Washington University in St. Louis

JAMES MCKELVEY SCHOOL OF ENGINEERING

Fall 2020 MEMS 4050 Vibrations Laboratory

Lab 4: Vibrational Analysis of a Multi Degree-of-Freedom System

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Group T (Friday 2PM)

We hereby certify that the lab report herein is our original academic work, completed in accordance to the McKelvey School of Engineering and Undergraduate Student academic integrity policies, and submitted to fulfill the requirements of this assignment:

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ABSTRACT *This experiment aims to explore and validate a simplified multi-degree of freedom vibration system modeling the vibration of a wing of the Boeing 787 Dreamliner for NyQuist Consulting Labs. The experiment consisted of determining the representative spring stiffness, finding the natural frequencies from the unforced response of a five mass system, and finding the mode shapes from forced responses of the system at each natural frequency. The spring stiffness of the representative springs is found to be 43.31 N/m, while the equivalent stiffness of the system is 28.88 N/m. Investigation of the overall system was found to have natural frequencies at 1.18 Hz, 2.17 Hz, 2.85 Hz, 3.85 Hz, and 4.25 Hz. The forced responses of the system at each natural frequency showed nodes with only minor error for each mode shape. The theoretical behavior of the system was confirmed to be an accurate model for simulating the system.*

INTRODUCTION

The purpose of this lab is to conduct a vibrations analysis on the wing of a Boeing 787 Dreamliner. The wing is modeled as a multiple degree-of-freedom (MDOF) system with masses and springs. The goal is to determine the natural frequencies and modes shapes of the model through forced and unforced oscillations as well as identifying the node locations in each mode.

In order to find the theoretical mode shapes and natural frequencies of the system the equation of motion for an undamped MDOF system is used [1],

$$[M]\ddot{\vec{x}} + [k]\vec{x} = \vec{F(t)} \tag{1}$$

where M is the mass matrix [kg], $\ddot{x}$ is the acceleration of the mass [m/s^2], k is the spring matrix [N/m], x is displacement [m], and F(t) is the forcing function. It is important to note here that when F(t) is 0 [N], then the system is unforced. Also, the spring constant (k), can be found from the equation below:

$$k = \frac{F}{x} \tag{2}$$

where F is the force on the spring and x is the displacement of the spring [m] from equilibrium.

From Eq. 1, assuming the system is unforced, the natural frequencies and mode shapes can be determined by assuming a function that is a harmonic function of time multiplied by a mode shape,

$$\vec{x} = \vec{\phi} a_0 e^{i\omega t} \tag{3}$$

where a_0 is a constant [m], ϕ is the mode shape, ω is a vector of natural frequencies [rad/s], and t is time [s]. It is important to note here that the ϕ is the shape of the system and $a_0 e^{i\omega t}$ represents the oscillation of the system. ω and ϕ are determined by solving the eigenvalue problem that results from taking the second derivative of Eq. 3 and plugging into Eq. 1:

$$[|k| - \omega^2|M|]\vec{\phi} = F(t) \quad (4)$$

The eigenvalues represent the different squared natural frequencies , ω^2 [rad/s]^2, of the system and the eigenvectors represent the corresponding modal shapes, ϕ, at those frequencies. These modal shapes are what shows the maximum displacement of the different masses at a certain natural frequency [2]. With this in mind, the free response of a mass in the system can then be represented as,

$$x_n = a_1\phi_{n1}e^{i\omega_1 t} + a_2\phi_{n1}e^{i\omega_2 t} + \dots + a_n\phi_{n1}e^{i\omega_n t} \quad (5)$$

It is important here to see that the different frequencies and modes contribute to the displacement response of that specific mass. An example of what the modal matrix, ϕ, could look like for a given frequency is in Fig. 1 below.

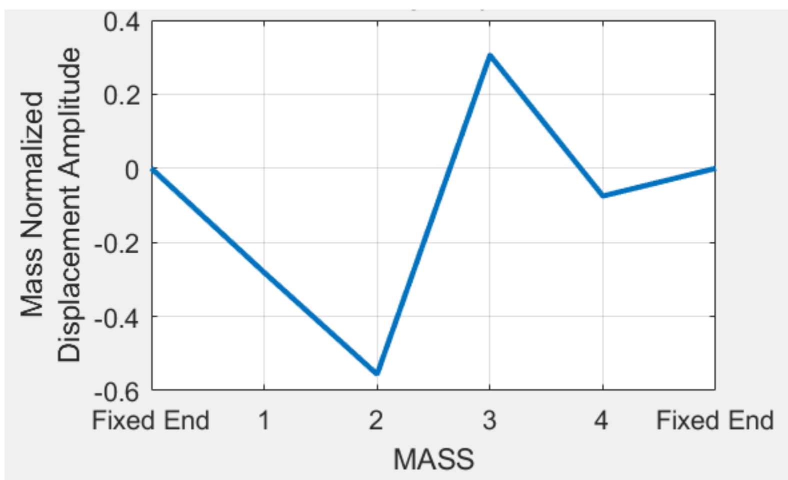

Figure 1 Mode shapes of 4 masses at a certain frequency [3].

When there is a forced response the modal matrix from the unforced response can be used to decouple the mass and spring matrices from Eq. 1. To decouple the mass and spring matrices in the system, $x = \Phi a(t)$, its second derivative is substituted into the equation of motion (Eq. 1) causing Eq. 1 to become:

$$M\Phi\ddot{a} + k\Phi a = f(t) \quad (6)$$

Then the transpose of Φ is multiplied to both sides, resulting in:

$$\Phi^T M \Phi \ddot{a} + \Phi^T k \Phi a = \Phi^T f(t) \tag{7}$$

This is done to separate the different modes and allow the equation of motion to become decoupled and turn into separate first order ordinary differential equation instead of a partial differential equation. Equation 7 can be simplified to:

$$\bar{M}\ddot{a} + \bar{k}a = \Phi^T f(t) \tag{8}$$

where,

$$\bar{M} = \Phi^T M \Phi = \begin{bmatrix} \alpha_1 & 0 & 0 & \dots \\ 0 & \alpha_2 & 0 & \dots \\ 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \alpha_n \end{bmatrix} \tag{9}$$

where α is the modal mass of each mode. $\bar{k}$ is found below:

$$\bar{k} = \Phi^T k \Phi = \begin{bmatrix} \alpha_1\omega_1^2 & 0 & 0 & \dots \\ 0 & \alpha_2\omega_2^2 & 0 & \dots \\ 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \alpha_n\omega_n^2 \end{bmatrix} \tag{10}$$

From Eq. 8 the different modes of vibrations can now be solved for as first order differential equations. An example first order equation is:

$$\ddot{x}_1 + \omega^2 x_1 = \frac{\vec{\Phi}_1^T f(t)}{\alpha_1} \tag{11}$$

where α_1 would be the first modal mass found from equation 9. It is important to note here that the total displacement of the system would be the sum of all the modal mass displacements. It can then be observed that the maximum displacement of a mass would be when the driving frequency is at a natural frequency of the system. As the the driving frequency changes from one natural frequency to the next, one of the masses displacement will dominate the system. For example as the driving frequency (Ω) approaches ω_1 the displacement from $x-1$ would dominate the system. Also, to find the nth mode shape of the system the forcing function would have to approach the nth natural frequency of the system.

METHODS

Apparatus. To analyze the response of a multi-degree of freedom mass and stiffness system, five sleds, depicted in Fig. 2, were placed on an Air Track. From the left to right, the carts weighed .280 kg, .180 kg, .280 kg, .180 kg, .305 kg respectively. Each cart was connected to any adjacent cart with a spring of stiffness 43 N/m. The springs on the extreme left and right ends of the apparatus were each anchored to a stationary wall. A string was attached to cart five and made to curve around a pulley wheel, redirecting, in the horizontal direction, vertical forces from the vibration generator on the ground. The string connected to two springs in series which then connected to the vibration generator. A function generator supplied a signal to the vibration generator. An air blower forced air through tiny holes on the air track, allowing the carts to hover, significantly lowering friction during cart motion. An accelerometer on mass 5 measured the accelerations of cart five. The signal from the accelerometer was sent to the signal conditioner which in turn sent its output to the Quattro. The Quattro analyzed the signal in the frequency and time domains, displaying its analysis on the PC via the SignalCalc application (See Fig. 2 and Table 1).

Table 1 Equipment used to analyze the forced and free responses for this MDOF system

Equipment	Make	Model	Serial #	Calibration Constant
Laptop	Lenovo	Thinkpad	N/A	N/A
SignalCal 240 Dynamic Signal Analyzer	N/A	N/A	N/A	N/A
Quattro SignalCalc Ace	Data Physics Corporation	N/A	10857	N/A
Function Generator	Pasco	N/A	P00138505	N/A
Vibration Generator	Frederiksen	2185.00		N/A
Accelerometer	PCB	353B33	LW203549	10.4 $\frac{mV}{s^2}$
Signal Conditioner 2	PCB	494A	816 or 813	10 mV
5 Sleds	Pasco	N/A	A-E	N/A
Air Track	PASCO Scientific	N/A	SYSTEM B	N/A
Air Blower	Frederiksen	1970.61	N/A	N/A
8 Springs	N/A	N/A	N/A	43 N/M

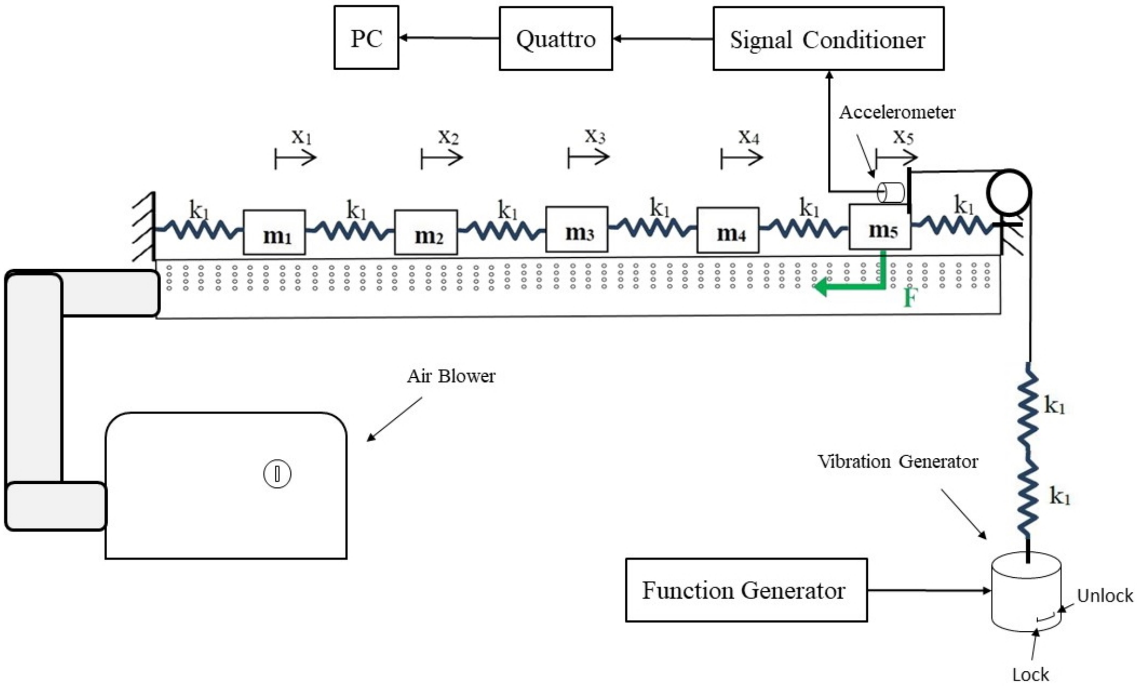

Figure 2 Diagram of multi-degree of freedom apparatus for analyzing the forced and unforced response of a mass and stiffness system [1].

Procedure. To begin the analysis of the behaviour of the multiple degree-of-freedom mass and stiffness system, SignalCalc was initialized to Fspan of *10 Hz*, Lines of 200, Trigger to *Free Run*, Avg. to *Off* and Auto End to *1 Record*. The air blower was turned on so that the carts hovered above the track without making any contact. The position of both ends of each cart, relative to the track, was recorded after turning on the blower. The track had measuring tape along its edge, making the measurement process simple. The spring constant was measured by producing a mass vs displacement curve on two identical springs that were representative of the type of spring used on the experimental apparatus, as seen in figure 3. The curve was generate by placing eight different masses between .020 and .500 kg on each spring and measuring the corresponding displacements.

The natural frequencies of the system were obtained by analyzing the free response of the system. During this analysis, the vibration generator was placed in "lock" mode. To begin, cart five was displaced to the right approximately five centimeter. The cart was released. Immediately after release, the operator triggered SignalCalc which analyzed the accelerations of cart five as the system returned to steady state. The five natural frequencies, which were obtained from the frequency domain SignalCalc plot, were recorded. Each of the five carts was displaced in the same manner and the frequency analysis of the accelerations of cart five were recorded. Before releasing any cart, the system was allowed to come to steady state.

To observe both the nodes of the system and the node locations of these modes the system was excited at each of the five natural frequencies observed in the previous operation. To force the system at its natural frequencies, the vibration generator was unlocked and excited with the function generator. The function generator output a signal amplitude of 6.5 V which was large enough to observe the response of the system to the vibrations but small enough to comply with the 1 A current limit of the vibration generator. The frequency domain resolution of the Quattro analysis was 0.05 Hz, so the natural frequencies that were measured were not necessarily that true natural frequencies of the system. Accordingly, the function generator was set to force the system at the measured natural frequency and then tuned until the system appeared to behave as though forced at a natural frequency. Once the system came to equilibrium a SignalCalc run was triggered. Also, a slow motion video was taken of the oscillations of each cart as well as of the locations of each node. This process was repeated for each natural frequency, waiting the system to reach steady state.

Analysis. The natural frequencies and modes of the system were determined theoretically and experimentally. The theoretical natural frequencies were given by the eigenvalues of Eq. 4 and the modes were solved for using the same equation. MATLAB was used to solve for the eigenvalues and eigenvectors of the system. The spring constants were given by the average of the slopes of the mass verse displacement plots generated experimentally (Fig. 3 and Eq. 2). When determining the slope for each spring, all points corresponding to forces beneath the pre-load of the springs was neglected. The details of this problem can be seen in Appendix B. All MATLAB code is in Appendix D.

The five natural frequencies were determined by using the the Quattro to generate spectral density plots of the accelerometer output for the system free response. The five natural frequencies were given as the five peaks present in the spectral plot (Fig. 5). A total of five free response spectral plots were generated, one for the free response to a displacement of each mass. So, each of the five natural frequencies was given as the average of the corresponding frequency from each plot.

The mode shapes were determined by measuring the peak to peak displacement of each cart for each natural frequency, using the recorded slow motion videos. The peak displacement was given as the difference between the greatest displacement of a cart to the right and the greatest displacement of the same mass to the left. Displacement measurements were taken for the both sides of the cart and the peak amplitude was the average of the two sides. The arithmetic was performed in Excel and mode shapes were plottied in MATLAB for each natural frequency. MATLAB was also used to plot the experimental mode shapes against the theoretical mode shapes. (Figs. 11 - 15).

Sources of Error. Our theoretical model assumes that the system experiences no friction. In reality the carts experience viscous damping from air drag. This is good because it keeps the system from blowing up when excited at natural frequencies, but it will also cause our experimental results to differ slightly from the theoretical values. Also, all measurements were acquired by eyeballing the position of the cart relative to the measuring tape on the air track. A more accurate form of measurement could be implemented to improve the accuracy of the displacement measurements.

RESULTS & DISCUSSION

First, the two springs were displaced using known weights to determine preload and spring constant. The slope of the line of best fit is equivalent to the spring constant, while the y-intercept is equivalent to the preload. A plot of the two spring behaviors is shown in Fig. 3 below.

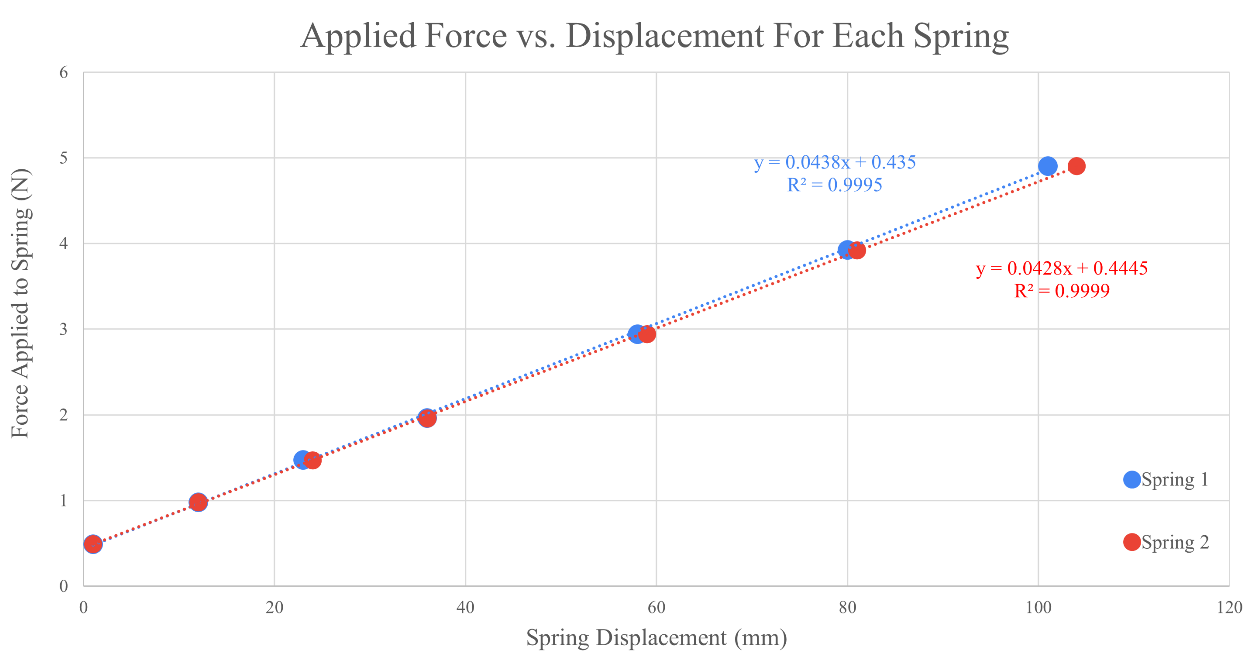

Figure 3 Linear spring displacement based on added weight to one end of a spring.

The preload and slope of each spring is nearly identical, and the springs can be assumed to be the same. Therefore, all six springs in series and the two attached to the function generator (in series with each other but in parallel to the other six) in the rest of the experiments can be assumed to have the same qualities, $k = 43.31\ N/m$ and a preload of about $0.44\ N$. Therefore, the cumulative stiffness can be calculated as $k_{eq} = 28.88\ N/m$.

Free Response. The free response of the system was measured, and the behavior was analyzed for an initial displacement of $0.036\ m$. A time series plot for mass five is displayed below in Fig. 4.

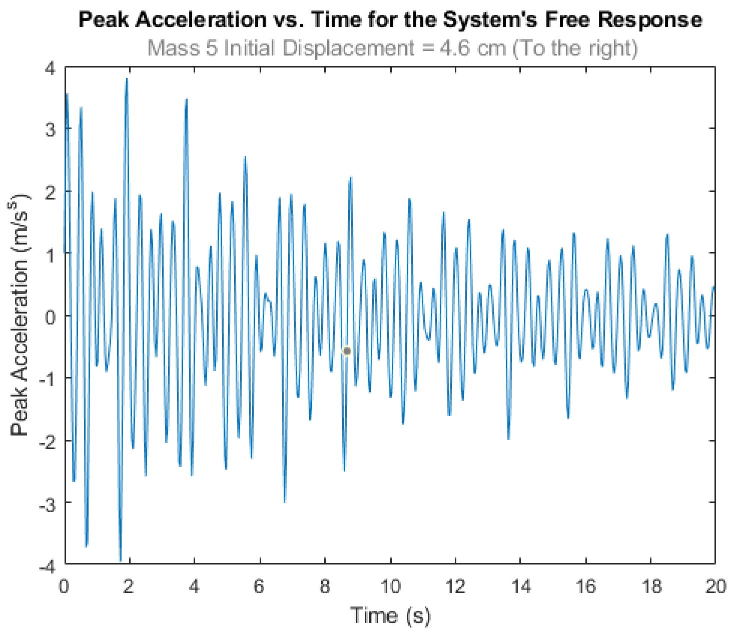

Figure 4 Experimental time series plot of the system's free response to an initial displacement of 4.6 cm to mass 5.

The peak accelerations decay linearly over time, as is expected for the experimental free response with air resistance. The combined waveform varies in amplitude over time, as the five relevant vibration modes combine and cancel the acceleration amplitudes from one another.

The spectral density plot for the same response can be seen below in Fig. 5.

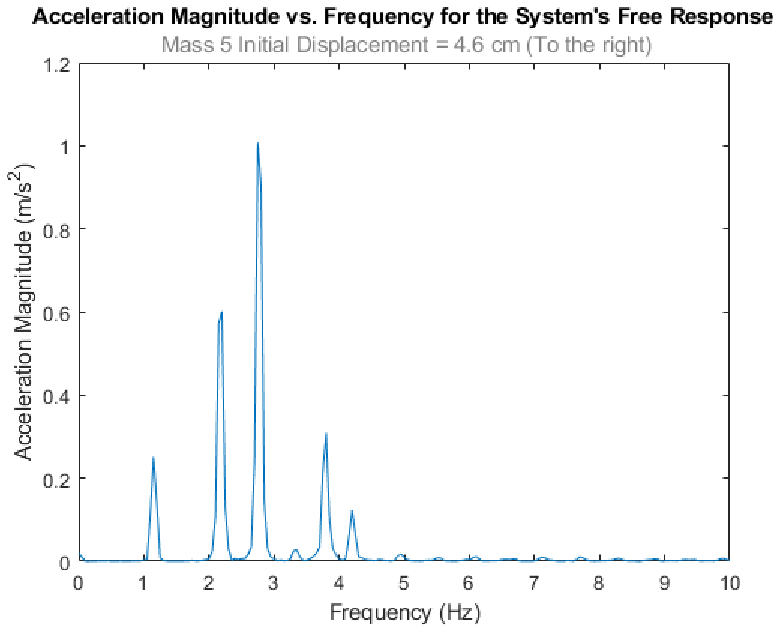

Figure 5 Experimental spectral density plot of the system's free response to an initial displacement of 4.6 cm to mass 5.

This plot displays clear peaks for all five modes of vibration. The peaks occur at *1.18 Hz*, *2.17 Hz*, *2.85 Hz*, *3.85 Hz*, and *4.25 Hz* respectively for each vibration mode. Mass five clearly displays modes two and three in its free response, as their acceleration magnitudes are significantly higher than the other mode peaks.

Steady-State Forced Response. The system was then forced into a steady-state response to excitation. Each of the five natural frequencies of the modes of vibration were applied from the function generator such that isolated mode responses could be observed. The first excitation frequency represents the first mode of vibration, the spectral density plot for which can be seen below in Fig. 6.

The peak is very robust and isolated, meaning that the first mode of vibration is expected to be seen with clarity. The other minor peaks may cause slight discrepancies in the mode shapes from theoretical expectations.

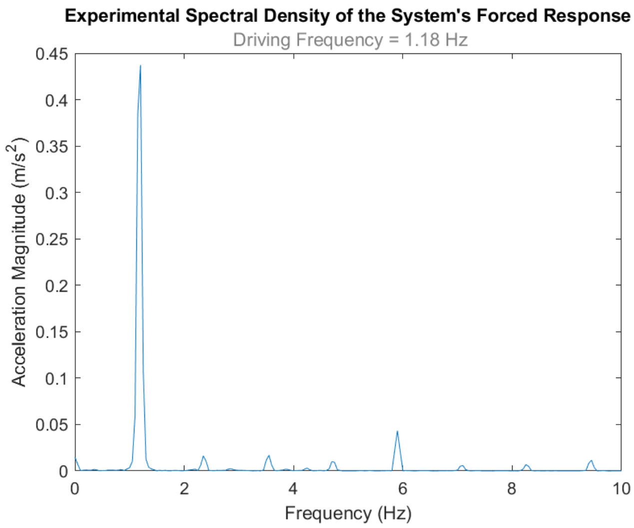

Figure 6 Experimental acceleration magnitude vs. frequency for the system's forced response to a driving frequency of 1.18 Hz.

The second excitation frequency represents the second mode of vibration, and the spectral density plot for the mode can be seen below in Fig. 7.

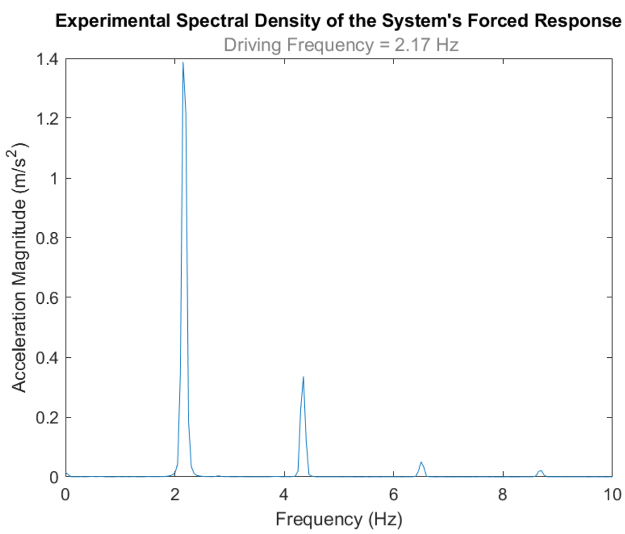

Figure 7 Experimental acceleration magnitude vs. frequency for the system's forced response to a driving frequency of 2.17 Hz.

While the peak is quite robust, a minor peak slightly above *4 Hz* could cause error in the mode shape amplitudes for each mass.

The third excitation frequency represents the third mode of vibration, the spectral density plot for which can be seen below in Fig. 8.

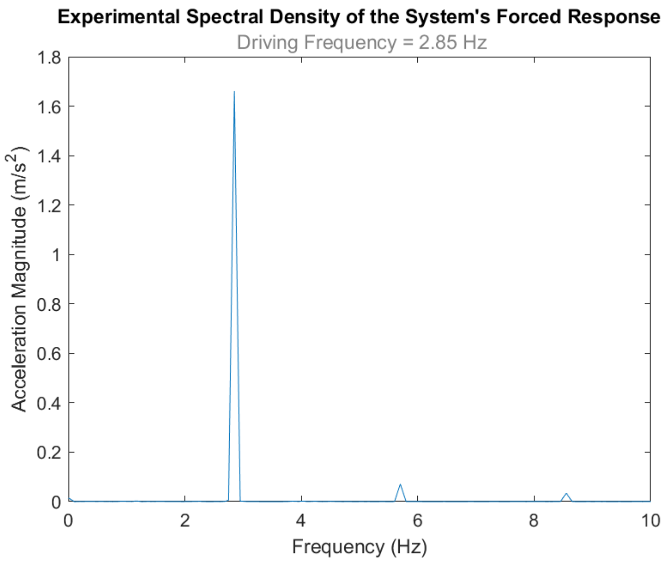

Figure 8 Experimental acceleration magnitude vs. frequency for the system's forced response to a driving frequency of 2.85 Hz.

The peak for this frequency is very robust and other peaks are quite minor, meaning the mode shape can be expected to be very accurate to theoretical expectations.

The fourth mode of vibration has a frequency of *3.85 Hz*. The spectral density plot of the system, excited at this frequency, can be seen in Fig. 9 below.

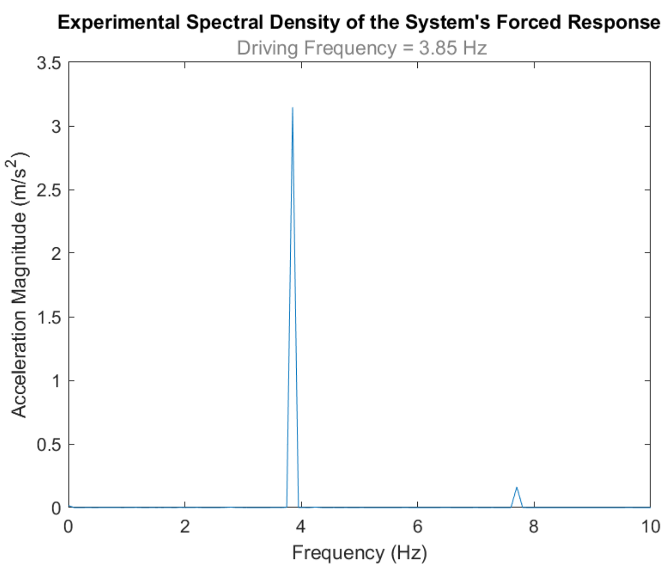

Figure 9 Experimental acceleration magnitude vs. frequency for the system's forced response to a driving frequency of 3.85 Hz.

The peak for the main frequency is very robust and other peaks are small, meaning the mode shape should be very accurate to theoretical expectations.

The fifth and final mode of vibration has a frequency of *4.25 Hz*. The spectral density plot of the system, excited at this frequency, can be seen in Fig. 10 below.

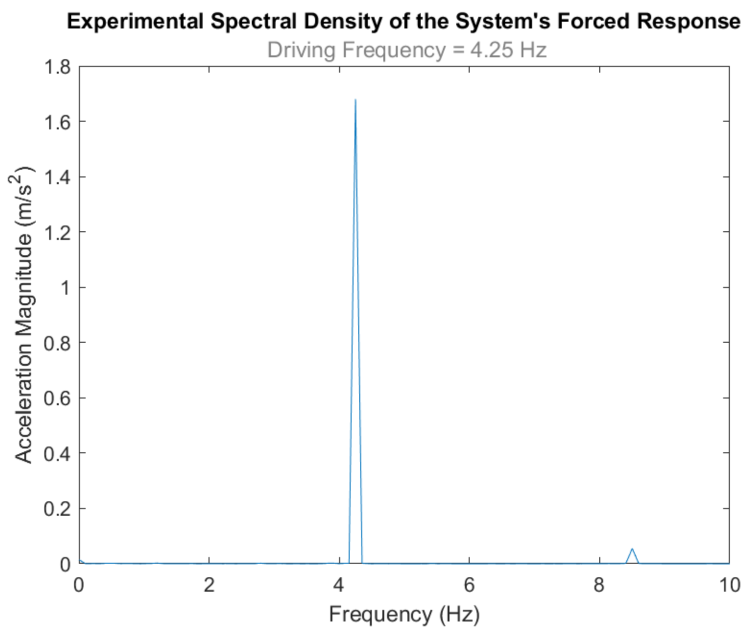

Figure 10 Experimental acceleration magnitude vs. frequency for the system's forced response to a driving frequency of 4.25 Hz.

The final spectral density plot also shows a single robust peak, such that the mode shape should display strong agreement with theoretical expectations.

The mode shapes for each forced response are of particular interest, so the theoretical nodes and displacement behaviors are compared to the measured response. The first experimental mode shape is compared to theory below in Fig. 11.

The theoretical mode shape matches the experimental quite well. Neither shape had a node, while both saw maximum displacement amplitude in mass three. There is very little error in the mass normalized displacement amplitudes.

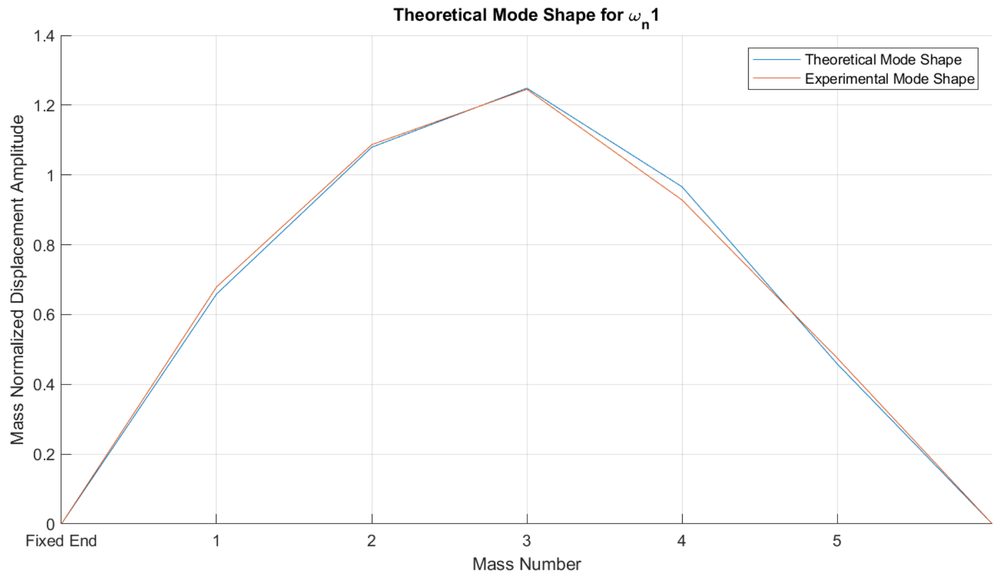

Figure 11 Mode shape 1 for the experimental (orange) and theoretical (blue) data.

Experimental mode shape two is expected to have one node at mass three and can be seen below in Fig. 12.

The theoretical mode shape matches the experimental well. The node is near mass three, despite slight amplitude disparity with the theoretical expectation. This variance could be attributed to asymmetries in the experimental model, such as inconsistent lift by the air platform causing friction. Additionally, other minor frequency peaks in Fig. 7 could contribute to the disparities.

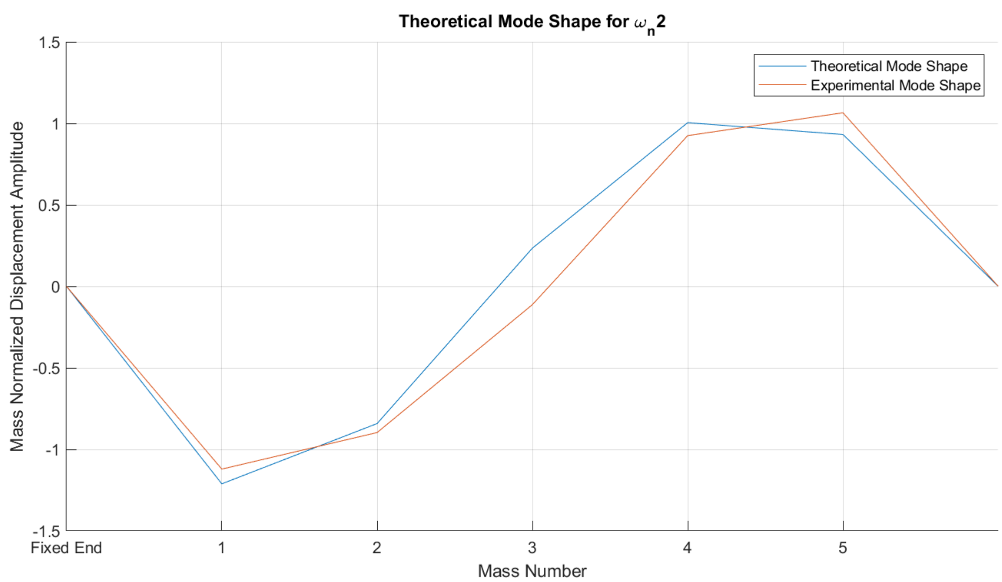

Figure 12 Mode shape 2 for the experimental (orange) and theoretical (blue) data.

Experimental mode shape three will have two nodes and three amplitude peaks. It is plotted against the theoretical expectations below in Fig. 13.

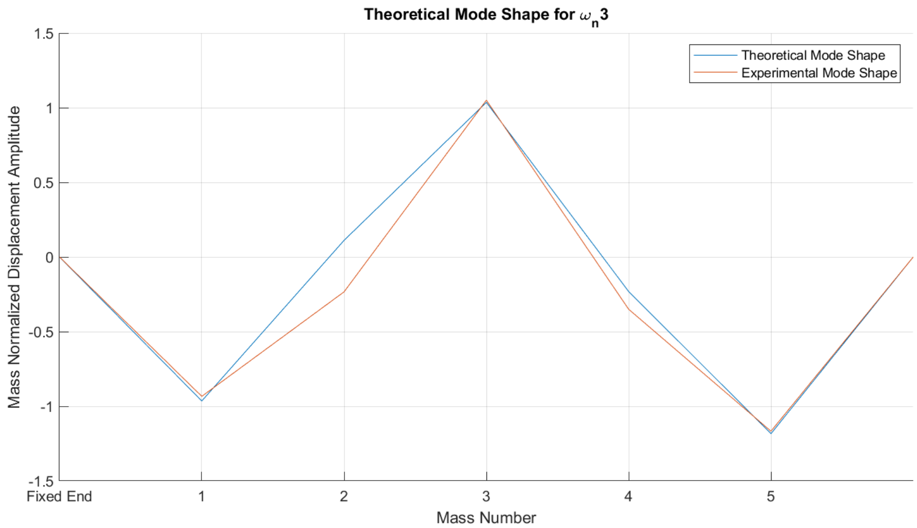

Figure 13 Mode shape 3 for the experimental (orange) and theoretical (blue) data.

The two curves match quite well, except for disparity in the node amplitude for mass two. This error can be attributed to similar irregularities in the testing apparatus or a source of friction. The minor spectral peaks in Fig. 8 could also contribute to the disparities.

Experimental mode shape four will have three nodes and four amplitude peaks. It is plotted against the theoretical expectations below in Fig. 14.

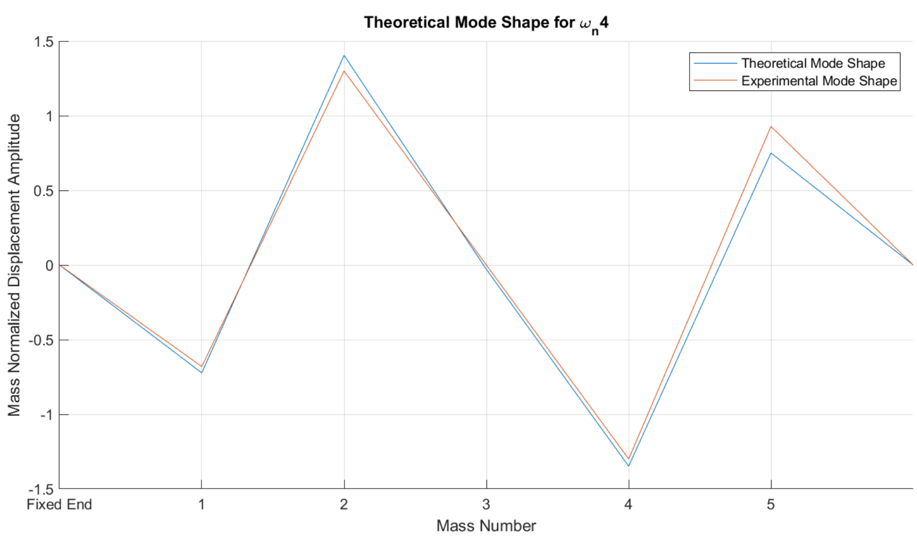

Figure 14 Mode shape 4 for the experimental (orange) and theoretical (blue) data.

The experimental peaks and nodes match the theoretical expectations well. The amplitude at masses one and five are lower than those of masses two and four, while the nodes lie between masses one and two, four and five, and at mass three.

Finally, experimental mode shape five will have four nodes and five amplitude peaks. It is plotted against the theoretical expectations below in Fig. 15.

This experimental plot matches the theoretical expectations very well. There is little discrepancy between the peaks and the nodes occur between each mass as expected.

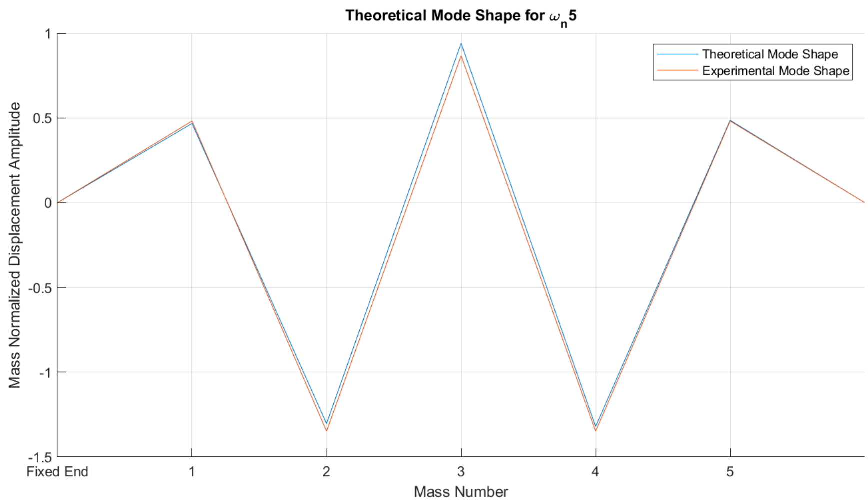

Figure 15 Mode shape 5 for the experimental (orange) and theoretical (blue) data.

All of the mode shapes from the forced responses matched the theoretical expectations quite well. Small errors were attributable to minor irregularities and do not impact the validity of the conclusions.

CONCLUSION

The exploration of multi-degree of freedom vibration systems gave experience measuring and analyzing natural frequencies and mode shapes of a system composed of discrete masses and springs. Participants gained hands-on experience analyzing vibration modes for complex systems. The results of the forced responses generally matched all theoretical expectations, and the theoretical mode responses matched quite well with the experimental data. The comparative accuracy means that the analytical model for a multi degree of freedom vibration system is accurate and acceptable for use by NYQuist Consulting Labs in the vibration analysis of the wing from a Boeing 787 Dreamliner.

Error in the lab that contributed to the results of the data can be attributed to multiple factors including friction on mass five of the system. This was due to inconsistent lift from the air track on the mass compared to the other masses. Other sources of error include general viscous damping from air resistance, slight inconsistencies in the spring stiffnesses, and importantly, relevant minor spectral peaks in the forced responses of modes two and three. The experimental setup could be improved by employing smart, variable force air tracks, such that masses of different weights could all be lifted, and lifted consistent amounts. However, this is likely an over-complicated and unreasonable endeavour.

References

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A Glossary

- **Mass-normalization**

  The method of normalizing eigenvectors of a system's mode shape. This method produces a unit value of a generalized mass by scaling each eigenvector [4].

- **Mode shape**:

  A vector describing the relative motions of the masses in a multi-degree of freedom system [5].

- **Modal frequency**

  A vector describing the natural frequencies of the multi-degree of freedom system [6].

- **Node of vibration**

  A node of vibration is a location at which a vibrating object will undergo no displacement [7].

- **Lumped mass system**

  A method in which all the system masses are lumped together into a singular matrix equally into each degree of freedom such that it can represent the overall masses on any node on a rigid object [8].

- **Overcurrent**

  A condition in which an excessive current flows in a circuit due to a short circuit or an overload of the system [9].

B Sample Calculations

The system of 5 masses can be modeled as shown in Fig. B.1, with 5 solid masses and 6 massless springs. Damping will be neglected for the sake of this derivation. The three springs at the right side of the system will be simpified into one equivalent spring: k_6.

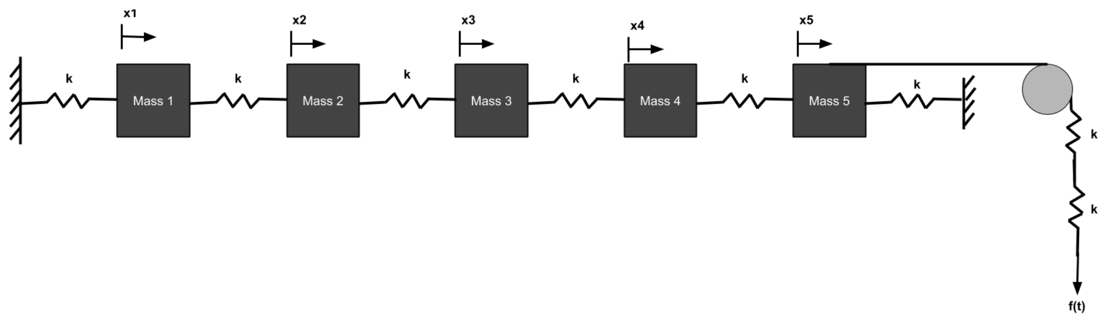

Figure B.1 Model of the experiment system setup, with relevant quantities labeled.

In order to derive the equations of motion, first a free body diagram for every component must be drawn. These free body diagrams follow in Fig. B.2.

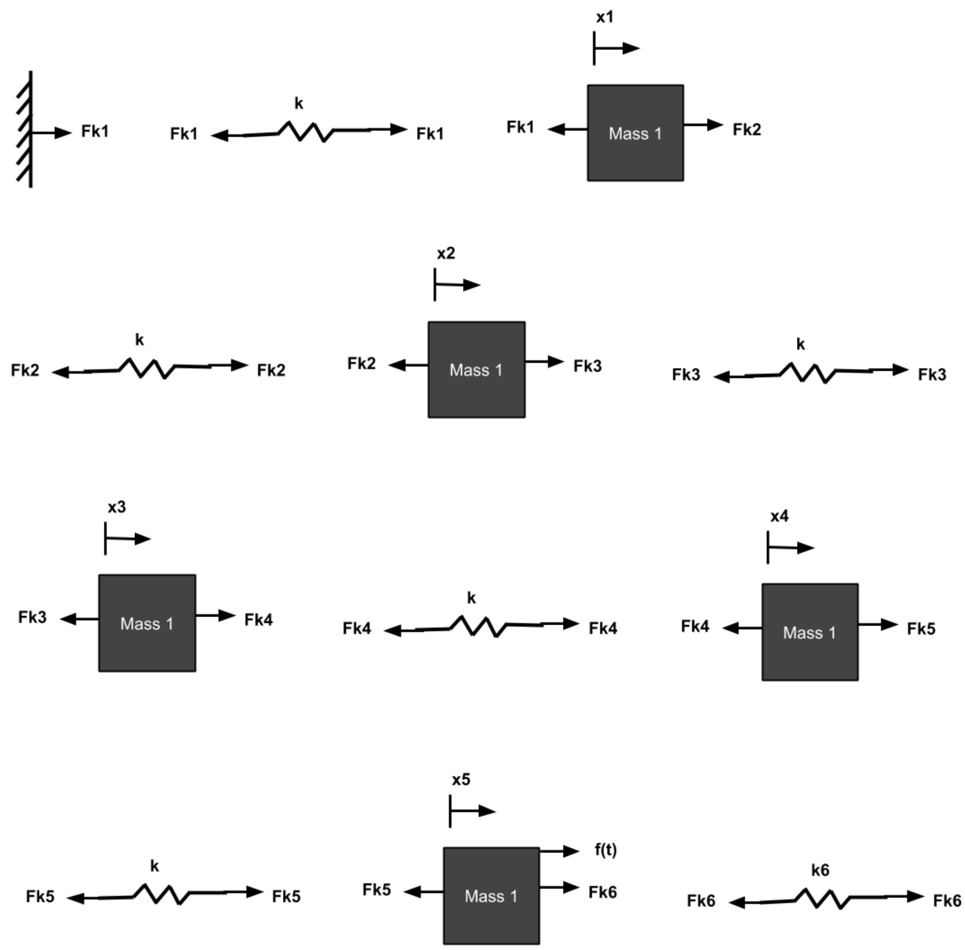

Figure B.2 Free body diagrams needed to derive the equations of motion for the system.

Applying Newton's second law to each of the masses shown above yields the following five equations, where $F_{k,n}$ refers to the spring force from the n^{th} spring, m_k refers to the mass of the k^{th}

mass, $f(t)$ refers to the force applied from the shaker, and x_k refers to the displacement of the k^{th} mass.

$$m_1\ddot{x}_1 = (F_{k2} - F_{k1}) \tag{12}$$

$$m_2\ddot{x}_2 = (F_{k3} - F_{k2}) \tag{13}$$

$$m_3\ddot{x}_3 = (F_{k4} - F_{k3}) \tag{14}$$

$$m_4\ddot{x}_4 = (F_{k5} - F_{k4}) \tag{15}$$

$$m_5\ddot{x}_5 = (F_{k6} - F_{k5}) + f(t) \tag{16}$$

In order to find the spring forces in equations 12 to 16, Hooke's law can be used for each spring. Every spring is assumed to have the same spring constant, k, except for the sixth spring, which has a combined spring constant of k_6.

$$F_{k1} = k(x_1) \tag{17}$$

$$F_{k2} = k(x_2 - x_1) \tag{18}$$

$$F_{k3} = k(x_3 - x_2) \tag{19}$$

$$F_{k4} = k(x_4 - x_3) \tag{20}$$

$$F_{k5} = k_6(-x_5) \tag{21}$$

Equations 17 through 21 can be plugged into equations 12 through 16 to generate the equation of motion for each mass, as follows.

$$m_1\ddot{x}_1 = k(x_2 - 2x_1) \tag{22}$$

$$m_2\ddot{x}_2 = k(x_3 - 2x_2 + x_1) \tag{23}$$

$$m_3\ddot{x}_3 = k(x_4 - 2x_3 + x_2) \tag{24}$$

$$m_4\ddot{x}_4 = k(x_5 - 2x_4 + x_3) \tag{25}$$

$$m_5\ddot{x}_5 = k(-x_5 + x_4) - k_6x_5 + f(t) \tag{26}$$

These equations can be rearranged and written in matrix form, as is natural for systems of liner equations, as follows. This will aid in generating a computer model of the system. Luckily, there are a lot of zeros, and the two key matrices are symmetric, which will make solving the problem much easier on both the engineer and the computer.

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \end{Bmatrix} + \begin{bmatrix} 2k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k+k_6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f(t) \end{Bmatrix} \tag{27}$$

This can be summarized in the following way.

$$\Big[M\Big]\ddot{\vec{x}} + \Big[K\Big]\vec{x} = \vec{f} \tag{28}$$

In order to find the homogeneous response, the force is set to be zero, yielding the following equation of motion for the system.

$$\left[M\right]\ddot{\vec{x}} + \left[K\right]\vec{x} = \vec{0} \tag{29}$$

In order to solve this equation for the displacement of each mass, the complex substitute problem is utilized as follows:

$$\vec{x} = \vec{\phi}a_0 e^{i\omega t} \tag{30}$$

$$\ddot{\vec{x}} = -\omega^2\vec{\phi}a_0 e^{i\omega t} \tag{31}$$

Inserting equations 30 and 31 into equation 29 yields the following equation.

$$\left(\left[K\right] - \omega^2\left[M\right]\right)\vec{\phi} = \vec{0} \tag{32}$$

This is an eigenvalue problem, with ω^2 as the eigenvalue and $\vec{\phi}$ as a matrix of eigenvectors! It is natural to solve such problems with the aid of computer software, as computers are much better at crunching through matrix operations than engineering students. To this end, MATLAB was used to find the eigenvectors and eigenvalues of the system, where the eigenvectors are the mode shapes for each of the natural frequencies, which are the eigenvalues. The full MATLAB script can be found in Appendix D.

For this experiment, the stiffness of each spring was found to be 43.31 $\frac{N}{m}$, m_1 is 0.28kg, m_2 = 0.18kg, m_3 = 0.28kg, m_4 = 0.18kg, and m_5 = 0.305kg. With these parameters, the modal matrix $\vec{\phi}$ was found to be:

$$\vec{\phi} = \begin{bmatrix} 0.6589 & -1.2114 & -0.9643 & -0.7225 & 0.4669 \\ 1.0793 & -0.8415 & 0.1115 & 1.4043 & -1.3031 \\ 1.2487 & 0.2344 & 1.0357 & -0.0291 & 0.9401 \\ 0.9661 & 1.0044 & -0.2313 & -1.3476 & -1.3205 \\ 0.4588 & 0.9316 & -1.1838 & 0.7504 & 0.4858 \end{bmatrix}$$

The natural frequencies were found to be:

$$\vec{\omega}_n = \begin{Bmatrix} 7.4821 \\ 14.2097 \\ 18.0907 \\ 24.6993 \\ 27.2229 \end{Bmatrix} \frac{rad}{s}$$

In order to plot mode shapes, the modal matrix needs to be mass normalized, in order to compare between mode shapes effectively. This can be achieved according to the following formula, where $\vec{\phi}_n$ is the n^{th} mode shape.

$$\vec{\phi}_{normalized,n} = \frac{\vec{\phi}_n}{\sqrt{\vec{\phi}_n^{\,T}[M]\vec{\phi}_n}} \tag{33}$$

This equation was also used in the MATLAB code referenced earlier in order to generate the theoretical mode shapes.

C Raw Data

Table C.1 shows the maximum and minimum displacement of each cart for each natural frequency. It should be noted that positive refers to the masses moving in the same direction as mass one while negative refers to the masses that move in the opposite direction of mass one and stationary is when the mass does not move or has minimal movement and is not in either direction compared to mass one.

Table C.1 Displacement each side of the cart for each mode shape, along with the phase.

Mode Shape	Left Tab Displacement(cm)		Right Tab Displacement(cm)		Phase
	Minimum	Maximum	Minimum	Maximum	
1	30.3	33.3	42.9	45.9	Positive
	63.1	67.9	75.7	80.5	Positive
	97.1	102.6	109.7	115.2	Positive
	132	136.1	144.5	148.6	Positive

Table C.1 Continued.

Mode Shape	Left Tab Displacement(cm)		Right Tab Displacement(cm)		Phase
	Minimum	Maximum	Minimum	Maximum	
1	167.2	169.3	179.8	181.9	Positive
2	30.8	32.8	43.4	45.4	Positive
	64.6	66.2	75.3	78.9	Positive
	99.7	99.9	112.3	112.5	Stationary
	133.2	134.9	145.8	147.4	Negative
	167.3	169.2	179.9	181.8	Negative
3	30.3	33.3	42.9	45.9	Positive
	63.1	67.9	75.7	80.5	Stationary
	97.1	102.6	109.7	115.2	Negative
	132	136.1	144.5	148.6	Stationary
	167.2	169.3	179.8	181.9	Positive
4	31.3	32.4	43.8	44.9	Positive
	74.4	76.5	77.0	79.1	Negative
	99.8	99.8	111.8	111.8	Stationary
	133	135.1	145.5	147.6	Positive
	167.7	169.2	179.8	181.3	Negative
5	31.5	32	44.1	44.6	Positive
	64.8	66.2	76.9	78.3	Negative
	99.4	100.3	111.9	112.8	Positive
	133.3	134.7	145.8	147.2	Negative
	168	168.5	180.5	181	Positive

D MATLAB Code

The MATLAB script below was used to import data from the experimental SignalCalc run data files and plot the proper data. The plotting was done iteratively, changing various titles, x- and y-axis labels, and the intended run number depending on the data that was being plotted.

```
%%
clear all; close all; clc;
%% Graphing run function for either file path

%First File path (STRICTLY for local file path of stored files)
% mainFilepath = 'D:\WashU\Classes\Fall 2020\Vibrations LAB\Lab 4\Run 1 Files';
%Second File path (STRICTLY for local file path of stored files)
mainFilepath = 'D:\WashU\Classes\Fall 2020\Vibrations LAB\Lab 4\Run 2 Files';

% Input for the run number that is inteded to be graphed
startRun = 15;
finalRun = 15;

for currentRun = (startRun:finalRun)
    localPath = strcat(mainFilepath,'/Run');
    if currentRun < 10
        localPath = strcat(localPath,'0000');
    end
    if currentRun ≥ 10 && currentRun < 100
        localPath = strcat(localPath,'000');
    end
    if currentRun ≥ 100 && currentRun < 1000
        localPath = strcat(localPath,'00');
    end
    localPath = strcat(localPath,int2str(currentRun));
    localPath = strcat(localPath,'/MATLAB/DPsv00000.mat');
    load(localPath);

    [Apk_Splot,fpk_Splot] = findpeaks(abs(S1(:,2)));
    [acceleration_Splot, frequency_Splot]=findPeaksGreaterThan(Apk_Splot,S1, ...
                                                                   fpk_Splot,0.01);
```

```
    [Accel_Xplot,time_Xplot] = findpeaks(X1(:,2));
    [acceleration_Xplot, frequency_Xplot]= findPeaksGreaterThan(Accel_Xplot,X1,...
                                                              time_Xplot,0.01);
    %
end
%% Part 1 Results Plots
% figure(1)
% plot(abs(S1(:,1)),abs(S1(:,2)));
% xlabel('Frequency (Hz)');
% ylabel('Acceleration Magnitude (m/s^2)');
% title_1 = {'Acceleration Magnitude vs. Frequency for the System''s Free ...
   Response';
%            '\color{gray}\rm Mass 5 Initial Displacement = 4.6 cm (To the ...
   right)'};
% title(title_1);
%
% figure(2)
% plot(X1(:,1),X1(:,2));
% xlabel('Time (s)');
% ylabel('Peak Acceleration (m/s^s)');
% title_2 = {'Peak Acceleration vs. Time for the System''s Free Response';
%            '\color{gray}\rm Mass 5 Initial Displacement = 4.6 cm (To the ...
   right)'};
% title(title_2);

%% Part 2 Results Plots
figure(3)
plot(abs(S1(:,1)),abs(S1(:,2)));
xlabel('Frequency (Hz)');
ylabel('Acceleration Magnitude (m/s^2)');
title_1 = {'Experimental Spectral Density of the System''s Forced Response';
           '\color{gray}\rm Driving Frequency = 1.18 Hz'};
title(title_1);

%% Finding Peak Function
function [peaks, locations]= findPeaksGreaterThan(pks,arr,locs,val)
    currentindex = 1;
    arrSizea = size(pks);
```

```
    arrSize = arrSizea(1);
    peaks = zeros(arrSize,1);
    locations = zeros(arrSize,1);
    for i=1:arrSize
        if(pks(i) ≥ val)
            peaks(currentindex) = pks(i);
            locations(currentindex) = arr(locs(i),1);
            currentindex = currentindex + 1;
        end
    end
    peaks = peaks(peaks ≠ 0);
    locations =  locations(locations ≠ 0);
end
```

Shown below is the MATLAB script used to calculate the theoretical modes and frequencies, as well as plot the mass normalized mode shapes for both the theoretical and experimental data.

```
close all; clear all; clc;

m1 = 0.28;  %kg
m2 = 0.18;  %kg
m3 = 0.28;  %kg
m4 = 0.18;  %kg
m5 = 0.305; %kg
k  = (43.83286991 + 42.7941428)/2; %N/m

K = [2*k -k 0 0 0; -k 2*k -k 0 0; 0 -k 2*k -k 0; 0 0 -k 2*k -k; 0 0 0 -k 2*k]
M = diag([m1,m2,m3,m4,m5])
[phi omegaSquared] = eig(K,M)
omega_n = ((omegaSquared)).^(1/2)

% Mass normalization:
an = (phi')*M*(phi)
phi_normalized = real(phi/(an.^(1/2)))


mode1_experimental = [3 4.8 5.5 4.1 2.1 ];
```

```
mode2_experimental = [2 1.6 0.2 -1.65 -1.9];
mode3_experimental = -1*[0.8 0.2 -0.9 0.3 1];
mode4_experimental = [1.1 -2.1 0 2.1 -1.5];
mode5_experimental = -1*[0.5 -1.4 0.9 -1.4 0.5];
phi_experimental = ...
    [mode1_experimental',mode2_experimental',mode3_experimental',mode4_experimental',mode5_ex
an_experimental = diag(diag((phi_experimental')*M*(phi_experimental)))
phi_experimental_normalized = real(phi_experimental/an_experimental.^(1/2))

% Plotting Mode Shapes:
for i = 1:size(phi_normalized,1)
    figure(i)
    hold on
    plot(1:(size(phi_normalized,2) + 2), [0;phi_normalized(:,i);0])
    plot(1:(size(phi_experimental_normalized,2) + 2), ...
        [0;phi_experimental_normalized(:,i);0])
    title(strcat('Theoretical Mode Shape for \omega_n',int2str(i)))
    xlabel('Mass Number')
    ylabel('Mass Normalized Displacement Amplitude')
    set(gca,'XTick',[0 1 2 3 4 5],'XTickLabel',{'Fixed End','Fixed ...
        End','1','2','3','4','Fixed End'});
    grid on
    filename = strcat('omega_n',int2str(i)),'.png'
    saveas(gcf,filename)
    hold off
end
```