Problem: Equity Engineering Group

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1 Problem Setting

We have the following equation that governs the heat propagation in the material Ω:

$$c_p \rho \frac{\partial T}{\partial t} - \operatorname{div}(k \nabla T) = f(x,t) \text{ in } \Omega.$$

Here $f(x,t)$ is a given piecewise defined time-dependent forcing. We divide the boundary into 3 sections: $\partial\Omega = G_{Amb} \cup G_{Ins} \cup G_{Proc}$. We have the following Neumann (Newton's law of cooling) boundary conditions

$$\begin{aligned} -k\nabla T \cdot n &= h_{Amb}(T - T_{Amb}) && \text{on } G_{Amb} \\ -k\nabla T \cdot n &= 0 && \text{on } G_{Ins} \\ -k\nabla T \cdot n &= h_{Proc}(T - T_{Proc}) && \text{on } G_{Proc}. \end{aligned}$$

Multiplying the heat equation by an arbitrary v, and integrating by parts we obtain:

$$\begin{aligned} c_p\rho \int_\Omega \frac{\partial T}{\partial t} v\,dx - \int_\Omega \operatorname{div}(k\nabla T) v\,dx &= \int_\Omega f v\,dx \to \\ c_p\rho \int_\Omega \frac{\partial T}{\partial t} v\,dx + k\int_\Omega \nabla T \nabla v\,dx + \int_{\partial\Omega} (-k\nabla T\cdot n) v\,ds &= \int_\Omega f v dx \end{aligned}$$

Using the boundary conditions on $\partial\Omega$ we obtain:

$$c_p\rho \int_\Omega \frac{\partial T}{\partial t} v\,dx + k\int_\Omega \nabla T \nabla v\,dx + \int_{\partial G_{Amb}} h_{Amb}(T - T_{Amb}) v\,ds + \int_{\partial G_{Proc}} h_{Proc}(T - T_{Proc}) v\,ds = \int_\Omega f v\,dx$$

Now moving the data to the right hand side we obtain:

$$\begin{aligned} c_p\rho \int_\Omega \frac{\partial T}{\partial t} v\,dx + k\int_\Omega \nabla T \nabla v\,dx + \int_{\partial G_{Amb}} h_{Amb} T v\,ds + \int_{\partial G_{Proc}} h_{Proc} T v ds\,ds \\ = \int_\Omega f v\,dx + \int_{\partial G_{Amb}} h_{Amb} T_{Amb} v\,ds + \int_{\partial G_{Proc}} h_{Proc} T_{Proc} v ds \end{aligned}$$

For compactness in notation denote:

$$F(v;t) = \int_\Omega f v\,dx + \int_{\partial G_{Amb}} h_{Amb} T_{Amb} v\,ds + \int_{\partial G_{Proc}} h_{Proc} T_{Proc} v ds,$$

and

$$a(T,v) = k\int_\Omega \nabla T \nabla v\,dx + \int_{\partial G_{Amb}} h_{Amb} T v\,ds + \int_{\partial G_{Proc}} h_{Proc} T v ds\,ds.$$

Thus, we obtain

$$c_p\rho \int_\Omega \frac{\partial T}{\partial t} v\,dx + a(T,v) = F(v;t).$$

We must now choose how to discretize in time. We propose a backwards Euler method with $\frac{\partial T}{\partial t} \approx \frac{T^{n+1} - T^n}{\Delta t}$, and so:

$$c_p\rho \int_\Omega \left(\frac{T^{n+1} - T^n}{\Delta t} \right) v\,dx + a(T^{n+1}, v) = F(v; t^{n+1}).$$

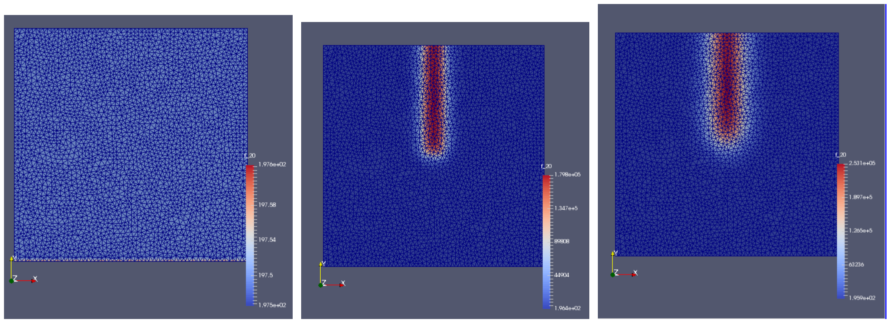

Figure 1: Plots of Heat Profile Square

With a little more manipulation we obtain:

$$c_p\rho \int_\Omega T^{n+1} v\, dx + \Delta t a(T^{n+1}, v) = \Delta t F(v; t^{n+1}) + c_p\rho \int_\Omega T^n v\, dx.$$

The above problem is now an elliptic problem that can be solved via FENICS.

2 Implementation

2.1 Square Geometry, Periodic Forcing

To test the method without worrying about the geometry, we first consider the problem on a square domain. We take the initial condition to be the average of the Ambient and the Process Temperatures, and all the material parameters as given in the problem. We take domain to be the 8×8 square, and apply insulating boundary conditions on the sides, Ambient conditions on the top, and Process conditions on the bottom. The forcing we define to be

$$f(x,t) = 2700 * 0.5 * (1 + \cos((t+5) * \pi/5))$$

for $.45 * 8 \leq x \leq .55 * 8$ and $y \geq .5 * 8$, and zero elsewhere. The initial condition is assumed to be $T_{int} = .5(T_{Amb} + T_{Proc})$. We choose the number of time steps to be 1000 for $t \in (0, 10)$, and choose a sufficiently refined mesh for both. We plot the mesh and solution at $t = dt$, $t = 5s$ and $t = 10s$ in Figure 1.

2.2 Weld Tap Geometry

First we choose $f = 0$ and $T_{init} = T_{Amb}$ and $f = 0$ and $T_{init} = T_{Proc}$ to do a sanity check on the boundary conditions. This is in Figure 2.

Now we choose the real forcing and all parameters. **NOTE:** We must convert units of some of the parameters. We choose $T_{init} = T_{Amb}$, and all parameters as above. We plot the mesh and solution at $t = dt$, $t = 5s$ and $t = 10s$ in Figure 3.

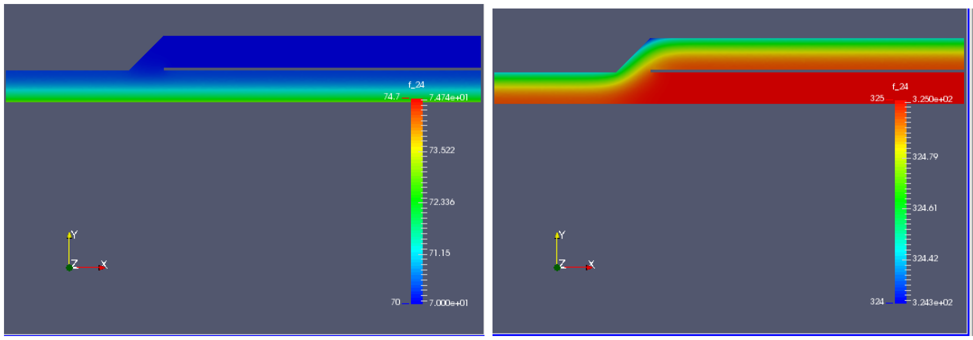

Figure 2: Plots of Heat Profile, Weld Tap Geometry No Forcing, $T_{init} = T_{Amb}$ and $T_{init} = T_{Proc}$

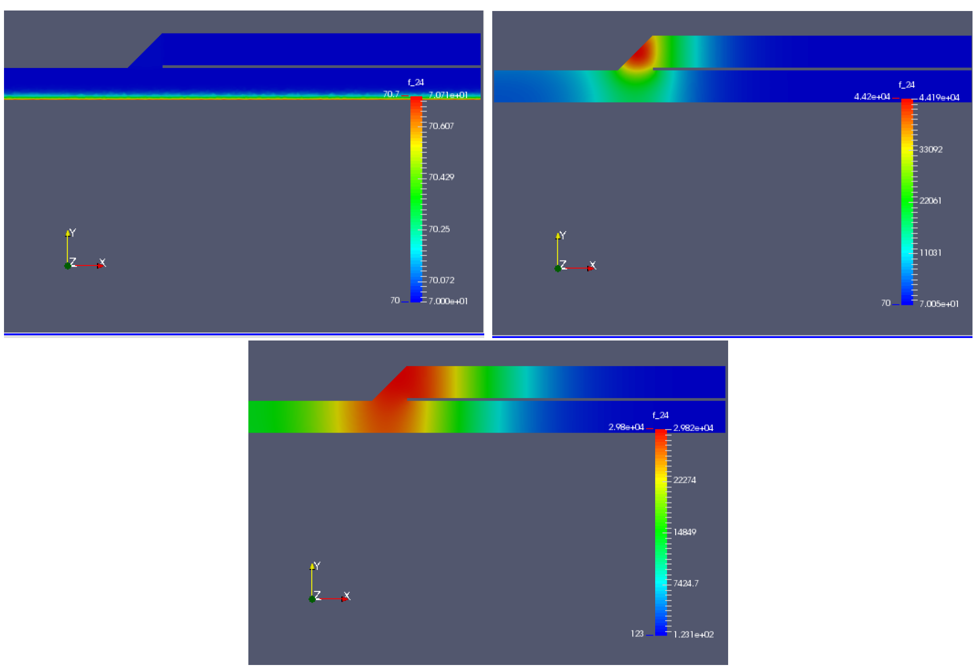

Figure 3: Plots of Heat Profile, Weld Tap Geometry with Forcing, $T_{init} = T_{Amb}$