# Debiased Kernel Estimation of Spot Volatility in the Presence of Infinite Variation Jumps\*

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#### Abstract

Volatility estimation is a central problem in financial econometrics, but becomes particularly challenging when jump activity is high, a phenomenon observed empirically in highly traded financial securities. In this paper, we revisit the problem of spot volatility estimation for an Itô semimartingale with jumps of unbounded variation. We construct truncated kernel-based estimators and debiased variants that extend the efficiency frontier for spot volatility estimation in terms of the jump activity index Y, raising the previous bound Y < 4/3 to Y < 20/11, thereby covering nearly the entire admissible range Y < 2. Compared with earlier work, our approach attains smaller asymptotic variances through the use of unbounded kernels, is simpler to implement, and has broader applicability under more flexible model assumptions. A comprehensive simulation study confirms that our procedures substantially outperform competing methods in finite samples.

# 1 Introduction

Spot volatility estimation is a central problem in financial econometrics, underpinning numerous critical applications ranging from short-horizon risk management to derivatives pricing and hedging. Formally, the core task is to estimate the coefficient  $\sigma_{\tau}$  of the Brownian component of an Itô semimartingale X at a fixed point of time  $\tau$  given high-frequency observations. For recent developments, we refer to [15, 7, 8, 19], as well as the monographs [10, 2] for in-depth treatments.

Widely observed empirically in high-frequency financial data, jumps are a salient feature of price dynamics, separate from fluctuations captured by diffusive movements in X, and must be carefully accounted for when estimating volatility and related functionals of X. When jump activity is high, however, standard estimation methods face inherent limitations, as estimation difficulty increases with the degree of jump activity of the underlying process. Three qualitatively distinct regimes can be distinguished: jumps of finite activity, infinite

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activity jumps of bounded variation, and infinite activity jumps of unbounded variation, the latter posing the greatest challenge, and the setting of the present work.

A fundamental approach to handling jumps is through truncation, introduced by [16]. Establishing jump-robustness for truncation-based methods classically amounts to two steps: suppose  $\hat{\theta}_n(\Delta_1^n X, \dots, \Delta_n^n X)$  is an estimator of some functional  $\theta$  of the continuous component  $X^c$  of X, where  $\Delta_i^n U := U_{t_i} - U_{t_{i-1}}$  denotes the  $i^{th}$  increment of a generic process U sampled at times  $0 < t_0 < \dots < t_n = T$  during a finite time period [0, T]. In the first step, one shows that the (infeasible) estimator  $\hat{\theta}_n^{\text{Cont}} := \hat{\theta}_n(\Delta_1^n X^c, \dots, \Delta_n^n X^c)$  is consistent and satisfies a central limit theorem (CLT) with, say, rate  $r_n$ :

$$\frac{1}{r_n} \left( \hat{\theta}_n(\Delta_1^n X^c, \dots, \Delta_n^n X^c) - \theta \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_\theta). \tag{1}$$

Second, it is shown that, by choosing an appropriate threshold level  $v_n \searrow 0$ , the truncated or thresholded version  $\hat{\theta}_n^{Trunc} := \hat{\theta}_n(\Delta_1^n X \mathbf{1}_{\{|\Delta_1^n X| \leq v_n\}}, \dots, \Delta_n X^n \mathbf{1}_{\{|\Delta_n^n X| \leq v_n\}})$  is close enough to  $\hat{\theta}^{Cont}$  so that

$$\frac{1}{r_n} \left( \hat{\theta}_n^{\text{Trunc}} - \hat{\theta}_n^{\text{Cont}} \right) = o_P(1). \tag{2}$$

An emblematic example of this approach arises in the estimation of the integrated variance or volatility  $IV := \int_0^T \sigma_s^2 ds$ , where one passes from the quantity

$$\widehat{IV}^{\text{Cont}} := \sum_{i=1}^{n} (\Delta_i^n X^c)^2, \tag{3}$$

to its truncated version

$$\widehat{IV}^{\text{Trunc}} := \sum_{i=1}^{n} (\Delta_i^n X)^2 \mathbf{1}_{\{|\Delta_i^n X| \le v_n\}}.$$
(4)

By (essentially) employing the steps (1)-(2), it has been established that in the bounded variation case [17, 5], an efficient CLT (see, e.g., [11]) is possible:

$$\sqrt{n}\left(\widehat{IV}^{\mathrm{Trunc}} - IV\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, 2\int_0^T \sigma_s^4 ds\right).$$

However, with unbounded-variation jumps, this approach alone falls short. It was proved by [5, 18] that  $\sqrt{n}(\widehat{IV}^{\text{Trunc}} - IV) \xrightarrow{P} \infty$  in the presence of an infinite variation symmetric stable Lévy process, precisely because the bias of  $\widehat{IV}^{\text{Trunc}}$  – and thus the difference  $(\widehat{IV}^{\text{Trunc}} - \widehat{IV}^{\text{Cont}})$  – vanishes too slowly.

Two methodological advances have since moved the field forward. The first is due to Jacod and Todorov [12, 13], who proposed an approach for integrated variance estimation based on the empirical characteristic function of the rescaled increments  $\Delta_i^n X/\sqrt{\Delta_n}$ , where  $\Delta_n$  is the time span of the increment. After applying a suitable logarithmic transformation and debiasing procedure, this approach admits a CLT with optimal rate and variance, even in the case of unbounded variation jumps. However, with this method, full (rate and variance) efficiency is attainable only under specific conditions that require either the jump activity index Y to satisfy Y < 3/2, or under additional restrictive jump symmetry assumptions. The second major advance was introduced more recently by [3], who showed that a debiasing method similar to [12] applied to the realized truncated variations (4) leads to a fully efficient estimator for any  $Y \le 8/5$  without any symmetry conditions.

A natural next question is how these insights extend to the problem of spot volatility estimation. A standard and widely used approach to constructing spot volatility estimators is to apply kernel smoothing to an estimator of integrated variance,

$$\hat{\sigma}_{\tau}^2 = \int K_b(t - \tau) d\widehat{IV}_t,$$

where  $\widehat{IV}_t$  denotes an estimator of  $IV_t = \int_0^t \sigma_s^2 ds$  and  $K_b(u) := K(u/b)/b$  for a chosen kernel K and a bandwidth b > 0. For instance, using the realized quadratic variation  $\widehat{IV}_t = \sum_{i=1}^{[nt]} (\Delta_i^n X)^2$ , this reduces to

$$\hat{\sigma}_{\tau}^{2} = \sum_{i=1}^{n} K_{b}(t_{i-1} - \tau)(\Delta_{i}^{n} X)^{2}, \tag{5}$$

as proposed in [14, 6].

Spot volatility estimators differ markedly from their integrated volatility counterparts, with their local nature altering the efficiency picture in subtle but important ways: on the one hand, optimal convergence rates are of order  $n^{-1/4}$  (compared with  $n^{-1/2}$  in the integrated volatility case); on the other hand, kernel-based estimators enjoy a degree of built-in robustness against jumps, since "large" jumps are relatively rare and are unlikely to fall in a narrow neighborhood of  $\tau$  where  $K_b$  places most of its mass. For instance, [10, Theorem 13.3.3] shows that in the case of uniform kernels (i.e.,  $K(u) = \mathbf{1}_{[0,1)}(u)$  or  $K(u) = \mathbf{1}_{(-1,0]}(u)$ ) the estimator (5) admits a rate-optimal CLT even in the case of unbounded variation jumps, provided  $Y < 4/3^1$ . However, high levels of jump activity can still degrade performance, and the rate becomes suboptimal for  $Y \ge 4/3$ : in general, for any  $a \in (0,1/2]$  such that Y < 2/(1+a), one can find a bandwidth  $b_n$  such that  $n^{a/2}(\hat{\sigma}_{\tau}^2 - \sigma_{\tau}^2) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_{\sigma})$ , which yields a rate that is slower than  $n^{-\frac{2-Y}{2Y}}$ . Thus, if, for instance,  $Y \le 3/2$ , the rate is slower than  $n^{-1/6}$ . Surprisingly, following the two-step approach (1)-(2), even when thresholding is applied to (5), i.e.,

$$\hat{\sigma}_{\tau}^{2} = \sum_{i=1}^{n} K_{b}(t_{i-1} - \tau)(\Delta_{i}^{n} X)^{2} \mathbf{1}_{\{|\Delta_{i}^{n} X| \le v_{n}\}},$$
(6)

there is no apparent asymptotic benefit: the optimal rate of  $n^{-1/4}$  remains attainable only if Y < 4/3, and the best possible rate when  $Y \ge 4/3$  remains bounded by  $n^{-\frac{2-Y}{2Y}}$ , which is strictly slower.

Given these limitations, one may wonder if estimation performance can be improved by modifying the kernel K. This question was addressed in [8], which extended the approach in [10] to two-sided kernels of unbounded support. Although the convergence rate remains unchanged, they established that the asymptotic variance can be substantially reduced. For example, using the kernel  $K(u) = .5e^{-|u|}$  leads to an asymptotic variance that is one-quarter of that obtained by a uniform kernel in the suboptimal rate regime  $(Y \ge 4/3)$ . Moreover, in the optimal rate regime (Y < 4/3), this kernel was shown to be variance-optimal, highlighting the utility of unbounded kernels in general. However, we note that the results in both [10] and [8] are based on the two-step approach (1)-(2), which we depart from in the present work.

These findings motivate a fundamental question: can one develop methods that attain better rates – possibly optimal – beyond the boundary Y < 4/3? A recent step in this direction was taken by [15], who developed a localized variant of the characteristic function approach of [12]. Assuming the infinite-variation jump component is of the form  $\int_0^t \chi_s dL_s$ , where L is a strictly Y-stable symmetric Lévy process, and while restricting to the class of compactly-supported kernels that are continuously differentiable, they establish that their estimator satisfies

$$\sqrt{\frac{b_n}{\Delta_n}} \frac{\hat{\sigma}_{\tau}^2 - \sigma_{\tau}^2}{\sqrt{2\sigma_{\tau}}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \int K^2(u)du\right),\tag{7}$$

for any  $Y \leq 3/2$ , but with a suboptimal convergence rate (i.e.,  $\sqrt{\frac{\Delta_n}{b_n}} \gg n^{-1/4}$ ). By applying an additional debiasing step similar to that in [12], they further showed that associated tuning parameters can be chosen so that (7) holds for any Y < 2, however, still with a suboptimal rate and under the jump symmetry condition described above. Consequently, the boundary Y < 4/3 has long marked the limit of efficiency in spot volatility estimation, and the regime Y > 4/3 has remained largely underexplored without symmetry assumptions.

In this work, we push this efficiency boundary substantially forward by introducing appropriately debiased versions of the estimator (6). First, by exploiting high-order expansions of the truncated moments of  $\Delta_i X$ , rather

<sup>&</sup>lt;sup>1</sup>The case of jumps of bounded variation corresponds to Y < 1, while the jumps are of unbounded variation when Y > 1.

than relying on the coarser two-step framework (1)-(2), which does not explicitly account for higher-order bias, we show the estimator (6) actually achieves rate-optimality for any  $Y \leq 3/2$  (namely,  $n^{1/4}(\hat{\sigma}_{\tau}^2 - \sigma_{\tau}^2) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_{\sigma})$ ) under standard assumptions and without additional adjustment. Second, by introducing debiasing steps similar to those in [12, 3], we construct fully efficient estimators that cover the range 0 < Y < 20/11. This extension is practically relevant, as empirical studies indicate that values of Y are typically between 1.5 and 1.8 for liquid stocks (see [3] for a brief supporting empirical analysis and further references). Our method stands in sharp contrast to the results in [15], offering rate-optimal estimators over a wider range of Y, broader applicability through more flexible model assumptions, reduced asymptotic variances enabled by the use of unbounded kernels, and simpler estimators that are easier to implement. To validate our results, we perform a comprehensive simulation study and find that our methods significantly outperform those of [15] in terms of both bias and variance over finite samples. Our findings further demonstrate that the advantages of kernels with unbounded support extend to the debiasing framework developed here, providing substantial improvements over widely-used uniform kernels, which are often the default choice in practice, especially in inference problems that require estimation of spot volatility as a preliminary step.

The paper is organized as follows. Section 2 introduces the setting, assumptions, and preliminary results. Section 3 presents the main results and their proofs. Section 4 contains simulations assessing the performance of our estimators. Some additional technical proofs are collected in Appendix A.

# 2 Setting and background

Consider a 1-dimensional Itô semimartingale  $X = (X_t)_{t \in \mathbb{R}_+}$  defined on a complete filtered probability space  $(\Omega, \mathscr{F}, (\mathscr{F})_{t \in \mathbb{R}_+}, \mathbb{P})$  that can be decomposed as

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \chi_{s-} dJ_s + \int_0^t \int \delta(s, z) \mathfrak{p}(ds, dz), \quad t \in [0, \infty), \tag{8}$$

where  $W := (W_t)_{t \in \mathbb{R}_+}$  is a Wiener process,  $\sigma$  is an adapted càdlàg process,  $J := (J_t)_{t \in \mathbb{R}_+}$  is an independent pure-jump Lévy process with Lévy triplet (b,0,v), and  $\mathfrak{p}$  is a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}$  with an intensity  $\mathfrak{q}(ds,dz) = ds \otimes \lambda(dz)$ , such that the  $\lambda$  is a  $\sigma$ -finite measure on  $\mathbb{R}$ , possibly dependent with J. The Lévy measure v is assumed to admit a density  $s : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$  of the form

$$s(x) := \frac{dv}{dx} := \left( C_{+} \mathbf{1}_{(0,\infty)}(x) + C_{-} \mathbf{1}_{(-\infty,0)}(x) \right) q(x) |x|^{-1-Y}. \tag{9}$$

We have the following standing assumptions:

#### Assumption 1.

- 1.  $C_{\pm} > 0$  and  $Y \in (0,2) \setminus \{1\}$ .
- 2.  $q: \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$  is a bounded Borel-measurable function such that:
  - (a)  $q(x) \to 1$ , as  $x \to 0$ .
  - (b) There exist  $a_{\pm}$  such that

$$\int_0^1 |q(x) - 1 - a_+ x | x^{-Y - 1} dx + \int_{-1}^0 |q(x) - 1 - a_- x | |x|^{-Y - 1} dx < \infty.$$
 (10)

3. The process  $\chi$  is given as

$$\chi_t = \chi_0 + \int_0^t b_x^{\chi} ds + \int_0^t \Sigma_s^{\chi} dB_s. \tag{11}$$

- 4. The processes W, B are standard Brownian motions with correlation  $d\langle W, B \rangle_t = \rho_t dt$  for an adapted, locally bounded càdlàg process  $\{\rho_t\}_{t>0}$ . W, B are independent of  $(J, \mathfrak{p})$ ; J is independent of  $\sigma$ .
- 5. The processes  $\Sigma^{\chi}$ , b,  $b^{\chi}$  are càdlàg adapted, and  $\delta$  is predictable. There exist a sequence  $\{\tau_n\}_{n\geq 1}$  of stopping times increasing to infinity, nonnegative  $\lambda(dz)$ -integrable function H and a positive sequence  $\{M_n\}_{n\geq 1}$  such that

$$t \le \tau_n \Rightarrow \begin{cases} |\sigma_t| + |b_t| + |b_t^{\chi}| + |\Sigma_t^{\chi}| \le M_n, \\ (|\delta(t, z)| \land 1)^r \le M_n H(z), \end{cases}$$

for some  $r \in [0, 1 \wedge Y)$ .

6. The spot variance process  $c_t := \sigma_t^2$  is assumed to follow the dynamics:

$$c_t = c_0 + \int_0^t \tilde{\mu}_s ds + \int_0^t \tilde{\sigma}_s dB_s, \tag{12}$$

where  $B := (B_t)_{t\geq 0}$  is as in point 3 above. Here  $\{\tilde{\mu}\}_{t\geq 0}$  and  $(\tilde{\sigma}_t)_{t\geq 0}$  are adapted càdlàg locally bounded processes.

Remark 1. The conditions above are essentially the same as those in [3]. [15] considered a similar but more restrictive component of unbounded variation by taking J to be a symmetric strictly stable Lévy process, while here we considered a type of tempered stable Lévy process. The most technical (but still relatively mild) condition is that in (10), which is used to apply a density transformation or change of probability measure under which the process J becomes stable. We refer to [3] for further details.

We assume we have at our disposal n evenly-spaced discrete observations  $X_{t_i}$  during a fixed time interval [0,T]. For simplicity, we further assume T=1 and, thus,  $t_i=i/n$ . Denote  $\Delta_n=1/n$ ,  $\mathcal{F}_i^n:=\mathcal{F}_{t_i}$ , and  $\Delta_i^n A=A_{i\Delta_n}-A_{(i-1)\Delta_n}$  for any process A, where we often omit the superscript n. Our estimation target is  $c_{\tau}=\sigma_{\tau}^2$  for a fixed time  $\tau\in(0,T)$ . We consider the spot volatility estimator

$$\hat{c}_n(m_n, v_n) := \frac{\sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau)(\Delta_i^n X)^2 \mathbf{1}_{\{|\Delta_i^n X| \le v_n\}}}{\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)},$$
(13)

where K is a kernel function, and  $m_n$  controls the bandwidth  $b_n := m_n \Delta_n$  such that  $m_n \to \infty$  and  $m_n \Delta_n \to 0$ , and  $K_b(x) := K(x/b)/b$ . We assume the following conditions on K, which are standard:

**Assumption 2.** The kernel function  $K : \mathbb{R} \to [0, \infty)$  is bounded and Lipschitz and piecewise  $C^1$  on  $(-\infty, \infty)$  such that  $\int K(x)dx = 1$ ,  $\int |K(x)x|dx < \infty$ ,  $\int |K'(x)|dx < \infty$ , and  $K(y)y^2 \to 0$  as  $|y| \to \infty$ .

The estimator (13) without the denominator factor, truncation, or any debiasing was studied in [6] and [14] (see also additional references in the introduction). The denominator, motivated by the standard Nadarayan-Watson nonparametric regression estimator, and noted in passing in [14], serves a dual role: it provides edge correction in finite samples and simplifies several proofs, notably leading to more relaxed technical assumptions in the optimal bandwidth regime (see Remark 2 below). In contrast, when the denominator is omitted, our results remain valid, but stricter conditions on the bandwidth  $b_n = m_n \Delta_n$  and its relationship with Y are required; however, in the suboptimal rate regime (see Remark 2, case (ii), below), the denominator can be dispensed of entirely without any loss of generality. The truncated version in (13) was studied in [7] and [8], also without debiasing. Finally, Assumption 2 permits kernels of unbounded support, which, as discussed in the introduction, enables estimators that are more variance efficient in both the optimal and suboptimal convergence regimes.

**Notation:** Throughout,  $A_n \lesssim B_n$  and  $a_n \ll b_n$  mean  $A_n = O_P(B_n)$  and  $a_n = o(b_n)$ , respectively. For any process V, when unambiguous we write  $\Delta_i V$  in place of  $\Delta_i^n V$ .

# 3 Main results

The following moment expansions play key roles in our analysis. The proofs are based on arguments in analogous results in [3] (see Lemmas 2, 3, 4 and 5 therein) and are given in Appendix A.1.5 for completeness.

**Proposition 1.** Suppose that  $Y \in (0,2) \setminus \{1\}$ . Let  $\Delta_n^{\frac{1}{2}-s} \ll v_n \ll \Delta_n^{\frac{1}{4-Y}}$  for a fixed  $s \in (0,1/2)$ . Then, the following statements hold:

1. For any integer  $p \ge 1$ , we have

$$\mathbb{E}((\Delta_{i}X)^{2p}\mathbf{1}_{\{|\Delta_{i}X|\leq v_{n}\}}|\mathcal{F}_{i-1}) = (2p-1)!!\sigma_{t_{i-1}}^{2p}\Delta_{n}^{p} + C_{p,i}\Delta_{n}v_{n}^{2p-Y} + o_{P}(\Delta_{n}v_{n}^{2p-Y}) + o_{P}(\Delta_{n}^{p}),$$
(14)  
where  $C_{p,i} := \frac{(C_{+}+C_{-})|\chi_{t_{i-1}}|^{Y}}{2p-Y}.$ 

2. Furthermore, if p = 1, the following high-order expansion holds:

$$\mathbb{E}\left((\Delta_{i}X)^{2}\mathbf{1}_{\{|\Delta_{i}X|\leq v_{n}\}}|\mathcal{F}_{i-1}\right) = \sigma_{t_{i-1}}^{2}\Delta_{n} + C_{1,i}\Delta_{n}v_{n}^{2-Y} + D_{1,i}\Delta_{n}^{2}v_{n}^{-Y} + O_{P}(\Delta_{n}^{3}v_{n}^{-2-Y}) + o_{P}(\Delta_{n}^{5/4}), (15)$$
where  $D_{1,i} = \frac{(C_{+}+C_{-})(Y+1)(Y+2)}{2Y}\sigma_{t_{i-1}}^{2}|\chi_{t_{i-1}}|^{Y}$ . Additionally, for any  $\zeta > 1$ , we have

$$\mathbb{E}((\Delta_{i}X)^{2}\mathbf{1}_{\{v_{n}<|\Delta_{i}X|\leq\zeta v_{n}\}}|\mathcal{F}_{i-1}) = C_{1,i}\Delta_{n}(\zeta^{2-Y}-1)v_{n}^{2-Y} + D_{1,i}\Delta_{n}^{2}(\zeta^{-Y}-1)v_{n}^{-Y} + O_{P}(\Delta_{n}^{3}v_{n}^{-2-Y}) + o_{P}(\Delta_{n}^{\frac{3}{4}}v_{n}^{\frac{4-Y}{2}}).$$

$$(16)$$

3. For any p > 1 (not necessarily an integer), we have

$$\mathbb{E}(|\Delta_i X|^{2p} \mathbf{1}_{\{|\Delta_i X| \le v_n\}} | \mathcal{F}_{i-1}) = O_P(\sigma_{t_{i-1}}^{2p} \Delta_n^p) + O_P(C_{p,t_{i-1}} \Delta_n v_n^{2p-Y}). \tag{17}$$

To analyze the asymptotic behavior of the estimator  $\hat{c}_n(m_n, v_n)$  defined in Eq. (13), we consider the following decomposition:

$$\hat{c}_{n}(m_{n}, v_{n}) = \left(\hat{c}_{n}(m_{n}, v_{n}) - \frac{\Delta_{n} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \sigma_{t_{i-1}}^{2}}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}} (t_{j-1} - \tau)}\right) + \left(\frac{\Delta_{n} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \sigma_{t_{i-1}}^{2}}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}} (t_{j-1} - \tau)}\right) =: \hat{c}_{n,1}(m_{n}, v_{n}) + \hat{c}_{n,2}(m_{n}).$$

$$(18)$$

Our next result is central to our analysis and underpins our main debiasing procedure. It establishes the joint asymptotic behavior of the terms in (18), accounting for appropriate bias terms, and provides the framework for separating qualitatively distinct asymptotic regimes for  $m_n$ . In addition, on its own, it yields feasible CLTs for  $\hat{c}_n(m_n, v_n)$  in certain settings that sharpen existing results in the literature (see Remark 2).

**Theorem 1.** Suppose that  $\Delta_n^{\beta} \ll v_n \ll \Delta_n^{\beta'}$  with  $\frac{1}{4-Y} < \beta' < \beta < \frac{1}{2}$ ,  $\sqrt{\log(n)} \ll m_n \ll \Delta_n^{-4} v_n^{4+2Y}$ , and  $m_n = O(\Delta_n^{-\frac{1}{2}})$ . Let

$$\tilde{Z}_n(m_n, v_n) := \sqrt{m_n} \Big( \hat{c}_{n,1}(m_n, v_n) - A(v_n, m_n) \Big),$$

$$\tilde{Z}'_n(m_n) := \frac{1}{\sqrt{m_n \Delta_n}} \Big( \hat{c}_{n,2}(m_n) - \sigma_\tau^2 \Big),$$

where, using the notation in (15),

$$A(v_n, m_n) := \frac{\sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) C_{1,i} \Delta_n v_n^{2-Y}}{\Delta_n \sum_{i=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)} + \frac{\sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) D_{1,i} \Delta_n^2 v_n^{-Y}}{\Delta_n \sum_{i=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)}.$$
 (19)

Then, under Assumptions 1 and 2, as  $n \to \infty$ ,

$$(\tilde{Z}_n(m_n, v_n), \tilde{Z}'_n(m_n)) \xrightarrow{st} (Z_1, Z_2),$$
 (20)

where  $Z_1 \sim N\left(0, 2\sigma_{\tau}^4 \int K^2(x) dx\right)$  and  $Z_2 \sim N(0, \tilde{\sigma}_{\tau}^2 \int L^2(t) dt)$  with  $L(t) = \int_t^{\infty} K(u) du \mathbf{1}_{t \geq 0} - \int_{-\infty}^t K(u) du \mathbf{1}_{t < 0}$ , and  $Z_1$  and  $Z_2$  are independent.

**Remark 2.** Taking  $m_n \to \infty$  at different rates alters the balance of the errors appearing on the right-hand side of (20). Three distinct regimes can be distinguished: (i)  $m_n \sqrt{\Delta_n} \to \theta \in (0, \infty)$ , (ii)  $m_n \sqrt{\Delta_n} \to 0$ , and (iii)  $m_n \sqrt{\Delta_n} \to \infty$ , which we remark on below.

Case (i):  $m_n\sqrt{\Delta_n} \to \theta \in (0,\infty)$ . In this regime, the orders of  $\sqrt{m_n}$  and  $1/\sqrt{m_n\Delta_n}$  match, in which case Theorem 1 gives

$$\Delta_n^{-1/4} \left( \hat{c}_n(m_n, v_n) - A(v_n, m_n) - \sigma_\tau^2 \right) \xrightarrow{st} \theta^{-1/2} Z_1 + \theta^{1/2} Z_2, \tag{21}$$

or, equivalently,

$$\sqrt{m_n} \left( \hat{c}_n(m_n, v_n) - A(v_n, m_n) - \sigma_\tau^2 \right) \xrightarrow{st} Z_1 + \theta Z_2. \tag{22}$$

Note that the bias term  $A(v_n, m_n)$  is  $O_P(v_n^{2-Y})$ . Thus, if  $\sqrt{m_n}v_n^{2-Y} \ll 1$ , then (21) and (22) remain true even with the term  $A(v_n, m_n)$  omitted, in which case we obtain a valid feasible CLT for the estimation error  $\hat{c}_n(m_n, v_n) - \sigma_\tau^2$  with convergence rate  $\Delta_n^{1/4}$  (which is optimal). While the condition  $\sqrt{m_n}v_n^{2-Y} \ll 1$  can be met in principle by taking small enough  $v_n$ , Theorem 1 additionally requires  $v_n \gg \Delta_n^{1/2}$ , and thus implicitly  $\Delta_n^{-1/4}\Delta_n^{(2-Y)/2} \ll 1$ , which can happen only if Y < 3/2. Nevertheless, this result significantly improves upon [8] (and also [10]), where a rate-optimal CLT is shown to be valid only when Y < 4/3. Theorem 1 also yields stronger results than the characteristic function approach taken in [15], in which a similar CLT is is shown to be valid, under substantially more restrictive model assumptions<sup>2</sup>, for Y < 3/2 at a convergent rate strictly slower than  $\Delta_n^{1/4}$  (due to an assumption that  $m_n \sqrt{\Delta_n} \to 0$ ).

Further, under case (i), the conditions on  $v_n$  take the form  $\Delta_n^{\beta} \vee \Delta_n^{\frac{7}{4Y+8}} \ll v_n \ll \Delta_n^{\beta'}$   $(\frac{1}{4-Y} < \beta' < \beta < \frac{1}{2})$ , which can hold only if  $\frac{7}{4Y+8} > \frac{1}{4-Y}$ , which requires  $Y < \frac{20}{11}$ . Thus, this rate-optimal case (i) covers nearly the entire admissible range of Y observed in applications, and ultimately underpins our feasible efficient CLT in this range (Theorem 3).

Case (ii):  $m_n \sqrt{\Delta_n} \to 0$ . This regime corresponds to setting  $\theta = 0$  in (22). Indeed,

$$\sqrt{m_n} \left( \hat{c}_n(m_n, v_n) - A(v_n, m_n) - \sigma_\tau^2 \right) = \tilde{Z}_n(m_n, v_n) + m_n \sqrt{\Delta_n} \tilde{Z}'_n(m_n, v_n), \tag{23}$$

and, from (20), the first term above converges to  $Z_1$ , while the second is  $o_P(1)$ . In this case, the attained convergence rate,  $1/\sqrt{m_n}$ , is suboptimal, i.e., slower than the  $\Delta^{1/4}$  rate achieved when  $\theta \in (0, \infty)$ . However, this asymptotic regime has the practical advantage that any eventual estimation of the volatility-of-volatility  $\tilde{\sigma}_{\tau}^2$  is not needed when constructing confidence intervals, since  $Z_2$  is absent from the limit. If we also require  $\sqrt{m_n}v_n^{2-Y} \ll 1$ , then the bias  $A(v_n, m_n)$  can be omitted in (23) leading to a feasible CLT for  $\hat{c}_n(m_n, v_n)$  centered at  $\sigma_{\tau}^2$ . Under this constraint, the condition  $v_n \gg \Delta_n^{\frac{1}{2}}$  and the asymptotic condition of case (ii) together imply that the convergence rate  $m_n^{-1/2}$  must be strictly slower than  $\Delta_n^{\frac{1}{4}\wedge\frac{2-Y}{2}}$ , yielding a convergence rate that can be taken arbitrarily close to  $\Delta_n^{1/4}$  when Y < 3/2 but deteriorates as Y increases to 2.

Case (iii):  $m_n\sqrt{\Delta_n} \to \infty$ . Though technically excluded from our hypotheses, our results can also cover the regime  $m_n\sqrt{\Delta_n} \to \infty$ , but at the expense of an upper bound of Y < 8/5 in the jump activity index. In this setting, the attained convergence rate is  $\sqrt{m_n\Delta_n}$ , which is not as fast as the analogous rates when  $\theta \in [0,\infty)$ .

<sup>&</sup>lt;sup>2</sup>For instance, they assume that the process J in (8) is a symmetric stable Lévy process

Remark 3. Beyond finite-sample edge correction, the denominator in (13) (which converges to 1) relaxes the technical conditions required for our results; without it, additional restrictions on Y and  $m_n$  are necessary. Specifically, define  $\tilde{c}_n(m_n, v_n) := \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) (\Delta_i X)^2 \mathbf{1}_{\{|\Delta_i X| \leq v_n\}}$ , and let  $\tilde{Z}_n(m_n, v_n)$ ,  $\tilde{Z}'_n(m_n)$ , and  $A(v_n, m_n)$  denote the corresponding analogs without the denominator. Then the limit (20) still holds under the hypotheses of Theorem 1, provided also that  $m_n \gg \Delta_n^{-1/3}$ . Under this additional condition on  $m_n$ , (22) remains valid for Y < 20/11 when  $m_n \sqrt{\Delta_n} \to \theta \in (0, \infty)$ . However, a much more substantive difference arises when  $m_n \sqrt{\Delta_n} \to 0$ : if we also impose  $\sqrt{m_n} v_n^{2-Y} \ll 1$  to obtain a feasible CLT (as discussed in Remark 2), the constraint becomes  $\Delta_n^{-1/3} \ll m_n \ll v_n^{2(Y-2)} \ll \Delta_n^{Y-2}$ , which can only be satisfied if Y < 5/3, illustrating the benefit of including the denominator factor in (13).

Proof of Theorem 1. Denote

$$Y_{i} := \sqrt{m_{n}} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau)(\Delta_{i}X)^{2} \mathbf{1}_{\{|\Delta_{i}X| \leq v_{n}\}},$$

$$Y'_{i} := \frac{\Delta_{n}}{\sqrt{m_{n}\Delta_{n}}} \times \begin{cases} 0, & \text{if } i = 1, \\ -\left(\sum_{l=1}^{i-1} K_{m_{n}\Delta_{n}}(t_{l-1} - \tau)\right)\Delta_{i}B, & \text{if } 2 \leq i \leq \lceil \tau/\Delta_{n} \rceil, \\ \left(\sum_{l=i}^{n} K_{m_{n}\Delta_{n}}(t_{l-1} - \tau)\right)\Delta_{i}B, & \text{if } \lceil \tau/\Delta_{n} \rceil < i \leq n. \end{cases}$$

Consider the following decompositions:

$$\tilde{Z}_{n}(m_{n}, v_{n}) = \left(\frac{\sum_{i=1}^{n} Y_{i}}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} - \frac{\sum_{i=1}^{n} \mathbb{E}(Y_{i} | \mathcal{F}_{i-1})}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)}\right) + \left(\frac{\sum_{i=1}^{n} \mathbb{E}(Y_{i} | \mathcal{F}_{i-1})}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} - \sqrt{m_{n}} \Delta_{n} \sum_{i=1}^{n} \frac{K_{m_{n} \Delta_{n}}(t_{i-1} - \tau)}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} \sigma_{t_{i-1}}^{2} - \sqrt{m_{n}} A(v_{n}, m_{n})\right) = : T_{1} + T_{2},$$
(24)

and

$$\tilde{Z}'_{n}(m_{n}) = \tilde{\sigma}_{\tau} \frac{\sum_{i=1}^{n} Y'_{i}}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} + \left(\tilde{Z}'_{n}(m_{n}) - \frac{\tilde{\sigma}_{\tau} \sum_{i=1}^{n} Y'_{i}}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)}\right) 
= \tilde{\sigma}_{\tau} \frac{\sum_{i=1}^{n} Y'_{i}}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} 
+ \left(\frac{\Delta_{n}}{\sqrt{m_{n} \Delta_{n}}} \sum_{i=1}^{n} \frac{K_{m_{n} \Delta_{n}}(t_{i-1} - \tau)(\sigma_{t_{i-1}}^{2} - \sigma_{\tau}^{2})}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} - \tilde{\sigma}_{\tau} \frac{\sum_{i=1}^{n} Y'_{i}}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)}\right) 
=: T_{3} + T_{4}.$$
(25)

We now proceed to show  $(T_1, T_3) \xrightarrow{st} (Z_1, \tilde{\sigma}_{\tau} Z_2)$  by verifying the conditions of Theorem 2.2.15 in [10], and in addition that  $T_2$  and  $T_4$  are asymptotically negligible.

a) We start with  $T_3$ . Clearly,  $\mathbb{E}(Y_i'|\mathcal{F}_{i-1}) = 0$  and, using the moment formula for Gaussian distributions and Lemma 3.1 in [9], we have

$$\sum_{i=1}^{n} \operatorname{Var}(Y_{i}'|\mathcal{F}_{i-1}) = \frac{\Delta_{n}^{2}}{m_{n}\Delta_{n}} \sum_{i=1}^{n} \left( \left( -\sum_{l=1}^{i-1} K_{m_{n}\Delta_{n}}(t_{l-1} - \tau) \right)^{2} \Delta_{n} \mathbf{1}_{\tau \geq t_{i}} + \left( \sum_{l=i}^{n} K_{m_{n}\Delta_{n}}(t_{l-1} - \tau) \right)^{2} \Delta_{n} \mathbf{1}_{\tau < t_{i}} \right) \\
= \frac{1}{m_{n}\Delta_{n}} \int_{0}^{1} \left( \int_{v}^{1} K_{m_{n}\Delta_{n}}(s - \tau) ds \mathbf{1}_{\tau \geq v} - \int_{0}^{v} K_{m_{n}\Delta_{n}}(s - \tau) ds \mathbf{1}_{\tau < v} \right)^{2} dv + o_{P}(1) \\
\xrightarrow{P} \int \left( \int_{t}^{\infty} K(u) du \mathbf{1}_{t \geq 0} - \int_{-\infty}^{t} K(u) du \mathbf{1}_{t < 0} \right)^{2} dt,$$

and

$$\sum_{i=1}^{n} \mathbb{E}(Y_{i}^{'4}|\mathcal{F}_{i-1}) = \frac{3\Delta_{n}^{4}}{m_{n}^{2}\Delta_{n}^{2}} \sum_{i=1}^{n} \left( \left( \sum_{l=1}^{i-1} K_{m_{n}\Delta_{n}}(t_{l-1} - \tau) \right)^{4} \Delta_{n}^{2} \mathbf{1}_{\tau \geq t_{i}} + \left( \sum_{l=i}^{n} K_{m_{n}\Delta_{n}}(t_{l-1} - \tau) \right)^{4} \Delta_{n}^{2} \mathbf{1}_{\tau < i} \right)$$

$$= \frac{3}{m_{n}} \int \left( \int_{t}^{\infty} K(u) du \mathbf{1}_{t \geq 0} - \int_{-\infty}^{t} K(u) du \mathbf{1}_{t < 0} \right)^{4} dt + o_{P}(1) \xrightarrow{P} 0.$$

On the other hand, by Assumption 2 and Lemma 3.1 in [9], with  $f \equiv 1$  and m = 1 therein, we have

$$\Delta_n \sum_{j=1}^n K_{m_n \Delta_n} (t_{j-1} - \tau) - 1 = O(m_n^{-1}), \tag{26}$$

which vanishes as  $n \to \infty$ . Therefore,

$$\frac{\sum_{i=1}^{n} \operatorname{Var}(Y_{i}'|\mathcal{F}_{i-1})}{\left(\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)\right)^{2}} \xrightarrow{P} \int \left(\int_{t}^{\infty} K(u) du \mathbf{1}_{t \geq 0} - \int_{-\infty}^{t} K(u) du \mathbf{1}_{t < 0}\right)^{2} dt,$$
and
$$\frac{\sum_{i=1}^{n} \mathbb{E}(Y_{i}'^{4}|\mathcal{F}_{i-1})}{\left(\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)\right)^{4}} \to 0.$$

To conclude  $T_3 \xrightarrow{st} \tilde{\sigma}_{\tau} Z_2$ , from Theorem 2.2.15 in [10] it suffices to show the following technical condition:

$$\sum_{i=1}^{n} \mathbb{E}_{i-1} (Y_i' \Delta_i M) \to 0, \tag{27}$$

where  $\mathbb{E}_{i-1}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{i-1})$  and M is either W or B or is in the set  $\mathcal{N}$  containing all bounded martingales orthogonal to W and B. When M = B, we have

$$\sum_{i=1}^{n} \mathbb{E}(Y_i' \Delta_i B | \mathcal{F}_{i-1}) = \frac{\Delta_n}{\sqrt{m_n \Delta_n}} \sum_{i=1}^{n} \left( -\left(\sum_{l=1}^{i-1} K_{m_n \Delta_n}(t_{l-1} - \tau)\right) \Delta_n \mathbf{1}_{\tau \ge t_i} + \left(\sum_{l=i}^{n} K_{m_n \Delta_n}(t_{l-1} - \tau)\right) \Delta_n \mathbf{1}_{\tau < t_i} \right)$$

$$= \frac{1}{\sqrt{m_n \Delta_n}} \int_0^1 \left( \int_v^1 K_{m_n \Delta_n}(s - \tau) ds \mathbf{1}_{\tau \ge v} - \int_0^v K_{m_n \Delta_n}(s - \tau) ds \mathbf{1}_{\tau < v} \right) dv + o(1)$$

$$= \sqrt{m_n \Delta_n} \int \left( \int_t^\infty K(u) du \mathbf{1}_{t \ge 0} - \int_{-\infty}^t K(u) du \mathbf{1}_{t < 0} \right) dt + o(1) \xrightarrow{P} 0.$$

When  $M \in \mathcal{N}$  or M = W, since  $\mathbb{E}(\Delta_i B \Delta_i W | \mathcal{F}_{i-1}) = O_P(\Delta_n)$ , similar to the case with M = B, we obtain

$$\sum_{i=1}^{n} \mathbb{E} \big( Y_i' \Delta_i M | \mathcal{F}_{i-1} \big) \lesssim \frac{1}{\sqrt{m_n \Delta_n}} \int \Big( \int_t^{\infty} K(u) du \mathbf{1}_{\tau \geq 0} - \int_{-\infty}^t K(u) du \mathbf{1}_{\tau < 0} \Big) dt \to 0.$$

**b)** Next, we work with the term  $T_4$  of (25). Let  $i' \in \{1, \ldots, n\}$  be such that  $\tau \in (t_{i'-1}, t_{i'}]$ . Notice that

$$\sum_{i=1}^{n} Y_{i}' = -\sum_{l=1}^{i'-1} \left( \sum_{i=l+1}^{i'} \Delta_{i} B \right) K_{m_{n} \Delta_{n}} (t_{l-1} - \tau) + \sum_{l=i'+1}^{n} \left( \sum_{i=i'+1}^{l} \Delta_{i} B \right) K_{m_{n} \Delta_{n}} (t_{l-1} - \tau)$$
(28)

$$= \sum_{l=1}^{i'-1} K_{m_n \Delta_n} (t_{l-1} - \tau) (B_{t_l} - B_{t_{i'}}) + \sum_{l=i'+1}^n K_{m_n \Delta_n} (t_{l-1} - \tau) (B_{t_l} - B_{t_{i'}}).$$
 (29)

In light of (26), we can then write

$$|T_4| \lesssim \left| \frac{1}{\sqrt{m_n \Delta_n}} \left( \Delta_n \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \left( \int_{\tau}^{t_{i-1}} (\tilde{\sigma}_s - \tilde{\sigma}_\tau) dB_s + \int_{\tau}^{t_{i-1}} \tilde{\mu}_s ds \right) \right) \right|$$

$$+ \left| \frac{1}{\sqrt{m_n \Delta_n}} \Delta_n \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \left( \int_{t_{i-1}}^{t_i} \tilde{\sigma}_\tau dB_s + \int_{\tau}^{t_{i'}} \tilde{\sigma}_\tau dB_s \right) \right| =: |T_{4,1}| + |T_{4,2}|.$$

$$(30)$$

Since  $|B_{t_{i'}} - B_{\tau}| = O_P(\Delta_n^{1/2})$  and  $\sup_i |B_{t_i} - B_{t_{i-1}}| = O_P(\Delta_n^{1/2}\log(n)^{1/2})$ , the second term  $T_{4,2}$  has order  $O_P(\frac{\sqrt{\Delta_n \log(n)}}{\sqrt{m_n \Delta_n}})$  and vanishes given that  $m_n \gg \sqrt{\log(n)}$  as  $n \to \infty$ . We just need to worry about the first term. Let

$$\eta_i := \int_{\tau}^{t_{i-1}} (\tilde{\sigma}_{\tau} - \tilde{\sigma}_s) dB_s + \int_{\tau}^{t_{i-1}} \tilde{\mu}_s ds, \quad \text{and} \qquad \rho(i) := \frac{1}{t_{i-1} - \tau} \mathbb{E}\Big(\int_{\tau}^{t_{i-1}} |\tilde{\sigma}_s - \tilde{\sigma}_{\tau}|^2 ds\Big).$$

Note that, following the same argument as in the proof of (13.3.37) for j=6 in [10] (see also the proof of (A.6) for l=4 in [8]), and using that  $\tilde{\sigma}$  is càdlàg and bounded, for any positive sequence  $N_n \to \infty$  with  $m_n \Delta_n N_n \to 0$  as  $n \to \infty$ , it holds that

$$\mathbb{E}(\eta_i^2 1_{i \in \mathcal{I}_n}) \le c|t_{i-1} - \tau|\rho(i) \quad \text{and} \quad \sup_{i \in \mathcal{I}_n} \rho(i) \to 0,$$

where  $\mathcal{I}_n := \{i \in \{1, ..., n\} : t_i \in (\tau - Nm_n\Delta_n, \tau + Nm_n\Delta_n)\}$ . Additionally, using the Itô isometry and boundedness of  $\tilde{\sigma}$  and  $\tilde{\mu}$ , it is easily seen that  $\mathbb{E}\eta_i^2 \lesssim |\tau - t_i|$  for any i. Thus,

$$\mathbb{E}|T_{4,1}| \lesssim \frac{1}{\sqrt{m_n \Delta_n}} \left( \Delta_n \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) (\mathbf{1}_{i \in \mathcal{I}_n} + \mathbf{1}_{i \notin \mathcal{I}_n}) \sqrt{\mathbb{E}\eta_i^2} \right)$$

$$\lesssim \frac{1}{\sqrt{m_n \Delta_n}} \left( \Delta_n \sum_{i=1}^n \frac{1}{m_n \Delta_n} \mathbf{1}_{i \in \mathcal{I}_n} \sqrt{|t_{i-1} - \tau| \rho_i} \right)$$

$$+ \frac{1}{\sqrt{m_n \Delta_n}} \left( \Delta_n \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau)) \mathbf{1}_{i \notin \mathcal{I}_n} \sqrt{|t_{i-1} - \tau|} \right)$$

$$\lesssim \frac{m_n \Delta_n}{\sqrt{m_n \Delta_n} m_n \Delta_n} N_n \sqrt{N_n m_n \Delta_n \sup_{i \in \mathcal{I}_n} \rho_i} + \int_{|u| > N} K(u) \sqrt{|u|} du \right).$$

Then, we see that  $T_4 = o_P(1)$  by taking  $N_n \to \infty$  slow enough such that  $N_n^{3/2} \sup_{i \in \mathcal{I}_n} \rho_i \to 0$  (such an  $N_n$  is possible since as  $N_n$  diverges more slowly, the quantity  $\sup_{i \in \mathcal{I}_n} \rho_i$  vanishes at a faster rate). Now, we conclude

$$\frac{1}{\sqrt{m_n \Delta_n}} \tilde{Z}'_n(m_n) \to N(0, \tilde{\sigma}_{\tau}^2 L^2(t)).$$

c) We now proceed to consider  $T_1$  and  $T_2$  of Eq. (24). First, by Proposition 1, for any p > 2, we can write

$$\begin{split} \mathbb{E}(Y_{i}|\mathcal{F}_{i-1}) &= \sqrt{m_{n}} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \left(\sigma_{t_{i-1}}^{2} \Delta_{n} + C_{1,i}\Delta_{n} v_{n}^{2-Y} + D_{1,i}\Delta_{n}^{2} v_{n}^{-Y} + O_{P}(\Delta_{n}^{3} v_{n}^{-2-Y}) + o_{P}(\Delta_{n}^{5/4})\right), \\ \mathbb{E}(Y_{i}^{p}|\mathcal{F}_{i-1}) &\lesssim m_{n}^{p/2} K_{m_{n}\Delta_{n}}^{p}(t_{i-1} - \tau) \left(\Delta_{n}^{p} \sigma_{t_{i-1}}^{2p} + \Delta_{n} v_{n}^{2p-Y}\right), \\ \operatorname{Var}(Y_{i}|\mathcal{F}_{i-1}) &= m_{n} K_{m_{n}\Delta_{n}}^{2}(t_{i-1} - \tau) \left(2\Delta_{n}^{2} \sigma_{t_{i-1}}^{4} + O_{P}(\Delta_{n}^{2} v_{n}^{2-Y}) + O(\Delta_{n} v_{n}^{4-Y})\right) \\ &= m_{n} K_{m_{n}\Delta_{n}}^{2}(t_{i-1} - \tau) \left(2\Delta_{n}^{2} \sigma_{t_{i-1}}^{4} + O_{P}(\Delta_{n} v_{n}^{4-Y})\right). \end{split}$$

Then,

$$s_{1n}^2 := \sum_{i=1}^n \operatorname{Var}(Y_i | \mathcal{F}_{i-1}) = m_n \sum_{i=1}^n K_{m_n \Delta_n}^2 (t_{i-1} - \tau) \left( 2\Delta_n^2 \sigma_{t_{i-1}}^4 + O_P(\Delta_n v_n^{4-Y}) \right).$$

Given  $v_n \ll \Delta_n^{\frac{1}{4-Y}}$  and (26), we obtain

$$\frac{s_{1n}^2}{(\Delta_n \sum_{j=1} K_{m_n \Delta_n} (t_{j-1} - \tau))^2} \to 2\sigma_{\tau}^4 \int K^2(x) dx.$$

Moreover, we have

$$\sum_{i=1}^{n} \mathbb{E}(Y_{i}^{p} | \mathcal{F}_{i-1}) \lesssim \frac{1}{m_{n}^{p/2-1} \Delta_{n}^{p-1}} \int K^{p}(x) dx (\sigma_{\tau}^{2p} \Delta_{n}^{p-1} + O_{P}(v_{n}^{2p-Y})).$$

If we rewrite  $p = (2\ell + 1)/\ell$  for some  $\ell > 0$ , the right-hand side of the above inequality vanishes if and only if

$$m_n^{-\frac{1}{2\ell}}\Delta_n^{-1-\frac{1}{\ell}}v_n^{\frac{4\ell+2}{\ell}-Y} = \left((\Delta_n^{-1}v_n^{4-Y})^{\ell}(m_n^{-\frac{1}{2}}\Delta_n^{-1}v_n^2)\right)^{\frac{1}{\ell}} = \Delta_n^{-1}v_n^{4-Y}\left(m_n^{-\frac{1}{2}}\Delta_n^{-1}v_n^2\right)^{\frac{1}{\ell}} \to 0.$$

Since  $\Delta_n^{-1} v_n^{4-Y} \ll \Delta_n^{\beta'(4-Y)-1} \ll \Delta_n^s$  for small enough s>0, the above vanishes by picking  $\ell$  large enough. In light of (26), we also have  $\frac{\sum_{i=1}^n \mathbb{E}(Y_i^p | \mathcal{F}_{i-1})}{(\Delta_n \sum_{j=1} K_{m_n \Delta_n}(t_{j-1}-\tau))^p} \to 0$  as  $n\to\infty$  for such an  $\ell$ . To show  $T_2=o_P(1)$ , recall  $m_n=O(\Delta_n^{-\frac{1}{2}})$  and, thus, by Proposition 1 and (26),

$$\sum_{i=1}^{n} \mathbb{E}(Y_{i}|\mathcal{F}_{i-1}) = \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \Delta_{n} \sigma_{t_{i-1}}^{2} + \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) C_{1,i} \Delta_{n} v_{n}^{2-Y}$$

$$+ \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) D_{1,i} \Delta_{n}^{2} v_{n}^{-Y} + O_{P}(\sqrt{m_{n}} \Delta_{n}^{2} v_{n}^{-2-Y}) + o_{P}(1).$$
(31)

Hence,

$$T_{2} = \frac{\sum_{i=1}^{n} \mathbb{E}(Y_{i}|\mathcal{F}_{i-1})}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} - \sqrt{m_{n}} \Delta_{n} \sum_{i=1}^{n} \frac{K_{m_{n} \Delta_{n}}(t_{i-1} - \tau)}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} \sigma_{t_{i-1}}^{2} - \sqrt{m_{n}} A(v_{n}, m_{n})$$

$$= O_{P}(\sqrt{m_{n}} \Delta_{n}^{2} v_{n}^{-2-Y}),$$

which vanishes due to the condition  $m_n \ll \Delta_n^{-4} v_n^{4+2Y}$ . To obtain  $T_1 \xrightarrow{st} Z_1$ , by Theorem 2.2.15 in [10], it suffices to check the condition:

$$\sum_{i=1}^{n} \mathbb{E}_{i-1}(Y_i \Delta_i M) \to 0, \tag{32}$$

 $\Diamond$ 

where M is either W or B or is in the set  $\mathcal{N}$  containing all bounded martingales orthogonal to W and B. The proof of (32) is technical and is deferred to Appendix A.2. This concludes the proof for  $\tilde{Z}_n(m_n, v_n) \xrightarrow{st} Z_1$ .

d) To establish the joint convergence (20) and the asymptotic independence of  $T_1, T_2$ , it is sufficient to show that

$$\sup_{i} \left| \mathbb{E} \left( (\Delta_i X)^2 \mathbf{1}_{\{|\Delta_i X| \le v_n\}} \Delta_i B | \mathcal{F}_{i-1} \right) \right| = o_P(\Delta_n^{3/2}), \tag{33}$$

which is shown in Appendix A.2. Indeed, from (33) we obtain:

$$\sum_{i=1}^{n} \left| \mathbb{E}(Y_{i}Y_{i}'|\mathcal{F}_{i-1}) \right| \\
= \frac{\sqrt{m_{n}}\Delta_{n}}{\sqrt{m_{n}}\Delta_{n}} \sum_{i=1}^{n} K_{m_{n}}\Delta_{n}(t_{i-1} - \tau) \cdot \left| \left( -\sum_{l=1}^{i-1} K_{m_{n}}\Delta_{n}(t_{l-1} - \tau) \right) \mathbf{1}_{\tau \geq t_{i}} + \left( \sum_{l=i}^{n} K_{m_{n}}\Delta_{n}(t_{l-1} - \tau) \right) \mathbf{1}_{\tau < t_{i}} \right| \\
\cdot \left| \mathbb{E}\left( (\Delta_{i}X)^{2} \mathbf{1}_{\{|\Delta_{i}X| \leq v_{n}\}} \Delta_{i}B|\mathcal{F}_{i-1} \right) \right| \\
= \Delta_{n}^{2} \sum_{i=1}^{n} K_{m_{n}}\Delta_{n}(t_{i-1} - \tau) \cdot \left( \left( \sum_{l=1}^{i-1} K_{m_{n}}\Delta_{n}(t_{l-1} - \tau) \right) \mathbf{1}_{\tau \geq t_{i}} + \left( \sum_{l=i}^{n} K_{m_{n}}\Delta_{n}(t_{l-1} - \tau) \right) \mathbf{1}_{\tau < t_{i}} \right) o_{P}(1) \\
= \int_{0}^{1} K_{m_{n}}\Delta_{n}(v - \tau) \left( \int_{0}^{v} K_{m_{n}}\Delta_{n}(s - \tau) ds \mathbf{1}_{\tau \geq v} + \int_{v}^{1} K_{m_{n}}\Delta_{n}(s - \tau) ds \mathbf{1}_{\tau < v} \right) dv \cdot o_{P}(1) \xrightarrow{P} 0.$$

In view of (27) and (32), one can finally apply Theorem 2.2.15 in [10] to conclude the proof.

The next result is the second key ingredient needed for our debiasing method. For an arbitrary fixed  $\zeta > 1$ , it establishes the convergence rate of the difference  $\tilde{Z}_n(m_n, \zeta v_n) - \tilde{Z}_n(m_n, v_n)$ , i.e. of  $\hat{c}_{n,1}(m_n, \zeta v_n) - \hat{c}_{n,1}(m_n, v_n)$  subject to a bias correction, which we exploit in our debiasing procedure.

**Theorem 2.** Suppose that 1 < Y < 2,  $\sqrt{\log(n)} \ll m_n$ ,  $m_n = O(\Delta_n^{-\frac{1}{2}})$ ,  $\Delta_n^{\beta} \ll v_n \ll \Delta_n^{\beta'}$  with  $\frac{1}{4-Y} < \beta' < \beta < \frac{1}{2}$ , and  $m_n \to \infty$  such that  $v_n^{\gamma} \Delta_n^{-1} \ll m_n \ll \Delta_n^{-5} v_n^{8+Y}$  with some  $\gamma < Y$ . Then, under Assumptions 1 and 2, for an arbitrary fixed  $\zeta > 1$ , we have

$$u_n^{-1}\left(\tilde{Z}_n(m_n,\zeta v_n) - \tilde{Z}_n(m_n,v_n)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{(C_+ + C_-)|\chi_\tau|^Y}{4 - Y}(\zeta^{4-Y} - 1)\int K^2(x)dx\right),\tag{34}$$

as  $n \to \infty$ , where  $u_n := \Delta_n^{-1/2} v_n^{2-Y/2} \to 0$ .

Remark 4. The conditions on  $m_n$  in Theorem 2 are stricter than those in Theorem 1 due to the condition  $\Delta_n^{-1}v_n^{\gamma} \ll m_n \ll \Delta_n^{-5}v_n^{8+Y}$  (note  $\Delta_n^{-5}v_n^{8+Y} \ll \Delta_n^{-4}v_n^{4+2Y}$ , which was the the upper bound for  $m_n$  in Theorem 1). As we shall see, the optimal rate of convergence of the estimation error of our ultimate debiased estimator can be achieved by taking  $m_n = \theta \Delta_n^{-1/2}$  for any constant  $\theta > 0$  (cf. Remark 2). In this bandwidth regime, the additional constraint on  $m_n$  in Theorem 2 reduces to  $\Delta_n^{\frac{9}{16+2Y}} \ll v_n \ll \min(\Delta_n^{\frac{1}{4-Y}}, \Delta_n^{\frac{1}{2\gamma}})$ , whose upper bound and lower bound can be made compatible only if Y < 20/11.

Proof. Denote

$$Z_{i} = \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2}} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) (\Delta_{i} X)^{2} \left( \mathbf{1}_{\{|\Delta_{i} X| \leq \zeta v_{n}\}} - \mathbf{1}_{\{|\Delta_{i} X| \leq v_{n}\}} \right),$$

and note that

$$u_n^{-1}(\tilde{Z}_n(m_n, \zeta v_n) - \tilde{Z}_n(m_n, v_n)) = \frac{\sum_{i=1}^n Z_i}{\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)} - u_n^{-1} \sqrt{m_n} \left( A(\zeta v_n, m_n) - A(v_n, m_n) \right)$$

$$=: \frac{\sum_{i=1}^n \xi_i}{\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)}$$

$$+ \left( \frac{\sum_{i=1}^n \mathbb{E}(Z_i | \mathcal{F}_{i-1})}{\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)} - u_n^{-1} \sqrt{m_n} \left( A(\zeta v_n, m_n) - A(v_n, m_n) \right) \right),$$

where  $\xi_i := Z_i - \mathbb{E}(Z_i|\mathcal{F}_{i-1})$ . The second term above is  $o_P(1)$ . Indeed, since  $m_n = O(\Delta_n^{-\frac{1}{2}})$ , by (16) in Proposition 1,

$$\frac{\sum_{i=1}^{n} \mathbb{E}(Z_{i}|\mathcal{F}_{i-1})}{\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)} - u_{n}^{-1} \sqrt{m_{n}} \left( A(\zeta v_{n}, m_{n}) - A(v_{n}, m_{n}) \right) \\
\lesssim \frac{m_{n}^{1/2} \Delta_{n}^{2} v_{n}^{-2-Y}}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2}} + \frac{m_{n}^{1/2} o_{P} \left( \Delta_{n}^{-1/4} v_{n}^{2-Y/2} \right)}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2}} \\
= m_{n}^{1/2} \Delta_{n}^{5/2} v_{n}^{-4-Y/2} + o_{P}(1),$$

which vanishes given our assumption  $m_n \ll \Delta_n^{-5} v_n^{8+Y}$ . Next we obtain the asymptotic behavior for  $\sum_{i=1}^n \xi_i$  by verifying the conditions of Theorem 2.2.15 in [10]. Expression (14) in Proposition 1 yields

$$\begin{aligned} \operatorname{Var}(\xi_{i}|\mathcal{F}_{i-1}) &= \frac{m_{n}}{\Delta_{n}^{-1}v_{n}^{4-Y}}K_{m_{n}\Delta_{n}}^{2}(t_{i-1}-\tau)\Big((\zeta^{4-Y}-1)C_{2,i}\Delta_{n}v_{n}^{4-Y} + o_{P}(\Delta_{n}v_{n}^{4-Y}) \\ &\qquad \qquad - \left\{(\zeta^{2-Y}-1)C_{1,i}\Delta_{n}v_{n}^{2-Y} + (\zeta^{-Y}-1)D_{1,i}\Delta_{n}^{2}v_{n}^{-Y} + O_{P}(\Delta_{n}^{3}v_{n}^{-2-Y})\right\}^{2}\Big) \\ &= \frac{m_{n}}{\Delta_{n}^{-1}v_{n}^{4-Y}}K_{m_{n}\Delta_{n}}^{2}(t_{i-1}-\tau)\Big((\zeta^{4-Y}-1)C_{2,i}\Delta_{n}v_{n}^{4-Y} + o_{P}(\Delta_{n}v_{n}^{4-Y})\Big), \end{aligned}$$

where  $C_{2,i} = \frac{(C_+ + C_-)|\chi_{t_{i-1}}|^Y}{4-Y}$ . Therefore, with 1 < Y < 2 and  $v_n \ll \Delta_n^{1/(4-Y)}$ , we obtain

$$s_{2n}^{2} := \sum_{i=1}^{n} \operatorname{Var}(\xi_{i} | \mathcal{F}_{i-1})$$

$$= m_{n} \Delta_{n}^{2} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}^{2} (t_{i-1} - \tau) \frac{(C_{+} + C_{-}) |\chi_{t_{i-1}}|^{Y}}{4 - Y} (\zeta^{4-Y} - 1) + o_{P}(1)$$

$$\longrightarrow \frac{(C_{+} + C_{-}) |\chi_{\tau}|^{Y}}{4 - Y} (\zeta^{4-Y} - 1) \int K^{2}(x) dx.$$

And consequently, with (26), we have

$$\frac{s_{2n}^2}{\left(\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)\right)^2} \to \frac{(C_+ + C_-)|\chi_\tau|^Y}{4 - Y} (\zeta^{4-Y} - 1) \int K^2(x) dx. \tag{35}$$

Similarly, for any p > 2, we have<sup>3</sup>

$$\frac{\sum_{i=1}^{n} \mathbb{E}(\xi_{i}^{p} | \mathcal{F}_{i-1})}{\left(\Delta_{n} \sum_{j=1}^{n} K_{m_{n} \Delta_{n}}(t_{j-1} - \tau)\right)^{p}} \lesssim \frac{m_{n}^{p/2}}{\Delta_{n}^{-p/2} v_{n}^{2p-pY/2}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}^{p}(t_{i-1} - \tau)(\Delta_{n} v_{n}^{2p-Y} + \Delta_{n}^{p})$$

$$\lesssim \frac{v_{n}^{(p/2-1)Y}}{m_{n}^{p/2-1} \Delta_{n}^{p/2-1}} \int K^{p}(x) dx + \frac{\Delta_{n}^{p/2}}{m_{n}^{p/2-1} v_{n}^{2p-pY/2}} \int K^{p}(x) dx. \tag{36}$$

In (36) the first term vanishes for any p>2 since  $m_n^{-1}\Delta_n^{-1}v_n^Y\to 0$ . For the second term, note that it can be written as  $\left(m_n^{1/p}\Delta_nv_n^{-2}\Delta_n^{-1/2}m_n^{-1/2}v_n^{Y/2}\right)^p$ . By taking  $p\downarrow 2$ , the second term vanishes given  $\Delta_nv_n^{-2}\lesssim \Delta_n^{1-2\beta}\to 0$  with  $\beta<\frac{1}{2}$  and  $m_n^{-1}\Delta_n^{-1}v_n^Y\to 0$ .

In view of (36) and (35), what remains is to check the martingale technical condition from Theorem 2.2.15 in [10,]. Specifically, due to (26), it requires us to show that as  $n \to \infty$ ,

$$\sum_{i=1}^{n} \mathbb{E}_{i-1} \left[ \xi_i \Delta_n M \right] = \sum_{i=1}^{n} \mathbb{E}_{i-1} \left[ Z_i \Delta_n M \right] \stackrel{P}{\to} 0,$$

when M = W or M is a bounded martingale orthogonal to W. Equivalently, it suffices to prove that

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i X)^2 f(\Delta_i X) \Delta_i M \right] = o_P(1),$$

where  $f(x) := \mathbf{1}_{\{v_n < |x| \le \zeta v_n\}}$ . Similar to the proof of Lemma 12 in [3], there exists a  $C^2$  smooth approximation  $f_n$  of f such that for any  $\eta > 0$ ,

$$\mathbf{1}_{\{v_n(1+\frac{2}{3}v_n^{\eta})<|x|<\zeta v_n(1-\frac{2}{3}v_n^{\eta})\}} \le f_n(x) \le \mathbf{1}_{\{v_n(1+\frac{1}{3}v_n^{\eta})<|x|<\zeta v_n(1-\frac{1}{3}v_n^{\eta})\}},$$
$$|f_n'(x)| \le \frac{C}{v_n^{1+\eta}}, \quad |f_n''(x)| \le \frac{C}{v_n^{2+2\eta}}.$$

Then, the result follows from the following two limits

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i X)^2 (f - f_n) (\Delta_i X) \Delta_i M \right] \to 0, \tag{37}$$

and

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i X)^2 f_n(\Delta_i X) \Delta_i M \right] = o_P(1). \tag{38}$$

<sup>&</sup>lt;sup>3</sup>By a standard localization argument, we may assume that  $\sigma$  is bounded.

While the proof of (37) is relatively straightforward, the proof of (38) is nontrivial and lengthy. This is shown in Appendix A.3.

 $\Diamond$ 

Finally we are ready to formally construct our debiasing estimators. Our debiasing steps follow the same idea as in [12]. Write  $\vec{\zeta}_k = (\zeta_1, \dots, \zeta_k)$ , where  $\zeta_1, \dots, \zeta_k > 1$  are fixed constants. Let

$$\widetilde{c}_{n}^{(k)}(m_{n}, v_{n}, \vec{\zeta}_{k}) := \widetilde{c}_{n}^{(k-1)}(m_{n}, v_{n}, \vec{\zeta}_{k-1}) \\
- \frac{\left(\widetilde{c}_{n}^{(k-1)}(m_{n}, \zeta_{k}v_{n}, \vec{\zeta}_{k-1}) - \widetilde{c}_{n}^{(k-1)}(m_{n}, v_{n}, \vec{\zeta}_{k-1})\right)^{2}}{\widetilde{c}_{n}^{(k-1)}(m_{n}, \zeta_{k}^{2}v_{n}, \vec{\zeta}_{k-1}) - 2\widetilde{c}_{n}^{(k-1)}(m_{n}, \zeta_{k}v_{n}, \vec{\zeta}_{k-1}) + \widetilde{c}_{n}^{(k-1)}(m_{n}, v_{n}, \vec{\zeta}_{k-1})},$$
(39)

for each  $k \geq 1$ , with  $\widetilde{c}_n^{(0)}(m_n, v_n) = \hat{c}_n(m_n, v_n)$ . Our next theorem is the second main result of this paper; it establishes a CLT for the feasible estimator  $\widetilde{c}_n^{(2)}(m_n, v_n, \vec{\zeta}_2)$ .

#### Theorem 3. Let

$$\tilde{Z}_n^{(2)}(m_n, v_n) := \sqrt{m_n} \Big( \tilde{c}_n^{(2)}(m_n, v_n, \vec{\zeta}_2) - \hat{c}_{n,2}(m_n) \Big).$$

Then, under the assumptions and notation of Theorems 1 and 2, as  $n \to \infty$ ,

$$(\tilde{Z}_n^{(2)}(m_n, v_n), \tilde{Z}_n'(m_n)) \xrightarrow{st} (Z_1, Z_2). \tag{40}$$

Furthermore, if we set

$$m_n \sqrt{\Delta_n} \to \theta$$
, with  $\theta \in [0, \infty)$ ,

we have the following stable convergence in law, as  $n \to \infty$ :

$$\sqrt{m_n}(\tilde{c}_n^{(2)}(m_n, v_n, \vec{\zeta_2}) - \sigma_\tau^2) \xrightarrow{st} Z_1 + \theta Z_2. \tag{41}$$

Remark 5. Recall that Remark 2 identifies two bandwidth regimes of primary importance: (i)  $0 < \theta < \infty$ , and (ii)  $\theta = 0$ , both with a convergence rate dictated by  $m_n^{-1/2}$ . In case (i), a rate-optimal (feasible) CLT for  $\tilde{c}_n^{(2)}(m_n, v_n, \vec{\zeta}_2)$  is attained in (41) since  $m_n^{-1/2} \times \Delta_n^{1/4}$ , provided  $1 \le Y < 20/11$ , but no CLT is available in case (i) when  $Y \ge 20/11$ . In case (ii), the rate is strictly slower than  $\Delta_n^{1/4}$  (see Remark 2 for details), but a CLT still holds for  $\tilde{c}_n^{(2)}(m_n, v_n, \vec{\zeta}_2)$  even when  $20/11 \le Y < 2$  at the expense of a deteriorating rate. Indeed, since  $m_n \ll \Delta_n^{-5} v_n^{8+Y}$  and  $v_n \ll \Delta_n^{\frac{1}{4-Y}}$ , we have  $m_n \ll \Delta_n^{(6Y-12)/(4-Y)}$  and hence  $m_n^{-1/2} \gg \Delta_n^{(6-3Y)/(4-Y)}$ , which tends to zero more slowly as Y gets closer to 2.

**Remark 6.** In Theorem 3, we proceeded with a two-step debiasing procedure to iteratively remove bias terms in  $A(v_n, m_n)$  one at a time. However, if the second-order term in (19) is negligible when multiplied by  $u_n^{-1}\sqrt{m_n}$  (and, thus, (34) is valid replacing  $A(v_n, m_n)$  with only the first-order term  $\sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau)C_{1,i}\Delta_n v_n^{2-Y}$ ), then only one debiasing step is needed. Specifically, for one-step debiasing it suffices that

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) C_{1,i} \Delta_n^2 v_n^{-Y} \ll 1.$$

Therefore, we only require that  $u_n^{-1}\sqrt{m_n}\Delta_n v_n^{-Y}\ll 1$  or, equivalently,  $m_n\ll \Delta_n^{-3}v_n^{Y+4}$ . In the optimal-rate bandwidth regime  $m_n\asymp \Delta_n^{-1/2}$ , this imposes the relation  $\Delta_n^{\frac{5}{2(Y+4)}}\ll v_n$ , but since  $v_n\ll \Delta_n^{\frac{1}{4-Y}}$ , this yields the restriction Y<12/7. Similarly as in Remark 5, if  $Y\geq 12/7$  and  $\theta=0$ , the one-step debiased estimator  $\tilde{c}_n^{(1)}(m_n,v_n,\zeta_1)$  will still enjoy a CLT centered at  $\sigma_\tau^2$ , but with a rate strictly slower than  $\Delta_n^{\frac{4-2Y}{4-Y}}$ , which tends to zero more slowly as Y gets closer to 2.

*Proof.* Denote  $\mathcal{I} = \{(1,1),(2,1)\}$  and recall that

$$\tilde{Z}_n(m_n, v_n) = \sqrt{m_n} \left( \hat{c}_n(m_n, v_n) - \frac{\Delta_n \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \sigma_{t_{i-1}}^2}{\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)} - A(v_n, m_n) \right), \tag{42}$$

where

$$A(v_n, m_n) = \frac{\sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) C_{1,i} \Delta_n v_n^{2-Y}}{\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)} + \frac{\sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) D_{1,i} \Delta_n^2 v_n^{-Y}}{\Delta_n \sum_{j=1}^n K_{m_n \Delta_n}(t_{j-1} - \tau)} =: \sum_{(i,j) \in \mathcal{I}} a_{i,j}(v_n).$$

We introduce the following notation:

$$\eta_{i,j}(\zeta) := \zeta^{2-2(i-j)-jY} - 1, 
\Psi_n := u_n^{-1} (\tilde{Z}_n(\zeta_1^2 m_n, v_n) - 2\tilde{Z}_n(\zeta_1 m_n, v_n) + \tilde{Z}_n(m_n, v_n)) = O_P(1), 
\Phi_n := u_n^{-1} (\tilde{Z}_n(\zeta m_n, v_n) - \tilde{Z}_n(m_n, v_n)) = O_P(1),$$

where  $O_P(1)$  is a consequence of Theorem 2. Note that, for any (i,j) we have

$$a_{i,j}(\zeta v_n) - a_{i,j}(v_n) = \eta_{i,j}(\zeta) a_{i,j}(v_n),$$

$$a_{i,j}(\zeta^2 v_n) - 2a_{i,j}(\zeta v_n) + a_{i,j}(v_n) = \eta_{i,j}^2(\zeta) a_{i,j}(v_n).$$
(43)

Recall  $\Delta_n v_n^{-2} \ll 1$  gives the order comparison:

$$a_{1,1}(v_n) \gg a_{2,1}(v_n).$$
 (44)

In the first stage of debiasing, the leading order term  $a_{1,1}(v_n)$  is removed.

# First step: removal of $a_{1,1}(v_n)$

Let

$$\widetilde{c}_{n}^{(1)}(m_{n}, v_{n}, \zeta_{1}) = \widehat{c}_{n}(m_{n}, v_{n}) - \frac{\left(\widehat{c}_{n}(m_{n}, \zeta_{1}v_{n}) - \widehat{c}_{n}(m_{n}, v_{n})\right)^{2}}{\widehat{c}_{n}(m_{n}, \zeta_{1}^{2}v_{n}) - 2\widehat{c}_{n}(m_{n}, \zeta_{1}v_{n}) + \widehat{c}_{n}(m_{n}, v_{n})}$$

$$=: \widehat{c}_{n}(m_{n}, v_{n}) - \widehat{a}_{1,1}(v_{n}). \tag{45}$$

Note that, due to (42)-(43),

$$\hat{c}_n(m_n, \zeta v_n) - \hat{c}_n(m_n, v_n) = m_n^{-1/2} \left( \tilde{Z}_n(m_n, \zeta v_n) - \tilde{Z}_n(m_n, v_n) \right) + A_n(\zeta v_n, m_n) - A_n(v_n, m_n)$$

$$= m_n^{-1/2} u_n \Phi_n + A_n(\zeta v_n, m_n) - A_n(v_n, m_n).$$

Similarly, we can write:

$$\hat{c}_n(m_n, \zeta^2 v_n) - 2\hat{c}_n(m_n, \zeta v_n) + \hat{c}_n(m_n, v_n) = m_n^{-1/2} u_n \Psi_n + A_n(\zeta^2 v_n, m_n) - 2A_n(\zeta v_n, m_n) + A_n(v_n, m_n).$$

Then, in the light of Proposition 1 and (43), we may expand  $\hat{a}_{1,1}(v_n)$  as follows:

$$\hat{a}_{1,1}(v_n) = \frac{\left(\sum_{(r,s)\in\mathcal{I}} \eta_{r,s} a_{r,s}(v_n) + m_n^{-1/2} u_n \Phi_n\right)^2}{\sum_{(r,s)\in\mathcal{I}} \eta_{r,s}^2 a_{r,s}(v_n) + m_n^{-1/2} u_n \Psi_n}$$

$$= a_{1,1}(v_n) + O_P(m_n^{-1/2} u_n)$$

$$+ \eta_{2,1} a_{2,1}(v_n) \frac{(2\eta_{1,1} - \eta_{2,1}) a_{1,1} + \eta_{2,1} a_{2,1}(v_n)}{\eta_{1,1}^2 a_{1,1}(v_n) + \eta_{2,1}^2 a_{2,1}(v_n) + m_n^{-1/2} u_n \Psi_n},$$
(46)

where  $O_P(m_n^{-1/2}u_n)$  includes the cross products between  $\sum_{(r,s)\in\mathcal{I}}\eta_{r,s}a_{r,s}(v_n)$  and  $m_n^{-1/2}u_n\Phi_n=o_P(1)$ , along with  $(m_n^{-1/2}u_n\Phi_n)^2$  and  $m_n^{-1/2}u_n\Psi_n a_{1,1}$ , all divided by  $\sum_{(r,s)\in\mathcal{I}}\eta_{r,s}^2a_{r,s}(v_n)+m_n^{-1/2}u_n\Psi_n$ .

To simplify the last term in (46), note that for any variable  $N = o_p(a_{1,1}^2(v_n))$ ,

$$\frac{N}{\sum_{(r,s)\in\mathcal{I}} a_{r,s}(v_n) + O_p(m_n^{-1/2}u_n)} - \frac{N}{\sum_{(r,s)\in\mathcal{I}} a_{r,s}(v_n)}$$

$$= -\frac{N}{\sum_{(r,s)\in\mathcal{I}} a_{r,s}(v_n)} \cdot \frac{O_P(m_n^{-1/2}u_n)}{\left(\sum_{(r,s)\in\mathcal{I}} a_{r,s}(v_n) + O_P(m_n^{-1/2}u_n)\right)}$$

$$= \frac{-N}{a_{1,1}^2(v_n)(1+o_p(1))} \cdot O_P(m_n^{-1/2}u_n)$$

$$= o_P(m_n^{-1/2}u_n), \tag{47}$$

where above, for simplicity, we omitted the coefficients associated with  $a_{i,j}(v_n)$ . The above shows that we can drop the term  $m_n^{-1/2}u_n\Psi_n$  in the denominator and can write (46) as follows

$$\begin{split} \hat{a}_{1,1}(v_n) &= a_{1,1}(v_n) + O_P(m_n^{-1/2}u_n) \\ &+ \eta_{2,1}a_{2,1}(v_n) \frac{(2\eta_{1,1} - \eta_{2,1})a_{1,1} + \eta_{2,1}a_{2,1}(v_n)}{\eta_{1,1}^2a_{1,1}(v_n) + \eta_{2,1}^2a_{2,1}(v_n)} \\ &= a_{1,1}(v_n) + \tilde{\eta}_{2,1}a_{2,1} + \check{\eta}_{2,1}\frac{a_{2,1}^2(v_n)}{\eta_{1,1}^2a_{1,1}(v_n) + \eta_{2,1}^2a_{2,1}(v_n)} + O_P(m_n^{-1/2}u_n), \end{split}$$

for some constants  $\tilde{\eta}_{2,1}$  and  $\tilde{\eta}_{2,1}$ . Note that the third term above is  $O_P(a_{2,1}^2/a_{1,1})$  and, thus, this term is  $o_P(m_n^{-1/2}u_n)$  due to our assumption  $m_n \ll \Delta_n^{-5}v_n^{8+Y}$ . We conclude that

$$\hat{a}_{1,1}(v_n) = a_{1,1}(v_n) + \tilde{\eta}_{2,1}a_{2,1}(v_n) + O_P(m_n^{-1/2}u_n). \tag{48}$$

We now show the first debiasing step eliminates  $a_{1,1}(v_n)$ . Indeed, denoting  $\tilde{a}_{2,1}(v_n) := (1 - \tilde{\eta}_{2,1})a_{2,1}(v_n)$ , in light of (42), (45), and (48) and recalling that  $A(v_n, m_n) = a_{1,1} + a_{1,2}$ , we see that

$$\begin{split} \tilde{Z}_{n}^{(1)}(m_{n},v_{n}) &:= \sqrt{m_{n}} \Big( \tilde{c}_{n}^{(1)}(m_{n},v_{n},\zeta_{1}) - \tilde{a}_{2,1}(v_{n}) - \Delta_{n} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \Big) \\ &= \sqrt{m_{n}} \Big( \hat{c}_{n}(m_{n},v_{n}) - \hat{a}_{1,1}(v_{n}) - \tilde{a}_{2,1}(v_{n}) - \Delta_{n} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \Big) \\ &= \tilde{Z}_{n}(m_{n},v_{n}) + O_{P}(u_{n}). \end{split}$$

Therefore, (20) implies that

$$(\tilde{Z}_n^{(1)}(m_n, v_n), \tilde{Z}_n'(m_n)) \xrightarrow{st} (Z_1, Z_2). \tag{49}$$

Second step: removal of  $a_{2,1}(v_n)$ 

Similar to (45), we can define

$$\widetilde{c}_{n}^{(2)}(m_{n}, v_{n}, \zeta_{2}, \zeta_{1}) = \widetilde{c}_{n}^{(1)}(m_{n}, v_{n}, \zeta_{1}) - \frac{\left(\widetilde{c}_{n}^{(1)}(m_{n}, \zeta_{2}v_{n}, \zeta_{1}) - \widetilde{c}_{n}^{(1)}(m_{n}, v_{n}, \zeta_{1})\right)^{2}}{\widetilde{c}_{n}^{(1)}(m_{n}, \zeta_{2}^{2}v_{n}, \zeta_{1}) - 2\widetilde{c}_{n}^{(1)}(m_{n}, \zeta_{2}v_{n}, \zeta_{1}) + \widetilde{c}_{n}^{(1)}(m_{n}, v_{n}, \zeta_{1})}$$

$$=: \widetilde{c}_{n}^{(1)}(m_{n}, \zeta_{2}v_{n}, \zeta_{1}) - \hat{a}_{2,1}(v_{n}). \tag{50}$$

Similar to (46), we may expand  $\hat{a}_{2,1}(v_n)$  as follows:

$$\hat{a}_{2,1}(v_n) = \tilde{a}_{2,1}(v_n) + O_P(m_n^{-1/2}u_n). \tag{51}$$

Then we see that

$$\tilde{Z}_{n}^{(2)}(m_{n}, v_{n}) := \sqrt{m_{n}} \left( \tilde{c}_{n}^{(2)}(m_{n}, v_{n}, \zeta_{2}, \zeta_{1}) - \Delta_{n} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \right) 
= \sqrt{m_{n}} \left( \tilde{c}_{n}^{(1)}(m_{n}, v_{n}, \zeta_{2}, \zeta_{1}) - \hat{a}_{2,1}(v_{n}) - \Delta_{n} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \right) 
= \sqrt{m_{n}} \left( \tilde{c}_{n}^{(1)}(m_{n}, v_{n}, \zeta_{2}, \zeta_{1}) - \tilde{a}_{2,1}(v_{n}) - \Delta_{n} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \right) + O_{P}(u_{n}) 
= \tilde{Z}_{n}^{(1)}(m_{n}, v_{n}) + O_{P}(u_{n}).$$
(52)

Again, (49) now implies that

$$(\tilde{Z}_n^{(2)}(m_n, v_n), \tilde{Z}_n'(m_n)) \xrightarrow{st} (Z_1, Z_2). \tag{53}$$

To conclude the last two statements of the theorem, we follow the same arguments as in Remark 2.  $\Diamond$ 

#### Simulation 4

In the numerical experiments below, we evaluate the performance of our debiased estimators under a Heston-type volatility model in two different settings. First, for comparison, we replicate the parameter setting and sampling scheme used in [15], which incorporates a symmetric stable jump component, although this configuration is arguably unrealistic for most financial applications (e.g., the overall annualized volatility is set to 1). Second, we consider a parameter setting that better reflects the empirical features of financial data. In this case, we also truncate the increments of the stable Lévy process, as is commonly done in applications, and investigate the impact of different choices of the threshold parameter  $v_n$ .

#### 4.1 Simulation with Stable Jump Component

Following [15], we consider the following model:

$$X_{t} = X_{0} + \int_{0}^{t} b_{t}dt + \int_{0}^{t} \sqrt{V_{s}} dW_{s} + J_{t},$$

$$V_{t} = V_{0} + \int_{0}^{t} \kappa(\theta - V_{s})ds + \xi \int_{0}^{t} \sqrt{V_{s}} dB_{s},$$
(54)

where B and B' are correlated standard Brownian motions with correlation  $\rho$ , J is a strictly symmetric stable Lévy process <sup>4</sup> with Blumenthal-Getoor index Y and scale parameter 1 (note [15] includes an additional finiteactivity component which we omit for simplicity). In this section, we take the same parameter values as in [15]:

$$X_0 = 1$$
,  $V_0 = 0$ ,  $\rho = 0$ ,  $b_t \equiv 1$ ,  $\kappa = 0.03$ ,  $\theta = 1$ ,  $\xi = 1.5$ .

In this simulation, we compare the performance of our estimator  $\hat{c}(m_n, v_n)$  defined in (18),  $\tilde{c}^{(1)}(m_n, v_n, \vec{\zeta}_1)$ and  $\tilde{c}_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$  defined in (39) to the estimators  $\hat{\sigma}_{\tau,n}^2(u_n,h)$  and  $\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$  introduced in [15], which are based on the empirical characteristic approach of [12]. Specifically,  $\hat{\sigma}_{\tau,n}^2(u_n,h)^5$  and  $\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$  are defined as

$$\hat{\sigma}_{\tau,n}^{2}(u_{n},h) = \bar{\sigma}_{\tau,n}^{2}(u_{n},h) - \frac{2\Delta_{n}}{u_{n}^{2}h}(\sinh(\bar{\sigma}_{\tau,n}^{2}(u_{n},h)))^{2},$$

$$\tilde{\sigma}_{\tau,n}^{2}(\lambda, u_{n},h) = \hat{\sigma}_{\tau,n}^{2}(u_{n},h) - \tilde{B}_{\tau,n}(\lambda, u_{n},h),$$

<sup>&</sup>lt;sup>4</sup>The rstable package in R was used to simulate the Y-stable portion of the jump component in all cases.  $\bar{\sigma}_{\tau,n}^2(u_n,h) = \bar{\sigma}_{\tau,n}^2(u_n,h) - \frac{2h}{u_n^2\Delta_n}(\sinh(\bar{\sigma}_{\tau,n}^2(u_n,h)))^2$  for simulation. The difference lies in the reciprocal placement of h and  $\Delta_n$  in the bias correction term. We interpret the version in [15] as a likely typo and instead follow the formulation consistent with [12].

where

$$\bar{\sigma}_{\tau,n}^{2}(u_{n},h) = \frac{-2}{u_{n}^{2}} \log \left( S_{\tau,n}(u_{n},h) \vee \sqrt{\frac{\Delta_{n}}{h}} \right),$$

$$S_{\tau,n}(u_{n},h) = \Delta_{n} \sum_{i=1}^{n} K_{h}(i\Delta_{n} - \tau) \cos \left( \frac{u_{n}\Delta_{i}X}{\sqrt{\Delta_{n}}} \right),$$

and  $\tilde{B}_{\tau,n}(\lambda, u_n, h)$  is a bias-correction term<sup>6</sup>

$$\tilde{B}_{\tau,n}(\lambda,u_n,h) = \frac{\sum_{i=1}^{\lfloor \frac{1}{m_n\Delta_n}\rfloor} \left[ \hat{\sigma}_{(i-1)m_n\Delta_n,n}^2(\lambda pu_n,h) - \hat{\sigma}_{(i-1)m_n\Delta_n,n}^2(pu_n,h) \right] \cdot \left[ (\hat{\sigma}_{\tau,n}^2(\lambda u_n,h) - \hat{\sigma}_{\tau,n}^2(u_n,h)) \wedge 0 \right]}{\sum_{i=1}^{\lfloor \frac{1}{m_n\Delta_n}\rfloor} \left( \hat{\sigma}_{(i-1)m_n\Delta_n,n}^2(\lambda^2 pu_n,h) - 2\hat{\sigma}_{(i-1)m_n\Delta_n,n}^2((\lambda pu_n,h) + \hat{\sigma}_{(i-1)m_n\Delta_n,n}^2(pu_n,h)) \right)}$$

which aggregates spot estimators across t for improved finite-sample performance. This approach was also used in [12].

For comparison, for our one- and two-step debiased estimators  $\tilde{c}_{m,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$  and  $\tilde{c}_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$ , we also adopt the debiasing adjustments of [15] and [12] described above. Specifically, denoting  $\tilde{c}_{n,\tau}^{(k)}(m_n,v_n,\vec{\zeta_k})$  the estimator  $\tilde{c}_n^{(k)}(m_n, v_n, \vec{\zeta}_k)$  at time  $\tau$ , we consider

$$\begin{split} \widetilde{c}_{n,\tau}^{(k)}(m_n, v_n, \vec{\zeta}_k) &:= \widetilde{c}_{n,\tau}^{(k-1)}(m_n, v_n, \vec{\zeta}_{k-1}) \\ &- \left(A_n^{(k)}(m_n, v_n, \vec{\zeta}_k) \vee 0\right) \left(\widetilde{c}_{n,\tau}^{(k-1)}(m_n, \zeta_k v_n, \vec{\zeta}_{k-1}) - \widetilde{c}_{n,\tau}^{(k-1)}(m_n, v_n, \vec{\zeta}_{k-1})\right) \vee 0, \end{split}$$

where  $A_n^{(k)} := A_n^{(k)}(m_n, v_n, \vec{\zeta}_k)$  is given by

$$A_n^{(k)} = \pm \frac{\sum_{i=1}^{\lfloor \frac{1}{m_n \Delta_n} \rfloor} \left( \widetilde{c}_{n,im_n \Delta_n}^{(k-1)}(m_n, \zeta_k p_k v_n, \vec{\zeta}_{k-1}) - \widetilde{c}_{n,im_n \Delta_n}^{(k-1)}(m_n, p_k v_n, \vec{\zeta}_{k-1}) \right)}{\sum_{i=1}^{\lfloor \frac{1}{m_n \Delta_n} \rfloor} \left( \widetilde{c}_{n,im_n \Delta_n}^{(k-1)}(m_n, \zeta_k^2 p_k v_n, \vec{\zeta}_{k-1}) - 2\widetilde{c}_{n,im_n \Delta_n}^{(k-1)}(m_n, \zeta_k p_k v_n, \vec{\zeta}_{k-1}) + \widetilde{c}_{n,im_n \Delta_n}^{(k-1)}(m_n, p_k v_n, \vec{\zeta}_{k-1}) \right)},$$

where the sign of the bias terms are chosen as in [3] and  $c_{n,\tau}^{(0)}(m_n,v_n,\vec{\zeta_0})=\hat{c}(m_n,v_n)$  at  $\tau$ , i.e.,  $A_n^{(1)}$  is positive and  $A_n^{(2)}$  is negative, so as to ensure compatibility with the theoretical sign of the terms in  $A(v_n, m_n)$ .

To evaluate estimation performance, we simulate M independent paths and aggregate various pathwise error measures computed along each simulated path for each of the estimators considered above. Denoting the true value of the spot variance at  $t_i$  in the j-th path as  $\sigma_{t_i,j}^2 := V_{t_i,j}$  and a given estimator as  $\hat{\sigma}_{t_i,j}^2$ , we consider an estimate of the pathwise root mean squared error (RMSE) on the time grid  $\{l_i = i \lfloor n/100 \rfloor\}_{i=10,...,90}$  as follows:

$$\widehat{RMSE} := \sqrt{\frac{1}{M} \sum_{j=1}^{M} MSE_j} \,,$$

where  $MSE_j := \frac{1}{81} \sum_{i=10}^{90} (\hat{\sigma}_{l_i,j}^2 - \sigma_{l_i,j}^2)^2$ . Similarly, we also consider the mean absolute relative error  $\widehat{ARE}$  and the mean relative error  $\widehat{R}\widehat{E}$ :

$$\widehat{ARE} := \frac{1}{M} \sum_{j=1}^{M} AE_j \text{ and } \widehat{RE} := \frac{1}{M} \sum_{j=1}^{M} E_j,$$

where  $AE_j := \frac{1}{81} \sum_{i=10}^{90} |\hat{\sigma}_{l_i,j}^2 - \sigma_{l_i,j}^2| / \sigma_{l_i,j}^2$  and  $E_j := \frac{1}{81} \sum_{i=10}^{90} (\hat{\sigma}_{l_i,j}^2 - \sigma_{l_i,j}^2) / \sigma_{l_i,j}^2$ . For our estimators  $\hat{c}_n(m_n, \Delta_n)$ ,  $\tilde{c}_{m,\tau}^{(1)}(m_n, v_n, \vec{\zeta}_1)$  and  $\tilde{c}_{n,\tau}^{(2)}(m_n, v_n, \vec{\zeta}_2)$ , we use the exponential kernel  $K(x) = \frac{1}{2} \sum_{i=10}^{90} (\hat{\sigma}_{l_i,j}^2 - \sigma_{l_i,j}^2) / \sigma_{l_i,j}^2$  $\exp(-|x|)/2$  as well as the two-sided uniform kernel  $G(x) = \mathbf{1}_{-1 \le |x| \le 1}/2$ , and set  $m_n = \Delta_n^{-1/2}$ ,  $v_n = \sqrt{BV}\Delta_n^{5/12}$ , where  $BV = \frac{\pi}{2} \sum_{i=2}^{n} |\Delta_i X| |\Delta_{i-1} X| / T$  is the bipower estimator for the integrated volatility. For the estimators

<sup>&</sup>lt;sup>6</sup>The formula of  $\tilde{B}_{\tau,n}(\lambda, u_n, h)$  in [15] forces each summand in the denominator to be nonpositive. However, in our experiments, we found that omitting control of the sign of these terms significantly improved the performance of the estimators.

 $\hat{\sigma}_{\tau,n}^2(u_n,h)$  and  $\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$  of [15], we take the kernel  $K_3(x)=\frac{15}{16}(1-x^2)^2\mathbf{1}_{|x|\leq 1}$  as specified on [15, p. 1964]<sup>7</sup> We also use the best-performing bandwidth as reported in their simulations,  $h=\Delta_n^{0.51}$  and  $u_n=\frac{\Delta_n^{0.0025}}{\sqrt{BV}}^8$ . The remaining tuning parameters  $\vec{\zeta}_1$ ,  $\vec{\zeta}_2$ ,  $\lambda$ , p,  $p_1$ ,  $(p_1,p_2)$  are selected via a grid search that minimizes  $\widehat{RMSE}^9$ . The grid for  $\zeta$  and  $\lambda$  ranged from 1.1 to 1.9 with increments of 0.05, while the grid for p spanned from 0.1 to 0.9 with increments of 0.05. For reproducibility, where applicable, we also report the auxiliary tuning parameter values found in our grid search for each considered estimator.

As in [15], we take T=1 month and n=8580, roughly corresponding to a 5-minute sampling frequency (assuming 24 hours of trading). For additional comparison, we also include the results for n=42900, corresponding to roughly 1-minute frequency data in this setup. Using M=1000 iterations, we report the results in Tables 1 and 2. We consider the values  $Y \in \{0.8, 1.2, 1.6, 1.75\}$ , which correspond to the cases where, in principle, no debiasing is needed ( $Y \in \{0.8, 1.2\}$ ), one step of debiasing is needed (Y = 1.6), and two steps are needed (Y = 1.75) to attain asymptotic efficiency according to our theory, though we may observe very different finite sample behavior.

As shown in Tables 1 and 2, the two-step debiasing estimator  $\tilde{c}_n^{(2)}$  consistently delivers the best performance, with the lowest RMSE and ARE, and generally well-controlled error. The advantage of two-step debiasing becomes especially pronounced as Y increases: While the performance of  $\hat{\sigma}_n^2$  and  $\tilde{\sigma}_n^2$  deteriorates sharply, the one- and two-step debiased estimators exhibit markedly greater robustness against increased jump activity. Notably, two-step debiasing yields improvement even where it is not strictly needed for asymptotically efficient estimation (Y = 0.8, 1.2). The pattern is similar at both 5-minute and 1-minute frequencies, where the two-step debiasing method is the only one that remains stable across all considered settings, offering reductions in RMSE often by a factor of 2 to 5 relative to the considered benchmarks.

Across kernel choices for  $\tilde{c}_n^{(i)}$ , the exponential kernel systematically outperforms the uniform kernel across all values of Y and at both 1- and 5-minute sampling frequencies. This finding is consistent with the established theoretical optimality of exponential kernels in related settings where debiasing is not required [8], and more broadly underscores the advantages of using unbounded kernels in spot volatility estimation. Overall, the two-step estimator with the exponential kernel achieves the strongest and most reliable performance among all configurations considered.

# 4.2 Simulation with truncated stable jump component

In this section, we consider a parameter setting that more closely reflects the empirical features of financial data. We also take a truncated stable Lévy process as the jump component, which is common in applications. We also study the effect of the truncation level  $v_n$ .

For this experiment, we fix a time unit of 1 year (i.e., t is measured in years). The increments of the observed process (corresponding to an asset's log return over  $(t_{i-1}, t_i]$ ) are given by

$$\Delta_i X = X_{t_i} - X_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \sqrt{V_s} \, dW_s + (\Delta_i J \wedge 0.005), \tag{55}$$

where J is a symmetric Y-stable Lévy process with scale parameter 0.5 and V follows the same Heston model

 $<sup>^{7}</sup>K_{3}$  is the best-performing kernel among those considered in [15] that satisfies the assumptions of their results (in particular, that is continuously differentiable as assumed by their main theorem).

<sup>&</sup>lt;sup>8</sup>Instead of using bipower variation for  $u_n$ , the authors of [15] employed a scaled local bipower variation defined as  $BV_{\tau} = \frac{\pi}{2(i_2-i_1)\Delta_n} \sum_{i=i_1}^{i_2} |\Delta_i X| |\Delta_{i+1} X|$ , where  $i_1 = \left\lfloor \frac{\tau-h}{\Delta_n} + 1 \right\rfloor \vee 1$  and  $i_2 = \left\lfloor \frac{\tau+h}{\Delta_n} + 1 \right\rfloor \wedge n$ . However, in our experiments, this choice lead to large RMSE when aggregating errors across various  $\tau$ , particularly for Y = 1.2. In contrast, the estimators demonstrated greater stability with our proposed choice of  $u_n$ .

<sup>&</sup>lt;sup>9</sup>Specifically, to each point in the grid, we simulated M = 100 independent paths to estimate the  $\widehat{RMSE}$  for the associated parameter settings.

Table 1: Estimation performance at a 5-minute sampling frequency, based on a set of M=1000 paths. The simulated data were generated at over a T=1 month horizon using the model setup of [15]. The indicated tuning parameters are from the grid search and correspond to the values of  $\lambda$ , p for  $\tilde{\sigma}_{\tau,n}^2$  and to the values of  $\vec{\zeta}, \vec{p}$  for  $c_{n,\tau}^{(i)}$ , i=1,2.

	Y = 0.8			Y = 1.2				
Estimator	RMSE	ARE	RE	Tuning Param.	RMSE	ARE	RE	Tuning Param.
$\hat{\sigma}_{\tau,n}^2(u_n,h)$	0.2024	0.1074	0.0070	-	0.2365	0.1168	0.0451	-
$\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$	0.2355	0.1109	0.0094	$\lambda = 1.6, \ p = 0.9$	0.2288	0.121	-0.0030	$\lambda = 1.25, p = 0.2$
$\hat{c}_{n,\tau,\text{exp}}(m_n,v_n)$	0.2805	0.1738	-0.1704	-	0.2509	0.1539	-0.1519	-
$\tilde{c}_{n,\tau,\exp}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.2806	0.1740	-0.1704	$\zeta = 1.8, \ p = 0.6$	0.2513	0.1540	-0.1522	$\zeta = 1.7, \ p = 0.85$
$\tilde{c}_{n,\tau,\exp}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.1215	0.0608	0.0099	$\vec{\zeta} = (1.7, 1.8),$	0.1421	0.0740	0.0234	$\vec{\zeta} = (1.9, 1.25),$
				$\vec{p} = (0.5, 0.85)$				$\vec{p} = (0.5, 0.65)$
$\hat{c}_{n,\tau,\mathrm{unif}}(m_n,v_n)$	0.2932	0.1768	-0.1722	-	0.2634	0.1568	-0.1527	-
$\tilde{c}_{n,\tau,\mathrm{unif}}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.2934	0.1769	-0.1726	$\zeta = 1.85, \ p = 0.85$	0.2635	0.1568	-0.1528	$\zeta = 1.65, p = 0.8$
$\tilde{c}_{n,\tau,\mathrm{unif}}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.1824	0.0861	0.0106	$\vec{\zeta} = (1.6, 1.9),$	0.1983	0.1046	0.0631	$\vec{\zeta} = (1.9, 1.25),$
				$\vec{p} = (0.5, 0.9)$				$\vec{p} = (0.6, 0.15)$
		Y = 1.6			Y = 1.75			
Estimator	RMSE	ARE	RE	Tuning Param.	RMSE	ARE	RE	Tuning Param.
$\hat{\sigma}_{\tau,n}^2(u_n,h)$	0.4551	0.2604	0.2534	-	0.7784	0.4830	0.4827	-
$\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$	0.3264	0.1760	0.1013	$\lambda = 1.9, \ p = 0.1$	0.5361	0.2984	0.2590	$\lambda = 1.9, p = 0.25$
$\hat{c}_{n,\tau,\text{exp}}(m_n,v_n)$	0.1121	0.0595	-0.0036	-	0.2785	0.1702	0.1688	-
$\tilde{c}_{n,\tau,\exp}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.1121	0.0595	-0.0036	$\zeta = 1.75, \ p = 0.75$	0.1783	0.0978	0.0706	$\zeta = 1.65, p = 0.1$
$\tilde{c}_{n,\tau,\exp}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.1121	0.0595	-0.0036	$\vec{\zeta} = (1.9, 1.75),$	0.1640	0.0885	0.0412	$\vec{\zeta} = (1.8, 1.55),$
				$\vec{p} = (0.7, 0.25)$				$\vec{p} = (0.2, 0.35)$
$\hat{c}_{n,\tau,\mathrm{unif}}(m_n,v_n)$	0.1490	0.0796	-0.0012	-	0.2992	0.1749	0.1684	-
$\tilde{c}_{n,\tau,\mathrm{unif}}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.1490	0.0796	-0.0012	$\zeta = 1.9, \ p = 0.65$	0.2265	0.1240	0.0892	$\zeta = 1.8, \ p = 0.1$
$\tilde{c}_{n,\tau,\mathrm{unif}}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.1490	0.0796	-0.0011	$\vec{\zeta} = (1.9, 1.9),$	0.2323	0.1279	0.0983	$\vec{\zeta} = (1.9, 1.9),$
				$\vec{p} = (0.6, 0.15)$				$\vec{p} = (0.1, 0.2)$

Table 2: Estimation performance at 1-minute sampling frequency, based on a set of M=1000 paths. The simulated data were generated at over a T=1 month horizon using the model setup of [15]. The indicated tuning parameters are from the grid search and correspond to the values of  $\lambda$ , p for  $\tilde{\sigma}_{\tau,n}^2$  and to the values of  $\vec{\zeta}, \vec{p}$  for  $c_{n,\tau}^{(i)}$ , i=1,2.

	Y = 0.8			Y = 1.2				
Estimator	RMSE	ARE	RE	Tuning Param.	RMSE	ARE	RE	Tuning Param.
$\hat{\sigma}_{\tau,n}^2(u_n,h)$	0.1373	0.0725	0.0028	-	0.1434	0.0754	0.0222	-
$\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$	0.1384	0.0729	0.0043	$\lambda = 1.85, \ p = 0.8$	0.1441	0.0758	0.0242	$\lambda = 1.85, \ p = 0.9$
$\hat{c}_{n,\tau,\exp}(m_n,v_n)$	0.1682	0.1016	-0.0998	-	0.1494	0.0888	-0.0873	-
$\tilde{c}_{n,\tau,\exp}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.1684	0.1016	-0.0990	$\zeta = 1.55, \ p = 0.9$	0.1495	0.0888	-0.0874	$\zeta = 1.7, \ p = 0.8$
$\tilde{c}_{n,\tau,\exp}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.0760	0.0401	0.0050	$\vec{\zeta} = (1.7, 1.9),$	0.0859	0.0455	0.0251	$\vec{\zeta} = (1.9, 1.25),$
				$\vec{p} = (0.5, 0.75)$				$\vec{p} = (0.6, 0.5)$
$\hat{c}_{n,\tau,\mathrm{unif}}(m_n,v_n)$	0.1788	0.1043	-0.1002	-	0.1610	0.0920	-0.0871	-
$\tilde{c}_{n,\tau,\mathrm{unif}}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.1792	0.1044	-0.1002	$\zeta = 1.65, \ p = 0.8$	0.1610	0.0920	-0.0871	$\zeta = 1.65, \ p = 0.75$
$\tilde{c}_{n,\tau,\mathrm{unif}}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.1060	0.0567	0.0050	$\vec{\zeta} = (1.7, 1.9),$	0.1184	0.0625	0.0293	$\vec{\zeta} = (1.9, 1.75),$
				$\vec{p} = (0.5, 0.75)$				$\vec{p} = (0.4, 0.8)$
	Y = 1.6							
			Y = 1.6				Y = 1.75	<u> </u>
Estimator	RMSE	ARE	Y = 1.6 <b>RE</b>	Tuning Param.	RMSE	ARE	Y = 1.75 <b>RE</b>	Tuning Param.
Estimator $\hat{\sigma}_{\tau,n}^2(u_n,h)$	<b>RMSE</b> 0.3157	ARE 0.1853			<b>RMSE</b> 0.6148	<b>ARE</b> 0.3959		
			RE				RE	
$\hat{\sigma}_{\tau,n}^{2}(u_{n},h)$ $\tilde{\sigma}_{\tau,n}^{2}(\lambda,u_{n},h)$ $\hat{c}_{n,\tau,\exp}(m_{n},v_{n})$	0.3157	0.1853	<b>RE</b> 0.1823	Tuning Param.	0.6148	0.3959	<b>RE</b> 0.3950	Tuning Param.
$\hat{\sigma}_{\tau,n}^{2}(u_{n},h)$ $\tilde{\sigma}_{\tau,n}^{2}(\lambda,u_{n},h)$ $\hat{c}_{n,\tau,\exp}(m_{n},v_{n})$	0.3157 0.2456	0.1853 0.1324	<b>RE</b> 0.1823 0.0910	Tuning Param.	0.6148 0.5286	0.3959 0.3575	<b>RE</b> 0.3950 0.0907	Tuning Param.
$\hat{\sigma}_{\tau,n}^{2}(u_{n},h)$ $\hat{\sigma}_{\tau,n}^{2}(\lambda,u_{n},h)$ $\hat{c}_{n,\tau,\exp}(m_{n},v_{n})$ $\tilde{c}_{n,\tau,\exp}^{(1)}(m_{n},v_{n},\vec{\zeta}_{1})$	0.3157 0.2456 0.0965	0.1853 0.1324 0.0533	RE 0.1823 0.0910 0.0406	Tuning Param. $\lambda = 1.85, \ p = 0.1$	0.6148 0.5286 0.3364	0.3959 0.3575 0.2209	RE 0.3950 0.0907 0.2209	Tuning Param. $\lambda = 1.85, \ p = 0.55$ -
$\hat{\sigma}_{\tau,n}^{2}(u_{n},h)$ $\tilde{\sigma}_{\tau,n}^{2}(\lambda,u_{n},h)$ $\hat{c}_{n,\tau,\exp}(m_{n},v_{n})$	0.3157 0.2456 0.0965 <b>0.0823</b>	0.1853 0.1324 0.0533 <b>0.0440</b>	RE 0.1823 0.0910 0.0406 0.0036	Tuning Param. $\lambda = 1.85, \ p = 0.1$ $\zeta = 1.9, \ p = 0.1$	0.6148 0.5286 0.3364 0.1563	0.3959 0.3575 0.2209 0.0882	RE 0.3950 0.0907 0.2209 0.0703	Tuning Param. $\lambda = 1.85, \ p = 0.55$ $\zeta = 1.75, \ p = 0.25$
$\hat{\sigma}_{\tau,n}^{2}(u_{n},h)$ $\tilde{\sigma}_{\tau,n}^{2}(\lambda,u_{n},h)$ $\hat{c}_{n,\tau,\exp}(m_{n},v_{n})$ $\tilde{c}_{n,\tau,\exp}^{(1)}(m_{n},v_{n},\vec{\zeta_{1}})$ $\tilde{c}_{n,\tau,\exp}^{(2)}(m_{n},v_{n},\vec{\zeta_{2}})$ $\hat{c}_{n,\tau,\mathrm{unif}}(m_{n},v_{n})$	0.3157 0.2456 0.0965 <b>0.0823</b>	0.1853 0.1324 0.0533 <b>0.0440</b>	RE 0.1823 0.0910 0.0406 0.0036	Tuning Param. $\lambda = 1.85, \ p = 0.1$ $\zeta = 1.9, \ p = 0.1$ $\vec{\zeta} = (1.9, 1.65),$	0.6148 0.5286 0.3364 0.1563	0.3959 0.3575 0.2209 0.0882	RE 0.3950 0.0907 0.2209 0.0703	Tuning Param. $\lambda = 1.85, \ p = 0.55$ $\zeta = 1.75, \ p = 0.25$ $\vec{\zeta} = (1.4, 1.9),$
$ \hat{\sigma}_{\tau,n}^{2}(u_{n},h) $ $ \tilde{\sigma}_{\tau,n}^{2}(\lambda,u_{n},h) $ $ \hat{c}_{n,\tau,\exp}(m_{n},v_{n}) $ $ \tilde{c}_{n,\tau,\exp}^{1}(m_{n},v_{n},\vec{\zeta_{1}}) $ $ \tilde{c}_{n,\tau,\exp}^{(2)}(m_{n},v_{n},\vec{\zeta_{2}}) $ $ \hat{c}_{n,\tau,\exp}^{(2)}(m_{n},v_{n},\vec{\zeta_{2}}) $ $ \hat{c}_{n,\tau,\mathrm{unif}}^{n}(m_{n},v_{n}) $ $ \tilde{c}_{n,\tau,\mathrm{unif}}^{(1)}(m_{n},v_{n},\vec{\zeta_{1}}) $	0.3157 0.2456 0.0965 0.0823 0.0823	0.1853 0.1324 0.0533 0.0440 0.0440	RE 0.1823 0.0910 0.0406 <b>0.0036</b> <b>0.0036</b>	Tuning Param. $\lambda = 1.85, \ p = 0.1$ $\zeta = 1.9, \ p = 0.1$ $\vec{\zeta} = (1.9, 1.65),$	0.6148 0.5286 0.3364 0.1563 <b>0.1423</b>	0.3959 0.3575 0.2209 0.0882 <b>0.0776</b>	RE 0.3950 0.0907 0.2209 0.0703 0.0253	Tuning Param. $\lambda = 1.85, \ p = 0.55$ $\zeta = 1.75, \ p = 0.25$ $\vec{\zeta} = (1.4, 1.9),$
	0.3157 0.2456 0.0965 <b>0.0823</b> <b>0.0823</b>	0.1853 0.1324 0.0533 0.0440 0.0440	RE 0.1823 0.0910 0.0406 0.0036 0.0036	Tuning Param. $\lambda = 1.85, \ p = 0.1$ $\zeta = 1.9, \ p = 0.1$ $\vec{\zeta} = (1.9, 1.65),$ $\vec{p} = (0.1, 0.3)$	0.6148 0.5286 0.3364 0.1563 <b>0.1423</b>	0.3959 0.3575 0.2209 0.0882 <b>0.0776</b>	RE 0.3950 0.0907 0.2209 0.0703 0.0253	Tuning Param. $\lambda = 1.85, \ p = 0.55$ $\zeta = 1.75, \ p = 0.25$ $\vec{\zeta} = (1.4, 1.9),$ $\vec{p} = (0.3, 0.2)$

(54) with parameter values as in [3]:

$$\rho = -0.5, \quad \kappa = 5, \quad \xi = 0.5, \quad \theta = 0.16.$$

In particular, we note the annualized average volatility is  $\sqrt{.16} = .4$ , which is more realistic for financial data. Assuming 252 trading days per year, and a 6.5 hour trading day, we set  $\Delta_n = (252*6.5*12)^{-1}$ , corresponding to a sampling frequency of 5 minutes. We consider a time horizon of 3 months  $(T = \frac{1}{4})$ , yielding a sample size of  $n = T/\Delta_n = 4914$ . In our experiments, we examine performance in the cases Y = 1.6 and Y = 1.75, corresponding to 1 and 2 steps of debiasing required for efficiency, and use an exponential kernel for our estimators  $\hat{c}_n(m_n, \Delta_n)$ ,  $\tilde{c}_{m,\tau}^{(1)}(m_n, v_n, \vec{\zeta}_1)$  and  $\tilde{c}_{n,\tau}^{(2)}(m_n, v_n, \vec{\zeta}_2)$ . The tuning parameters are similar to the previous section, with  $\vec{\zeta}_1$ ,  $\vec{\zeta}_2$ ,  $\lambda$ , p,  $p_1$ , and  $(p_1, p_2)$  being selected via a grid search that minimizes  $\widehat{RMSE}$  over 100 iterations (using the same grid as described in previous section), but here we take  $v_n = \sqrt{BV}v_0$ , for different  $v_0$  to assess the sensitivity of our methods to the threshold choice. We report the performance for our estimators across various threshold choices  $v_0$  in Table 3 (Y = 1.6) and in Table 4 (Y = 1.75) as measured by  $\widehat{RMSE}$ ,  $\widehat{ARE}$  and  $\widehat{RE}$  using M = 1000 paths. In each table, we also include the performance of the estimators of [15] for comparison. We also report the auxiliary tuning parameter values found in our grid search.

As shown in Tables 3 and 4, for both Y=1.6 and Y=1.75 the two-step estimator  $\tilde{c}_n^{(2)}$  again provides the most reliable performance, delivering the lowest or near-lowest RMSE and ARE across all  $v_0$  considered. The advantage of debiasing is most pronounced at the moderate threshold choices  $v_0 = \Delta_n^{20/48} = \Delta_n^{5/12}$  and  $v_0 = \Delta_n^{21/48}$ , where the two-step estimators offer reductions in RMSE over  $\hat{\sigma}_n^2$  and  $\tilde{\sigma}_n^2$  by roughly a factor of 2, and these improvements are observed consistently for both Y=1.6 and Y=1.75. Remarkably, the choice  $v_0 = \Delta_n^{5/12}$  is the same as suggested by the theory for two-step debiasing in the integrated volatility case (see [3]). Broadly, these findings highlight the benefit of two-step debiasing beyond specific threshold choices, offering reliable nontrivial reductions in MSE across a variety of jump activity levels.

Table 3: Estimation performance across  $v_0$  in a realistic parameter setting with Y=1.6. Results based on M=1000 paths at a 5-minute sampling frequency using the exponential kernel. The indicated tuning parameters are from the grid search and correspond to the values of  $\lambda$ , p for  $\tilde{\sigma}_{\tau,n}^2$  and to the values of  $\vec{\zeta}, \vec{p}$  for  $c_{n,\tau}^{(i)}$ , i=1,2.

Threshold	Estimator	RMSE	ARE	RE	Tuning Param.
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.03296	0.22079	0.21617	-
$v_0 = \Delta_n^{19/48}$	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.0291	0.18878	0.16621	$\zeta = 1.4, \ p = 0.2$
	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.0295	0.1789	0.12536	$\vec{\zeta} = (1.4, 1.9), \ \vec{p} = (0.3, 0.2)$
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.02342	0.14157	0.08644	-
$v_0 = \Delta_n^{20/48}$	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.02342	0.14157	0.08644	$\zeta = 1.9, \ p = 0.9$
	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.02408	0.15699	0.13156	$\vec{\zeta} = (1.9, 1.7), \vec{p} = (0.8, 0.2)$
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.03956	0.17759	-0.12086	-
$v_0 = \Delta_n^{21/48}$	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.03956	0.17759	-0.12086	$\zeta = 1.7, \ p = 0.9$
	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.03956	0.17759	-0.12086	$\vec{\zeta}(1.5, 1.2), \ \vec{p} = (0.9, 0.4)$
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.07364	0.37008	-0.36356	-
$v_0 = \Delta_n^{22/48}$	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.07364	0.37008	-0.36356	$\zeta = 1.8, \ p = 0.8$
	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.05129	0.34341	0.34314	$\vec{\zeta} = (1.8, 1.9), \ \vec{p} = (0.7, 0.9)$
	$\hat{\sigma}_{\tau,n}^2(u_n,h)$	0.04669	0.28256	0.27899	-
_	$\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$	0.04187	0.2466	0.22257	$\lambda = 1.9, p = 0.35$

Table 4: Estimation performance across  $v_0$  in a realistic parameter setting with Y=1.75. Results based on M=1000 paths at a 5-minute sampling frequency. The indicated tuning parameters are from the grid search and correspond to the values of  $\lambda$ , p for  $\tilde{\sigma}_{\tau,n}^2$  and to the values of  $\vec{\zeta}$ ,  $\vec{p}$  for  $c_{n,\tau}^{(i)}$ , i=1,2.

Threshold	Estimator	RMSE	ARE	RE	Tuning Param.
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.05626	0.38748	0.3872	-
$v_0 = \Delta_n^{19/48}$	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.04713	0.31793	0.30345	$\zeta = 1.5, \ p = 0.3$
$v_0 = \Delta_n$	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.04713	0.31793	0.30345	$\vec{\zeta} = (1.5, 1.9), \ \vec{p} = (0.3, 0.2)$
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.03883	0.26904	0.25687	-
$v_0 = \Delta_n^{20/48}$	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.03826	0.21271	0.08458	$\zeta = 1.7, p = 0.3$
$v_0 = \Delta_n$	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.03464	0.22508	0.0588	$\vec{\zeta} = (1.7, 1.9), \ \vec{p} = (0.2, 0.2)$
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.03203	0.16966	0.02212	-
	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.03203	0.16966	0.02212	$\zeta = 1.6, \ p = 0.9$
$v_0 = \Delta_n^{21/48}$	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.02906	0.16739	0.06315	$\vec{\zeta} = (1.7, 1.9), \ \vec{p} = (0.8, 0.2)$
	$\hat{c}_{n,\tau}(m_n,v_n)$	0.06220	0.28966	-0.2564	-
	$c_{n,\tau}^{(1)}(m_n,v_n,\vec{\zeta}_1)$	0.06220	0.28966	-0.2564	$\zeta = 1.8, \ p = 0.8$
$v_0 = \Delta_n^{22/48}$	$c_{n,\tau}^{(2)}(m_n,v_n,\vec{\zeta}_2)$	0.03363	0.1677	-0.00909	$\vec{\zeta} = (1.9, 1.2), \ \vec{p} = (0.9, 0.6)$
	$\hat{\sigma}_{\tau,n}^2(u_n,h)$	0.07123	0.46975	0.43293	-
_	$\tilde{\sigma}_{\tau,n}^2(\lambda,u_n,h)$	0.06628	0.46931	0.42743	$\lambda = 1.9, \ p = 0.1$

# A Additional technical proofs

Throughout the proofs below, we make use of the following notation:

$$x_{i} := b_{t_{i-1}} \Delta_{n} + \sigma_{t_{i-1}} \Delta_{i} W + \chi_{t_{i-1}} \Delta_{i} J, \quad \hat{x}_{i} := \sigma_{t_{i-1}} \Delta_{i} W + \chi_{t_{i-1}} \Delta_{i} J,$$

$$\varepsilon_{i} := \Delta_{i} X - x_{i} = \int_{t_{i-1}}^{t_{i}} \sigma_{t} dW_{t} - \sigma_{t_{i-1}} \Delta_{i} W + \int_{t_{i-1}}^{t_{i}} \chi_{t} dJ_{t} - \chi_{t_{i-1}} \Delta_{i} J + \int_{t_{i-1}}^{t_{i}} b_{t} dt - b_{t_{i-1}} \Delta_{n}. \tag{56}$$

In what follows we often refer to the following standard moment estimates of the error terms (see Lemma 3 in [4]):

$$\mathbb{E}\left[\left|\varepsilon_{i}\right|^{p}\middle|\mathcal{F}_{t_{i-1}}\right] \leq C\Delta_{n}^{\frac{(2\wedge p)+p}{2}},\tag{57}$$

for any p > 0 and a constant C.

# A.1 Proof of Proposition 1

Similar to [3] and [4], we approximate the original process  $X_t$  by

$$X'_{t} := \int_{0}^{t} b_{s} \, ds + \int_{0}^{t} \sigma_{s} \, dW_{s} + \int_{0}^{t} \chi_{s} \, dJ_{s}^{\infty}, \tag{58}$$

where, with  $\delta_0 \in (0,1)$  being chosen such that q(x) > 0 for all  $|x| \le \delta_0$  and denoting the jump measure of J and its compensated version as N and  $\bar{N}$ , respectively, the process  $J^{\infty}$  is defined as

$$J_t^{\infty} := \left(\bar{b} + \int_{\delta_0 < |x| \le 1} x \,\nu(dx)\right) t + \int_0^t \int_{|x| \le \delta_0} x \,\bar{N}(ds, dx) + \check{J}_t. \tag{59}$$

The term  $\check{J}_t$  is a pure-jump Lévy process, independent of J, with characteristic triplet  $(0, 0, \check{\nu})$ , where its Lévy measure,  $\check{\nu}$ , is given by

$$\breve{\nu}(dx) := e^{-|x|^p} \left( C_+ \mathbf{1}_{(0,\infty)}(x) + C_- \mathbf{1}_{(-\infty,0)}(x) \right) \mathbf{1}_{|x| > \delta_0} |x|^{-1-Y} \ dx,$$

for a fixed constant  $p < 1 \land Y$ . Additionally, we write

$$J_t^0 := J_t - J_t^\infty = \int_0^t \int_{|x| > \delta_0} x N(ds, dx) - \breve{J}_t, \tag{60}$$

and note that

$$X_t - X_t' = \int_0^t \int \delta(s, z) \mathfrak{p}(ds, dz) + \int_0^t \chi_s dJ_s^0.$$

In conclusion, the process X' contains the continuous part and the jumps of the infinite variation, and X - X' only includes the jumps of bounded variation.

# **A.1.1** Proof of (14)

This expansion follows from the one obtained in [3, Lemma 4]. Indeed, the condition Y < 8/5 stated in the hypotheses of that lemma is not used in its proof.

### **A.1.2** Proof of (15)

The proof of (15) follows along the lines of the proof of Lemma 2 and Lemma 5 in [3]. The main difference is that [3] has the constraint  $v_n \gg \Delta_n^{\frac{3}{2(2+Y)}}$  as well as the condition  $Y \leq 8/5$ . However, these restrictions were imposed to achieve an error term of order  $o_P(\Delta_n^{3/2})$  in the expansion therein, while, in our case, it suffices to have an error of order  $o_P(\Delta_n^{\frac{5}{4}}) + O_P(\Delta_n^3 v_n^{-2-Y})$ . Under this more relaxed constraint, we will show the statement (15) holds under the entire range  $Y \in (0,1) \cup (1,2)$ .

Note that the proof of Lemma 2 in [3] remains valid for 0 < Y < 1, without relying on the condition  $v_n \gg \Delta_n^{\frac{3}{2(2+Y)}}$ . Hence, Lemma 2 holds for 0 < Y < 1 under the weaker condition  $\Delta_n^{1/2} \ll v_n \ll \Delta_n^{1/(4-Y)}$ , which in turn implies (15). In the following, we focus on the proof for the case 1 < Y < 2.

First, we show that the expansion (15) holds when X is a sum of a Brownian motion and the tempered stable Lévy process  $J^{\infty}$  defined in (59). This fact actually follows directly from Proposition 2 in [4]. Indeed, it was shown there that for such a process the error, excluding exponentially decaying terms, is of order  $O_P(\Delta_n^3 v_n^{-Y-2}) + O_P(\Delta_n^2 v_n^{2-2Y}) + O_P(\Delta_n v_n^{2-\bar{\delta}}) + O_P(\Delta_n^{3/2} v_n^{1-Y/2})$ . However, we observe that both  $\Delta_n v_n^{2-\bar{\delta}}$  and  $\Delta_n^{\frac{3}{2}} v_n^{1-Y/2}$  are  $o_P(\Delta_n^{\frac{3}{4}})$  and  $\Delta_n^3 v_n^{-Y-2} \gg \Delta_n^2 v_n^{2-2Y}$  since  $v_n \ll \Delta_n^{1/(4-Y)}$ . Therefore, the expansion of Proposition 2 in [4] is valid for any  $Y \in (1,2)$  with the errors being of order  $O_P(\Delta_n^3 v_n^{-2-Y}) + o_P(\Delta_n^{5/4})$ .

As in [3], we next extend the expansion for the process X' defined in (58), which broadly replaces J with  $J^{\infty}$  and omits the last bounded variation Poisson term in X. This is done in Lemma 2 in [3]. To conclude that the expansion therein holds, but with an adjusted error of order  $O_P(\Delta_n^3 v_n^{-2-Y}) + o_P(\Delta_n^{5/4})$ , it suffices to show the terms  $R_{i,1}$  and  $R_{i,4}$  appearing in the proof of [3, Lemma 2] are of order  $o_P(\Delta_n^{5/4})$ . We only show  $R_{i,4} = o_P(\Delta_n^{5/4})$  since the argument for  $R_{i,1}$  is similar. It suffices to bound  $D_{i,1}$  and  $D_{i,2}$  as defined on [3, p.20]. To obtain  $D_{i,1} = o_P(\Delta_n^{5/4})$ , under the constraint 1 < Y < 2, the analog of [3, eq. (B.12)] reads

$$\Delta_n v_n^{1-Y} \delta \ll \Delta_n^{5/4} \quad \Longleftrightarrow \quad \delta \ll \Delta_n^{1/4} v_n^{Y-1}. \tag{61}$$

To obtain  $D_{i,2} = o_P(\Delta_n^{5/4})$  for 1 < Y < 2, it suffices to bound the terms  $\mathcal{V}_{i,2}$  and  $\mathcal{V}_{i,3}$  defined on [3, p. 21]. Following the same argument leading to [3, eq. (B.13)], to obtain  $\mathcal{V}_{i,1} = o_P(\Delta_n^{5/4})$ , we require, for arbitrarily small s', s'' > 0,

$$\Delta_n^{5/2-s'} v_n^{1-Y-s''} \delta^{-1} \ll \Delta_n^{5/4} \quad \Longleftrightarrow \quad \delta \gg \Delta_n^{5/4-s'} v_n^{1-Y-s''}. \tag{62}$$

The conditions (61) and (62) are consistent for any 1 < Y < 2, since

$$\Delta_n^{5/4-s'} v_n^{1-Y-s''} \ll \Delta_n^{1/4} v_n^{Y-1} \Leftrightarrow \Delta_n^{\frac{1-s'}{2Y-2+s''}} \ll v_n,$$

which holds for sufficiently small s', s'' under our condition  $\Delta_n^{\frac{1}{2}-s} \ll v_n$ . Finally, for the term  $\mathcal{V}_{i,3}$ , since  $v_n \ll \Delta_n^{\frac{1}{4-Y}}$ , the condition  $\delta \gg \Delta_n^{\frac{1}{2}} v_n^{\frac{Y}{2}}$  imposed at the end of the proof of [3, Lemma 2] is consistent with (61). The condition  $\delta \gg \Delta_n^{\frac{1}{2}} v_n^{\frac{Y}{2}}$  enables us to apply [4, Lemma 4] and conclude that  $\mathbb{E}(V_{i,3}) = O(\Delta_n^2 v_n^{2-2Y})$ , which is obviously also  $O(\Delta_n^3 v_n^{-Y-2})$ , as desired.

To conclude the proof, with the updated error  $O_P(\Delta_n^3 v_n^{-2-Y}) + o_P(\Delta_n^{5/4})$  and the more general process X, it remains to establish an analog of Lemma 5 in [3], which allows us to add the last bounded variation term to X and go from  $J^{\infty}$  to the more general tempered stable Lévy process J. This is detailed in Section A.1.4 below.

### **A.1.3** Proof of (16)

We first note that the expansion (16) holds when X is a sum of a Brownian motion and the independent tempered stable Lévy process  $J^{\infty}$  on account of [4, Proposition 2], which implies that that the error terms in (16) are  $O_P(\Delta_n^3 v_n^{-Y-2}) + O_P(\Delta_n^2 v_n^{2-2Y}) + O_P(\Delta_n v_n^{2-\bar{\delta}}) + O_P(\Delta_n^{3/2} v_n^{1-Y/2})$ . As with the proof of (15), we can similarly check that  $O_P(\Delta_n^3 v_n^{-2-Y})$  and  $O_P(\Delta_n^3 v_n^{2-\bar{\delta}})$  dominate the higher order terms  $\Delta_n^2 v_n^{2-2Y}$ ,  $\Delta_n^{\frac{3}{2}} v_n^{1-\frac{Y}{2}}$  and  $\Delta_n v_n^{2-\bar{\delta}}$  therein.

Next, similarly to [3, Lemma 3], we argue that the expansion (16) holds for the process X' defined in (58). Indeed, following the proof of [3, Lemma 3], it suffices to obtain bounds for the terms  $\overline{\mathcal{D}}_{i,1}$  and  $\overline{\mathcal{D}}_{i,2}$  on [3, p.22]. First to show  $\overline{\mathcal{D}}_{i,1} = o_P(\Delta_n^{\frac{3}{4}} v_n^{\frac{4-Y}{2}})$ , we arrive at the constraint analogous to [3, eq. (B.17)]:

$$\Delta_n v_n^{1-Y} \delta \ll \Delta_n^{\frac{3}{4}} v_n^{\frac{4-Y}{2}} \quad \Longleftrightarrow \quad \delta \ll \Delta_n^{-\frac{1}{4}} v_n^{1+\frac{Y}{2}}. \tag{63}$$

Similarly, to obtain  $\overline{\mathcal{D}}_{i,2} = o_P(\Delta_n^{\frac{3}{4}} v_n^{\frac{4-Y}{2}}) + O_P(\Delta_n^3 v_n^{-Y-2})$ , we first recall that (see [3, eq. (B.19)]), under the constraint  $\delta \gg \Delta_n^{1/2} v_n^{Y/2}$ ,

$$\overline{\mathcal{D}}_{i,2} = O_P(\Delta_n^{5/2-s'} v_n^{1-Y-s''} \delta^{-1}) + O_P(\Delta^2 v_n^{2-2Y}).$$

The second term above is  $O_P(\Delta_n^3 v_n^{-Y-2})$  because  $v_n \ll \Delta_n^{1/(4-Y)}$ . For the first term above to be  $o_P(\Delta_n^{\frac{3}{4}} v_n^{\frac{4-Y}{2}})$ , we need:

$$\Delta_n^{5/2-s'} v_n^{1-Y-s''} \delta^{-1} \ll \Delta_n^{\frac{3}{4}} v_n^{\frac{4-Y}{2}} \iff \Delta_n^{\frac{7}{4}-s'} v_n^{-1-\frac{Y}{2}-s''} \ll \delta, \tag{64}$$

which is compatible with (63) if  $\Delta_n^{\frac{7}{4}}v_n^{-1-\frac{Y}{2}}\ll \Delta_n^{-\frac{1}{4}}v_n^{1+\frac{Y}{2}}$  or, equivalently,  $\Delta_n^{\frac{2}{2+Y}}\ll v_n$ . The latter obviously holds for all 0< Y<2 since  $v_n\gg \Delta_n^{\frac{1}{2}}$ . On the other hand, (63) is also compatible with the constraint  $\delta\gg \Delta_n^{1/2}v_n^{Y/2}$  required for [3, eq. (B.19)] since  $\Delta_n^{1/2}\ll v_n$ . The remaining part of the proof is the same as Lemma 3 in [3]. To carry the expansion of [3, Lemma 3], with the updated error  $O_P(\Delta_n^3v_n^{-2-Y})+o_P(\Delta_n^{\frac{3}{4}}v_n^{\frac{4-Y}{2}})$ , to the most general process X, it remains to establish an appropriate analog of Lemma 5 in [3]. This is detailed in Section A.1.4 below.

# A.1.4 Analog of Lemma 5 in [3]

With  $r = r_0 \in (0, Y \land 1)$ , Lemma 5 in [3] states that for any  $\alpha \ge 1$ ,

$$|(\Delta_{i}X)^{2p}\mathbf{1}_{\{|\Delta_{i}X| \leq v_{n}\}} - (\Delta_{i}X')^{2p}\mathbf{1}_{\{|\Delta_{i}X'| \leq v_{n}\}}|$$

$$= O_{P}(\Delta_{n}v_{n}^{2p-\alpha r_{0}}) + O_{P}(\Delta_{n}v_{n}^{2p-Y-1+\alpha}).$$
(65)

The above equation still holds under the hypotheses of Proposition 1. To establish (14), it is further required that the above terms are  $o_P(\Delta_n v_n^{2p-Y})$ , which can be ensured by choosing  $\alpha$  close to 1 from above, since  $r_0 < Y$ ,  $\Delta_n v_n^{-r_1\alpha} \ll \Delta_n v_n^{-2} \ll 1$  under our condition  $r_0 \in [0, Y \wedge 1)$ .

For (15) and (16), the corresponding requirements for the errors in (65) are  $o_P(\Delta_n^{5/4})$  and  $o_P(\Delta_n^{3/4}v_n^{2-Y/2})$ , respectively. Since  $\Delta_n^{3/4}v_n^{2-Y/2} \ll \Delta_n^{5/4}$  under  $v_n \ll \Delta_n^{1/(4-Y)}$ , it suffices to establish that the terms are  $o_P(\Delta_n^{3/4}v_n^{2-Y/2})$ , which occurs if the following hold:

- 1.  $\Delta_n v_n^{2-\alpha r_0} \ll \Delta_n^{3/4} v_n^{2-Y/2}$ , which holds if  $\alpha < (Y+1)/(2r_0)$ . This is obtained by observing  $\Delta_n v_n^{2-\alpha r_0} \ll \Delta_n^{3/4} v_n^{2-Y/2} \Leftrightarrow \Delta_n \ll v_n^{4\alpha r_0-2Y}$  and, since we required  $\Delta_n^{1/2} \ll v_n$ , we only need to impose that  $v_n^2 \ll v_n^{4\alpha r_0-2Y}$ , which is satisfied if  $\alpha < (Y+1)/(2r_0)$ .
- 2.  $\Delta_n v_n^{2-Y-1+\alpha} \ll \Delta_n^{3/4} v_n^{2-Y/2}$  if  $\alpha > 1/2+Y/2$ . This is obtained by observing  $\Delta_n v_n^{2-Y-1+\alpha} \ll \Delta_n^{3/4} v_n^{2-Y/2} \Leftrightarrow \Delta_n^{1/(2Y+4-4\alpha)} \ll v_n$  and comparing  $\Delta_n^{1/(2Y+4-4\alpha)} \ll \Delta_n^{1/2}$ .

In conclusion, we need to pick  $\alpha$  such that

$$\frac{1+Y}{2} < \alpha < \frac{1+Y}{2r_0},$$

which is always possible given  $r_0 \in [0, 1)$ .

### **A.1.5** Proof of (17)

As with the proofs of (15) and (16), we proceed in two steps: first we show (17) for the process X' defined in (58) and then, we show the analog of Lemma 5 in [3]; i.e., we show that, for any real number p > 1,

$$|(\Delta_i X)^{2p} \mathbf{1}_{\{|\Delta_i X| < v_n\}} - (\Delta_i X')^{2p} \mathbf{1}_{\{|\Delta_i X'| < v_n\}}| = O_P(\sigma_{t_{i-1}}^{2p} \Delta_n^p) + O_P(\Delta_n v_n^{2p-Y}).$$
(66)

Note that (66) allows non-integer moment and does not directly follow from Lemma 5 in [3].

For simplicity, we re-use the notation in (56), except with  $J_t^{\infty}$  in place of  $J_t$ . To prove (17), we need to show that, for any p > 1, the following two estimates hold:

$$\mathbb{E}_{i-1}|x_i \mathbf{1}_{\{|x_i| \le v_n\}}|^{2p} = O_P(\sigma_{t_{i-1}}^{2p} \Delta_n^p) + O_P(\Delta_n v_n^{2p-Y}), \tag{67}$$

and

$$\mathbb{E}_{i-1} |(\Delta_i X) \mathbf{1}_{\{|\Delta_i X| < v_n\}}|^{2p} - \mathbb{E}_{i-1} |x_i \mathbf{1}_{\{|x_i| < v_n\}}|^{2p} = O_P(\sigma_{t_{i-1}}^{2p} \Delta_n^p) + O_P(\Delta_n v_n^{2p-Y}).$$
 (68)

We begin with (67) and proceed along the lines of Proposition 3 in [4]. It suffices to show

$$\mathbb{E}_{i-1}(|\sigma_{t_{i-1}}\Delta_i W + \chi_{t_{i-1}}\Delta_i J^{\infty}|^{2p} \mathbf{1}_{\{|\sigma_{t_{i-1}}\Delta_i W + \chi_{t_{i-1}}\Delta_i J^{\infty}| \le v_n\}}) = O_P(\sigma_{t_{i-1}}^{2p}\Delta_n^p) + O_P(\Delta_n v_n^{2p-Y}).$$
(69)

Note that

$$\mathbb{E}_{i-1}(\left|\sigma_{t_{i-1}}\Delta_{i}W + \chi_{t_{i-1}}\Delta_{i}J^{\infty}\right|^{2p}\mathbf{1}_{\{|\sigma_{t_{i-1}}\Delta_{i}W + \chi_{t_{i-1}}\Delta_{i}J^{\infty}| \leq v_{n}\}})$$

$$\lesssim \mathbb{E}_{i-1}|\sigma_{t_{i-1}}\Delta_{i}W|^{2p} + \mathbb{E}_{i-1}(\left|\Delta_{i}J^{\infty}\right|^{2p}\mathbf{1}_{\{|\sigma_{t_{i-1}}\Delta_{i}W + \chi_{t_{i-1}}\Delta_{i}J^{\infty}| \leq v_{n}\}}),$$

and, by self-similarity,  $\mathbb{E}_{i-1}|\sigma_{t_{i-1}}\Delta_i W|^{2p} = O_P(\sigma_{t_{i-1}}^{2p}\Delta_n^p)$ . Thus, to show (69), it suffices that

$$A_n:=\mathbb{E}_{i-1}\big(|\chi_{t_{i-1}}\Delta_i J^\infty|^{2p}\mathbf{1}_{\{|\sigma_{t_{i-1}}\Delta_i W+\chi_{t_{i-1}}\Delta_i J^\infty|\leq v_n\}}\big)\lesssim \Delta_n v_n^{2p-Y}+\Delta_n^p.$$

Note that  $\mathbf{1}_{\{|\sigma_{t_{i-1}}\Delta_iW+\chi_{t_{i-1}}\Delta_iJ^{\infty}|\leq v_n\}} \leq \mathbf{1}_{|\sigma_{t_{i-1}}\Delta_iW|\leq v_n}\mathbf{1}_{\{|\chi_{t_{i-1}}\Delta_iJ^{\infty}|\leq 2v_n\}} + \mathbf{1}_{|\sigma_{t_{i-1}}\Delta_iW|>v_n}$ . Then we may further write

$$A_{n} \leq \mathbb{E}_{i-1}(|\chi_{t_{i-1}}\Delta_{i}J^{\infty}|^{2p}\mathbf{1}_{\{|\chi_{t_{i-1}}\Delta_{i}J^{\infty}|\leq 2v_{n}\}}) + \mathbb{E}_{i-1}(|\chi_{t_{i-1}}\Delta_{i}J^{\infty}|^{2p}\mathbf{1}_{\{|\sigma_{t_{i-1}}\Delta_{i}W+\chi_{t_{i-1}}\Delta_{i}J^{\infty}|\leq v_{n}\}} \cdot \mathbf{1}_{|\sigma_{t_{i-1}}\Delta_{i}W|>v_{n}})$$

$$=: V_{1} + V_{2}.$$

For  $V_1$ , by (B.38) in [4], we obtain that

$$V_1 \lesssim \Delta_n v_n^{2p-Y} + \Delta_n^p. \tag{70}$$

For  $V_2$ , let K be the smallest positive integer such that K > p. And further assume  $\sigma_{t_{i-1}} \neq 0$ , otherwise  $V_2 = 0$ . By Hölder's inequality and (B.37) in [4], we have

$$V_{2} \leq \mathbb{E}_{i-1} \left( |\chi_{t_{i-1}} \Delta_{i} J^{\infty}|^{2K} \mathbf{1}_{\{|\sigma_{t_{i-1}} \Delta_{i} W + \chi_{t_{i-1}} \Delta_{i} J^{\infty}| \leq v_{n}\}} \right)^{\frac{p}{K}} P \left( |\sigma_{t_{i-1}} \Delta_{i} W| > v_{n} \right)^{\frac{K-p}{K}}$$

$$\lesssim \left( \Delta_{n} v_{n}^{2K-Y} + \Delta_{n}^{2} v_{n}^{2K-Y-2} + \Delta_{n}^{2K-Y/2} \right)^{\frac{p}{K}} \exp \left( -\frac{K-p}{2K\sigma_{t_{i-1}}^{2}} (\Delta_{n}^{-1} v_{n}^{2}) \right), \tag{71}$$

which vanishes faster than any power of  $\Delta_n$  given  $\Delta_n^{-1}v_n^2 \gg \Delta_n^{-2s}$  and hence  $V_2 = o_P(\Delta_n v_n^{2p-Y})$ . Combining (70) and (71) yields (69).

Now we proceed to establish (68). Note that for any q > 1 and real numbers a, b, we have  $^{10}$ 

$$||a|^q - |b|^q| \le q||a| - |b||(|a|^{q-1} + |b|^{q-1}).$$
 (72)

We utilize the inequality (72) to bound (68) by taking q = 2p,  $a = (\Delta_i X) \mathbf{1}_{\{|\Delta_i X| \leq v_n\}}$ , and  $b = x_i \mathbf{1}_{\{|x_i| \leq v_n\}}$ . Also, recalling that  $\Delta_i X = x_i + \varepsilon_i$ , we have

$$||a| - |b|| \le |a - b| \le |\varepsilon_i| \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} + |x_i| (\mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n < |x_i|\}} + \mathbf{1}_{\{|x_i| \le v_n < |x_i + \varepsilon_i|\}}).$$

Using the two inequalities above together with  $|x + \varepsilon|^{2p-1} \le 2^{2p-2}(|x|^{2p-1} + |\varepsilon|^{2p-2})$ , we obtain:

$$\mathbb{E}_{i-1} | (\Delta_{i}X) \mathbf{1}_{\{|\Delta_{i}X| \leq v_{n}\}}|^{2p} - \mathbb{E}_{i-1} | x_{i} \mathbf{1}_{\{|x_{i}| \leq v_{n}\}}|^{2p} \\
\lesssim \mathbb{E}_{i-1} \Big\{ \Big[ |\varepsilon_{i} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}}| + |x_{i}| (\mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}<|x_{i}|\}} + \mathbf{1}_{\{|x_{i}| \leq v_{n}<|x_{i}+\varepsilon_{i}|\}}) \Big] \\
\cdot \Big[ |x_{i}+\varepsilon_{i}|^{2p-1} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}} + |x_{i}|^{2p-1} \mathbf{1}_{\{|x_{i}| \leq v_{n}\}} \Big] \Big\} \\
\lesssim \mathbb{E}_{i-1} |\varepsilon_{i}^{2p} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}}| + \mathbb{E}_{i-1} |\varepsilon_{i}x_{i}^{2p-1} (\mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}} + \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}} + \mathbf{1}_{\{|x_{i}| \leq v_{n}\}}) \Big| \\
+ \mathbb{E}_{i-1} |x_{i}^{2p} (\mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}<|x_{i}|\}} + \mathbf{1}_{\{|x_{i}| \leq v_{n}<|x_{i}|\}}) (\mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}} + \mathbf{1}_{\{|x_{i}| \leq v_{n}\}}) \Big| \\
= \mathbb{E}_{i-1} |\varepsilon_{i}^{2p} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}<|x_{i}|\}} + \mathbb{E}_{i-1} |\varepsilon_{i}x_{i}^{2p-1} (\mathbf{1}_{|x_{i}+\varepsilon_{i}| \leq v_{n}<|x_{i}|} + \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n},|x_{i}| \leq v_{n}\}}) \Big| \\
+ \mathbb{E}_{i-1} |x_{i}^{2p} (\mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}<|x_{i}|\}} + \mathbf{1}_{\{|x_{i}| \leq v_{n}\leq|x_{i}+\varepsilon_{i}|\}}) \Big| \\
=: R_{i,1} + R_{i,2} + R_{i,3}. \tag{73}$$

By (57), we obtain  $R_{i,1} = O_P(\Delta_n^{1+p}) = o_P(\Delta_n v_n^{2p-Y})$  given  $\Delta_n^{1/2} \ll v_n$  and Y > 0. Now we analyze  $R_{i,3}$ . For  $\mathbb{E}_{i-1}|x_i|^{2p}\mathbf{1}_{\{|x_i|\leq v_n\leq |x_i+\varepsilon_i|\}}$  we can directly apply (67) and consequently  $\mathbb{E}_{i-1}|x_i|^{2p}\mathbf{1}_{\{|x_i|\leq v_n\leq |x_i+\varepsilon_i|\}}$  is of order  $O_P(\sigma_{t-1}^{2p}\Delta_n^p) + O_P(\Delta_n v_n^{2p-Y})$ . For the remaining term in  $R_{i,3}$ , we consider the following decomposition:

$$\mathbb{E}_{i-1}|x_{i}\mathbf{1}_{\{|x_{i}+\varepsilon_{i}|\leq v_{n}\leq |x_{i}|\}}|^{2p} \leq \mathbb{E}_{i-1}|x_{i}\mathbf{1}_{\{|x_{i}+\varepsilon_{i}|\leq v_{n}\leq |x_{i}|\}} \cdot \mathbf{1}_{\{|\varepsilon_{i}|\leq |v_{n}|\}}|^{2p} + \mathbb{E}_{i-1}|x_{i}\mathbf{1}_{\{|x_{i}+\varepsilon_{i}|\leq v_{n}\leq |x_{i}|\}} \cdot \mathbf{1}_{\{|\varepsilon_{i}|>|v_{n}|\}}|^{2p} =: \tilde{D}_{i,1} + \tilde{D}_{i,2}.$$

Note that  $|x_i + \varepsilon_i| \leq v_n$  and  $|\varepsilon_i| \leq |v_n|$  implies  $|x_i| \leq 2v_n$  and, we can again, simply apply (67) to obtain  $\tilde{D}_{i,1} \lesssim \Delta_n v_n^{2p-Y} + \Delta_n^p$ . Regarding  $\tilde{D}_{i,2}$ , for some q > 1, Hölder's inequality yields

$$\mathbb{E}_{i-1}|x_i\mathbf{1}_{\{|x_i+\varepsilon_i|\leq v_n\leq |x_i|\}}\cdot \mathbf{1}_{\{|\varepsilon_i|>|2v_n|\}}|^{2p} \leq \left(\mathbb{E}_{i-1}|x_i\mathbf{1}_{\{|x_i+\varepsilon_i|\leq v_n\}}|^{2pq}\right)^{1/q} P(|\varepsilon_i|>|2v_n|)^{(q-1)/q},$$

which is of order  $o_P(\Delta_n v_n^{2p-Y})$  by taking q such that pq is integer, as  $\mathbb{E}_{i-1}|x_i \mathbf{1}_{\{|x_i+\varepsilon_i|\leq v_n\}}|^{2pq}$  is finite by (14), while  $P(|\varepsilon_i| > |2v_n|)$  vanishes faster than any power of  $\Delta_n$  by (57) given that  $\Delta_n^{\beta} \ll v_n$  with  $\beta < \frac{1}{2}$ . Combining  $\tilde{D}_{i,1}$  and  $\tilde{D}_{i,2}$ , we obtain  $R_{i,3} = O_P(\Delta_n v_n^{2p-Y})$ .

<sup>10</sup>Indeed, for some  $\theta \in (0,1)$ ,  $||b|^q - |a|^q| = q||b| - |a|||(\theta|a| + (1-\theta)|b|)|^{q-1} \le q||b| - |a||(|a|^{q-1} + |b|^{q-1})$  because clearly  $\theta|a| + (1-\theta)|b| \le \max\{|a|, |b|\}$ .

Lastly, we turn to  $R_{i,2}$  in (73). By Hölder's inequality,

$$\begin{split} R_{i,2} & \leq \left(\mathbb{E}_{i-1}|\varepsilon_{i}|^{2p}\right)^{\frac{1}{2p}} \left(\mathbb{E}_{i-1}|x_{i}|^{2p} \mathbf{1}_{|x_{i}+\varepsilon_{i}| \leq v_{n} < |x_{i}|}\right)^{\frac{2p-1}{2p}} + \left(\mathbb{E}_{i-1}|\varepsilon_{i}|^{2p}\right)^{\frac{1}{2p}} \left(\mathbb{E}_{i-1}|x_{i}|^{2p} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}, |x_{i}| \leq v_{n}\}}\right)^{\frac{2p-1}{2p}} \\ & \lesssim \Delta_{n}^{\frac{1+p}{2p}} \left(\Delta_{n} v_{n}^{2p-Y}\right)^{\frac{2p-1}{2p}} + \Delta_{n}^{\frac{1+p}{2p}} \left(\Delta_{n} v_{n}^{2p-Y}\right)^{\frac{2p-1}{2p}}, \end{split}$$

where the second line follows from (57), (67), and the bound we obtained for  $\tilde{D}_{i,2}$  above. We can then conclude that  $R_{i,2} = o_P(\Delta_n v_n^{2p-Y})$  as  $\Delta_n^{1+p} = o_P(\Delta_n v_n^{2p-Y})$ .

Combining the results for  $R_{i,1}$ ,  $R_{i,2}$ , and  $R_{i,3}$ , we establish (68) and conclude the proof of (17) for  $X'_t$ . More specifically, we have obtained

$$\mathbb{E}(|\Delta_i X'|^{2p} \mathbf{1}_{\{|\Delta_i X| < v_n\}} | \mathcal{F}_{i-1}) = O_P(\sigma_{t_{i-1}}^{2p} \Delta_n^p) + O_P(C_{p,t_{i-1}} \Delta_n v_n^{2p-Y}).$$
(74)

Now we will show (66). Following the idea of Lemma 5 in [3], for any p > 1, let

$$D_{2p} := \left| (\Delta_i X)^{2p} \mathbf{1}_{\{|\Delta_i X| \le v_n\}} - (\Delta_i X')^{2p} \mathbf{1}_{\{|\Delta_i X'| \le v_n\}} \right|.$$

Denote  $V_t = X_t - X_t'$ , where recalling  $J^0$  as defined in (60).

$$V_t := \int_0^t \int \delta(s, z) \mathfrak{p}(ds, dz) + \int_0^t \chi_s dJ_s^0 =: Y_t + \int_0^t \chi_s dJ_s^0.$$

For any fixed  $p \ge 1$ , by decomposing the indicator  $\mathbf{1}_{\{|\Delta_i X| \le v_n\}}$  and  $\mathbf{1}_{\{|\Delta_i X'| \le v_n\}}$ , we can obtain

$$D_{2p} = \left| (\Delta_i X)^{2p} - (\Delta_i X')^{2p} \right| \mathbf{1}_{\{|\Delta_i X| \le v_n, |\Delta_i X'| \le v_n\}}$$

$$+ (\Delta_i X)^{2p} \mathbf{1}_{\{|\Delta_i X| \le v_n, |\Delta_i X'| > v_n\}}$$

$$+ (\Delta_i X')^{2p} \mathbf{1}_{\{|\Delta_i X| > v_n, |\Delta_i X'| \le v_n\}}$$

$$=: T_1 + T_2 + T_3.$$

We start with  $T_1$ . By Taylor expansion of  $(x'+v)^{2p}$  at v=0, we obtain

$$|(x'+v)^{2p}-(x')^{2p}| \le K|v|\left(|x'|^{2p-1}+|v|^{2p-1}\right),$$

which applied to  $T_1$  yields

$$T_1 \le K \left( |\Delta_i V|^{2p} + |\Delta_i V| |\Delta_i X'|^{2p-1} \right) \mathbf{1}_{\{ |\Delta_i X| \le v_n, |\Delta_i X'| \le v_n \}}. \tag{75}$$

We first bound  $\Delta_i V$ . By Corollary 2.1.9 in [10], for each  $l \geq 1$  and some constant K > 0, we have:

$$\mathbb{E}_{i-1}\left(\left|\frac{\Delta_i Y}{v_n} \wedge 1\right|^l\right) \le K\Delta_n v_n^{-r_0}. \tag{76}$$

On the other hand, since  $J^0$  is compound Poisson, if we let  $N^0(ds, dx)$  be its jump measure, for some K > 0, we have

$$\mathbb{E}_{i-1}\left(\left|\frac{1}{v_n}\int_{t_{i-1}}^{t_i}\chi_s dJ_s^0 \wedge 1\right|^p\right) \leq \mathbb{P}(N^0([t_{i-1},t_i],\mathbb{R}) > 0) \leq K\Delta_n.$$

Together with (76), we obtain

$$\mathbb{E}_{i-1}\left(|\Delta_i V|^l \mathbf{1}_{\{|\Delta_i V| \le v_n\}}\right) \le K \Delta_n v_n^{l-r_0}.$$

Since  $|x+v| \le v_n$  and  $|x| \le v_n$  imply  $|v| \le 2v_n$ , applying the above inequality to  $T_1$  we get:

$$\mathbb{E}_{i-1}T_1 \le K\left(\mathbb{E}_{i-1}[|\Delta_i V|^{2p} \mathbf{1}_{\{|\Delta_i V| \le 2v_n\}}] + v_n^{2p-1} \mathbb{E}_{i-1}[|\Delta_i V| \mathbf{1}_{\{|\Delta_i V| \le 2v_n\}}]\right) \le K\Delta_n v_n^{2p-r_0}. \tag{77}$$

Now we move to  $T_2$ . For some  $\alpha \geq 1$  to be later specified, we may decompose  $T_2$  as follows:

$$T_2 = |\Delta_i X|^{2p} \mathbf{1}_{\{|\Delta_i X| \le v_n, |\Delta_i X'| > v_n + v_n^{\alpha}\}} + |\Delta_i X|^{2p} \mathbf{1}_{\{|\Delta_i X| \le v_n < |\Delta_i X'| \le v_n + v_n^{\alpha}\}} =: T_2' + T_2''$$

$$(78)$$

Notice that  $|x'+v| \leq v_n$  and  $|x'| > v_n + v_n^{\alpha}$  implies  $|v| > v_n^{\alpha}$ . For  $T_2'$  we obtain the bound

$$\mathbb{E}_{i-1}T_2' \le v_n^{2p} \mathbb{P}_{i-1}(|\Delta_i V| > v_n^{\alpha}, |\Delta_i X'| > v_n + v_n^{\alpha})$$

$$\le K v_n^{2p} \left(\Delta_n v_n^{-\alpha r_0}\right), \tag{79}$$

which follows from the bound  $\mathbb{P}_{i-1}(|\Delta_i V| > u) \leq \mathbb{E}_{i-1}\left(\left|\frac{\Delta_i V}{u} \wedge 1\right|^l\right) \leq K\Delta_n u^{-r_0}$ . For  $T_2''$ , notice that  $|x'+v| \leq v_n$  and  $|x'| \leq v_n + v_n^{\alpha}$  implies  $|v| \leq 2v_n + v_n^{\alpha}$ . We have:

$$\mathbb{E}_{i-1} T_2'' = \mathbb{E}_{i-1} \left[ |\Delta_i X|^{2p} \mathbf{1}_{\{|\Delta_i X| \le v_n < |\Delta_i X'| \le v_n + v_n^{\alpha}\}} \right] 
\leq K \mathbb{E}_{i-1} \left[ |\Delta_i X'|^{2p} + |\Delta_i V|^{2p} \right] \mathbf{1}_{\{|\Delta_i X| \le v_n < |\Delta_i X'| \le v_n + v_n^{\alpha}\}} 
\leq \mathbb{E}_{i-1} |\Delta_i X'|^{2p} \mathbf{1}_{\{|\Delta_i X'| < v_n + v_n^{\alpha}\}} + O_P(\Delta_n v_n^{2p - r_0}).$$
(80)

Since  $\alpha \geq 1$ , by (74), we obtain

$$\mathbb{E}_{i-1}|\Delta_i X'|^{2p} \mathbf{1}_{\{|\Delta_i X'| < v_n + v_n^{\alpha}\}} = O_P(\sigma_{t_{i-1}}^{2p} \Delta_n^p) + O_P(\Delta_n v_n^{2p-Y}),$$

which combined with (79) implies

$$\mathbb{E}_{i-1}T_2 = O_P(\sigma_{t_{i-1}}^{2p} \Delta_n^p) + O_P(\Delta_n v_n^{2p-Y}) + O_P(\Delta_n v_n^{2p-\alpha r_0}).$$

Lastly for  $T_3$ , similarly we may write

$$T_3 = (\Delta_i X')^{2p} \mathbf{1}_{\{|\Delta_i X| > v_n, |\Delta_i X'| \le v_n - v_n^{\alpha}\}} + (\Delta_i X')^{2p} \mathbf{1}_{\{v_n - v_n^{\alpha} < |\Delta_i X'| \le v_n\}} = T_3' + T_3''.$$

Since  $|x'+v| > v_n$  and  $|x'| \le v_n - v_n^{\alpha}$  imply  $|v| > v_n^{\alpha}$ , the same arguments as for  $T_2', T_2''$  apply to  $T_3', T_3''$ . In conclusion, we have the following bound

$$\mathbb{E}_{i-1}T_1 + \mathbb{E}_{i-1}T_2 + \mathbb{E}_{i-1}T_3 \lesssim \Delta_n v_n^{2p-r_0} + \Delta_n v_n^{2p-\alpha r_0} + \sigma_{t_{i-1}}^{2p} \Delta_n^p + \Delta_n v_n^{2p-Y}. \tag{81}$$

To show (66), it suffices to show

$$\Delta_n v_n^{2p-r_0} + \Delta_n v_n^{2p-\alpha r_0} \lesssim \sigma_{t_{i-1}}^{2p} \Delta_n^p + \Delta_n v_n^{2p-Y}.$$

Indeed, notice that  $r \in (0, Y \wedge 1)$ . It follows that  $\Delta_n v_n^{2p-r_0} = o(\Delta_n v_n^{2p-Y})$ . Thus, if we take  $\alpha = \frac{Y}{2} + \frac{Y}{2r}$ , we obtain  $\alpha r < Y$ , and Since Y < 2, we have

$$\Delta_n v_n^{2p - \alpha r_0} = o(\Delta_n v_n^{2p - Y}),$$

which concludes the proof.

# A.2 Proofs of some technical identities in Theorem 1

In view of (17) and Lemma 5 in [3], in the following, we assume that the discontinuous part of X is comprosed only of Lévy-driven term of infinite variation and may omit the jumps of finite variation. Specifically, we may assume that X takes the form:

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \chi_s dJ_s.$$

Proof of (32). As in [3], the key idea is to discretize the integrals involved in  $Y_i$  as in (56). We treat each term in the expansion  $\Delta_i X^2 = (x_i + \varepsilon_i)^2 = x_i^2 + 2x_i\varepsilon_i + \varepsilon_i^2$  separately. For the third term  $\varepsilon_i^2$ , by Cauchy-Schwarz, we obtain

$$m_{n} \Big| \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \varepsilon_{i}^{2} \mathbf{1}_{\{|\Delta_{i} X| \leq v_{n}\}}(\Delta_{i} M) \Big) \Big|^{2}$$

$$\leq m_{n} \Big| \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1}(\varepsilon_{i}^{4})^{1/2} \mathbb{E}_{i-1}(\Delta_{i} M^{2})^{1/2} \Big|^{2}$$

$$\leq m_{n} \Big( \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1}(\varepsilon_{i}^{4}) \Big) \Big( \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1}(\Delta_{i} M^{2}) \Big)$$

$$\leq m_{n} \Delta_{n}^{-1} O_{P}(\Delta_{n}^{3}) O_{P} \Big( \frac{1}{m_{n} \Delta_{n}} \Big) = O_{P}(\Delta_{n}) \to 0, \tag{82}$$

where the second inequality follows from Cauchy-Schwarz concerning the summation and the last inequality is due to (57) and

$$\mathbb{E}\left(\sum_{i=1}^{n} K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1}(\Delta_i M^2)\right) \lesssim \frac{1}{m_n \Delta_n} \sum_{i=1}^{n} \mathbb{E}_{i-1}(\Delta_i M^2) = O_P\left(\frac{1}{m_n \Delta_n}\right),\tag{83}$$

which follows from the boundedness of  $K(\cdot)$  and the fact that M is square integrable.

For the term  $2x_i\varepsilon_i$ , when  $|\varepsilon_i| > v_n$ , since  $\Delta_n^{1/2}/v_n \ll \Delta_n^{1/2-\beta} \ll 1$ , we can use Markov's inequality to conclude that

$$P(|\varepsilon_i| > v_n) \le K\Delta_n^{1+(1/2-\beta)r},$$

for all r > Y according to (57). By taking large enough r and applying Hölder's inequality, for some  $q_1, q_2, q_3 > 1$  and  $\sum_{i=1}^{3} 1/q_i = 1/2$ , as  $\mathbb{E}_{i-1}[|x_i|^{q_1}]$  and  $\mathbb{E}_{i-1}[|\varepsilon_i|^{q_2}]$  is finite, we can show that

$$\sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |x_{i}\varepsilon_{i}| \mathbf{1}_{\{|\Delta_{i}X| \leq v_{n}\}} \mathbf{1}_{\{|\varepsilon_{i}| > v_{n}\}}(\Delta_{i}M) \right] 
\leq \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} [|x_{i}|^{q_{1}}]^{1/q_{1}} \mathbb{E}_{i-1} [|\varepsilon_{i}|^{q_{2}}]^{1/q_{2}} \mathbb{E}_{i-1}(\Delta_{i}M^{2})^{1/2} P_{i-1}(|\varepsilon_{i}| > v_{n})^{1/q_{3}} 
\leq \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} [|x_{i}|^{q_{1}}]^{1/q_{1}} \mathbb{E}_{i-1} [|\varepsilon_{i}|^{q_{2}}]^{1/q_{2}} \mathbb{E}_{i-1}(\Delta_{i}M^{2})^{1/2} \Delta_{n}^{1/q_{3} + (1/2 - \beta)r/q_{3}} 
= o_{P}(1).$$
(84)

When  $|\varepsilon_i| \leq v_n$ , note  $|\Delta_i X| = |x_i + \varepsilon_i| \leq v_n$  implies that  $|x_i| \leq 2v_n$ . Then by applying Cauchy-Schwarz twice, we obtain:

$$m_{n} \left| \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\varepsilon_{i}x_{i}| \mathbf{1}_{\{|x_{i}| \leq 2v_{n}\}} \mathbf{1}_{\{|\varepsilon_{i}| \leq v_{n}\}} |\Delta_{i}M| \right] \right|^{2}$$

$$\leq m_{n} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \varepsilon_{i}^{2} x_{i}^{2} \mathbf{1}_{\{|x_{i}| \leq 2v_{n}\}} \right] \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_{i}M)^{2} \right]$$

$$\leq m_{n} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) O_{P}(\Delta_{n}^{3/2}) O_{P}(\Delta_{n}) \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_{i}M)^{2} \right]$$

$$= m_{n} O_{P}(\Delta_{n}^{3/2}) O_{P} \left( \frac{1}{m_{n}\Delta_{n}} \right) = O_{P}(\Delta_{n}^{1/2}) \xrightarrow{P} 0, \tag{85}$$

where in the second inequality we used Proposition 1 and (57), and in the last line, we used (83). All that remains is the term  $x_i^2$ . Next, we want to show that

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ x_i^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}}(\Delta_i M) \right] \xrightarrow{P} 0.$$
 (86)

Firstly we start with replacing  $x_i$  with  $b_{t_{i-1}}\Delta_n$  in (86) and arbitrary M. By the boundness of  $b_t$ , K and Cauchy-Schwarz, it follows that

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ b_{t_{i-1}}^2 \Delta_n^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}}(\Delta_i M) \right] 
\lesssim \frac{\sqrt{m_n} \Delta_n^2}{m_n \Delta_n} \sum_{i=1}^n \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{1/2} 
\leq \frac{\sqrt{m_n} \Delta_n^2}{m_n \Delta_n^{3/2}} \left( \sum_{i=1}^n \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right] \right)^{1/2} \xrightarrow{P} 0,$$

where we used  $\sum_{i=1}^{n} \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{1/2} \leq \Delta_n^{-1/2} \left( \sum_{i=1}^{n} \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right] \right)^{1/2}$ . Besides, by Cauchy-Schwarz twice, we also have

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ b_{t_{i-1}} \Delta_n \hat{x}_i \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}}(\Delta_i M) \right] 
\lesssim \frac{\sqrt{m_n} \Delta_n}{m_n \Delta_n} \left( \sum_{i=1}^n \mathbb{E}_{i-1} \left[ \hat{x}_i^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} \right] \right)^{1/2} \left( \sum_{i=1}^n \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right] \right)^{1/2} \xrightarrow{P} 0,$$

Consequently, it is sufficient to prove (86) with  $x_i = \sigma_{t_{i-1}} \Delta_i W + \chi_{t_{i-1}} \Delta_i J$ .

Now we prove (86) when M = W and we start with the case when  $x_i$  is replaced with  $\sigma_{t_{i-1}}\Delta_i W$ . Similar to last inequality, by combining Cauchy-Schwarz and Proposition 1, we can write

$$\sqrt{m_{n}} \left| \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \mathbb{E}_{i-1} \left[ (\Delta_{i}W)^{2} \mathbf{1}_{\{|x_{i} + \varepsilon_{i}| \leq v_{n}\}} \Delta_{i}M \right] \right| \\
= \sqrt{m_{n}} \left| \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \mathbb{E}_{i-1} \left[ (\Delta_{i}W)^{3} \mathbf{1}_{\{|x_{i} + \varepsilon_{i}| > v_{n}\}} \right] \right| \\
\leq \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \mathbb{E}_{i-1} \left[ |\Delta_{i}W|^{3} |x_{i} + \varepsilon_{i}| \right] / v_{n} \\
\leq C \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_{i}W)^{6} \right]^{1/2} \mathbb{E}_{i-1} \left[ |x_{i} + \varepsilon_{i}|^{2} \right]^{1/2} / v_{n} \\
\leq C \sqrt{m_{n}} n \Delta_{n}^{3/2} \Delta_{n}^{1/2} / v_{n} \to 0, \tag{87}$$

where the last limit follows from  $m_n \Delta_n \to 0$  and  $\Delta_n^{1/2} \ll v_n$ .

When  $x_i$  and M are replaced with  $\chi_{t_{i-1}}\Delta_i J$  and M=W in (86), we need to show that

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} |\Delta_i W| \right] \xrightarrow{P} 0.$$
 (88)

Note that when  $|\Delta_i W| > v_n$  or  $|\varepsilon_i| > v_n$ , by Hölder's inequality, and that  $P(|\varepsilon_i| > v_n)$  and  $P(|\Delta_i W| > v_n)$  decay faster than any power of  $\Delta_n$  since  $\Delta_n^{1/2}/v_n \ll \Delta_n^{1/2-\beta}$ , by similar techniques as in (84), we obtain

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} |\Delta_i W| \left( \mathbf{1}_{\{|\Delta_i W| > v_n\}} + \mathbf{1}_{\{|\varepsilon_i| > v_n\}} \right) \right] = o_P(1). \tag{89}$$

So we only focus on the case when  $|\Delta_i W| \leq v_n$  and  $|\varepsilon_i| \leq v_n$  with M = W. Note that  $|x_i + \varepsilon_i| \leq v_n$  implies that  $|\Delta_i J| \leq C v_n$ , for some positive constant C that we assume for simplicity is 1. In that case, for (88), it suffices to show

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|\Delta_i J| \le v_n\}} |\Delta_i W| \right] \xrightarrow{P} 0.$$

By Hölder's inequality and Lemma 17 in [3], for any p, q > 1 such that 1/p + 1/q = 1,

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|\Delta_i J| \le v_n\}} |\Delta_i W| \right] 
\leq \sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^{2p} \mathbf{1}_{\{|\Delta_i J| \le v_n\}} \right]^{1/p} \mathbb{E}_{i-1} \left[ |\Delta_i W|^q \right]^{1/q} 
\lesssim \sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) (\Delta_n v_n^{2p-Y})^{1/p} \Delta_n^{1/2} \lesssim \sqrt{m_n} n \Delta_n^{1/p+1/2} v_n^{2-Y/p}.$$

So, making p close to 1 (and q large) and combining with (89), we get the convergence to 0 for (88) as  $m_n \Delta_n \to 0$  and  $\Delta_n^{1/2} \ll v_n$ . Combining (87) and (88), we establish (86) when M = W.

Next, we work on (86) with M = B and we start with the case where  $x_i$  is replaced with  $\Delta_i W$ . Notice that

$$\sqrt{m_n} \left| \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \sigma_{t_{i-1}}^2 \mathbb{E}_{i-1} \left[ (\Delta_i W)^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} \Delta_i B \right] \right| \\
\leq \sqrt{m_n} \left| \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \sigma_{t_{i-1}}^2 \mathbb{E}_{i-1} \left[ (\Delta_i W)^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \ge v_n\}} \Delta_i B \right] \right| \\
+ \sqrt{m_n} \left| \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \sigma_{t_{i-1}}^2 \mathbb{E}_{i-1} \left[ (\Delta_i W)^2 \Delta_i B \right] \right|.$$

For the first term, by Hölder's inequality and Proposition 1, we have

$$\sqrt{m_{n}} \left| \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \mathbb{E}_{i-1} \left[ (\Delta_{i} W)^{2} \mathbf{1}_{\{|x_{i} + \varepsilon_{i}| \leq v_{n}\}} \Delta_{i} B \right] \right| \\
\leq C \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_{i} W)^{8} \right]^{1/4} \mathbb{E}_{i-1} \left[ (\Delta_{i} B)^{4} \right]^{1/4} \mathbb{E}_{i-1} \left[ |x_{i} + \varepsilon_{i}|^{2} \right]^{1/2} / v_{n} \\
\lesssim \sqrt{m_{n}} n \Delta_{n}^{3/2} \Delta_{n}^{1/2} / v_{n}, \tag{90}$$

which vanishes under  $v_n \gg \Delta_n^{1/2}$  and  $m_n \Delta_n \to 0$ .

For the second term, given  $m_n \Delta_n \to 0$ , it suffices to show  $\sup_i \mathbb{E}_{i-1}[(\Delta_i W)^2 \Delta_i B] = o_P(\Delta_n^{3/2})$ . Denote  $\rho_s = d\langle W, B \rangle_s / ds$ .  $\rho_s$  is càdlàg and bounded on the interval  $[t_{i-1}, t_i]$ . By Itô's Lemma, Cauchy-Schwarz inequality, and Doob's inequality, we have

$$\sup_{i} \mathbb{E}_{i-1}[(\Delta_i W)^2 \Delta_i B] \lesssim \Delta_n^{3/2} \sup_{i} \sqrt{\mathbb{E}_{i-1}(\rho_{t_i} - \rho_{t_{i-1}})^2}.$$

We notice that  $\rho$  is right-continuous and uniformly bounded on [0,1] and thus,  $\mathbb{E}_{i-1}(\rho_{t_i}-\rho_{t_{i-1}})^2 \to 0$  Next, we check the case where  $x_i$  and M are replaced with  $\Delta_i J$  and B in (86), respectively. Similar to (88), we need to show that

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} |\Delta_i B| \right] \xrightarrow{P} 0.$$
 (91)

Following the similar technique as in (88) and (84), note that when  $|\Delta_i W| > v_n$  or  $|\varepsilon_i| > v_n$ , by Hölder's inequality, and that  $P(|\varepsilon_i| > v_n)$  and  $P(|\Delta_i W| > v_n)$  decay faster than any power of  $\Delta_n$  due to  $\Delta_n^{1/2}/v_n \ll \Delta_n^{1/2-\beta}$ , we obtain

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} |\Delta_i B| \left( \mathbf{1}_{\{|\Delta_i W| > v_n\}} + \mathbf{1}_{\{|\varepsilon_i| > v_n\}} \right) \right] = o_P(1).$$

So, when  $|\Delta_i W| \leq v_n$  and  $|\varepsilon_i| \leq v_n$ , note  $|x_i + \varepsilon_i| \leq v_n$  implies that  $|\Delta_i J| \leq C v_n$ , for some positive constant C. Then, for (91), it suffices to show

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|\Delta_i J| \le v_n\}} |\Delta_i B| \right] \xrightarrow{P} 0.$$

By Hölder's inequality and Lemma 17 in [3], for any p, q > 1 such that 1/p + 1/q = 1,

$$\sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|\Delta_i J| \le v_n\}} |\Delta_i B| \right] \\
\leq \sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^{2p} \mathbf{1}_{\{|\Delta_i J| \le v_n\}} \right]^{1/p} \mathbb{E}_{i-1} \left[ |\Delta_i B|^q \right]^{1/q} \\
\lesssim \sqrt{m_n} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) (\Delta_n v_n^{2p-Y})^{1/p} \Delta_n^{1/2} \lesssim \sqrt{m_n} n \Delta_n^{1/p+1/2} v_n^{2-Y/p}.$$

So, again, making p close to 1 (and q large), we get the convergence to 0, establishing (86) with M=B.

Lastly, we want to show (86) for any bounded martingale M which is orthogonal to W. We first consider the case when  $x_i$  is replaced with  $\Delta_i J$ . As before, we can deal with the cases  $|\Delta_i W| > v_n$  or  $\varepsilon_i > v_n$ , since  $P(|\varepsilon_i| > v_n)$  and  $P(|\Delta_i W| > v_n)$  decays fast. So, we assume that  $|\Delta_i W| \le v_n$  and  $|\varepsilon_i| \le v_n$ , so that, together with  $|x_i + \varepsilon_i| \le v_n$ , we have  $|\Delta_i J| \le 3v_n$ . Then, we proceed as follows:

$$\sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i}J|^{2} \mathbf{1}_{\{|\Delta_{i}J| \leq 3v_{n}\}} |\Delta_{i}M| \right] 
\leq \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i}J|^{4} \mathbf{1}_{\{|\Delta_{i}J| \leq 3v_{n}\}} \right]^{1/2} \mathbb{E}_{i-1} \left[ |\Delta_{i}M|^{2} \right]^{1/2} 
\leq \sqrt{m_{n}} \left( \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i}J|^{4} \mathbf{1}_{\{|\Delta_{i}J| \leq 3v_{n}\}} \right] \right)^{1/2} \left( \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i}M|^{2} \right] \right)^{1/2} 
\lesssim \sqrt{m_{n}} (n\Delta_{n}v_{n}^{4-Y})^{1/2} \left( \frac{1}{m_{n}^{1/2}\Delta_{n}^{1/2}} \right) = (\Delta_{n}^{-1}v_{n}^{4-Y})^{1/2} \ll 1,$$

which follows from  $v_n \ll \Delta_n^{1/(4-Y)}$ . For the cross-product  $\sigma_{t_{i-1}}\Delta_i W \Delta_i J$ , when expanding  $x_i^2$ , and again

assuming  $|\Delta_i J| \leq 3v_n$ , we have, for any p > 1:

$$\begin{split} &\sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i} J| |\Delta_{i} W | \mathbf{1}_{\{|\Delta_{i} J| \leq 3v_{n}\}} |\Delta_{i} M | \right] \\ &\leq \sqrt{m_{n}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i} J|^{2p} \mathbf{1}_{\{|\Delta_{i} J| \leq 3v_{n}\}} \right]^{\frac{1}{2p}} \mathbb{E}_{i-1} \left[ |\Delta_{i} W|^{\frac{2p}{p-1}} \right]^{\frac{p-1}{2p}} \mathbb{E}_{i-1} \left[ |\Delta_{i} M|^{2} \right]^{\frac{1}{2}} \\ &\leq \sqrt{m_{n}} \Delta_{n}^{1/2} \left( \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \left( \mathbb{E}_{i-1} \left[ |\Delta_{i} J|^{2p} \mathbf{1}_{\{|\Delta_{i} J| \leq 3v_{n}\}} \right] \right)^{1/p} \right)^{1/2} \\ &\times \left( \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i} M|^{2} \right] \right)^{1/2} \\ &\lesssim m_{n}^{1/2} \Delta_{n}^{1/2} (n \Delta_{n}^{1/p} v_{n}^{2-Y/p})^{1/2} \left( \frac{1}{m_{n}^{1/2} \Delta_{n}^{1/2}} \right). \end{split}$$

By taking p close to 1, the above vanishes given  $v_n \to 0$ . Finally, we consider the case when  $x_i$  is replaced with  $\sigma_{t_{i-1}}\Delta_i W$ . Note that

$$(\Delta_i W)^2 = 2 \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) dW_s + \Delta_n,$$

and since M is orthogonal to W,

$$\mathbb{E}_{i-1}\left[\left((\Delta_i W)^2 - \Delta_n\right) \Delta_i M\right] = 0,$$

which implies that  $\mathbb{E}_{i-1}\left[(\Delta_i W)^2 \Delta_i M\right] = 0$ . Thus, for any p, q > 0 such that 1/p + 1/q = 1/2, by orthogonality between M and W, we have

$$\sqrt{m_{n}} \left| \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \mathbb{E}_{i-1} \left[ (\Delta_{i} W)^{2} \mathbf{1}_{\{|x_{i} + \varepsilon_{i}| \leq v_{n}\}} \Delta_{i} M \right] \right| \\
= \sqrt{m_{n}} \left| - \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \sigma_{t_{i-1}}^{2} \mathbb{E}_{i-1} \left[ (\Delta_{i} W)^{2} \mathbf{1}_{\{|x_{i} + \varepsilon_{i}| \geq v_{n}\}} \Delta_{i} M \right] \right| \\
\leq \sqrt{m_{n}} v_{n}^{-1} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i} W|^{2} |x_{i} + \varepsilon_{i}| |\Delta_{i} M | \right] \\
\leq \sqrt{m_{n}} v_{n}^{-1} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i} W|^{2p} \right]^{1/p} \mathbb{E}_{i-1} \left[ |x_{i} + \varepsilon_{i}|^{q} \right]^{1/q} \mathbb{E}_{i-1} \left[ |\Delta_{i} M|^{2} \right]^{1/2} \\
\leq \sqrt{m_{n}} v_{n}^{-1} \left( \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i} W|^{2p} \right]^{2/p} \mathbb{E}_{i-1} \left[ |x_{i} + \varepsilon_{i}|^{q} \right]^{2/q} \right)^{1/2} \\
\times \left( \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_{i} M|^{2} \right] \right)^{1/2} \\
\lesssim \sqrt{m_{n}} v_{n}^{-1} \left( n \Delta_{n}^{2} \Delta_{n}^{2/q} \right)^{1/2} \times \left( \frac{1}{m_{n}^{1/2} \Delta_{n}^{1/2}} \right) \\
\leq (m_{n} \Delta_{n})^{1/2} \Delta_{n}^{1/q} v_{n}^{-1} \times \left( \frac{1}{m_{n}^{1/2} \Delta_{n}^{1/2}} \right) = \Delta_{n}^{1/q} v_{n}^{-1}, \tag{92}$$

which vanishes by taking q close enough to 2 and p large enough given that  $v_n \gg \Delta_n^{\beta}$  with  $\beta < 1/2$ .

Proof of (33). Recall that  $(\Delta_i X)^2 = x_i^2 + 2x_i\varepsilon_i + \varepsilon_i^2$ . We analyze  $\mathbb{E}_{i-1}((x_i + 2x_i\varepsilon_i + \varepsilon_i^2)\mathbf{1}_{\{|\Delta_i X| \leq v_n\}}\Delta_i B)$  term by term. For the third term, by Cauchy-Schwarz and (57), we have

$$\sup_{i} |\mathbb{E}_{i-1}(\varepsilon_{i}^{2} \mathbf{1}_{\{|\Delta_{i}X| \leq v_{n}\}} \Delta_{i}B)| \leq \sup_{i} (\mathbb{E}_{i-1}\varepsilon_{i}^{4} \mathbb{E}_{i-1} \Delta_{i}B^{2})^{1/2} = O_{P}(\Delta_{n}^{3/2})(\Delta_{n}^{1/2}) = o_{P}(\Delta_{n}^{3/2}). \tag{93}$$

For the second term, similar to (84), when  $|\varepsilon_i| > v_n$ , by Markov's inequality and (57), we can obtain that

$$P(|\varepsilon_i| > v_n) \le K\Delta_n^{1+(1/2-\beta)r}$$

for all r > 0. By Holder's inequality, for  $p_1, p_2, p_3 > 1$  such that  $\sum_{i=1}^{3} 1/p_i = 1$ , we may write

$$\sup_{i} |\mathbb{E}_{i-1}(\varepsilon_{i}x_{i}\mathbf{1}_{|\Delta_{i}X|\leq v_{n},|\varepsilon_{i}|>v_{n}}\Delta_{i}B)| \leq \sup_{i} \mathbb{E}_{i-1}|\varepsilon_{i}^{p_{1}}|^{1/p_{1}}\mathbb{E}_{i-1}|x_{i}^{p_{2}}\mathbf{1}_{\{|\Delta_{i}X|\leq v_{n}\}}|^{1/p_{2}}P(|\varepsilon_{i}|>v_{n})^{1/p_{3}}.$$

As all the quantities on the right hand side are bounded but  $P(|\varepsilon_i| > v_n)$  vanishes faster than any fixed power of  $\Delta_n$ . By taking large enough r, we can obtain

$$\sup_{i} |\mathbb{E}_{i-1}(|\varepsilon_i x_i| \mathbf{1}_{|\Delta_i X| \le v_n, |\varepsilon_i| > v_n} \Delta_i B)| = o_P(\Delta_n^{3/2}).$$

Then what remains for the second term is the case that  $|\varepsilon_i| \leq v_n$ . Since  $|\varepsilon_i| \leq v_n$  and  $|\Delta_i X| \leq v_n$  implies  $|x_i| \leq 2v_n$ . By Cauchy-Schwarz, we obtain

$$\sup_{i} \mathbb{E}_{i-1} \left[ |\varepsilon_{i} x_{i}| \mathbf{1}_{\{|x_{i}| \leq 2v_{n}\}} \mathbf{1}_{\{|\varepsilon_{i}| \leq v_{n}\}} |\Delta_{i} B| \right] \leq \sup_{i} \left( \mathbb{E}_{i-1} \left[ \varepsilon_{i}^{2} x_{i}^{2} \mathbf{1}_{\{|x_{i}| \leq 2v_{n}\}} \right] \mathbb{E}_{i-1} \left[ (\Delta_{i} B)^{2} \right] \right)^{1/2}$$

$$\leq O_{P}(\Delta_{n}^{3/4}) O_{P}(\Delta_{n}^{1/2}) \Delta_{n}^{1/2} = o_{P}(\Delta_{n}^{3/2}), \tag{94}$$

Then what remains is the first term involving  $x_i^2$ . We aim to show

$$\sup_{i} \mathbb{E}_{i-1} \left[ x_i^2 \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} \Delta_i B \right] = o_P(\Delta_n^{3/2}). \tag{95}$$

We start by replacing  $x_i^2$  with  $(\sigma_{t_{i-1}}\Delta_i W)^2$ . Note that when  $|\Delta_i W| > v_n$  or  $|\varepsilon_i| > v_n$ , by Hölder's inequality, and that  $P(|\varepsilon_i| > v_n)$  and  $P(|\Delta_i W| > v_n)$  decay faster than any power of  $\Delta_n$  since  $\Delta_n^{1/2}/v_n \ll \Delta_n^{1/2-\beta}$ , using similar techniques as in (89) and the boundness of  $\sigma$ , we obtain

$$\sup_{i} \mathbb{E}_{i-1} \left[ \sigma_{t_{i-1}} (\Delta_{i} W)^{2} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}} |\Delta_{i} B| \left( \mathbf{1}_{\{|\Delta_{i} W| > v_{n}\}} + \mathbf{1}_{\{|\varepsilon_{i}| > v_{n}\}} \right) \right] = o_{P}(\Delta_{n}^{3/2}). \tag{96}$$

And further notice that when  $|\Delta_i W| \leq v_n$  and  $\varepsilon_i \leq v_n$ ,  $|x_i + \varepsilon_i| \leq v_n$  implies  $|\Delta_i J| \leq C v_n$  for some positive constant C that we assume for simplicity is 1. Then equivalently, we may write

$$\sup_{i} \mathbb{E}_{i-1} \left[ \sigma_{t_{i-1}} (\Delta_{i} W)^{2} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}} \Delta_{i} B \mathbf{1}_{\{|\Delta_{i} W| \leq v_{n}\}} \mathbf{1}_{\{|\varepsilon_{i}| \leq v_{n}\}} \right]$$

$$= \sup_{i} \mathbb{E}_{i-1} \left[ \sigma_{t_{i-1}} (\Delta_{i} W)^{2} \mathbf{1}_{\{|\Delta_{i} J| \leq v_{n}\}} \Delta_{i} B \right]$$

$$= \sup_{i} \sigma_{t_{i-1}} \mathbb{E}_{i-1} \left[ (\Delta_{i} W)^{2} \Delta_{i} B \right] P(|\Delta_{i} J| \leq v_{n}). \tag{97}$$

Then it suffices to show  $\sup_i \mathbb{E}_{i-1}[(\Delta_i W)^2 \Delta_i B] = o_P(\Delta_n^{3/2})$ . Similar to how we handle the second term in (90), recall  $\rho_s = d\langle W, B \rangle_s/ds$ . By Itô's Lemma, Cauchy-Schwarz inequality, and Doob's inequality, we have

$$\sup_{i} \mathbb{E}_{i-1}[(\Delta_i W)^2 \Delta_i B] \lesssim \Delta_n^{3/2} \sup_{i} \sqrt{\mathbb{E}_{i-1}(\rho_{t_i} - \rho_{t_{i-1}})^2}.$$

Since  $\rho$  is right-continuous and uniformly bounded on [0,1],  $\mathbb{E}_{i-1}(\rho_{t_i}-\rho_{t_{i-1}})^2\to 0$  and (95) holds with  $x_i^2=\sigma_{t_{i-1}}^2(\Delta_i W)^2$ .

Now replace  $x_i^2$  with  $(\Delta_i J)^2$  in (95). We need to show

$$\sup_{i} \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|x_i + \varepsilon| \le v_n\}} \Delta_i B \right] = o_P(\Delta_n^{3/2}). \tag{98}$$

Note that when  $|\Delta_i W| > v_n$  or  $|\varepsilon_i| > v_n$ , by Hölder's inequality, and that  $P(|\varepsilon_i| > v_n)$  and  $P(|\Delta_i W| > v_n)$  decay faster than any power of  $\Delta_n$  since  $\Delta_n^{1/2}/v_n \ll \Delta_n^{1/2-\beta}$ , by similar techniques as in (89), we obtain

$$\sup_{i} \mathbb{E}_{i-1} \left[ (\Delta_{i} J)^{2} \mathbf{1}_{\{|x_{i}+\varepsilon_{i}| \leq v_{n}\}} |\Delta_{i} B| \left( \mathbf{1}_{\{|\Delta_{i} W| > v_{n}\}} + \mathbf{1}_{\{|\varepsilon_{i}| > v_{n}\}} \right) \right] = o_{P}(\Delta_{n}^{3/2}). \tag{99}$$

And further notice that when  $|\Delta_i W| \leq v_n$  and  $\varepsilon_i \leq v_n$ ,  $|x_i + \varepsilon_i| \leq v_n$  implies  $|\Delta_i J| \leq Cv_n$  for some positive constant C that we assume for simplicity is 1. Then by Holder inequality and Lemma 17 in [3], for any p, q > 1 such that 1/p + 1/q = 1, we obtain

$$\sup_{i} \mathbb{E}_{i-1} \left[ (\Delta_{i} J)^{2} \mathbf{1}_{\{|\Delta_{i} J| \leq v_{n}\}} |\Delta_{i} B| \right] \leq \sup_{i} \mathbb{E}_{i-1} \left[ (\Delta_{i} J)^{2p} \mathbf{1}_{\{|\Delta_{i} J| \leq v_{n}\}} \right]^{1/p} \mathbb{E}_{i-1} \left[ |\Delta_{i} W^{q}| \right]^{1/q}$$

$$\lesssim (\Delta_{n} v_{n}^{2p-Y})^{1/p} \Delta_{n}^{1/2}.$$
(100)

By making p close to 1 and q large, the above bound is of order  $o_P(\Delta_n^{3/2})$ .

What remains to show is to replace  $x_i^2$  with  $\Delta_i W \Delta_i J$  in (95). Similar to the case for  $(\Delta_i J)^2$ . By Hölder's inequality, and that  $P(|\varepsilon_i| > v_n)$  and  $P(|\Delta_i W| > v_n)$  decay faster than any power of  $\Delta_n$  since  $\Delta_n^{1/2}/v_n \ll \Delta_n^{1/2-\beta}$ , with similar techniques as in (89), we obtain

$$\sup_{i} \sigma_{t_{i-1}} \mathbb{E}_{i-1} \left[ |\Delta_i J| |\Delta_i W | \mathbf{1}_{\{|x_i + \varepsilon_i| \le v_n\}} |\Delta_i B| \left( \mathbf{1}_{\{|\Delta_i W| > v_n\}} + \mathbf{1}_{\{|\varepsilon_i| > v_n\}} \right) \right] = o_P(\Delta_n^{3/2}). \tag{101}$$

And further notice that when  $|\Delta_i W| \leq v_n$  and  $\varepsilon_i \leq v_n$ ,  $|x_i + \varepsilon_i| \leq v_n$  implies  $|\Delta_i J| \leq Cv_n$  for some positive constant C that we assume for simplicity is 1. Then by Holder inequality and Lemma 17 in [3], for any p, q > 1 such that 1/p + 1/q = 1/2, we obtain

$$\sup_{i} \sigma_{t_{i-1}} \mathbb{E}_{i-1} \left[ |\Delta_{i} J| |\Delta_{i} W | \mathbf{1}_{\{|\Delta_{i} J| \leq v_{n}\}} |\Delta_{i} B| \right] \leq \sup_{i} \sigma_{t_{i-1}} \mathbb{E}_{i-1} \left[ (\Delta_{i} J)^{2} \mathbf{1}_{\{|\Delta_{i} J| \leq v_{n}\}} \right]^{1/2} \mathbb{E}_{i-1} \left[ |\Delta_{i} W^{q}| \right]^{1/q} \mathbb{E}_{i-1} \left[ |\Delta_{i} B^{q}| \right]^{1/q} \\
\lesssim (\Delta_{n} v_{n}^{2-Y})^{1/2} \Delta_{n} = o_{P}(\Delta_{n}^{3/2}). \tag{102}$$

 $\Diamond$ 

Then we have proved (33) and  $Z_2 \perp Z_1$ .

# A.3 Proof of technical limits in Theorem 2

*Proof of* (37). It suffices to prove the following two terms:

$$A_1 := \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i X)^2 \mathbf{1}_{\{v_n \le |\Delta_i X| \le v_n (1 + v_n^{\eta})\}} |\Delta_i M| \right] = o_P(1),$$

and

$$A_2 := \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i X)^2 \mathbf{1}_{\{\zeta v_n (1 - v_n^{\eta}) \le |\Delta_i X| \le \zeta v_n\}} |\Delta_i M| \right] = o_P(1).$$

Firstly we work on  $A_1$ . By Cauchy-Schwarz and Proposition 1, we obtain:

$$|A_{1}| \leq \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_{i} X)^{4} \mathbf{1}_{\{v_{n} \leq |\Delta_{i} X| \leq v_{n}(1+v_{n}^{\eta})\}} \right]^{\frac{1}{2}} \mathbb{E}_{i-1} \left[ (\Delta_{i} M)^{2} \right]^{\frac{1}{2}}$$

$$\lesssim \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2}} \left( \Delta_{n} v_{n}^{4-Y} v_{n}^{\eta} \right)^{\frac{1}{2}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_{i} M)^{2} \right]^{\frac{1}{2}}.$$

$$(103)$$

Since M is square integrable, it yields  $\sum_{i=1}^{n} (\Delta_i M)^2 = O_P(1)$ . Then we find:

$$\sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_{i}M)^{2} \right]^{\frac{1}{2}} \leq \left( \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}^{2}(t_{i-1} - \tau) \right)^{1/2} \left( \sum_{i=1}^{n} (\Delta_{i}M)^{2} \right)^{1/2} \\
\lesssim \frac{1}{\sqrt{m_{n}\Delta_{n}}}.$$
(104)

Along with  $m_n \ll \Delta_n^{-1}$  and the boundedness of K, we conclude that

$$|A_1| \lesssim \frac{1}{\Delta_n^{1/2} v_n^{2-Y/2}} \Delta_n^{1/2} v_n^{2-Y/2} v_n^{\frac{\eta}{2}} = v_n^{\frac{\eta}{2}} = o_P(1).$$

A similar argument shows that  $A_2 = o_P(1)$ .

Proof of (38). For convenience, we rewrite (38) here:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i X)^2 f_n(\Delta_i X) \Delta_i M \right] = o_P(1). \tag{105}$$

The proof follows along the lines of the proof of an analogous identity in [3] (see Lemma 7 and Eq. (D.10) in the supplemental material of [4] posted online). More specifically, it was proved there in that

$$\frac{1}{v_n^{2-Y/2}} \sum_{i=1}^n \mathbb{E}_{i-1} \left[ (\Delta_i X)^2 f_n(\Delta_i X) \Delta_i M \right] = o_P(1). \tag{106}$$

 $\Diamond$ 

Note that (105) does not directly follow from (106) and the boundedness of K because  $m_n \Delta_n \to 0$ . So, we need to account for the localizing effect of the kernel, for which we applied some different bounds.

Using the notation introduced in (56), consider the decomposition

$$(\Delta_i X)^2 = (x_i + \varepsilon_i)^2 = x_i^2 + 2x_i \varepsilon_i + \varepsilon_i^2.$$

We begin by analyzing the second and third terms on the right-hand side by Cauchy-Schwarz. In particular, applying (57), we obtain that

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \varepsilon_i^2 f_n(\Delta_i X) |\Delta_i M| \right] 
\leq \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \varepsilon_i^4 \right]^{\frac{1}{2}} \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{\frac{1}{2}} 
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \Delta_n^{\frac{3}{2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{\frac{1}{2}} = o_P(1),$$

since, by (104),  $\sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{m_n \Delta_n}}$  and, recalling that  $v_n \gg \Delta_n^{1/2}$ ,

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}}\Delta_n^{3/2}\frac{1}{\sqrt{m_n}\Delta_n} = \Delta_n v_n^{Y/2-2} \ll \Delta_n^{Y/4} \ll 1.$$

For the term involving  $x_i \varepsilon_i$ , note that when  $|\varepsilon_i| > v_n$ , by Lemma 3 in [4], for some constant K > 0, we have

$$P_{i-1}(|\varepsilon_i| > v_n) \le \frac{\mathbb{E}_{i-1}[|\varepsilon_i|^r]}{v^r} \le K\Delta_n^{1+r(\frac{1}{2}-\beta)},\tag{107}$$

which holds for sufficiently large r due to (57) and  $v_n \gg \Delta_n^{\beta}$ . Similar to (84), using Hölder's inequality and Proposition 1, we conclude that:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\varepsilon_i x_i| \mathbf{1}_{\{|x_i| \le Cv_n\}} |\Delta_i M| \mathbf{1}_{\{|\varepsilon_i| > v_n\}} \right] = o_P(1).$$

When  $|\varepsilon_i| \leq v_n$ , observe that  $|\Delta_i X| \leq \zeta v_n$  implies  $|x_i| = |\Delta_i X - \varepsilon_i| \leq (1 + \zeta)v_n =: Cv_n$ . Then, by Proposition 3 in [4], (57), and (104), we get:

$$\begin{split} &\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\varepsilon_i x_i| \mathbf{1}_{\{|x_i| \le Cv_n\}} |\Delta_i M| \right] \\ & \le \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \varepsilon_i^4 \right]^{\frac{1}{4}} \mathbb{E}_{i-1} \left[ x_i^4 \mathbf{1}_{\{|x_i| \le Cv_n\}} \right]^{\frac{1}{4}} \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{\frac{1}{2}} \\ & \lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \Delta_n^{\frac{3}{4}} \Delta_n^{\frac{1}{2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{\frac{1}{2}} \\ & = \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \Delta_n^{\frac{3}{4}} \Delta_n^{\frac{1}{2}} \frac{1}{\sqrt{m_n \Delta_n}} = \Delta_n^{\frac{3}{4}} v_n^{Y/2-2} \ll \Delta_n^{\frac{Y-1}{4}} \ll 1, \end{split}$$

since  $\Delta_n^{1/2} \ll v_n$  and Y > 1.

Then we just need to focus on the last term involving  $x_i^2$ . We need to show that

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ x_i^2 f_n(\Delta_i X) \Delta_i M \right] = o_P(1). \tag{108}$$

Let's start by replacing  $x_i$  with  $b_{t_{i-1}}\Delta_n$  in (108) with arbitrary M. By the boundedness of K, b and Cauchy-Schwarz, we have

$$\frac{m_n^{1/2} \Delta_n^2}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ b_{t_{i-1}}^2 f_n(\Delta_i X) \Delta_i M \right] 
\lesssim \frac{m_n^{1/2} \Delta_n^2}{\Delta_n^{-1/2} v_n^{2-Y/2} m_n \Delta_n} \sum_{i=1}^n \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{1/2} 
\lesssim \frac{m_n^{1/2} \Delta_n^{3/2}}{v_n^{2-Y/2} m_n \Delta_n} \left( \sum_{i=1}^n \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right] \right)^{1/2} 
\lesssim \frac{\Delta_n^{1/2}}{v_n^{2-Y/2} \Delta_n^{-1/2} v_n^{Y/2}} \to 0$$
(109)

which the last line follows from  $m_n \gg \Delta_n^{-1} v_n^{\gamma}$  with  $\gamma < Y$  and  $v_n \gg \Delta_n^{1/2}$ . Besides, recall that  $\hat{x}_i = \sigma_{t_{i-1}} \Delta_i W + \chi_{t_{i-1}} \Delta_i J$  and by applying Cauchy-Schwarz twice and boundness of K, we obtain

$$\frac{m_n^{1/2}\Delta_n}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ b_{t_{i-1}}\hat{x}_i f_n(\Delta_i X) \Delta_i M \right] 
\lesssim \frac{m_n^{1/2}\Delta_n}{\Delta_n^{-1/2}v_n^{2-Y/2}m_n\Delta_n} \left( \sum_{i=1}^n \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right] \right)^{1/2} \left( \sum_{i=1}^n \mathbb{E}_{i-1} \left[ \hat{x}_i^2 f_n^2(\Delta_i X) \right] \right)^{1/2} 
\lesssim \frac{\Delta_n^{1/2}}{v_n^{2-Y/2}m_n^{1/2}} \lesssim \frac{\Delta_n^{1/2}}{v_n^{2-Y/2}\Delta_n^{-1/2}v_n^{Y/2}} \to 0,$$
(110)

where the last line follows from  $m_n \gg \Delta_n^{-1} v_n^{\gamma}$  with  $\gamma < Y$  and  $v_n \gg \Delta_n^{1/2}$ .

Now we just need to work on the case that  $x_i = \hat{x}_i = \sigma_{t_{i-1}} \Delta_i W + \Delta_i J$ . We first prove this for M = W.

Replacing  $x_i$  with  $\sigma_{t_{i-1}}\Delta_i W$  in (108), for any q, p > 1 such that  $\frac{1}{q} + \frac{1}{p} = 1$ , we obtain:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \left| \sum_{i=1}^n \sigma_{t_{i-1}}^2 K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i W)^2 f_n(\Delta_i X) \Delta_i M \right] \right| \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_i W|^3 \mathbf{1}_{\{|\Delta_i X| > v_n\}} \right] \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i W)^{3p} \right]^{\frac{1}{p}} P_{i-1} \left[ |\Delta_i X| > v_n \right]^{\frac{1}{q}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \Delta_n^{-1} \Delta_n^{\frac{3}{2}} (\Delta_n v_n^{-Y})^{\frac{1}{q}}, \tag{111}$$

where we used Lemma 2 in [4] and Hölder's inequality in the third line. Notice that

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}}\Delta_n^{1/2}(\Delta_n v_n^{-Y})^{\frac{1}{q}} \ll v_n^{-2+Y/2}(\Delta_n v_n^{-Y})^{\frac{1}{q}}.$$

Moreover, since  $\frac{3}{4+Y} > \frac{1}{2}$ , we have

$$\Delta_n^{3/2} v_n^{-\frac{4+Y}{2}} \ll 1 \quad \Leftrightarrow \quad \Delta_n^{3/(4+Y)} \ll v_n.$$

Then (111) vanishes by taking q close to 1.

Now, replacing  $x_i$  with  $\chi_{t_{i-1}}\Delta_i J$  with M=W, we need to show that:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 f_n(\Delta_i X) |\Delta_i W| \right] = o_P(1). \tag{112}$$

When  $|\Delta_i W| > v_n$  or  $|\varepsilon_i| > v_n$ , recall (107) that  $P(|\varepsilon_i| > v_n)$  and  $P(|\Delta_i W| > v_n)$  both decay faster than any power of  $\Delta_n$ . By Hölder's inequality, we can derive:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 f_n(\Delta_i X) |\Delta_i W| (\mathbf{1}_{\{|\Delta_i W| > v_n\}} + \mathbf{1}_{\{|\varepsilon_i| > v_n\}}) \right] = o_P(1).$$

Thus, when  $|\Delta_i W| \leq v_n$ ,  $|\varepsilon_i| \leq v_n$ , and  $|\Delta_i X| = |x_i + \varepsilon_i| \leq \zeta v_n$ , we have  $|\Delta_i J| \leq C v_n$  for some constant C. Therefore, for (112) to hold, it suffices to show that:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{\{|\Delta_i J| \le C v_n\}} |\Delta_i W| \right] \stackrel{P}{\to} 0.$$

Using Hölder's inequality and Lemma 17 in [3], for any p,q>1 such that  $\frac{1}{p}+\frac{1}{q}=1$ , we have:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^{2p} \mathbf{1}_{\{|\Delta_i J| \le 2v_n\}} \right]^{\frac{1}{p}} \mathbb{E}_{i-1} \left[ |\Delta_i W|^q \right]^{\frac{1}{q}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \left( \Delta_n v_n^{2p-Y} \right)^{\frac{1}{p}} \Delta_n^{1/2}.$$

Thus, by taking p very close to 1 (and q very large), we get convergence to 0 since

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \left( \Delta_n v_n^{2-Y} \right) \Delta_n^{1/2} \lesssim v_n^{Y/2 - 2} v_n^{2-Y} \Delta_n^{1/2} \to 0,$$

which is due to  $\Delta_n^{1/Y} \ll \Delta_n^{1/2} \ll v_n$ .

Now, we prove (108) for any bounded martingale M orthogonal to W. The cross-product term when expanding  $x_i^2$  can be analyzed similarly by assuming  $|\Delta_i J| \leq Cv_n$ . To this end, we need to show that:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_i J| |\Delta_i W | \mathbf{1}_{\{|\Delta_i J| \le C v_n\}} |\Delta_i M| \right] = o_P(1).$$

Using Hölder's inequality and (104), for any p, q > 2 such that 1/p + 1/q = 1/2 we conclude that:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_i J| |\Delta_i W | \mathbf{1}_{\{|\Delta_i J| \le C v_n\}} |\Delta_i M| \right] \\
\leq \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^p \mathbf{1}_{\{|\Delta_i J| \le C v_n\}} \right]^{\frac{1}{p}} \mathbb{E}_{i-1} \left[ (\Delta_i W)^q \right]^{\frac{1}{q}} \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right]^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} (\Delta_n v_n^{p-Y})^{1/p} \Delta_n^{1/2} \frac{1}{\sqrt{m_n \Delta_n}} = \frac{(\Delta_n v_n^{p-Y})^{1/p}}{v_n^{2-Y/2}}, \tag{113}$$

which vanishes by taking p close to 2 since

$$\frac{(\Delta_n v_n^{2-Y})^{1/2}}{v_n^{2-Y/2}} = \Delta_n^{1/2} v_n^{-1} \ll \Delta_n^{1/2-\beta} \ll 1.$$

Next we consider the case when  $x_i$  is replaced with  $\sigma_{t_{i-1}}\Delta_i W$ . Since  $f_n(x) \leq \mathbf{1}_{|\Delta_i X| \geq v_n}$ , it suffices to show

$$D_n := \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \sigma_{t_{i-1}} \mathbb{E}_{i-1} \left[ |\Delta_i W|^2 \mathbf{1}_{\{|\Delta_i J| \le Cv_n\}} |\Delta_i M| \right] = o_P(1).$$

Notice that  $|\Delta_i X| = |x_i + \varepsilon_i| \ge v_n$  implies that either  $|\Delta_i W| > v_n/3$ ,  $|\varepsilon_i| > v_n/3$  or  $|\Delta_i J| > v_n/3$ . The first two cases are straightforward as the tails decay faster than any power of  $\Delta_n$ . For the remaining case we utilize Hölder's inequality. For any p > 0, q > 0 such that 1/p + 1/q = 1/2, we obtain

$$\begin{split} D_n &\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \sigma_{t_{i-1}} \mathbb{E}_{i-1} \left[ |\Delta_i W|^{2p} \right]^{1/p} \\ &\qquad \times \mathbb{P}_{i-1} \left[ |\Delta_i J| > v_n/3 \right]^{1/q} \mathbb{E}_{i-1} \left[ |\Delta_i M|^2 \right]^{1/2} \\ &\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \Delta_n (\Delta_n v_n^{-Y})^{1/q} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ |\Delta_i M|^2 \right]^{1/2} \\ &\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \Delta_n (\Delta_n v_n^{-Y})^{1/q} \frac{1}{\sqrt{m_n \Delta_n}} = \frac{1}{\Delta_n^{1/2} v_n^{2-Y/2}} \Delta_n (\Delta_n v_n^{-Y})^{1/q}, \end{split}$$

where we used (104). Using  $v_n \gg \Delta_n^{\beta}$  and taking q close to 2 from above, the above equation vanishes since

$$\frac{1}{\Delta_n^{1/2} v_n^{2-Y/2}} \Delta_n (\Delta_n v_n^{-Y})^{1/2} = \Delta_n v_n^{-2} \ll 1.$$

Finally, we consider the case where  $x_i$  is replaced with  $\chi_{t_{i-1}}\Delta_i J$  in (108). It requires to show:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 f_n(\Delta_i X) \Delta_i M \right] = o_P(1). \tag{114}$$

Note that

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \left| f_n(\Delta_i X) - f_n(\chi_{t_{i-1}} \Delta_i J) \right| \left| \Delta_i M \right| \right] = o_P(1), \tag{115}$$

which is shown at the end of the appendix. Hence it suffices to prove:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 f_n(\chi_{t_{i-1}} \Delta_i J) \Delta_i M \right] = o_P(1). \tag{116}$$

In what follows, we assume  $\chi = 1$ ; the general case follows from analogous arguments. Hereafter, we assume that the following Lévy–Itô decomposition for J:

$$J_t = \int_0^t \int x \mathbf{1}_{\{|x| \le 1\}} \bar{N}(ds, dx) + \int_0^t \int x \mathbf{1}_{\{|x| > 1\}} N(ds, dx),$$

where  $\bar{N}(ds, dx) = N(ds, dx) - \nu(dx)ds$  is the compensated jump measure of J. Additionally, similar to expression (80) in [1], M admits the decomposition:

$$M_t = M_t' + \int_0^t \int \delta(s, x) \bar{N}(ds, dx),$$

where M' is a martingale orthogonal to both W and N, and  $\delta$  is a bounded function satisfying:

$$E\left[\int_{t}^{u} \int \delta^{2}(s,x)\nu(dx)ds \middle| \mathcal{F}_{t}\right] \leq E\left[(M_{u}-M_{t})^{2}\middle| \mathcal{F}_{t}\right] < \infty.$$

For future reference, define  $N^i(ds, dx) := N(t_i + ds, dx), \ \delta_i(s, x) := \delta(t_i + s, x), \ \text{and}$ 

$$p_n(x) = x^2 f_n(x), \quad t_n(y,x) = p_n(x+y) - p_n(y), \quad w_n(y,x) = p_n(x+y) - p_n(y) - xp'_n(y) \mathbf{1}_{\{|x| < 1\}}.$$

Applying Itô's formula to  $Y_u^{i-1} := L_{t_{i-1}+u} - L_{t_{i-1}}$ , we obtain:

$$p_n(Y_u^{i-1}) = \int_0^u p_n'(Y_{s-}^{i-1})dY_s^{i-1} + \int_0^u \int \left(p_n(Y_{s-}^{i-1} + x) - p_n(Y_{s-}^{i-1}) - p_n'(Y_{s-}^{i-1})x\right)N^{i-1}(ds, dx)$$

$$= \int_0^u \int_{|x| \le 1} p_n'(Y_{s-}^{i-1})x\bar{N}^{i-1}(ds, dx) + \int_0^u \int w_n(Y_{s-}^{i-1}, x)N^{i-1}(ds, dx). \tag{117}$$

Now, let  $Z_u^{i-1} := M_{t_{i-1}+u} - M_{t_{i-1}}$ . Applying Itô's formula to  $p_n(Y_u^{i-1})Z_u^{i-1}$ , we get:

$$p_n(Y_u^{i-1})Z_u^{i-1} = \int_0^u p_n(Y_s^{i-1})dZ_s^{i-1} + \int_0^u Z_s^{i-1}d\left(p_n(Y_s^{i-1})\right) + \sum_{s \le u} \Delta(p_n(Y_s^{i-1}))\Delta Z_s^{i-1}$$

$$= \operatorname{Mrtg} + \int_0^u \int \delta_i(s,x)t_n(Y_s^{i-1},x)\nu(dx)ds + \int_0^u \int w_n(Y_s^{i-1},x)Z_s^{i-1}\nu(dx)ds, \tag{118}$$

where Mrtg denotes the martingale part generated from the Itô's formula. Fixing  $u = \Delta_n$  in (118) and taking expectations, we obtain:

$$\mathbb{E}_{i-1}\left[p_n(\Delta_i J)\Delta_i M\right] = \mathbb{E}_{i-1}\left[\int_0^{\Delta_n} \int \delta_i(s,x) t_n(Y_s,x) \nu(dx) ds\right] + \mathbb{E}_{i-1}\left[\int_0^{\Delta_n} \int w_n(Y_s,x) Z_s \nu(dx) ds\right]. \tag{119}$$

In the above equation, we omit the superscript i-1 in Y and Z for simplicity. For the second term in (119), we use the following bound (see (D.22) and (D.23) in [4]):

$$\mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |w_n(Y_s, x)| |Z_s| \nu(dx) ds \right] \leq K(v_n^{2-Y(1+\eta)} \vee v_n) \Delta_n \mathbb{E}_{i-1}(|\Delta_i M|) + Kv_n^{2-Y(1+\eta)} \Delta_n^2 v_n^{\eta-Y}.$$

Combining the above equation with the second term in (119) and plugging it back in (116), we need to show the negligibility of the two terms below:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \left( v_n^{2-Y(1+\eta)} \vee v_n \right) \Delta_n \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} (|\Delta_i M|) 
+ \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} v_n^{2-Y(1+\eta)} \Delta_n^2 v_n^{\eta-Y} \Delta_n^{-1}.$$
(120)

The first term converges to 0 since  $v_n^{Y/2-2}v_n^{2-Y(1+\eta)}\Delta_n^{1/2}\ll 1$  and  $v_n^{Y/2-2}v_n\Delta_n^{1/2}\ll 1$  with Y>1 if  $\eta$  is small enough and

$$\Delta_n^{1/2} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} (|\Delta_i M|) 
\leq \left( \Delta_n \sum_{i=1}^n K_{m_n \Delta_n}^2 (t_{i-1} - \tau) \right)^{1/2} \left( \sum_{i=1}^n \mathbb{E}_{i-1} (|\Delta_i M|^2) \right)^{1/2} = O_P(m_n^{-1/2} \Delta_n^{-1/2}).$$

The second term in (120) if of order  $m_n^{1/2}v_n^{-3Y/2+(1-Y)\eta}\Delta_n^{3/2}$ , which vanishes by taking  $\eta$  small enough due to the condition  $m_n \ll \Delta_n^{-5}v_n^{8+Y}$ , since  $\Delta_n^{-5}v_n^{8+Y} \ll v_n^{3Y}\Delta_n^{-3}$  with  $v_n \ll \Delta_n^{1/(4-Y)}$ . Thus, the two terms in (120) converge to zero, and we conclude that:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1}\left(\int_0^{\Delta_n} \int |w_n(Y_s, x)| |Z_s| \nu(dx) ds\right) = o_P(1).$$

It remains to show that the contribution of the first term in (119) is negligible:

$$F_n := \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int \delta_{i-1}(s, x) t_n(Y_s, x) \nu(dx) ds \right] = o_P(1).$$

As in (D.25) in [4], we separately consider the cases where  $|Y_s| \le v_n$ ,  $|Y_s| > \zeta v_n$ ,  $v_n(1+v_n^{\eta}) < |Y_s| < \zeta v_n(1-v_n^{\eta})$ ,  $v_n \le |Y_s| \le v_n(1+v_n^{\eta})$ , and  $\zeta v_n \le |Y_s| \le \zeta v_n(1+v_n^{\eta})$ .

First, suppose that  $|Y_s| \le v_n$ , so that  $f_n(Y_s) = 0$ , and thus  $t_n(Y_s, x) = (Y_s + x)^2 f_n(Y_s + x)$ , with  $|x| \ge \frac{1}{3} v_n^{1+\eta}$  (otherwise  $f_n(Y_s + x) = 0$ ). Notice that  $|t_n(Y_s, x)| \le Y_s^2 + 2|xY_s| + x^2$ . For  $Y_s^2$ , using Cauchy-Schwarz and the bound

$$\mathbb{E}_{i-1}\left[Y_s^{2k} 1_{\{|Y_s| \le v_n\}}\right] \le K\Delta_n v_n^{2k-Y},\tag{121}$$

valid for all  $0 < s < \Delta_n$  (see Lemma 12 in [4]),

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)| Y_s^2 \mathbf{1}_{\{|Y_s| \le v_n\}} \mathbf{1}_{\{|x| \ge \frac{1}{3}v_n^{1+\eta}\}} \nu(dx) ds \right) \\
\leq \frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( \int_0^{\Delta_n} \int Y_s^4 \mathbf{1}_{\{|Y_s| \le v_n\}} \mathbf{1}_{\{|x| \ge \frac{1}{3}v_n^{1+\eta}\}} \nu(dx) ds \right)^{\frac{1}{2}} \\
\times \mathbb{E}_{i-1} \left( \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)|^2 \nu(dx) ds \right)^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} (v_n^{-Y(1+\eta)} \Delta_n^2 v_n^{4-Y})^{1/2} \\
\times \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( \int_{t_{i-1}}^{t_i} \int |\delta(u, x)|^2 \nu(dx) du \right)^{\frac{1}{2}} \\
\lesssim m_n^{1/2} \Delta_n^{3/2} v_n^{-Y(1+\eta)/2} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( (\Delta_i M)^2 \right)^{\frac{1}{2}} \\
\lesssim \Delta_n^{1/2} v_n^{-Y(1+\eta)/2}, \tag{122}$$

where the last line follows from  $\sum_{i=1}^{n} K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( (\Delta_i M)^2 \right)^{\frac{1}{2}} = O_P(m_n^{-1/2} \Delta_n^{-1})$  (see (104)) and  $\Delta_n m_n \ll 1$ . The bound (122) tends to zero under our assumption  $v_n \gg \Delta_n^{\beta}$  with small enough  $\eta$ . Next, we focus on  $2Y_s x$ . Using again (121) and Cauchy-Schwarz,

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)| |Y_s x| 1_{\{|Y_s| \le v_n\}} 1_{\{|x| \ge \frac{1}{3}v_n^{1+\eta}\}} \nu(dx) ds \right] \\
\leq \frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)|^2 \nu(dx) ds \right)^{\frac{1}{2}} \\
\times \mathbb{E}_{i-1} \left( \int_0^{\Delta_n} \int Y_s^2 x^2 1_{\{|Y_s| \le v_n\}} 1_{\{|x| \ge \frac{1}{3}v_n^{1+\eta}\}} \nu(dx) ds \right)^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \left( \Delta_n^2 v_n^{2-Y} \right)^{\frac{1}{2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_{t_{i-1}}^{t_i} \int |\delta(u, x)|^2 \nu(dx) du \right]^{\frac{1}{2}} \\
= \frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^2} \Delta_n \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \left( \mathbb{E}_{i-1} \left[ (\Delta_i M)^2 \right] \right)^{\frac{1}{2}} = \Delta_n^{\frac{1}{2}} v_n^{-1} O_P(1), \tag{123}$$

which again is  $o_P(1)$  since  $v_n \gg \Delta_n^{\frac{1}{2}}$ .

It remains to analyze the term corresponding to  $x^2$ . First, since  $|Y_s + x| \leq \zeta v_n$  when  $f_n(Y_s + x) > 0$ , we

have  $|x| \leq (1+\zeta)v_n =: Cv_n$  due to our assumption  $|Y_s| \leq v_n$ . Then, for any sequence  $\alpha_n > 0$  such that  $\alpha_n \to 0$ ,

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int_{|\delta_{i-1}(s,x)| \leq \alpha_n} |\delta_{i-1}(s,x)| x^2 \mathbf{1}_{\{|x| \leq Cv_n\}} \nu(dx) ds \right] \\
\leq \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int_{|x| \leq Cv_n} x^4 \nu(dx) ds \right]^{\frac{1}{2}} \\
\times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int_{|\delta_{i-1}(s,x)| \leq \alpha_n} |\delta_{i-1}(s,x)|^2 \nu(dx) ds \right]^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \left( \Delta_n v_n^{4-Y} \right)^{\frac{1}{2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_{t_{i-1}}^{t_i} \int_{|\delta(s,x)| \leq \alpha_n} |\delta(s,x)|^2 \nu(dx) ds \right]^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \left( \Delta_n v_n^{4-Y} \right)^{\frac{1}{2}} \left[ \sum_{i=1}^n K_{m_n\Delta_n}^2(t_{i-1} - \tau) \right]^{1/2} \left[ \sum_{i=1}^n \mathbb{E}_{i-1} \left[ \int_{t_{i-1}}^{t_i} \int_{|\delta(s,x)| \leq \alpha_n} |\delta(s,x)|^2 \nu(dx) ds \right] \right]^{1/2} \\
\lesssim m_n^{1/2} \Delta_n [m_n^{-1} \Delta_n^{-2}]^{1/2} \left[ \sum_{i=1}^n \mathbb{E}_{i-1} \left[ \int_{t_{i-1}}^{t_i} \int_{|\delta(s,x)| \leq \alpha_n} |\delta(s,x)|^2 \nu(dx) ds \right] \right]^{1/2} = o_P(1), \tag{124}$$

since as  $\alpha_n \to 0$ ,

$$\mathbb{E}\left[\sum_{i=1}^n \mathbb{E}_{i-1}\left(\int_{t_{i-1}}^{t_i} \int_{|\delta(s,x)| \leq \alpha_n} |\delta(s,x)|^2 \nu(dx) ds\right)\right] = \mathbb{E}\left[\int_0^T \int_{|\delta(s,x)| \leq \alpha_n} |\delta(s,x)|^2 \nu(dx) ds\right] \to 0.$$

On the complementary region, we proceed as follows:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int_{|\delta_{i-1}(s,x)| > \alpha_n} |\delta_{i-1}(s,x)| x^2 \mathbf{1}_{\{|x| \le Cv_n\}} \nu(dx) ds \right] \\
\leq \frac{C}{\alpha_n} \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} v_n^2 \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int_{|\delta_{i-1}(s,x)| > \alpha_n} |\delta_{i-1}(s,x)|^2 \nu(dx) ds \right] \\
\leq \frac{C}{\alpha_n} \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} v_n^2 \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_{t_{i-1}}^{t_i} \int |\delta(s,x)|^2 \nu(dx) ds \right] \\
\lesssim \frac{C}{\alpha_n} \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} v_n^2 \frac{1}{m_n \Delta_n} = \frac{v_n^{\frac{Y}{2}}}{\alpha_n m_n^{1/2} \Delta_n^{1/2}}, \tag{125}$$

which is  $o_P(1)$  under our assumption  $m_n \Delta_n \gg v_n^{\gamma}$  with  $\gamma < Y$ , since  $\frac{v_n^{\frac{\gamma}{2}}}{\alpha_n m_n^{1/2} \Delta_n^{1/2}} \ll \frac{v_n^{\frac{\gamma-\gamma}{2}}}{\alpha_n}$  and we can take  $\alpha_n \to 0$  such that  $\frac{v_n^{(Y-\gamma)/2}}{\alpha_n} \to 0$ . The proof for the negligibility of  $F_n$  in the case  $|Y_s| \le v_n$  is complete. For the case of  $|Y_s| \ge \zeta v_n$ , we have  $f_n(Y_s) = 0$  and  $t_n(Y_s, x) = (Y_s + x)^2 f_n(Y_s + x) \le (\zeta v_n)^2$ . We observe

that when  $f_n(Y_s + x) > 0$ , we have  $|Y_s + x| \le \zeta v_n (1 - \frac{1}{3}v_n^{\eta})$ , implying  $|x| > \zeta v_n^{1+\eta}/3$ . We then can write that

$$\begin{split} \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} & \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \\ & \times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s,x)| (Y_s + x)^2 f_n(Y_s + x) \mathbf{1}_{\{|Y_s| \ge \zeta v_n\}} \mathbf{1}_{\{|x| \ge C v_n^{1+\eta}\}} \nu(dx) ds \right] \\ & \lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} v_n^2 \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s,x)|^2 \nu(dx) ds \right]^{\frac{1}{2}} \\ & \times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int \mathbf{1}_{\{|Y_s| \ge \zeta v_n\}} \mathbf{1}_{\{|x| \ge C v_n^{1+\eta}\}} \nu(dx) ds \right]^{\frac{1}{2}} \\ & \lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} v_n^2 \left( \Delta_n^2 v_n^{-Y - (1+\eta)Y} \right)^{\frac{1}{2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_{t_{i-1}}^{t_i} \int |\delta(u,x)|^2 \nu(dx) du \right]^{\frac{1}{2}} \stackrel{(104)}{\lesssim} \Delta_n^{\frac{1}{2}} v_n^{-\frac{(1+\eta)Y}{2}}, \end{split}$$

which vanishes by taking small  $\eta > 0$  due to  $v_n \gg \Delta_n^{\beta}$ . We have then proved the negligibility of  $F_n$  in the case  $|Y_s| \geq \zeta v_n$ .

For the case when  $v_n(1+v_n^{\eta}) < |Y_s| < \zeta v_n(1-v_n^{\eta})$ , when  $|x| \ge \frac{1}{3}v_n^{1+\eta}$ , we can use the bound  $|t_n(Y_s,x)| \le Cv_n^2$  and follow a similar proof as in (E.27). When  $|x| \le \frac{1}{3}v_n^{1+\eta}$ ,  $f_n(x+Y_s) = 1 = f(Y_s)$ , implying  $t_n(Y_s,x) = x^2 + 2xY_s$ . We consider the contribution of each term separately below. For the case of  $x^2$ , we have:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \\
\times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)| x^2 1_{v_n(1+v_n^{\eta}) < |Y_s| < \zeta v_n(1-v_n^{\eta})} 1_{\{|x| \le Cv_n^{1+\eta}\}} \nu(dx) ds \right] \\
\leq \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \\
\times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)|^2 \nu(dx) ds \right]^{\frac{1}{2}} E_{i-1} \left[ \int_0^{\Delta_n} \int x^4 1_{\{|Y_s| \ge v_n\}} \nu(dx) ds \right]^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \left( v_n^{(4-Y)(1+\eta)} \Delta_n^2 v_n^{-Y} \right)^{\frac{1}{2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \left( E_{i-1} [(\Delta_i M)^2] \right)^{\frac{1}{2}} \lesssim v_n^{\frac{\eta(4-Y)-Y}{2}} \Delta_n^{\frac{1}{2}} \ll 1,$$

for all  $\eta > 0$ , where the last line follows from (104). For the term corresponding to  $2|xY_s|$ , since  $|Y_s| < \zeta v_n (1-v_n^{\eta})$ ,

for some C,

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \\
\times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)| |xY_s| 1_{v_n(1+v_n^{\eta}) < |Y_s| < \zeta v_n(1-v_n^{\eta})} 1_{\{|x| \le C v_n^{1+\eta}\}} \nu(dx) ds \right] \\
\leq \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)|^2 \nu(dx) ds \right]^{\frac{1}{2}} \\
\times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int x^2 Y_s^2 1_{\{|Y_s| \le \zeta v_n\}} \nu(dx) ds \right]^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \left( v_n^{(2-Y)(1+\eta)} \Delta_n^2 v_n^{2-Y} \right)^{\frac{1}{2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \left( \mathbb{E}_{i-1} [(\Delta_i M)^2] \right)^{\frac{1}{2}} \\
= v_n^{-\frac{Y}{2} + \frac{\eta(2-Y)}{2}} \Delta_n^{\frac{1}{2}} O_P(1) = o_P(1), \tag{126}$$

where again the last line follows from (104) for all  $\eta > 0$ . This proves the negligibility of  $F_n$  for the case  $v_n(1+v_n^{\eta}) < |Y_s| < \zeta v_n(1-v_n^{\eta})$ .

The only cases left are when  $v_n \leq |Y_s| \leq v_n(1+v_n^{\eta})$  and  $\zeta v_n \leq |Y_s| \leq \zeta v_n(1+v_n^{\eta})$ . It suffices to consider the first case as the other case is proved similarly. If  $|x| \geq v_n^{1+\eta}$ , consider the same argument as in (D.26) in [4] and suppose that  $|x| \leq v_n^{1+\eta}$ . Note that  $|p'_n(y)| \leq 2|yf_n(y)| + |y^2f'_n(y)| \leq Cv_n^{1-\eta}$  when  $|y| \leq Cv_n$ . Consequently,  $|t_n(Y_s,x)| \leq |p'_n(Y_s,x)||x| \leq Cv_n^{1-\eta}|x|$ . Thus,  $F_n$  is bounded by

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}} v_n^{1-\eta} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)| |x| 1_{\{v_n \le |Y_s| \le v_n (1+v_n^{\eta})\}} 1_{\{|x| \le v_n^{1+\eta}\}} \nu(dx) ds \right] \\
\le \frac{m_n^{1/2}}{\Delta_n^{-1/2}} v_n^{1-\eta} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int |\delta_{i-1}(s, x)|^2 \nu(dx) ds \right]^{\frac{1}{2}} \\
\times \mathbb{E}_{i-1} \left[ \int_0^{\Delta_n} \int_{|x| \le C v_n^{1+\eta}} x^2 1_{\{v_n \le |Y_s| \le v_n (1+v_n^{\eta})\}} \nu(dx) ds \right]^{\frac{1}{2}} \\
\lesssim \frac{m_n^{1/2}}{\Delta_n^{-1/2}} v_n^{1-\eta} \left( v_n^{(2-Y)(1+\eta)} \Delta_n^2 v_n^{\eta-Y} \right)^{\frac{1}{2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \left( \mathbb{E}_{i-1} [(\Delta_i M)^2] \right)^{\frac{1}{2}} \\
= v_n^{-\frac{Y}{2} + \frac{\eta(1-Y)}{2}} \Delta_n^{\frac{1}{2}} O_P(1), \tag{127}$$

where we used (D.24) in [4] and (104). By taking  $\eta$  small enough, it follows that the above expression is  $o_P(1)$ . This concludes the proof.

Proof of (115). We start by noting that

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n} (t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{|\chi_{t_{i-1}} \Delta_i J| > 4\zeta v_n} \left| f_n(\Delta_i X) - f_n(\chi_{t_{i-1}} \Delta_i J) \right| |\Delta_i M| \right] = o_P(1).$$
(128)

Indeed, note that

$$\mathbf{1}_{|\chi_{t_{i-1}}\Delta_{i}J|>4\zeta v_{n}} \left| f_{n}(\Delta_{i}X) - f_{n}(\chi_{t_{i-1}}\Delta_{i}J) \right| \leq \mathbf{1}_{|\chi_{t_{i-1}}\Delta_{i}J|>4\zeta v_{n}} f_{n}(\Delta_{i}X) \leq \mathbf{1}_{|b_{t_{i-1}}\Delta_{n}+\sigma_{t_{i-1}}\Delta_{i}W+\varepsilon_{i}|>3\zeta v_{n}}, \quad (129)$$

and, thus, by Hölder's and Markov's inequalities along with  $v_n \gg \Delta_n^{\beta}$  and (57), for large enough r, we have:

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2}v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n\Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 |\Delta_i M| (\mathbf{1}_{|b_{t_{i-1}}\Delta_n| > \zeta v_n} + \mathbf{1}_{|\sigma_{t_{i-1}\Delta_i W| > \zeta v_n}} + \mathbf{1}_{|\varepsilon_i| > \zeta v_n}) \right] = o_P(1),$$

which implies (128) and (129). Consequently, it suffices to prove that

$$\frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{|\chi_{t_{i-1}} \Delta_i J| \le 4\zeta v_n} \left| f_n(\Delta_i X) - f_n(\chi_{t_{i-1}} \Delta_i J) \right| |\Delta_i M| \right] = o_P(1).$$
(130)

First, we observe that

$$|f_n(x+y) - f_n(y)| \le \mathbf{1}_{|x| \ge v_n^{1+\eta}/3} + \mathbf{1}_{v_n \le |y| \le (1+v_n^{\eta}) \text{ or } \zeta v_n (1-v_n^{\eta}) \le |y| \le \zeta v_n} \frac{|x|}{v_n^{1+\eta}},\tag{131}$$

where  $y = \chi_{t_{i-1}} \Delta_i J$  and  $x = \Delta_i X - \chi_{t_{i-1}} \Delta_i J = b_{t_{i-1}} \Delta_n + \sigma_{t_{i-1}} \Delta_i W + \varepsilon_i =: z_{i,1} + z_{i,2} + z_{i,3}$ . We start with the first term in (131). We proceed to verify that

$$B_l := \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{|\chi_{t_{i-1}} \Delta_i J| \le 4\zeta v_n} \mathbf{1}_{|z_{i,l}| \ge v_n^{1+\eta}/9} |\Delta_i M| \right] = o_P(1),$$

for l = 1, 2, 3. Take the case l = 2 for instance. Assuming  $\chi = \sigma = 1$  for simplicity, by Holder's inequality we obtain

$$B_2 \leq \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( \Delta_i J^8 \mathbf{1}_{|\Delta_i J| \leq 4\zeta v_n} \right)^{1/4} P_{i-1} \left( |\Delta_i W| \geq v_n^{1+\eta}/9 \right)^{1/4} \mathbb{E}_{i-1} \left( |\Delta_i M|^2 \right)^{1/2}.$$

Notice that for any  $\alpha > 0$ , for some constant C, we have  $P_{i-1}(|\Delta_i W| \geq v_n^{1+\eta}/9) \lesssim \mathbb{E}(|\Delta_i W|^{\alpha})/v_n^{\alpha(1+\eta)} \leq \Delta_n^{\alpha}/v_n^{\alpha(1+\eta)}$ . Therefore, along with Proposition 1 and (104), we have

$$B_{2} \lesssim \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2}} \Delta_{n}^{1/4} v_{n}^{(8-Y)/4} \Delta_{n}^{\alpha/4} / v_{n}^{\alpha(1+\eta)/4} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}} (t_{i-1} - \tau) \mathbb{E}_{i-1} (|\Delta_{i} M|^{2})^{1/2}$$

$$\lesssim \frac{m_{n}^{1/2} v_{n}^{Y/4}}{\Delta_{n}^{1/4} v_{n}^{2-Y/2}} \Delta_{n}^{\alpha/4} / v_{n}^{\alpha(1+\eta)/4}.$$

Since  $v_n \gg \Delta_n^{\beta}$ , for small enough  $\eta > 0$ ,  $\frac{1}{2} - \beta(1+\eta) > 0$ . Then by taking  $\alpha$  large enough we obtain  $B_2 = o_P(1)$ .  $B_1$  can be shown  $o_P(1)$  using the similar argument. For  $B_3$ , by (57)

$$B_{3} \leq \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2}v_{n}^{2-Y/2}} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left(\Delta_{i} J^{8} \mathbf{1}_{|\Delta_{i}J| \leq 4\zeta v_{n}}\right)^{1/4} \mathbb{E}\left(|\varepsilon_{i}|^{4\alpha}\right)^{1/4} / v_{n}^{\alpha(1+\eta)/4} \mathbb{E}_{i-1} \left(|\Delta_{i}M|^{2}\right)^{1/2}$$

$$\leq \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2}v_{n}^{2-Y/2}} \Delta_{n}^{1/4} v_{n}^{(8-Y)/4} \Delta_{n}^{(1+\alpha/2)/4} / v_{n}^{\alpha(1+\eta)/4} \sum_{i=1}^{n} K_{m_{n}\Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left(|\Delta_{i}M|^{2}\right)^{1/2},$$

which tends to zero in probability by taking  $\eta$  small enough and  $\alpha$  large enough similar to  $B_2$ . Next, we analyze the second term in (131). We only consider  $\{v_n \leq |y| \leq v_n(1+v_n^{\eta})\}$  as the other case is similar. To this end, we need to show that

$$C_l := \frac{m_n^{1/2}}{\Delta_n^{-1/2} v_n^{2-Y/2+1+\eta}} \sum_{i=1}^n K_{m_n \Delta_n}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left[ (\Delta_i J)^2 \mathbf{1}_{v_n \le |\chi_{t_{i-1}} \Delta_i J| \le v_n (1+v_n^{\eta})} |\Delta_i M| |z_{i,l}| \right] = o_P(1),$$

for l = 1, 2, 3. Assume for simplicity of notation that  $\chi = \sigma = 1$ . By (104), Holder's inequality and Lemma 17 in [3], for any p, q > 1 such that 1/p + 1/q = 1/2,

$$C_{2} \lesssim \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2+1+\eta}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) \mathbb{E}_{i-1} \left( \Delta_{i} J^{2p} \mathbf{1}_{v_{n} \leq |\Delta_{i} J| \leq v_{n} (1+v_{n}^{\eta})} \right)^{1/p} \mathbb{E}_{i-1} \left( |\Delta_{i} W|^{q} \right)^{1/q} \mathbb{E}_{i-1} (|\Delta_{i} M|^{2})^{1/2}$$

$$\lesssim \frac{m_{n}^{1/2}}{\Delta_{n}^{-1/2} v_{n}^{2-Y/2+1+\eta}} \sum_{i=1}^{n} K_{m_{n} \Delta_{n}}(t_{i-1} - \tau) (\Delta_{n} v_{n}^{2p-Y+\eta})^{1/p} \Delta_{n}^{1/2} \mathbb{E}_{i-1} \left( |\Delta_{i} W|^{q} \right)^{1/q} \mathbb{E}_{i-1} (|\Delta_{i} M|^{2})^{1/2}$$

$$\lesssim \frac{(\Delta_{n} v_{n}^{2p-Y+\eta})^{1/p}}{v_{n}^{2-Y/2+1+\eta}}.$$

Taking p close to 2 such that 1/p = 1/2 - s for some small s > 0, the last line has order

$$\frac{(\Delta_n v_n^{2p-Y+\eta})^{1/p}}{v_n^{2-Y/2+1+\eta}} = v_n^{(Y-4)/2} \Delta_n^{1/2-s} v_n^{2-Y/2+\eta/2+s(Y-\eta)} = \Delta_n^{1/2-s} v_n^{\eta/2-1+s(Y-\eta)} \leq \Delta_n^{1/2-s} v_n^{\eta/2-1},$$

which vanishes since  $v_n \gg \Delta_n^{\beta}$  and  $\Delta_n^{1/2-s-(1-\eta/2)\beta} \ll 1$  given  $\eta$  and s are small enough. A similar argument works for  $C_1$  and  $C_3$  and establishes (115).

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