Washington University in St. Louis

JAMES MCKELVEY SCHOOL OF ENGINEERING

Fall 2020 MEMS 4050 Vibrations Laboratory

Lab 2: Basic Vibration Measurements

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Group T (Friday 2PM)

We hereby certify that the lab report herein is our original academic work, completed in accordance to the McKelvey School of Engineering and Undergraduate Student academic integrity policies, and submitted to fulfill the requirements of this assignment:

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ABSTRACT: *This experiment aims to explore a simplified single degree of freedom vibration system modeling a pedestrian overpass for NYQuist Consulting Labs. Investigation of eddy-current viscous damping and coulomb friction damping are performed. The theoretical behaviors of the damped systems were confirmed and relevant values for the damping were obtained. The damping coefficient for eddy-current viscous damping with magnets at a height of 8 mm is 0.2307 Ns/m. The coefficient for the magnets at a height of 3 mm is 1.471 Ns/m. Finally, the friction force for a coulomb damping case is found to be 12.87 N. These findings were accurate based on theoretical expectations for the decay rates and time-series decay shapes. However, observed ambient damping likely caused the calculated damping coefficients and friction force to be too high. The data was processed using SignalCalc software and plots were generated using MATLAB and Microsoft Excel.*

INTRODUCTION

The purpose of this experiment is to "observe and model transient and forced vibrations in a single degree of freedom system and investigate damping on that system" [1]. This is done by measuring the acceleration of a cart, that is attached to springs while the motion is unforced. The measurement was then repeated with a forced vibration and a coulomb or viscous damping force.

Before talking about the two types of damping it is important to understand the motion of the cart. Since this is a single degree of freedom system, the ordinary differential equation below describes the cart's motion:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k_{eq}}{m}x = \frac{F_0 Cos(\Omega t)}{m} \tag{1}$$

where on the left side of the equation $\ddot{x}$ is the acceleration $[m/s^2]$, $\dot{x}$ is the velocity $[m/s]$, x is the displacement [m] of the cart, c is the damping coefficient [kg/s] from dampers, k_{eq} is the equivalent spring coefficient [N/m] and m is the mass of the cart [kg]. On the right side is the forcing function, where F_0 is the force applied [N], ω is the angular velocity [rad/s], t is the time [s], and m is still the mass of the cart [kg]. It should be noted here that if F_0 is 0 then the right side will become 0 and the system is have to have no forced vibration applied [1].

If there is only one spring in the system then k_{eq} is simply the spring constant of that spring.

However if there is more than one spring then the equivalent spring coefficient can be found from the two equations below:

$$k_{eq}^{-1} = \sum k_n^{-1} \tag{2}$$

$$k_{eq} = \sum k_n \tag{3}$$

where k_n [N/m] is the spring constant for each spring in the system. Equation 2 is for springs in series while equation 3 is for springs in parallel. The difference between springs in parallel and in series is seen in figure 1 below:

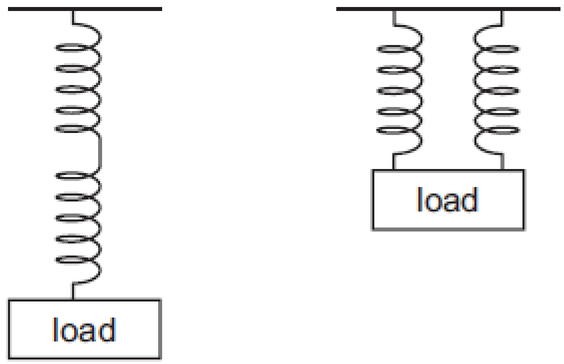

Figure 1 Schematic of springs in series on the left and springs in parallel on the left [2].

If one assumes the solution takes the form of:

$$x(t) = X_0 e^{-i\Omega t} \tag{4}$$

Where $x(t)$ [m] is the displacement of the system, X_0 [m] is the amplitude of the system, Ω [rads/s] is the angular velocity, and t is the amount of time, then the solution can be found to be:

$$x(t) = X_0 * Cos(\Omega t - \angle X) \tag{5}$$

where $\angle X$ [rad] is the angle between the real part of X_0 and the imaginary part from Eq. 6 below:

$$X_0 = \frac{(F_0/m)}{(\omega_n^2 - \Omega^2) + i2\zeta\omega\Omega} \tag{6}$$

where ω_n is the angular velocity and $i2\zeta\omega\Omega\ \frac{c}{m}$. Both can be found below in Eq. 12 and 14 were ζ is the damping coefficient. The magnitude of X_0 is then calculated from the complex substitution method, yielding a magnitude and a phase angle of:

$$|X_0| = \frac{(F_0/m)}{\sqrt{(\frac{k_{eq}}{m}) - \Omega^2)^2 + (2\zeta\omega\Omega)^2}} \tag{7}$$

$$\angle X = arctan(\frac{B_1}{A_1}) - arctan(\frac{B_2}{A_2}) \tag{8}$$

where A_1 and B_1 are the real and imaginary parts of the numerator, respectively, and A_2 and B_2 are the real and imaginary parts of the denominator, respectively.

It is also important to note that given the second derivative of Eq. 4 the displacement amplitude of the function is found to be.

$$x_0 = \frac{\ddot{x}}{\omega^2} \tag{9}$$

where $\ddot{x}$ $[m/s^2]$ is the second derivative and ω [rad/s] is the angular frequency.

Viscous damping is defined as damping that casuses a non linear decrease from peak to peak. The natural frequency and angular velocity of the system can be found by looking at the peaks in the time domain of the displacement or even of the acceleration:

$$f = 1/T \tag{10}$$

were T [s] is the period of the wave in second and f [rad/s] is the frequency. The frequency can then be found by using equation 11 below:

$$\omega = 2\pi f \tag{11}$$

it should also be noted that the natural frequency, ω_n, can be solved for by:

$$\omega = \sqrt{\frac{k_{eq}}{m}} \tag{12}$$

When dealing with a viciously damped system it is important to look at how the damping coefficient, C, is calculated. To calculate C a damping ratio, ζ is calculated using the log-decrement method:

$$\zeta_i = \frac{1}{2\pi} ln(\frac{x_i}{x_{i+1}}) \tag{13}$$

where x_i [m] is a time domain peak and x_{i+1} [m] is the next time domain peak. From ζ, the damping coefficient, C can be calculated by:

$$C = 2\zeta\omega\Omega \tag{14}$$

where ω is the angular frequency [rad/s] and m [kg] is the mass of the system. From ζ the damped frequency can the be found from equation 15 below:

$$\omega_d = \omega_n * \sqrt{1 - \zeta^2} \tag{15}$$

Coulomb damping is the decrease in displacement from a frictional force. This force comes from two bodies sliding past one-another while in contact [3]. To find the coulomb force, the slope given by the difference between adjacent peaks on the time curve is calculated. This slope will be linear under coulomb damping. This slope between peaks is given by Eq. 16 below:

$$M = \frac{x_i - x_{i+1}}{t_i - t_{i+1}} \tag{16}$$

where M is the slope [m/s], x [m] and t [s] are the displacement amplitude and time at those displacement amplitudes respectively. It is important to note here that x_i [m] and t_i [s] represent a peak in the time domain and x_{i+1} [m] and t_{i+1} [s] represents the next consecutive peak. It should also be mentioned here that the coulomb force can also be equated to the slope of the decrease in peaks through equation 17:

$$M = \frac{4\mu N}{k_{eq}} \tag{17}$$

where μ is the coefficient of kinetic friction, which is unit less, and N is the normal force [N]. If equations 16 and 17 are combined, the coulomb force is found to be:

$$f_C = \frac{(x_i - x_{i+1}) * k_{eq}}{4(T_i - T_{i+1})} \tag{18}$$

It is important to note here that f_c is the product of μ and the normal force (N).

METHODS

Apparatus. The equipment used to study the behavior of a mass, spring, and damper system is catalogued in Table 1. A wheeled cart, with low friction axles, was placed on a flat track. Two springs were attached to each side of the cart with spring constants as noted in Fig. 2. The springs on the right side of the cart were anchored to a stationary wall. The springs on the left side were attached to a mechanical oscillator capable of forcing the system at some frequency Ω with eccentricity e. The mechanical oscillator's motor was powered with a DC power source. Cart oscillations were damped either with a magnetic viscous damping attachment or a drop down Coulomb damper. Damping spacers were used to provide accurate offsets of the viscous damper from the track. The cart accelerations were measured with an accelerometer. The voltage signal output by the accelerometer was sent to the signal conditioner which sent its output to channel one of the Quattro. In each case, the signal was transmitted by BNC cable. The Quattro then analysed the signal in the frequency and time domains, rendering its results on the PC by the SignalaAlc application. In turn, the SignalCalc application controlled the settings of the Quattro (Fig 2).

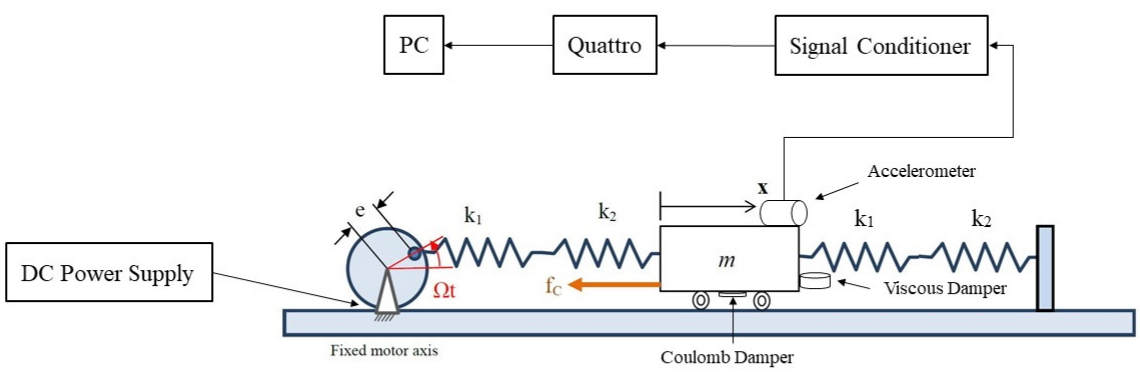

Figure 2 Singe Degree of Freedom Oscillating Cart Set-up [1].

Table 1 Equipment used to observe the oscillatory characteristics of the mass and stiffness system (Fig. 2) [4].

Equipment	Make	Model	Serial #	Calibration Constant
Laptop	Lenovo	Thinkpad	N/A	N/A
SignalCal 240 Dynamic Signal Analyzer	N/A	N/A	N/A	N/A
Quattro SignalCalc Ace	Data Physics Corporation	N/A	000010657	N/A
DC Power Supply	Pasco	PI-9880	P00085452	N/A
Mechacnical Oscillator/Driver	Pasco	ME-8750	P00090765	N/A
Accelerometer	PCB	353B33	LW203550	10.4 $\frac{mV}{s^2}$
Signal Conditioner 2	PCB	494A	816	10 mV
Cart	PASCar	ME-6950	System A	N/A
Track	PASCO	Unknown	System A	N/A
Magnetic Damping	Pasco	ME-6828	N/A	N/A
Damping Spacers	N/A	N/A	N/A	N/A

Procedure. In order to analyze the oscillations of the mass, stiffness, damper system, the spring constants of the springs had to be determined. This was accomplished by hanging each spring vertically and attaching masses to the free end of each spring. Every spring was made to support eight masses between .010 and .500 kg. The displacement response of the springs for each mass was recorded using a ruler. With these measurements, the spring constants were calculated.

The cart weighed weighed 0.286 kg, the viscous damping magnet weighed .073 kg, and accelerometer weighed .384 kg for a total system mass of .384 kg. The cart was .715 m in length. The eccentricity of the mechanical oscillator was (.01 ± .001) m.

To measure the osculations of the system, SignalCalc was set up as follows: Fspan was between 8 Hz and 20 Hz, Lines to 200, Trigger to Input, Avg. to Off, Window to *Hanning*, and Auto End to 1 Record. Channel one had a calibration constant of 100.4 mV/EU and the signal conditioner provided a gain of 10 to the accelerometer output. When modifying the system, the DC voltage source was turned off. Additionally, the DC source was never allowed to output more that 12 V.

To begin analysis of the system, the viscous damping magnets were placed .008 m above the track using the spacers as guide. The cart was then displaced from its equilibrium position by hand and then released, being careful not to displace the cart so far as to take the pre-load off any of the springs.

The cart was allowed to oscillate freely until coming to rest at its equilibrium position. SignalCalc was set to auto trigger after cart release. This process was performed a total of four times, releasing the card from a different location relative to its equilibrium position each time. These measurements were used to determine the natural frequency, ω_n, of the system.

Using the calculated ω_n, the mechanical oscillator was made to oscillate the system between the frequencies $.25\omega_n$ and $1.25\omega_n$. SignalCalc was used to analyse the state of the system at fifteen different frequencies across the given frequency range. Before each measurement the system was allowed to come to steady state.

The free oscillation measurement process was repeated four more times, but this time the viscous damping magnets were set .003 m above the track. From this data, ω_n was calculated once again and SignalCalc was used to take fifteen more measurements of the system at unique frequencies between $.25\omega_n$ and $1.25\omega_n$. Before each measurement was taken, the system was allowed to come to steady state.

Finally, the viscous damping magnets were raised approximately .016 m above the track and the coulomb damping brush was put in contact with the track. The friction from the brush was set by screwing the brush down until it just touched the track, and then turning the screw half a turn farther downward so that the brush was pre-loaded against the track. The free oscillation process was repeated four more times.

Analysis. In lab, the natural frequency of the of the system under viscous damping was calculated using Eq. 15. Determining ζ will be discussed later in this section. ω_d was given as ω from Eq. 11 and f was calculated from Eq. 10 were T was the average time between acceleration peeks generated by SignalCalc. Microsoft Excel was used to implement these equations to determine an average ω_n for each free response run with viscous damping.

The effective stiffness of the model was calculated two different ways. The first method was to generate displacement vs force graph of each spring in Excel (Fig 3). Each spring constant was given as the line of best fit of the measured values neglecting those points that were clearly measured before the springs began to displace. Excel also generated the R^2 value of the fit. The equivalent stiffness of the system was then determined by solving for the equivalent spring stiffness, K_{eq}, of the system. The equivalent spring stiffnesses of both the right side and left side of the cart (see Fig. 2) were given by Eq.2. These equivalent springs were then summed using Eq. 3 to find the total effective stiffness of the system. In method two, the effective stiffness was determined using Eq. 12. ω_n was the average of the natural frequencies calculated for each free oscillation run during lab, disregarding the outliers. See Appendix B for more detail.

The frequency vs displacement plot for the system with viscous damping was determined by analyzing the SignalCalc measurements of the cart oscillations at each recorded frequency for viscous damping when the damping magnets were .003 m above the track (Fig. 3). Each frequency point on the plot was given by the frequency at the peak amplitude in the frequency domain as measured by the Quattro's fast Fourier transform. Displacement amplitude was given by Eq. 9 where ω was the frequency peak value [rad/sec] and $\ddot{x}$ was the acceleration amplitude of the peak. The same process was repeated to generate the frequency vs displacement plot for the system with viscous dampers raised .008 ma above the track. The two curves were plotted together (Fig. 4).

Two different average eddy-current viscous damping coefficients,C_v, were calculated, one for the system with damper magnets .003 m above the track and the other for the system with damper magnets .008 m above the track. C_v was calculated by Eq. 14. ζ was determined using Eq. 13. x_i was given as the domain acceleration peak as measured by the Quattro. x_{i+1} would be the value at the next consecutive time domain acceleration peak. Excel was used to calculated multiple ζs for each free oscillation trial. These ζ values were then averaged. This average value was used to calculate C_v. The coulomb force was calculated with Eq. 18 where t_i and t_{i+1} were the times at the corresponding acceleration peaks. The friction coefficient, μ was found by dividing the friction force by the weight of the cart.

Finally, a sample run from each free response case (small viscous damping, large viscous damping, coulomb damping) was chosen. The SignalCalc time domain acceleration values were converted to displacement values through MATLAB using Eq. 9. Then, MATLAB plots were generated of each sample against the mathematical model developed in Appendix B (Figs 5 - 7). All MATLAB code is in Appendix D

Sources of Error. The cart used in this experiment appeared to have bent axle, causing it to roll on three wheels instead of four. The moments this placed on other wheels may have caused more friction than the mathematical model accounted for. All displacement measurements were accomplished by estimating the displacement relative to a ruler, which is not the most accurate way to measure. Additionally, when the cart reached its natural frequency, the oscillations were large enough to allow half of the springs to relax all the way, reducing the force on these springs beneath their pre-load. This would cause the spring constants to be non-linear at points, violating the linear stiffness assumption of the mathematical model.

RESULTS & DISCUSSION

First, each of the four linear spring constant responses are determined based on added weights and the corresponding displacement. The slope of the curve relating applied force as a function of displacement corresponds to the spring constant (k), in Newtons per millimeter, and the y-intercept relates to the pre-load that is necessary for the spring to begin it's displacement. These curves are displayed below in Fig. 3.

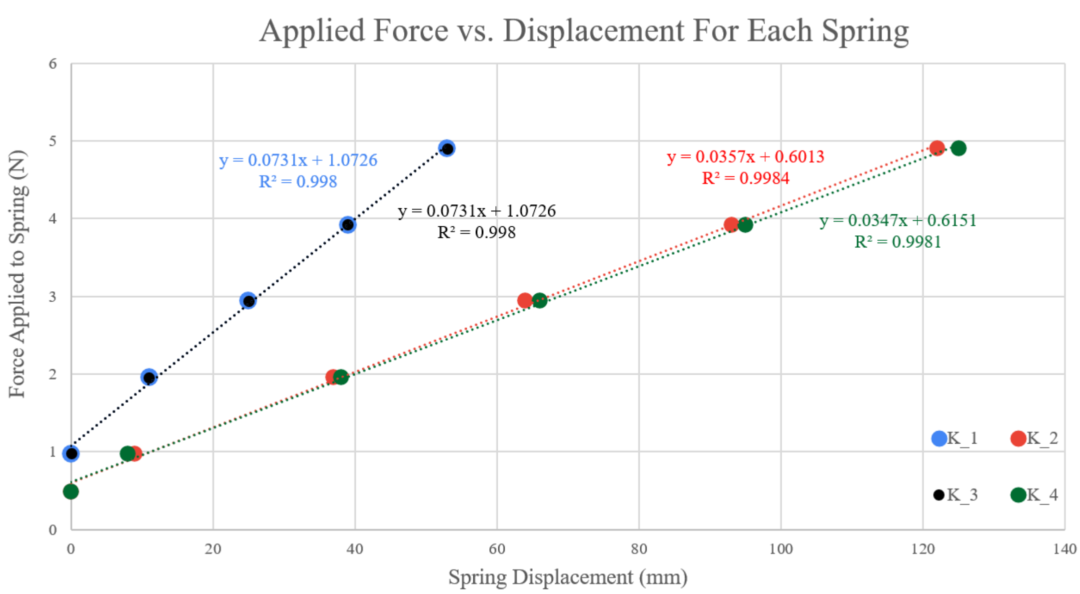

Figure 3 Linear spring displacement based on added weight to one end of four different springs.

Each of the best fit curves have their related R^2 values displayed, with each value being greater than 0.99, meaning the curves fit each series of points with great confidence. The spring constants for the first and third spring are approximately 73.1 N/m, while springs two and four have constants 35.7 N/m and 34.7 N/m respectively.

With the system parameters understood, various responses were measured. The data from these tests is summarized below.

The system vibration amplitude was measured under forced excitation, with damping magnets at two heights above the track, 8 mm and 3 mm. The displacement amplitudes, as a function of frequency are plotted below in Fig. 4.

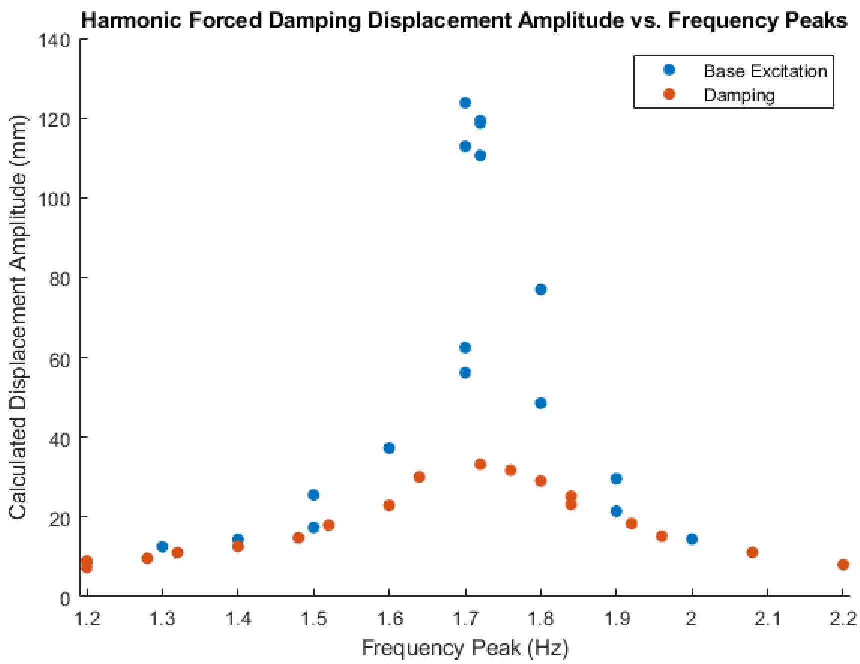

Figure 4 Displacement amplitude as a function of frequency for steady state forced harmonic oscillation with two damping conditions.

The amplitude for the higher magnet height peaks at nearly 1.7 Hz, which corresponds to the system's natural resonant frequency. At this frequency, the vibration amplitude is much greater than at surrounding frequencies. However, when eddy-current viscous damping is more significant at the lower magnet height, the amplitude peak at the natural frequency is significantly reduced. The local maximum of the amplitude still occurs at 1.7 Hz, but the variation between the resonant amplitude and those of surrounding frequencies is much lower.

Next, the system is measured under coulomb friction damping. The important damping ratios, coefficients, and forces from each damping case are displayed below in Table 2.

Table 2 The calculated damping ratio and damping coefficients for the the base case and the viscous damping case, as well as the coulomb friction coefficient and coulomb friction force.

Base Damping Ratio (unitless)	**Viscous Damping Ratio (unitless)**	**Coulomb Friction Coefficient (unitless)**
0.0275	0.5593	3.416
Base Damping Coefficient ($\frac{Ns}{m}$)	**Viscous Damping Coefficient ($\frac{Ns}{m}$)**	**Coulomb Friction Force (N)**
0.2307	1.471	12.87

The viscous damping coefficient is much higher than the base damping coefficient, which is consistent with expectations for lowering the height of the magnet.

Next, the time-series plots for various damping conditions are overlaid with a theoretical free response plot for the system, derived in the Appendix. In Fig. 5, the free response of the system with base damping at the high magnet height is compared to the theoretical free response.

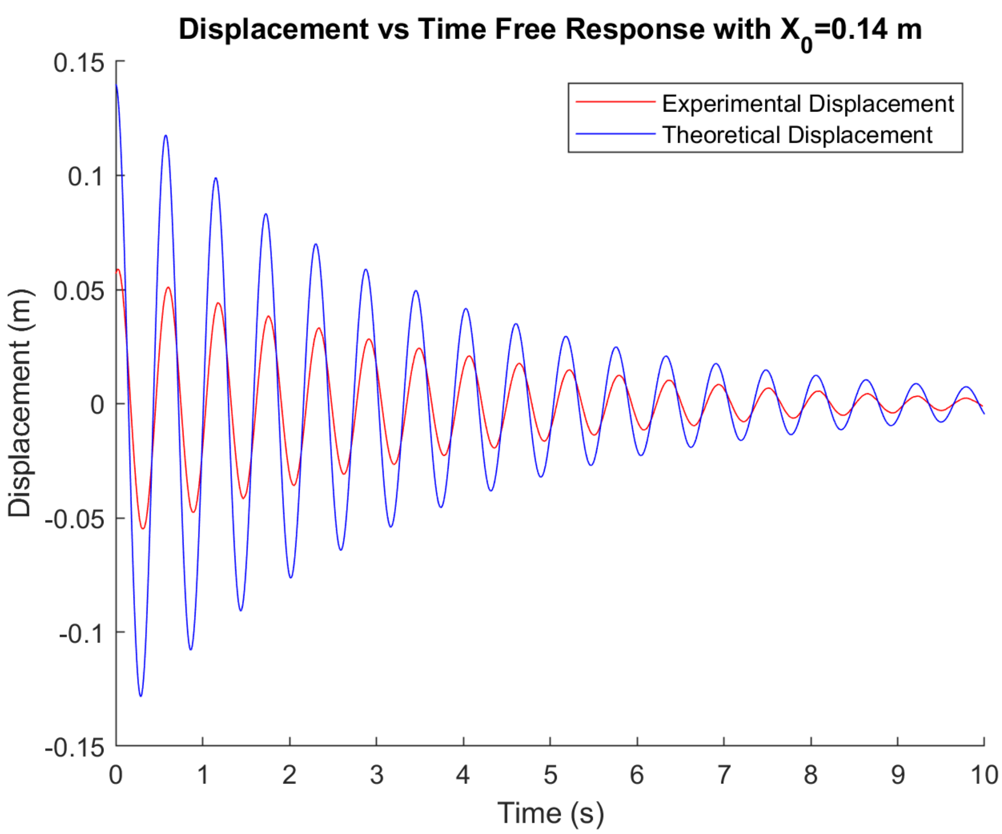

Figure 5 The theoretical base free response is shown in blue, and the actual base free response is shown in red.

The response is notably lower in amplitude than the theoretical response. This is likely due to natural friction in the cart wheels' rotation, as well as the minor interactions between the magnets and track at the higher height. The initial disparity is likely due to a delay in the trigger during measurement. The period is also consistent between the theoretical and experimental curves, implying an accurate and well resolved time series plot. The increased viscous damping response is then compared to the theoretical response in Fig. 6 below.

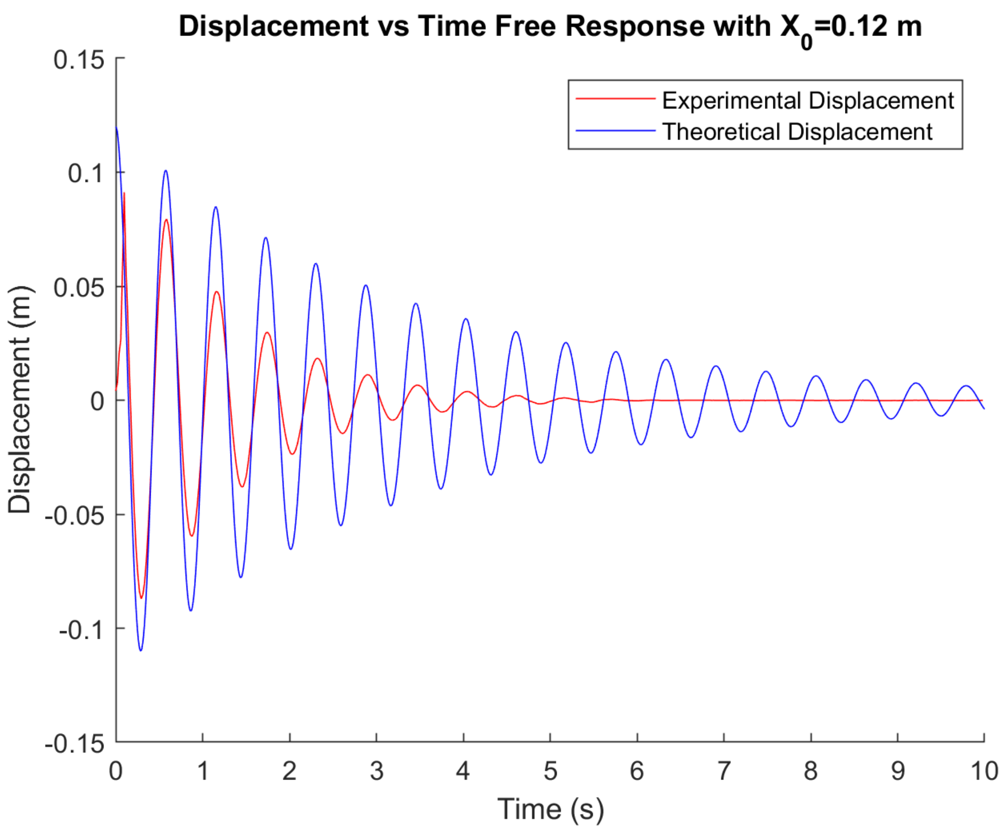

Figure 6 The theoretical base free response is shown in blue, and the actual free response with increased viscous damping is shown in red.

This response starts with high amplitude and very quickly decays to minimal displacement. This is consistent with theoretical expectations of logarithmic decay as the initial displacement is consistent with the theoretical free response, but the decay rate is significantly higher and then slows down to an asymptote at zero amplitude. Finally, the response with coulomb damping is compared to the theoretical expectation in Fig. 7.

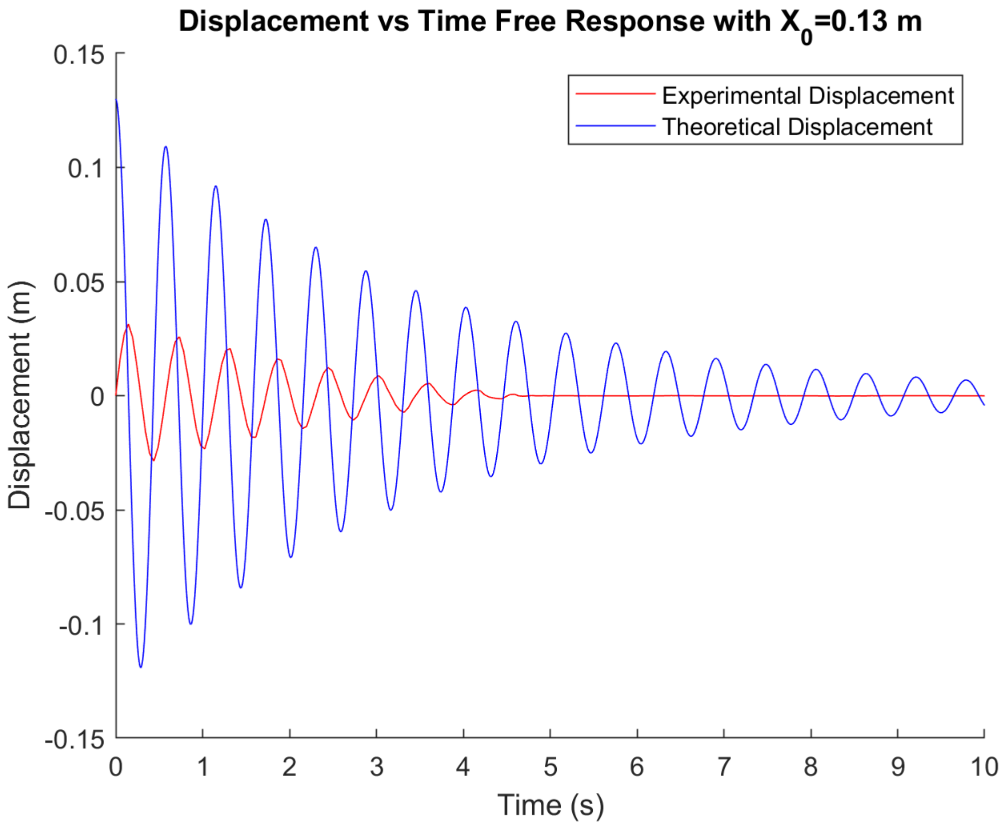

Figure 7 The theoretical base free response is shown in blue, and the actual free response with coulomb damping is shown in red.

This curve has significantly lower amplitude than the theoretical curve from the beginning, which is not consistent with expectations. This error is likely due to a delayed trigger when recording the data. However, this error did not impact the shape of the curve, which exhibits linear decay as expected. Additionally, it likely did not impact the calculation of the damping coefficient as the decay rate should be constant over time. The plot could be corrected by shifting the experimental displacement plot in the positive x direction by the amount of the delay in the trigger, which is unknown.

CONCLUSION

This experiment provided participants valuable experience measuring and processing transient and forced vibrations as digital data. In addition, the theoretical expectations for the system behavior were generally consistent with the experimental findings for each kind of damping. This means that the analytical model for a single degree of freedom vibration system is accurate and acceptable for use by NYQuist Consulting Labs in the design of the pedestrian overpass across Forest Park Parkway.

The effects of damping on the model were found to be consistent with theoretical expectations. Eddy-current viscous damping caused logarithmic decay in the amplitude of vibration, while coulomb damping caused linear decay. The damping coefficient of the viscous damping at the high magnet height of 8 mm was found to be 0.2307 Ns/m, while at the low magnet height of 3 mm the viscous damping coefficient was 1.471 Ns/m. Finally, the coulomb friction force was calculated to be 12.87 N, causing the linear decay.

Error induced by ambient conditions such as friction as the cart wheels rotate and air resistance likely limited the accuracy of the measurements in this experiment. They would cause additional damping, which would skew the calculated damping coefficients and friction forces to be too large. The experimental procedure could be improved by applying a friction minimizing surface treatment to the cart wheel axles/bearings. This change would allow for more accurate measurement of damping coefficients isolated from ambient damping.

References

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A Glossary

- **Tension spring preload**:

  The preload is the tension that must be overcome in order to stretch a spring. Once this tension is overcome, the spring amount of force needed to further stretch the spring will increase proportionally to the distance it has been stretched. [5]

- **Rotating unbalance**:

  A rotating unbalance is when something is rotating such that the moment it exerts on a spring varies with time. It could be a wheel that is not homogeneous, a rotating pendulum, and so on.[6]

- **Resonant frequency**:

  The resonant frequency is the frequency at which it is easiest to vibrate an object or system, as determined by the physical parameters of the system. [7] Vibrating it at this frequency will cause the maximum amplitude of oscillation. For a one degree of freedom system, there will be only one resonant frequency, but more generally there will be as many resonant frequencies as there are degrees of freedom.

- **Logarithmic decrement**:

  The logarithmic decrement represents the rate at which the acceleration and position amplitudes of a viscously damped system decreases. Because it is known that the amplitude will decay logarithmically, a damping ratio (and thus damping coefficient, natural frequency, and more) can be calculated by analysing the time domain graph of a system's acceleration or position. [8]

- **Coulomb damping**:

  Coulomb damping occurs when an oscillating object experiences a sliding friction force in the direction opposite its motion. This sliding friction force is known as coulomb friction, and it will create a damped oscillation that decreases the position amplitude linearly, rather than logarithmically as with viscous damping.[9]

B Sample Calculations

The work for finding the equivalent spring stiffness of the system used in this experiment is shown below, and was done via two different methods: simplifying a network of springs using parallel and series relations, and solving for the spring stiffness using the natural frequency and mass of the system. The first method found the equivalent stiffness to be 46.002 $\frac{N}{m}$, while the second method found it to be 45.769 $\frac{N}{m}$. There was a percent error of 0.506% between these two methods, which means they agree with each other very well. The damping constant was found to be 0.2306 $\frac{Ns}{m}$, as shown below.

The equivalent spring stiffness of the system (k_{eq}) was found from a spring stiffness network as shown below.

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2} + \frac{k_3 k_4}{k_3 + k_4}$$

$$k_1 = 70.071 \left[\frac{N}{m}\right] \; k_2 = 34.780 \left[\frac{N}{m}\right] \; k_3 = 70.071 \left[\frac{N}{m}\right] \; k_4 = 33.707 \left[\frac{N}{m}\right]$$

$$k_{eq} = \frac{70.071 * 34.780}{70.071 + 34.780} + \frac{70.071 * 33.707}{70.071 + 33.707} = 23.243 + 22.759 = 46.002 \left[\frac{N}{m}\right]$$

This k_{eq} was also found using the natural frequency and mass of the system as shown below.

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} \longrightarrow k_{eq} = \omega_n^2 m$$

$$\omega_n = 10.91747 \left[\frac{rad}{s}\right] \; m = 385g = 0.384kg \; \zeta = 0.02751$$

$$k_{eq} = 10.91747^2 * 0.384 = 45.769 \left[\frac{N}{m}\right] = k_{eq}$$

The percent difference between them was found to be 0.506%, as shown below

$$\%\ difference = \frac{46.002 - 45.769}{46.002} * 100 = 0.506\%\ difference$$

The system damping constant was found as follows.

$$c = 2\zeta\omega_n m = 2 * 0.02751 * 10.91747 * 0.384 = 0.23066 \left[\frac{Ns}{m}\right] = c$$

Shown below are the results of finding a function for the position of the cart as a function of time and driving frequency, for the forced response and the free response. These equations were combined to generate a function for the full response of the system. Note that this function simplifies down to the homogeneous response in the case where there is no force, and to the forced response in the case where there is no initial condition.

The forcing function was set up as shown below.

$$\delta = e\cos\omega t \;\; \longrightarrow \;\; F = k_{12}\delta \longrightarrow F = k_{12}e\cos\omega t$$

The equation of motion of the one degree of freedom system is as follows.

$$m\ddot{x} + c\dot{x} + k_{eq}x = F \longrightarrow \;\; \ddot{x} + \frac{c}{m}\dot{x} + \frac{k_{eq}}{m}x = \frac{F}{m} = \frac{k_{12}e}{m}\cos\omega t$$

$$x = Xe^{i\omega t} \;\; \dot{x} = i\omega Xe^{i\omega t} \;\; \ddot{x} = \;\; -\omega^2 Xe^{i\omega t} \;\; \overline{F} = \frac{k_{12}e}{m}e^{i\omega t}$$

The complex substitute method was used to find the particular response of the system under the forcing conditions described above.

$$X\left(-\omega^2 + \frac{c\omega}{m}i + \frac{k_{eq}}{m}\right) = \frac{k_{12}e}{m} \longrightarrow X = \frac{\frac{k_{12}e}{m}}{\left(\frac{k_{eq}}{m} - \omega^2 + \frac{c\omega}{m}i\right)}$$

$$|X| = \frac{\frac{k_{12}e}{m}}{\sqrt{\left(\frac{k_{eq}}{m} - \omega^2\right)^2 + \left(\frac{c\omega}{m}\right)^2}}$$

$$\angle \mathrm{X}{=}0{-}\mathrm{arctan}\left(\frac{\frac{c\omega}{m}}{\frac{k_{eq}}{m} - \omega^2}\right)$$

$$x\,(t,\omega) = \frac{0.06052}{\sqrt{\left(119.7968 - \omega^2\right)^2 + 0.3608\omega^2}}\cos\left(\omega t - arctan\left(\frac{0.6007\omega}{119.19 - \omega^2}\right)\right)$$

The unit response method was used to find the homogeneous response of the system.

$$x\,(t) = x_D\,(t)\,x_0 + x_V\,(t)\,v_0$$

$$x_D\,(t) = e^{-\zeta\omega_n t}\left[\cos\,(w_d t) \;\; + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\,(\omega_d t)\right]$$

$$x_V\,(t) = \frac{e^{-\zeta\omega_n t}\sin\,(\omega_d t)}{\omega_d}$$

$$x\,(t) = \;e^{-\zeta\omega_n t}\left[\cos\,(w_d t) \;\; + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\,(\omega_d t)\right]x_0 + \frac{e^{-\zeta\omega_n t}\sin\,(\omega_d t)}{\omega_d}v_0$$

$$x(t) = e^{-.3003t}\left[\cos(10.9133t) \ + 0.02752\sin(10.9133t)\ \right]x_0 + \frac{e^{-0.3003t}\sin(10.9133t)}{10.9133}v_0$$

The full response of the system was found as follows.

$$x(t) = x_p(t) + \left(x_0 - x_p(0)\right)x_D(t) + \left(v_0 - \dot{x}_P(0)\right)x_V(t)$$

$$x(t) = \frac{0.06052}{\sqrt{\left(119.7968 - \omega^2\right)^2 + 0.3608\omega^2}}\cos\left(\omega t - arctan\left(\frac{0.6007\omega}{119.19 - \omega^2}\right)\right)$$

$$+ \left(x_0 - x_P(0)\right)e^{-.3003t}\left[\cos(10.9133t) \ + 0.02752\sin(10.9133t)\ \right]$$

$$+ \frac{e^{-0.3003t}\sin(10.9133t)}{10.9133}\left(v_0 - \ \dot{x}_P(0)\right)$$

Figure B.1 shows a summary of the system parameters found for this theoretical analysis, and a labeled diagram of the system.

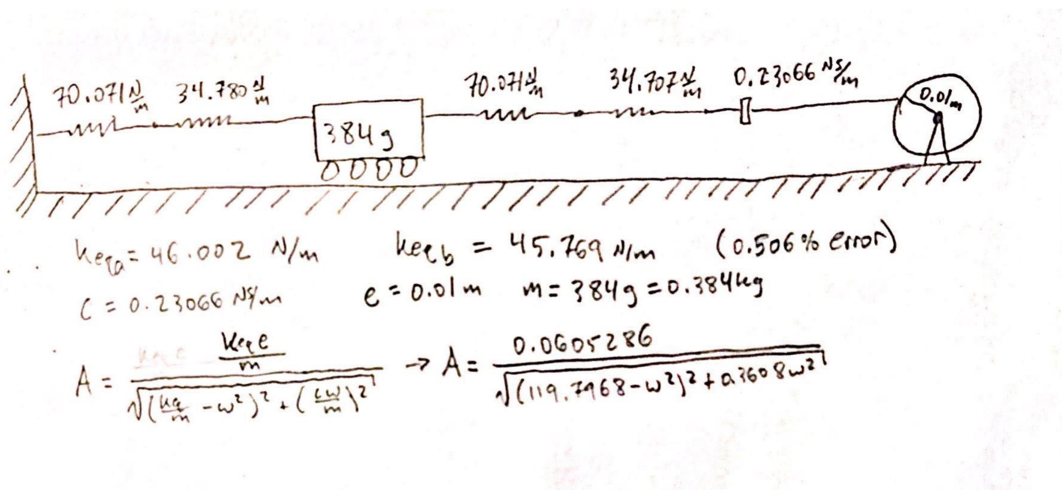

Figure B.1 Summary of parameters.

C Raw Data

Shown below in Table C.1 is the data used to determine the spring constants.

Table C.1 Spring Constant Calculation Data

Spring Number	Masses (g)	Initial Length (cm)	Displaced Length (cm)
K_1	10	8	8
	20	8	8
	50	8	8
	100	8	8
	200	8	9.1
	300	8	10.5
	400	8	11.9
	500	8	13.3
K_2	10	7.9	7.9
	20	7.9	7.9
	50	7.9	7.9
	100	7.9	8.8
	200	7.9	11.6
	300	7.9	14.3
	400	7.9	17.2
	500	7.9	20.1
K_3	10	8.1	8.1
	20	8.1	8.1
	50	8.1	8.1
	100	8.1	8.1
	200	8.1	9.2
	300	8.1	10.6
	400	8.1	12
	500	8.1	13.4
K_4	10	8	8
	20	8	8
	50	8	8
	100	8	8.8
	200	8	11.8
	300	8	14.6
	400	8	17.5
	500	8	20.5

Table C.2 below are the time and amplitude peaks of the acceleration for each run for free vibration with viscous damping from the magnets being set to 8 mm above the track.

Table C.2 Free response with slight viscous damping from magnets set to 8 mm above the track.

Run Number	Displacement (cm)	Time (s)	Acceleration Peaks ($\frac{m}{s^2}$)
1	8.7	1.05	2.138
		1.62	1.794
		2.21	1.513
		2.79	1.240
		3.36	1.030
		3.93	0.8309
2	13.7	1.17	4.72
		1.76	4.109
		2.34	3.541
		2.91	3.047
		3.50	2.601
		4.06	2.201
3	8.7	0.703	2.232
		1.27	1.868
		1.86	1.579
		2.44	1.310
		3.01	1.070
		3.57	0.9077
4	5.7	0.703	0.703
		1.29	1.582
		1.86	1.329
		2.44	1.095
		3.01	0.934
		3.59	0.7431
23	8.7	0.703	2.617
		1.27	2.234

Table C.2 Continued.

Run Number	Displacement (cm)	Time (s)	Acceleration Peaks ($\frac{m}{s^2}$)
23	8.7	1.86	1.900
		2.42	1.593
		3.01	1.326
		3.57	1.082
24	13.7	1.17	5.047
		1.76	4.399
		2.34	3.796
		2.91	3.245
		3.50	2.779
		4.06	02.391
25	8.7	1.02	3.086
		1.60	2.649
		2.17	2.243
		2.75	1.909
		3.32	1.594
		3.91	1.340
26	5.7	1.04	2.260
		1.60	1.943
		2.19	1.621
		2.75	1.362
		3.32	1.126
		3.91	0.9366

Table C.3 shows data that was used to determine the frequency response of the base system. The voltage supplied to the motor is recorded, as well as the acceleration amplitude and oscillation frequency. The first trial had faulty data, so no acceleration amplitude or frequency are reported.

Table C.3 Forced response data for base system

Run Number	Motor Voltage (V)	Frequency (Hz)	Acceleration Amplitude ($\frac{m}{s^2}$)
5	6.00	N/A	N/A
6	6.00	1.2	0.417
7	6.30	1.2	0.516
8	6.60	1.3	0.651
9	6.90	1.3	0.747
10	7.20	1.4	1.115
11	7.50	1.5	1.511
12	7.80	1.5	1.979
13	8.10	1.6	3.759
14	8.40	1.7	6.761
15	8.70	1.7	13.33
16	8.80	1.8	8.954
17	9.00	1.8	6.272
18	9.20	1.9	3.802
19	9.50	1.9	2.895
20	8.55	1.7	12.780
21	8.30	1.7	5.876
22	9.90	2.0	2.215
56	8.70	1.72	12.33
57	8.75	1.72	13.53
58	8.85	1.72	13.44

Table C.6 below are the time and amplitude peaks of the acceleration for each run for free vibration with viscous damping from the magnets being set to 3 mm above the track.

Table C.4 Free response with viscous damping from magnets set to 3 mm above the track.

Run Number	Displacement (cm)	Time (s)	Acceleration Peaks ($\frac{m}{s^2}$)
27	9.5	0.645	0.2996
		1.19	0.1433
		1.72	0.0879
		2.21	0.03116
		2.58	0.009204
28	13.5	0.50	0.5105
		1.07	0.280

Table C.6 Continued.

Run Number	Displacement (cm)	Time (s)	Acceleration Peaks ($\frac{m}{s^2}$)
28	13.5	1.62	0.213
	13.5	2.17	0.05361
		2.68	0.005959
29	11.5	0.586	9.069
		1.15	5.440
		1.74	3.408
		2.32	2.106
		2.89	1.276
		3.48	0.7532
30	12.5	0.0977	4.901
		0.664	0.2844
		1.19	0.1281
		1.82	0.05399
		2.36	0.003968
31	10.5	0.0781	0.6893
		0.664	0.4015
		1.23	0.2379
		1.78	0.1021
		2.27	0.03537

Table C.5 below shows the data that was used to determine the frequency response of the damped system, with the magnets set to 3mm above the track. The voltage supplied to the motor is recorded, as well as the acceleration amplitude and frequency of oscillation.

Table C.5 Forced response data for viscous damping from magnets set 3 mm above the track.

Run Number	Motor Voltage (V)	Frequency (Hz)	Acceleration Amplitude ($\frac{m}{s^2}$)
32	6.00	1.20	0.3807
33	6.30	1.20	0.507
34	6.30	1.20	0.490
35	6.30	1.20	0.486
36	6.60	1.28	0.6321
37	6.90	1.32	0.7381
38	7.20	1.40	0.9915
39	7.50	1.48	1.155
40	7.80	1.52	1.658
41	8.10	1.60	2.176
42	8.40	1.64	2.990
43	8.70	1.72	3.891
44	9.00	1.80	3.637
45	9.15	1.84	3.220
46	9.30	1.84	2.894
47	9.50	1.92	2.598
48	9.80	1.96	2.252
49	10.10	2.08	1.726
50	10.70	2.20	1.573
51	8.85	1.76	3.821

Table C.6 below shows the free response data from the Coulomb damping case. The run number and cart initial displacement are recorded, as well as the acceleration peaks and the time at which each peak occurs.

Table C.6 Coulomb Damping with magnets at 16 mm above the track and the friction pad set with a half turn pre-load.

Run Number	Displacement (cm)	Time (s)	Acceleration Peaks ($\frac{m}{s^2}$)
52	9.5	0.146	4.345
		0.732	3.355
		1.27	2.607
		1.86	1.865
		2.44	1.183
		2.98	0.5264

Table C.6 Continued.

Run Number	Displacement (cm)	Time (s)	Acceleration Peaks ($\frac{m}{s^2}$)
		3.42	0.1317
53	11.5	0.488	3.556
		1.07	2.815
		1.61	2.126
53	11.5	2.2	1.599
		2.78	1.078
	11.5	3.37	0.573
		3.86	0.1701
		0.146	3.838
		0.732	3.164
54	12.5	1.32	2.252
		1.86	1.972
		2.44	1.515
		3.03	1.073
		3.61	0.06476
		0.439	1.387
		1.03	0.9467
55	8.5	1.61	0.5132
		2.10	0.1273

D MATLAB Code

This MATLAB code was used to plot the frequency response of the system with base conditions and with coulomb damping.

```
%%
clear all; close all; clc;
%% Graphing run function for either file path

%First File path (STRICTLY for local file path of stored files)
% mainFilepath = 'D:\WashU\Classes\Fall 2020\Vibrations LAB\Lab 3\Run 1 Files';
%Second File path (STRICTLY for local file path of stored files)
mainFilepath = 'D:\WashU\Classes\Fall 2020\Vibrations LAB\Lab 3\Run 2 Files';

% Part 2 free
%     Run 1-4,23-26
% Part 3 forced
%     Run 6-22,56-58
% Part 4 free
%     Run 27-31
% Part 4 forced
%     Run 32-51

% Input for the run number that is inteded to be graphed
startRun = 6;
finalRun = 22;

frequencies = zeros(1,finalRun-startRun+1+(58-56+1));
mean_acceleration = zeros(1,finalRun-startRun+1+(58-56+1));

for currentRun = (startRun:finalRun)
    localPath = strcat(mainFilepath,'/Run');
    if currentRun < 10
        localPath = strcat(localPath,'0000');
    end
    if currentRun ≥ 10 && currentRun < 100
        localPath = strcat(localPath,'000');
```

```
    end
    if currentRun ≥ 100 && currentRun < 1000
        localPath = strcat(localPath,'00');
    end
    localPath = strcat(localPath,int2str(currentRun));
    localPath = strcat(localPath,'/MATLAB/DPsv00000.mat');
    load(localPath);
    [Apk_Splot,fpk_Splot] = findpeaks(abs(S1(:,2)));
    [acceleration_Splot, frequency_Splot]=findPeaksGreaterThan(Apk_Splot,S1, ...
                                                        fpk_Splot,0.1);
    [Accel_Xplot,time_Xplot] = findpeaks(X1(:,2));
    [acceleration_Xplot, frequency_Xplot]= findPeaksGreaterThan(Accel_Xplot,X1,...
                                                        time_Xplot,0.1);
    % Storing the frequencies and acceleration peaks
    for i = (currentRun-startRun+1)
        frequencies(:,i) = frequency_Splot;
    end
    for i = (currentRun-startRun+1)
        mean_acceleration(:,i) = mean(acceleration_Xplot);
    end
end
%% Second Calcs for displacement and Freq.
startRun2 = 56;
finalRun2 = 58;
for currentRun2 = startRun2:finalRun2
    localPath = strcat(mainFilepath,'/Run');
    if currentRun2 < 10
        localPath = strcat(localPath,'0000');
    end
    if currentRun2 ≥ 10 && currentRun2 < 100
        localPath = strcat(localPath,'000');
    end
    if currentRun2 ≥ 100 && currentRun2 < 1000
        localPath = strcat(localPath,'00');
    end
    localPath = strcat(localPath,int2str(currentRun2));
    localPath = strcat(localPath,'/MATLAB/DPsv00000.mat');
    load(localPath);
```

```
    [Apk_Splot,fpk_Splot] = findpeaks(abs(S1(:,2)));
    [acceleration_Splot, frequency_Splot]=findPeaksGreaterThan(Apk_Splot,S1, ...
                                                              fpk_Splot,0.1);
    [Accel_Xplot,time_Xplot] = findpeaks(X1(:,2));
    [acceleration_Xplot, frequency_Xplot]= findPeaksGreaterThan(Accel_Xplot,X1,...
                                                              time_Xplot,0.1);
    % Storing the frequencies and the acceleration peaks
    for i = ((currentRun2)-startRun2+1)
        frequencies(:,i) = frequency_Splot;
    end
    for i = (currentRun2-startRun2+1)
        mean_acceleration(:,i) = mean(acceleration_Xplot);
    end
end

%% Plotting 1st forced damping

% displacement amplitude calculation @ sin(x)=1
displacement_amplitude = (mean_acceleration./((frequencies.*2*pi).^2))*1000; %mm

figure(1)
scatter(frequencies,displacement_amplitude,'filled');
hold on

%% Second Calcs for displacement and Freq.
startRun3 = 32;
finalRun3 = 51;
frequencies3 = zeros(1,finalRun3-startRun3+1);
mean_acceleration3 = zeros(1,finalRun3-startRun3+1);
for currentRun3 = startRun3:finalRun3
    localPath = strcat(mainFilepath,'/Run');
    if currentRun3 < 10
        localPath = strcat(localPath,'0000');
    end
    if currentRun3 ≥ 10 && currentRun3 < 100
        localPath = strcat(localPath,'000');
    end
    if currentRun3 ≥ 100 && currentRun3 < 1000
```

```
        localPath = strcat(localPath,'00');
    end
    localPath = strcat(localPath,int2str(currentRun3));
    localPath = strcat(localPath,'/MATLAB/DPsv00000.mat');
    load(localPath);
    [Apk_Splot,fpk_Splot] = findpeaks(abs(S1(:,2)));
    [acceleration_Splot, frequency_Splot]=findPeaksGreaterThan(Apk_Splot,S1, ...
                                                                    fpk_Splot,0.1);
    [Accel_Xplot,time_Xplot] = findpeaks(X1(:,2));
    [acceleration_Xplot, frequency_Xplot]= findPeaksGreaterThan(Accel_Xplot,X1,...
                                                                    time_Xplot,0.1);
    % Storing the frequencies and the acceleration peaks
    for i = (currentRun3-startRun3+1)
        frequencies3(:,i) = frequency_Splot;
    end
    for i = (currentRun3-startRun3+1)
        mean_acceleration3(:,i) = mean(acceleration_Xplot);
    end
end

%% Plotting 2nd forced damping

% displacement amplitude calculation @ sin(x)=1
displacement_amplitude2 = ...
    (mean_acceleration3./((frequencies3.*2*pi).^2))*1000; %mm

figure(1)
scatter(frequencies3,displacement_amplitude2,'filled');
xlim([1.19 2.21])
xlabel('Frequency Peak (Hz)')
ylabel('Calculated Displacement Amplitude (mm)')
title_1 = {'Harmonic Forced Damping Displacement Amplitude vs. Frequency Peaks'};
title(title_1);
legend('Base Excitation','Damping');

%% Finding Peak Function
function [peaks, locations]= findPeaksGreaterThan(pks,arr,locs,val)
    currentindex = 1;
```

```
    arrSizea = size(pks);
    arrSize = arrSizea(1);
    peaks = zeros(arrSize,1);
    locations = zeros(arrSize,1);
    for i=1:arrSize
        if(pks(i) ≥ val)
            peaks(currentindex) = pks(i);
            locations(currentindex) = arr(locs(i),1);
            currentindex = currentindex + 1;
        end
    end
    peaks = peaks(peaks ≠ 0);
    locations =  locations(locations ≠ 0);
end
```

This MATLAB code was used to generate the time domain graphs for the free response of the system under base conditions, coulomb damping, and viscous damping.

```
clear all;
close all;
clc;

%Setting up x(t) theoretical base model
x_0 = 0.083;
v_0 = 0;

t = 0:0.01:10;
x_D = exp(-0.3003.*t).*(cos(10.9133.*t) + 0.02752.*sin(10.9133.*t))*x_0;
x_V = exp(-0.3003.*t).*(sin(10.9133.*t)/10.9113)*v_0;
x = x_D + x_V;


currentRun = 29;

%file accessing code from earlier
mainFilepath = 'C:\Users\mitry\Google Drive\Lab 3 Run 2 Files';
cd(mainFilepath)
```

```
localPath = strcat(mainFilepath,'/Run');
if currentRun < 10
    localPath = strcat(localPath,'0000');
end
if currentRun ≥ 10 && currentRun < 100
    localPath = strcat(localPath,'000');
end
if currentRun ≥ 100 && currentRun < 1000
    localPath = strcat(localPath,'00');
end
localPath = strcat(localPath,int2str(currentRun));
localPath = strcat(localPath,'/MATLAB/DPsv00000.mat');
load(localPath);

figure(2)
plot(abs(S1(:,1)),abs(S1(:,2)));


%Finding the frequency peak from the FFT
[Apk,fpk] = findpeaks(abs(S1(:,2)));
[Acceleration, frequency]=findPeaksGreaterThan(Apk,S1,fpk,0.03);
w_d = 2*pi()*(frequency(1));

%Actually making the graph
figure(1)
hold on;
plot(X1(:,1),(X1(:,2)/(w_d^2)),'r');
plot(t,x,'b');
xlabel('Time (s)');
ylabel('Displacement (m)');
title(strcat('Displacement vs Time Free Response with X_0=',num2str(x_0),' m'));
xlim([0 10])
legend('Experimental Displacement','Theoretical Displacement');

%find peaks
function [peaks, locations]= findPeaksGreaterThan(pks,arr,locs,val)
    currentindex = 1;
    arrSizea = size(pks);
```

```
    arrSize = arrSizea(1);
    peaks = zeros(arrSize,1);
    locations = zeros(arrSize,1);
    for i=1:arrSize
        if(pks(i) ≥ val)
            peaks(currentindex) = pks(i);
            locations(currentindex) = arr(locs(i),1);
            currentindex = currentindex + 1;
        end
    end
    peaks = peaks(peaks ≠ 0);
    locations =  locations(locations ≠ 0);
end
```