Washington University in St. Louis

James McKelvey School of Engineering

Fall 2020 MEMS 4050 Vibrations Laboratory

Pre-Lab 4: Vibrational Analysis of a Multi Degree-of-Freedom System

Lab Instructor: Dr. Bayly

Group T (Friday 2PM)

We hereby certify that the lab report herein is our original academic work, completed in accordance to the McKelvey School of Engineering and Undergraduate Student academic integrity policies, and submitted to fulfill the requirements of this assignment:

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Mitry Anderson	*Quality Control Engineer*
Aidan Murphy	*Test Engineer II*

1 Question 1

(*4pts*) Figure 1 below shows an undamped four degree of freedom system fixed on both ends.

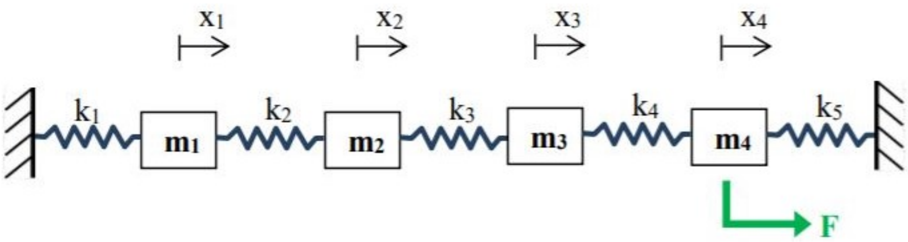


Figure 1 Four degree of freedom undamped system [1].

a) Draw free-body diagrams of the different components and determine the equation of motion of each.

Figure 2 below depicts each free body diagram for all of the components in the undamped four degree-or-freedom system from Fig. 1. Each of the 5 springs and each of the 4 masses have their respective free body diagrams displayed.

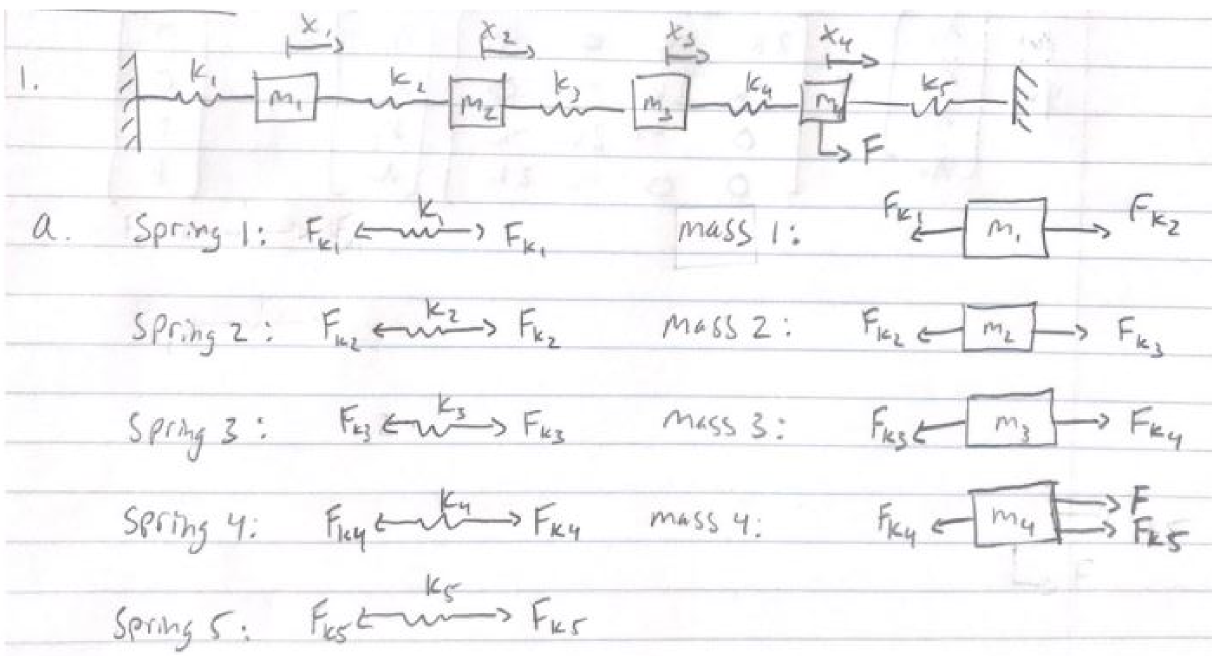


Figure 2 Free body diagrams for each component in the the system provided (Fig. 1).

b) Fill in the equation of motion of the system.

Figure 3 represents the equation of motion for this undamped system in matrix form.

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F \end{bmatrix}$$

Figure 3 Matrix representation for the equation of motion for this system.

c. Assume that all the masses are equal (i.e. all of mass m [kg]) and all the springs have the same spring constant k [N/m]. Plug in and simplify the equation of motion. Identify the mass and stiffness matrices of the system.

The following figure represent a simplified version of the equation of motion from Fig. 3 with each spring constant and mass being equal (Fig. 4).

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F \end{bmatrix}$$

Figure 4 Simplified equation of motion for this undamped system

Figure 5 below is the same simplified equation of motion but further simplified with annotations for the spring and mass matrices for this system.

$$\underbrace{\begin{bmatrix} m I_4 \end{bmatrix}}_{\text{mass matrix}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & 2k \end{bmatrix}}_{\text{spring matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F \end{bmatrix}$$

Figure 5 Fully simplified equation of motion with call outs for the spring and mass matrices.

2 Question 2

(3pts) Review the MATLAB script excitation.m. Modify the base system. Each mass should weigh 555g and each spring has stiffness of k = 170 N/m.

a) Summarize the code.

The code takes in the mass and stiffness parameters of a system of four springs connected in series, as well as a driving frequency. It then finds the eigenmodes of the system, which yields the natural frequencies of the system in Hz. It diagonalizes the system using the modal matrix, in order to solve the now decoupled system for a forcing function with the supplied driving frequency. Then it plots the displacement of each mass as a function of both time and frequency.

b) What are the system's natural frequencies?

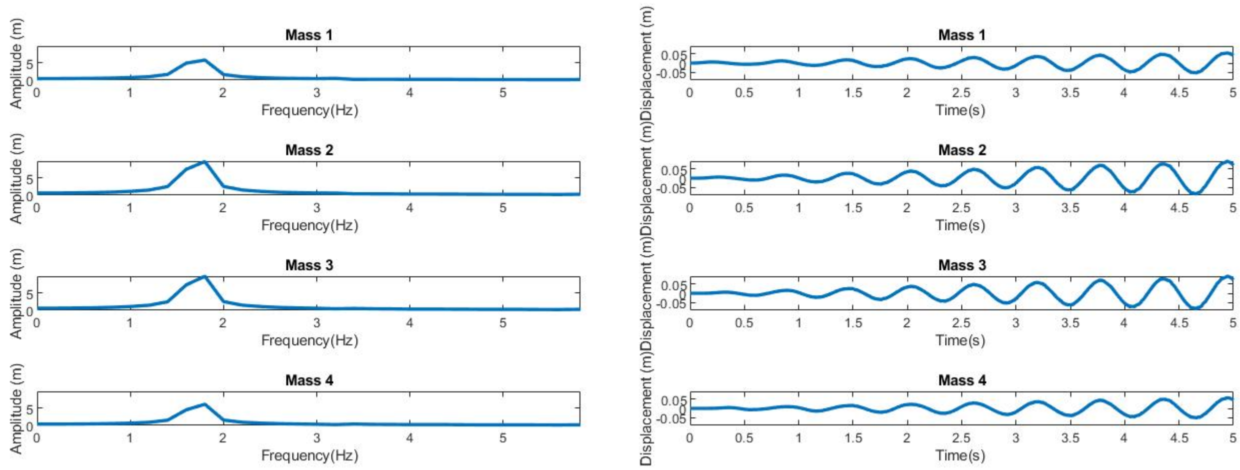
$\omega_n = [10.8166, 20.5744, 28.3182, 33.2900]\ [\frac{\text{rad}}{\text{sec}}]$

c) Excite the model at your calculated natural frequencies. Does the system respond differently to a random driving frequency? Describe your observations.

At a natural frequency, the time series displacement graphs are smooth sine waves with one dominant frequency for every mass, with an amplitude that increases continuously for all time. The frequency spectrum plot peaks at just one frequency. At a random frequency that isn't a natural frequency, the time series displacement plots for each mass don't look like single sine functions, and the frequency spectrum plot shows peaks at multiple frequencies.

d) Include the plots of frequency and displacement responses for the masses at the observed natural frequencies.

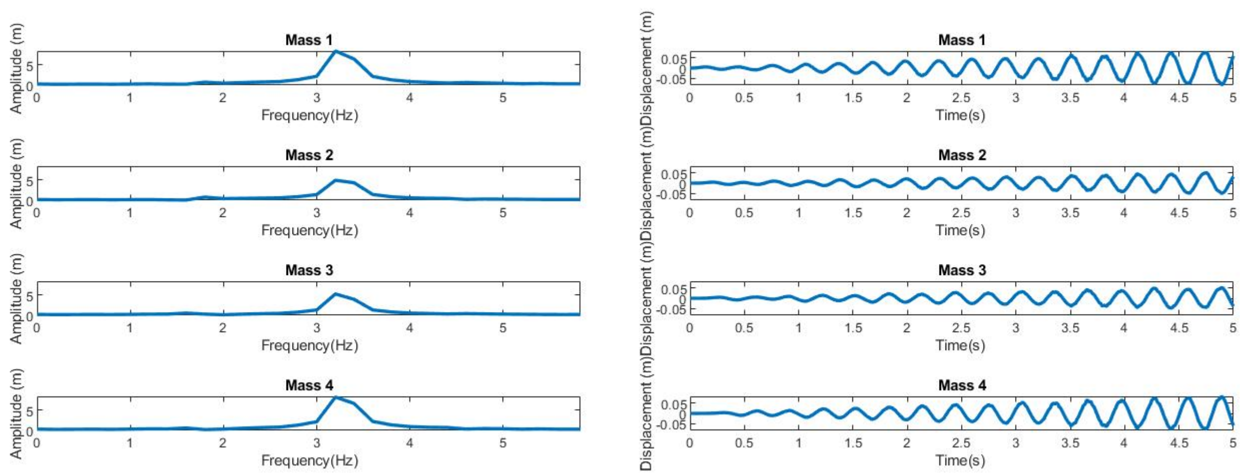
Each figure below represents the frequency and displacement responses at $\omega_1, \omega_2, \omega_3$, and ω_4. Figures 6 to 9 were generated using the provided MATLAB script with inputted values for this specific undamped system [1].



(a) Frequency plot for ω_n of 10.8166 $[\frac{\text{rad}}{\text{sec}}]$

(b) Time plot for ω_n of 10.8166 $[\frac{\text{rad}}{\text{sec}}]$

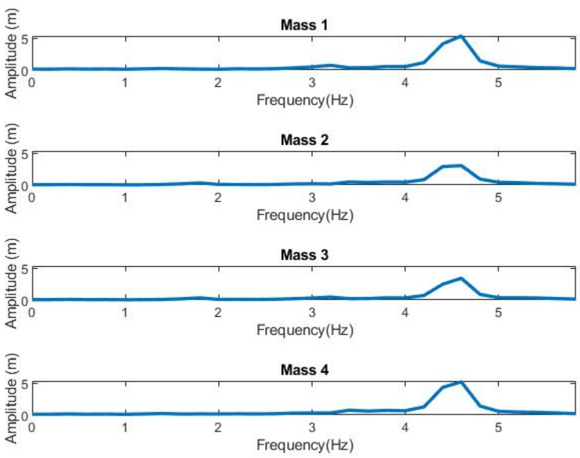
Figure 6 Frequency and time plots for the first natural frequency



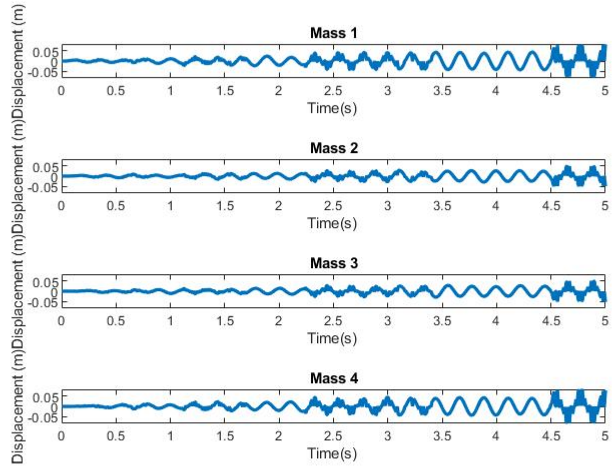
(a) Frequency plot for ω_n of 20.5744 $[\frac{\text{rad}}{\text{sec}}]$

(b) Time plot for ω_n of 20.5744 $[\frac{\text{rad}}{\text{sec}}]$

Figure 7 Frequency and time plots for the second natural frequency

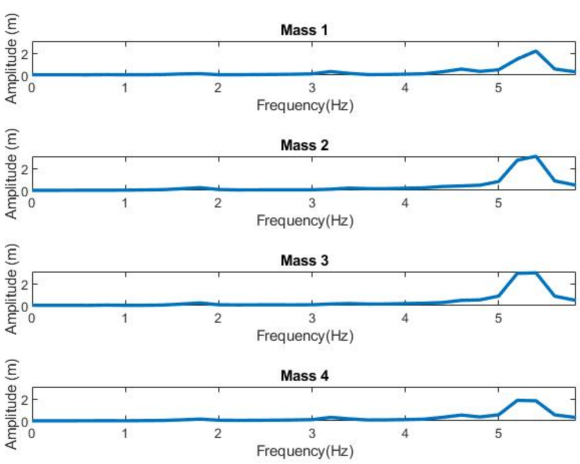


(a) **Frequency plot for ω_n of 28.3182 [$\frac{\text{rad}}{\text{sec}}$]**

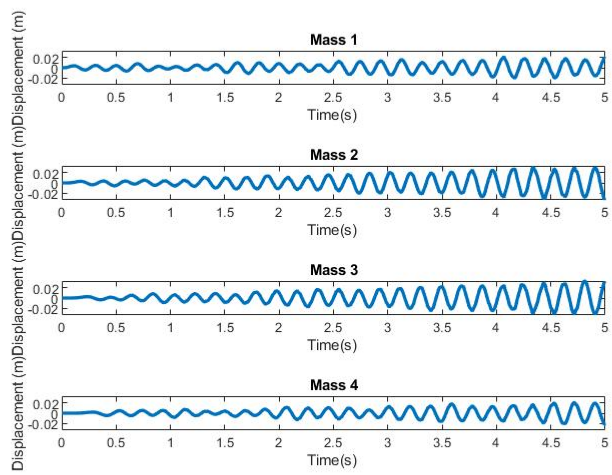


(b) **Time plot for ω_n of 28.3182 [$\frac{\text{rad}}{\text{sec}}$]**

Figure 8 Frequency and time plots for the third natural frequency



(a) **Frequency plot for ω_n of 33.2900 [$\frac{\text{rad}}{\text{sec}}$]**



(b) **Time plot for ω_n of 33.2900 [$\frac{\text{rad}}{\text{sec}}$]**

Figure 9 Frequency and time plots for the fourth natural frequency

3 Question 3

(3pts) Review the MATLAB script **mode_shapes.m**. Modify the base system. Each mass should weigh 255g and each spring has stiffness of k = 1050 N/m.

a) Summarize the code.

This code inputs mass and stiffness information about a multi-degree of freedom system, performs modal analysis to determine the displacement modes for each mass, and normalizes the displacements by mass to determine the relative responses.

b) What are the system's natural frequencies? How will the natural frequencies of the MDOF system change if you double all the masses? By what factor? How will the natural frequencies of the MDOF system change if you double the spring stiffness?

The natural frequencies of the masses are:

- $f_1 = 1.4112Hz$
- $f_2 = 1.8113Hz$
- $f_3 = 2.2630Hz$
- $f_4 = 3.2734Hz$

for masses 1, 2, 3, and 4 respectively.

The natural frequencies of the system will decrease by a factor of $\frac{\sqrt{2}}{2}$ if the masses are each doubled. However, if the spring stiffnesses are doubled, the natural frequencies will increase by a factor of $\sqrt{2}$.

c) Model a different system where m1 = 195g, m2 = 250g, m3 = 300g, and m4 = 250g and k1 = 1500 N/m, k2 = 2200 N/m, k3 = 1100 N/m, k4 = 2050 N/m, and k5 = 1100 N/m. What are the system's natural frequencies? Include plots of the mass-normalized mode shapes of your base system versus your modified system (i.e. one plot for each mode type). Provide your generated code for this question.

The natural frequencies of the new masses are:

- $f_1' = 7.2713Hz$
- $f_2' = 13.4512Hz$
- $f_3' = 21.9956Hz$
- $f_4' = 26.1248Hz$

for masses 1, 2, 3, and 4 respectively.

Below are the plots of the mass-normalized mode shapes for both the base system and the modified system (Fig. 10 and 11).

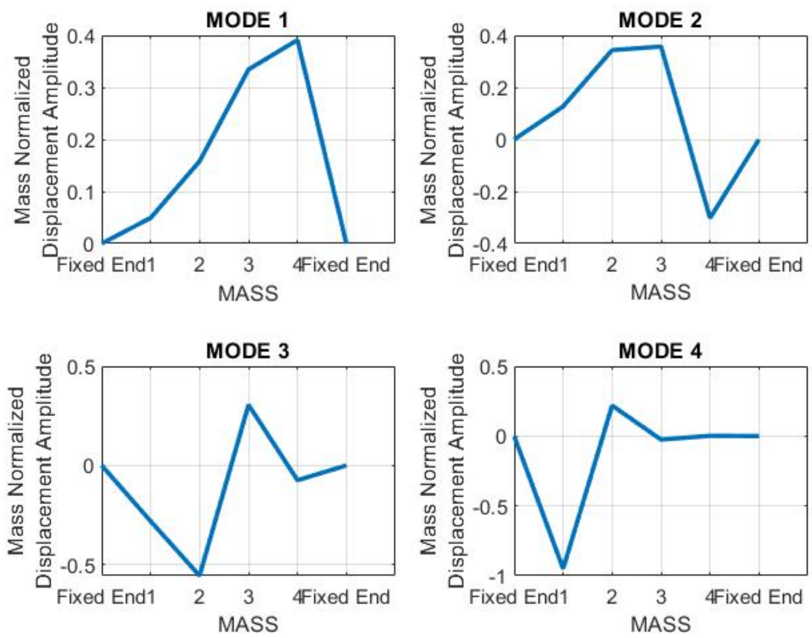


Figure 10 Mass-normalized mode shapes for base system

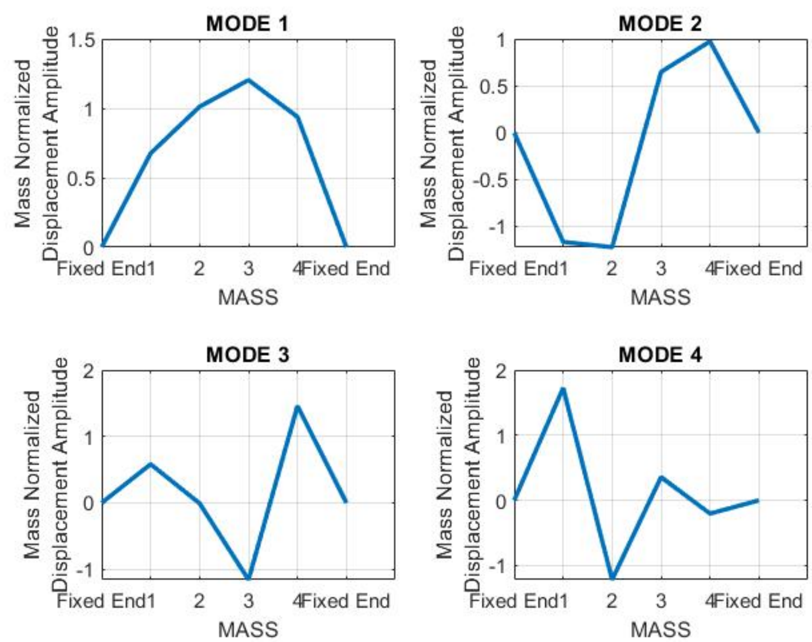


Figure 11 Mass-normalized mode shapes for modified system

The provided MATLAB code which has been modified is included below [1]:

```
% The code provided below is only a template
% Review each line and plug in known values from the prelab

% Note that modal analysis of Mx''+Kx = Fsin(wt) reduces to
% the homogenous case  Mx''+ Kx = 0 to determine the mode shapes

% Modified 2020

clc
clear all
close all
Mb = [1 0 0 0;
     0 2 0 0;
     0 0 3 0;
     0 0 0 4];
M = [.195 0 0 0;
     0 .25 0 0;
     0 0 .3 0;
     0 0 0 .25];             % Mass Matrix(kg)

k = 100; % Spring stiffness(N/m)
Kb = k*[4 -1 0 0;
       -1 4 -1 0;
        0 -1 4 -1;
        0 0 -1 4];
K = k*[37 -22 0 0;
      -22 33 -11 0;
      0 -11 31.5 -20.5;
      0 0 -20.5 31.5];            % Stiffness Matrix(N/m)

F0 = 0;                      % Applied force (N)
F = F0*[0;0;0;1];

[phi,wn_sq]=eig(K,M);
[phib,wn_sqb]=eig(Kb,Mb);       % Solve the eigenvalue problem
```

```
wn = sqrt(diag(wn_sq))      % Natural frequencies
f = wn/(2*pi)               % Natural frequencies
wnb = sqrt(diag(wn_sqb))      % Natural frequencies
fb = wnb/(2*pi)               % Natural frequencies


alpha = phi'*M*phi;                  % Diagonal matrix
phiNormalized = phi/sqrt(alpha);     % Mass-normalization
alphab = phib'*Mb*phib;                  % Diagonal matrix
phiNormalizedb = phib/sqrt(alphab); % Mass-normalization

%PLOT MODE SHAPES - BASE SYSTEM
modeShapeb = [zeros(1,4);real(phiNormalizedb);zeros(1,4)]';
massPointb = 0:1:5;
figure(1)
 for x = 1:4
   subplot(2,2,x)
   plot(massPointb, modeShapeb(x,:),'DisplayName', ['Mode ' ...
       num2str(x)],'LineWidth',2)
   title(['MODE ' num2str(x)])
   xlabel({'MASS',''})
   ylabel({'Mass Normalized','Displacement Amplitude'})
   set(gca,'XTick',[0 1 2 3 4 5],'XTickLabel',{'Fixed ...
       End','1','2','3','4','Fixed End'});
   grid on

   hold all
 end

 %PLOT MODE SHAPES - MODIFIED SYSTEM

modeShape = [zeros(1,4);real(phiNormalized);zeros(1,4)]';
massPoint = 0:1:5;
figure(2)
 for x = 1:4
   subplot(2,2,x)
   plot(massPoint, modeShape(x,:),'DisplayName', ['Mode ' ...
       num2str(x)],'LineWidth',2)
```

```
    title(['MODE ' num2str(x)])
    xlabel({'MASS',''})
    ylabel({'Mass Normalized','Displacement Amplitude'})
    set(gca,'XTick',[0 1 2 3 4 5],'XTickLabel',{'Fixed ...
        End','1','2','3','4','Fixed End'});
    grid on

    hold all
end
```

References

[1] Dr. Philip Bayly, Prof. Louis Woodhams, and Prof. Chiamaka Asinugo, 2020, "PRE-LAB 4: VIBRATIONAL ANALYSIS OF A MULTI DEGREE-OF-FREEDOM SYSTEM," FL2020.E37.MEMS.4050.01 - Vibrations Lab.