Limits Cheatsheet

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Contents

1 Fundamental Limits of Elementary Functions

$$\lim_{x \to \infty} c = c$$

$$\lim_{x \to \pm\infty} x = \pm\infty$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to (\frac{\pi}{2})^-} \tan x = -\infty \qquad \lim_{x \to (\frac{\pi}{2})^+} \tan x = +\infty$$

$$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2} \qquad \lim_{x \to +\infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x \to 0} \log x = -\infty \qquad \lim_{x \to +\infty} \log x = +\infty$$

2 Indeterminate Forms

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

3 Notable Limits

3.1 Notable Limits of Exponential and Logarithmic Functions

Natural Logarithm

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{f(x) \to 0} \frac{\ln(1+f(x))}{f(x)} = 1$$

$$\lim_{x \to 0} \frac{x}{\ln(1+x)} = \lim_{f(x) \to 0} \frac{f(x)}{\ln(1+f(x))} = 1$$

$$\lim_{x \to 0} \frac{\ln(1+nx)}{mx} = \lim_{f(x) \to 0} \frac{\ln(1+nf(x))}{mf(x)} = \frac{n}{m}$$

$$\lim_{x \to 0} \frac{mx}{\ln(1+nx)} = \lim_{f(x) \to 0} \frac{mf(x)}{\ln(1+nf(x))} = \frac{m}{n}$$

$$\lim_{x \to 0} \frac{\ln(1+ax)}{\ln(1+bx)} = \frac{a}{b}$$

$$\lim_{x \to 0} \frac{\ln x - \ln a}{x - a} = \lim_{f(x) \to 0} \frac{\ln f(x) - \ln a}{f(x) - a} = \frac{1}{a}$$

$$\lim_{x \to 0} \frac{x - a}{\ln x - \ln a} = \lim_{f(x) \to 0} \frac{f(x) - a}{\ln f(x) - \ln a} = a$$

Logarithmic

$$\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \lim_{f(x) \to 0} \frac{\log_a(1+f(x))}{f(x)} = \frac{1}{\ln(a)}$$

$$\lim_{x \to 0} \frac{x}{\log_a(1+x)} = \lim_{f(x) \to 0} \frac{f(x)}{\log_a(1+f(x))} = \ln(a)$$

Power

$$\lim_{x\to 1}\frac{x^\alpha-1}{x-1}=\alpha$$

$$\lim_{x\to+\infty}\frac{x^n+a^{bx^n}}{x^n}=ab$$

Natural Exponential

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{f(x)\to 0}\frac{e^{f(x)}-1}{f(x)}=1$$

$$\lim_{x\to 0}\frac{x}{e^x-1}=\lim_{f(x)\to 0}\frac{f(x)}{e^{f(x)}-1}=1$$

$$\lim_{x\to 0}\frac{e^{ax}-1}{bx}=\lim_{f(x)\to 0}\frac{e^{af(x)}-1}{bf(x)}=\frac{a}{b}$$

$$\lim_{x\to 0}\frac{bx}{e^{ax}-1}=\lim_{f(x)\to 0}\frac{bf(x)}{e^{af(x)}-1}=\frac{b}{a}$$

Exponential with Arbitrary Base

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{f(x)\to 0}\frac{a^{f(x)}-1}{f(x)}=\ln(a)$$

$$\lim_{x\to 0}\frac{x}{a^x-1}=\lim_{f(x)\to 0}\frac{f(x)}{a^{f(x)}-1}=\frac{1}{\ln(a)}$$

$$\lim_{x\to 0}\frac{a^{mx}-1}{nx}=\lim_{f(x)\to 0}\frac{a^{mf(x)}-1}{nf(x)}=\frac{m\ln(a)}{n}$$

$$\lim_{x\to 0}\frac{nx}{a^{mx}-1}=\lim_{f(x)\to 0}\frac{nf(x)}{a^{mf(x)}-1}=\frac{n}{m\ln(a)}$$

$$\lim_{x\to 0}\frac{a^x-1}{x-1}=a^x\ln(a)$$

Euler's Number

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{f(x)\to\infty}\left(1+\frac{1}{f(x)}\right)^{f(x)}=e$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to\infty}\left(1+\frac{1}{x+a}\right)^x \qquad a\in\mathbb{R}$$

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{nx}=e^{kn}$$

$$\lim_{x\to a}\frac{e^x-e^a}{x-a}=e^a$$

$$\lim_{x\to\infty}(1+x)^{\frac{1}{x}}=e$$

$$\lim_{x\to\infty}(1+cx^n)^{\frac{k}{cx^n}}=e^{kc}$$

Power with Difference

$$\lim_{x\to 0}\frac{(1+x)^c-1}{x}=\lim_{f(x)\to 0}\frac{(1+f(x))^c-1}{f(x)}=c$$

3.2 Notable Limits of Trigonometric Functions

Sine Function

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{f(x)\to 0}\frac{\sin f(x)}{f(x)}=1$$

$$\lim_{x\to 0}\frac{\sin^n x}{\sin x^n}=\lim_{f(x)\to 0}\frac{\sin^n f(x)}{\sin f^n(x)}=1$$

$$\lim_{x\to 0}\frac{\sin x^n}{\sin^n x}=\lim_{f(x)\to 0}\frac{\sin f^n(x)}{\sin^n f(x)}=1$$

$$\lim_{x\to 0}\frac{\sin nx}{mx}=\frac{n}{m}\qquad \lim_{x\to 0}\frac{mx}{\sin nx}=\frac{m}{n}$$

Cosine Function

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{f(x)\to 0}\frac{1-\cos f(x)}{f^2(x)}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{f(x)\to 0}\frac{1-\cos f(x)}{f(x)}=0$$

Tangent Function

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{f(x)\to 0}\frac{\tan f(x)}{f(x)}=1$$

$$\lim_{x\to 0}\frac{x}{\tan x}=\lim_{f(x)\to 0}\frac{f(x)}{\tan f(x)}=1$$

$$\lim_{x\to 0}\frac{\tan^n x}{x^n}=\lim_{x\to 0}\frac{x^n}{\tan^n x}=1$$

$$\lim_{x\to 0}\frac{\tan mx}{nx}=\frac{m}{n}$$

$$\lim_{x\to 0}\frac{nx}{\tan mx}=\frac{n}{m}$$

Arcsine Function

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{f(x)\to 0}\frac{\arcsin f(x)}{f(x)}=1$$

$$\lim_{x\to 0}\frac{x}{\arcsin x}=\lim_{f(x)\to 0}\frac{f(x)}{\arcsin f(x)}=1$$

Arccosine Function

$$\lim_{x\to 0}\frac{\arccos^2(1-x)}{x}=2$$

Arctangent Function

$$\lim_{x\to 0}\frac{\arctan x}{x}=\lim_{f(x)\to 0}\frac{\arctan f(x)}{f(x)}=1$$

$$\lim_{x\to 0}\frac{x}{\arctan x}=\lim_{f(x)\to 0}\frac{f(x)}{\arctan f(x)}=1$$