Adaptive composite estimation in small population areas

Andrius Čiginas

Vilnius University

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Design-based vs model-based estimation

Suppose there are estimation domains (areas) with sample sizes too small to yield sufficiently accurate direct estimates.

- That is, design-based direct estimation is inefficient for these domains.
- Traditional design-based (synthetic and) composite estimators improve efficiency and are desirable, though they present challenges in inference.
- By contrast, model-based estimators, such as EBLUPs, offer greater flexibility in estimation and have well-developed MSE estimation methods.

Aim: To re-evaluate classical design-based composite estimators and methods for estimating their MSEs.

Direct estimation in domains

- $\mathcal{U} = \{1, \ldots, N\}$ represents a finite survey population.
- There are M areas $\mathcal{U}_1, \ldots, \mathcal{U}_M$ with sizes $N_1, \ldots, N_M$, such that $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_M = \mathcal{U}$ and $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ for $i \neq j$.
- y is the study variable with fixed values $y_1, \ldots, y_N$ in $\mathcal{U}$.
- We aim to estimate the domain parameters θ_i, $i = 1, \ldots, M$.
- The sample $s \subset \mathcal{U}$, of size $n < N$, is drawn according to a sampling design $p(\cdot)$.
- Let $\hat{\theta}_i^{\mathrm{d}}$ be an approximately design unbiased direct estimator of θ_i based on the sample $s_i = s \cap \mathcal{U}_i$ of size n_i.
- If n_i is small, we get a large design variance $\psi_i = \mathrm{var}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{d}})$.
- Direct estimators $\hat{\psi}_i^{\mathrm{d}}$ of ψ_i also have high variances for small samples s_i.

Direct estimation in domains: an example

- Let us estimate the domain means (or domain *proportions*)

$$\theta_i = \frac{1}{N_i} \sum_{k \in \mathcal{U}_i} y_k, \qquad i = 1, \ldots, M,$$

where the values N_i are assumed to be known.

- Let $\pi_k = \mathrm{P_p}\{k \in s\} > 0$. The weighted sample means

$$\hat{\theta}_i^{\mathrm{d}} = \frac{1}{\widehat{N}_i} \sum_{k \in s_i} \frac{y_k}{\pi_k} \quad \text{where} \quad \widehat{N}_i = \sum_{k \in s_i} \frac{1}{\pi_k}, \qquad i = 1, \ldots, M,$$

are approximately unbiased direct estimators of θ_i.

- The direct estimators (Särndal et al., 1992)

$$\hat{\psi}_i^{\mathrm{d}} = \frac{1}{\widehat{N}_i^2} \sum_{k \in s_i} \sum_{l \in s_i} \left(1 - \frac{\pi_k \pi_l}{\pi_{kl}}\right) \frac{(y_k - \hat{\theta}_i^{\mathrm{d}})(y_l - \hat{\theta}_i^{\mathrm{d}})}{\pi_k \pi_l}$$

of ψ_i, where $\pi_{kl} = \mathrm{P_p}\{k, l \in s\} > 0$, can have high variances.

Synthetic estimation

- The synthetic estimator $\hat{\theta}_i^{\mathrm{S}}$ of θ_i uses the sample of a larger area through an implicit linking model. This model relies on the ***synthetic assumption*** that the small domain $\mathcal{U}_i$ has the same characteristics as the large area (Rao and Molina, 2015). Consequently, $\hat{\theta}_i^{\mathrm{S}}$ has a smaller design variance than $\hat{\theta}_i^{\mathrm{d}}$, but it is biased.
- Similarly, the estimators $\hat{\psi}_i^{\mathrm{d}}$ of $\psi_i = \mathrm{var}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{d}})$ are smoothed using the generalized variance function approach (Wolter, 2007). This yields a smoothed (synthetic) estimator $\hat{\psi}_i^{\mathrm{s}}$.

Synthetic estimation: an example

- Let $\mathbf{z}_i = (z_{1i}, z_{2i}, \ldots, z_{Pi})'$ be auxiliary data for $\mathcal{U}_i$, and let $\hat{\psi}_i$ be any estimator of ψ_i. The regression-synthetic estimator

$$\hat{\theta}_i^{\mathrm{S}} = \hat{\theta}_i^{\mathrm{S}}(\hat{\psi}_i) = \mathbf{z}_i' \hat{\boldsymbol{\beta}} \quad \text{with} \quad \hat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{M} \frac{\mathbf{z}_i \mathbf{z}_i'}{\hat{\psi}_i} \right)^{-1} \sum_{i=1}^{M} \frac{\mathbf{z}_i \hat{\theta}_i^{\mathrm{d}}}{\hat{\psi}_i}$$

  of θ_i is derived from the basic area-level model for EBLUP, ignoring random area effects (Rao and Molina, 2015).

- It may be preferable to take $\hat{\psi}_i = \hat{\psi}_i^{\mathrm{s}}$ instead of the direct estimators $\hat{\psi}_i^{\mathrm{d}}$, where, for domain *proportions* θ_i, the quantities $\hat{\psi}_i^{\mathrm{s}}$ smooth $\hat{\psi}_i^{\mathrm{d}}$ using the assumption $\psi_i \approx K N_i^{\gamma}$. Here, $K > 0$ and $\gamma \in \mathbb{R}$ are estimated through a log-log regression model, similar to the approach in Dick (1995).

Design-based composite estimation

The design-based linear composition

$$\tilde{\theta}_i^{\mathrm{C}} = \tilde{\theta}_i^{\mathrm{C}}(\lambda_i) = \lambda_i \hat{\theta}_i^{\mathrm{d}} + (1 - \lambda_i)\hat{\theta}_i^{\mathrm{S}}$$

with weight $0 \leqslant \lambda_i \leqslant 1$ provides a trade-off between the larger variance of the direct estimator $\hat{\theta}_i^{\mathrm{d}}$ and the bias of the synthetic estimator $\hat{\theta}_i^{\mathrm{S}}$. In other words, it balances the unbiasedness of $\hat{\theta}_i^{\mathrm{d}}$ with the smaller variance of $\hat{\theta}_i^{\mathrm{S}}$.

Question: How should we choose the weights λ_i, $i = 1, \ldots, M$?

By minimizing the function $\mathrm{MSE}_{\mathrm{p}}(\tilde{\theta}_i^{\mathrm{C}}(\lambda_i))$ with respect to λ_i, we obtain the optimal weight λ_i^*, which can be approximated by

$$\lambda_i^* \approx \frac{\mathrm{MSE}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{S}})}{\mathrm{MSE}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{d}}) + \mathrm{MSE}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{S}})},$$

as shown in Rao and Molina (2015).

Approximations to optimal compositions

1. A straightforward estimation of the optimal weight λ_i^* leads to estimators such as

$$\hat{\lambda}_i = \frac{\mathrm{mse_u}(\hat{\theta}_i^{\mathrm{S}})}{\hat{\psi}_i^{\mathrm{s}} + \mathrm{mse_u}(\hat{\theta}_i^{\mathrm{S}})},$$

with an approximately design unbiased estimator (Gonzalez and Waksberg, 1973)

$$\mathrm{mse_u}(\hat{\theta}_i^{\mathrm{S}}) = (\hat{\theta}_i^{\mathrm{S}} - \hat{\theta}_i^{\mathrm{d}})^2 - \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{S}} - \hat{\theta}_i^{\mathrm{d}}) + \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{S}})$$

of $\mathrm{MSE_p}(\hat{\theta}_i^{\mathrm{S}})$, where $\hat{\sigma}^2(\cdot)$ denotes an estimator of $\mathrm{var_p}(\cdot)$.

2. Use of a common weight across all domains by minimizing the total MSE (Purcell and Kish, 1979).
3. Application of the James–Stein method, as described in Rao and Molina (2015).
4. Sample-size-dependent estimation, where $\hat{\lambda}_i$ adjusts to the domain sample size (Drew et al., 1982).

Estimation of mean square errors

General method. Treating the composite estimator $\hat{\theta}_i^{\mathrm{C}} = \tilde{\theta}_i^{\mathrm{C}}(\hat{\lambda}_i)$ as synthetic, one can use

$$\mathrm{mse_u}(\hat{\theta}_i^{\mathrm{C}}) = (\hat{\theta}_i^{\mathrm{C}} - \hat{\theta}_i^{\mathrm{d}})^2 - \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{C}} - \hat{\theta}_i^{\mathrm{d}}) + \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{C}})$$

to estimate $\mathrm{MSE_p}(\hat{\theta}_i^{\mathrm{C}})$. This estimator is approximately unbiased but has a large variance and can yield negative values (Rao and Molina, 2015).

Alternative method. Assuming $\hat{\theta}_i^{\mathrm{C}} = \tilde{\theta}_i^{\mathrm{C}}(\hat{\lambda}_i)$ approximates the optimal combination $\hat{\theta}_i^{\mathrm{opt}} = \tilde{\theta}_i^{\mathrm{C}}(\lambda_i^*)$ well, we can use

$$\mathrm{mse_b}(\hat{\theta}_i^{\mathrm{C}}) = \hat{\lambda}_i(1 - \hat{\lambda}_i)\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{C}})$$

to estimate $\mathrm{MSE_p}(\hat{\theta}_i^{\mathrm{C}})$, where $\hat{\sigma}^2(\hat{\theta}_i^{\mathrm{C}})$ estimates $\mathrm{var_p}(\hat{\theta}_i^{\mathrm{C}})$. This MSE estimator is non-negative but may be biased (Čiginas, 2023).

Sample-size-dependent estimation

According to Drew et al. (1982), the estimators of the weights λ_i are taken in the form

$$\hat{\lambda}_i = \hat{\lambda}_i(\delta) = \begin{cases} 1 & \text{if } \widehat{N}_i/N_i \geqslant \delta, \\ \widehat{N}_i/(\delta N_i) & \text{otherwise.} \end{cases}$$

These weights depend on a single, subjectively chosen parameter δ for all domains, with a default value of $\delta = 1$.

However, the choice of δ may vary by survey. If very small domains dominate, an appropriate δ is often significantly greater than 1 (Čiginas, 2020).

Adaptive sample-size-dependent estimator

To select the value of δ for the composition $\tilde{\theta}_i^{\mathrm{C}}(\delta) = \tilde{\theta}_i^{\mathrm{C}}(\hat{\lambda}_i(\delta))$, we numerically minimize the sample-based function

$$r(\delta) = \frac{1}{M} \sum_{i=1}^{M} \mathrm{mse_u}(\tilde{\theta}_i^{\mathrm{C}}(\delta))$$

with respect to δ. The adaptive composite estimators of the domain parameters θ_i are then defined by (Čiginas, 2020)

$$\hat{\theta}_i^{\mathrm{SSD}} = \tilde{\theta}_i^{\mathrm{C}}(\hat{\delta}^*) \quad \text{where} \quad \hat{\delta}^* = \arg\min_{\delta > 0} r(\delta).$$

To evaluate the MSEs of these compositions, we apply the estimators

$$\mathrm{mse_b}(\hat{\theta}_i^{\mathrm{SSD}}) = \hat{\lambda}_i(\hat{\delta}^*)(1 - \hat{\lambda}_i(\hat{\delta}^*))\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{SSD}}).$$

Self-adapting (two-step) composite estimator

The composition is built in two steps (Čiginas, 2023).

1. To estimate the optimal coefficient λ_i^*, use the estimator

$$\hat{\lambda}_i^{(1)} = \frac{\hat{\sigma}^2(\hat{\theta}_i^{\mathrm{S}})}{\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{S}})},$$

   and $\hat{m}_i^{(1)} = \mathrm{mse}_{\mathrm{b}}(\tilde{\theta}_i^{\mathrm{C}}(\hat{\lambda}_i^{(1)}))$ is the MSE estimator for the composition $\tilde{\theta}_i^{\mathrm{C}}(\hat{\lambda}_i^{(1)})$.

2. Since $\hat{\lambda}_i^{(1)} < \lambda_i^*$ is expected, treat $\tilde{\theta}_i^{\mathrm{C}}(\hat{\lambda}_i^{(1)})$ as the synthetic estimator and construct the new composition

$$\hat{\theta}_i^{\mathrm{Cb}} = \hat{\lambda}_i^{(2)}\hat{\theta}_i^{\mathrm{d}} + (1-\hat{\lambda}_i^{(2)})\tilde{\theta}_i^{\mathrm{C}}(\hat{\lambda}_i^{(1)}) \quad \text{where} \quad \hat{\lambda}_i^{(2)} = \frac{\hat{m}_i^{(1)}}{\hat{\psi}_i^{\mathrm{s}} + \hat{m}_i^{(1)}},$$

   and $\mathrm{mse}_{\mathrm{b}}(\hat{\theta}_i^{\mathrm{Cb}}) = \hat{\lambda}_i^{(2)}(1-\hat{\lambda}_i^{(2)})\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{Cb}})$ is the MSE estimator.

EBLUP based on the Fay–Herriot model

The EBLUP of the domain parameter θ_i is expressed as the linear combination (Fay and Herriot, 1979)

$$\hat{\theta}_i^{\mathrm{FH}} = \hat{\gamma}_i \hat{\theta}_i^{\mathrm{d}} + (1 - \hat{\gamma}_i)\mathbf{z}_i' \hat{\boldsymbol{\beta}} \quad \text{where} \quad \hat{\gamma}_i = \frac{\hat{\sigma}_v^2}{\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}_v^2},$$

and

$$\hat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{M} \frac{\mathbf{z}_i \mathbf{z}_i'}{\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}_v^2} \right)^{-1} \sum_{i=1}^{M} \frac{\mathbf{z}_i \hat{\theta}_i^{\mathrm{d}}}{\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}_v^2},$$

where $\hat{\sigma}_v^2$ is an estimator of the variance σ_v^2 of random domain effects. Here, we use the estimator $\hat{\sigma}_v^2$ of σ_v^2 based on the method of moments, as originally proposed by Fay and Herriot (1979).

Mean square error estimation for EBLUP

An approximately unbiased estimator of the MSE of $\hat{\theta}_i^{\mathrm{FH}}$ was derived in Datta et al. (2005):

$$\begin{aligned}
\mathrm{mse}(\hat{\theta}_i^{\mathrm{FH}}) = \hat{\gamma}_i \hat{\psi}_i^{\mathrm{s}} + (1-\hat{\gamma}_i)^2 \Bigg[& \mathbf{z}_i' \left(\sum_{j=1}^{M} \frac{\mathbf{z}_j \mathbf{z}_j'}{\hat{\psi}_j^{\mathrm{s}} + \hat{\sigma}_v^2} \right)^{-1} \mathbf{z}_i \\
& + \frac{4M}{\hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}_v^2} \left(\sum_{j=1}^{M} \frac{1}{\hat{\psi}_j^{\mathrm{s}} + \hat{\sigma}_v^2} \right)^{-2} \\
& - 2\hat{\sigma}_v^2 \left(\sum_{j=1}^{M} \hat{\gamma}_j \right)^{-3} \left\{ M \sum_{j=1}^{M} \hat{\gamma}_j^2 - \left(\sum_{j=1}^{M} \hat{\gamma}_j \right)^2 \right\} \Bigg].
\end{aligned}$$

For *comparison*, if we ignore the covariance term, we obtain

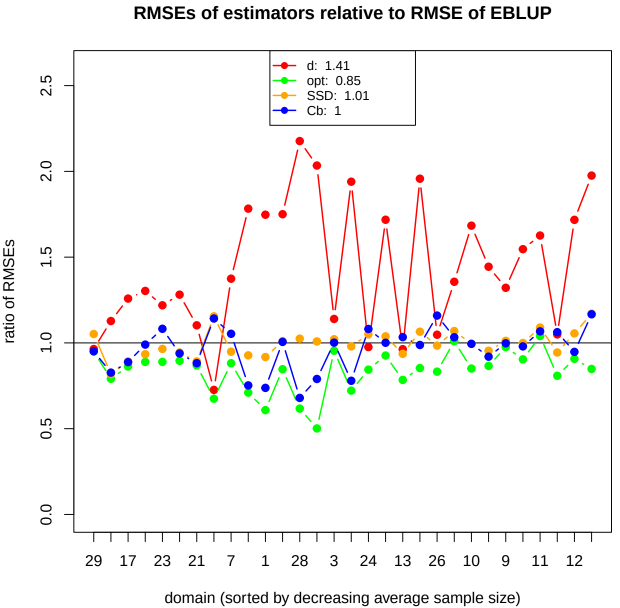
$$\mathrm{mse}_{\mathrm{b}}(\hat{\theta}_i^{\mathrm{C}}) \approx \hat{\lambda}_i \hat{\psi}_i^{\mathrm{s}} + (1-\hat{\lambda}_i)^2 \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{S}})$$

for the design-based composite estimators.

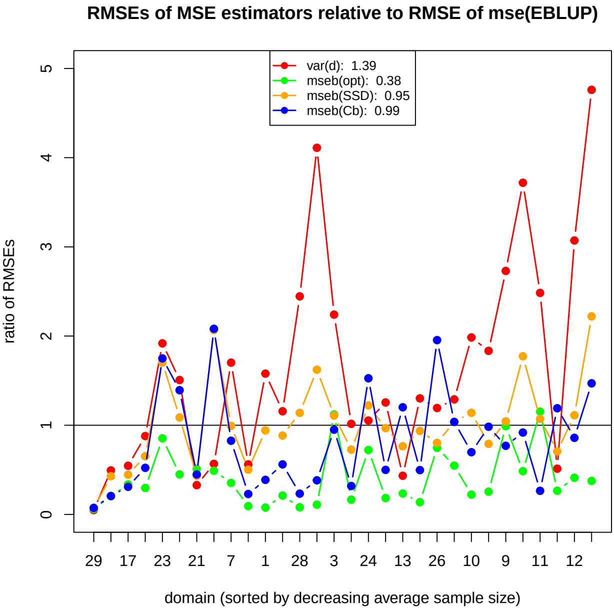
Simulation using the Labor Force Survey data

- The parameters θ_i are the *proportions* of unemployed and employed individuals in Lithuanian municipalities.
- The artificial population $\mathcal{U}$ has $N = 1\,396\,763$ individuals and $M = 30$ domains.
- The proportions in $\mathbf{z}_i = (1, z_{2i}, z_{3i}, z_{4i}, z_{5i}, z_{6i})'$ are defined as follows: z_{2i} is registered unemployment, z_{3i} represents individuals paying social contributions, z_{4i} denotes males, and z_{5i} and z_{6i} correspond to age groups 26–40 and 41–55.
- The estimators $\hat{\theta}_i^{\mathrm{d}}$ and $\hat{\theta}_i^{\mathrm{S}}$ are used as in the examples.
- We draw $R = 1\,000$ samples, each with $n \approx 7\,667$ individuals, by selecting $n' = 3\,700$ households with unequal probabilities.
- We calculate the ratios between RMSEs for $\hat{\theta}_i^{\mathrm{d}}$, $\hat{\theta}_i^{\mathrm{opt}}$, $\hat{\theta}_i^{\mathrm{SSD}}$, $\hat{\theta}_i^{\mathrm{Cb}}$, and the RMSE of EBLUP $\hat{\theta}_i^{\mathrm{FH}}$.
- We evaluate the ratios between RMSEs for $\hat{\psi}_i^{\mathrm{d}}$, $\mathrm{mse}_{\mathrm{b}}(\hat{\theta}_i^{\mathrm{opt}})$, $\mathrm{mse}_{\mathrm{b}}(\hat{\theta}_i^{\mathrm{SSD}})$, $\mathrm{mse}_{\mathrm{b}}(\hat{\theta}_i^{\mathrm{Cb}})$, and the RMSE of $\mathrm{mse}(\hat{\theta}_i^{\mathrm{FH}})$.

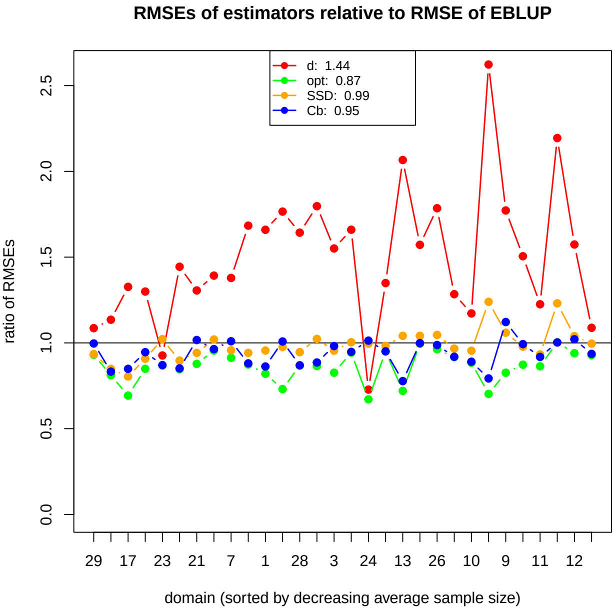
Estimation of unemployed



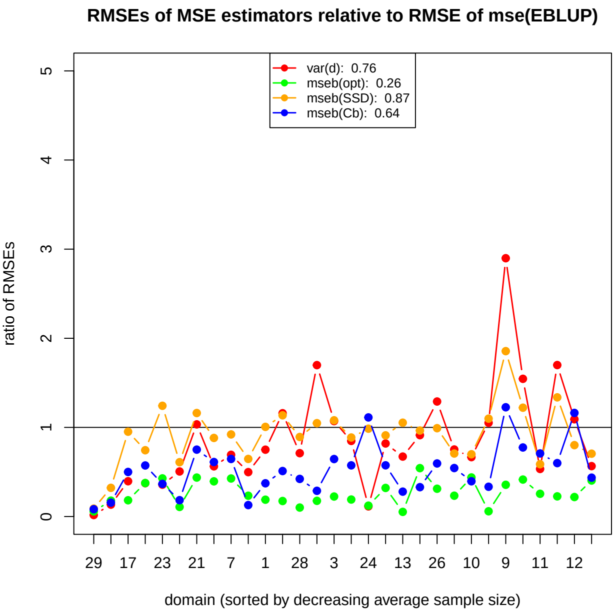
Estimation of MSEs for unemployed



Estimation of employed



Estimation of MSEs for employed



Conclusions

- The proposed adaptive design-based composite estimators provide viable alternatives to the model-based EBLUP, as demonstrated in the presented simulation study and observed in other experiments.
- These compositions are applicable to direct and synthetic estimators based on unit-level auxiliary data and can be used to estimate various domain parameters.
- The proposed design-based MSE estimator is well-suited for any design-based composition aimed at estimating the optimal one. This estimator is simple and ensures non-negative values.

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