Washington University in St. Louis
James McKelvey School of Engineering

Fall 2020 MEMS 4050 Vibrations Laboratory

Lab 5: Design Your Own Experiment (Beam and Modal Analysis)

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Group T (Friday 2 PM)

We hereby certify that the lab report herein is our original academic work, completed in accordance to the McKelvey School of Engineering and Undergraduate Student academic integrity policies, and submitted to fulfill the requirements of this assignment:

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ABSTRACT: *This experiment aims to determine material characteristics of samples given to us by NyQuist Consulting Lab. In order to determine the material samples the Young's Modulus and density were found from modal analysis. The experiment consisted of the two unknown samples clamped to a table where the the natural frequencies and mode shapes of the beams were recorded when struck by a force hammer. The base natural frequency for beam 2 was found to be 68.4867* [$\frac{rad}{s}$]*. This allowed us to calculate the density and Young's Modulus of the sample to be 1.3838* [$\frac{g}{cm^3}$] *and 3.72* [GPa]*, respectively. The second beam, beam 7, had a base natural frequency of 127.5487 [rad/s]which lead to a density and Young's Modulus of 9.1489* [$\frac{g}{cm^3}$] *and 87.44* [GPa]*, respectively. This allowed us to narrow down the material for beam 2 as a plastic and beam 7 as a copper alloy. Theoretical analysis, using MATLAB and SOLIDWORKS, of the mode shapes for both beams showed that both models correlate well to one another through plotting. For the metal beam (beam 7), our experimental data confirms that the simulation and theoretical models relate well through graphical evidence.*

INTRODUCTION

The purpose of this experiment is to identify the material of two cantilever beams, by using vibration analysis to measure the natural frequencies and mode shapes of each cantilever beam. Material properties of the cantilever beams can then be found by using the base natural frequencies of each beam. A cantilever beam has an infinite amount of natural frequencies, however the lower natural frequencies are usually the ones of interest and that dominate the response [1]. The natural frequencies can be found from the the equation of motion:

$$\rho A\ddot{y} - EIy'''' = F(x) \tag{1}$$

where $\rho[kg/m^3]$ is the density of the material, A $[m^2]$ is the cross sectional area of the beam, E $[Pa]$ is the Modulus of Elasticity of the beam and I $[m^4]$ is the area moment of inertia of the beam. Each of these quantities is assumed to be constant over the beam. F(x) [N/m] is the force applied to the beam. It should be noted that Y is the displacement function as a function of time (t) and space (x). $\ddot{y}$ is the second derivative with respect to time and y"" is the fourth derivative in respect to space.

It can then be assumed that y(x,t) = Y(x)T(t), where Y(x) [m] represents the position dependent shape and T(t) [s] represents the time dependent oscillation of the beam. T(t) is assumed to have the form $ae^{i\omega t}$ [2]. Assuming no forcing function, Y(x)T(t) can be substituted for y , and equation 1 becomes,

$$y'''' = \frac{\rho A\omega^2}{EI}Y = \beta^4 Y \tag{2}$$

where β^4 is $\frac{\rho A\omega^2}{EI}$. From this, the general solution is found to be,

$$Y(x) = a_1 sin(\beta x) + a_2 cos(\beta x) + a_3 sinh(\beta x) + a_4 cosh(\beta x) \tag{3}$$

where a_1, a_2, a_3, a_4 are constants with units of meters. Then, applying the boundary conditions for a cantilever beam, seen in Eq. 4, an exact solution to Y is determined to be Eq. 5.

$$\text{Boundary Conditions:} \qquad Y(0) = 0; \qquad Y'(0) = 0; \qquad Y''(L) = 0; \qquad Y'''(L) = 0 \tag{4}$$

$$Y_n(x) = C[sin\beta_n x - sinh\beta_n x - \alpha_n(cos\beta_n x - cosh(\beta_n x)2] \tag{5}$$

where C is a constant and n represents the nth theoretical mode shape [2]. With this known the natural frequency is found to be:

$$\omega_n = \beta_n^2 * \sqrt{\frac{EI}{\rho A}} \tag{6}$$

For example the first natural frequency of the system would be written as:

$$\omega_1 = \beta_1^2 * \sqrt{\frac{EI}{\rho A}} \tag{7}$$

where β_1 is equal to 1.875 [1]. This value comes from experiments done From this, the density of the material can then be found by rearranging Eq. 7 to get:

$$\rho = \frac{\beta_1^4 EI}{\omega_n^2 A L^4} \tag{8}$$

It is important to note here that this assumes that there is no added mass on the beam at this point.

Another property that can be found is the Elastic Modulus of the beam. This can be done if the mass of the beam is negligible. In order to make it negligible it is important to note that as mass is added to the end of the beam the inertia increase which will decrease the natural frequency of the system. The system can be modeled as a spring with a stiffness K_{eq} if the added mass is much greater than the mass of the beam so that the mass of the beam contributes a small amount to the natural frequency of the system. It can then be said that the mass of the beam is negligible. In the case that the mass of the beam is negligible the spring stiffness can be written as Eq. 9 and the natural frequency is then seen in Eq. 10 [3].

$$k_{eq} = \frac{3EI}{L^3} \tag{9}$$

$$\omega_1 = \sqrt{\frac{k_{eq}}{M}} \tag{10}$$

Rearranging equations 9 and 10 to solve for the modulus of elasticity (E) gives: [3].

$$E = \frac{mw_n^2 L^3}{3I} \tag{11}$$

where m [kg] is the added mass, not the mass of the beam.

Lastly, it should be noted that the masses used were normalized. In order to normalize the masses the density is taken and multiplied by the volume and put into a 20 by 20 matrix. This puts a portion of the mass in each element in the matrix.

$$[M] = (\rho * L * B * H/20) * I \tag{12}$$

where L, B, and H are the length base and height (in meters), respectively, and I is the identity matrix. Alpha is then determined as the decoupled mass matrix when the mass matrix is is multiplied by the modal matrix and the transpose of the modal matrix. This can be seen below in Eq. 13

$$\alpha = U' * M * U \tag{13}$$

where U (unitless) is the modal matrix and ' represents the transpose of that matrix. Final the mass was normalized by dividing the the modal matrix by the square root of the alpha matrix.

$$U_{norm} = \frac{U}{\sqrt{\alpha}} \tag{14}$$

METHODS

Apparatus. In order to determine the physical properties of the two beams, the beams and measuring equipment were set up as shown in Fig. 1. Beam 2, made of the provided material 2, was clamped to the table so that 0.562 m of the beam extended off of the table. Beam 7, made of the provided material 7, was clamped so that 0.384 m extended off the table. For each beam, one clamp was placed where the table ended and another at the leftmost end of the beam. An accelerometer was attached to the bottom of the rightmost end of the beam using sticky putty. The accelerometer was attached so that it measured accelerations in the vertical direction. The signal output from the accelerometer was sent to signal conditioner 2. The loose portion of the BNC cable, adjacent to the beam, that connected the BNC to the signal conditioner was taped to the beam with Scotch tape. The conditioner added a gain of ten to the accelerometer output and sent the amplified signal to the Quattro. An impulse hammer with a plastic/vinyl tip was used to excite the beam and a force transducer within the hammer recorded the impulse applied by the hammer.

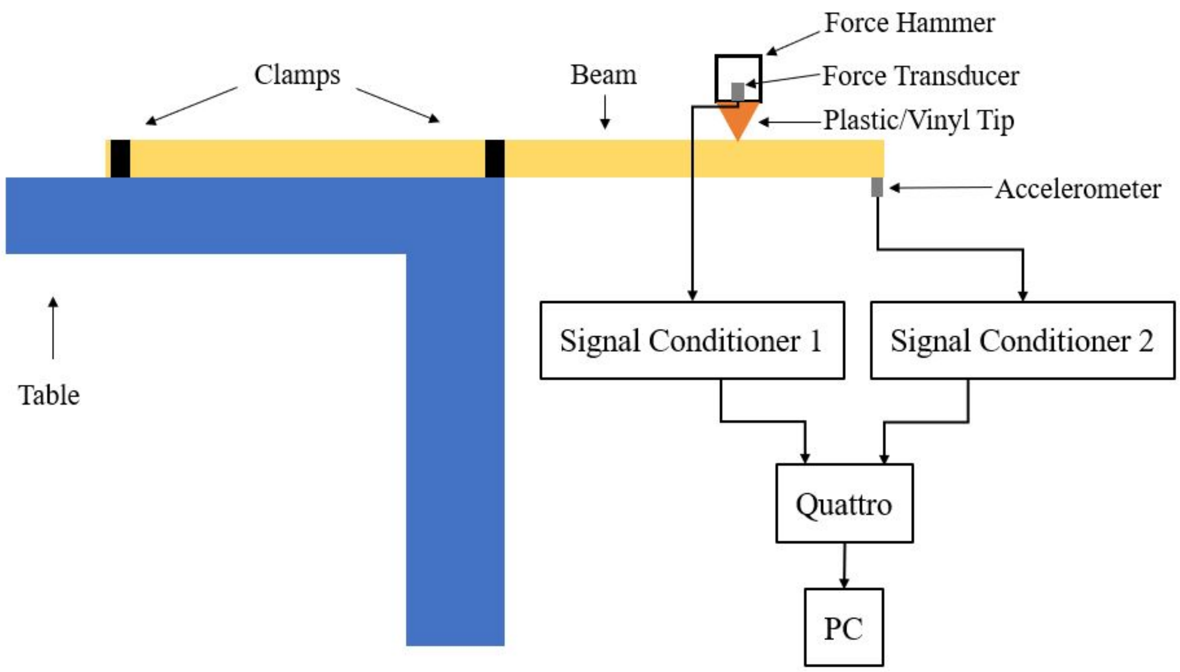

Figure 1   Apparatus set up used to measure the physical properties of the two beams.

The signal from the force transducer was sent to signal conditioner 1 , which passed on the signal to the Quattro with a gain of ten. The Quattro analyzed the input signals in the frequency and time domains, presenting its analysis through the SignalCalc program on the PC. Masses up to 1 kg were available to attach to the rightmost end of the beam. The masses were made to stick with sticky putty. All the equipment used during this lab was recorded in Table 1.

Table 1 Equipment used to measure the physical properties of the two beams [4].

Equipment	Make	Model	Serial #	Calibration Constant
Laptop	Lenovo	Thinkpad	N/A	N/A
SignalCalc 240 Dynamic Signal Analyzer	N/A	N/A	N/A	N/A
Quattro SignalCalc Ace	Data Physics Corporation	N/A	10857	N/A
Signal Conditioner 1	PCB	494A	816	10 mV
Signal Conditioner 2	PCB	494A	813	10 mV
Accelerometer	PCB	352C65	LW209194	10.15 $\frac{mV}{s^2}$
Impulse Hammer	PCB	086C03	LW35851	N/A
Beam 7	N/A	N/A	N/A	N/A
Beam 2	N/A	N/A	N/A	N/A
Masses	N/A	N/A	16751	N/A
Sticky Putty	N/A	N/A	N/A	N/A
Scotch Tape				
2 Clamps	N/A	N/A	N/A	N/A
Imperial Measuring Tape	N/A	N/A	N/A	N/A

Procedure. To acquire the measurements necessary to determine the modulus of elasticity, density, and total mass of each of the two beams, the dimensions of each beam were first measured with a tape measure (Table 1). The bar and measurement equipment were then organized as described in the apparatus section. A .trf test file was opened on SignalCalc. For each run, accelerometer data was recorded in the frequency and time domains. Impulse hammer data was recorded in the time domain and the imaginary portion of the transfer function was also recorded, both as input and as a magnitude. The trigger was set to *Input* and Avg. to 3. The trigger level was set to .0100 V. F Span and Lines were set to acquire a relatively high frequency resolution (dF of .03906 Hz). The accelerometer channel had a calibration constant of 10.15 $\frac{\mathrm{mV}}{\mathrm{s}^2}$.

To determine the material properties of the beam, the free response of the bar was analyzed in SignalCalc. Each run in SignlaCalc was an average of three trials. A trial was triggered by the force transducer output from a hammer strike and lasted the duration of the SignalCalc analysis time span. Therefore, to measure the free response of the beam, a run was begun by striking the table with the hammer. Then, the beam was manually displaced and allowed to vibrate. After the time span elapsed the beam was stabilized, the table struck again, and the beam made to oscillate freely once more. This process was repeated a final time to complete the full SignalCalc run. Between each strike, SignalCalc acquired data and then notified the user when it was done collecting data. Upon the next strike, SignalCalc began recording again. SignalCalc averaged the three strikes, so each SignalCalc measurement represented an average of three strikes. Using the SignalCalc output, the modulus of elasticity was calculated in real time. After the bar had been tested at all three locations, a mass was added to the end of the bar and the process repeated. The mass was increased and the bar re-tested until the calculated modulus of elasticity ceased to be effected by the increase of mass. This process was applied to both bars.

To determine the dynamic response of beam 7 to vibrations, all SignalCalc settings were kept the same except F Span and Lines were set to 200 and 800, respectively. Then, using scotch tape and a sharpie, that bar was divided into 21 evenly spaced sections, each 0.03175 m long. The response of the bar to an impulse from a hammer strike at an arbitrary location was recorded. Finally, the response of the bar to a strike from the impulse hammer was measured at each of these locations. The process for measuring the response of the bar to the hammer was identical to the process described previously, meaning that the measured response of the bar at each strike location was the average of three hammer strikes.

Analysis. The material properties of each of the two beams tested were determined with the exact same analysis method. Therefore, the analysis for determining the material properties of a beam is only described for one beam but is applicable to both beams.

The modulus of elasticity, E, of the beam was determined using Eq. 11. m was the mass of the object added to the end of the beam and not the beam's mass as Eq. 11 assumes that the beam mass is negligible. The base natural frequency of the beam/mass system was ω_n and was recorded from the frequency plot of the accelerometer input, and given as the average of the base frequencies measured by SignalCalc at each of the three striking locations along the beam. The base fre Using Excel, the E of the beam was calculated for each mass added to the end of the beam (See Appendix B for sample calculations). More mass was added to the end of the beam until adding additional mass had a negligible effect on the calculated value of E. At this point, the negligible beam mass assumption was valid and the calculated value of E could be taken as the true modulus of elasticity of the beam.

With the known value of E, the density of the beam was determined with Excel using Eq. 8. The ω_n in Eq. 8 was the base natural frequency of the beam with no added mass and was also given by the average of the base frequencies measured by SignalCalc at each of the three striking locations along the beam. The mass of the beam was calculated by multiplying the density of the beam by its volume (See Appendix B) for sample calculations. The volume was determined from the measured dimensions of the beam. The percent difference of the experimental values from the published values was determined. The results were can be seen in Table 2.

The experimental mass normalized modes were only determined for beam 7. Using SignalCalc measurements of the arbitrary beam strike described in the modal analysis portion of the procedure, the first three natural frequencies of the beam were taken to be the frequencies of the first three peaks picked by SignalCalc on the absolute value plot of the beam/hammer transfer function. Then, the amplitude of the transfer function at each of the three natural frequencies was recorded for each of the 20 strike locations along the beam. With MATLAB, these amplitudes were mass normalized (Eq. 14) and then plotted against the length normalized strike locations, resulting in mass and length normalized plots of the first three modes of the beam. The length normalized modes of the beam were also extracted from a SOLIDWORKS model of the beam. These modes were imported into MATLAB and mass normalized. Finally, in MATLAB, the mode shapes were determined using Eq. 5 with the first three β values for a cantilevered beam [2]. These modes shapes were then mass normalized and length normalized. The comparison plots of the SOLIDWORKS, MATLAB, and experimental analysis of the mode shapes can be seen in Figs. 5 - 7.

Modal analysis of beam 2 was also performed using a SOLIDWORKS model and Eq. 5. The MATLAB comparison plots of these analyses can be seen in Figs. 8 - 10. All MATLAB code is available in Appendix D.

Three significant assumptions were made in the setup of the cantilevered beam. The first was that the left end of the beam was fully constrained. The clamps did a good job of connecting the beam to the table, but the clamps and even the table were unable to stay perfectly motionless while the beam experienced vibration. The vibrations of the table and clamp, while minimal, absorbed some energy from the beam, reducing the amplitude of the mode shapes. The energy absorption also caused some damping, which reduced the measured natural frequency, thereby reducing the measured modulus of elasticity and density (Eqs. 11 and 8).

The second significant assumption was that, while measuring the modulus of elasticity, the mass of the beam was said to be negligible. This assumption was verified by adding mass to the end of the beam until and increase in mass had a negligible effect on the calculated modulus of elasticity. However, the mass of the beam was never truly negligible and caused the calculated modulus of elasticity to be lower than its true value. This can be seen from Eq. 11, where the ratio between the added mass, m and the true mass, of which ω_n is a function, is always less than one. Accordingly, the calculated density values will be less than the true values.

Finally, our experimental setup assumed that the experimenter was capable of striking the beam in exactly the same place three time in a row. However, no experimenter, no matter how consistent, is capable of striking the exact same place three times in a row for every trial run of the experiment. Inconsistencies in striking location skewed the mode, making the mode value at each location too big or too small depending on whether the experimenter struck a location on the beam with a greater transfer function magnitude or smaller transfer function magnitude than the true transfer function magnitude of the beam.

RESULTS & DISCUSSION

Modal analysis permits the determination of material properties for the beams. Therefore, the response of material seven to impulse excitation by the force hammer at three theoretical vibration nodes were recorded and documented. The response is documented by the time series plots of the force hammer and the accelerometer, as well as the local mode shape value across the frequency spectrum. These plots for node five are given below in Fig. 2.

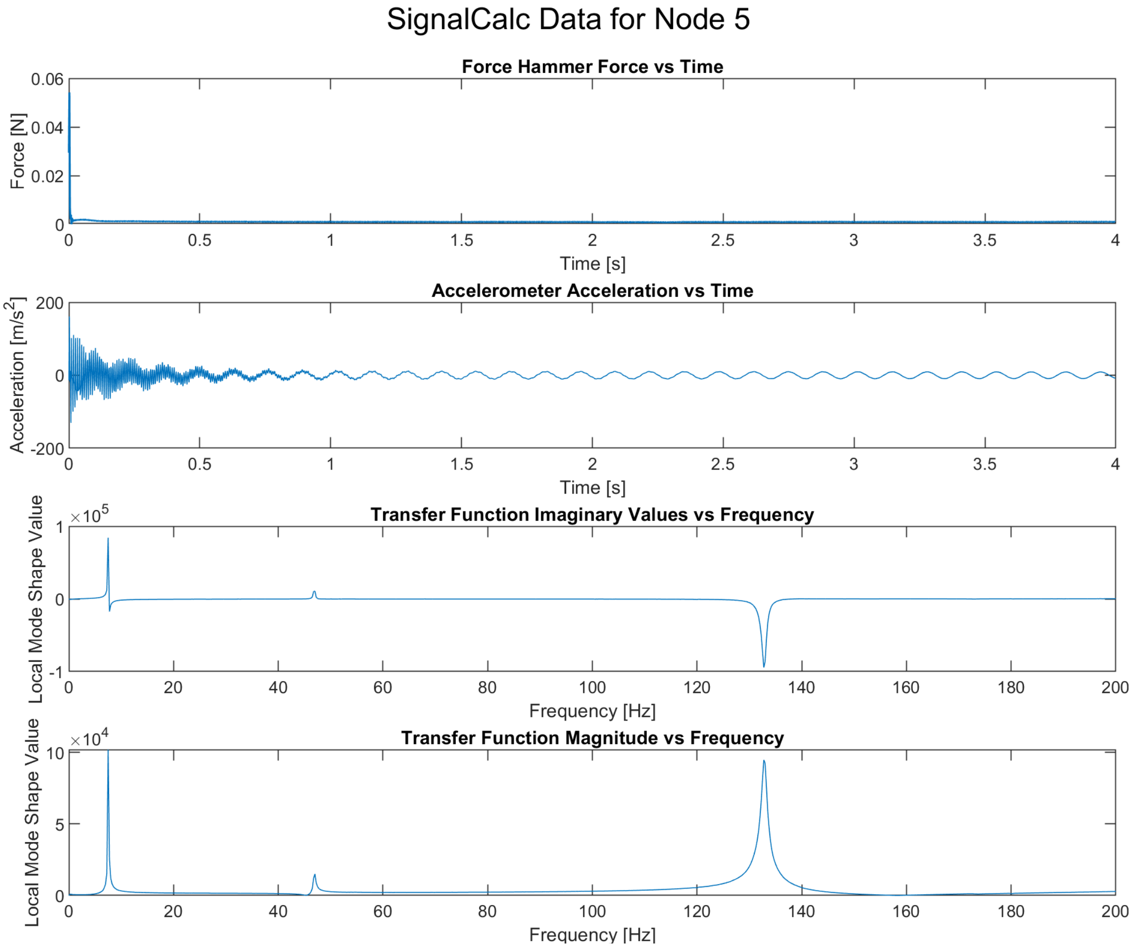

Figure 2 Accelerometer Acceleration, Force Hammer Force, and Transfer Function Plots for a hammer strike at node 5 (5 inches from the tip of the beam).

These plots show a damped response to the impulse which is consistent with theoretical expectations for a cantilever beam. Additionally, the mode shape values show three natural frequencies at about 7.5 Hz, 46.8 Hz, and 133 Hz. These are the first three modal frequencies for the system with an added tip mass. The first and third frequencies having large modal amplitudes are consistent with the expected response for the location of the impulse excitation.

Next, the time series and frequency plots for an impulse applied at node ten are given in Fig. 3 below.

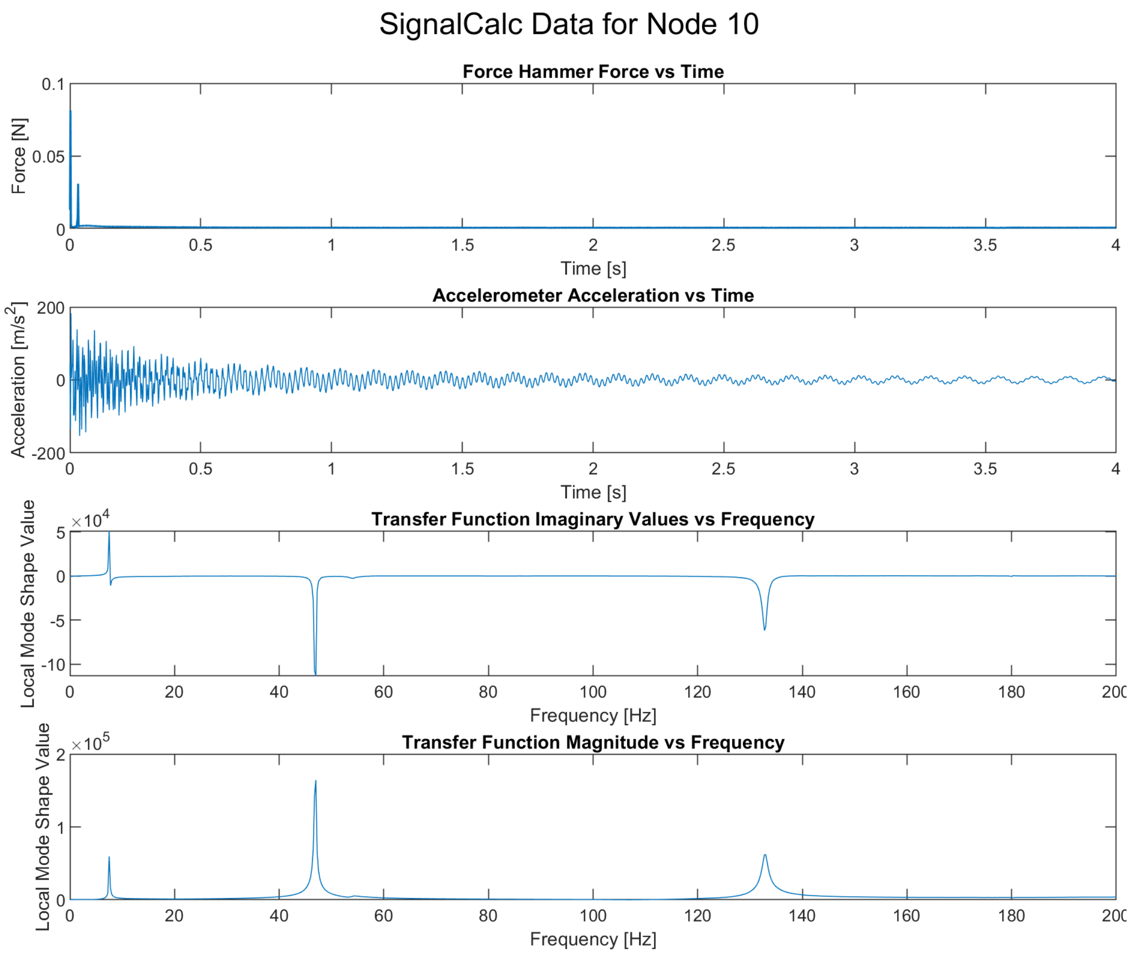

Figure 3 Accelerometer Acceleration, Force Hammer Force, and Transfer Function Plots for a hammer strike at node 10 (11.25 inches from the tip of the beam).

A damped response to the impulse can be seen in the time series plots, which is a consistent result for the expectations for a cantilever beam. Additionally, the frequency spectra show the same first three modal frequency values. The second frequency has the largest modal amplitude, while the other two are have moderate amplitudes. This is consistent with the expected response for the location of the impulse excitation.

Finally, the time series and frequency plots for an impulse applied at node fifteen are given in Fig. 4 below.

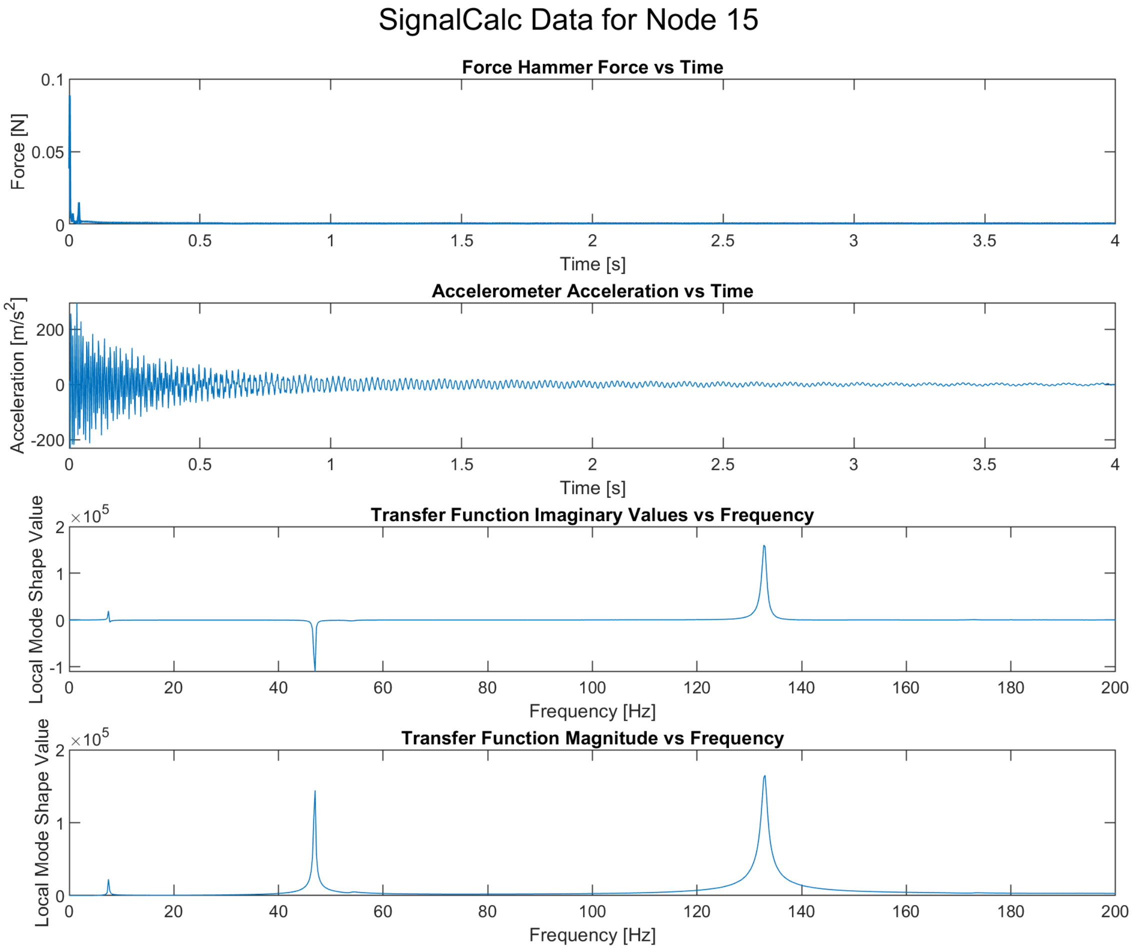

Figure 4 Accelerometer Acceleration, Force Hammer Force, and Transfer Function Plots for a hammer strike at node 15 (17.5 inches from the tip of the beam).

Again, a damped response to the impulse can be seen in the time series plots. In addition, the frequency spectra show the same first three modal frequency values. The third frequency has the largest modal amplitude, while the second frequency has a moderate amplitude and the first frequency has a very small amplitude. This is consistent with the expected response for an impulse excitation towards the free end of the beam.

Next, the mode shapes for the beam made of material seven are found and plotted theoretically using both Matlab and Solidworks Finite Element Analysis. The experimental mode shapes from the force hammer impulse response are plotted and overlaid with the theoretical curves. Mode shape one at 7.5 Hz for beam seven is shown below in Fig. 5.

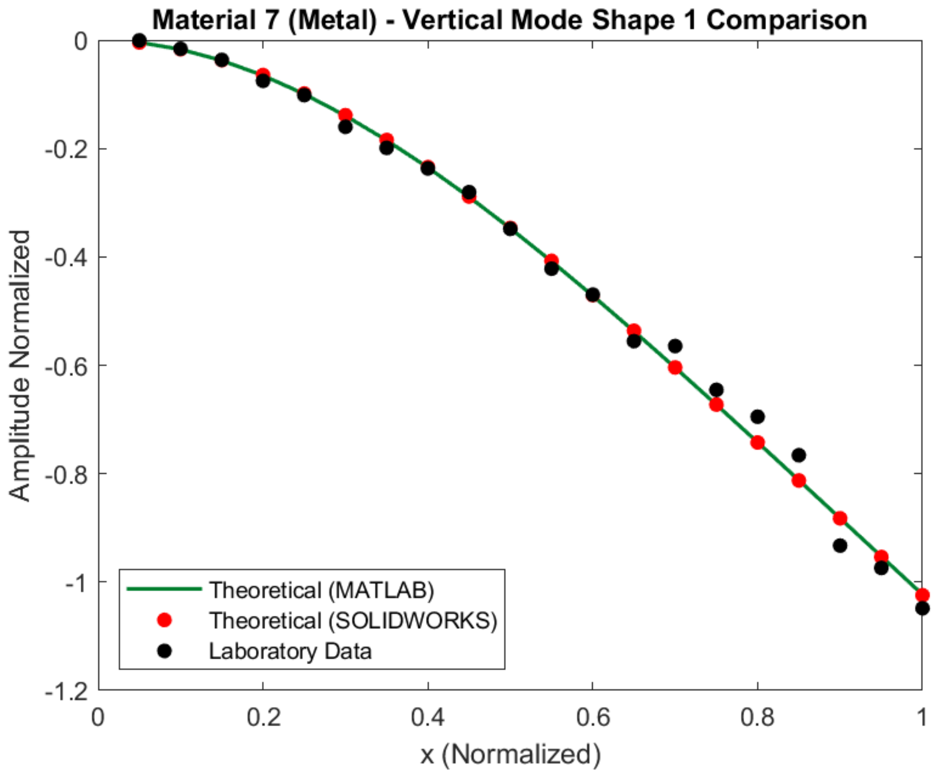

Figure 5 Comparison of vertical mode shape 1 between the SOLIDWORKS, theoretical, and experimental models for the metal beam.

The two theoretical curves show near exact agreement, while the experimental data nearly matches, with slight error. The error in magnitude deviates higher and lower than the curves at different x values, and can be explained by noise in the signal as well as imperfect material shape and composition.

Mode shape two at 46.8 Hz for beam seven is displayed in Fig. 6 below. The shape of both theoretical curves shows reasonable agreement with the experimental data. The experimental curve seems to be shifted slightly left from the theoretical model. This could be explained by the distance between the accelerometer and the points of greater error. As the points approach the fixed cantilevered end of the beam the error in the amplitude increases slightly. Mode shape three at 133 Hz for beam seven is displayed in Fig. 7 below.

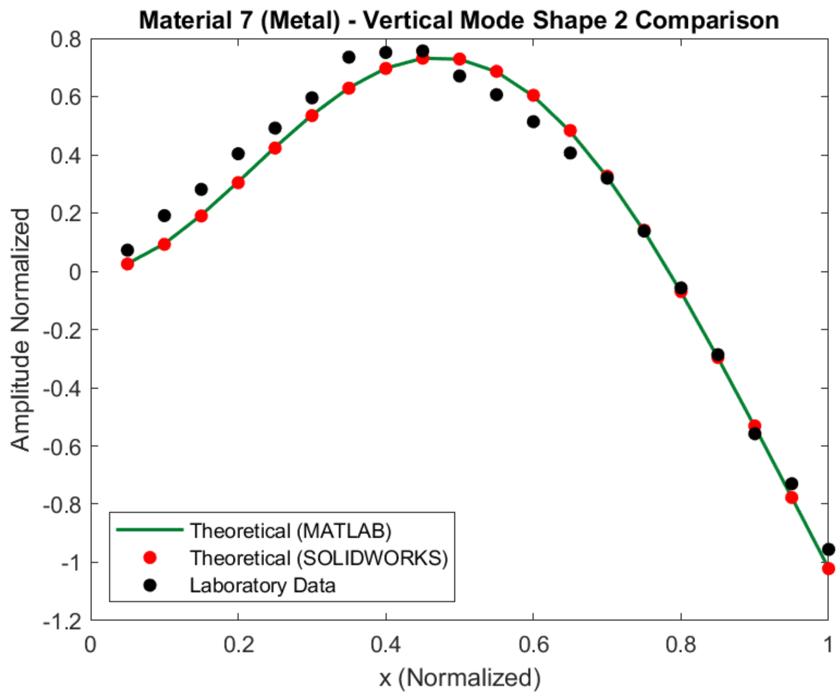

Figure 6 Comparison of vertical mode shape 2 between the SOLIDWORKS, theoretical, and experimental models for the metal beam.

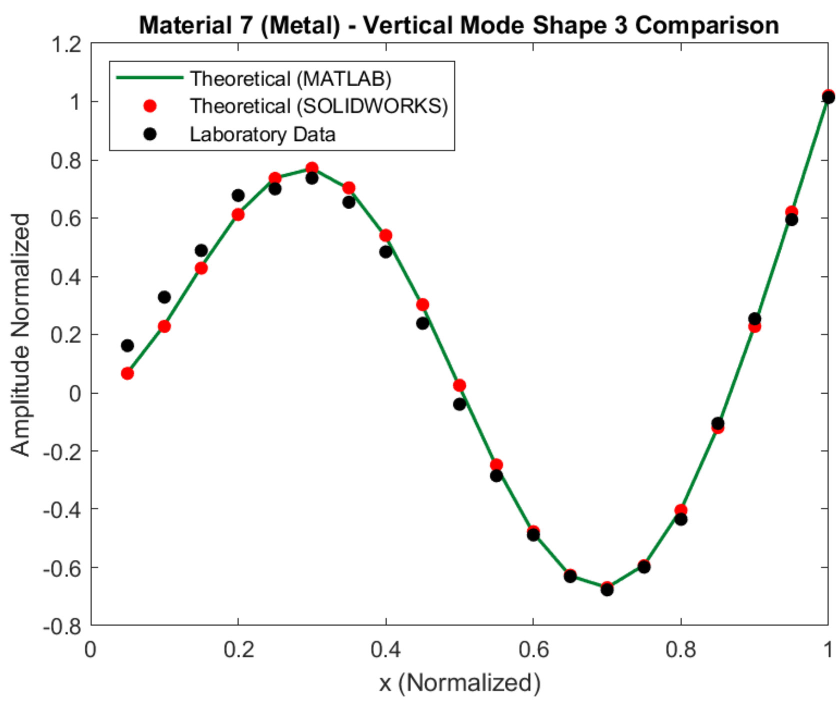

Figure 7 Comparison of vertical mode shape 3 between the SOLIDWORKS, theoretical, and experimental models for the metal beam.

The shape of both theoretical curves shows reasonable agreement with the experimental data. Again, the experimental curve seems to be shifted slightly left from the theoretical model. The error is minor, and the proximity to the accelerometer can again explain the minor error in the amplitude values.

Next, the theoretical mode shapes for the beam made of material two are plotted using both Matlab and Solidworks Finite Element Analysis. Mode shape one is plotted below for beam two in Fig. 8.

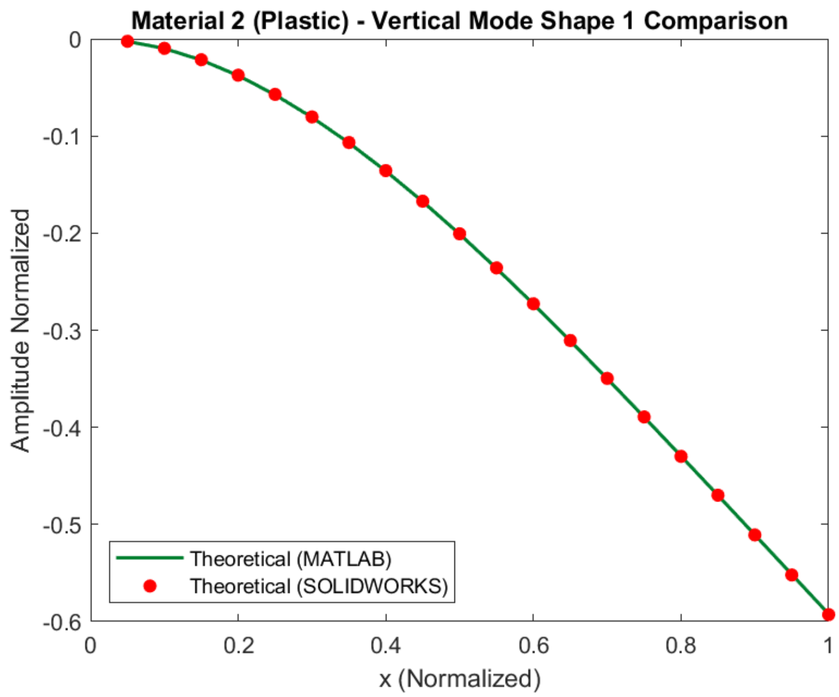

Figure 8 Comparison of vertical mode shape 1 between the SOLIDWORKS and theoretical models for the plastic beam.

The amplitude of the vibration in beam two is significantly less than that of beam seven. This can be explained by the significantly different material properties of the beam.

The second mode shape for beam two is plotted below in Fig. 9. This mode shape has lower amplitude than that of beam seven, which is consistent for the material property disparity between the beams. Finally, mode shape three for beam two is displayed in Fig. 10 below.

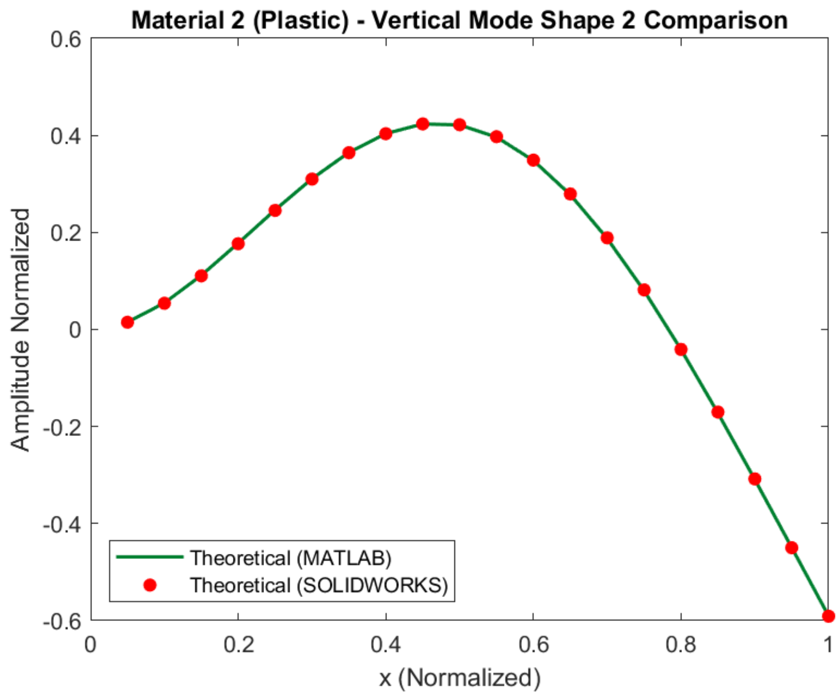

Figure 9 Comparison of vertical mode shape 2 between the SOLIDWORKS and theoretical models for the plastic beam.

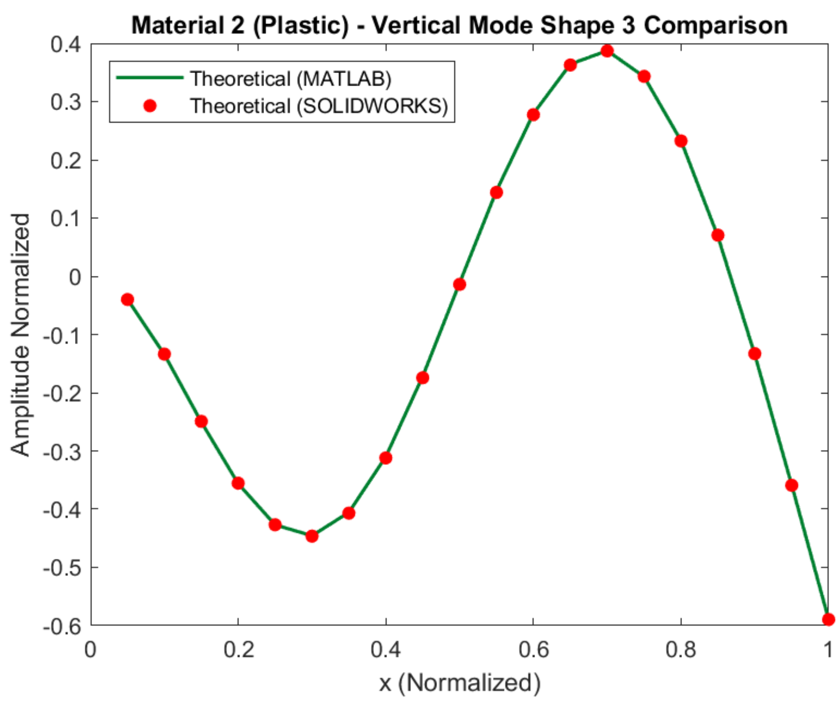

Figure 10 Comparison of vertical mode shape 3 between the SOLIDWORKS and theoretical models for the plastic beam.

This mode shape is consistent with the lesser amplitude expected when compared to that of beam seven.

Equations 11 and 8 are used to find the modulus of elasticity and density for each beam from the modal analysis which are presented in Table 2 below. The values for PET [5] and Bronze [6] were found on matweb.com.

Table 2 Material Properties and Simple Statistics for Material 2 and Material 7 Tests.

Material	**Average Young's Modulus** (GPa)	**Average Density** ($\frac{g}{cm^3}$)
Material 2	3.72	1.3838
PET (Theroetical)	3.1	1.28
Percent Difference	20.00%	8.11%
Material 7	87.44	9.1489
Bronze (Theroetical)	112	8.38
Percent Difference	21.93%	9.18%

These material properties provide enough insight to make conclusions about the composition of each beam. The Ashby plot below, Fig. 11, is used to estimate materials which match the properties determined through modal analysis.

The density for material two is 1383.8 kg/m^3 while the Young's modulus is 3.72 GPa. This data falls in a cluster of candidate materials within the Polymer region. Therefore, material two is almost definitely a polymer. According to our additional research, we predict that it is likely made out of a form of PET, although it could be ABS or Nylon as well. Additionally, the density for material seven is 9148.9 kg/m^3 while the Young's modulus is 87.44 GPa. This data is in the small range of densities for copper alloys, but significantly low for the Young's modulus of most copper alloys. Therefore, material seven is definitely a metal, and likely a copper alloy such as brass or bronze.

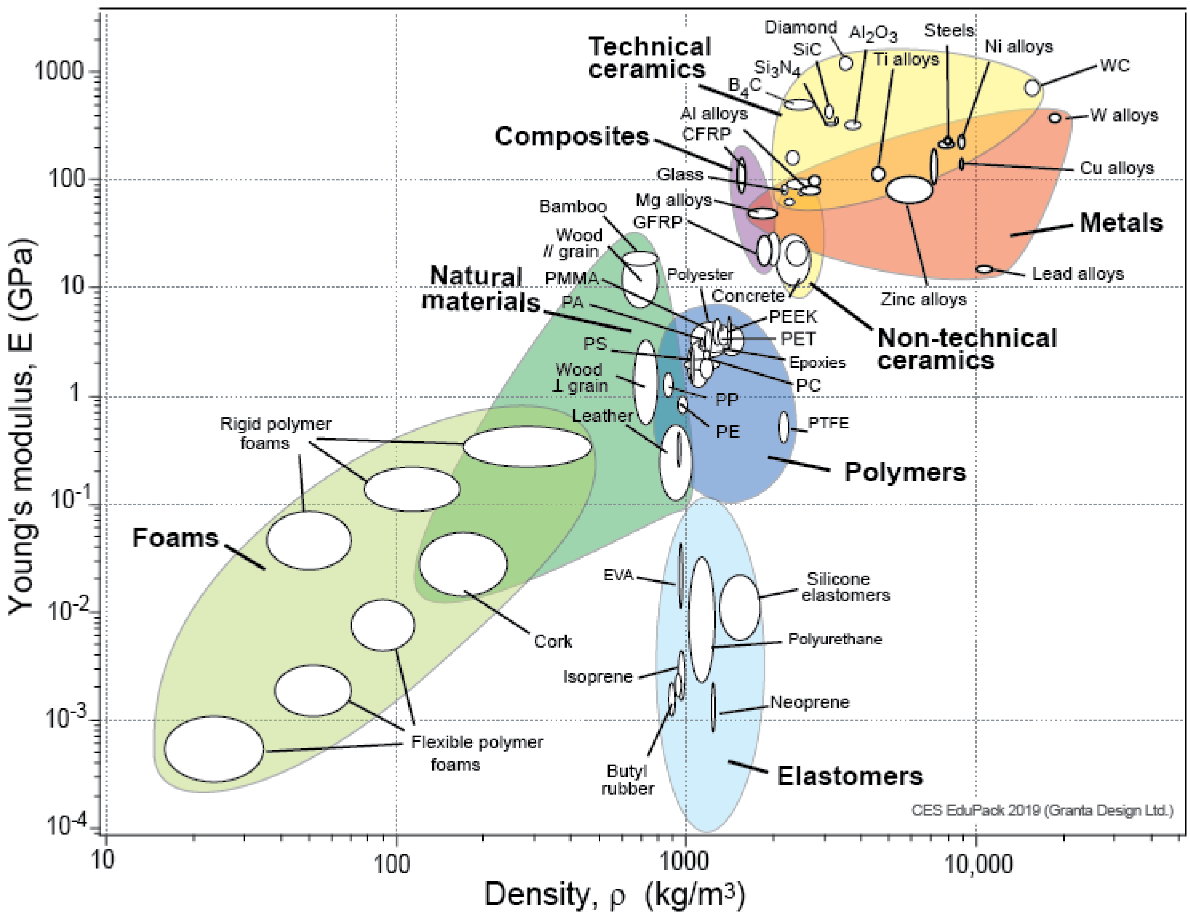

Figure 11 Ashby chart [7] comparing mechanical properties of common materials.

CONCLUSION

This experiment provided valuable experience to participants in designing and carrying out an experiment to find unknown information about materials. Additionally, it provided experience with modal analysis and the derivation of material properties from vibration data. The theoretical expectations for the mode shapes were quite consistent with the experimental data, indicating that the conclusions of the report are valid.

The modal analysis allowed for the derivation of a Young's modulus of 3.72 GPa and a density of 1383.8 kg/m^3 for material two. This would indicate that material two is almost definitely a polymer such as PET, ABS, or Nylon. Additionally, the Young's modulus for material seven was found to be 87.44 GPa and the density was found to be 9148.9 kg/m^3. This material is definitely a metal and is likely a copper alloy such as a dense brass or a weak bronze.

The small amounts of error in the modal analysis data could be attributed to noise in the signal measurements or inconsistencies in accelerometer data at locations closer to the built in end of the cantilevered beam. This experimental procedure could be improved by using longer, larger beams, with larger force impulses, such that material inconsistencies and the weight of the accelerometer have less significant impacts on the vibration response.

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A Glossary

- **Small angle approximation:** When an angle is considerably small ($\theta \approx 0$), trigonometric estimates can be made [8]:

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1 \qquad \tan\theta \approx \theta$$

- **Euler-Bernoulli beam theory:** A method of structural analysis of beams under the assumption that the beam cross-section "is infinitely rigid in it's own plane (no deformations occur in the plane of the cross-section)." Some other assumptions of this theory is that during deformation, the beam's "cross-section ... remains plane and normal to the deformed axis of the beam [9]."

- **Finite element analysis:** A method of simulating the reaction of a part or assembly, on a computer, based on the governing boundary conditions and applied loads. This is done to gain an understanding of how this part or assembly will react and the behavior of this reaction [10].

- **Modulus of elasticity:** The measurement of a material's resistance to elastic deformation, or elasticity. It is also the linear relationship between stress and strain that a material undergoes in response to a load [11].

- **Ashby plots:** Charts that display 2 material properties, one on the x-axis and one on the y-axis, along with ranges that different materials can be on the plots. They are mainly used for material selection, through selecting the range of 2 material properties and determining the corresponding materials that lie in that range on the plot [7].

B Sample Calculations

In order to find the area moment of inertia of the beam, the following standard equation was used with a base width of 0.026m and a height of 0.013m.

$$I = \frac{1}{12}bh^3 = \frac{1}{12} * 0.026 * (0.013)^3 = 0.0000000047602m^4 \tag{15}$$

With this calculated, the Young's modulus can be calculated. For one trial, the natural frequency was 24.5673 rad/s with an added mass of 0.5 kg.

$$E = \frac{m(\omega_n^2)(L^3)}{3I} = \frac{0.5 * 24.5673^2 * 0.5619^3}{3 * 0.0000000047602} = 3750514339Pa \tag{16}$$

This process was repeated for each trial with added mass. Then, once the Young's Modulus was known, the density was found as follows. This sample is from a trial with a natural frequency of 68.4867 rad/s and a calculated average Young's Modulus of 3.719252080Pa.

$$\rho = \frac{(1.875^4)EI}{(\omega_n^2 * A * L^4)} = \frac{(1.875^4)3.719252080 * 0.0000000047602}{(68.4867^2 * 0.026 * 0.013 * 0.5619^4) = 1383.84 kg/m^3} \tag{17}$$

This process was used for each beam to find the average Young's Modulus and the average density.

C Raw Data

The data from the trials used to identify the young's modulus of each beam is shown in Table C.1, along with the calculated values for the area moment of inertia and Young's Modulus.

Table C.1 Young's Modulus Test Data

Bar number	ω_n (with mass) ($\frac{rad}{s}$)	added mass (kg)	I (m^4)	E (GPa)
2	36.82	0.2	0.000000004760	3.370
2	24.57	0.5	0.000000004760	3.751
2	24.57	0.5	0.000000004760	3.751
2	17.15	1.0	0.000000004760	3.657
7	83.57	0.2	0.000000000468	5.640
7	83.57	0.2	0.000000000468	5.640
7	61.39	0.5	0.000000000468	7.609
7	61.39	0.5	0.000000000468	7.609
7	46.62	1.0	0.000000000468	8.778
7	46.62	1.0	0.000000000468	8.778
7	37.82	1.5	0.000000000468	8.667
7	38.01	1.5	0.000000000468	8.754

The data from the trials used to identify the density of each beam is shown in Table C.2, along with the calculated values for the area moment of inertia, Young's Modulus, and density.

Table C.2 Density Test Data

Bar number	ω_n (no mass) ($\frac{rad}{s}$)	I (m^4)	E (GPa)	rho ($\frac{kg}{m^3}$)
2	68.49	0.000000004760	3.719	1384
2	68.49	0.000000004760	3.719	1384
7	127.5	0.0000000004680	8.744	9149
7	127.5	0.0000000004680	8.744	9149

The data from the trials used to identify the mode shapes of the metal beam is shown in Table C.3 below. The beam was struck starting at the tip and working back toward the base, so node 1 is at the tip. The data for node 25 is assumed from the fact that node 25 was clamped down (no hammer blow was actually cast there).

Table C.3 Density Test Data

Hit location (in)	Hit/Node	ϕ_1	ϕ_2	ϕ_3	f_1 (Hz)	f_2 (Hz)	f_3 (Hz)
0	1	127	178	220	7.5	46.8	133
1.25	2	118	136	129	7.5	46.8	133
2.5	3	113	104	55.1	7.5	46.8	133
3.75	4	92.8	53.4	-22.8	7.5	47	133
5	5	84.2	10.7	-94.4	7.5	47	133
6.25	6	78.2	-25.7	-130	7.5	47	133
7.5	7	68.4	-59.6	-147	7.5	47	133
8.75	8	67.3	-75.7	-137	7.5	47	133
10	9	56.9	-95.7	-106	7.5	47	133
11.25	10	51.1	-113	-61.9	7.5	47	133
12.5	11	42.2	-125	-8.61	7.5	47	133
13.75	12	34	-141	51.7	7.5	47	133
15	13	28.7	-140	105	7.5	47	133
16.25	14	24.1	-137	142	7.5	47	133
17.5	15	19.4	-111	160	7.5	47	133
18.75	16	12.3	-91.6	152	7.5	47	133
20	17	9.12	-75.2	147	7.5	47	133
21.25	18	4.42	-52.4	106	7.5	47	133
22.5	19	1.95	-35.6	71.2	7.5	47	133
23.75	20	0.0624	-13.5	35.1	7.5	47	133
25	21	0	0	0	N/A	N/A	N/A

D MATLAB Code

The MATLAB script below was used to import data from the experimental SignalCalc run data files and plot the proper data. The plotting was done iteratively, changing various titles, x- and y-axis labels, and the intended run number depending on the data that was being plotted.

```
%%
clear all;
clc;
close all;

%%
% mainFilepath = 'D:\WashU\Classes\Fall 2020\Vibrations LAB\Lab 1\Run Files';
mainFilepath = 'C:\Users\mitry\Google Drive\NEW Data Week 1';

% excellFilename = '/first_runs.xlsx';
excellFilename = '/MassEstimation.xlsx';

cd(mainFilepath)
A = {'Run number','S-plot peak (m/s^2)','S-plot frequency(Hz)', ...
     'X-plot avg Acceleration peak (m/s^2)','X-plot avg Force peak (N)'};
xlswrite(strcat(mainFilepath,excellFilename),A)

firstRun = 49-10;
lastRun = 49-10;
% numFolders = 73;
height = 2;

for currentRun = firstRun:lastRun
    localPath = strcat(mainFilepath,'/Run');
    if currentRun < 10
        localPath = strcat(localPath,'0000');
    end
    if currentRun ≥ 10 && currentRun < 100
        localPath = strcat(localPath,'000');
    end
    if currentRun ≥ 100 && currentRun < 1000
```

```
        localPath = strcat(localPath,'00');
    end
    localPath = strcat(localPath,int2str(currentRun));
    localPath = strcat(localPath,'/MATLAB/DPsv00000.mat');
    load(localPath);

    figure('Position', [10 100 900 700]);
    a = num2str(currentRun - 34);
    sgtitle(['SignalCalc Data for Node ' a]);

    subplot(4,1,1);
    plot(abs(X1(:,1)),abs(X1(:,2)),'LineWidth',1);
    title('Force Hammer Force vs Time');
    xlabel('Time [s]');
    ylabel('Force [N]');

    subplot(4,1,2);
    plot(X2(:,1),X2(:,2));
    title('Accelerometer Acceleration vs Time');
    xlabel('Time [s]');
    ylabel('Acceleration [m/s^2]');

    subplot(4,1,3);
    plot(H1_2(:,1),imag(H1_2(:,2)));
    title('Transfer Function Imaginary Values vs Frequency');
    xlabel('Frequency [Hz]');
    ylabel('Local Mode Shape Value');

    subplot(4,1,4);
    plot(abs(H1_2(:,1)),abs(H1_2(:,2)));
    title('Transfer Function Magnitude vs Frequency');
    xlabel('Frequency [Hz]');
    ylabel('Local Mode Shape Value');
end

function [peaks, locations]= findPeaksGreaterThan(pks,arr,locs,val)
    currentindex = 1;
    arrSizea = size(pks);
```

```
    arrSize = arrSizea(1);
    peaks = zeros(arrSize,1);
    locations = zeros(arrSize,1);
    for i=1:arrSize
        if(pks(i) ≥ val)
            peaks(currentindex) = pks(i);
            locations(currentindex) = arr(locs(i),1);
            currentindex = currentindex + 1;
        end
    end
    peaks = peaks(peaks ≠ 0);
    locations =  locations(locations ≠ 0);
end
```

The following MATLAB script was used to find the theoretical mode shapes, mass normalize them, and then plot them against mass normalized mode shapes from SolidWorks and, for beam 7, from experimental data.

```
clear all;
close all;
clc;

%% Beam 1 PLASTIC
E_1 = 3719252080;       % young's modulus (Pa)
rho_1 = 1383.838614;      % density (kg/m^3)
L_1 = 22.125;         % length (m)
b_1 = 0.026;       % width (m)
h_1 = 0.013;      % thickness (m)

I_1  =  b_1*h_1^3/12; % area moment of inertia
A_1 = b_1*h_1;         % cross section area

B1L_1 = 1.875;     % eignevalue 1
B2L_1 = 4.694;     % eigenvalue 2
B3L_1 = 7.855;     % eigenvalue 3

% COMPUTE NATURAL FREQUENCIES
```

```
BL_1 = [B1L_1;B2L_1;B3L_1];
wn_1 = (BL_1.^2)*sqrt(E_1*I_1/(rho_1*A_1*L_1^4)); % natural freq [rad/sec]
fn_1 = wn_1/(2*pi);                              % natural freq [Hz]

% COMPUTE MODE SHAPES
alpha1_1 = (sin(BL_1) + sinh(BL_1))./(cos(BL_1) + cosh(BL_1));
dx_1 = L_1/20;
x_1=dx_1:dx_1:L_1;
W1_1 = ((sin(BL_1(1)*x_1/L_1) - sinh(BL_1(1)*x_1/L_1) -alpha1_1(1)* ...
        (cos(BL_1(1)*x_1/L_1) - cosh(BL_1(1)*x_1/L_1))))';
W2_1 = ((sin(BL_1(2)*x_1/L_1) - sinh(BL_1(2)*x_1/L_1) -alpha1_1(2)* ...
        (cos(BL_1(2)*x_1/L_1) - cosh(BL_1(2)*x_1/L_1))))';
W3_1 = ((sin(BL_1(3)*x_1/L_1) - sinh(BL_1(3)*x_1/L_1) -alpha1_1(3)* ...
        (cos(BL_1(3)*x_1/L_1) - cosh(BL_1(3)*x_1/L_1))))';

% SCALE SO W(L)=1
W1_1  = W1_1/W1_1(end);
W2_1  = W2_1/W2_1(end);
W3_1  = W3_1/W3_1(end);

% MASS NORMALIZATION OF MODE SHAPES
M_1 = ((rho_1*L_1*b_1*h_1)/20)*eye(20);

alpha1_1 = W1_1'*M_1*W1_1;
W11_norm = W1_1/sqrt(alpha1_1);

alpha2_1 = W2_1'*M_1*W2_1;
W21_norm = W2_1/sqrt(alpha2_1);

alpha3_1 = W3_1'*M_1*W3_1;
W31_norm = W3_1/sqrt(alpha3_1);


%% Beam 2 METAL
E_2 = 5.25e9;       % young's modulus (Pa)
rho_2 = 889;      % density (kg/m^3)
L_2 = 25;         % length (m)
b_2 = 0.026;       % width (m)
```

```
h_2 = 0.006;      % thickness (m)

I_2  =  b_2*h_2^3/12; % area moment of inertia
A_2 = b_2*h_2;         % cross section area

B1L_2 = 1.875;     % eignevalue 1
B2L_2 = 4.694;     % eigenvalue 2
B3L_2 = 7.855;     % eigenvalue 3

% COMPUTE NATURAL FREQUENCIES
BL_2 = [B1L_2;B2L_2;B3L_2];
wn_2 = (BL_2.^2)*sqrt(E_2*I_2/(rho_2*A_2*L_2^4)); % natural freq [rad/sec]
fn_2 = wn_2/(2*pi);                          % natural freq [Hz]

% COMPUTE MODE SHAPES
alpha1_2 = (sin(BL_2) + sinh(BL_2))./(cos(BL_2) + cosh(BL_2));
dx_2 = L_2/20;
x_2=dx_2:dx_2:L_2;
W1_2 = ((sin(BL_2(1)*x_2/L_2) - sinh(BL_2(1)*x_2/L_2) -alpha1_2(1)* ...
        (cos(BL_2(1)*x_2/L_2) - cosh(BL_2(1)*x_2/L_2))))';
W2_2 = ((sin(BL_2(2)*x_2/L_2) - sinh(BL_2(2)*x_2/L_2) -alpha1_2(2)* ...
        (cos(BL_2(2)*x_2/L_2) - cosh(BL_2(2)*x_2/L_2))))';
W3_2 = ((sin(BL_2(3)*x_2/L_2) - sinh(BL_2(3)*x_2/L_2) -alpha1_2(3)* ...
        (cos(BL_2(3)*x_2/L_2) - cosh(BL_2(3)*x_2/L_2))))';

% SCALE SO W(L)=1
W1_2  = W1_2/W1_2(end);
W2_2  = W2_2/W2_2(end);
W3_2  = W3_2/W3_2(end);

% MASS NORMALIZATION OF MODE SHAPES
M_2 = ((rho_2*L_2*b_2*h_2)/20)*eye(20);

alpha1_2 = W1_2'*M_2*W1_2;
W12_norm = W1_2/sqrt(alpha1_2);

alpha2_2 = W2_2'*M_2*W2_2;
W22_norm = W2_2/sqrt(alpha2_2);
```

```
alpha3_2 = W3_2'*M_2*W3_2;
W32_norm = W3_2/sqrt(alpha3_2);

%% PLOT NORMALIZED MODE SHAPES FROM SOLIDWORKS
Beam1_MS1 = xlsread('Beam2_modeshape1.csv'); % Beam 1 Mode Shape 1
Beam1_MS2 = xlsread('Beam2_modeshape2.csv'); % Beam 1 Mode Shape 2
Beam1_MS3 = xlsread('Beam2_modeshape3.csv'); % Beam 1 Mode Shape 3

alpha1_1SW = Beam1_MS1(:,3)'*M_1*Beam1_MS1(:,3);
W11_normSW = (Beam1_MS1(:,3)/sqrt(alpha1_1SW))';

alpha2_1SW = Beam1_MS2(:,3)'*M_1*Beam1_MS2(:,3);
W21_normSW = (Beam1_MS2(:,3)/sqrt(alpha2_1SW))';

alpha3_1SW = Beam1_MS3(:,3)'*M_1*Beam1_MS3(:,3);
W31_normSW = (Beam1_MS3(:,3)/sqrt(alpha3_1SW))';

%% Beam 2
Beam2_MS1 = xlsread('Beam7_modeshape1.csv'); % Beam 2 Mode Shape 1
Beam2_MS2 = xlsread('Beam7_modeshape2.csv'); % Beam 2 Mode Shape 2
Beam2_MS3 = xlsread('Beam7_modeshape3.csv'); % Beam 2 Mode Shape 3

alpha1_2SW = Beam2_MS1(:,3)'*M_2*Beam2_MS1(:,3);
W12_normSW = (Beam2_MS1(:,3)/sqrt(alpha1_2SW))';

alpha2_2SW = Beam2_MS2(:,3)'*M_2*Beam2_MS2(:,3);
W22_normSW = (Beam2_MS2(:,3)/sqrt(alpha2_2SW))';

alpha3_2SW = Beam2_MS3(:,3)'*M_2*Beam2_MS3(:,3);
W32_normSW = (Beam2_MS3(:,3)/sqrt(alpha3_2SW))';

%% PLOT NORMALIZED MODE SHAPES FROM LAB
% Beam 2 - Metal beam
Beam2_MS1_lab = xlsread('Beam7_modeshape1_experimental.csv');
Beam2_MS2_lab = xlsread('Beam7_modeshape2_experimental.csv');
Beam2_MS3_lab = xlsread('Beam7_modeshape3_experimental.csv');
```

```
alpha1_2_lab = Beam2_MS1_lab(:,2)'*M_2*Beam2_MS1_lab(:,2);
W12_norm_lab = (Beam2_MS1_lab(:,2)/sqrt(alpha1_2_lab))';

alpha2_2_lab = Beam2_MS2_lab(:,2)'*M_2*Beam2_MS2_lab(:,2);
W22_norm_lab = (Beam2_MS2_lab(:,2)/sqrt(alpha2_2_lab))';

alpha3_2_lab = Beam2_MS3_lab(:,2)'*M_2*Beam2_MS3_lab(:,2);
W32_norm_lab = (Beam2_MS3_lab(:,2)/sqrt(alpha3_2_lab))';

%% Beam 1 comparison - Material 2 (Plastic)
figure(1)
plot(x_1/L_1(end),W11_norm*(-1),'color', [0 0.5 0.19], 'LineWidth', 1.5)
hold on
plot(Beam1_MS1(:,1)./20,W11_normSW,'r.', 'MarkerSize', 20),
legend('Theoretical (MATLAB)',...
       'Theoretical (SOLIDWORKS)',...
       'Location','southwest');
xlabel('x (Normalized)');
ylabel('Amplitude Normalized');
title('Material 2 (Plastic) - Vertical Mode Shape 1 Comparison');
saveas(gcf,'Material2_MS1.png')

figure(2)
plot(x_1/L_1(end),W21_norm*(-1),'color', [0 0.5 0.19], 'LineWidth', 1.5)
hold on
plot(Beam1_MS2(:,1)./20,W21_normSW,'r.', 'MarkerSize', 20)
legend('Theoretical (MATLAB)',...
       'Theoretical (SOLIDWORKS)',...
       'Location','southwest');
xlabel('x (Normalized)');
ylabel('Amplitude Normalized');
title('Material 2 (Plastic) - Vertical Mode Shape 2 Comparison');
saveas(gcf,'Material2_MS2.png')

figure(3)
plot(x_1/L_1(end),W31_norm*(-1),'color', [0 0.5 0.19], 'LineWidth', 1.5)
hold on
plot(Beam1_MS3(:,1)./20,W31_normSW,'r.', 'MarkerSize', 20)
```

```
legend('Theoretical (MATLAB)',...
        'Theoretical (SOLIDWORKS)',...
        'Location','northwest');
xlabel('x (Normalized)');
ylabel('Amplitude Normalized');
title('Material 2 (Plastic) - Vertical Mode Shape 3 Comparison');
saveas(gcf,'Material2_MS3.png')

%% Beam 2 comparison - Material 7 (Metal)
figure(4)
plot(x_2/L_2(end),W12_norm*(-1),'color', [0 0.5 0.19], 'LineWidth', 1.5)
hold on
plot(Beam2_MS1(:,1)./20,W12_normSW,'r.', 'MarkerSize', 20)
hold on
plot(Beam2_MS1_lab(:,1)./20,W12_norm_lab*(-1),'k.', 'MarkerSize', 20)
legend('Theoretical (MATLAB)',...
        'Theoretical (SOLIDWORKS)',...
        'Laboratory Data',...
        'Location','southwest');
xlabel('x (Normalized)');
ylabel('Amplitude Normalized');
title('Material 7 (Metal) - Vertical Mode Shape 1 Comparison');
saveas(gcf,'Material7_MS1.png')

figure(5)
plot(x_2/L_2(end),W22_norm*(-1),'color', [0 0.5 0.19], 'LineWidth', 1.5)
hold on
plot(Beam2_MS2(:,1)./20,W22_normSW,'r.', 'MarkerSize', 20)
hold on
plot(Beam2_MS2_lab(:,1)./20,W22_norm_lab*(-1),'k.', 'MarkerSize', 20)
legend('Theoretical (MATLAB)',...
        'Theoretical (SOLIDWORKS)',...
        'Laboratory Data',...
        'Location','southwest');
xlabel('x (Normalized)');
ylabel('Amplitude Normalized');
title('Material 7 (Metal) - Vertical Mode Shape 2 Comparison');
saveas(gcf,'Material7_MS2.png')
```

```
figure(6)
plot(x_2/L_2(end),W32_norm,'color', [0 0.5 0.19], 'LineWidth', 1.5)
hold on
plot(Beam2_MS3(:,1)./20,W32_normSW,'r.', 'MarkerSize', 20)
hold on
plot(Beam2_MS3_lab(:,1)./20,W32_norm_lab,'k.', 'MarkerSize', 20)
legend('Theoretical (MATLAB)',...
       'Theoretical (SOLIDWORKS)',...
       'Laboratory Data',...
       'Location','northwest');
xlabel('x (Normalized)');
ylabel('Amplitude Normalized');
title('Material 7 (Metal) - Vertical Mode Shape 3 Comparison');
saveas(gcf,'Material7_MS3.png')
```