Assignment 0 Feb 8.

1. a. $F(x) = x - 1$

yes, This function is differentiable because it has a Derivative and is continous

b. $F(x) = \begin{cases} x-1 & x \geq 1 \\ 0 & x < 1 \end{cases}$

This function is differentiable because The $\lim_{x \to 1^-}$ and $\lim_{x \to 1^+}$ both would equal 0.

which means it is continuous, and they both have a first Derivative

c. $F(x) = |x|$

This function is not Differentiable because it is not continuous.

2. a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & \pi \end{bmatrix} \rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & e-\lambda & 0 \\ 0 & 0 & \pi-\lambda \end{bmatrix}$ $A - \lambda I = 0$

$(1-\lambda) \begin{vmatrix} e-\lambda & 0 \\ 0 & \pi-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)(e-\lambda)(\pi-\lambda) = 0$

$\boxed{\lambda = 1, \ \lambda = e, \ \lambda = \pi}$ → eigen values.

Let $\lambda = 1$:

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & e-1 & 0 \\ 0 & 0 & \pi-1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$(e-1)y = 0$ $(\pi-1)z = 0$

$y = 0$ $z = 0$

Let $\lambda = e$: $\begin{bmatrix} 1-e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \pi - e \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$(1-e)x = 0$ $\quad$ $(\pi - e)z = 0$

$x = 0$ $\quad$ $z = 0$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Let $\lambda = \pi$ $\begin{bmatrix} 1-\pi & 0 & 0 \\ 0 & e-\pi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$(1-\pi)x = 0$ $\quad$ $(e-\pi)y = 0$

$x = 0$ $\quad$ $y = 0$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

b. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$(-\lambda)(-\lambda) + 1 = 0 \rightarrow \lambda^2 = -1 \rightarrow \boxed{\lambda = \pm i}$ eigen values.

Let $\lambda = -i$: $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$ix - y = 0$ $\quad$ $x + iy = 0$ $\quad$ $x = iy$

$y = ix$ $\quad$ $x + i(ix) = 0$

$x - x = 0$

$0 = 0$

$\begin{pmatrix} 1 \\ i \end{pmatrix}$

Let $\lambda = i$ $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-ix - y = 0$ $\quad$ $x - iy = 0$ $\quad$ $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$y = -ix$ $\quad$ $x = iy$

c. $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1-\lambda & 0 & 0 \\ -3 & 3-\lambda & 0 \\ 3 & 2 & 2-\lambda \end{bmatrix} = 0$

$(1-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = 0 \to (1-\lambda)(3-\lambda)(2-\lambda) = 0$.

$\boxed{\lambda = 1,\ \lambda = 3,\ \lambda = 2}$ eigen values.

Let $\lambda = 1$: $\begin{bmatrix} 0 & 0 & 0 \\ -3 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-3x + 2y = 0$ $\qquad$ $3x + 2y + z = 0$

$y = \frac{3}{2}x$ $\qquad$ $\begin{pmatrix} 1 \\ 3/2 \\ -6 \end{pmatrix} \longrightarrow \begin{pmatrix} -1/6 \\ -1/4 \\ 1 \end{pmatrix}$

Let $\lambda = 3$: $\begin{bmatrix} -2 & 0 & 0 \\ -3 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-2x = 0$ $\qquad$ $-3x = 0$ $\qquad$ $3x + 2y - z = 0$

$x = 0$ $\qquad$ $2y - z = 0$

$z = 2y$

$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Let $\lambda = 2$: $\begin{bmatrix} -1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-x = 0$ $\qquad$ $-3x + y = 0$ $\qquad$ $3x + 2y = 0$

$x = 0$ $\qquad$ $y = 0$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

3. $u \in R^n$ $\quad u^T u = 1$ $\quad H^{n \times n} = I - 2uu^T$

a. $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\quad u^T = [u_1 \; u_2 \dots u_n]$

$$u^T u = [u_1 \; u_2 \dots u_n] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1^2 + u_2^2 + \dots + u_n^2 = 1$$

So only 1 variable can = 1 all others are 0.

$$uu^T = \begin{bmatrix} u_1 \\ u_2 \\ \\ u_n \end{bmatrix} [u_1 u_2 \dots u_n] = \begin{bmatrix} u_1^2 & u_1 u_2 & \dots & u_1 u_n \\ u_1 u_2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ u_n u_n & - - & - - & u_n^2 \end{bmatrix}$$

Since only 1 variable can = 1 we get all 0 except for one of the Diagnols.

Then $H = I - 2uu^T = I - 2\begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & & \vdots \\ 0 & \dots & 0 & \vdots \end{bmatrix}$

<u>This makes H Symmetric</u> b/c all points are 0 except for the Diagnols.

$H^T H = I \rightarrow$ means orthogonal

$$\begin{bmatrix} H_{11} & H_{21} & H_{31} \\ H_{12} & \ddots & \\ H_{13} & & H_{nn} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & \ddots & \\ H_{31} & & H_{nn} \end{bmatrix} =$$ results in only the Diagnols as 1, which makes ~~H~~ I.

<u>So H is orthogonal</u>

ex: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

3b. $Hu = (I - 2uu^T)u = Iu - 2uu^Tu = Iu - 2u$

$= -u$ Since This results in a Scalar multiple u is an eigen vector w/ eigen value $= -1$

c. $v \in \mathbb{R}^n$ v is orthogonal to u $\rightarrow u^T v = 0$

So $Hu = (I - 2uu^T)v = Iv - 2uu^Tv = Iv = v$

Since This results in a Scalar multiple v is an eigen vector w/ eigen value $= 1$

4. a. $f(x) = u^T x + c$; $u, x \in \mathbb{R}^n$ $c \in \mathbb{R}$

$$= [u_1 \; u_2 \; \dots \; u_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [u_1 x_1 + u_2 x_2 + u_3 x_3 + \dots + u_n x_n]$$

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_1}{\partial x_n} \right] = u^T$$

b. $f(x) = Ax$ $A \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$

$$= \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1} & \dots & & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11} x_1 + A_{12} x_2 + \dots A_{1n} x_n \\ A_{21} x_1 + A_{22} x_2 + \dots \\ \vdots \\ A_{n1} x_1 + A_{n2} x_2 + \dots A_{nn} x_n \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_n}{\partial x_p} & & \frac{\partial f_n}{\partial x_n} \end{bmatrix} = A$$

c. $f(x) = \|x\|^2 = x^T x \rightarrow \frac{\partial f(x)}{\partial x} = 2x$