Assignment 1 4/14

1. attached

2. $X = [\,5.4,\ -1.2 \quad 0.0 \quad 3.2\,]^T \qquad y = [-1 \quad 2 \quad 3.0 \quad 0.2\,]^T$

a. 1-norm: $\|x\|_1 = \sum_{i=1}^{4} |x_i| = |5.4| + |-1.2| + |0.0| + |3.2|$

$$\|X\|_1 = 9.8$$

2-norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{4} x_i^2} = \sqrt{5.4^2 + (1.2)^2 + 0.0^2 + 3.2^2}$

$$\|X\|_2 = \sqrt{40.8} = 6.4$$

3-norm: $\|x\|_3 = \left(\sum_{i=1}^{4} |x_i|^3\right)^{1/3} = \left(|5.4|^3 + |-1.2|^3 + |0.0|^3 + |3.2|^3\right)$

$$\|X\|_3 = \sqrt[3]{191.96} = 5.8$$

inf-norm: $\|X\|_\infty = \max_{1 \le i \le 4} |x_i| = 5.4$

Largest ⟷ Smallest			
1-norm	2-norm	3-norm	inf.-norm
(9.8)	(6.4)	(5.8)	(5.4)

b. $d(x,y) = \|x-y\|$

1-norm: $\|x-y\|_1 = \|[\,5.4 \ -1.2 \ 0.0 \ 3.2\,]^T - [-1 \ 2 \ 3.0 \ 0.2\,]^T\|_1$

$= \|[6.4 \ -3.2 \ -3.0 \ 3.0]^T\|_1$

$= |6.4| + |-3.2| + |-3.0| + |3.0| = \boxed{15.6}$

2-norm: $\|x-y\|_2 = \|[6.4 \ -3.2 \ -3.0 \ 3.0]^T\|_2$

$= \sqrt{6.4^2 + (-3.2)^2 + (-3.0)^2 + 3.0^2} = \sqrt{64.2} = \boxed{8.3}$

inf-norm: $\|x-y\|_\infty = \|[6.4 \ -3.2 \ -3.0 \ 3.0]\|_\infty =$

$= \max_{1 \le i \le 4} |[6.4 \ -3.2 \ -3.0 \ 3.0]^T| = \boxed{6.4}$

3. $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$

a. $A - \lambda I = 0 \longrightarrow \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} = 0$

$(5-\lambda)(5-\lambda) - 9 = 0$

$\lambda^2 - 10\lambda + 25 - 9 = 0$

$\lambda^2 - 10\lambda + 16 = 0$

$(\lambda - 2)(\lambda - 8) = 0$

$\boxed{\lambda = 2, \quad \lambda = 8 \quad \leftarrow \text{eigen values}}$

Let $\lambda = 2$: $AV_1 = \lambda V_1$

$[A - \lambda I] V_1 = 0$

$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} V_1 = 0 \longrightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$3v_1 + 3v_2 = 0$ $\quad v_1 = -v_2$

$3v_1 + 3v_2 = 0$

$\boxed{V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Let $\lambda = 8$ $\begin{bmatrix} 5-8 & 3 \\ 3 & 5-8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$v_1 = v_2$ $\quad \boxed{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

eigen vectors $\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$A = Q \Lambda Q^{-1}$

3b $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

note: Q is Defined as Matrix of eigen vectors (V_1, V_2)

also $Q^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(-1)(1) - (1)(1)} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$= -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

3c. Code is ~~attached separately~~ attached to Solve - Matlab.

eigen values $= -1, 3, -1$

eigen vector $= \begin{bmatrix} -0.512 & -0.408 & -0.02 \\ 0.384 & 0.408 & -0.417 \\ 0.768 & 0.816 & 0.908 \end{bmatrix}$

3d. A is Positive Definite ~~b/c~~ because it's eigen values are positive.

B is indefinite because it has negative and positive eigen values.

4. $f(x) = a^T x + b$

Def. of Convexity: $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$

$$f(tx + (1-t)y) = a^T(tx + (1-t)y) + b$$

$$\leq ta^Tx + (1-t)a^Ty + b$$

$$= ta^Tx + (1-t)a^Ty + tb + (1-t)b$$

$$= tf(x) + (1-t)f(y)$$

So $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ ✓

5. a. $f(x,y) = x^2 + y^2 - xy$

$$\nabla f(x,y) = \begin{bmatrix} 2x - y \\ 2y - x \end{bmatrix} \qquad Hf(x,y) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

finding definitness: $\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = 0$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 3$$

So $Hf(x,y) > 0 \longrightarrow$ Positive definite

b. Python code attached