Exercise 7

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1 Connection between Stochastic Differential Equations and Fokker-Planck equation

Pen-and-paper exercise Consider the one dimensional *Ito* Stochastic Differential Equation (SDE)

$$dx_t = f(x_t, t)dt + \sigma(x_t, t)dW_t \,, \tag{1}$$

and the equivalent *Stratonovich* SDE

$$dx_t = f(x_t, t)dt - \frac{1}{2}\frac{\partial \sigma}{\partial x}\sigma(x_t, t)dt + \sigma(x_t, t) \circ dW_t \,, \tag{2}$$

where $f, \sigma : \mathbb{R} \to \mathbb{R}$ and W_t is a Wiener process.

Pen-and-paper exercise

1. Derive the Fokker-Planck equation corresponding to the Ito SDE defined in eq. 1.
2. Derive the Fokker-Planck equation corresponding to the Stratonovich SDE defined in eq. 1. **Hint:**
   (a) Introduce a test function $g(x)$ and derive the Ito's formula

$$dg_t = \frac{dg}{dx}f(x_t, t)dt + \frac{1}{2}\frac{d^2 g}{dx^2}\sigma(x_t, t)^2 dt + \frac{dg}{dx}\sigma(x_t, t)dW_t \,. \tag{3}$$

   (b) Use the differential rules:

$$dW_t^2 = dt \,; \quad dt^2 = 0 \,; \quad dW_t dt = 0 \,. \tag{4}$$

2 The Ornstein-Uhlenbeck process

Consider the Ito SDE of the Ornstein-Uhlenbeck process

$$dx_t = -\frac{dV(x_t)}{dx}\, dt + \sigma(x_t)\, dW_t \,, \tag{5}$$

where

$$V(x) = \frac{1}{2}k(x - x_0)^2 \,, \tag{6}$$

is the potential energy function of the harmonic oscillator and the position-dependent volatility is written as

$$\sigma(x) = \sqrt{\frac{2k_B T}{m\gamma(x)}} \tag{7}$$

with

$$\gamma(x) = \frac{1}{\arctan(x) + \frac{3}{2}} \,. \tag{8}$$

Pen-and-paper exercise Find the analytical solution of eq. 5.

Computational exercise Simulate the Ornstein-Uhlenbeck process using the following integration schemes based on Ito and Stratonovich rules.

2.1 Ito integration scheme

Consider a time interval $[0, \tau]$, and a discretization into N equal sub-intervals $\Delta t = t_{k+1} - t_k$, with $t_k = 0$ and $t_N = \tau$. The SDE (eq. 1) is discretized, according to Ito, as

$$x_{k+1} = x_k + f(x_k, t_k)\Delta t + \sigma(x_k, t_k)[W_{k+1} - W_k]\,. \tag{9}$$

Note that $\sigma(x_t, t)$ is estimated at time t_k, i.e. at the left point of the interval $[t_k, t_{k+1}]$ as required by Ito. The increments $W_{t+\Delta t} - W_t$ are drawn from a Gaussian distribution with mean 0 and variance Δt:

$$W_{t+\Delta t} - W_t \sim \mathcal{N}(0, \Delta t)\,, \tag{10}$$

then the increments $W_{t+\Delta t} - W_t$ can be approximated as

$$W_{t+\Delta t} - W_t \sim \mathcal{N}(0, 1)\sqrt{\Delta t} \sim \eta\sqrt{\Delta t}\,, \tag{11}$$

with η random number drawn from a standard Gaussian distribution with mean 0 and variance 1. This integration scheme is called *Euler-Maruyama integration scheme.*

2.2 Stratonovich integration scheme

The Stratonovich SDE (eq. 2) can be discretized as

$$x_{k+1} = x_k + f(x_k, t_k)\Delta t - \frac{1}{2}\frac{\partial\sigma}{\partial x}\sigma\left(\frac{x_{k+1} + x_k}{2}, \frac{t_{k+1} + t_k}{2}\right)\Delta t + \sigma(x_k, t_k)[W_{k+1} - W_k]\,. \tag{12}$$

This equation cannot be directly solved as the term x_{k+1} appears on both the sides. Here, we use a predictor-corrector method:

1. We first obtain the an approximation of $\tilde{x}_{k+1}$ with the Euler-Maruyama integration scheme (prediction)

$$\tilde{x}_{k+1} = x_k + f(x_k, t_k)\Delta t - \frac{1}{2}\frac{\partial\sigma}{\partial x}\sigma\left(x_k, t_k\right)\Delta t + \sigma(x_k, t_k)[W_{k+1} - W_k] \tag{13}$$

2. We use the predicted x_{k+1} in a Stratonovich integration step (correction)

$$x_{k+1} = x_k + f(x_k, t_k)\Delta t - \frac{1}{2}\frac{\partial\sigma}{\partial x}\sigma\left(\frac{\tilde{x}_{k+1} + x_k}{2}, \frac{t_{k+1} + t_k}{2}\right)\Delta t + \sigma(x_k, t_k)[W_{k+1} - W_k]\,. \tag{14}$$

The increments $W_{t+\Delta t} - W_t$ are approximated by η random numbers drawn from a standard Gaussian distribution with mean 0 and variance 1. This integration scheme is called *Heun integration scheme.*