Baire category theorem and its consequences

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Abstract

In this notes we talk about the Baire category theorem and its consequences: the Banach-Steinhaus theorem, the open mapping theorem and the closed graph theorem.

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1 Introduction

The notion of what it means for a subset E of a space X to be "small" varies from context to context. In measure theory, when $X = (X, \mu)$ is a measure space, one useful notion of a "small" set is that of a null set: a set E of measure zero. By countable additivity, countable unions of null sets are null. Taking contrapositives, we obtain the following Lemma:

Lemma 1.1 (Pigeonhole principle for measure spaces)**.** *Let $E_1, E_2, \ldots$ be a countable sequence of measurable subsets of a measure space X. If $\bigcup_{n=1}^{\infty} E_n$ has positive measure, then at least one of the E_n has positive measure.*

Now suppose that X was a Euclidean space $\mathbb{R}^d$ with Lebesgue measure λ. One can prove that having positive measure is equivalent to being "dense" in certain balls:

Proposition 1.2. *Let E be a measurable subset of $\mathbb{R}^d$. Then the following are equivalent:*

1. *E has positive measure.*
2. *For any $\alpha < 1$, there exists a ball B such that $\lambda(E \cap B) > \alpha\lambda(B)$.*

Proof. It is clear that 2) implies 1). To prove the converse suppose then that there is an $\alpha < 1$ such that $\lambda(E \cap B) \leq \alpha\lambda(B)$ for any ball B. By definition of Lebesgue measure there are disjoint open

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intervals I_n such that $\bigcup_{n=1}^{\infty} I_n \supseteq E$ and $\lambda(\bigcup_{n=1}^{\infty} I_n) < \lambda(E) + \varepsilon$. By assumption on α we have that $\lambda(E \cap I_n) \leq \alpha\lambda(I_n)$ for all n. Hence

$$\begin{aligned}\lambda(E) &= \lambda\left(E \cap \bigcup_{n=1}^{\infty} I_n\right) \\ &= \sum_{n=1}^{\infty} \lambda(E \cap I_n) \\ &\leq \alpha \sum_{n=1}^{\infty} \lambda(I_n) \\ &= \alpha\lambda\left(\bigcup_{n=1}^{\infty} \lambda(I_n)\right) \\ &\leq \alpha\left(\lambda(E) + \varepsilon\right).\end{aligned}$$

By squeezing ε, we get $\lambda(E) \leq \alpha\lambda(E)$ and therefore, since $\alpha < 1$, we have $\lambda(E) = 0$. $\square$

Thus one can think of a null set as a set which is "nowhere dense" in some measure-theoretic sense. It turns out that there are analogues of these results when the measure space $X = (X, \mu)$ is replaced by a complete metric space $X = (X, \rho)$. Here, the appropriate notion of a "small" set is not a null set any more, but rather that of a nowhere dense set:

Definition 1.3. *A set S in a metric space X is nowhere dense if $\overline{S}$ has an empty interior.*

We then have the following important result, which we can loot at as the topological counterpart of Lemma 1.1:

Theorem 1.4 (Baire category theorem, version 1)**.** *Let (X, ρ) be a complete metric space, and let $C_1, C_2, \ldots$ be a countable collection of closed subsets of X such that $\bigcup_{n=1}^{\infty} C_n = X$. Then at least one of the E_n contains an open ball, i.e., there exist n, $x \in X$, and $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq C_n$.*

In this notes, we present in Section 2 the proof of the Baire category theorem and the different forms it takes. In Section 3, we derive some important results as consequence of the Baire category theorem: the Banach-Steinhaus theorem, the open mapping theorem and the closed graph theorem. We assume the reader is familiar with the concepts of metric, Banach and Hilbert spaces.

2 Baire category theorem

In this section we present the Baire category theorem in three different forms. We start with the proof of the one presented in the previous Section:

Proof of Theorem 1.4. We argue by contradiction. If none of the C_n contains an open ball, we are able to construct a Cauchy sequence $\{x_n\}$ which stays away from each C_n so that its limit point x doesn't belong to any C_n, thereby contradicting the statement $\bigcup_{n=1}^{\infty} C_n = X$.

Suppose then that each C_n does not contain any open ball. This means that any open ball B in X contains a point from $X \setminus C_n$, and so $B \cap (X \setminus C_n)$ contains a nontrivial closed ball because $X \setminus C_n$ is open.

Applying this with B equal to a ball of radius 1, we obtain $x_1 \in X$ and $r_1 \in (0, 1)$ such that $\overline{B_{r_1}(x_1)} \subseteq X \setminus C_1$. Applying now the result to $B_{r_1}(x_1)$ allows us to obtain $x_2 \in X$ and $r_2 \in (0, \frac{1}{2})$ such that $\overline{B_{r_2}(x_2)} \subseteq B_{r_1}(x_1) \cap (X \setminus C_2)$. By iterating this process, we get a sequence of balls $B_{r_n}(x_n)$ such that $r_n \in (0, \frac{1}{n})$ and $\overline{B_{r_n}(x_n)} \subseteq B_{r_{n-1}}(x_{n-1}) \cap (X \setminus C_n)$.

By construction we have $x_n \in \overline{B_{r_N}(x_N)}$ for $n > N$ and therefore for $n, m > N$

$$\rho(x_n, x_m) \leq \rho(x_n, x_N) + \rho(x_m, x_N) \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} \to 0$$

as $N \to \infty$. Hence $\{x_n\}$ is Cauchy, and by completeness of X, there is $x \in X$ such that $x_n \to x$ in X. By closedness, we have $x \in \overline{B_{r_n}(x_n)}$ for all n, but $\overline{B_{r_n}(x_n)} \subseteq X \setminus C_n$ and therefore $x \notin C_n$ for all n, thus contradicting the hypothesis $\bigcup_{n=1}^{\infty} C_n = X$. □

The word "category " comes from the following terminology used by R. Baire:

Definition 2.1. *Any countable union of nowhere dense sets is called a set of first category. All other subsets are of second category (Baire's terminology).*

We can then restate version 1 of the Baire category theorem:

Theorem 2.2 (Baire category theorem, version 2)**.** *A complete metric space is of second category, i.e., is never the union of nowhere dense sets.*

Proof. Suppose X is the complete metric space and $X = \bigcup_{n=1}^{\infty} S_n$ where the S_n are nowhere dense. Applying the Baire category theorem (version 1) to the sets $C_n = \overline{S_n}$ shows that one of the $\overline{S_n}$ must contain an open ball which is a contradiction since all S_n are nowhere dense. □

Finally, we can also restate the Baire category theorem in the the following way:

Theorem 2.3 (Baire category theorem, version 3)**.** *If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.*

Proof. It is enough to take complements in version 2 of the Baire category theorem. □

In practice, one rarely uses the Baire category theorem directly but rather one of the consequences we present in the next Section. We should point out the the concept of Baire category has proven to be one of the basic tools of analysis ans is especially important in infinite dimensional spaces.

3 Consequences of the Baire category theorem

The Baire category theorem leads to three fundamental results:

1. Banach-Steinhaus theorem,
2. open mapping theorem,
3. closed graph theorem.

We start by proving the uniform boundedness principle which we will then use to prove the Banach-Steinhaus theorem.

Theorem 3.1 (Uniform boundedness principle)**.** *Let X be a complete metric space, and let F be a collection of continuous functions $f : X \to [0, \infty)$ such that*

$$\sup_{f \in F} f(x) < \infty$$

for each $x \in X$. Then there is a nonempty open set $B \subseteq X$ such that

$$\sup_{x \in B} \sup_{f \in F} f(x) < \infty.$$

Proof. The sets

$$C_n = \bigcap_{f \in F} \{x \in X : f(x) \leq n\},$$

are closed by continuity of each $f \in F$ and $X = \bigcup_{n=1}^{\infty} C_n$ since $\sup_{f \in F} f(x) < \infty$ for each $x \in X$. So by the Baire category theorem (version 1) at least one of the C_n contains an open ball and we are done. □

Remark 3.2. *The uniform boundedness principle tells us that pointwise boundedness of continuous functions on a complete metric space implies uniform boundedness on a nonempty open set.*

Theorem 3.3 (Banach-Steinhaus). *Let X and Y be a Banach and normed spaces, respectively, and let $\mathcal{T}$ be a collection of bounded linear operators $T : X \to Y$ such that*

$$\sup_{T\in\mathcal{T}} \|Tx\| < \infty$$

for each $x \in X$. Then we have

$$\sup_{T\in\mathcal{T}} \|T\| < \infty$$

Proof. For each $T \in \mathcal{T}$, define the function $f_T : X \to [0, \infty)$ by $f_T(x) = \|Tx\|$. Applying Theorem 3.1 allows us to conclude that there is a ball $B_\varepsilon(x_0)$ with $\varepsilon > 0$ and $x_0 \in X$ such that

$$\alpha := \sup_{x\in B_\varepsilon(x_0)} \left(\sup_{T\in\mathcal{T}} \|Tx\| \right) < \infty.$$

Now if $x \in B_\varepsilon(0)$, we can bound $\|Tx\|$ by using the triangle inequality in the following way:

$$\|Tx\| = \|T(x_0 + x) - Tx_0\| \le \|T(x_0 + x)\| + \|Tx_0\| \le 2\alpha$$

Finally, for arbitrary $x \in X$, a simple scaling argument yields

$$\|Tx\| = \frac{2\|x\|}{\varepsilon} \left\| T\left(\frac{\varepsilon x}{2\|x\|}\right) \right\| \le \frac{4\alpha}{\varepsilon}\|x\|,$$

from where we deduce that $\|T\| \le 4\alpha/\varepsilon$ independent of $T \in \mathcal{T}$. □

Remark 3.4. *The Banach-Steinhaus theorem tells us that pointwise boundedness of linear operators on a Banach space implies uniform boundedness.*

As a typical application of this Theorem we have the following Corollary.

Corollary 3.5. *Let X and Y be Banach spaces and let $B(\cdot, \cdot)$ be a separately continuous bilinear mapping from $X \times Y$ to $\mathbb{C}$, that is, for each fixed $x \in X$, $B(x, \cdot)$ is a bounded linear transformation, and for each fixed $y \in Y$, $B(\cdot, y)$ is a bounded linear transformation. Then $B(\cdot, \cdot)$ is jointly continuous, that is, if $x_n \to 0$ and $y_n \to 0$ then $B(x_n, y_n) \to 0$.*

Proof. Let $T_n(y) = B(x_n, y)$. Since $B(x_n, \cdot)$ is continuous, each T_n is bounded. Moreover, since $x_n \to 0$ and $B(\cdot, y)$ is bounded, $\{\|T_n(y)\|\}$ is bounded for each fixed y. We can the apply the Banach-Steinhaus theorem to the collection $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$, which tells us that there exists a constant $C > 0$ such that

$$\|T_n(y)\| \le C\|y\|$$

for all n. Thus

$$\|B(x_n, y_n)\| = \|T_n(y_n)\| \le C\|y_n\| \to 0$$

as $n \to \infty$ since $y_n \to 0$. □

We now proceed to proving the open mapping theorem. We start with a definition and simple proposition.

Definition 3.6. *A mapping is called open if it sends open sets to open sets.*

Proposition 3.7. *An open linear mapping $T : X \to Y$ is surjective.*

Proof. Let V be an open neighbourhood of $0 \in X$. Then, since T is open and linear, $U = T(V)$ is an open neighbourhood of $0 \in Y$. Let $y \in Y$. Then there is $\alpha \ne 0$ such that $\alpha y \in U$. Hence, since $U = T(V)$, there is $x \in X$ such that $Tx = \alpha y$. Finally, since T is linear, we can rewrite this as $T(\frac{1}{\alpha}x) = y$ and so we are done. □

The following theorem, known as open mapping theorem, tells us that the converse of the above proposition is also true when the two spaces are complete.

Theorem 3.8 (Open mapping theorem)**.** *Let $T : X \to Y$ be a bounded linear operator between two Banach spaces. Then T is surjective if and only if it is open.*

Proof. Suppose T is surjective. We need to show that for every neighbourhood V of $x \in X$, $T(V)$ is a neighbourhood of Tx. Since $T(x + V) = T(x) + T(V)$, we only need to show this for a particular $x \in X$. Moreover, since neighbourhood contain balls it is sufficient to show that for all $r > 0$ $B_{r'}(x) \subseteq T(B_r(x))$ for some $r' > 0$. Additionally, $T(B_r(x)) = rT(B_1(x))$ and therefore it is enough to show for some $r > 0$.

Since by assumption T is surjective, $Y = \bigcup_{n=1}^{\infty} \overline{T(B_n(0))}$ and hence by the Baire category theorem (version 1), there is a nonempty ball $B_\delta(y) \subseteq Y$ and $n \geq 1$ such that $B_\delta(y) \subseteq \overline{T(B_n(0))}$. By choosing $x \in X$ such that $y = Tx$ and $\varepsilon > 0$ so large that $B_\varepsilon(x) \supseteq B_n(0)$, we can guarantee $B_\delta(Tx) \subseteq \overline{T(B_\varepsilon(x))}$. By linearity, with $\alpha = \delta/\varepsilon$ we have $B_{\alpha r}(Tx) \subseteq \overline{T(B_r(x))}$ for all $x \in X$ and all $r > 0$. If the inclusion did not have the closure in the right hand side, this statement is exactly what we wanted.

Now we shall remove the closure. Let $x \in B_{\alpha r}(Tx)$, and fix some $\varepsilon \in (0, 1)$. Then there is $x_0 \in B_r(x)$ such that $\|z - Tx_0\| < \alpha\varepsilon$, which implies that $z \in B_{\alpha\varepsilon}(T(x_0) \subseteq \overline{T(B_\varepsilon(x_0))}$. This means that there is $x_1 \in B_\varepsilon(x_0)$ such that $\|z - Tx_1\| < \alpha\varepsilon^2$. Iterating this process allows us to get a sequence $\{x_n\}$ in X satisfying $\|x_n - x_{n-1}\| < \varepsilon^n$ and $\|z - Tx_n\| < \alpha\varepsilon^n$ for $n \in \mathbb{N}$. From that first property we deduce that $\{x_n\}$ is a Cauchy sequence and since X is complete, there is $x_* \in X$ such that $x_* = \lim x_n$. From the same property we have that

$$
\begin{aligned}
\|x_* - x\| &\leq \|x_* - x_n\| + \|x_n\| \\
&\leq \|x_* - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - x\| \\
&\leq \|x_* - x_n\| + \varepsilon^n + \|x_{n-1} - x\| \\
&\leq \|x_* - x_n\| + \sum_{i=1}^{n} \varepsilon^i + \|x_0 - x\| \\
&\leq \|x_* - x_n\| + \varepsilon\frac{1 - \varepsilon^n}{1 - \varepsilon} + r.
\end{aligned}
$$

By letting $n \to \infty$, we get that $\|x_* - x\| < r_* := r + \varepsilon/(1 - \varepsilon)$. From the second property of the sequence $\{x_n\}$ we have $z = Tx_*$. Hence $B_{\alpha r}(Tx) \subseteq T(B_{r_*}(x))$. By squeezing ε we can get the result with $r_* = r$, but what we have is already sufficient for establishing the theorem. □

Since continuity is equivalent to boundedness for linear operators we have the following Corollary.

Corollary 3.9 (Inverse mapping theorem)**.** *Let $T : X \to Y$ be an invertible bounded linear operator between two Banach spaces. Then the inverse $T^{-1} : Y \to X$ is bounded.*

Proof. The open mapping theorem implies that if the inverse T^{-1} exists, then it must be continuous. Since continuity is equivalent to boundedness for linear operators on normed spaces, we are done. □

Definition 3.10. *Given a map $T : X \to Y$ we define its graph to be the set*

$$\operatorname{graph}(T) = \{(x, Tx) : x \in X\} \subseteq X \times Y.$$

Suppose that (X, ρ) and (Y, σ) are complete metric spaces and equip $X \times Y$ with the metric $\rho + \sigma$. If T is continuous, obviously $\operatorname{graph}(T)$ is closed, since $(x_n, Tx_n) \to (x, y)$ in $X \times Y$ implies $y = Tx$. In the linear world the conserve is also true.

Theorem 3.11 (Closed graph theorem)**.** *Let $T : X \to Y$ be a linear operator between two Banach spaces. Then T is bounded if and only if its graph is closed.*

Proof. Suppose the graph(T) is closed in $X \times Y$, i.e., that it is a Banach space. Define the two (projection) operators $\pi_1 : \text{graph}(T) \to X$ and $\pi_2 : \text{graph}(T) \to Y$ by $\pi(x, y) = x$ and $\pi_2(x, y) = y$, respectively. It is clear that both operators are bounded linear, and that π_1 is invertible. Then by the inverse mapping theorem, π_1^{-1} os bounded. Since we have $T = \pi_2\pi_1^{-1}$, we are done. □

Remark 3.12. *Consider the three following statements:*

a) x_n *converges to some element* x.

b) Tx_n *converges to some element* y.

c) $Tx = y$.

A priori to prove that T *is continuous one must show that* a) *implies* b) *and* c). *What the close graph say is that it is sufficient to prove that* a) *and* b) *imply* c).

The following Corollary has important consequences in Mathematical Physics.

Corollary 3.13 (Hellinger-Toeplizt theorem)**.** *Let* T *be a everywhere defined linear operator on a Hilbert space* H *with* $\langle x, Ty\rangle = \langle Tx, y\rangle$ *for all* $x, y \in H$. *Then* T *is bounded.*

Proof. We will prove that graph(T) is closed and the result follows from the closed graph theorem. Suppose that $(x_n, Tx_n) \to (x, y)$. We need only to prove that $(x, y) \in graph(T)$, that is that $y = Tx$. But, for any $z \in H$

$$\begin{aligned} \langle z, y\rangle &= \lim_{n\to\infty} \langle z, Tx_n\rangle \\ &= \lim_{n\to\infty} \langle Tz, x_n\rangle \\ &= \langle Tz, x\rangle \\ &= \langle z, Tx\rangle \end{aligned}$$

Thus $y = Tx$ and graph(T) is closed. □

This theorem is the cause of much technical pain because in quantum mechanics there are operators (like the energy) which are unbounded but which we want to obey $\langle x, Ty\rangle = \langle Tx, y\rangle$ in some sense. The Hellinger-Toeplizt theorem tells us that such operators cannot be everywhere defined. Thus such operators are defined on subspaces $D(T)$ of H and defining what one means by $T + S$ or TS may be difficult. For example $T + S$ is a priori only defined in $D(T) \cap D(S)$ which may by equal $\{0\}$ even in the case where both $D(T)$ and $D(S)$ are dense.

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