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预积分学和图优化模型

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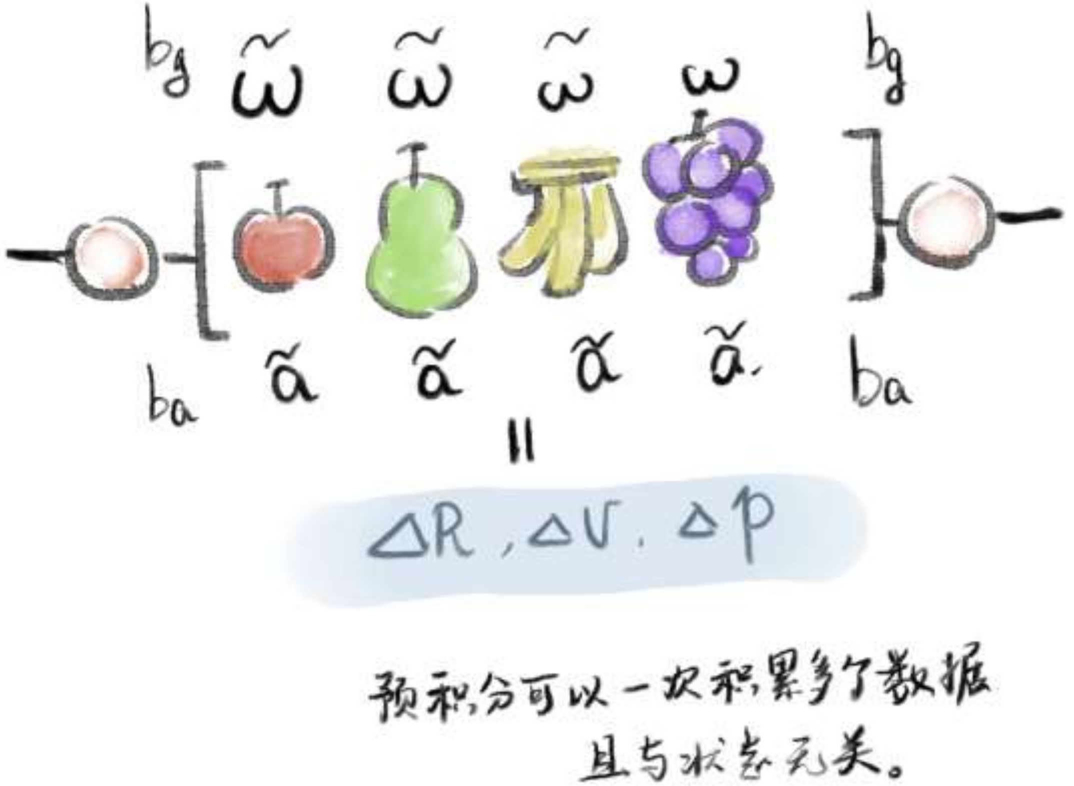
章节内容

- IMU状态的预积分学
- 测量模型和噪声模型
- 图优化模型和雅可比矩阵
- 代码实现和比较



1. IMU状态的预积分模型

IMU状态的预积分模型

- 预积分是另一种处理更长时间IMU累计数据的方式。

ESKF	预积分
• 每个IMU数据通过运动模型积分至名义状态变量 • 积分结果与名义状态变量的当前取值有关 • 每处理一个数据，就要计算一次均值和协方差矩阵	• 每个IMU数据通过运动模型积分至名义状态变量 • 积分结果与名义状态变量的当前取值无关 • 可以一次处理任意多个数据
一口一口吃菜	将菜放到一个小碗中，再一口气吃掉

IMU状态的预积分模型

回顾

- 前两章的运动学模型（连续形式和离散形式）

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\boldsymbol{\omega}^{\wedge}, \qquad \boldsymbol{R}(t+\Delta t) = \boldsymbol{R}(t)\mathrm{Exp}(\boldsymbol{\omega}(t)\Delta t),$$

$$\dot{\boldsymbol{p}} = \boldsymbol{v}, \qquad \boldsymbol{v}(t+\Delta t) = \boldsymbol{v}(t) + \boldsymbol{a}(t)\Delta t,$$

$$\dot{\boldsymbol{v}} = \boldsymbol{a}. \qquad \boldsymbol{p}(t+\Delta t) = \boldsymbol{p}(t) + \boldsymbol{v}(t)\Delta t + \frac{1}{2}\boldsymbol{a}(t)\Delta t^2.$$

$$\tilde{\boldsymbol{\omega}}(t) = \boldsymbol{\omega}(t) + \boldsymbol{b}_g(t) + \boldsymbol{\eta}_g(t),$$

$$\tilde{\boldsymbol{a}}(t) = \boldsymbol{R}^{\top}(\boldsymbol{a}(t) - \boldsymbol{g}) + \boldsymbol{b}_a(t) + \boldsymbol{\eta}_a(t),$$

IMU测量与噪声

使用测量值表达的离散时间运动学:

$$\boldsymbol{R}(t+\Delta t) = \boldsymbol{R}(t)\mathrm{Exp}\left((\tilde{\boldsymbol{\omega}} - \boldsymbol{b}_g(t) - \boldsymbol{\eta}_{gd}(t))\Delta t\right),$$

$$\boldsymbol{v}(t+\Delta t) = \boldsymbol{v}(t) + \boldsymbol{g}\Delta t + \boldsymbol{R}(t)(\tilde{\boldsymbol{a}} - \boldsymbol{b}_a(t) - \boldsymbol{\eta}_{ad}(t))\Delta t,$$

$$\boldsymbol{p}(t+\Delta t) = \boldsymbol{p}(t) + \boldsymbol{v}(t)\Delta t + \frac{1}{2}\boldsymbol{g}\Delta t^2 + \frac{1}{2}\boldsymbol{R}(t)(\tilde{\boldsymbol{a}} - \boldsymbol{b}_a(t) - \boldsymbol{\eta}_{ad}(t))\Delta t^2,$$

IMU状态的预积分模型

- 进一步将多个时刻的IMU累计起来

$$\boldsymbol{R}_j = \boldsymbol{R}_i \prod_{k=i}^{j-1} \left(\operatorname{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,k} - \boldsymbol{\eta}_{gd,k}\right) \Delta t\right)\right),$$

$$\boldsymbol{v}_j = \boldsymbol{v}_i + \boldsymbol{g}\Delta t_{ij} + \sum_{k=i}^{j-1} \boldsymbol{R}_k (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k})\Delta t,$$

$$\boldsymbol{p}_j = \boldsymbol{p}_i + \sum_{k=i}^{j-1} \boldsymbol{v}_k \Delta t + \frac{1}{2}\sum_{k=i}^{j-1} \boldsymbol{g}\Delta t^2 + \frac{1}{2}\sum_{k=i}^{j-1} \boldsymbol{R}_k (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k})\Delta t^2,$$

说明:

- 注意该式只是单纯的累积;
- 在连续时间上为积分形式，在离散时间上为求和形式;
- 使用该式，可以从 i 时刻的状态变量推算至 j 时刻;
- 但该式依然需要知道 i 时刻的状态估计。

IMU状态的预积分模型

- 对该式稍加修改，将状态变量放到等式一侧，将读数放到另一侧：

$$\Delta \boldsymbol{R}_{ij} \doteq \boldsymbol{R}_i^\top \boldsymbol{R}_j = \prod_{k=i}^{j-1} \operatorname{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,k} - \boldsymbol{\eta}_{gd,k}\right) \Delta t\right),$$

$$\Delta \boldsymbol{v}_{ij} \doteq \boldsymbol{R}_i^\top (\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}) = \sum_{k=i}^{j-1} \Delta \boldsymbol{R}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k})\Delta t,$$

$$\begin{aligned}\Delta \boldsymbol{p}_{ij} &\doteq \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i \Delta t_{ij} - \frac{1}{2}\sum_{k=i}^{j-1} \boldsymbol{g}\Delta t^2\right), \\ &= \sum_{k=i}^{j-1} \left[\Delta \boldsymbol{v}_{ik}\Delta t + \frac{1}{2}\Delta \boldsymbol{R}_{ik}\left(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k}\right)\Delta t^2\right].\end{aligned}$$

$$\boldsymbol{R}_j = \boldsymbol{R}_i \prod_{k=i}^{j-1} (\operatorname{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,k} - \boldsymbol{\eta}_{gd,k}\right) \Delta t\right),$$

$$\boldsymbol{v}_j = \boldsymbol{v}_i + \boldsymbol{g}\Delta t_{ij} + \sum_{k=i}^{j-1} \boldsymbol{R}_k(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k})\Delta t,$$

$$\boldsymbol{p}_j = \boldsymbol{p}_i + \sum_{k=i}^{j-1} \boldsymbol{v}_k \Delta t + \frac{1}{2}\sum_{k=i}^{j-1} \boldsymbol{g}\Delta t^2 + \frac{1}{2}\sum_{k=i}^{j-1} \boldsymbol{R}_k(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k})\Delta t^2,$$

注意：

- 这三个定义是累加的；
- 右侧只和IMU读数有关，与状态无关；
- 如果零偏发生变化，该式理论上仍需重新计算，但也可以先假定零偏不变，再用一阶修正；
- 尽管写成R, v, p形式，但并不是直接的物理量；
- 可以视为虚拟的测量模型，进而讨论其噪声模型。

IMU状态的预积分模型

- 分离测量值和噪声值

$$\Delta \boldsymbol{R}_{ij} \doteq \boldsymbol{R}_i^\top \boldsymbol{R}_j = \prod_{k=i}^{j-1} \operatorname{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,k} - \boldsymbol{\eta}_{gd,k} \right) \Delta t \right),$$

$$\Delta \boldsymbol{v}_{ij} \doteq \boldsymbol{R}_i^\top \left(\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g} \Delta t_{ij} \right) = \sum_{k=i}^{j-1} \Delta \boldsymbol{R}_{ik} \left(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k} \right) \Delta t,$$

$$\begin{aligned} \Delta \boldsymbol{p}_{ij} &\doteq \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i \Delta t_{ij} - \frac{1}{2} \sum_{k=i}^{j-1} \boldsymbol{g} \Delta t^2 \right), \\ &= \sum_{k=i}^{j-1} \left[\Delta \boldsymbol{v}_{ik} \Delta t + \frac{1}{2} \Delta \boldsymbol{R}_{ik} \left(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,k} - \boldsymbol{\eta}_{ad,k} \right) \Delta t^2 \right]. \end{aligned}$$

- 这个形式中，IMU传感器噪声会以非线性方式合入测量模型；
- 我们希望把噪声项分离出来，方便分析它们的形式和大小。

IMU状态的预积分模型

- **旋转部分**

$$\Delta \boldsymbol{R}_{ij} = \prod_{k=i}^{j-1} \underbrace{\operatorname{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,k} - \boldsymbol{\eta}_{gd,k}\right)\Delta t\right)}_{\text{利用 BCH:}\approx \operatorname{Exp}((\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,i})\Delta t)\operatorname{Exp}(-\boldsymbol{J}_{r,k}\boldsymbol{\eta}_{gd,k}\Delta t)},$$

$$\approx \prod_{k=i}^{j-1} \left[\operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,i})\Delta t\right)\operatorname{Exp}\left(-\boldsymbol{J}_{r,k}\boldsymbol{\eta}_{gd,k}\Delta t\right)\right].$$

定义旋转测量读数:

$$\Delta \tilde{\boldsymbol{R}}_{ij} = \prod_{k=i}^{j-1} \operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,i})\Delta t\right).$$

不断利用BCH分离噪声项:

$$\Delta \boldsymbol{R}_{ij} = \underbrace{\operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_i - \boldsymbol{b}_{g,i})\Delta t\right)}_{\Delta \tilde{\boldsymbol{R}}_{i,i+1}}\operatorname{Exp}\left(-\boldsymbol{J}_{r,i}\boldsymbol{\eta}_{gd,i}\Delta t\right)\underbrace{\operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_{i+1} - \boldsymbol{b}_{g,i})\Delta t\right)}_{\Delta \tilde{\boldsymbol{R}}_{i+1,i+2}}\operatorname{Exp}\left(-\boldsymbol{J}_{r,i+1}\boldsymbol{\eta}_{gd,i}\Delta t\right)\ldots,$$

$$= \Delta \tilde{\boldsymbol{R}}_{i,i+1}\underbrace{\operatorname{Exp}\left(-\boldsymbol{J}_{r,i}\boldsymbol{\eta}_{gd,i}\Delta t\right)\Delta \tilde{\boldsymbol{R}}_{i+1,i+2}}_{=\Delta \tilde{\boldsymbol{R}}_{i+1,i+2}\operatorname{Exp}(-\Delta \tilde{\boldsymbol{R}}_{i+1,i+2}^{\top}\boldsymbol{J}_{r,i}\boldsymbol{\eta}_{gd,i}\Delta t)}\operatorname{Exp}\left(-\boldsymbol{J}_{r,i+1}\boldsymbol{\eta}_{gd,i}\Delta t\right)\ldots,$$

$$= \Delta \tilde{\boldsymbol{R}}_{i,i+2}\operatorname{Exp}(-\Delta \tilde{\boldsymbol{R}}_{i+1,i+2}^{\top}\boldsymbol{J}_{r,i}\boldsymbol{\eta}_{gd,i}\Delta t)\operatorname{Exp}\left(-\boldsymbol{J}_{r,i+1}\boldsymbol{\eta}_{gd,i}\Delta t\right)\Delta \tilde{\boldsymbol{R}}_{i+2,i+3}\ldots.$$

IMU状态的预积分模型

最后可以得到旋转部分的噪声项定义:

$$
\begin{aligned}
\Delta \boldsymbol{R}_{ij} &= \Delta \tilde{\boldsymbol{R}}_{ij} \prod_{k=i}^{j-1} \operatorname{Exp}\left(-\Delta \tilde{\boldsymbol{R}}_{k+1,j}^{\top} \boldsymbol{J}_{r,k} \boldsymbol{\eta}_{gd,k} \Delta t\right), \\
&\doteq \Delta \tilde{\boldsymbol{R}}_{ij} \operatorname{Exp}(-\delta \boldsymbol{\phi}_{ij}).
\end{aligned}
$$

注意: 它也是累加形式的。

IMU状态的预积分模型

- **速度部分**

$$\begin{aligned}
\Delta \boldsymbol{v}_{ij} &= \sum_{k=i}^{j-1} \Delta \boldsymbol{R}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i} - \boldsymbol{\eta}_{ad,k}) \Delta t, \\
&= \sum_{k=i}^{j-1} \Delta \tilde{\boldsymbol{R}}_{ik} \underbrace{\mathrm{Exp}(-\delta \boldsymbol{\phi}_{ik})}_{\approx \boldsymbol{I} - \delta \boldsymbol{\phi}_{ik}^{\wedge}} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i} - \boldsymbol{\eta}_{ad,k}) \Delta t, \\
&= \sum_{k=i}^{j-1} \Delta \tilde{\boldsymbol{R}}_{ik} (\boldsymbol{I} - \delta \boldsymbol{\phi}_{ik}^{\wedge}) (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i} - \boldsymbol{\eta}_{ad,k}) \Delta t.
\end{aligned}$$

可以把前面的旋转噪声代进去，
同时可以舍掉二阶小量

速度读数为：

$$\Delta \tilde{\boldsymbol{v}}_{ij} = \sum_{k=i}^{j-1} \Delta \tilde{\boldsymbol{R}}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i}) \Delta t.$$

那么，速度读数和噪声的关系为：

$$\begin{aligned}
\Delta \boldsymbol{v}_{ij} &= \sum_{k=i}^{j-1} \left[\underbrace{\Delta \tilde{\boldsymbol{R}}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i}) \Delta t}_{\text{累加此项}} + \Delta \tilde{\boldsymbol{R}}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^{\wedge} \delta \boldsymbol{\phi}_{ik} \Delta t - \Delta \tilde{\boldsymbol{R}}_{ik} \boldsymbol{\eta}_{ad,k} \Delta t \right], \\
&= \Delta \tilde{\boldsymbol{v}}_{ij} + \sum_{k=i}^{j-1} \left[\Delta \tilde{\boldsymbol{R}}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^{\wedge} \delta \boldsymbol{\phi}_{ik} \Delta t - \Delta \tilde{\boldsymbol{R}}_{ik} \boldsymbol{\eta}_{ad,k} \Delta t \right], \\
&= \Delta \tilde{\boldsymbol{v}}_{ij} - \delta \boldsymbol{v}_{ij}.
\end{aligned}$$

IMU状态的预积分模型

- **平移部分：**

$$
\begin{aligned}
\Delta \boldsymbol{p}_{ij} &= \sum_{k=i}^{j-1}\left[\Delta \boldsymbol{v}_{ik}\Delta t + \frac{1}{2}\Delta \boldsymbol{R}_{ik}\left(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i} - \boldsymbol{\eta}_{ad,k}\right)\Delta t^2\right], \\
&= \sum_{k=i}^{j-1}\left[(\Delta \tilde{\boldsymbol{v}}_{ik} - \delta \boldsymbol{v}_{ik})\Delta t + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}\underbrace{\operatorname{Exp}(-\delta \boldsymbol{\phi}_{ik})}_{\boldsymbol{I}-\delta \boldsymbol{\phi}_{ik}^{\wedge}}\left(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i} - \boldsymbol{\eta}_{ad,k}\right)\Delta t^2\right], \\
&\approx \sum_{k=i}^{j-1}\left[(\Delta \tilde{\boldsymbol{v}}_{ik} - \delta \boldsymbol{v}_{ik})\Delta t + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}(\boldsymbol{I} - \delta \boldsymbol{\phi}_{ik}^{\wedge})(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})\Delta t^2 - \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}\boldsymbol{\eta}_{ad,k}\Delta t^2)\right], \\
&\approx \sum_{k=i}^{j-1}\left[\Delta \tilde{\boldsymbol{v}}_{ik}\Delta t + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})\Delta t^2 - \delta \boldsymbol{v}_{ik}\Delta t + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^{\wedge}\delta \boldsymbol{\phi}_{ik}\Delta t^2 - \right. \\
&\qquad \left. \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}\boldsymbol{\eta}_{ad,k}\Delta t^2\right].
\end{aligned}
$$

注意代入速度和旋转部分噪声项

定义位移读数:

$$
\Delta \tilde{\boldsymbol{p}}_{ij} = \sum_{k=i}^{j-1}\left[(\Delta \tilde{\boldsymbol{v}}_{ik}\Delta t) + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})\Delta t^2\right].
$$

位移读数与噪声的关系为:

$$
\begin{aligned}
\Delta \boldsymbol{p}_{ij} &= \Delta \tilde{\boldsymbol{p}}_{ij} + \sum_{k=i}^{j-1}\left[-\delta \boldsymbol{v}_{ik}\Delta t + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^{\wedge}\delta \boldsymbol{\phi}_{ik}\Delta t^2 - \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}\boldsymbol{\eta}_{ad,k}\Delta t^2\right], \\
&\doteq \Delta \tilde{\boldsymbol{p}}_{ij} - \delta \boldsymbol{p}_{ij}.
\end{aligned}
$$

IMU状态的预积分模型

- **整理旋转测量读数、速度读数、位移读数：**

$$\Delta\tilde{\boldsymbol{R}}_{ij} = \boldsymbol{R}_i^\top \boldsymbol{R}_j \mathrm{Exp}(\delta\boldsymbol{\phi}_{ij}),$$

$$\Delta\tilde{\boldsymbol{v}}_{ij} = \boldsymbol{R}_i^\top (\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}) + \delta\boldsymbol{v}_{ij},$$

$$\Delta\tilde{\boldsymbol{p}}_{ij} = \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i\Delta t_{ij} - \frac{1}{2}\boldsymbol{g}\Delta t_{ij}^2\right) + \delta\boldsymbol{p}_{ij}.$$

实际可读的数值　　状态变量的运算　　未知的噪声

$$\Delta\tilde{\boldsymbol{R}}_{ij} = \prod_{k=i}^{j-1} \mathrm{Exp}\left((\tilde{\boldsymbol{\omega}}_k - \boldsymbol{b}_{g,i})\Delta t\right).$$

$$\Delta\tilde{\boldsymbol{v}}_{ij} = \sum_{k=i}^{j-1} \Delta\tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})\Delta t.$$

$$\Delta\tilde{\boldsymbol{p}}_{ij} = \sum_{k=i}^{j-1} \left[(\Delta\tilde{\boldsymbol{v}}_{ik}\Delta t) + \frac{1}{2}\Delta\tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})\Delta t^2\right].$$

这三个的定义式

下面给出噪声项的简化形式（累加形式）。

2. 预积分的噪声模型

预积分的噪声模型

- **旋转部分的噪声模型**

$$\mathrm{Exp}(-\delta\boldsymbol{\phi}_{ij}) = \prod_{k=i}^{j-1} \mathrm{Exp}(-\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\boldsymbol{\eta}_{gd,k}\Delta t).$$

分析它和IMU噪声之间的关系，两侧取Log:

$$\delta\boldsymbol{\phi}_{ij} = -\mathrm{Log}\left(\prod_{k=i}^{j-1} \mathrm{Exp}(-\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\boldsymbol{\eta}_{gd,k}\Delta t)\right)$$

可以近似为:

$$\delta\boldsymbol{\phi}_{ij} \approx \sum_{k=i}^{j-1} \Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\boldsymbol{\eta}_{gd,k}\Delta t.$$

预积分的噪声模型

将它写成累加形式:

$$
\begin{aligned}
\delta\boldsymbol{\phi}_{ij} &\approx \sum_{k=i}^{j-1} \Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top} \boldsymbol{J}_{r,k} \boldsymbol{\eta}_{gd,k} \Delta t, \\
&= \sum_{k=i}^{j-2} \Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top} \boldsymbol{J}_{r,k} \boldsymbol{\eta}_{gd,k} \Delta t + \underbrace{\Delta\boldsymbol{R}_{j,j}^{\top}}_{=\boldsymbol{I}} \boldsymbol{J}_{r,j-1} \boldsymbol{\eta}_{gd,j-1} \Delta t, \\
&= \sum_{k=i}^{j-2} \underbrace{\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}}_{(\Delta\tilde{\boldsymbol{R}}_{k+1,j-1}\Delta\tilde{\boldsymbol{R}}_{j-1,j})^{\top}} \boldsymbol{J}_{r,k} \boldsymbol{\eta}_{gd,k} \Delta t + \boldsymbol{J}_{r,j-1} \boldsymbol{\eta}_{gd,j-1} \Delta t, \\
&= \Delta\tilde{\boldsymbol{R}}_{j-1,j}^{\top} \sum_{k=i}^{j-2} \Delta\tilde{\boldsymbol{R}}_{k+1,j-1}^{\top} \boldsymbol{J}_{r,k} \boldsymbol{\eta}_{gd,k} \Delta t + \boldsymbol{J}_{r,j-1} \boldsymbol{\eta}_{gd,j-1} \Delta t, \\
&= \Delta\tilde{\boldsymbol{R}}_{j-1,j}^{\top} \delta\boldsymbol{\phi}_{i,j-1} + \boldsymbol{J}_{r,j-1} \boldsymbol{\eta}_{gd,j-1} \Delta t.
\end{aligned}
$$

这实际上是一个线性变换

$$
\boldsymbol{\Sigma}_j = \Delta\tilde{\boldsymbol{R}}_{j-1,j}^{\top} \boldsymbol{\Sigma}_{j-1} \Delta\tilde{\boldsymbol{R}}_{j-1,j} + \boldsymbol{J}_{r,j-1} \boldsymbol{\Sigma}_{\boldsymbol{\eta}_{gd}} \boldsymbol{J}_{r,j-1}^{\top} \Delta t^2.
$$

上一时刻变换　　本时刻合入

预积分的噪声模型

- **速度噪声**

$$\delta \boldsymbol{v}_{ij} \approx \sum_{k=i}^{j-1} \left[-\Delta \tilde{\boldsymbol{R}}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^\wedge \delta \boldsymbol{\phi}_{ik} \Delta t + \Delta \tilde{\boldsymbol{R}}_{ik} \boldsymbol{\eta}_{ad,k} \Delta t \right].$$

它的累加形式:

$$\begin{aligned}
\delta \boldsymbol{v}_{ij} &= \sum_{k=i}^{j-1} \left[-\Delta \tilde{\boldsymbol{R}}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^\wedge \delta \boldsymbol{\phi}_{ik} \Delta t + \Delta \tilde{\boldsymbol{R}}_{ik} \boldsymbol{\eta}_{ad,k} \Delta t \right], \\
&= \sum_{k=i}^{j-2} \left[-\Delta \tilde{\boldsymbol{R}}_{ik} (\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^\wedge \delta \boldsymbol{\phi}_{ik} \Delta t + \Delta \tilde{\boldsymbol{R}}_{ik} \boldsymbol{\eta}_{ad,k} \Delta t \right] \\
&\quad - \Delta \tilde{\boldsymbol{R}}_{i,j-1} (\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i})^\wedge \delta \boldsymbol{\phi}_{i,j-1} \Delta t + \Delta \tilde{\boldsymbol{R}}_{i,j-1} \boldsymbol{\eta}_{ad,j-1} \Delta t, \\
&= \delta \boldsymbol{v}_{i,j-1} - \Delta \tilde{\boldsymbol{R}}_{i,j-1} (\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i})^\wedge \delta \boldsymbol{\phi}_{i,j-1} \Delta t + \Delta \tilde{\boldsymbol{R}}_{i,j-1} \boldsymbol{\eta}_{ad,j-1} \Delta t.
\end{aligned}$$

预积分的噪声模型

- **位移噪声**

$$
\begin{aligned}
\delta \boldsymbol{p}_{ij} &= \sum_{k=i}^{j-1}\left[\delta \boldsymbol{v}_{ik}\Delta t - \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^{\wedge}\delta \boldsymbol{\phi}_{ik}\Delta t^2 + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}\boldsymbol{\eta}_{ad,k}\Delta t^2\right], \\
&= \sum_{k=i}^{j-2}\left[\delta \boldsymbol{v}_{ik}\Delta t - \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i})^{\wedge}\delta \boldsymbol{\phi}_{ik}\Delta t^2 + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{ik}\boldsymbol{\eta}_{ad,k}\Delta t^2\right] \\
&\quad + \delta \boldsymbol{v}_{i,j-1}\Delta t - \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{i,j-1}(\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i})^{\wedge}\delta \boldsymbol{\phi}_{i,j-1}\Delta t^2 + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{i,j-1}\boldsymbol{\eta}_{ad,j-1}\Delta t^2, \\
&= \delta \boldsymbol{p}_{i,j-1} + \delta \boldsymbol{v}_{i,j-1}\Delta t - \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{i,j-1}(\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i})^{\wedge}\delta \boldsymbol{\phi}_{i,j-1}\Delta t^2 + \frac{1}{2}\Delta \tilde{\boldsymbol{R}}_{i,j-1}\boldsymbol{\eta}_{ad,j-1}\Delta t^2.
\end{aligned}
$$

预积分的噪声模型

将它们整理成矩阵形式:

$$\boldsymbol{\eta}_{ik} = \begin{bmatrix} \delta\boldsymbol{\phi}_{ik} \\ \delta\boldsymbol{v}_{ik} \\ \delta\boldsymbol{p}_{ik} \end{bmatrix}, \qquad \boldsymbol{\eta}_{d,j} = \begin{bmatrix} \boldsymbol{\eta}_{gd,j} \\ \boldsymbol{\eta}_{ad,j} \end{bmatrix},$$

那么: $\boldsymbol{\eta}_{ij} = \boldsymbol{A}_{j-1}\boldsymbol{\eta}_{i,j-1} + \boldsymbol{B}_{j-1}\boldsymbol{\eta}_{d,j-1},$

系数矩阵为: $$\boldsymbol{A}_{j-1} = \begin{bmatrix} \Delta\tilde{\boldsymbol{R}}_{j-1,j}^\top & \boldsymbol{0} & \boldsymbol{0} \\ -\Delta\tilde{\boldsymbol{R}}_{i,j-1}(\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i})^\wedge \Delta t & \boldsymbol{I} & \boldsymbol{0} \\ -\frac{1}{2}\Delta\tilde{\boldsymbol{R}}_{i,j-1}(\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i})^\wedge \Delta t^2 & \Delta t\boldsymbol{I} & \boldsymbol{I} \end{bmatrix}, \boldsymbol{B}_{j-1} = \begin{bmatrix} \boldsymbol{J}_{r,j-1}\Delta t & \boldsymbol{0} \\ \boldsymbol{0} & \Delta\tilde{\boldsymbol{R}}_{i,j-1}\Delta t \\ \boldsymbol{0} & \frac{1}{2}\Delta\tilde{\boldsymbol{R}}_{i,j-1}\Delta t^2 \end{bmatrix}.$$

协方差部分: $\boldsymbol{\Sigma}_{i,k+1} = \boldsymbol{A}_{k+1}\boldsymbol{\Sigma}_{i,k}\boldsymbol{A}_{k+1}^\top + \boldsymbol{B}_{k+1}\mathrm{Cov}(\boldsymbol{\eta}_{d,k})\boldsymbol{B}_{k+1}^\top$

预积分的噪声模型

零偏的更新

- 预积分模型假设积分过程中零偏是不变的，但实际上零偏是状态变量，可以有更新。

将预积分观测视为零偏的函数，然后求它们对零偏的一阶导数：

$$
\begin{aligned}
\Delta\tilde{\boldsymbol{R}}_{ij}(\boldsymbol{b}_{g,i}+\delta\boldsymbol{b}_{g,i}) &= \Delta\tilde{\boldsymbol{R}}_{ij}(\boldsymbol{b}_{g,i})\mathrm{Exp}\left(\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ij}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}\right),\\
\Delta\tilde{\boldsymbol{v}}_{ij}(\boldsymbol{b}_{g,i}+\delta\boldsymbol{b}_{g,i},\boldsymbol{b}_{a,i}+\delta\boldsymbol{b}_{a,i}) &= \Delta\tilde{\boldsymbol{v}}_{ij}(\boldsymbol{b}_{g,i},\boldsymbol{b}_{a,i})+\frac{\partial\Delta\tilde{\boldsymbol{v}}_{ij}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}+\frac{\partial\Delta\tilde{\boldsymbol{v}}_{ij}}{\partial\boldsymbol{b}_{a,i}}\delta\boldsymbol{b}_{a,i},\\
\Delta\tilde{\boldsymbol{p}}_{ij}(\boldsymbol{b}_{g,i}+\delta\boldsymbol{b}_{g,i},\boldsymbol{b}_{a,i}+\delta\boldsymbol{b}_{a,i}) &= \Delta\tilde{\boldsymbol{p}}_{ij}(\boldsymbol{b}_{g,i},\boldsymbol{b}_{a,i})+\frac{\partial\Delta\tilde{\boldsymbol{p}}_{ij}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}+\frac{\partial\Delta\tilde{\boldsymbol{p}}_{ij}}{\partial\boldsymbol{b}_{a,i}}\delta\boldsymbol{b}_{a,i}.
\end{aligned}
$$

然后，当零偏发生更新时，按照一阶导数来更新预积分取值。

预积分的噪声模型

旋转部分对零偏的导数:

$$
\begin{aligned}
\Delta\tilde{\boldsymbol{R}}_{ij}(\boldsymbol{b}_{g,i}+\delta\boldsymbol{b}_{g,i}) &= \prod_{k=i}^{j-1}\operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_k-(\boldsymbol{b}_{g,i}+\delta\boldsymbol{b}_{g,i}))\Delta t\right),\\
&= \prod_{k=i}^{j-1}\operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_k-\boldsymbol{b}_{g,i})\Delta t\right)\operatorname{Exp}(-\boldsymbol{J}_{r,k}\delta\boldsymbol{b}_{g,i}\Delta t),\\
&= \underbrace{\operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_i-\boldsymbol{b}_{g,i})\Delta t\right)}_{\Delta\tilde{\boldsymbol{R}}_{i,i+1}}\operatorname{Exp}(-\boldsymbol{J}_{r,i}\delta\boldsymbol{b}_{g,i}\Delta t)\underbrace{\operatorname{Exp}\left((\tilde{\boldsymbol{\omega}}_{i+1}-\boldsymbol{b}_{g,i})\Delta t\right)}_{\Delta\tilde{\boldsymbol{R}}_{i+1,i+2}}\\
&\qquad \operatorname{Exp}(-\boldsymbol{J}_{r,i+1}\delta\boldsymbol{b}_{g,i}\Delta t)\ldots,\\
&= \Delta\tilde{\boldsymbol{R}}_{i,i+1}\Delta\tilde{\boldsymbol{R}}_{i+1,i+2}\operatorname{Exp}(-\Delta\tilde{\boldsymbol{R}}_{i+1,i+2}^{\top}\boldsymbol{J}_{r,i}\delta\boldsymbol{b}_{g,i}\Delta t)\ldots,\\
&= \Delta\tilde{\boldsymbol{R}}_{ij}\prod_{k=i}^{j-1}\operatorname{Exp}\left(-\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\delta\boldsymbol{b}_{g,i}\Delta t\right),\\
&\approx \Delta\tilde{\boldsymbol{R}}_{ij}\operatorname{Exp}\left(-\sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\Delta t\delta\boldsymbol{b}_{g,i}\right).
\end{aligned}
$$

显然: $$\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ij}}{\partial\boldsymbol{b}_{g,i}} = -\sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\Delta t.$$

它的递推形式:

$$
\begin{aligned}
\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ij}}{\partial\boldsymbol{b}_{g,i}} &= -\sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\Delta t,\\
&= -\sum_{k=i}^{j-2}\Delta\tilde{\boldsymbol{R}}_{k+1,j}^{\top}\boldsymbol{J}_{r,k}\Delta t-\Delta\tilde{\boldsymbol{R}}_{j,j}^{\top}\boldsymbol{J}_{r,j-1}\Delta t,\\
&= -\sum_{k=i}^{j-2}(\Delta\tilde{\boldsymbol{R}}_{k+1,j-1}\Delta\tilde{\boldsymbol{R}}_{j-1,j})^{\top}\boldsymbol{J}_{k,r}\Delta t-\boldsymbol{J}_{r,j-1}\Delta t,\\
&= \Delta\tilde{\boldsymbol{R}}_{j-1,j}^{\top}\frac{\partial\Delta\tilde{\boldsymbol{R}}_{i,j-1}}{\partial\boldsymbol{b}_{g,i}}-\boldsymbol{J}_{r,j-1}\Delta t.
\end{aligned}
$$

预积分的噪声模型

速度部分对零偏的导数:

$$
\begin{aligned}
\Delta\tilde{\boldsymbol{v}}_{ij}(\boldsymbol{b}_i+\delta\boldsymbol{b}_i) &= \sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{ik}(\boldsymbol{b}_{g,i}+\delta\boldsymbol{b}_{g,i})(\tilde{\boldsymbol{a}}_k-\boldsymbol{b}_{a,i}-\delta\boldsymbol{b}_{a,i})\Delta t,\\
&= \sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{ik}\mathrm{Exp}\left(\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ik}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}\right)(\tilde{\boldsymbol{a}}_k-\boldsymbol{b}_{a,i}-\delta\boldsymbol{b}_{a,i})\Delta t,\\
&\approx \sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{ik}\left(\boldsymbol{I}+\left(\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ik}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}\right)^{\wedge}\right)(\tilde{\boldsymbol{a}}_k-\boldsymbol{b}_{a,i}-\delta\boldsymbol{b}_{a,i})\Delta t,\\
&\approx \Delta\tilde{\boldsymbol{v}}_{ij}-\sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{ik}\Delta t\delta\boldsymbol{b}_{a,i}-\sum_{k=i}^{j-1}\Delta\tilde{\boldsymbol{R}}_{ik}(\tilde{\boldsymbol{a}}_k-\boldsymbol{b}_{a,i})^{\wedge}\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ik}}{\partial\boldsymbol{b}_{g,i}}\Delta t\delta\boldsymbol{b}_{g,i},\\
&= \Delta\tilde{\boldsymbol{v}}_{ij}+\frac{\partial\Delta\boldsymbol{v}_{ij}}{\partial\boldsymbol{b}_{a,i}}\delta\boldsymbol{b}_{a,i}+\frac{\partial\Delta\boldsymbol{v}_{ij}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}.
\end{aligned}
$$

预积分的噪声模型

平移部分对零偏的导数:

$$
\begin{aligned}
\Delta\tilde{\boldsymbol{p}}_{ij}(\boldsymbol{b}_i+\delta\boldsymbol{b}_i) \approx \sum_{k=i}^{j-1}\Bigg[&\left(\Delta\tilde{\boldsymbol{v}}_{ik}+\frac{\partial\Delta\boldsymbol{v}_{ik}}{\partial\boldsymbol{b}_{a,i}}\delta\boldsymbol{b}_{a,i}+\frac{\partial\Delta\boldsymbol{v}_{ik}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}\right)\Delta t+ \\
&\frac{1}{2}\Delta\tilde{\boldsymbol{R}}_{ik}\left(\boldsymbol{I}+\left(\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ik}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}\right)^{\wedge}\right)(\tilde{\boldsymbol{a}}_k-\boldsymbol{b}_{a,i}-\delta\boldsymbol{b}_{a,i})\Delta t^2\Bigg], \\
\approx \Delta\tilde{\boldsymbol{p}}_{ij}+&\sum_{k=i}^{j-1}\left[\frac{\partial\Delta\boldsymbol{v}_{ik}}{\partial\boldsymbol{b}_{a,i}}\Delta t-\frac{1}{2}\Delta\tilde{\boldsymbol{R}}_{ik}\Delta t^2\right]\delta\boldsymbol{b}_{a,i}+ \\
&\sum_{k=i}^{j-1}\left[\frac{\partial\Delta\boldsymbol{v}_{ik}}{\partial\boldsymbol{b}_{g,i}}\Delta t-\frac{1}{2}\Delta\tilde{\boldsymbol{R}}_{ik}\left(\tilde{\boldsymbol{a}}_k-\boldsymbol{b}_{a,i}\right)^{\wedge}\frac{\partial\Delta\tilde{\boldsymbol{R}}_{ik}}{\partial\boldsymbol{b}_{g,i}}\Delta t^2\right]\delta\boldsymbol{b}_{g,i}, \\
= \Delta\tilde{\boldsymbol{p}}_{ij}+&\frac{\partial\Delta\tilde{\boldsymbol{p}}_{ij}}{\partial\boldsymbol{b}_{a,i}}\delta\boldsymbol{b}_{a,i}+\frac{\partial\Delta\tilde{\boldsymbol{p}}_{ij}}{\partial\boldsymbol{b}_{g,i}}\delta\boldsymbol{b}_{g,i}.
\end{aligned}
$$

预积分的噪声模型

整理得到:

普通形式

$$\frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}} = -\sum_{k=i}^{j-1}\left[\Delta \tilde{\boldsymbol{R}}_{k+1,j}^{\top} \boldsymbol{J}_{r,k} \Delta t\right],$$

$$\frac{\partial \Delta \tilde{\boldsymbol{v}}_{ij}}{\partial \boldsymbol{b}_{a,i}} = -\sum_{k=i}^{j-1} \Delta \tilde{\boldsymbol{R}}_{ik} \Delta t,$$

$$\frac{\partial \Delta \tilde{\boldsymbol{v}}_{ij}}{\partial \boldsymbol{b}_{g,i}} = -\sum_{k=i}^{j-1} \Delta \tilde{\boldsymbol{R}}_{ik} \left(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i}\right)^{\wedge} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{ik}}{\partial \boldsymbol{b}_{g,i}} \Delta t,$$

$$\frac{\partial \Delta \tilde{\boldsymbol{p}}_{ij}}{\partial \boldsymbol{b}_{a,i}} = \sum_{k=i}^{j-1}\left[\frac{\partial \Delta \tilde{\boldsymbol{v}}_{ik}}{\partial \boldsymbol{b}_{a,i}} \Delta t - \frac{1}{2} \Delta \tilde{\boldsymbol{R}}_{ik} \Delta t^2\right],$$

$$\frac{\partial \Delta \tilde{\boldsymbol{p}}_{ij}}{\partial \boldsymbol{b}_{g,i}} = \sum_{k=i}^{j-1}\left[\frac{\partial \Delta \tilde{\boldsymbol{v}}_{ik}}{\partial \boldsymbol{b}_{g,i}} \Delta t - \frac{1}{2} \Delta \tilde{\boldsymbol{R}}_{ik} \left(\tilde{\boldsymbol{a}}_k - \boldsymbol{b}_{a,i}\right)^{\wedge} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{ik}}{\partial \boldsymbol{b}_{g,i}} \Delta t^2\right].$$

累加形式

$$\frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}} = \Delta \tilde{\boldsymbol{R}}_{j-1,j}^{\top} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{i,j-1}}{\partial \boldsymbol{b}_{g,i}} - \boldsymbol{J}_{r,k} \Delta t,$$

$$\frac{\partial \Delta \tilde{\boldsymbol{v}}_{ij}}{\partial \boldsymbol{b}_{a,i}} = \frac{\partial \Delta \tilde{\boldsymbol{v}}_{i,j-1}}{\partial \boldsymbol{b}_{a,i}} - \Delta \tilde{\boldsymbol{R}}_{i,j-1} \Delta t,$$

$$\frac{\partial \Delta \tilde{\boldsymbol{v}}_{ij}}{\partial \boldsymbol{b}_{g,i}} = \frac{\partial \Delta \tilde{\boldsymbol{v}}_{i,j-1}}{\partial \boldsymbol{b}_{g,i}} - \Delta \tilde{\boldsymbol{R}}_{i,j-1} \left(\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i}\right)^{\wedge} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{i,j-1}}{\partial \boldsymbol{b}_{g,i}} \Delta t,$$

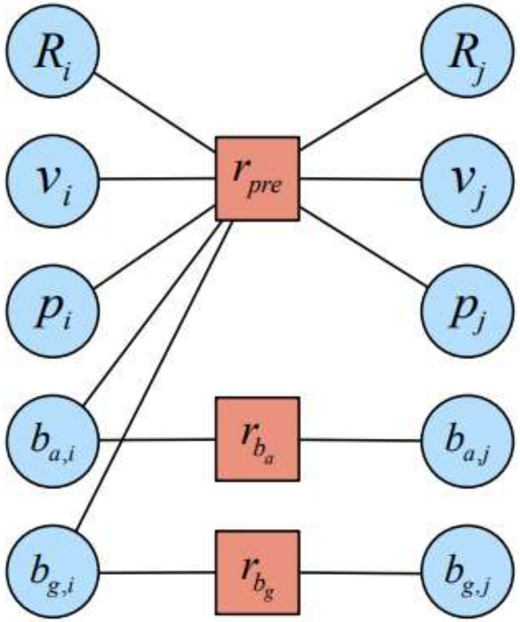
$$\frac{\partial \Delta \tilde{\boldsymbol{p}}_{ij}}{\partial \boldsymbol{b}_{a,i}} = \frac{\partial \Delta \tilde{\boldsymbol{p}}_{i,j-1}}{\partial \boldsymbol{b}_{a,i}} + \frac{\partial \Delta \tilde{\boldsymbol{v}}_{i,j-1}}{\partial \boldsymbol{b}_{a,i}} \Delta t - \frac{1}{2} \Delta \tilde{\boldsymbol{R}}_{i,j-1} \Delta t^2,$$

$$\frac{\partial \Delta \tilde{\boldsymbol{p}}_{ij}}{\partial \boldsymbol{b}_{g,i}} = \frac{\partial \Delta \tilde{\boldsymbol{p}}_{i,j-1}}{\partial \boldsymbol{b}_{g,i}} + \frac{\partial \Delta \tilde{\boldsymbol{v}}_{i,j-1}}{\partial \boldsymbol{b}_{g,i}} \Delta t - \frac{1}{2} \Delta \tilde{\boldsymbol{R}}_{i,j-1} \left(\tilde{\boldsymbol{a}}_{j-1} - \boldsymbol{b}_{a,i}\right)^{\wedge} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{i,j-1}}{\partial \boldsymbol{b}_{g,i}} \Delta t^2.$$

3. 图优化模型和雅可比矩阵

图优化模型和雅可比矩阵

- 我们将预积分模型转换为图优化中的变量和因子
- 变量部分即状态变量：$\boldsymbol{x}_k = [\boldsymbol{R}, \boldsymbol{p}, \boldsymbol{v}, \boldsymbol{b}_a, \boldsymbol{b}_g]_k \in \mathcal{X},$



$$\Delta\tilde{\boldsymbol{R}}_{ij} = \boldsymbol{R}_i^\top \boldsymbol{R}_j \mathrm{Exp}(\delta\boldsymbol{\phi}_{ij}),$$
$$\Delta\tilde{\boldsymbol{v}}_{ij} = \boldsymbol{R}_i^\top (\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}) + \delta\boldsymbol{v}_{ij},$$
$$\Delta\tilde{\boldsymbol{p}}_{ij} = \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i\Delta t_{ij} - \frac{1}{2}\boldsymbol{g}\Delta t_{ij}^2\right) + \delta\boldsymbol{p}_{ij}.$$

预积分测量值关联了两个时刻的PVQ和零偏，
同时我们还需要建模零偏的随机游走。

图优化模型和雅可比矩阵

- 从测量模型定义残差：

$$
\begin{aligned}
\Delta\tilde{\boldsymbol{R}}_{ij} &= \boldsymbol{R}_i^\top \boldsymbol{R}_j \mathrm{Exp}(\delta\boldsymbol{\phi}_{ij}), \\
\Delta\tilde{\boldsymbol{v}}_{ij} &= \boldsymbol{R}_i^\top (\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}) + \delta\boldsymbol{v}_{ij}, \\
\Delta\tilde{\boldsymbol{p}}_{ij} &= \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i\Delta t_{ij} - \frac{1}{2}\boldsymbol{g}\Delta t_{ij}^2\right) + \delta\boldsymbol{p}_{ij}.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{r}_{\Delta\boldsymbol{R}_{ij}} &= \mathrm{Log}\left(\Delta\tilde{\boldsymbol{R}}_{ij}^\top \left(\boldsymbol{R}_i^\top \boldsymbol{R}_j\right)\right), \\
\boldsymbol{r}_{\Delta\boldsymbol{v}_{ij}} &= \boldsymbol{R}_i^\top (\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}) - \Delta\tilde{\boldsymbol{v}}_{ij}, \\
\boldsymbol{r}_{\Delta\boldsymbol{p}_{ij}} &= \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i\Delta t_{ij} - \frac{1}{2}\boldsymbol{g}\Delta t_{ij}^2\right) - \Delta\tilde{\boldsymbol{p}}_{ij}.
\end{aligned}
$$

注：

1. 残差部分实际对应噪声部分，噪声的协方差即残差的协方差；
2. 由于测量值内部还包含了零偏的线性变换，所以残差实际和零偏也相关（只是公式中省略了）。

图优化模型和雅可比矩阵

- 各残差项对状态变量雅可比矩阵：

$$
\begin{aligned}
\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}(\boldsymbol{R}_i \mathrm{Exp}(\boldsymbol{\phi}_i)) &= \mathrm{Log}\left(\Delta \tilde{\boldsymbol{R}}_{ij}^\top ((\boldsymbol{R}_i \mathrm{Exp}(\boldsymbol{\phi}_i))^\top \boldsymbol{R}_j\right), \\
&= \mathrm{Log}\left(\Delta \tilde{\boldsymbol{R}}_{ij}^\top \mathrm{Exp}(-\boldsymbol{\phi}_i) \boldsymbol{R}_i^\top \boldsymbol{R}_j\right), \\
&= \mathrm{Log}\left(\Delta \tilde{\boldsymbol{R}}_{ij}^\top \boldsymbol{R}_i^\top \boldsymbol{R}_j \mathrm{Exp}(-\boldsymbol{R}_j^\top \boldsymbol{R}_i \boldsymbol{\phi}_i)\right), \\
&= \boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}} - \boldsymbol{J}_r^{-1}(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}) \boldsymbol{R}_j^\top \boldsymbol{R}_i \boldsymbol{\phi}_i.
\end{aligned}
$$

i时刻旋转

$$
\begin{aligned}
\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}(\boldsymbol{R}_j \mathrm{Exp}(\boldsymbol{\phi}_j)) &= \mathrm{Log}\left(\Delta \tilde{\boldsymbol{R}}_{ij}^\top \boldsymbol{R}_i^\top \boldsymbol{R}_j \mathrm{Exp}(\boldsymbol{\phi}_j)\right), \\
&= \boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}} + \boldsymbol{J}_r^{-1}(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}) \boldsymbol{\phi}_j.
\end{aligned}
$$

j时刻旋转

图优化模型和雅可比矩阵

- 对陀螺零偏:

$$
\begin{aligned}
\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}(\boldsymbol{b}_{g,i}+\delta \boldsymbol{b}_{g,i}+\tilde{\delta} \boldsymbol{b}_{g,i}) &= \operatorname{Log}\left(\left(\Delta \tilde{\boldsymbol{R}}_{ij} \operatorname{Exp}\left(\frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}}(\delta \boldsymbol{b}_{g,i}+\tilde{\delta} \boldsymbol{b}_{g,i})\right)\right)^{\top} \boldsymbol{R}_i^{\top} \boldsymbol{R}_j\right), \\
&\overset{\text{BCH}}{\approx} \operatorname{Log}\left(\left(\underbrace{\Delta \tilde{\boldsymbol{R}}_{ij} \operatorname{Exp}(\frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}} \delta \boldsymbol{b}_{g,i})}_{\Delta \tilde{\boldsymbol{R}}_{ij}^{\prime}} \operatorname{Exp}(\boldsymbol{J}_{r,b} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}} \tilde{\delta} \boldsymbol{b}_{g,i})\right)^{\top} \boldsymbol{R}_i^{\top} \boldsymbol{R}_j\right), \\
&= \operatorname{Log}\left(\operatorname{Exp}\left(-\boldsymbol{J}_{r,b} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}} \tilde{\delta} \boldsymbol{b}_{g,i}\right) \underbrace{(\Delta \tilde{\boldsymbol{R}}_{ij}^{\prime})^{\top} \boldsymbol{R}_i^{\top} \boldsymbol{R}_j}_{\operatorname{Exp}\left(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime}\right)}\right), \\
&= \operatorname{Log}\left(\operatorname{Exp}\left(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime}\right) \operatorname{Exp}\left(-\operatorname{Exp}\left(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime}\right)^{\top} \boldsymbol{J}_{r,b} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}} \tilde{\delta} \boldsymbol{b}_{g,i}\right)\right), \\
&\approx \boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime} - \boldsymbol{J}_r^{-1}(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime}) \operatorname{Exp}\left(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime}\right)^{\top} \boldsymbol{J}_{r,b} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}} \tilde{\delta} \boldsymbol{b}_{g,i}.
\end{aligned}
$$

$$
\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}}{\partial \boldsymbol{b}_{g,i}} = -\boldsymbol{J}_r^{-1}(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime}) \operatorname{Exp}\left(\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}}^{\prime}\right)^{\top} \boldsymbol{J}_{r,b} \frac{\partial \Delta \tilde{\boldsymbol{R}}_{ij}}{\partial \boldsymbol{b}_{g,i}}.
$$

图优化模型和雅可比矩阵

- 速度观测的雅可比：

$$
\begin{aligned}
\boldsymbol{r}_{\Delta \boldsymbol{v}_{ij}}\left(\boldsymbol{R}_i \operatorname{Exp}(\delta \boldsymbol{\phi}_i)\right) &= \left(\boldsymbol{R}_i \operatorname{Exp}(\delta \boldsymbol{\phi}_i)\right)^\top \left(\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}\right) - \Delta \tilde{\boldsymbol{v}}_{ij}, \\
&= \left(\boldsymbol{I} - \delta \boldsymbol{\phi}_i^\wedge\right) \boldsymbol{R}_i^\top \left(\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}\right) - \Delta \tilde{\boldsymbol{v}}_{ij}, \\
&= \boldsymbol{r}_{\Delta \boldsymbol{v}_{ij}}(\boldsymbol{R}_i) + \left(\boldsymbol{R}_i^\top \left(\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}\right)\right)^\wedge \delta \boldsymbol{\phi}_i.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{v}_{ij}}}{\partial \boldsymbol{v}_i} &= -\boldsymbol{R}_i^\top, \\
\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{v}_{ij}}}{\partial \boldsymbol{v}_j} &= \boldsymbol{R}_i^\top.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}} &= \operatorname{Log}\left(\Delta \tilde{\boldsymbol{R}}_{ij}^\top \left(\boldsymbol{R}_i^\top \boldsymbol{R}_j\right)\right), \\
\boldsymbol{r}_{\Delta \boldsymbol{v}_{ij}} &= \boldsymbol{R}_i^\top \left(\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}\right) - \Delta \tilde{\boldsymbol{v}}_{ij}, \\
\boldsymbol{r}_{\Delta \boldsymbol{p}_{ij}} &= \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i \Delta t_{ij} - \frac{1}{2}\boldsymbol{g}\Delta t_{ij}^2\right) - \Delta \tilde{\boldsymbol{p}}_{ij}.
\end{aligned}
$$

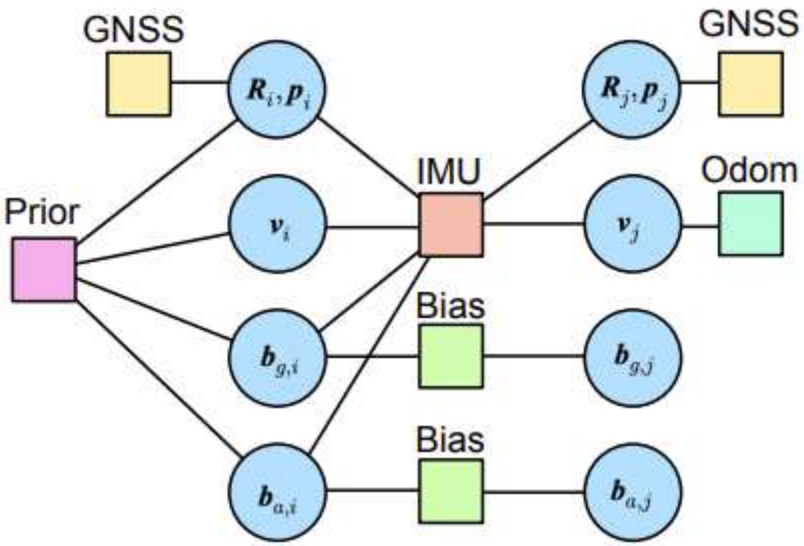
图优化模型和雅可比矩阵

- 平移部分:

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{p}_{ij}}}{\partial \boldsymbol{p}_i} &= -\boldsymbol{R}_i^\top, \\
\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{p}_{ij}}}{\partial \boldsymbol{p}_j} &= \boldsymbol{R}_i^\top, \\
\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{p}_{ij}}}{\partial \boldsymbol{v}_i} &= -\boldsymbol{R}_i^\top \Delta t_{ij}, \\
\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{p}_{ij}}}{\partial \boldsymbol{\phi}_i} &= \left(\boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i \Delta t_{ij} - \frac{1}{2}\boldsymbol{g}\Delta t_{ij}^2\right)\right)^\wedge.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{r}_{\Delta \boldsymbol{R}_{ij}} &= \mathrm{Log}\left(\Delta \tilde{\boldsymbol{R}}_{ij}^\top \left(\boldsymbol{R}_i^\top \boldsymbol{R}_j\right)\right), \\
\boldsymbol{r}_{\Delta \boldsymbol{v}_{ij}} &= \boldsymbol{R}_i^\top \left(\boldsymbol{v}_j - \boldsymbol{v}_i - \boldsymbol{g}\Delta t_{ij}\right) - \Delta \tilde{\boldsymbol{v}}_{ij}, \\
\boldsymbol{r}_{\Delta \boldsymbol{p}_{ij}} &= \boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i \Delta t_{ij} - \frac{1}{2}\boldsymbol{g}\Delta t_{ij}^2\right) - \Delta \tilde{\boldsymbol{p}}_{ij}.
\end{aligned}
$$

图优化模型和雅可比矩阵



本节演示使用预积分+图优化完成和上一章同样的任务

4. 代码实现与实验

习题

1. 推导以下等式(书中式4.48d)

$$\frac{\partial \boldsymbol{r}_{\Delta \boldsymbol{p}_{ij}}}{\partial \boldsymbol{\phi}_i} = \left(\boldsymbol{R}_i^\top \left(\boldsymbol{p}_j - \boldsymbol{p}_i - \boldsymbol{v}_i \Delta t_{ij} - \frac{1}{2} \boldsymbol{g} \Delta t_{ij}^2 \right) \right)^\wedge .$$

2. 实现由Odom数据触发的图优化（g2o）
3. 利用数值求导工具，验证本书实验中的雅可比矩阵的正确性