RESEARCH STATEMENT

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My main interest is in numerical methods for degenerate elliptic partial differential equations (PDEs), with emphasis on monotone schemes. These methods have been highly successful in building convergent schemes for the Monge-Ampère equation, including the case of Optimal Transportation [BFO14], which suggest they could be extended to more general geometric and optimal transportation type PDEs. During my PhD, I studied viscosity solutions and numerical methods for Hamilton-Jacobi equations (with special focus on the eikonal equation), the three dimensional 2-Hessian equation, and the affine curvature flow, under the supervision of Professor Adam Oberman. My work, building on the foundation of monotone schemes, has thus been in widening the class of equations covered and extending the theory and accuracy of the methods.

The theory of Barles and Souganidis [BS91] and its extension by Oberman [Obe06] are the foundations of my research. In short, it tells us that monotone, stable and consistent schemes converge to the unique viscosity solution of an elliptic PDE. I give a brief overview of this in Section 1. This result makes monotone schemes very desirable. However, the accuracy is less than ideal is some situations. To address this, filtered schemes were introduced in [FO13] in the context of the Monge-Ampère equation, where the goal was to overcome the reduction in accuracy based on the wide-stencil monotone scheme.

In my work, I apply filtered schemes to Hamilton-Jacobi equations. The idea is to blend a stable monotone convergent scheme with an accurate scheme and retain the advantages of both: stability and convergence of the former, and higher accuracy of the latter. The only requirement on the accurate scheme is consistency, thus allowing for a wide range of possibilities, e.g. ENO and WENO schemes in the specific context of Hamilton-Jacobi equations. I introduced a simpler construction using high order upwind schemes, which can be seen as a generalization of the first order monotone schemes. Due to their explicit nature and the particular construction of the filtered scheme, I was able to use a simple but effective fast sweeping method to compute the solutions and achieve high order accuracy in the smooth regions of the solution. Moreover, in the particular case of the one-dimensional eikonal equation, I proved high order of convergence even for non-smooth solutions. I will elaborate on this project in Section 2. This was a joint work with Professor Oberman, which resulted in a paper in a leading Numerical Analysis journal [OS15].

The 2-Hessian equation is a fully nonlinear PDE which is elliptic provided the solutions are restricted to a convex cone, which I called plane-subharmonic. I gave two different discretizations for the 2-Hessian equation in the three-dimensional case: a naive one obtained by simply using standard finite differences to discretize the Hessian and a monotone discretization that takes advantage of a characterization of the operator using a matrix inequality. The monotone discretization is provably convergent, but less accurate because it required the use of a wide stencil. More details can be seen in Section 3. The full details of this joint work with Professor Oberman and Professor Froese are available in [FOS16].

My latest project focused on building monotone schemes for the nonlinear PDE that governs the planar motion of level sets by affine curvature. It is closely related to mean curvature but exhibits instabilities not found in the latter. These instabilities and lack of regularity of the affine curvature operator posed unexpected and additional difficulties in building monotone schemes. A standard finite difference scheme is proposed and an example that illustrates its nonlinearly instability is given. We build convergent finite difference schemes using the theory of viscosity solutions. A preprint with the full details is also available in [OS16]. This was joint work with my Professor Oberman.

In Section 5, I briefly discuss my current research, a few projects I worked on during my PhD, and some projects I wish to work on in the future.

In my Master's thesis, I studied viscosity solutions and the theory of Barles and Souganidis discussed here in Section 1. The main contribution was a new proof for the rate of convergence of numerical schemes for the one dimensional time dependent Hamilton-Jacobi equation (a well known result in the literature) using the adjoint method.

Date: November 12, 2016.

1. On the convergence of monotone schemes

The convergence of the finite difference schemes proposed in my work relies on the framework developed by Barles and Souganidis [BS91] and on the extension provided by Oberman [Obe06].

The main result in [BS91] provides us with sufficient conditions for the convergence of approximation schemes to the unique viscosity solution of a PDE.

Theorem 1.1. *Consider an elliptic PDE that satisfies a comparison principle. A consistent, stable and monotone approximation scheme converges locally uniformly to the (unique) viscosity solution.*

Despite the clear requirements of the theory, building monotone schemes is a challenge since the definition does not provide any insight on how to build such schemes. Moreover, conditions to ensure monotonicity are different for first and second order equations, and for explicit and implicit schemes. The work in [Obe06] provides a unifying and easy-to-check definition.

Definition 1.2. *Given a finite difference scheme of the form*

$$F^h[u] = F^h(u_i, u_{j\in N(i)} - u_i)$$

where $N(i)$ is the list of neighbours of u_i, we say that F^h is elliptic if F is nondecreasing in each variable.

The following theorem, which can be found in [Obe06], addresses the equivalence of monotonicity and degenerate ellipticity.

Theorem 1.3. *A scheme is monotone and nonexpansive in the l^∞ norm if and only if it is degenerate elliptic.*

From the definition of ellipticity, the finite difference method emerges as the natural method to build elliptic schemes. In general, elliptic schemes are built by composing non-decreasing maps with elliptic terms (an example is the work described in Section 4). Moreover, for some nonlinear elliptic PDEs, the domain of ellipticity is restricted and thus we need to build non-decreasing extensions of the underlying functions (see Section 3 for an example).

The theory can be extended to parabolic PDEs of the form $u_t + F[u] = 0$. In [Obe06] it is shown that the Euler map $u \to u - \rho F^h[u]$ with constant $\rho \leq 1/K^h$ is monotone, and a (non-strict) contraction, where K^h is the Lipschitz constant of the elliptic scheme F^h. By adding an arbitrarily small multiple of u to the scheme, the comparison principle holds, and the Euler map is a strict contraction. The resulting scheme provides a convergent discretization of the solution of the parabolic PDE $u_t + F[u] = 0$, since the discretization is monotone.

2. Filtered schemes for Hamilton-Jacobi equations

In my work in [OS15], I considered Hamilton-Jacobi equations of the form

$$\begin{cases} H(x, \nabla u) = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \Gamma, \end{cases} \tag{1}$$

where ∇u is the gradient of the function u, Ω is an open set, Γ is the boundary of Ω, and the Hamiltonian H is a nonlinear Lipschitz continuous function. Hamilton-Jacobi equations appear in many applications, such as optimal control, differential games, image processing, computer vision, and geometric optics. In particular, I was interested in the special case of the eikonal equation ($H(x,p) = |p|$). As pointed out in [BLZ10], high order schemes are particularly important in high frequency wave propagation, where the eikonal equation is coupled with a transport equation through its gradient [QS99, SVST94].

I proposed the following filtered schemes

$$F^h[u] = \begin{cases} F^h_A[u], & \text{if } \left|F^h_A[u] - F^h_M[u]\right| \leq \sqrt{h}, \\ F^h_M[u], & \text{otherwise}, \end{cases} \tag{2}$$

where F^h_M denotes the monotone discretization of the operator on the grid with spacing h and F^h_A denotes an accurate discretization of the same operator. By construction, the filtered scheme is no longer monotone, but it is still consistent provided both underlying schemes are consistent. More importantly, it is almost monotone given that

$$F^h[u] = F^h_M[u] + \mathcal{O}(h^{1/2}).$$

It is based on this property that the proof in [BS91] can be modified to include the filtered schemes as the term $\mathcal{O}(h^{1/2})$ can be absorbed into the truncation error. This results in the following convergence theorem, which was proved in [FO13] in a more general setting.

Theorem 2.1. *Let u be the unique viscosity solution of (1). For each $h > 0$, let u^h be a stable solution of $F^h[u] = 0$, where the filtered scheme F^h is given by (2) and F^h_M is consistent and monotone. Then $u^h \to u$ locally uniformly as $h \to 0$.*

The only requirement of the theorem is the existence of stable solutions for the filtered schemes, which was proven in [FO13] for a slightly different form of the filtered scheme. Instead of (2), a continuous interpolation between the monotone and accurate scheme was used. This was required for the continuity argument in the proof of existence, and it was also of practical use for a Newton solver. Although discontinuous here, (2) has a simpler form that allows for explicit solution formulas. Consequently fast sweeping solvers were built, which are appropriate for Hamilton-Jacobi equations. For the purpose of the proof, a continuous filter is needed; however the practical advantages of the discontinuous filter outweigh the lack of rigor. In practice, the computational results are as good as could be expected.

Theorem 2.1 does not provide any information regarding the convergence rate. Proving higher order convergence requires additional efforts and is only possible in specific settings. In particular, for the one-dimensional eikonal equation, I proved higher order convergence for high order upwind schemes (without filtering). It is important to emphasize that this result holds even for non-smooth (but continuous) solutions. However, in the general case of Hamilton-Jacobi equations such a result, even with filtering, is neither available nor expected. Indeed, high order convergence was only observed numerically in smooth regions: solutions of (1) are typically piecewise smooth and so, in two dimensions, second order convergence rates were achieved in the smooth regions, but only first order overall convergence in the l^∞ norm.

Looking forward, I believe filtered schemes can be adapted to time dependent equations. This can be accomplished by using the filtered scheme of the spatial part of the operator and a standard time discretization (forward Euler or strong stability preserving time discretizations) for the time derivative. As needed, the filter can be applied to the time derivative as well. In this case, with minor modifications, the proof of convergence for the filtered scheme follows similarly by taking the time component to be an additional variable, a standard technique in viscosity solutions theory. This was recently done for time-dependent Hamilton-Jacobi equations [BFS16] and in front propagation [Sah16].

The filtered schemes discussed here are simple and easy to implement on Cartesian grids. However, the framework can be applied in a more general setting. An example of future work could then consider the popular discontinous Galerkin method together with a monotone scheme in the same setting. Recently, they were combined with monotone meshfree schemes for approximating functions of the eigenvalues of the Hessian in [Fro15].

3. Numerical Methods for the 2-Hessian Elliptic Partial Differential Equation

In this work I studied numerical approximations of the 2-Hessian equation in three dimensions, a fully nonlinear PDE of the form

$$S_2[u] = \sigma_2(\lambda(D^2u)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \tag{2H}$$

where $\lambda(D^2u) = (\lambda_1, \lambda_2, \lambda_3)$ denotes the eigenvalues of D^2u. This is a particular instance of a much larger k-Hessian family of PDEs in n-dimensional space, that include the Laplace equation and the Monge-Ampère equation.

Geometric PDEs have been proven to be especially useful in image analysis [Sap06a]. In particular, the Monge-Ampère equation in the context of Optimal Transportation has been used in three dimensional volume based image registration [HZTA04]. The 2-Hessian equation is closely related to the scalar curvature operator, which provides an intrinsic curvature for a three dimensional manifold. Thus, one would expect that scalar curvature equations would have been used in these contexts, which is not the case. Reasons for this may include a lack of effective PDE solvers for this operator. The 2-Hessian operator also appears in conformal mapping problems. Conformal surface mappings have been used for two dimensional image registration [AHTK99, GWC$^+$04], but they do not generalize directly to three dimensions. Quasi-conformal maps have been used in three dimensions [WWJ$^+$07, ZG11], however these methods are still being developed.

In [FOS15], I focused on the Dirichlet problem

$$\begin{cases} S_2[u] = f, & \text{in } \Omega,\\ u = g, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a rectangular (three dimensional box) domain, which is natural for prescribed curvature problems. For other geometries, different boundary conditions need to be used. For the torus, periodic boundary conditions could be used. For the sphere, it is more complicated, but it is possible to patch together several cubic domains to obtain this topology.

When working with the 2-Hessian equation one of the difficulties comes from the fact that the operator is not elliptic, unless an additional constraint is imposed. By assuming $f > 0$, this constraint is reduced to $\lambda(D^2u) \in \Gamma$ where

$$\Gamma = \left\{ \lambda \in \mathbb{R}^3 \mid \lambda_1 + \lambda_2 > 0,\ \lambda_1 + \lambda_3 > 0,\ \lambda_2 + \lambda_3 > 0 \right\}.$$

Informally, we are requiring that the Laplacian restricted to every two-dimensional plane is positive. Hence the discretizations of the operator must also enforce this additional constraint. This means that either we are working with a family of inequality constraints, which makes the discretization very challenging, or we need to find a way to encode the constraints in the PDE itself. I pursued the second option for the monotone discretization.

I showed that the naive finite difference scheme together with a parabolic solver fails in general. I introduced a monotone discretization, which is provably convergent using the theory of Barles and Souganidis discussed in Section 1. In addition, and despite being less accurate, it has the advantage of yielding a globally consistent monotone discretization of the operator, meaning that we can apply it to non-admissible functions. This is useful because it circumvents the need for special initial data, and allows for the parabolic (time-dependent) operator to be defined on an unconstrained class of functions. In order to achieve this, the first step was to find a non-decreasing extension of σ_2 from Γ to $\mathbb{R}^3$ which was accomplished in the following lemma.

Lemma 3.1. *The function $\bar{\sigma} = f \circ \mathrm{sort}$ where* sort *denotes the sorting function and f is given by*

$$f(x, y, z) = x \max(y, |x|) + x \max(z, |x|) + \max(y, |x|) \max(z, |x|)$$

extends σ_2 on Γ and is non-decreasing in $\mathbb{R}^3$.

The second step consisted of using wide stencils and a rotated coordinated system in which the Hessian matrix becomes diagonal. Consequently, provided the coordinate system is found in a monotone way, we are able to construct a monotone discretization of the 2-Hessian operator. This was accomplished with the following result.

Lemma 3.2. *Let V be the set of all orthonormal bases of $\mathbb{R}^3$, i.e., $V = \left\{ (\nu_1, \nu_2, \nu_3) | \nu_i \in \mathbb{R}^3, \nu_i \perp \nu_j \text{ if } i \neq j, \|\nu_i\|_2 = 1 \right\}$. Then for $u \in C^2(\Omega)$*

$$S_2[u] = \min_{(\nu_1,\nu_2,\nu_3) \in V} \sigma_2 \left(\frac{\partial^2 u}{\partial \nu_1^2}, \frac{\partial^2 u}{\partial \nu_2^2}, \frac{\partial^2 u}{\partial \nu_3^2} \right).$$

Combining the above two Lemmas, the monotone discretization was then obtained:

$$S_2^M[u] = \min_{(\nu_1,\nu_2,\nu_3) \in \mathcal{G}} \bar{\sigma} \left(\mathcal{D}_{\nu_1\nu_1} u, \mathcal{D}_{\nu_2\nu_2} u, \mathcal{D}_{\nu_3\nu_3} u \right), \tag{$(2H)^M$}$$

where $\mathcal{D}_{\nu\nu}$ is the finite difference operator for the second directional derivative in the direction ν, being given by

$$\mathcal{D}_{\nu\nu} u(x_i) = \frac{1}{|\nu|^2 h^2} (u(x_i + h\nu) + u(x_i - h\nu) - 2u(x_i)),$$

and $\mathcal{G}$ is the set of orthogonal vectors that include the finite number of possible directions ν that lie on the grid. This leads to the introduction of a spatial resolution, h, due to the grid spacing and a directional resolution, $d\theta$, resulting from the restriction of ν to the set of directions available on the grid. I proved the following consistency result.

Lemma 3.3. *Let $x_0 \in \Omega$ be a reference point on the grid and ϕ be a twice continuously differentiable function that is defined in a neighborhood of the grid. Then, the scheme $S_2^M[\phi]$ defined in $(2H)^M$ approximates $(2H)$ with accuracy*

$$S_2^M[\phi] = S_2[\phi] + \mathcal{O}(h^2 + d\theta).$$

I introduced two heuristic methods to improve the naive finite difference schemes. These are explicit and semi-implicit solvers that performed better in the examples considered by enforcing the ellipticity constraint. However, they lack a proof of convergence. I also considered a Newton solver.

Numerical experiments showed that the two alternative solvers presented for the naive discretization, which enforced the plane-subharmonic restrictions, converged even for degenerate examples or with singular right-hand sides. Likewise, the Newton solver showed the same results, whenever initialized with a good initial guess. For smooth examples, second order convergence was obtained. As for the monotone discretization, a Newton solver was also implemented. Numerical examples show that the directional resolution easily dominates the spatial resolution, a natural consequence of the three dimensional setting.

In addition, filtered schemes to combine the monotone discretization with the naive finite difference discretization could have been implemented. With little additional efforts, this approach, as explained in Section 2, results in provably convergent scheme with greater accuracy than the monotone schemes. In this work, filtered schemes were not implemented, since the main goal was to compare the two different discretizations presented. Moreover, the accurate scheme by itself proved to be convergent for all the examples considered, including degenerate ones.

As mentioned earlier, the 2-Hessian equation is related to the scalar curvature in the sense that they are equal up to a constant when the gradient of the function vanishes. Therefore, a natural extension to the current work would be to build schemes for the prescribed scalar curvature of a three dimensional graph. Recently, I worked on an extension of the scheme for arbitrary dimensions.

4. Numerical Methods for Motion of Level Sets by Affine Curvature

The affine curvature evolution is one of the most fundamental geometric evolution equations, after the mean curvature evolution. It was introduced by Sapiro and Tannenbaum in [ST94] and [AST98] and has applications in mathematical morphology, edge detection, image smoothing, and image enhancement (see [Sap06b]). The work was motivated by recent work of Jeff Calder [CS16], which provides an application of the affine curvature PDE to the statistics of large data sets. The planar motion of level sets by affine curvature is governed by the nonlinear PDE

$$u_t = \mathit{Aff}[u] := |\nabla u| \left(k[u]\right)^{1/3}. \tag{AC}$$

Here $u = u(x,y) : \mathbb{R}^2 \to \mathbb{R}$, $\nabla u = (u_x, u_y)$ denotes the gradient of u, and $k[u]$ denotes the curvature of the level set of u

$$k[u] = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{u_{xx}u_y^2 - 2u_xu_yu_{xy} + u_{yy}u_x^2}{(u_x^2+u_y^2)^{3/2}}.$$

The affine curvature PDE is closely related to the well known PDE for motion of level sets by mean curvature

$$u_t = \Delta_1 u := |\nabla u| \, k[u] \tag{MC}$$

studied in the seminal article [OS88].

Currently, there are a large number of numerical methods for the level set mean curvature PDE (MC), see the review papers [DDE05] and [CMM11]. In particular, Catté and Dibos proposed a morphological scheme that satisfies the comparison principle [CDK95] and Oberman proposed a convergent wide stencil finite difference scheme [Obe04] that uses a median formula. For the affine curvature evolution, the recent article [ERT10] gives a Bence-Merriman-Osher [MBO94] thresholding scheme. It introduced a (different) regularization of the cube root, which was needed for theoretical purposes, but not in practice. Alvarez and Guichard proposed a local scheme which lacks the affine invariance property [Gui94]. A morphological scheme which generalized [CDK95] was proposed by Guichard and Morel for affine curvature [GM97]. This inf-sup scheme, although morphologically invariant, has some limitations on the speed at which the level set curves move. In [Moi98], a nonlocal geometric morphological scheme is presented.

One of the challenges in working with the affine curvature evolution is that the PDE (AC) exhibits instabilities not found in the mean curvature PDE (MC). In particular, I found an example where standard finite differences break and are nonlinearly unstable. Moreover, due to the cube root, the affine curvature operator is not Lipschitz continuous, a desirable property to build provably convergent monotone schemes.

The approach taken was the following. Write $\mathit{Aff}[u]$ as

$$\mathit{Aff}[u] = A(|\nabla u|, \Delta_1 u) = (|\nabla u|^2 \Delta_1 u)^{1/3},$$

where $A(p,q) = \left(p^2 q\right)^{1/3}$. The reason for this is that elliptic discretizations are available for $\Delta_1 u$, rather than for $k[u]$. However, to build an elliptic discretization for $\mathit{Aff}[u]$ it is not enough to simply insert the available elliptic discretizations for $|\nabla u|$ and $\Delta_1 u$. Elliptic schemes are built by composing non-decreasing maps with elliptic operators, and $A(p,q)$ fails to be non-decreasing. To overcome this, I show that $A(p,q)$ may be decomposed into the sum of two non-decreasing functions in terms of $\pm|p|$, q. Then the scheme can be defined in terms of elliptic schemes for $\pm|\nabla u|$ and $\Delta_1 u$.

As discussed in Section 1, in order to build an explicit time discretization we require Lipschitz continuous operators, and $A(p,q)$ fails to be Lipschitz continuous. The following regularization of $A(p,q)$ is thus introduced. Given $K = K(\delta), L = L(\delta) > 0$, we define $A^\delta(p,q)$ as

$$A^\delta(p,q) = \operatorname{sgn}(q)\min(|A(p,q)|, K|p|, L|q|).$$

Naturally, we also define the regularized operator $\mathit{Aff}^{\,\delta}[u] = A^\delta(|\nabla u|\,, \Delta_1 u)$, which is still a geometric PDE. Furthermore, viscosity solutions converge to solutions of the affine curvature PDE in the limit as the regularization parameter goes to zero [Gig06, Theorem 4.6.1].

The elliptic discretization for $\mathit{Aff}^{\,\delta}[u]$ is built in an identical way to the one for $\mathit{Aff}\,[u]$, since, just like $A(p,q)$, $A^\delta(p,q)$ fails to be non-decreasing. A careful choice of the parameters K, L allows us to prove the consistency of $\mathit{Aff}^{e,\delta}[u]$ with $\mathit{Aff}[u]$ and, since the schemes are Lipschitz continuous, we are able to obtain stable, convergent explicit solvers. Moreover, the resulting discretization is combined into a filtered scheme that achieves the higher accuracy of the otherwise unstable standard finite difference scheme.

Simulations demonstrated the preservation of several geometric properties of the PDE, including affine invariance, morphological properties, and accurate representation of the shrinking ellipses. In addition, simulations also validated the convergence of the elliptic scheme, and the improved accuracy of the filtered scheme.

5. Ongoing and future work

Currently, I am writing my PhD thesis. In the spring, I will be working with Professor Froese on a project which will have a similar theme to those mentioned earlier. For instance, in a recent article, Professor Froese introduced meshfree finite difference methods for approximating nonlinear elliptic operators that depend on second directional derivatives or the eigenvalues of the Hessian [Fro15]. Currently, there is a big setup cost in the algorithm which could potentially be overcome with the use of more structured grids while still allowing for adaptivity or complicated geometries - two of the main advantages of the method. Such possible structures could, for example, be a quadtree grid recently used in [OZ16].

As for future work, building monotone schemes is just one of the aspects in the study of numerical methods for elliptic PDEs. Building fast solvers can be seen as equally important and the next natural step. In general, only the parabolic solver is available, which is known to be very slow due to the CFL condition. In light of this, I have studied multigrid schemes and their possible application to monotone schemes. Positive results were obtained in very simple cases, hinting at the possibility of development and success towards more complex cases.

In the case of Hamilton-Jacobi equations, two fast solvers are available for monotone schemes: fast sweeping and fast marching. As mentioned in Section 2, I was able to devise fast sweeping solvers for the filtered schemes proposed. I am interested in extending the fast marching algorithm to filtered schemes for Hamilton-Jacobi equations and, as a more general goal, in building fast solvers for filtered schemes.

I am also interested in studying the relationship between filtered schemes and flux limiters in the context of conservations laws. In a preliminary result of that work, I proved a discrete maximum principle for flux limiters in the context of conversation laws. I am interested in extending these results by proving a comparison principle, which would ultimately result in a new proof of convergence for flux limiter schemes for conservation laws.

The framework for the convergence of finite difference schemes for nonlinear elliptic PDEs, discussed in Section 1, assumes the existence of a comparison principle to the PDE. Currently the PDE theory only provides us with a weak form of the comparison principle (the boundary conditions are assumed in a strong sense). Recently, in [Fro16] this gap was addressed for the equation of prescribed Gaussian curvature. Although the arguments used are very specific to the PDE in question, I am interested in developing a similar analysis for Hamilton-Jacobi equations.

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