

1 Smooth Manifolds made Concrete for Dumbasses

The intention behind this document is to provide an extended and comprehensive example of a smooth manifold for dumbasses like me who can't always grok the heavily abstract without a more concrete example, or as the OO framework types would say, a reification. The extended example is the prototypical/canonical stereographic projection considering the sphere S^2 as a manifold. Examples cover deriving the charts, finding the inverse functions between manifolds including deriving an ellipsoid stereographic manifold, and tangent spaces including both equivalence classes of curves and derivation based approaches, illustrating proofs of vector space closure and change of basis of tangent vectors. The examples are based on using the open source SageMath Computer Algebra System (CAS) to create an interactive notebook. Sage is available on most Linux distributions and the web site also has a Windows installer. The notebook has also been converted to a Latex/SageTex document which hides most of the Sage specific manipulation and shows just the mathematics for those who aren't interested in Sage (although you lose the ability to interact with the content including the graphs). Note Sage does have built in Manifold support, but this document won't be using it as the intention is to provide examples from first principles. Errors or misunderstandings on my part or also more than likely as I am a computer scientist who dabbles in mathematics, for Computer Vision, Machine Learning and Neural Networks, therefore any error corrections or suggestions for improvements would also be welcome.

1.1 The Stereographic Projection Atlas

The stereographic projection is the canonical example for a topological/smooth manifold. The example shown here will map S^2 to R^2 , although it can be extended to $S^n \rightarrow R^n$.

The sphere S^2 is seen as a manifold having an atlas with two open sets and associated charts. The first chart is defined on the open set defined by the full sphere excluding the north (top) pole $(0, 0, 1)$, while the second chart is defined on a set that excludes only the south (bottom) $(0, 0, -1)$ pole. Both charts project a point on the sphere surface to the plane at $z = 0$ by extending a line from the excluded pole through the point on the manifold (sphere) being charted onto the plane.

To derive the chart equation let $\vec{o}_n = (0, 0, 1)$, $\vec{p}_n = (x, y, z)$ be a point on the sphere and $\vec{s}_n = (u, v, 0)$ be the projection then (see the figure below):

$$\vec{o}_n - \vec{p}_n = l(\vec{o}_n - \vec{s}_n) \text{ which evaluates to: } (-x, -y, -z + 1) = (-lu, -lv, l)$$

Solving for the third component of the vector equation and substituting into the first two:

$$l = -z + 1$$
$$\left(u = \frac{x}{l}, v = \frac{y}{l}\right)$$

\therefore the charts with chart coordinates given by u and v are given by.

North pole:

$$\left(-\frac{x}{z-1}, -\frac{y}{z-1}\right)$$

South pole:

$$\left(-\frac{x}{z-1}, -\frac{y}{z-1}\right)$$

1.2 Visualization

To visualize the manifold and charts see Figure 1 which demonstrates manifold points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$ projected to the plane $z = 0$ (it does look a lot better in the original Jupyter notebook):

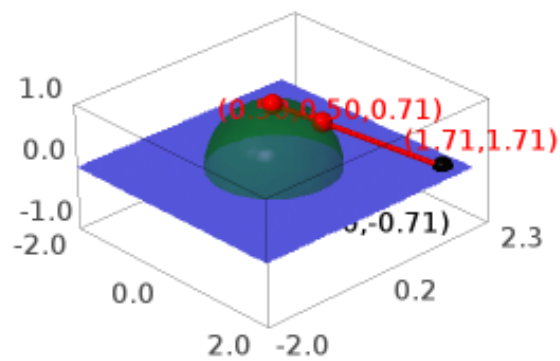


Figure 1: Visualization of stereographic projection.

1.3 Chart Inverses

The above charts are bijective and continuous within the domain $S^2 - (0, 0, 1), S^2 - (0, 0, -1)$ and codomain $(u, v) \in \mathbb{R}^2$, however to be valid charts they must also be bicontinuous, ie the inverse must be continuous too, that is they must be homeomorphic to \mathbb{R}^2 .

To find the inverse first find the unit vector in the direction of the line joining the pole to the projected coordinates, for example $\vec{d} = \frac{((0,0,1)-(u,v,0))}{|(0,0,1)-(u,v,0)|} = \left(-\frac{u}{\sqrt{u^2+v^2+1}}, -\frac{v}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}\right)$.

Now express the line from the pole to the projection as a vector equation:

$$(0, 0, 1) + l\vec{d} = \left(-\frac{lu}{\sqrt{u^2+v^2+1}}, -\frac{lv}{\sqrt{u^2+v^2+1}}, \frac{l}{\sqrt{u^2+v^2+1}} + 1\right)$$

The intersection of this line with the sphere occurs where $|(0, 0, 1) + l\vec{d} - (0, 0, 0)| = 1$, or more generally $|(0, 0, 1) + l\vec{d} - \vec{c}| = r$ where \vec{c} is the sphere centre and r is the radius. This can also be expressed as $[(0, 0, 1) + l\vec{d} - (0, 0, 0)] \cdot [(0, 0, 1) + l\vec{d} - (0, 0, 0)] = r^2$:

$$\frac{l^2 u^2}{u^2+v^2+1} + \frac{l^2 v^2}{u^2+v^2+1} + \left(\frac{l}{\sqrt{u^2+v^2+1}} + 1\right)^2 = 1$$

This quadratic will have one solution for $l = 0$ at the pole. The other solution will be the required (x, y, z) on the sphere manifold. Letting Sage do the PT:

$$\left[l = -\frac{2}{\sqrt{u^2+v^2+1}}, l = 0\right]$$

and substituting the appropriate l back into the line equation:

$$\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, -\frac{2}{u^2+v^2+1} + 1\right)$$

See also the Wikipedia page on sphere-line intersection (note their notation swops \vec{d} and l as used in the above).

Similarly the inverse for the south pole chart is given by (it could probably also be derived by symmetry):

$$\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{2}{u^2+v^2+1} - 1\right)$$

1.4 Verifying the Charts

The two charts described above are homeomorphic and are also differentiable with a differentiable inverse, therefore they are diffeomorphic too. Assume the charts are named c_n and c_s , then to verify the two charts define a compatible atlas it is required that the intersection of their domain open sets be empty ie $S^2 - (0, 0, 1) \cap S^2 - (0, 0, -1) = \emptyset$, (definitely not the case) or $c_s \circ c_n^{-1} : c_n(S^2 - (0, 0, 1) \cap S^2 - (0, 0, -1)) \rightarrow c_s(S^2 - (0, 0, 1) \cap S^2 - (0, 0, -1))$ and the opposite $(c_n \circ c_s^{-1})$ is diffeomorphic. After performing the composition we get:

$$\left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

which is homeomorphic and smooth over the specified domain.

1.5 Smooth Functions between Manifolds

A smooth map $F : M \rightarrow N$ can be described [Lee, 2012] as “if for every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U)$ to $\psi(V)$ ”.

We will start by defining a manifold E on an ellipsoid to use as the destination manifold corresponding to N above for the map. U will then correspond to either $S^2 - (0, 0, 1)$ or $S^2 - (0, 0, -1)$ depending on the source chart selection. N will similarly correspond to $E - (0, 0, 1)$ or $E - (0, 0, -1)$. Because E uses $(0, 0, 1)$ and $(0, 0, -1)$ the a and b ellipse axes are arbitrarily defined but $c = 1$ (the z semi-axis) to ensure that the poles are at $(0, 0, 1)$ and $(0, 0, -1)$.

1.5.1 The Ellipsoid as a Stereographical Manifold

The chart map from the ellipsoid to R^2 can be derived similarly to the sphere, resulting in $u = -\frac{x}{z-1}$, $v = -\frac{y}{z-1}$ for the north chart and $u = \frac{x}{z+1}$, $v = \frac{y}{z+1}$, but with z calculated for the ellipse. The inverse is a little more problematic however. As with the sphere start by describing the line from the pole to the projection in terms of a unit vector pointing in the direction of the line:

$$L = (0, 0, 1) + l \frac{(0, 0, 1) - (u, v, 0)}{\|(0, 0, 1) - (u, v, 0)\|} = \left(-\frac{lu}{\sqrt{u^2 + v^2 + 1}}, -\frac{lv}{\sqrt{u^2 + v^2 + 1}}, \frac{l}{\sqrt{u^2 + v^2 + 1}} + 1 \right) \quad (1)$$

Recall that a vector equation for an ellipsoid can be defined as $(\mathbf{x} - \mathbf{o})^T A (\mathbf{x} - \mathbf{o}) = 1$ where 0 is the origin coordinates and A is a positive definite matrix which in the general case may include rotation, but in our case contains only the ellipsoid axes:

$$\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The ellipsoid equation is then $L^T A L$:

$$\left(\frac{l}{\sqrt{u^2 + v^2 + 1}} + 1 \right)^2 + \frac{l^2 u^2}{(u^2 + v^2 + 1)a^2} + \frac{l^2 v^2}{(u^2 + v^2 + 1)b^2} = 1$$

with two solutions again with one at 0 corresponding to the pole:

$$\left[l = -\frac{2\sqrt{u^2 + v^2 + 1}a^2b^2}{a^2b^2 + b^2u^2 + a^2v^2}, l = 0 \right]$$

Substituting back into L (Equation 1):

$$\left(\frac{2a^2b^2u}{a^2b^2 + b^2u^2 + a^2v^2}, \frac{2a^2b^2v}{a^2b^2 + b^2u^2 + a^2v^2}, -\frac{2a^2b^2}{a^2b^2 + b^2u^2 + a^2v^2} + 1 \right)$$

Similarly for the south pole we get:

$$\left(\frac{2a^2b^2u}{a^2b^2 + b^2u^2 + a^2v^2}, \frac{2a^2b^2v}{a^2b^2 + b^2u^2 + a^2v^2}, \frac{2a^2b^2}{a^2b^2 + b^2u^2 + a^2v^2} - 1 \right)$$

1.6 Defining the Inter-Manifold Function

A linear transformation applied to a sphere results in an ellipse. In this case we use a linear transformation matrix:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Applied to $(x, y, \sqrt{-x^2 - y^2 + 1})$ resulting in $F : S \rightarrow E$ where S is the sphere manifold and E is the ellipsoid one:

$$F = (ax, by, \sqrt{-x^2 - y^2 + 1})$$

Let the ellipse chart be $(V, e_n) : V \subset E$ (or $(V, e_s) : V \subset E$) and the sphere chart be $(U, s_n) : U \subset S$ (or $(U, s_s) : U \subset S$), then

$$e_n \circ F \circ s_n^{-1}[s_n(U)]$$

should be a smooth function from R^2 (the stereographic projection coordinates for the sphere) to R^2 (the stereographic projection coordinates for the ellipse) (its probably best to take Sage's word for it unless you **really** enjoy Algebra):

$$\left(\frac{2au}{u^2+v^2-(u^2+v^2+1)\sqrt{\frac{u^4+v^4+2(u^2-1)v^2-2u^2+1}{u^4+v^4+2(u^2+1)v^2+2u^2+1}+1}}, \frac{2bv}{u^2+v^2-(u^2+v^2+1)\sqrt{\frac{u^4+v^4+2(u^2-1)v^2-2u^2+1}{u^4+v^4+2(u^2+1)v^2+2u^2+1}+1}} \right)$$

1.7 Tangent Vectors and Spaces

Confusingly there are three different approaches to defining tangent spaces on manifolds [Montgomery, 2016]. These examples will cover the geometrically based curve approach and the more algebraic derivation approach.

1.7.1 Tangent Spaces defined as Equivalence class of Curves

This approach starts with a curve with a domain on an interval of R^1 and the manifold M as the co-domain i.e $\lambda : I \subset R^1 \rightarrow M$. The simplest approach combines the co-domain of the curve with a chart map to define curves as being tangent at a point p if [see Isham, 1999, pp. 73]:

1. Values on the manifold correspond ie $\lambda_1(t_0) = \lambda_2(t_1) = p$,
2. The velocities at the point in a chart are the same ie $(c_n \circ \lambda_1)'|_{t=t_0} = (c_n \circ \lambda_2)'|_{t=t_1}$

The curve used in the example is a spiral on S^2 (from math.stackexchange)

$$\lambda_1 = (x, y, z) : (\sqrt{1-t^2} \cos(a\pi t), \sqrt{1-t^2} \sin(a\pi t), t)$$

To find the geometric tangent vector at a point we use the north pole chart to project a point from the manifold specified by the curve to the R^2 Euclidean space of the stereographic projection:

As a reminder, the north chart: $(-\frac{x}{z-1}, -\frac{y}{z-1})$

$$(c_n \circ \lambda_1) = \left(-\frac{\sqrt{-t^2+1} \cos(7\pi t)}{t-1}, -\frac{\sqrt{-t^2+1} \sin(7\pi t)}{t-1} \right)$$

Next we differentiate the above to get the tangent vector:

$$\left(\frac{7\pi \sqrt{-t^2+1} \sin(7\pi t)}{t-1} + \frac{t \cos(7\pi t)}{\sqrt{-t^2+1}(t-1)} + \frac{\sqrt{-t^2+1} \cos(7\pi t)}{(t-1)^2}, -\frac{7\pi \sqrt{-t^2+1} \cos(7\pi t)}{t-1} + \frac{t \sin(7\pi t)}{\sqrt{-t^2+1}(t-1)} + \frac{\sqrt{-t^2+1} \sin(7\pi t)}{(t-1)^2} \right)$$

Next to find another tangent vector with the necessary properties (having the same value in the manifold at t and equal velocity (derivative) in the chart), we find the inverse of the tangent line in the chart which results in a circle in the manifold (s is the parametric variable for the tangent line/circle) $(0.9270, -2.853) + s(65.32, 12.46)$:

$$\lambda_2 = \left(\frac{3 \left(63\pi s \sqrt{2\sqrt{5}+10} + 25s(\sqrt{5}-1) + 9\sqrt{5}-9 \right)}{2((3969\pi^2+625)s^2+450s+90)}, \frac{3 \left(63\pi s(\sqrt{5}-1) - (25s+9)\sqrt{2\sqrt{5}+10} \right)}{2((3969\pi^2+625)s^2+450s+90)}, \frac{(3969\pi^2+625)s^2+450s+72}{(3969\pi^2+625)s^2+450s+90} \right)$$

Plotting the results on the manifold (Figure 2):

Verifying the two curves satisfy the equivalence class criteria:

Value on Manifold:

(0.1854102, -0.5706339, 0.8000000)
 ((0.1854102, -0.5706339, 0.8000000))

Velocity in chart:

(65.31962, 12.46144)
 (65.31962, 12.46144)

1.8 Derivation based Approach

A derivation at a point in a C^∞ manifold is defined to be a linear map $D(C^\infty) \rightarrow R$,

$$(D_1 + D_2)(f) = D_1(f) + D_2(f)$$

$$(\lambda D)(f) = \lambda D(f) \text{ which also obeys the Leibnitz identity } D(fg) = D(f) \cdot g(x) + f(x) \cdot D(g).$$

Computer science types could see the derivation as defining an abstract interface, with the actual derivative being a concrete class implementing the abstract interface.

See Parzygnat [2017] (lecture 12 part 3) for a description of derivations and associated associative algebra (in a Euclidean manifold context, but most of what is covered also applies to general manifolds, other than the manifold chart map directions are inverted).

A tangent vector can then be applied to a C^∞ function f given a curve λ by $D(f \circ \lambda)[t]$ where $\lambda(t) = \vec{p} \in M$ where \vec{p} is the point in the manifold where the tangent is taken. The derivation based velocities defined this way should form a vector space similarly to the previous section tangent vectors which can fairly easily be shown to form a vector space by adding/multiplying in chart space [see Isham, 1999, pp. 76]. Proving a vector space using the $D(f \circ \lambda)[t]$ approach is a little more difficult. Additive closure is covered quite lengthily in lesson 9 of XylyXylyX [2017] and more concisely by Schuller [2015] lecture 5, while Schuller also covers scalar multiplication.

The above proofs are regurgitated here for documentary purposes as I couldn't find a book containing the proofs. Intermediate results in the proofs are also illustrated using the stereographic example.

Multivariable differentiation is denoted, as for derivations, with a D operator. For multivariable functions and maps the differentiation result will be a Jacobian, and in many cases with tangent spaces mapping to R^1 will be vectors, although transition maps between charts for example will be matrices. In some cases the differentiation operator D will be subscripted with the variables w.r.t. which the differentiation is taking place eg D_t . This is in contrast with many physics books which use the old fashioned multiple different chain rules to extensively utilize component indexing, instead of one chain rule using linear transformations with Jacobian matrices applying in all cases (as for example in Spivak [1965] or more colorfully in Ghrist [2016] and Ghrist [2019]). Where applicable the comparable 'standard' notation will also be shown.

We will continue to use the curves defined in the previous section. We also define $f(\vec{p}) : S \rightarrow R = 2x + 2y + 2z$ (where z is constrained to be on the manifold ie $z = \sqrt{1 - x^2 - y^2}$). Now $D(f \circ \lambda_1)$ and $D(f \circ \lambda_2)$ are velocities:

$$(f \circ \lambda_1)' = 14\pi \sqrt{-t^2+1} \cos(7\pi t) - 14\pi \sqrt{-t^2+1} \sin(7\pi t) - \frac{2t \cos(7\pi t)}{\sqrt{-t^2+1}} - \frac{2t \sin(7\pi t)}{\sqrt{-t^2+1}} + 2$$

$$(f \circ \lambda_2)' = \frac{4((3969\pi^2+625)s+225)}{(3969\pi^2+625)s^2+450s+90} - \frac{4((3969\pi^2+625)s^2+450s+72)((3969\pi^2+625)s+225)}{((3969\pi^2+625)s^2+450s+90)^2} - \frac{6(63\pi s(\sqrt{5}-1)-(25s+9)\sqrt{2\sqrt{5}+10})((3969\pi^2+625)s+225)}{((3969\pi^2+625)s^2+450s+90)^2} - \frac{6(63\pi s\sqrt{2\sqrt{5}+10}+25s(\sqrt{5}-1)+9\sqrt{5}-9)((3969\pi^2+625)s+225)}{((3969\pi^2+625)s^2+450s+90)^2} + \frac{3(63\pi(\sqrt{5}-1)-25\sqrt{2\sqrt{5}+10})}{(3969\pi^2+625)s^2+450s+90} + \frac{3(63\pi\sqrt{2\sqrt{5}+10}+25\sqrt{5}-25)}{(3969\pi^2+625)s^2+450s+90}$$

To demonstrate the curves form a vector space (for this example anyway), before providing a proof (of closure under addition and scalar multiplication - the rest is left as an exercise for the reader ☺) we first set $\lambda_3 = \lambda_1 + \lambda_2$ and then substituting the values for the intersection point specified by $s = 0$ and $t = \frac{4}{5}$ into the three different velocities.

$$(D(f \circ \lambda_1))[\frac{4}{5}] = 36.96466$$

$$(D(f \circ \lambda_2))[0] = 36.96466$$

$$(D(f \circ \lambda_3))[\frac{4}{5}] = 36.96466$$

1.8.1 Additive Closure Proof

The proof starts with a construction, which given two existing curves λ_1, λ_2 intersecting at point \vec{p} at parameter values t_1 and t_2 , constructs a third curve λ_3 also intersecting \vec{p} at a parameter t_3 ie $\lambda_1(t_1) = \lambda_2(t_2) = \lambda_3(t_3) = \vec{p}$:

$$\lambda_3(t) = c^{-1}\{(c \circ \lambda_1)[t_1 + t] + (c \circ \lambda_2)[t_2 + t] - (c \circ \lambda_1)[t_1]\}$$

At $t = 0$ this evaluates to $c^{-1}\{c(\vec{p}) + c(\vec{p}) - c(\vec{p})\} = c^{-1}(c(\vec{p})) = \vec{p}$ so $t_3 = 0$. Checking this construct results in the same values as the other two curves:

Value on Manifold: "(0.1854102, -0.5706339, 0.8000000)

Because the third curve parameter t_3 evaluates to 0, we can therefore write the velocity vector for λ_3 at \vec{p} as $(f \circ \lambda_3)'[t_3 = 0]$.

Inserting a chart and its inverse and using associativity results in

$$\{(f \circ c^{-1}) \circ (c \circ \lambda_3))'[0].$$

Using the chain rule with the $f \circ c^{-1}$ as the outer function:

$$D(f \circ c^{-1})[(c \circ \lambda_3)[0]) D(c \circ \lambda_3)[0]$$

Substituting the expression for λ_3 into the right side term:

$$D(c \circ \lambda_3)[0] = D(c \circ c^{-1}\{(c \circ \lambda_1)[t_1 + t] + (c \circ \lambda_2)[t_2 + t] - (c \circ \lambda_1)[t_1]\}) =$$

$D((c \circ c^{-1} \circ c \circ \lambda_1))[t_1 + t] + (c \circ c^{-1} \circ c \circ \lambda_2)[t_2 + t] - (c \circ c^{-1} \circ c \circ \lambda_1)[t_1])$ (distributivity, associativity)

The third term is not dependent on t so the derivative would be 0 \therefore is will disappear and the inner $c^{-1} \circ c$ in the other terms will cancel leaving:

$$D((c \circ \lambda_1))[t_1 + t] + (c \circ \lambda_2)[t_2 + t])[0] = D(c \circ \lambda_1)[t_1] + D(c \circ \lambda_2)[t_2]$$

The left hand term can also be simplified:

$$D(f \circ c^{-1})[(c \circ \lambda_3)[0]] \equiv D(f \circ c^{-1})[(c \circ \lambda_3[0])] = D(f \circ c^{-1})[c(\vec{p})]$$

as $\lambda_3[0] = \vec{p}$ by construction.

Recombining the simplified left and right sides:

$$D(f \circ c^{-1})[c(\vec{p})] \cdot (D(c \circ \lambda_1)[t_1] + D(c \circ \lambda_2)[t_2]) = D(f \circ c^{-1})[c(\vec{p})] D(c \circ \lambda_1)[t_1] + D(f \circ c^{-1})[c(\vec{p})] D(c \circ \lambda_2)[t_2] \quad (2)$$

Using the chain rule to differentiate $D(f \circ \lambda_1)[t_1]$ to show that it is the same as the first term above:

$$\begin{aligned} D(f \circ \lambda_1)[t_1] &= D((f \circ c^{-1}) \circ (c \circ \lambda_1))[t_1] = \\ D(f \circ c^{-1})[(c \circ \lambda_1)[t_1]] \cdot D(c \circ \lambda_1)[t_1] &= \\ D(f \circ c^{-1})[c(\vec{p})] \cdot D(c \circ \lambda_1)[t_1] \end{aligned}$$

which is the first term in the previous expression (Equation 2) (combining the simplified left and right sides).

Similarly the second term $D(f \circ c^{-1})[c(\vec{p})] D(c \circ \lambda_2)[t_2]$ can be shown by reversing the chain rule to be $D(f \circ \lambda_2)[t_2] \therefore$ the equation that shall remain unnumbered can be reduced to:

$$D(f \circ \lambda_1)[t_1] + D(f \circ \lambda_2)[t_2]$$

which proves closure under addition of velocity vectors in the tangent space. We can verify the above terms for our concrete example:

$$f \circ \lambda_{1,2,3} = 0.8295526, 0.8295526, 0.8295526$$

$$D(f \circ \lambda_1) + D(f \circ \lambda_2) = 36.96466 + 36.96466 = 73.92932 = D(f \circ \lambda_3) = 73.92932$$

1.8.2 Scalar Multiplication Closure

Proving closure under scalar multiplication is easier and does not involve any charts. Again start with a construction for the new curve λ_3 tangent at \vec{p} ie $\lambda_1(t_1) = \lambda_3(t_3) = \vec{p}$ in terms of λ_1 :

$$\lambda_3(t) = \lambda_1(at + t_1) = (\lambda_1 \circ u_a)[t_3] \quad a \in R, \quad u_a(t) : R \rightarrow R = at + t_1$$

$$\text{At } t = 0 \text{ this evaluates to } \lambda_3(0) = \lambda_1(0 \cdot a + t_1) = \lambda_1(t_1) = \vec{p} \text{ so } t_3 = 0.$$

$$D(f \circ \lambda_3)[t_3] = D(f \circ \lambda_3)[0] = D((f \circ \lambda_1) \circ u_a)[0]$$

Applying the chain rule:

$$D(f \circ \lambda_1)[u_a(0)] \cdot D(u_a)[0] = D(f \circ \lambda_1)[t_1] \cdot a = aD(f \circ \lambda_1)[t_1]$$

which proves closure under scalar multiplication.

1.9 Chart Realizations for Tangent Space

We start by introducing a chart c into a velocity vector applied to an arbitrary function f for a curve $\lambda(t_1) = \vec{p} \in U \subseteq M$:

$$D((f \circ c^{-1}) \circ (c \circ \lambda))[t_1]$$

Applying the chain rule once again:

$$D_{u,v}(f \circ c^{-1})[c \circ \lambda(t_1)] \cdot D_t(c \circ \lambda)[t_1] = D_t(c \circ \lambda)[t_1] \cdot D_{u,v}(f \circ c^{-1})[c(\vec{p})]$$

The representation of derivation based tangent space vectors w.r.t charts as used above in the proofs is quite complicated, so a simplified representation, otherwise known as syntactic sugar in the computer science community, is conventionally used, particularly in Physics, which is shown here along with an example.

We start by investigating what the left and right hand sides of the above look like for a concrete example using the north chart of the stereographic projection and the spiral curve on the sphere. To do this we take the Jacobian of both sides and then do matrix multiplication (or dot product as they are both vectors) on the jacobians:

$$LHS = \begin{pmatrix} \frac{7\pi\sqrt{-t^2+1}\sin(7\pi t)}{t-1} + \frac{t\cos(7\pi t)}{\sqrt{-t^2+1}(t-1)} + \frac{\sqrt{-t^2+1}\cos(7\pi t)}{(t-1)^2} & -\frac{7\pi\sqrt{-t^2+1}\cos(7\pi t)}{t-1} + \frac{t\sin(7\pi t)}{\sqrt{-t^2+1}(t-1)} + \frac{\sqrt{-t^2+1}\sin(7\pi t)}{(t-1)^2} \end{pmatrix} \left[t = \frac{4}{5} \right]$$

$$RHS = \begin{pmatrix} \frac{4(u+1)}{u^2+v^2+1} - \frac{4(u^2+v^2+2u+2v-1)u}{(u^2+v^2+1)^2} & \frac{4(v+1)}{u^2+v^2+1} - \frac{4(u^2+v^2+2u+2v-1)v}{(u^2+v^2+1)^2} \end{pmatrix} \left[(u,v) = (0.927051, -2.85317) \right]$$

The right hand side of the expression is simplified as follows:

1. As the derivative is being taken Leibnitz notation is used (odd - one would have thought physicists would prefer Newton), the D is replaced with a del i.e. ∂ ,
2. The function is used in the numerator and the chart is used in the denominator in the Leibnitz notation for differentiation i.e. $\frac{\partial f}{\partial c}$,
3. The coordinate index of the component being differentiated in the vector is used as a superscript in the denominator (the use of vectors and matrices was implied in the Jacobian differentiation operator used in most of this document) $\frac{\partial f}{\partial c^i}$,
4. Finally the point on the manifold is given as a subscript i.e. $(\frac{\partial f}{\partial c^i})_p$.

The left hand side is also simplified:

1. As the curve is being differentiated with respect to a parameter which can be interpreted as time, Newton finally arrives on the scene, with the Newtonian notation with a dot on top of the curve being used i.e. $\dot{\lambda}$,

2. As with the RHS the component index is used as a superscript (which also ties in with using Einstein summation for the entire expression) i.e. λ^i ,
3. The curve is used as a subscript i.e. λ_c^i
4. Finally the point in the manifold is added i.e. $\lambda_c^i(t_1)$

The entire shorthand expression for a component is expressed as an operator which operates on C^∞ functions: $\lambda_c^i(t_1)(\frac{\partial}{\partial c^i})_p f$

The $(\frac{\partial}{\partial c^i})_p$ part (or more accurately its full expansion) forms a basis for the vector space of derivations defining the tangent space. Isham [see 1999, pp. 82-86] provides a proof of this as well as the isomorphism between the derivation and the previously described geometric tangent space. This basis is known as a chart induced basis.

The example shows the basis dependent on u, v and the components on t :

$$\left(\begin{array}{l} 4 \left(\frac{7\pi \sqrt{-t^2+1} \sin(7\pi t)}{t-1} + \frac{t \cos(7\pi t)}{\sqrt{-t^2+1}(t-1)} + \frac{\sqrt{-t^2+1} \cos(7\pi t)}{(t-1)^2} \right) \left(\frac{u+1}{u^2+v^2+1} - \frac{(u^2+v^2+2u+2v-1)u}{(u^2+v^2+1)^2} \right) - \\ 4 \left(\frac{7\pi \sqrt{-t^2+1} \cos(7\pi t)}{t-1} - \frac{t \sin(7\pi t)}{\sqrt{-t^2+1}(t-1)} - \frac{\sqrt{-t^2+1} \sin(7\pi t)}{(t-1)^2} \right) \left(\frac{v+1}{u^2+v^2+1} - \frac{(u^2+v^2+2u+2v-1)v}{(u^2+v^2+1)^2} \right) \end{array} \right) = (36.9647)$$

1.10 Change of basis for Tangent Space Components

Given two charts (U, c_1) and (V, c_2) where $U \cap V \neq \emptyset$, where the tangent vector to a point $\vec{p} \in U, V$ is known relative to say U , then the equivalent basis vector for V can be derived:

1. Recall the expression for the basic component from the previous section and once again apply the idiom of using the inverse of a map after the map, in this case the map being the chart from V . $(\frac{\partial f}{\partial c_1^{i:1 \dots m}})_p = D(f \circ c_1^{-1})[c_1(\vec{p})] = D((f \circ c_2^{-1}) \circ (c_2 \circ c_1^{-1}))[c_1(\vec{p})]$
2. Utilise the chain rule again: $D_{u,v}(f \circ c_2^{-1})[c_2 \circ c_1^{-1}(c_1(\vec{p}))] \cdot D_{u,v}(c_2 \circ c_1^{-1})[c_1(\vec{p})] = D_{u,v}(c_2 \circ c_1^{-1})[c_1(\vec{p})] \cdot D_{u,v}(f \circ c_2^{-1})[c_2(\vec{p})]$.
3. The full expression for the tangent vector in U is given by: $\lambda_{c_1}^i(t_1)(\frac{\partial f}{\partial c_1^{i:1 \dots m}})_p = D(c_1 \circ \lambda)[t_1] \cdot D(f \circ c_1^{-1})[c_1(\vec{p})]$ Substituting the expression from 2: $D(c_1 \circ \lambda)[t_1] \cdot D(f \circ c_1^{-1})[c_1(\vec{p})] = D_t(c_1 \circ \lambda)[t_1] \cdot D_{u,v}(c_2 \circ c_1^{-1})[c_1(\vec{p})] \cdot D_{u,v}(f \circ c_2^{-1})[c_2(\vec{p})]$
4. The full expression for the tangent vector in V is given by: $\lambda_{c_2}^i(t_2)(\frac{\partial f}{\partial c_2^{i:1 \dots n}})_p = D(c_2 \circ \lambda)[t_2] \cdot D(f \circ c_2^{-1})[c_2(\vec{p})]$
5. As the curve on the manifold remains the same, the tangent vector is also the same, its just expressed differently in the two bases, therefore these expressions can be equated: $D_t(c_2 \circ \lambda)[t_2] \cdot D_{u,v}(f \circ c_2^{-1})[c_2(\vec{p})] = D_t(c_1 \circ \lambda)[t_1] \cdot D_{u,v}(c_2 \circ c_1^{-1})[c_1(\vec{p})] \cdot D_{u,v}(f \circ c_2^{-1})[c_2(\vec{p})]$
The components in V are thus $D_t(c_1 \circ \lambda)[t_1] \cdot D_{u,v}(c_2 \circ c_1^{-1})[c_1(\vec{p})]$

The $D_{u,v}(c_2 \circ c_1^{-1})[c_1(\vec{p})]$ (Jacobian of the overlap function between U and V) is $R^m \rightarrow R^n$ so the Jacobian will be a $n \times m$ matrix. This matrix is multiplied by a $m \times 1$ vector of the map from the curve to chart coordinates via the manifold.

Applying this to the example using a change of basis from the north to the south hemisphere of the stereographic projection (the point at which the tangent are found is the northern hemisphere test point, and the curve is the same as the one used in the previous tangent space examples):

$$\begin{aligned}
& \left(\begin{array}{cc} -\frac{2u^2}{(u^2+v^2)^2} + \frac{1}{u^2+v^2} & -\frac{2uv}{(u^2+v^2)^2} \\ -\frac{2uv}{(u^2+v^2)^2} & -\frac{2v^2}{(u^2+v^2)^2} + \frac{1}{u^2+v^2} \end{array} \right) \left(\begin{array}{c} \frac{7\pi\sqrt{-t^2+1}\sin(7\pi t)}{t-1} + \frac{t\cos(7\pi t)}{\sqrt{-t^2+1}(t-1)} + \frac{\sqrt{-t^2+1}\cos(7\pi t)}{(t-1)^2} \\ -\frac{7\pi\sqrt{-t^2+1}\cos(7\pi t)}{t-1} + \frac{t\sin(7\pi t)}{\sqrt{-t^2+1}(t-1)} + \frac{\sqrt{-t^2+1}\sin(7\pi t)}{(t-1)^2} \end{array} \right) \Big|_{t=0.8} = \\
& \left(\begin{array}{c} -\frac{1}{12} \left(63\pi\sqrt{2\sqrt{5}+10} + 25\sqrt{5}-25 \right) \left(\frac{2u^2}{(u^2+v^2)^2} - \frac{1}{u^2+v^2} \right) - \frac{(63\pi(\sqrt{5}-1)-25\sqrt{2\sqrt{5}+10})uv}{6(u^2+v^2)^2} \\ -\frac{1}{12} \left(63\pi(\sqrt{5}-1) - 25\sqrt{2\sqrt{5}+10} \right) \left(\frac{2v^2}{(u^2+v^2)^2} - \frac{1}{u^2+v^2} \right) - \frac{(63\pi\sqrt{2\sqrt{5}+10}+25\sqrt{5}-25)uv}{6(u^2+v^2)^2} \end{array} \right) \\
& \left(\begin{array}{c} -\frac{1}{12} \left(63\pi\sqrt{2\sqrt{5}+10} + 25\sqrt{5}-25 \right) \left(\frac{2u^2}{(u^2+v^2)^2} - \frac{1}{u^2+v^2} \right) - \frac{(63\pi(\sqrt{5}-1)-25\sqrt{2\sqrt{5}+10})uv}{6(u^2+v^2)^2} \\ -\frac{1}{12} \left(63\pi(\sqrt{5}-1) - 25\sqrt{2\sqrt{5}+10} \right) \left(\frac{2v^2}{(u^2+v^2)^2} - \frac{1}{u^2+v^2} \right) - \frac{(63\pi\sqrt{2\sqrt{5}+10}+25\sqrt{5}-25)uv}{6(u^2+v^2)^2} \end{array} \right) \left[\begin{array}{l} u = 0.92705 \\ v = -2.8532 \end{array} \right] = \\
& \left(\begin{array}{c} 6.685482 \\ 3.145821 \end{array} \right)
\end{aligned}$$

We can test the change of basis by calculating the component representation in the south chart directly:

$$\begin{aligned}
& D_{x,y,z} \left(\frac{x}{z+1}, \frac{y}{z+1} \right) \left[\begin{array}{l} x = \sqrt{-t^2+1}\cos(7\pi t) \\ y = \sqrt{-t^2+1}\sin(7\pi t) \\ z = t \end{array} \right] = \\
& \left(\begin{array}{c} -\frac{7\pi\sqrt{-t^2+1}\sin(7\pi t)}{t+1} - \frac{t\cos(7\pi t)}{\sqrt{-t^2+1}(t+1)} - \frac{\sqrt{-t^2+1}\cos(7\pi t)}{(t+1)^2} \\ \frac{7\pi\sqrt{-t^2+1}\cos(7\pi t)}{t+1} - \frac{t\sin(7\pi t)}{\sqrt{-t^2+1}(t+1)} - \frac{\sqrt{-t^2+1}\sin(7\pi t)}{(t+1)^2} \end{array} \right) \Big|_{t=0.8} = \\
& \left(\begin{array}{cc} 6.68548 & 3.14582 \end{array} \right)
\end{aligned}$$

Most (physics) textbooks use a shorthand notation for $D_{u,v}(c_2 \circ c_1^{-1})[c_1(\vec{p})] \cdot D_t(c_1 \circ \lambda)[t_1]$, namely $\frac{\partial c_2^j}{\partial c_1^i} \cdot X_{c_1}^i$ where the $\frac{\partial c_2^j}{\partial c_1^i}$ refers to the Jacobian matrix with the superscript indices corresponding to rows and columns from the matrix, while the $X_{c_1}^i$ term is a generic reference to the components in the convert-from chart minus any curve information.

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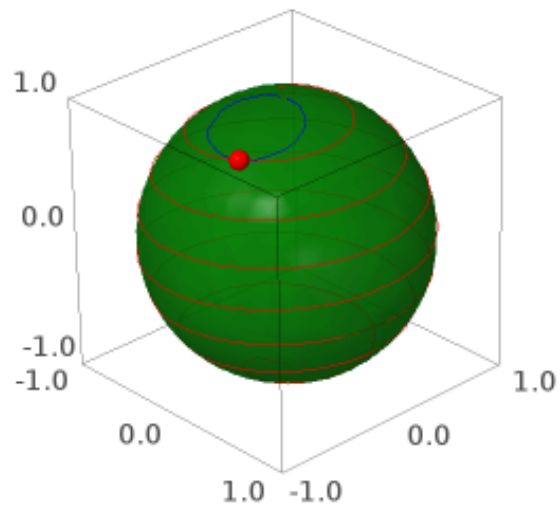


Figure 2: Visualization of tangent vector curves