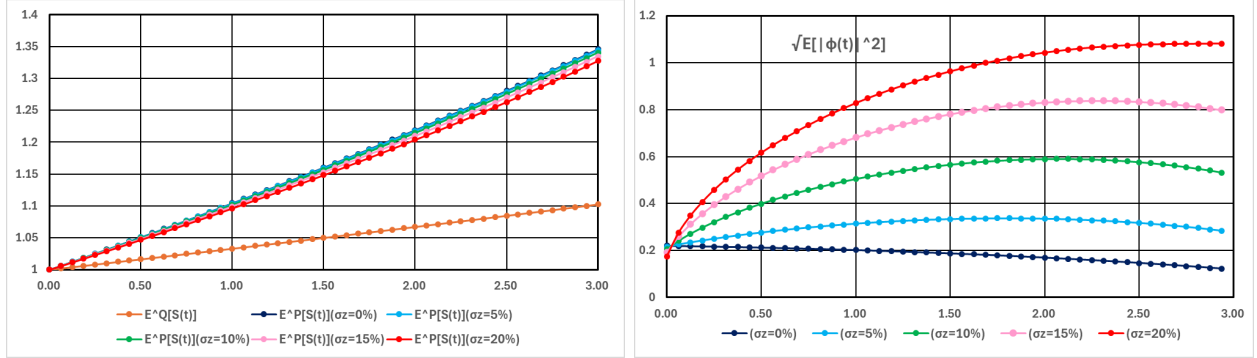


Figure 13: Left panel: Comparison of the expected value of $S(t_n)$ under the risk-neutral as well as the equilibrium price distributions with $\sigma_z = 0\%, 5\%, 10\%, 15\%, 20\%$. Right panel: Comparison of the trading volumes $\mathbb{E}^\mathbb{P}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$ with $\sigma_z := 0\%, 5\%, 10\%, 15\%, 20\%$.

In the left panel of Figure 13, we have plotted the evolution of the expected value of $S(t)$ for each case of σ_z . (The result for the risk-neutral measure is also plotted for reference.) From this result, we see that the size of excess return is almost unaffected by the volatility σ_z . This stems from the functional form of (4.4) and (4.5), as well as the fact that the expectation value of Z_n^i remains nearly identical across all cases. These results suggest that we can control trading volume by changing σ_z without significantly affecting the risk-premium. Although trading volume is also significantly influenced by the variation in \mathcal{F}_0^i -measurable random variables such as γ_i, ζ_i and ψ_i , these variables may simultaneously induce a large change in the risk-premium.

5 Concluding remarks and future research directions

In this work, we have proved the existence of the unique mean-field equilibrium for agents with exponential-type utilities and derived an explicit formula for equilibrium transition probabilities of the stock price by restricting its trajectories onto a recombining binomial tree. The agents are supposed to have stochastic terminal liabilities and incremental endowments, both dependent on unhedgeable common and idiosyncratic factors in addition to the past trajectory of the stock price. We also examined the impacts of external order flow. Finally, we provided numerical examples to illustrate qualitative effects of these components on the equilibrium price distributions. Our results clearly show that the equilibrium distributions can substantially change their shapes in response to these inputs. In particular, focusing on liabilities whose size changes countercyclically with market performance, we have found that the more levered financial firms and institutional investors are, the higher the risk premium is. The same observation also holds for cyclical endowments, i.e.,

non-tradable incomes originating from the firms' other business lines. Empirical analysis regarding this finding would constitute an important research topic. On the other hand, the trading volume per capita crucially depends on the variation in idiosyncratic factors.

Our method can also be applied to other asset classes, such as commodities and foreign exchanges, as long as they can be modeled by binomial trees. The Black-Derman-Toy model (BDG) [8], which is a famous short-rate model constructed on a binomial tree, may provide an interesting framework to analyze a mean-field equilibrium for a risk-free interest rates. Although many of them are becoming obsolete in today's financial markets, various techniques (such as "implied binomial trees") have been developed by practitioners for tree-based derivative pricing since its initial invention by Sharpe. These can now serve as valuable tools for investigating the mean-field equilibrium in our framework. For a comprehensive overview of general Markov processes for financial applications, see, for example, the textbooks by Bäuerle & Rieder [7] and Hernández-Lerma & Lasserre [25].

Extensions to general multinomial trees and multi-asset frameworks constitute interesting future research directions. Although our framework remains conceptually the same, there appear several hurdles to be overcome.

- Although it is not difficult to put appropriate assumptions so that there exists a unique optimal solution, its explicit form is generally unavailable.
- There are more degrees of freedom in the transition probabilities than are imposed by the market-clearing conditions. This remaining freedom must be fixed by imposing an appropriate dependence structure among the assets.

Due to these issues, although the second point might be beneficial for flexibility, numerical costs would be significantly higher than the single asset case, in particular, in the presence of common noises. The fact that the market-clearing condition alone does not uniquely determine the asset price process is already well known. (See, Karatzas & Shreve [30, Chapter 4].) This is because that one can build a equivalent set of mutual funds from the original stocks without affecting the market-clearing condition.

Constructing mean-field equilibrium among agents with other utilities, such as power type, remains one of the most challenging problems. This is a common issue, mirroring the challenge in the continuous-time setting. For utilities other than the exponential-type, the optimal trade position $\phi_n^{i,*}$ is, in general, dependent on the size of wealth at t_n . Since the wealth of each agent w_n^i at t_n depends on the trading strategy up to t_n , the mean-field equilibrium condition leads to a complex fixed-point problem involving the backward $\phi_n^{i,*}$ and forward w_n^i discrete processes. Although we can decouple the wealth process by deliberately constructing the model so that $\phi_n^{i,*} \equiv 0$, the resulting model allows no trading activity in the market and is thus clearly unrealistic.

Acknowledgements

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