Washington University in St. Louis

MEMS 205 Mechanics and Materials Science Laboratory

Lab 5: Rotational Inertia

Section: Group D (Tuesday 12 PM)

Lab Instructor: Dr. Sellers

Experiment Date: Tuesday 3/31/2020

Report Submission Date: Tuesday, 4/7/2020

We hereby certify that the lab report herein is our original academic work, completed in accordance with the McKelvey School of Engineering and Undergraduate Student academic integrity policies, and submitted to fulfill the requirements of this assignment.

Matthew Donaldson	Team Leader	
Isaac Sasser	Data Acquisition Manager	
Andrew Brown	Test Engineer	
Aidan Murphy	Quality Control Engineer	

Abstract

The purpose of this lab was to evaluate material properties for components for the Starling communication satellite for SpaceX. In this lab determined the mass moment of inertia of several different objects using three different methods then comparing the values we found for the moment of inertia. The objects used in this experiment were a solid disk, a black disk with an offset hole, gold-colored mass, a titanium clover mass, and the titanium drop object. The three methods were, horizontal rotary motion, a physical pendulum, and a CAD model of each part. For the horizontal rotary motion, we had a falling mass on a string applying a torque to the object. This torque caused the object to spin and the angular acceleration was recorded using a rotary motion sensor. The physical pendulum had the object attached to the rotary motion sensory. The object was then pushed to make small oscillations so that the small-angle approximation can be used. As the object swung back and forth the rotary motion sensor recorded the angle of the object. Lastly, a CAD model was used to measure the properties of each object.

From the horizontal rotary motion method we found that for the tall cylinder + silver disk, offset hole - center, holde side and solid side their mass moment of inertia is 0.0011093 ($kg * m^2$), 0.0002724 ($kg * m^2$), 0.0007728 ($kg * m^2$) and 0.0005682 ($kg * m^2$) respectively. For the physical pendulum was found that the mass moment of inertia for the offset hole- center, hole side and solid side, solid disk and rod with a gold weight are 0.0012 ($kg * m^2$), 0.00038 ($kg * m^2$), 0.00036 ($kg * m^2$), 0.00027 ($kg * m^2$) and 0.00137203 ($kg * m^2$) respectively. Our team believes that it is better to calculate a solid disk on a rotary machine due to its similarity with the SOLIDWORKS values and to calculate the offset hole using a physical pendulum test because of its similarity to the theoretical values.

Introduction

In order to predict and manipulate and control the behavior of the new Space X Starlink communications satellite, it is imperative that the mass moments of inertia of its components are known. Accordingly, we determine the mass moment of inertia of five objects to be on the satellite: a gold-colored mass, an aluminum clover object, an aluminum drop object, a black disk with an offset hole, and a solid disk. To determine the mass moment of inertia we have three techniques; the horizontal rotary motion technique, the physical pendulum technique, and simulation. We can also apply theory to provide insight into the accuracy of our data, we use as many techniques as possible to calculate the mass moment of inertia of each object.

Mass moment of inertia is the rotational analogy to mass [1]. Newton's second law for linear motion teaches that mass is the resistance to acceleration: $a = \frac{F}{m}$. Note that for any given force, F, the larger the mass, m, the smaller the acceleration, a. In the same way, an angular acceleration relationship to torque and moment of inertia is given by the following equation:

$$\alpha = \frac{\tau}{I} \tag{1}$$

α is angular acceleration, $\tau$is torque, and I is the mass moment of inertia. As I increases, τ also increases to maintain a constant angular acceleration.

I is also related to the distribution of the object's mass relative to the object's axis of rotation. Each particle of mass contributes to the mass moment of inertia proportional to the square of the particle's distance from the axis of rotation. Therefore, in integral form, the mass moment of inertia is represented by the equation:

$$I = \int r^2 dm \tag{2}$$

where r is some distance from the object's center of rotation corresponding to a differential mass on the object. In the same way that the integral sums differential portions of the moment of inertia to determine an object's mass moment of inertia, an object is often divided into multiple sections that are easy to analyze, and these sections are then summed to give the composite object's mass moment of inertia. However, it is often the case that the axis most suited to determining some component's moment of inertia is not the axis of rotation of the entire object. Rather, we determine the component's mass moment of inertia about the axis that passes through its center of mass. In this case, the moment of inertia of the composite object is determined with the parallel axis theorem [1]:

$$I = \sum I_{cm} + Md^2 \tag{3}$$

I_{cm} is the mass moment of inertia about some component's center of mass, M is the mass of the component and d is the distance from the component's center of mass to the axis of rotation of the object.

Horizontal Rotatory Motion

The horizontal rotatory motion method allows us to couple the angular acceleration of a rotating mass and the linear motion of a horizontally falling mass. The falling mass is attached to a horizontally rotating disk (Fig.1). Combining Newton's second law in linear motion form with Newton's second law in angular form (Eq. 1), we derive the equation of the mass moment of inertia of the horizontally spinning mass (Fig. B.2). The result is shown below:

$$I = \frac{mgr}{\alpha} - mr^2 \quad (4)$$

Where m is the mass of the falling object, g is the local gravitational constant, α is the angular acceleration of the rotating mass, and r is the distance between the applied force of the string on the plate and the plate's axis of rotation (Fig. 1). The moment of inertia of a test object is determined by performing the experiment once with just the horizontally spinning plate and then again with the plate and the test object above the plate. Subtracting these two values gives the moment of inertia of the test object about the horizontal plates center of rotation.

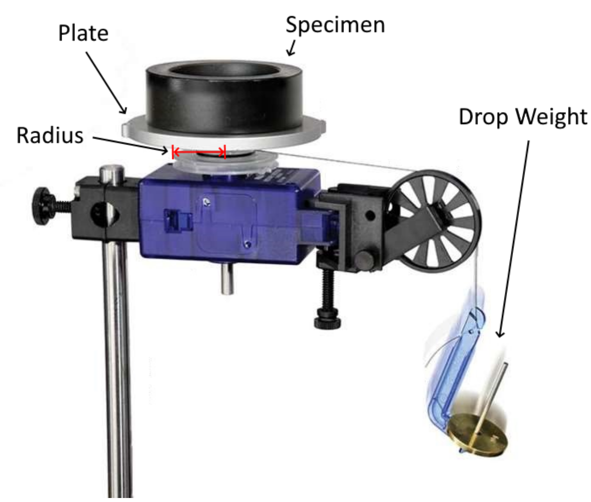

Fig. 1 Horizontal rotatory motion apparatus.

Physical pendulum

The physical pendulum test is another manipulation of Newton's second law. A specimen is hung from an axle by the extremities of its area (Fig. 2). The specimen is lifted and released. It oscillates as gravity, in the form of a force at the specimen's center of mass, provides a torque about the axis of rotation. This force is proportional to $gsin(\theta)$. However, Eq. 1 is much more easily manipulated using the small-angle approximation, $sin(\theta) = \theta$. With this approximation, the mass moment of inertia about the axis of rotation is [2]:

$$I = \frac{MglT^2}{4\pi^2}. \tag{5}$$

l is the distance between the center of mass and the axis of rotation and T is the specimen's period of oscillation. It is simple, with the proper equipment, to measure the variables in Eq. 4 and determine the mass moment of inertia, but note that the equation only holds for small angles ($\theta < 10^{\circ}$) [2].

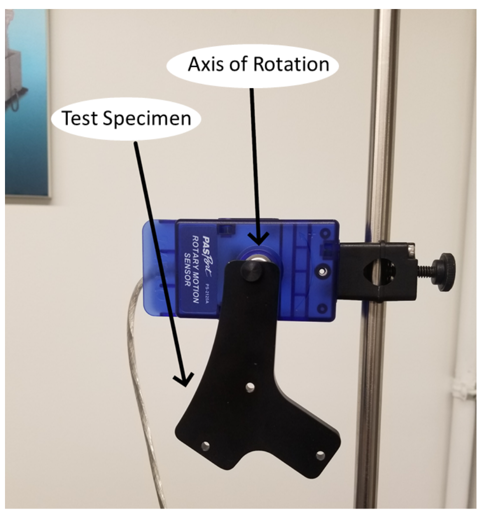

Fig. 2 Physical pendulum apparatus

Applying the Parallel Axis Theorem

The parallel axis theorem (Eq. 3) is broad, applying to objects of any geometry. We use the parallel axis theorem to calculate the mass moment of inertia about the varying axis of the black object with the offset whole. Applying the geometry of this object to Eq. 3 for the specific scenario of an object with a hole that offset from its center, the following equation is derived for the rotation of this specimen about the edge farthest from the offset hole:

$$I = M(2 * R^2 - \frac{r^2}{R^2}(r^2 + (R + a)^2)). \tag{6}$$

Applying geometric relations and altering the above equation for this same offset hole specimen, the following equation is derived for rotation about edge nearest the offset hole:

$$I = M(2 * R^2 - \frac{r^2}{R^2}(2 * r^2)) \tag{7}$$

Manipulating the equation again for rotation about the center of the large outer circle yields the following equation:

$$I = M(2 * R^2 - \frac{r^2}{R^2}(a^2)). \tag{8}$$

$M$is the mass of the object, R is the outer radius of the object, r is the radius of the offset whole, and a is the distance from the center of the offset whole to the center of the object (Fig. 3). The base case of rotation about point P is derived in Fig. B.1.

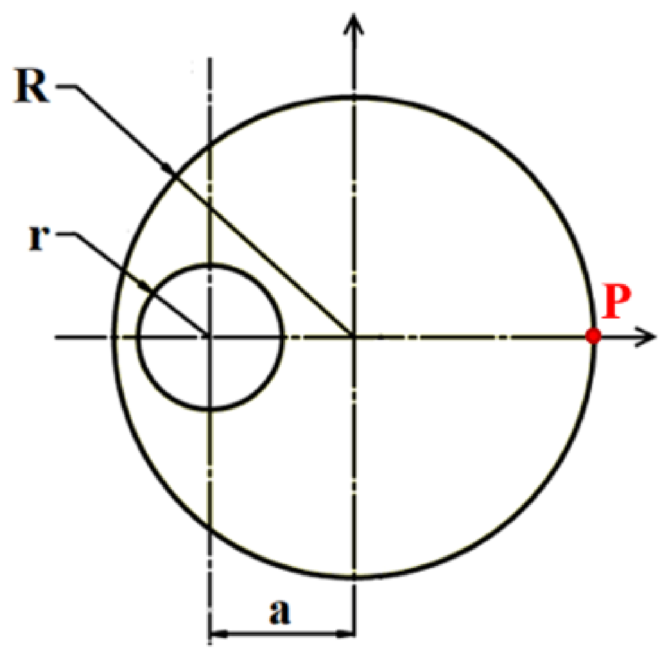

Fig. 3 Offset hole geometric relationships. (Given image)

Simulation

Finally, in order to provide confidence in our results, we constructed models of objects, for which we had dimensions, using SOLIDWORKS. This program provides us with mass moments of inertia through the objects' centers of mass, allowing us to support or experimental values.

Experimental Methods

The purpose of this lab was to measure the mass moment of inertia for several different objects. In this lab, we used three different methods and compared the specimen's values. These three methods were a horizontal rotatory motion, physical pendulum and a simulation using a CAD model.

We first carried out the horizontal rotatory motion method. This method involved applying a torque and measuring the angular acceleration of the object. The angular acceleration was taken from a rotary motion sensor. To make a torque we had a weight tied to a string and hung in mid-air using a black pulley. The other side of the string was wrapped around another clear pulley. This clear pulley was attached to the rotary motion sensor. The object of interest was attached to the top of the clear pulley with a screw. It is important to make sure that the pulley is oriented so that the string is tangent to the clear pulley. The object was attached to the motion sensor at one of the object's holes by a screw. Figure 3 below shows an example of the set up of the first method. To collect data we wound up the string and around the clear pulley, and let it go.

The physical pendulum was the second method to measure the mass moment of inertia. In order to measure the mass moment of inertia, we used a rotary motion sensor to record the angle of the object as it swung back and forth. We made sure that the object had small oscillations during the experiment to keep the angle small. Figure 4 shows the set up used for the pendulum.

The last method used a CAD model to measure the properties of each object. We created a 3D CAD drawing of each object and took a measurement of its properties using the mass properties button.

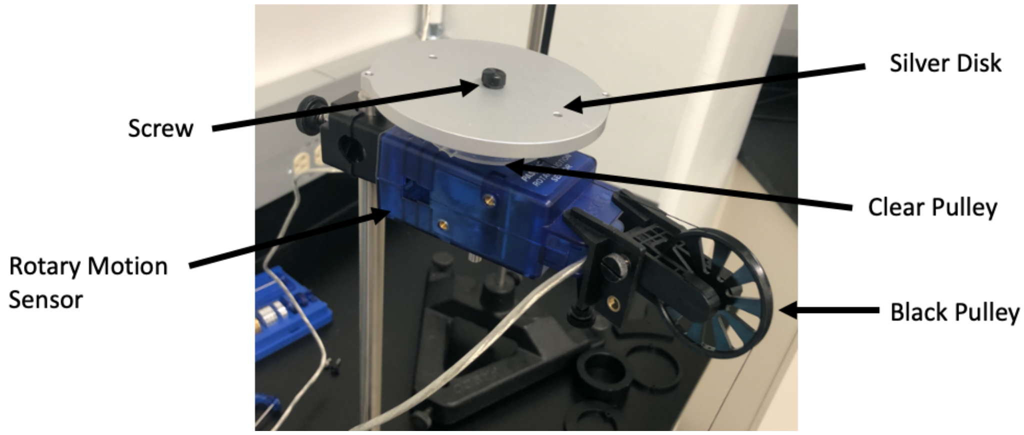

Fig. 3 Horizontal rotary motion method set up. (Given image)

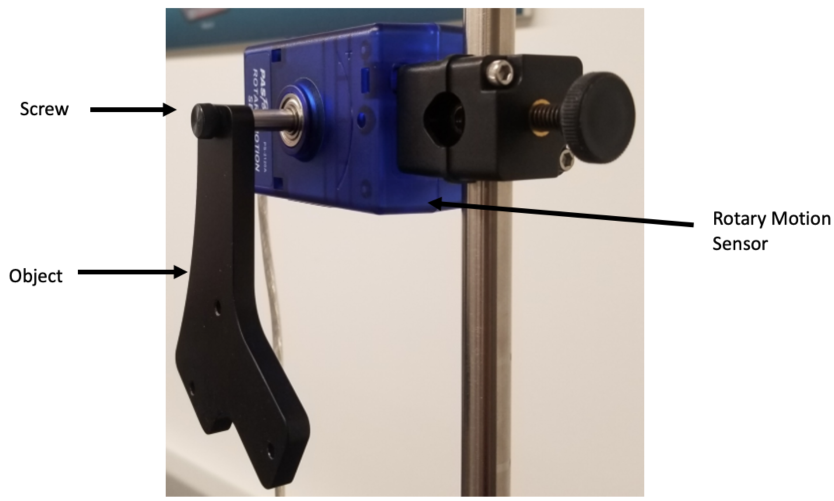

Fig. 4 Physical pendulum method set up. (Given image)

Results and Discussion

To evaluate material properties for components for the Starling communication satellite for SpaceX, the team analyzed three different experimental procedures to determine the moments of inertia for a solid disk, disk with offset hole, gold mass, titanium drop object, titanium clover object. The team performed a horizontal rotary motion test, a pendulum test, and a SOLIDWORKS simulation. When possible, data from the tests were compared to assess validity of the methods. In addition, Eq. 6-8 for analyzing the moment of inertia for the disk with an offset hole were derived in Appendix B (Fig. B.1) and the results for the offset hole were compared to the theoretical values.

Table 1 below has the masses of different objects that were required to complete the analysis. In addition, another important piece of information was the radius from the center to the contact point of the string, r = 0.05 m.

Table 1 **Masses for each of the tested specimens.**

Specimen	Mass (kg)
Offset Hole	0.101
Solid Disk	0.0885
Gold Mass	0.0751
Rod	0.0265
Carriage and Horizontal Pulley	0.0123
Carriage	0.005
Weight Added to Carriage	0.004
Pulley Weight	0.0073

Table 2 below shows the values for the moment of inertia as gathered from the horizontal rotary test, where the α value represents the slope of the linear portion of a plot of angular velocity versus time.

Table 2 **All calculated values that were used to determine the mass moment of inertia of each specific object under the state of rotatory motion.**

Specimen with the Specified Scenario of Rotatory Motion	Units and Values	
	α $(\frac{rad}{s^2})$	I $(kg * m^2)$
Base Case	216.699	-0.000002
Base Case for tall cylinder with silver disk on top of the horizontal pulley	14.167	0.0002891
Tall cylinder + silver disk	3.107	0.0011093
Offset Hole - Center	15.078	0.0002724
Offset Hole - Hole Side	5.566	0.0007728
Offset Hole - Solid Side	7.500	0.0005682
Solid Disk - Center	25.666	0.0001516
Solid Disk - Edge	9.210	0.0004589
Base case of rod with no weights added	5.620	0.000763
Rod (mounted in the center) with TWO gold weights, each 6 in. from the center	0.453	0.00895819

Note: Rotatory motion locations (i.e. solid side, hole side, etc.) are annotated in SOLIDWORKS Part drawings (Appendix A)

The horizontal rotary motion calculated values of mass moment of inertia are in table 2. In order to calculate the mass moment of inertia we used a falling weight to cause a torque and measure angular velocity. From this we graphed angular velocity as a function of time and took the slope of the line as the angular acceleration. We then calculated mass moment of inertia based on Eq. 4 and our derivations derived in appendix B.2.

The physical pendulum method gave the following values seen in Table 3. These values were taken at different distances from the center of mass of each object. To calculate the mass moment of inertia for each object our group graphed angle as a function of time. Then using interpolation found the first two x-intercepts on the graph. Since the x-intercepts represent the time at which the object was at the bottom of its rotation, the period could then be calculated by subtracting two x-intercepts. From this, the mass moment of inertia was calculated using Eq. 5.

Table 3 All calculated values that were used to determine the mass moment of inertia of each specific object under the state of an oscillatory pendulum motion.

Specimen with the Specified Scenario of Pendulum Motion	Units and Values			
	Mass (kg)	Length (m)	Period (s)	I ($kg * m^2$)
Offset hole CENTER	0.101	0.05	0.97586	0.0012
Offset hole HOLE SIDE	0.101	0.05	0.54933	0.00038
Offset hole SOLID SIDE	0.101	0.05	0.53627	0.00036
Solid disk EDGE	0.0885	0.05	0.49519	0.00027
Rod (mounted on the end) with gold weight 6in from center	0.0751	0.06	1.13231	0.00137203
Rod (mounted on the end) with NO weight.	0.0265	0.01	0.9824	0.000063

Looking at the data we can see that the mass moment of inertia for the offset hole changes when it was rotating about different points. From this, when the offset hole rotated at its center it had a lower moment of inertia, this tells us that it is easier for the offset hole to rotate about its center than at the offset hole side. You can also see that from Table 2 that the mass moment of inertia on either side of the offset hole is about the same, which means that no matter what side the offset hole rotates on the offset hole will have about the same resistance to rotation. Lastly, from the pendulum data we can see that the rod mounted with the gold colored mass had the highest mass moment of inertia and the rod without the gold colored mass had the lowest mass moment of inertia.

Comparing this to the mass moment of inertia calculated from the horizontal rotary motion method in Table 2, we see that the values do not agree. Looking at the offset hole you notice that the horizontal rotary method calculated a mass moment of inertia about four and a half that calculated in the pendulum method. However you can also see that the solid disk rotating about its edge has a mass moment of inertia that is about half in the pendulum method compared to the horizontal rotary motion.

Table 4 below depicts the calculated theoretical moment of inertia values (Eqs. 6 - 8) for the offset hole specimen rotating about different axes, refer to Fig. A.4 for notated locations of these rotation axes. The table also has percent errors from the calculated rotatory motion and pendulum values (Tables 2 & 3).

Table 4 Theoretically calculated values for the offset hole specimen at different axes of rotation

Offset Hole Specimen with Specific Rotation Axis	I (kg*m^2)	Percent Error from Rotatory Motion	Percent Error from Pendulum
Offset Hole - Solid Side	0.000348924	146.8884362	3.42626232
Offset Hole - Hole Side	0.000427372	149.4364281	11.3953291
Offset Hole - Center	0.000434936	30.05584216	174.75644

When looking at the theoretical values for the offset hole in Table 4, we see that the rotatory motion method produces large percent errors, above 100%, for rotations about the solid side and hole side axes, but a reasonable percent error, about 30%, for rotation about the central axis. The opposite relation is true for the percent errors for the pendulum method; smaller percent errors, less than 12% for oscillatory motion about the solid side and the hole side axes, but larger percent errors, nearly 175%, for oscillations about the central axis. We can infer from these data that the pendulum predicts better results for objects that rotate about an axis that is far away from its center of mass, producing better oscillations that are more consistent and smoother. This is in comparison to oscillation of an object about an axis that is much closer to the center of mass. For specimens that are rotated about an axis close to their center of mass, horizontal angular rotatory motion is a much better predictor of the actual moment of inertia of that specimen about the specific axis. This may be due to the specimens oscillating much easier about axes that are far from the center of masses and angularly rotating easier for axes that are near the center of mass.

SOLIDWORKS

The CAD simulation gave the following values seen in Table 5. This data was obtained through first modeling the object in SOLIDWORKS then evaluating the properties. The values of moment of inertia are the values around the center of mass in the y-direction (normal to the face of the object). Since the object was built on the top plane in SOLIDWORKS the y-direction is normal to the face of the object.

Table 5: Simulated data from SOLIDWORKS.

Object	Moment of Inertia	Units
Clover Drop	0.000088	$kg * m^2$
Drop Object	0.000097	$kg * m^2$
Solid Disk	0.00014	$kg * m^2$
Offset Hole	0.00012	$kg * m^2$

While there no other experimental data was available for the drop and the clover objects, the SOLIDWORKS simulation shows that the titanium objects, the clover and drop, have slightly lower moments of inertia than the solid disk and offset hole Table 6 shows the SOLIDWORKS data for the solid disk and the offset hole around their centers.

Table 6: The simulated data from SOLIDWORKS. This data was obtained through first modeling the object in SOLIDWORKS then evaluating the properties. The values of moment of inertia are the values around the center of the solid disk and the offset hole.

Object	Moment of Inertia	Units
Solid Disk	0.00014	$kg * m^2$
Offset Hole	0.00012	$kg * m^2$

From Table 5 we can compare the solidworks method of analysis to the pendulum, the horizontal rotatory motion, and the theoretical. The theoretical will not provide a good comparison as the solid works data is only about the center of mass whereas the theoretical is about specific mounting points. Both the pendulum and horizontal rotatory motion contain trials where the object was rotated around its center. Table 6 includes data from SOLIDWORKS of the moment of inertia about the center of the object. When considering only two significant figures, the values are not any different. Table 7 shows a comparison between the SOLIDWORKS and the other experimental methods.

Table 7 A comparison of the experimental methods and their results for the solid disk and offset hole.

Object	SOLIDWORKS	I of Pendulum Test $[kg * m^2]$	Difference	I of Rotatory Motion $[kg * m^2]$	Difference
Solid Disk	0.00014	N/a	N/a	0.00015	7%
Offset Hole	0.00012	0.0012	90%	0.00027	56%

The SOLIDWORKS values show a much better correlation to the rotary motion than the pendulum test. Data for the solid disk in the pendulum test was not provided. However, in Table 4 the theoretical values, from Eq. 6-8, were much more closely aligned with the offset hole. These two pieces of evidence led the team to conclude that the rotary motion test better predicts the moment of inertia for the solid disk. While the pendulum test better predicts the moment of inertia for the offset hole.

Conclusion

From our experimental data we observed that the rod gold with the mass had the highest moment of inertia (0.0090 $kg * m^2$), meaning that the rod with the gold mass had the highest resistance to rotation. We also observed that the solid disk with the offset whole, rotated about its center of areas, had the smallest moment of inertia (0.00012 $kg * m^2$), telling us that it has the lowest resistance to rotation. We also believe that the pendulum test is better for the offset hole when its axis of rotation is at its extremities. We believe this because the pendulum results agree with our theoretical results very well - within 3.3 %. We also believe that the horizontal rotary motion method is better at determining the mass moment of inertia for the solid disk because it lines up better with the solidworks calculation -within 7% [Table 7]. Finally, we note that while we were able to determine mass moments of inertia for the clover drop object and the drob object (.00088 and .00097 $kg * m^2$ respectively) we were unable to confirm these results because we were unable to obtain experimental data for these objects.

References

[1] Knight, K., 2016, Physics for Scientists and Engineers A Strategic Approach, Fourth ed., Boston, MA, Chap. 12.

[2] Knight, K., 2016, Physics for Scientists and Engineers A Strategic Approach, Fourth ed., Boston, MA, Chap. 15.

Appendix A

The following figures (Figs. A.1 - A.4) are the SOLIDWORKS drawings of the parts that were created for the moment of inertia property analysis.

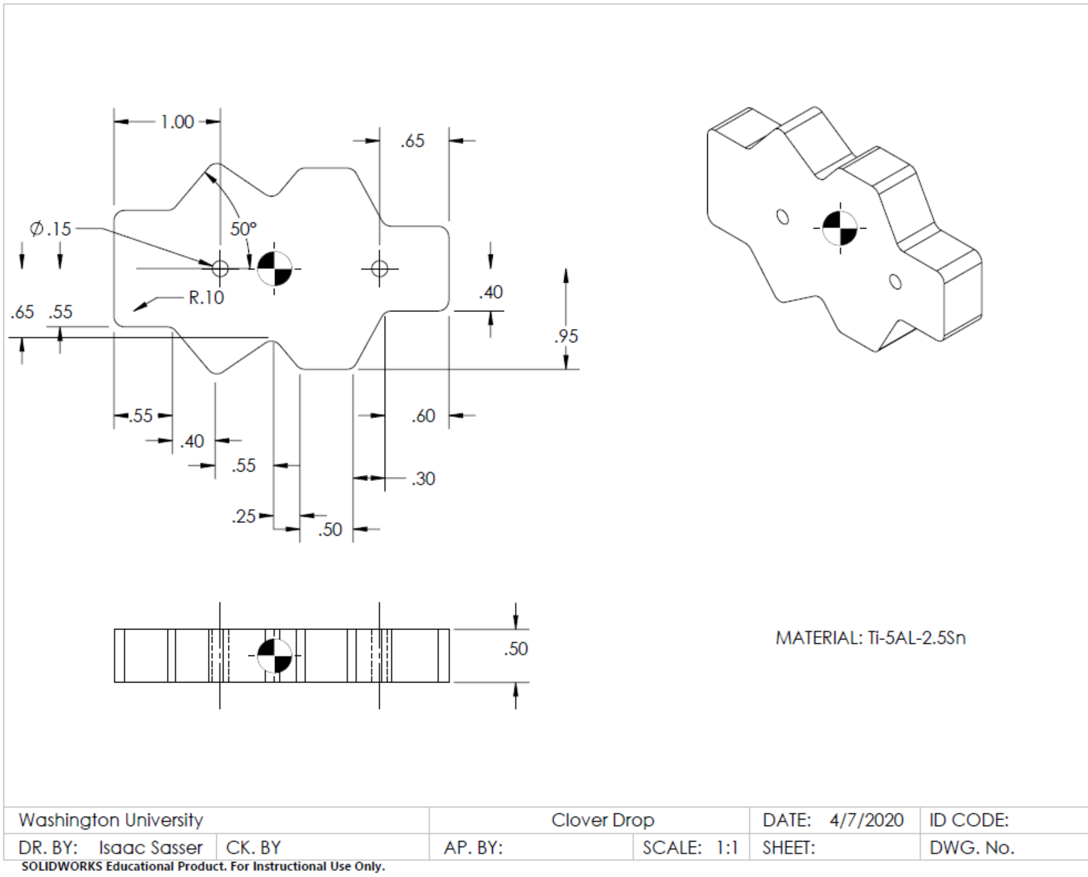

Fig. A.1 SOLIDWORKS drawing for the titanium clover object.

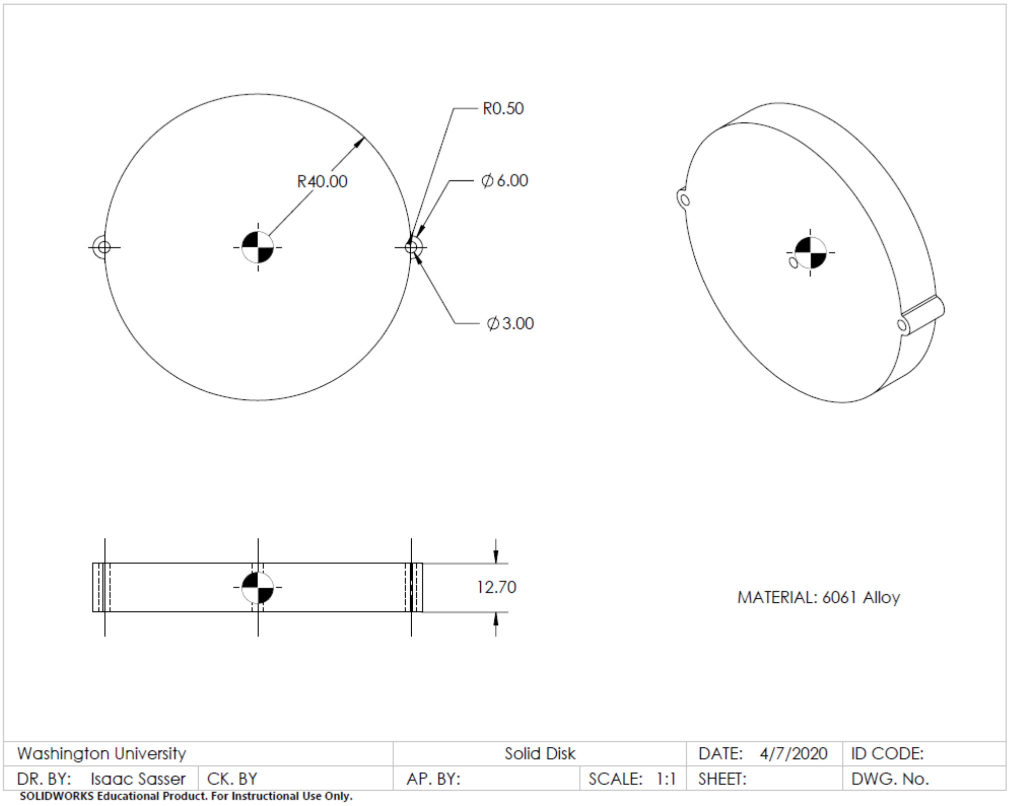

Fig. A.2 SOLIDWORKS drawing for the solid disk specimen.

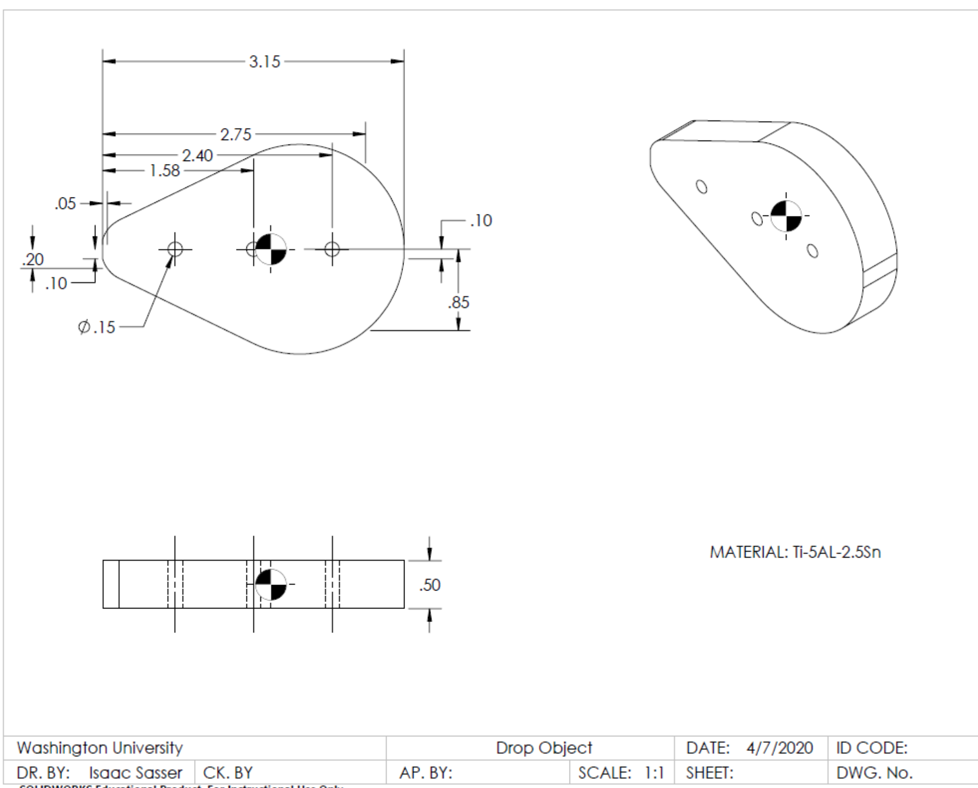

Fig. A.3 SOLIDWORKS drawing for the titanium drop object.

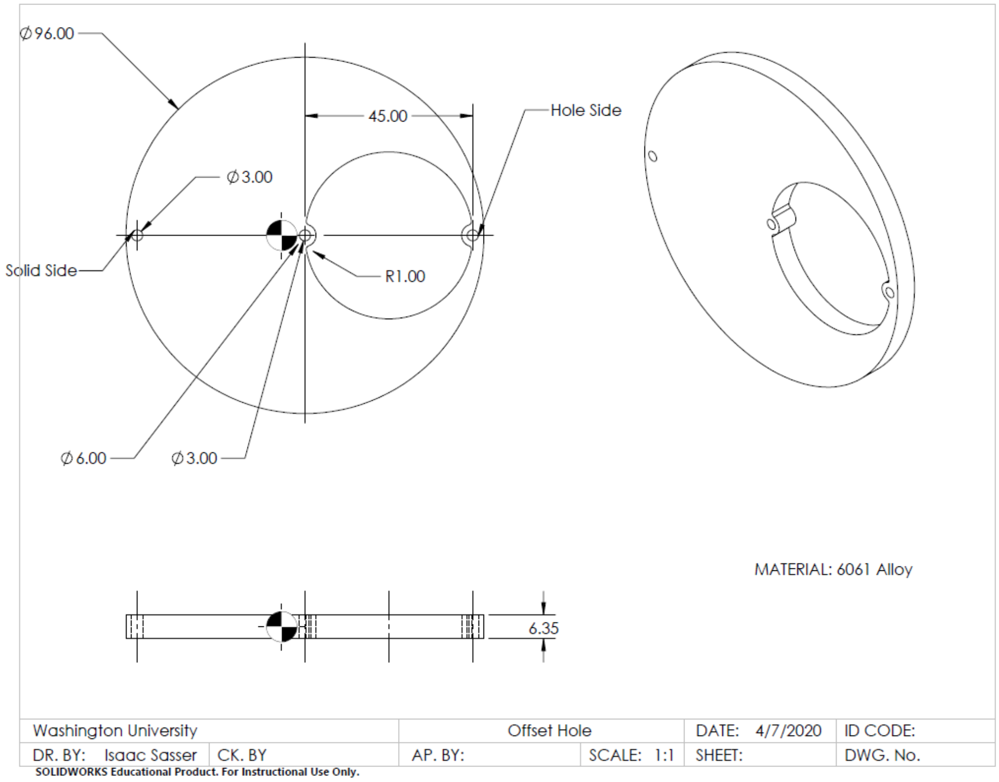

Fig. A.4 SOLIDWORKS drawing for the offset hole specimen.

Appendix B

The following figures of equations are the derivations of equations that were used to calculate the moments of inertia (Figs. B.1 & B.2).

$$I_{tot} = \left(I_{big} + Md_{big}^2\right) - \left(I_{hole} + M_{hole}d_{hole}^2\right)$$
$$d_{hole} = R + a$$
$$d_{big} = R$$
$$I_{tot} = \left(I_{big} + MR^2\right) - \left(I_{hole} + M_{hole}(R+a)^2\right)$$
$$I = M * r^2$$
$$I_{tot} = (MR^2 + MR^2) - \left(M_{hole} * r^2 + M_{hole}(R+a)^2\right)$$
$$I_{tot} = 2MR^2 - M_{hole}\left(r^2 + (R+a)^2\right)$$
$$\rho = const$$
$$\frac{M}{M_{hole}} = \frac{V}{V_{hole}}$$
$$M_{hole} = M * \frac{V_{hole}}{V}$$
$$thickness = const$$
$$M_{hole} = M * \frac{r^2}{R^2}$$
$$I_{tot} = 2MR^2 - M * \frac{r^2}{R^2}\left(r^2 + (R+a)^2\right)$$
$$I_{tot} = M\left(2R^2 - \frac{r^2}{R^2}\left(r^2 + (R+a)^2\right)\right)$$

Fig. B.1 Moment of inertia derivation for the offset hole object rotation about the solid side of the specimen (refer to Fig. A.4 for location of this rotation point).

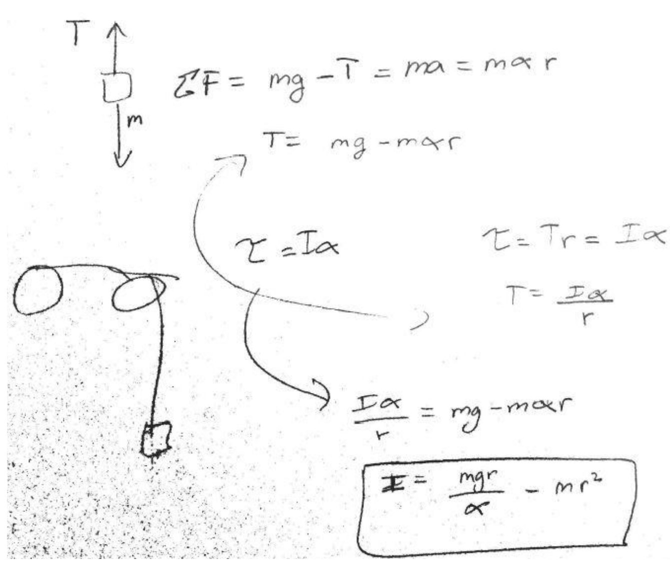

Fig. B.2 Moment of inertia derivation for the horizontal rotatory motion method