Derivatives and Integrals Cheatsheet

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Contents

1 The fundamental derivatives

Powers of x

$$\mathrm{D}\, k = 0$$
$$\mathrm{D}\, x^a = a x^{a-1}, \quad a \in \mathbb{R}$$
$$\mathrm{D}\, x = 1$$
$$\mathrm{D}\, \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$
$$\mathrm{D}\, \sqrt[n]{x} = \frac{1}{n\sqrt[n]{x^{n-1}}}, \quad x > 0,\ n \in \mathbb{N}$$
$$\mathrm{D}\, \frac{1}{x} = -\tfrac{1}{x^2}$$

Logarithmic and exponential functions

$$\mathrm{D}\, a^x = a^x \ln a, \quad a > 0$$
$$\mathrm{D}\, e^x = e^x$$
$$\mathrm{D}\, \log_a x = \frac{\log_a e}{x}, \quad x > 0$$
$$\mathrm{D}\, \ln x = \frac{1}{x}, \quad x > 0$$

Trigonometric functions

$$\mathrm{D}\, \sin x = \cos x$$
$$\mathrm{D}\, \cos x = -\sin x$$
$$\mathrm{D}\, \tan x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$
$$\mathrm{D}\, \cot x = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x)$$

Inverse trigonometric functions

$$\mathrm{D}\, \arctan x = \frac{1}{1+x^2}$$
$$\mathrm{D}\, \mathrm{arccot}\, x = -\frac{1}{1+x^2}$$
$$\mathrm{D}\, \arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\mathrm{D}\, \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

The rules of differentiation

$$\mathrm{D}\, [k \cdot f(x)] = k \cdot f'(x)$$
$$\mathrm{D}\, [f(x) + g(x)] = f'(x) + g'(x)$$
$$\mathrm{D}\, [f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
$$\mathrm{D}\, [f(x) \cdot g(x) \cdot z(x)] = f'(x) \cdot g(x) \cdot z(x) + f(x) \cdot g'(x) \cdot z(x) + f(x) \cdot g(x) \cdot z'(x)$$
$$\mathrm{D}\, [f(x)]^a = a[f(x)]^{a-1} \cdot f'(x), \quad a \in \mathbb{R}$$
$$\mathrm{D}\, \left[\frac{1}{f(x)}\right] = -\frac{f'(x)}{f^2(x)}$$
$$\mathrm{D}\, \left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$
$$\mathrm{D}\, [f(g(x))] = f'(z) \cdot g'(x), \quad z = g(x)$$
$$\mathrm{D}\, [f(g(z(x)))] = f'(u) \cdot g'(t) \cdot z'(x), \quad t = z(x),\ u = g(t)$$

$$\mathrm{D}\,[f(x)]^{g(x)} = [f(x)]^{g(x)}\left[g'(x)\ln f(x) + \frac{g(x)\cdot f'(x)}{f(x)}\right]$$

$$\mathrm{D}\,[f^{-1}(y)] = \frac{1}{f'(x)}, \quad x = f^{-1}(y)$$

2 Basic integrals

- **Potenze di x**

$$\int dx = x + c$$

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c \quad \forall\, p \in \mathbb{R}\setminus\{-1\}$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int \sqrt{x}\,dx = \frac{2}{3}\sqrt{x^3} + c$$

- **Logarithmic and exponential functions**

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c \quad \forall\, a \in \mathbb{R}\setminus\{1\}$$

- **Trigonometric functions**

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + c$$

$$\int \frac{1}{\sin^2 x}\,dx = -\cot x + c$$

- **Inverse trigonometric functions**

$$\int \frac{1}{1+x^2} = \arctan(x) + c = -\mathrm{arccot}(x) + c$$

$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x) + c = -\arccos(x) + c$$

- **Other more complex basic integrals**

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{|a|} + c$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{x}{\sqrt{x^2+a}}\,dx = \sqrt{x^2+a} + c$$

$$\int \frac{x}{\sqrt{a-x^2}}\,dx = -\sqrt{a-x^2} + c$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln|x+\sqrt{x^2\pm a^2}| + c$$

$$\int \frac{1}{\sin x}\,dx = \ln\left|\tan\frac{x}{2}\right| + c$$

$$\int \frac{1}{\cos x}\,dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$$

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x) + c$$

$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x\cos x) + c$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}(a^2\arcsin\frac{x}{a} + x\sqrt{a^2-x^2}) + c$$

3 Basic integrals of composite functions

- **Composite trigonometric functions**

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int \frac{f'(x)}{\cos^2 f(x)}\,dx = \tan f(x) + c$$

$$\int \frac{f'(x)}{\sin^2 f(x)}\,dx = -\cot f(x) + c$$

- **Composite inverse trigonometric functions**

$$\int \frac{f'(x)}{1+[f(x)]^2}\,dx = \begin{cases} \arctan f(x) + c \\ -\mathrm{arccot} f(x) + c \end{cases}$$

$$\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx = \begin{cases} \arcsin f(x) + c \\ -\arccos f(x) + c \end{cases}$$

- **Composite powers of x**

$$\int f^{\alpha}(x) f'(x)\,dx = \frac{f^{\alpha+1}}{\alpha+1} + c \quad \forall\, \alpha \in \mathbb{R} \setminus \{1\}$$

- **Composite logarithmic and exponential functions**

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$$

$$\int a^{f(x)} f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + c$$

- **More complex composite inverse trigonometric functions**

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \arcsin\frac{f(x)}{|a|} + c,\ a \neq 0$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\arctan\frac{f(x)}{a} + c,\ a \neq 0$$

4 Summary of derivatives and integrals of normal and composite trigonometric functions

$$\mathrm{D}\ \sin x = \cos x \quad\Longleftrightarrow\quad \int \cos x\,dx = \sin x + c$$

$$\mathrm{D}\ \cos x = -\sin x \quad\Longleftrightarrow\quad \int \sin x\,dx = -\cos x + c$$

$$\mathrm{D}\ \tan x = \frac{1}{\cos^2 x} \quad\Longleftrightarrow\quad \int \frac{1}{\cos^2 x}\,dx = \tan x + c$$

$$\mathrm{D}\ \cot x = -\frac{1}{\sin^2 x} \quad\Longleftrightarrow\quad \int \frac{1}{\sin^2 x}\,dx = -\cot x + c$$

$$\left.\begin{aligned} \mathrm{D}\ \arcsin x &= \frac{1}{\sqrt{1-x^2}} \\ \mathrm{D}\ \arccos x &= -\frac{1}{\sqrt{1-x^2}} \end{aligned}\right\} \quad\Longleftrightarrow\quad \int \frac{1}{\sqrt{1-x^2}} = \begin{cases} \arcsin(x) + c \\ -\arccos(x) + c \end{cases}$$

$$\left.\begin{aligned} \mathrm{D}\ \arctan x &= \frac{1}{1+x^2} \\ \mathrm{D}\,\mathrm{arccot}\ x &= -\frac{1}{1+x^2} \end{aligned}\right\} \quad\Longleftrightarrow\quad \int \frac{1}{1+x^2} = \begin{cases} \arctan(x) + c \\ -\mathrm{arccot}(x) + c \end{cases}$$

$$\mathrm{D}\ \sin f(x) = f'(x)\cos f(x) \quad\Longleftrightarrow\quad \int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\mathrm{D}\ \cos f(x) = -f'(x)\sin f(x) \quad\Longleftrightarrow\quad \int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\mathrm{D}\ \tan f(x) = \frac{f'(x)}{\cos^2 f(x)} \quad\Longleftrightarrow\quad \int \frac{f'(x)}{\cos^2 f(x)}\,dx = \tan f(x) + c$$

$$\mathrm{D}\ \cot f(x) = -\frac{f'(x)}{\sin^2 f(x)} \quad\Longleftrightarrow\quad \int \frac{f'(x)}{\sin^2 f(x)}\,dx = -\cot f(x) + c$$

$$\left.\begin{aligned} \mathrm{D}\ \arcsin f(x) &= \frac{f'(x)}{\sqrt{1-[f(x)]^2}} \\ \mathrm{D}\ \arccos f(x) &= -\frac{f'(x)}{\sqrt{1-[f(x)]^2}} \end{aligned}\right\} \quad\Longleftrightarrow\quad \int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx = \begin{cases} \arcsin f(x) + c \\ -\arccos f(x) + c \end{cases}$$

$$\left.\begin{aligned} \mathrm{D}\ \arctan f(x) &= \frac{f'(x)}{1+[f(x)]^2} \\ \mathrm{D}\,\mathrm{arccot}\ f(x) &= -\frac{f'(x)}{1+[f(x)]^2} \end{aligned}\right\} \quad\Longleftrightarrow\quad \int \frac{f'(x)}{1+[f(x)]^2}\,dx = \begin{cases} \arctan f(x) + c \\ -\mathrm{arccot} f(x) + c \end{cases}$$