Filtering Problem for Random Processes with Stationary Increments

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Abstract. This paper deals with the problem of optimal mean-square filtering of the linear functionals $A\xi = \int_0^\infty a(t)\xi(-t)dt$ and $A_T\xi = \int_0^T a(t)\xi(-t)dt$ which depend on the unknown values of random process $\xi(t)$ with stationary nth increments from observations of process $\xi(t) + \eta(t)$ at points $t \leq 0$, where $\eta(t)$ is a stationary process uncorrelated with $\xi(t)$. We propose the values of mean-square errors and spectral characteristics of optimal linear estimates of the functionals when spectral densities of the processes are known. In the case where we can operate only with a set of admissible spectral densities relations that determine the least favorable spectral densities and the minimax spectral characteristics are proposed.

Keywords: Random process with nth stationary increments, minimax-robust estimate, mean-square error, least favorable spectral density, minimax spectral characteristic.

1 Introduction

The problems of estimating unobserved values of random processes play an important role in both theoretical and applied probability. Widely developed methods of extrapolation, interpolation and filtering of stationary random processes and sequences were proposed by Kolmogorov [11] and Wiener [33]. Since that time a lot of generalizations of stationary processes have been established; some of which are discussed, for example, in books by Yaglom [34, 35]. One important generalization was proposed by Yaglom in paper [36], in which he considered random processes with stationary nth increments. He described spectral representation of stationary increments and canonical factorization of spectral density, solved extrapolation problem for unknown value of random process with stationary increments and discussed a few examples for given spectral densities. The further investigations of such random processes were made in articles by Pinsker [30], Yaglom and Pinsker [29], Yaglom [37].

The crucial assumption of most of the papers dedicated to one of problems of estimating values of random processes is that spectral densities of involved random processes are exactly

known. However, the established results can't be directly applied to practical issues, because we operate with estimates of spectral densities, not with exact views. Often instead of estimates researchers operate with sets of admissible spectral densities. In this case they can use minimax-robust approach to the estimation problem. This method provides us an estimate that minimizes the maximum of mean-square errors for all spectral densities from the set of admissible spectral densities simultaneously. Grenander [7] was the first one who proposed this method for solving extrapolation problem for stationary processes. Later Franke [8], Franke and Poor [9] applied minimax-robust method to extrapolation and interpolation problems. They used convex optimization for investigations. In the papers by Moklyachuk [18] -[25] minimax-robust extrapolation, interpolation and filtering problems are studied. The papers by Moklyachuk and Masyutka [22] -[27] are dedicated to minimax-robust extrapolation, interpolation and filtering problems for vector-valued stationary processes and sequences. Dubovetska, Moklyachuk and Masyutka [1] solved a problem of minimax-robust interpolation of another generalization of stationary processes – periodically correlated sequences. In the papers [2] -[5] Dubovetska and Moklyachuk solved minimax-robust extrapolation, interpolation and filtering problems for periodically correlated processes.

Minimax-robust extrapolation, interpolation and filtering problems for stochastic sequences and random processes with nth stationary increments are solved by Luz and Moklyachuk in [12] - [17]. In particular, minimax-robust filtering problem for such sequences is considerer in papers [14, 16]. Random processes with stationary increments with continuous time are considered in the article [15] by Luz and Moklyachuk, where authors investigate extrapolation problem.

In this article we propose solution to the problem of optimal mean-square estimation of the linear functionals $A_T \xi = \int_0^T a(t) \xi(-t) dt$ and $A \xi = \int_0^\infty a(t) \xi(-t) dt$ of unknown values of a random process $\xi(t)$ with stationary nth increments from observation of a process $\xi(t) + \eta(t)$ at points $t \leq 0$, where a process $\eta(t)$ is stationary uncorrelated with $\xi(t)$, we propose formulas for calculating the value of mean-square errors and the spectral characteristics of the optimal estimates of the functionals $A_T \xi$ and $A \xi$ under the condition that spectral densities of the processes are exactly known. In the case when spectral densities are not known, but sets of admissible spectral densities are available, relations which determine least favorable densities and minimax-robust spectral characteristics for different classes of spectral densities are found.

2 Stationary random increment process. Spectral representation

In this section we present a brief review of the spectral theory of random processes with stationary increment developed in [36].

Definition 2.1 For a given random process $\{\xi(t), t \in \mathbb{R}\}$ a process

$$\xi^{(n)}(t,\tau) = (1 - B_{\tau})^n \xi(t) = \sum_{l=0}^n (-1)^l C_n^l \xi(t - l\tau), \tag{1}$$

where B_{τ} is a backward shift operator with step $\tau \in \mathbb{R}$, such that $B_{\tau}\xi(t) = \xi(t-\tau)$, is called the random nth increment with step $\tau \in \mathbb{R}$.

Random *n*th increment process $\xi^{(n)}(t,\tau)$ satisfies relations:

$$\xi^{(n)}(t, -\tau) = (-1)^n \xi^{(n)}(t + n\tau, \tau), \tag{2}$$

$$and\xi^{(n)}(t,k\tau) = \sum_{l=0}^{(k-1)n} A_l \xi^{(n)}(t-l\tau,\tau), \quad \forall k \in \mathbb{N},$$
(3)

where coefficients $\{A_l, l=0,1,2,\ldots,(k-1)n\}$ are taken from a representation

$$(1+x+\ldots+x^{k-1})^n = \sum_{l=0}^{(k-1)n} A_l x^l.$$

Definition 2.2 Random nth increment process $\xi^{(n)}(t,\tau)$ generated by a random process $\{\xi(t), t \in R\}$ is wide sense stationary if the mathematical expectations

$$\mathsf{E}\xi^{(n)}(t_0,\tau) = c^{(n)}(\tau).$$

$$\mathsf{E}\xi^{(n)}(t_0+t,\tau_1)\xi^{(n)}(t_0,\tau_2) = D^{(n)}(t,\tau_1,\tau_2)$$

exist for all $t_0, \tau, t, \tau_1, \tau_2$ and do not depend on t_0 . The function $c^{(n)}(\tau)$ is called the mean value of the nth increment and the function $D^{(n)}(t, \tau_1, \tau_2)$ is called the structural function of the stationary nth increment (or the structural function of nth order of the random process $\{\xi(t), t \in \mathbb{R}\}$).

The random process $\{\xi(t), t \in \mathbb{R}\}$ which determines the stationary nth increment process $\xi^{(n)}(t,\tau)$ by formula (1) is called the process with stationary nth increments.

The following theorem provodes representations of mean value and structural function of stationary random nth increment.

Theorem 2.1 The mean value $c^{(n)}(\tau)$ and the structural function $D^{(n)}(m, \tau_1, \tau_2)$ of a random stationary nth increment process $\xi^{(n)}(t, \tau)$ can be represented in the following forms

$$c^{(n)}(\tau) = c\tau^n, \tag{4}$$

$$D^{(n)}(t,\tau_1,\tau_2) = \int_{-\infty}^{\infty} e^{i\lambda t} (1 - e^{-i\tau_1 \lambda})^n (1 - e^{i\tau_2 \lambda})^n \frac{(1 + \lambda^2)^n}{\lambda^{2n}} dF(\lambda), \tag{5}$$

where c is a constant, $F(\lambda)$ is a left-continuous nondecreasing bounded function with $F(-\infty) = 0$. The constant c and the function $F(\lambda)$ are determined uniquely by the increment process $\xi^{(n)}(t,\tau)$.

On the other hand, the function $c^{(n)}(\tau)$ which has the form (4) with a constant c and the function $D^{(n)}(t,\tau_1,\tau_2)$ which has the form (5) with a function $F(\lambda)$ which satisfies the indicated conditions are the mean value and the structural function of some stationary nth increment process $\xi^{(n)}(t,\tau)$.

From representation (5) of the structural function of the stationary nth increment process $\xi^{(n)}(t,\tau)$ and the Karhunen theorem (see Karhunen [10]), we obtain the following spectral representation of the stationary nth increment process $\xi^{(n)}(t,\tau)$:

$$\xi^{(n)}(t,\tau) = \int_{-\infty}^{\infty} e^{it\lambda} (1 - e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ(\lambda), \tag{6}$$

where $Z(\lambda)$ – is a random process with independent increments on \mathbb{R} connected with a spectral function $F(\lambda)$ from representation (5) by the relation

$$E|Z(t_2) - Z(t_1)|^2 = F(t_2) - F(t_1) < \infty \quad \text{for all } t_2 > t_1.$$
(7)

Without loss of generality we will consider random increments $\xi^{(n)}(t,\tau)$ with step $\tau > 0$ and mean value 0.

3 Filtering problem

Consider a random process $\{\xi(t), t \in \mathbb{R}\}$ which defines a stationary nth increment process $\xi^{(n)}(t,\tau)$ with an absolutely continuous spectral function $F(\lambda)$ which has a spectral density $f(\lambda)$. Assume that we observe another process $\zeta(t) = \xi(t) + \eta(t)$ on the time interval $t \leq 0$. Filtering problem means that we need to restore values of the original process $\xi(t)$ at points t < 0.

In this article we focus on finding mean-square optimal estimates of linear functionals $A_T \xi = \int_0^T a(t)\xi(-t)dt$ and $A\xi = \int_0^\infty a(t)\xi(-t)dt$ under the assumption that a noise process $\eta(t)$ is an uncorrelated with $\xi(t)$ stationary process with spectral density $g(\lambda)$.

For the further discussions we require the spectral densities $f(\lambda)$ and $g(\lambda)$ of the random processes $\xi(t)$ and $\eta(t)$ satisfy minimality conditions

$$\int_{-\infty}^{\infty} \frac{|\gamma(\lambda)|^2}{f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda)} d\lambda < \infty, \quad \int_{-\infty}^{\infty} \frac{|\gamma(\lambda)|^2 \lambda^{2n}}{|1 - e^{i\lambda\tau}|^{2n} (1 + \lambda^2)^n (f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda))} d\lambda < \infty \quad (8)$$

for some non-zero function of the exponential type $\gamma(\lambda) = \int_0^\infty \alpha(t)e^{i\lambda t}dt$. We also require a function $\mathbf{a}_{\tau}(t)$, $t \geq -\tau n$, being defined later satisfy conditions

$$\int_0^\infty |\mathbf{a}_{\tau}(t-\tau n)| dt < \infty, \quad \int_0^\infty t |\mathbf{a}_{\tau}(t-\tau n)|^2 dt. \tag{9}$$

To solve the filtering problem for the functional $A\xi$ and the random processes $\xi(t)$ and $\eta(t)$ described above we represent the functional $A\xi = \int_0^\infty a(t)\xi(-t)dt$ in the form

$$A\xi = A\zeta - A\eta,$$

where $A\zeta = \int_0^\infty a(t)\zeta(-t)dt$, $A\eta = \int_0^\infty a(t)\eta(-t)dt$. As we can find the exact value of the functional $A\zeta$ from the observations $\zeta(t)$ at points $t \leq 0$, it's sufficient to find an estimate $\widehat{A}\eta$ of the functional $A\eta$ in order to build the estimate $\widehat{A}\xi$:

$$\widehat{A}\xi = A\zeta - \widehat{A}\eta. \tag{10}$$

Moreover, the values of mean-square errors $\Delta(f, g; \widehat{A}\xi) = \mathsf{E}|A\xi - \widehat{A}\xi|^2$ and $\Delta(f, g; \widehat{A}\eta) = \mathsf{E}|A\eta - \widehat{A}\eta|^2$ of the estimates $\widehat{A}\xi$ and $\widehat{A}\eta$ respectively satisfy the equalities

$$\Delta\left(f,g;\widehat{A}\xi\right) = \mathsf{E}\left|A\xi - \widehat{A}\xi\right|^2 = \mathsf{E}\left|A\zeta - A\eta - A\zeta + \widehat{A}\eta\right|^2 = \mathsf{E}\left|A\eta - \widehat{A}\eta\right|^2 = \Delta\left(f,g,\widehat{A}\eta\right).$$

In this paper we do not try to find the optimal estimate $\widehat{A}\eta$ by direct minimizing mean-square error as a function of admissible estimates. We just use properties of minimal distances in Hilbert space $H = L_2(\Omega, \mathfrak{F}, \mathsf{P})$ of random variables of the second order. In this space the available observations $\{\xi^{(n)}(t,\tau) + \eta^{(n)}(t,\tau) : t \leq 0\}, \tau > 0$, generate the closed linear subspace $H^0(\xi_{\tau}^{(n)} + \eta_{\tau}^{(n)})$ of the Hilbert space $H = L_2(\Omega, \mathfrak{F}, \mathsf{P})$. Thus the optimal estimate $\widehat{A}\eta$ is a projection of the element $A\eta$ of the space H on the subspace $H^0(\xi_{\tau}^{(n)} + \eta_{\tau}^{(n)})$ and the value of mean-square error $\Delta(f, g; \widehat{A}\eta)$ is the minimal distance from the element $A\eta$ from the space H to the subspace $H^0(\xi_{\tau}^{(n)} + \eta_{\tau}^{(n)})$. These properties give us two conditions characterizing the optimal estimate $\widehat{A}\eta$:

1)
$$\widehat{A}\eta \in H^0\left(\xi_{\tau}^{(n)} + \eta_{\tau}^{(n)}\right);$$

2)
$$\left(A\eta - \widehat{A}\eta\right) \perp H^0\left(\xi_{\tau}^{(n)} + \eta_{\tau}^{(n)}\right)$$
.

The obtained conditions allow us to find a spectral characteristic $h_{\tau}(\lambda)$ of the estimate $\widehat{A}\eta$. To be able to use them we need to describe the spectral representations of the involved processes.

The stationary random process $\eta(t)$ with spectral function $G(\lambda)$ and the *n*th increment $\eta^{(n)}(t,\tau)$ admit the spectral representations

$$\eta(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ_{\eta}(\lambda)$$

and

$$\eta^{(n)}(t,\tau) = \int_{-\infty}^{\infty} e^{i\lambda t} (1 - e^{-i\lambda\tau})^n dZ_{\eta}(\lambda),$$

where $Z_{\eta}(\lambda)$ is a random process with independent increments defined on \mathbb{R} corresponding to the spectral function $G(\lambda)$. The stationary increment process $\zeta^{(n)}(t,\tau)$ admits the spectral representation

$$\zeta^{(n)}(t,\tau) = \int_{-\infty}^{\infty} e^{i\lambda t} (1 - e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda)$$
$$= \int_{-\infty}^{\infty} e^{i\lambda t} (1 - e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ_{\xi^{(n)}}(\lambda) + \int_{-\infty}^{\infty} e^{i\lambda t} (1 - e^{-i\lambda\tau})^n dZ_{\eta}(\lambda),$$

where $dZ_{\eta^{(n)}}(\lambda) = \frac{\lambda^{2n}}{(1+\lambda^2)^n} dZ_{\eta}(\lambda)$, $\lambda \in \mathbb{R}$. From the described representations we can obtain the spectral density $p(\lambda)$ of the random process $\zeta(t)$:

$$p(\lambda) = f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda).$$

Finally, the spectral representation of the estimate $\widehat{A}\eta$ is

$$\widehat{A}\eta = \int_{-\infty}^{\infty} h_{\tau}(\lambda) dZ_{\xi^{(n)} + \eta^{(n)}}(\lambda),$$

and then the spectral representation of the estimate $\widehat{A}\xi$ of the functional $A\xi$ is

$$\widehat{A}\xi = A\zeta - \int_{-\infty}^{\infty} h_{\tau}(\lambda) dZ_{\xi^{(n)} + \eta^{(n)}}(\lambda). \tag{11}$$

It comes from condition 2) that for all $t \leq 0$ the function $h_{\tau}(\lambda)$ satisfies the following equality:

$$\int_{-\infty}^{\infty} \left[A(\lambda) \frac{(-i\lambda)^n}{(1-i\lambda)^n} g(\lambda) - h_{\tau}(\lambda) \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right) \right] \frac{(1-i\lambda)^n}{(-i\lambda)^n} (1-e^{i\lambda\tau})^n e^{-i\lambda t} d\lambda = 0.$$
(12)

Let us define a function

$$C^{\tau}(\lambda) = \left[A(\lambda) \frac{(-i\lambda)^n}{(1-i\lambda)^n} g(\lambda) - h_{\tau}(\lambda) \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right) \right] \frac{(1-i\lambda)^n}{(-i\lambda)^n} (1 - e^{i\lambda\tau})^n, \quad \lambda \in \mathbb{R},$$

and its Fourier transformation

$$\mathbf{c}_{\tau}(t) = \int_{-\infty}^{\infty} C^{\tau}(\lambda) e^{-i\lambda t} d\lambda, \quad t \in \mathbb{R}.$$

Condition (12) let us conclude that the function $\mathbf{c}_{\tau}(t)$ is equal to 0 on $(-\infty; 0]$, which implies

$$C^{\tau}(\lambda) = \int_0^{\infty} \mathbf{c}_{\tau}(t) e^{i\lambda t} dt.$$

Hence the function $h_{\tau}(\lambda)$ has a form

$$h_{\tau}(\lambda) = \frac{A(\lambda)(-i\lambda)^n g(\lambda)}{(1-i\lambda)^n \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda)\right)} - \frac{(1-e^{i\lambda\tau})^{-n} (-i\lambda)^n C^{\tau}(e^{i\lambda})}{(1-i\lambda)^n \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda)\right)},$$
$$A(\lambda) = \int_0^{\infty} a(t)e^{-i\lambda t} dt.$$

Define a closed linear subspace $L_2^0\left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)$ of the Hilbert space $L_2\left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)$ generated by the set of functions

$$\left\{ e^{i\lambda t} (1 - e^{-i\lambda \tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} : t \le 0 \right\}.$$

Using condition 1) and isometry between the subspaces $L_2^0\left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)$ and $H^0(\xi_{\tau}^{(n)} + \eta_{\tau}^{(n)})$ we can obtain the following properties of the spectral characteristic $h_{\tau}(\lambda)$:

$$h_{\tau}(\lambda) = h(\lambda)(1 - e^{-i\lambda\tau})^n \frac{(1 + i\lambda)^n}{(i\lambda)^n}, \quad h(\lambda) = \int_{-\infty}^0 s(t)e^{i\lambda t}dt,$$

for some function $s(t) \in L_2^0$,

$$\int_{-\infty}^{\infty} |h_{\tau}(\lambda)|^2 |1 - e^{i\lambda\tau}|^{2n} \left(\frac{(1+\lambda^2)^n}{\lambda^{2n}} f(\lambda) + g(\lambda) \right) d\lambda < \infty, \quad \frac{(i\lambda)^n h_{\tau}(\lambda)}{(1+i\lambda)^n (1 - e^{-i\lambda\tau})^n} \in L_2^0.$$

Thus for all $s \geq 0$ the following equality holds true

$$\int_{-\infty}^{\infty} \left[\frac{A(\lambda)(1 - e^{-i\lambda\tau})^{-n}\lambda^{2n}g(\lambda)}{(1 + \lambda^2)^n \left(f(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^2)^n}g(\lambda)\right)} - \frac{|1 - e^{i\lambda\tau}|^{-n}\lambda^{2n}C^{\tau}(\lambda)}{(1 + \lambda)^{2n} \left(f(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^2)^n}g(\lambda)\right)} \right] e^{-i\lambda s} d\lambda = 0.$$

The last condition provides us with an equation defining the function $\mathbf{c}_{\tau}(t)$:

$$\int_{-\tau n}^{\infty} \mathbf{a}_{\tau}(t) \int_{-\infty}^{\infty} e^{-i\lambda(t+s)} \frac{\lambda^{2n} g(\lambda)}{|1 - e^{i\lambda\tau}|^{2n} \left((1 + \lambda^{2})^{n} f(\lambda) + \lambda^{2n} g(\lambda) \right)} d\lambda dt$$

$$= \int_{0}^{\infty} \mathbf{c}_{\tau}(t) \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \frac{\lambda^{2n}}{|1 - e^{i\lambda\tau}|^{2n} \left((1 + \lambda^{2})^{n} f(\lambda) + \lambda^{2n} g(\lambda) \right)} d\lambda dt, \quad s \ge 0, \tag{13}$$

where

$$\mathbf{a}_{\tau}(t) = \sum_{l=\max\{0, \left[-\frac{t}{\tau}\right]'\}}^{n} C_n^l(-1)^l a(t+\tau l), \quad t \ge -\tau n.$$
(14)

Here by [x]' we denote the least integer number among numbers which are greater than or equal to x. Equation (13) can be rewritten in terms of linear operators in Hilbert space L_2 . For functions $\mathbf{x}(t) \in L_2[-\tau n; \infty)$ and $\mathbf{y}(t), \mathbf{z}(t) \in L_2[0; \infty)$ define linear operators

$$(\mathbf{S}_{\infty}^{\tau}\mathbf{x})(s) = \int_{-\tau n}^{\infty} \mathbf{x}(t) \int_{-\infty}^{\infty} e^{-i\lambda(t+s)} \frac{\lambda^{2n}g(\lambda)}{|1 - e^{i\lambda\tau}|^{2n} \left((1 + \lambda^2)^n f(\lambda) + \lambda^{2n}g(\lambda)\right)} d\lambda dt, \ s \in [0, \infty),$$

$$\begin{split} (\mathbf{P}_{\infty}^{\tau}\mathbf{y})(s) &= \int_{0}^{\infty}\mathbf{y}(t)\int_{-\infty}^{\infty}e^{i\lambda(t-s)}\frac{\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^{2})^{n}f(\lambda)+\lambda^{2n}g(\lambda)\right)}d\lambda dt,\ s\in[0;\infty),\\ (\mathbf{Q}_{\infty}\mathbf{z})(s) &= \int_{0}^{\infty}\mathbf{z}(t)\int_{-\infty}^{\infty}e^{i\lambda(t-s)}\frac{(1+\lambda^{2})^{n}f(\lambda)g(\lambda)}{(1+\lambda^{2})^{n}f(\lambda)+\lambda^{2n}g(\lambda)}d\lambda dt,\ s\in[0;\infty). \end{split}$$

Then equation (13) can be written as

$$(\mathbf{S}_{\infty}^{\tau}\mathbf{a}_{\tau})(s) = (\mathbf{P}_{\infty}^{\tau}\mathbf{c}_{\tau})(s), \quad s \geq 0,$$

and its solution is

$$\mathbf{c}_{\tau}(s) = ((\mathbf{P}_{\infty}^{\tau})^{-1} \mathbf{S}_{\infty}^{\tau} \mathbf{a}_{\tau})(s), \quad s \ge 0.$$

The obtained function $\mathbf{c}_{\tau}(s)$ allows us to write a solution to filtering problem. The spectral characteristic $h_{\tau}(\lambda)$ of the optimal estimate $\widehat{A}\xi$ of the functional $A\xi$ is calculated by the formula

$$h_{\tau}(\lambda) = \frac{A(\lambda)(1+i\lambda)^n(-i\lambda)^n g(\lambda)}{(1+\lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda)} - \frac{(1+i\lambda)^n(-i\lambda)^n C^{\tau}(e^{i\lambda})}{(1-e^{i\lambda\tau})^n ((1+\lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda))},$$
(15)

where

$$C^{\tau}(\lambda) = \int_0^{\infty} ((\mathbf{P}_{\infty}^{\tau})^{-1} \mathbf{S}_{\infty}^{\tau} \mathbf{a}_{\tau})(t) e^{i\lambda t} dt.$$

The value of the mean-square error of the estimate $\hat{A}\xi$ of the functional $A\xi$ is calculated by the formula

$$\Delta \left(f, g; \widehat{A}\xi \right) = \Delta \left(f, g; \widehat{A}\eta \right) = \mathbb{E} \left| A\eta - \widehat{A}\eta \right|^{2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left| A(\lambda)(1 - e^{i\lambda\tau})^{n}(1 + \lambda^{2})^{n}f(\lambda) + \lambda^{2n}C^{\tau}(\lambda) \right|^{2}}{|1 - e^{i\lambda\tau}|^{2n}(1 + \lambda^{2})^{2n}(f(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^{2})^{n}}g(\lambda))^{2}} g(\lambda) d\lambda$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left| A(\lambda)(1 - e^{i\lambda\tau})^{n}(-i\lambda)^{n}g(\lambda) - (-i\lambda)^{n}C^{\tau}(\lambda) \right|^{2}}{|1 - e^{i\lambda\tau}|^{2n}(1 + \lambda^{2})^{n}(f(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^{2})^{n}}g(\lambda))^{2}} f(\lambda) d\lambda$$

$$= \langle \mathbf{S}_{\infty}^{\tau} \mathbf{a}_{\tau}, (\mathbf{P}_{\infty}^{\tau})^{-1} \mathbf{S}_{\infty}^{\tau} \mathbf{a}_{\tau} \rangle + \langle \mathbf{Q}_{\infty} \mathbf{a}, \mathbf{a} \rangle, \tag{16}$$

where a function $\mathbf{a}(t)$, $t \ge 0$, is defined as $\mathbf{a}(t) = a(t)$.

Summing up we can formulate the following theorem.

Theorem 3.1 Consider a random process $\{\xi(t), t \in \mathbb{R}\}$ with stationary nth increments $\xi^{(n)}(t, \tau)$ which has spectral density $f(\lambda)$. Let $\{\eta(t), t \in \mathbb{R}\}$ be a stationary process with spectral density $g(\lambda)$. Assume that these processes are uncorrelated and their spectral densities satisfy minimality conditions (8). If the function $\mathbf{a}_{\tau}(t)$ defined by (14) satisfies conditions (9), the optimal linear estimate $\widehat{A}\xi$ of the functional $A\xi$ of unknown values $\xi(t)$, $t \leq 0$, from observations of a process $\xi(t) + \eta(t)$ at points $t \leq 0$ is determined by formula (11). The spectral characteristic $h_{\tau}(\lambda)$ of the optimal estimate $\widehat{A}\xi$ can be calculated by formula (15). The value of the mean-square error $\Delta(f, g; \widehat{A}\xi)$ of the optimal estimate is calculated by formula (16).

As a particular case of Theorem 3.1 we can derive an estimate

$$\widehat{A}_T \xi = A_T \zeta - \int_{-\infty}^{\infty} h_{\tau,T}(\lambda) dZ_{\xi^{(n)} + \eta^{(n)}}(\lambda)$$
(17)

of the functional $A_T \xi = \int_0^T a(t) \xi(-t) dt$ from observations of the process $\xi(t) + \eta(t)$ at points $t \leq 0$. Consider a function a(t), $t \geq 0$, which is equal to 0 for t > T. In this case the formula for calculating the spectral characteristic $h_{\tau,T}(\lambda)$ of the linear estimate is the following:

$$h_{\tau,T}(\lambda) = \frac{A_T(\lambda)(1+i\lambda)^n(-i\lambda)^n g(\lambda)}{(1+\lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda)} - \frac{(1+i\lambda)^n(-i\lambda)^n C_T^{\tau}(e^{i\lambda})}{(1-e^{i\lambda\tau})^n ((1+\lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda))},$$
(18)

$$A_T(\lambda) = \int_0^T a(t)e^{-i\lambda t}dt, \quad C_T^{\tau}(\lambda) = \int_0^{\infty} ((\mathbf{P}_{\infty}^{\tau})^{-1}\mathbf{S}_T^{\tau}\mathbf{a}_{\tau,T})(t)e^{i\lambda t}dt,$$

where a linear operator S_T^{τ} is defined as

$$(\mathbf{S}_T^{\tau}\mathbf{x})(s) = \int_{-\tau n}^T \mathbf{x}(t) \int_{-\infty}^{\infty} e^{-i\lambda(t+s)} \frac{\lambda^{2n} g(\lambda)}{|1 - e^{i\lambda\tau}|^{2n} \left((1 + \lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda) \right)} d\lambda dt, \ s \in [0; \infty),$$

and a function $\mathbf{a}_{\tau,T}(t), t \in [-\tau n; T]$, is defined as

$$\mathbf{a}_{\tau,T}(t) = \sum_{l=\max\left\{0,\left[-\frac{t}{\tau}\right]'\right\}}^{\min\left\{n,\left[\frac{T-t}{\tau}\right]'\right\}} C_n^l(-1)^l a(t+\tau l), \quad t \in [-\tau n; T].$$

The mean-square error of the estimate $\widehat{A}_T \xi$ is calculated by the formula

$$\Delta\left(f,g;\widehat{A}_{T}\xi\right) = \Delta\left(f,g;\widehat{A}_{T}\eta\right) = \mathsf{E}\left|A_{T}\eta - \widehat{A}_{T}\eta\right|^{2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left|A_{T}(\lambda)(1 - e^{i\lambda\tau})^{n}(1 + \lambda^{2})^{n}f(\lambda) + \lambda^{2n}C_{T}^{\tau}(\lambda)\right|^{2}}{\left|1 - e^{i\lambda\tau}\right|^{2n}(1 + \lambda^{2})^{2n}(f(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^{2})^{n}}g(\lambda))^{2}}g(\lambda)d\lambda$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left|A_{T}(\lambda)(1 - e^{i\lambda\tau})^{n}(-i\lambda)^{n}g(\lambda) - (-i\lambda)^{n}C_{T}^{\tau}(\lambda)\right|^{2}}{\left|1 - e^{i\lambda\tau}\right|^{2n}(1 + \lambda^{2})^{n}(f(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^{2})^{n}}g(\lambda))^{2}}f(\lambda)d\lambda$$

$$= \langle \mathbf{S}_{T}^{\tau}\mathbf{a}_{\tau}, (\mathbf{P}_{\infty}^{\tau})^{-1}\mathbf{S}_{T}^{\tau}\mathbf{a}_{\tau,T}\rangle + \langle \mathbf{Q}_{T}\mathbf{a}_{T}, \mathbf{a}_{T}\rangle, \tag{19}$$

where a linear operator Q_T is defined as

$$(\mathbf{Q}_T \mathbf{z})(s) = \int_0^T \mathbf{z}(t) \int_{-\infty}^\infty e^{i\lambda(t-s)} \frac{(1+\lambda^2)^n f(\lambda)g(\lambda)}{(1+\lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda)} d\lambda dt, \ s \in [0, \infty),$$

and a function $\mathbf{a}_T(t)$, $t \in [0, T]$, is defined as $\mathbf{a}_T(t) = a(t)$.

The following theorem holds true.

Theorem 3.2 Consider a random process $\{\xi(t), t \in \mathbb{R}\}$ with stationary nth increments $\xi^{(n)}(t, \tau)$ which has spectral density $f(\lambda)$. Let $\{\eta(t), t \in \mathbb{R}\}$ be a stationary process with spectral density $g(\lambda)$. Assume that these processes are uncorrelated and their spectral densities satisfy minimality conditions (8). Then the optimal linear estimate $\widehat{A}_T\xi$ of the functional $A_T\xi$ of unknown values $\xi(t)$, $t \in [-T; 0]$, from observations of a process $\xi(t) + \eta(t)$ at points $t \leq 0$ is determined by formula (17). The spectral characteristic $h_{\tau,T}(\lambda)$ of the optimal estimate $\widehat{A}_T\xi$ can be calculated by formula (18). The value of the mean-square error $\Delta(f, g; \widehat{A}_T\xi)$ of the optimal estimate is calculated by formula (19).

4 Minimax-robust method of filtering

In the previous section we solved filtering problem for the functionals $A\xi$ and $A_T\xi$ of unknown values of the random process $\xi(t)$ based on observations of the random process $\xi(t)+\eta(t)$. Derived formulas (??) and (18) for spectral characteristics of the mean-square estimates $\widehat{A}\xi$ and $\widehat{A}_T\xi$ can be used only if we know spectral functions $f(\lambda)$ and $g(\lambda)$ of the random processes $\xi(t)$ and $\eta(t)$. In practice we do not the exact view of these spectral densities. Therefore in this section we propose the minimax (robust) approach to estimation of functionals which depend on the unknown values of random process with stationary increments. This method allows us to find an estimate that minimizes the maximum of mean-square errors of the estimates for all spectral densities from a given class $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ of admissible spectral densities.

Definition 4.1 For a given class of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ spectral densities $f^0(\lambda) \in \mathcal{D}_f$ and $g^0(\lambda) \in \mathcal{D}_g$ are called least favorable in the class \mathcal{D} for the optimal linear filtering of the functional $A\xi$ if the following relation holds true

$$\Delta(f^0, g^0) = \Delta\left(h(f^0, g^0); f^0, g^0\right) = \max_{(f,g) \in \mathcal{D}_f \times \mathcal{D}_g} \Delta\left(h(f,g); f, g\right).$$

Definition 4.2 For a given class of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ the spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A\xi$ is called minimax-robust if there are satisfied conditions

$$h^0(\lambda) \in H_{\mathcal{D}} = \bigcap_{(f,g) \in \mathcal{D}_f \times \mathcal{D}_g} L_2^0 \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right),$$

$$\min_{h \in H_{\mathcal{D}}} \max_{(f,g) \in \mathcal{D}_f \times \mathcal{D}_g} \Delta(h; f, g) = \max_{(f,g) \in \mathcal{D}_f \times \mathcal{D}_g} \Delta(h^0; f, g).$$

From the formulas proposed in the previous section and the introduced definitions we can obtain the following statement.

Lemma 4.1 Spectral densities $f^0 \in \mathcal{D}_f$, $g^0 \in \mathcal{D}_g$ satisfying minimality condition (8) are least favorable in the class $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ for the optimal linear filtering of the functional $A\xi$ if operators $(\mathbf{P}_{\infty}^{\tau})^0$, $(\mathbf{S}_{\infty}^{\tau})^0$, $(\mathbf{Q}_{\infty})^0$ determined by the Fourier coefficients of the functions

$$\frac{\lambda^{2n}|1-e^{i\lambda\tau}|^{-2n}}{(1+\lambda^2)^nf^0(\lambda)+\lambda^{2n}g^0(\lambda)}, \quad \frac{\lambda^{2n}g^0(\lambda)|1-e^{i\lambda\tau}|^{-2n}}{(1+\lambda^2)^nf^0(\lambda)+\lambda^{2n}g^0(\lambda)}, \quad \frac{(1+\lambda^2)^nf^0(\lambda)g^0(\lambda)}{(1+\lambda^2)^nf^0(\lambda)+\lambda^{2n}g^0(\lambda)}$$

determine a solution of the conditional extremum problem

$$\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} \left(\langle \mathbf{S}_{\infty}^{\tau} \mathbf{a}_{\tau}, (\mathbf{P}_{\infty}^{\tau})^{-1} \mathbf{S}_{\infty}^{\tau} \mathbf{a}_{\tau} \rangle + \langle \mathbf{Q}_{\infty} \mathbf{a}, \mathbf{a} \rangle \right) = \langle (\mathbf{S}_{\infty}^{\tau})^0 \mathbf{a}_{\tau}, ((\mathbf{P}_{\infty}^{\tau})^0)^{-1} (\mathbf{S}_{\infty}^{\tau})^0 \mathbf{a}_{\tau} \rangle + \langle (\mathbf{Q}_{\infty})^0 \mathbf{a}, \mathbf{a} \rangle.$$
(20)

The minimax spectral characteristic is determined as $h^0 = h_{\tau}(f^0, g^0)$ if $h_{\tau}(f^0, g^0) \in H_{\mathcal{D}}$.

The minimax-robust spectral characteristic h^0 and the least favorable spectral densities (f^0, g^0) form a saddle point of the function $\Delta(h; f, g)$ on the set $H_{\mathcal{D}} \times \mathcal{D}$. The saddle point inequalities

$$\Delta(h; f^0, g^0) \ge \Delta(h^0; f^0, g^0) \ge \Delta(h^0; f, g) \quad \forall f \in \mathcal{D}_f, \forall g \in \mathcal{D}_g, \forall h \in H_{\mathcal{D}}$$

hold true if $h^0 = h_{\tau}(f^0, g^0)$ and $h_{\tau}(f^0, g^0) \in H_{\mathcal{D}}$, where a pair (f^0, g^0) provides a solution to the following conditional extremum problem:

$$\begin{split} \widetilde{\Delta}(f,g) &= -\Delta \left(h_{\tau}(f^{0},g^{0});f,g\right) \rightarrow \inf, \quad (f,g) \in \mathcal{D}_{f} \times \mathcal{D}_{g}, \\ \Delta \left(h_{\tau}(f^{0},g^{0});f,g\right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left|A(\lambda)(1-e^{i\lambda\tau})^{n}(1+\lambda^{2})^{n}f^{0}(\lambda) + \lambda^{2n} \int_{0}^{\infty} (((\mathbf{P}_{\infty}^{\tau})^{0})^{-1}(\mathbf{S}_{\infty}^{\tau})^{0}\mathbf{a}_{\tau})(t)e^{i\lambda t}dt\right|^{2}}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^{2})^{2n}(f^{0}(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^{2})^{n}}g^{0}(\lambda))^{2}} g(\lambda)d\lambda \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left|A(\lambda)(1-e^{i\lambda\tau})^{n}(-i\lambda)^{n}g^{0}(\lambda) - (-i\lambda)^{n} \int_{0}^{\infty} (((\mathbf{P}_{\infty}^{\tau})^{0})^{-1}(\mathbf{S}_{\infty}^{\tau})^{0}\mathbf{a}_{\tau})(t)e^{i\lambda t}dt\right|^{2}}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^{2})^{n}(f^{0}(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^{2})^{n}}g^{0}(\lambda))^{2}} f(\lambda)d\lambda. \end{split}$$

This last extremum problem is equivalent to the unconditional extremum problem

$$\Delta_{\mathcal{D}}(f,g) = \widetilde{\Delta}(f,g) + \delta(f,g|\mathcal{D}_f \times \mathcal{D}_g) \to \inf,$$

where $\delta(f, g|\mathcal{D}_f \times \mathcal{D}_g)$ is the indicator function of the set $\mathcal{D}_f \times \mathcal{D}_g$. Solution (f^0, g^0) to this unconditional extremum problem is characterized by condition $0 \in \partial \Delta_{\mathcal{D}}(f^0, g^0)$. For more details we refer to the books [28, 31].

5 Least favorable spectral densities in the class $\mathcal{D}_f^0 imes \mathcal{D}_g^0$

Consider a set of admissible spectral densities $\mathcal{D} = \mathcal{D}_f^0 \times \mathcal{D}_q^0$, where

$$\mathcal{D}_f^0 = \left\{ f(\lambda) \middle| \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) d\lambda \le P_1 \right\}, \quad \mathcal{D}_g^0 = \left\{ g(\lambda) \middle| \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\lambda) d\lambda \le P_2 \right\}.$$

Assume that spectral densities $f^0 \in \mathcal{D}_f^0$, $g^0 \in \mathcal{D}_g^0$ and functions

$$h_{\tau,f}(f^{0},g^{0}) = \frac{\left| A(\lambda)(1 - e^{i\lambda\tau})^{n}(-i\lambda)^{n}g^{0}(\lambda) - (-i\lambda)^{n} \int_{0}^{\infty} (((\mathbf{P}_{\infty}^{\tau})^{0})^{-1}(\mathbf{S}_{\infty}^{\tau})^{0}\mathbf{a}_{\tau})(t)e^{i\lambda t}dt \right|}{|1 - e^{i\lambda\tau}|^{n}|1 + i\lambda|^{n}(f^{0}(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^{2})^{n}}g^{0}(\lambda))}, \quad (21)$$

$$h_{\tau,g}(f^{0},g^{0}) = \frac{\left| A(\lambda)(1 - e^{i\lambda\tau})^{n}(1 + \lambda^{2})^{n}f^{0}(\lambda) + \lambda^{2n} \int_{0}^{\infty} (((\mathbf{P}_{\infty}^{\tau})^{0})^{-1}(\mathbf{S}_{\infty}^{\tau})^{0}\mathbf{a}_{\tau})(t)e^{i\lambda t}dt \right|}{\left| 1 - e^{i\lambda\tau}\right|^{n}(1 + \lambda^{2})^{n}(f^{0}(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^{2})^{n}}g^{0}(\lambda))}$$
(22)

are bounded. Under these conditions the functional $\Delta(h_{\tau}(f^0, g^0); f, g)$ is continuous and bounded in $\mathcal{L}_1 \times \mathcal{L}_1$ space. The condition $0 \in \partial \Delta_{\mathcal{D}}(f^0, g^0)$ implies that least favorable densities $f^0 \in \mathcal{D}_f^0$, $g^0 \in \mathcal{D}_g^0$ satisfy the equation

$$\begin{vmatrix}
A(\lambda)(1 - e^{i\lambda\tau})^n (1 + \lambda^2)^n f^0(\lambda) + \lambda^{2n} \int_0^\infty (((\mathbf{P}_\infty^\tau)^0)^{-1} (\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \\
= \alpha_1 |1 - e^{i\lambda\tau}|^n \left((1 + \lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda) \right), \qquad (23)$$

$$\begin{vmatrix}
A(\lambda)(1 - e^{i\lambda\tau})^n (-i\lambda)^n g^0(\lambda) - (-i\lambda)^n \int_0^\infty (((\mathbf{P}_\infty^\tau)^0)^{-1} (\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \\
= \alpha_2 |1 - e^{i\lambda\tau}|^n |1 - i\lambda|^{-n} \left((1 + \lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda) \right), \qquad (24)$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are constants such that $\alpha_1 \neq 0$ if $(2\pi)^{-1} \int_{-\infty}^{\infty} f^0(\lambda) d\lambda = P_1$ and $\alpha_2 \neq 0$ if $(2\pi)^{-1} \int_{-\infty}^{\infty} g^0(\lambda) d\lambda = P_2$.

Thus, the following statements hold true.

Theorem 5.1 Let conditions (9) hold true, spectral densities $f^0(\lambda) \in \mathcal{D}_f^0$, $g^0(\lambda) \in \mathcal{D}_g^0$ satisfy conditions (8) and functions $h_{\tau,f}(f^0, g^0)$, $h_{\tau,g}(f^0, g^0)$ be bounded. The spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ are least favorable in the class $\mathcal{D} = \mathcal{D}_f^0 \times \mathcal{D}_g^0$ for the optimal linear estimation of the functional $A\xi$ if they are a solution of equations (23)-(24) and determine a solution to extremum problem (20). Minimax spectral characteristic $h_{\tau}(f^0, g^0)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).

Theorem 5.2 Let known spectral density $f(\lambda)$ and spectral density $g^0(\lambda) \in \mathcal{D}_g^0$ satisfy conditions (8). Let conditions (9) hold true and the function $h_{\tau,g}(f,g^0)$ be bounded. Then spectral density $g^0(\lambda)$ is least favorable in the class \mathcal{D}_g^0 for the optimal linear filtering of the functional $A\xi$ if it has the form

$$g^{0}(\lambda) = (1 + \lambda^{2})^{n} \lambda^{-2n} \max \left\{0, f_{1}(\lambda)\right\},$$

$$f_{1}(\lambda) = \frac{\left|A(\lambda)(1 - e^{i\lambda\tau})^{n}(1 + \lambda^{2})^{n} f(\lambda) + \lambda^{2n} \int_{0}^{\infty} (((\mathbf{P}_{\infty}^{\tau})^{0})^{-1}(\mathbf{S}_{\infty}^{\tau})^{0} \mathbf{a}_{\tau})(t) e^{i\lambda t} dt\right|}{\alpha_{2} |1 - e^{i\lambda\tau}|^{n} (1 + \lambda^{2})^{n}} - f(\lambda) \quad (25)$$

and the pair (f, g^0) determines a solution to extremum problem (20). Minimax spectral characteristic $h_{\tau}(f, g^0)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).

Theorem 5.3 Let spectral density $f^0(\lambda) \in \mathcal{D}_f^0$ and known spectral density $g(\lambda)$ satisfy conditions (8). Let conditions (9) hold true and the function $h_{\tau,f}(f^0,g)$ be bounded. Then spectral density $f^0(\lambda)$ is least favorable in the class \mathcal{D}_f^0 for the optimal linear filtering of the functional $A\xi$ if it has the form

$$f^{0}(\lambda) = (1 + \lambda^{2})^{n} \lambda^{-2n} \max \left\{0, g_{1}(\lambda)\right\},$$

$$g_{1}(\lambda) = \frac{\left|A(\lambda)(1 - e^{i\lambda\tau})^{n}(-i\lambda)^{n}g(\lambda) - (-i\lambda)^{n} \int_{0}^{\infty} (((\mathbf{P}_{\infty}^{\tau})^{0})^{-1}(\mathbf{S}_{\infty}^{\tau})^{0}\mathbf{a}_{\tau})(t)e^{i\lambda t}dt\right|}{\alpha_{1}|1 - e^{i\lambda\tau}|^{n}|1 + i\lambda|^{n}} - g(\lambda) \quad (26)$$

and the pair (f^0, g) determines a solution to extremum problem (20). Minimax spectral characteristic $h_{\tau}(f^0, g)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).

6 Least favorable densities in the class $\mathcal{D} = \mathcal{D}_u^v \times \mathcal{D}_{\varepsilon}$

Consider an optimal linear filtering of the functional $A\xi$ in case of the set $\mathcal{D} = \mathcal{D}_u^v \times \mathcal{D}_{\varepsilon}$ of admissible spectral densities, where

$$\mathcal{D}_{u}^{v} = \left\{ f(\lambda) | v(\lambda) \le f(\lambda) \le u(\lambda), \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) d\lambda \le P_{1} \right\},$$

$$\mathcal{D}_{\varepsilon} = \left\{ g(\lambda) | g(\lambda) = (1 - \varepsilon) g_{2}(\lambda) + \varepsilon w(\lambda), \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\lambda) d\lambda \le P_{2} \right\}.$$

Spectral densities $u(\lambda)$, $v(\lambda)$ and $g_2(\lambda)$ from the definition os the set $\mathcal{D} = \mathcal{D}_u^v \times \mathcal{D}_{\varepsilon}$ are known, fixed and in addition spectral densities $u(\lambda)$ and $v(\lambda)$ are bounded.

Let functions $h_{\tau,f}(f^0,g^0)$ and $h_{\tau,g}(f^0,g^0)$ determined by formulas (21), (22) are bounded for spectral densities $f^0 \in \mathcal{D}_u^v$, $g^0 \in \mathcal{D}_{\varepsilon}$. Condition $0 \in \partial \Delta_{\mathcal{D}}(f^0,g^0)$ imply equations that define least favorable spectral densities

$$\begin{vmatrix}
A(\lambda)(1 - e^{i\lambda\tau})^n (1 + \lambda^2)^n f^0(\lambda) + \lambda^{2n} \int_0^\infty (((\mathbf{P}_\infty^\tau)^0)^{-1} (\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \\
= |1 - e^{i\lambda\tau}|^n \left((1 + \lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda) \right) (\gamma_1(\lambda) + \gamma_2(\lambda) + \alpha_1), \tag{27}$$

$$\begin{vmatrix}
A(\lambda)(1 - e^{i\lambda\tau})^n (-i\lambda)^n g^0(\lambda) - (-i\lambda)^n \int_0^\infty (((\mathbf{P}_\infty^\tau)^0)^{-1} (\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \\
= |1 - e^{i\lambda\tau}|^n |1 - i\lambda|^{-n} \left((1 + \lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda) \right) (\varphi(\lambda) + \alpha_2), \tag{28}$$

where $\gamma_1 \leq 0$ and $\gamma_1 = 0$ if $f^0(\lambda) \geq v(\lambda)$; $\gamma_2(\lambda) \geq 0$ and $\gamma_2 = 0$ if $f^0(\lambda) \leq u(\lambda)$; $\varphi(\lambda) \leq 0$ and $\varphi(\lambda) = 0$ when $g^0(\lambda) \geq (1 - \varepsilon)g_1(\lambda)$.

The following statements hold true.

Theorem 6.1 Let conditions (9) hold true, spectral densities $f^0(\lambda) \in \mathcal{D}_u^v$, $g^0(\lambda) \in \mathcal{D}_{\varepsilon}$ satisfy condition (8) and functions $h_{\tau,f}(f^0,g^0)$, $h_{\tau,g}(f^0,g^0)$ determined by formulas (21), (22) be bounded. Spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ are least favorable in the class $\mathcal{D} = \mathcal{D}_u^v \times \mathcal{D}_{\varepsilon}$ for the optimal linear filtering of the functional $A\xi$ if they satisfy equations (27), (28) and determine a solution to extremum problem (20). Minimax spectral characteristic $h_{\tau}(f^0, g^0)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).

Theorem 6.2 Let known spectral density $f(\lambda)$ and spectral density $g^0(\lambda) \in \mathcal{D}_{\varepsilon}$ satisfy conditions (8). Let conditions (9) hold true and the function $h_{\tau,g}(f,g^0)$ be bounded. Spectral density $g^0(\lambda)$ is least favorable in the class $\mathcal{D}_{\varepsilon}$ for the optimal linear filtering of the functional $A\xi$ if it has the form

$$g^{0}(\lambda) = \max\left\{ (1 - \varepsilon)g_{2}(\lambda), (1 + \lambda^{2})^{n} \lambda^{-2n} f_{1}(\lambda) \right\},\,$$

where function $f_1(\lambda)$ is defined by formula (25), and the pair (f, g^0) determines a solution to extremum problem (20). Minimax spectral characteristic $h_{\tau}(f, g^0)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).

Theorem 6.3 Let spectral density $f^0(\lambda) \in \mathcal{D}_u^v$ and known spectral density $g(\lambda)$ satisfy conditions (8). Let conditions (9) hold true and the function $h_{\tau,f}(f^0,g)$ be bounded. Spectral density $f^0(\lambda)$ is least favorable in the class \mathcal{D}_u^v for the optimal linear filtering of the functional $A\xi$ if it has the form

$$f^{0}(\lambda) = \min \left\{ v(\lambda), \max \left\{ u(\lambda), (1 + \lambda^{2})^{n} \lambda^{-2n} g_{1}(\lambda) \right\} \right\},\,$$

where function $g_1(\lambda)$ is defined by formula (26), and the pair (f^0, g) determines a solution to extremum problem (20). Minimax spectral characteristic $h_{\tau}(f^0, g)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).

7 Conclusions

In this paper we derived two methods of solving a problem of optimal mean-square estimation of linear functionals $A\xi = \int_0^\infty a(t)\xi(-t)dt$ and $A_T\xi = \int_0^T a(t)\xi(-t)dt$ of unknown values of a random process $\xi(t)$ with nth stationary increments from the observation of a process $\xi(t) + \eta(t)$ at points $t \leq 0$. The first method provides us with formulas for calculating the values of mean-square errors and the spectral characteristics of the estimates of the functionals $A\xi$ and $A_T\xi$ when spectral densities of the processes are given. The second one, called minimax-robust method, let us solve the problem in the case when spectral densities are not known, but a set of admissible spectral densities is available.

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