Exercise on Vector Subspaces

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Vector Space

Definition 0.1. A *vector space over K* or *K-vector space* is a non-empty set V such that:

1. $\forall\, \vec{v}, \vec{w} \in V, \quad \vec{v} + \vec{w} \in V$;
2. $\forall\, \vec{v} \in V,\ \forall\, c \in K, \quad c\vec{v} \in V$.

satisfying the following properties:

- **VS1**: $\forall\, \vec{u}, \vec{v}, \vec{w} \in V, \quad (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$;
- **VS2**: $\forall\, \vec{v} \in V, \quad \vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$;
- **VS3**: $\forall\, \vec{v} \in V, \quad \vec{v} + (-1)\vec{v} = (-1)\vec{v} + \vec{v} = \vec{0}$;
- **VS4**: $\forall\, \vec{u}, \vec{v} \in V, \quad \vec{u} + \vec{v} = \vec{v} + \vec{u}$;
- **VS5**: $\forall\, \lambda \in K,\ \forall\, \vec{u}, \vec{v} \in V, \quad \lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$;
- **VS6**: $\forall\, \lambda, \mu \in K,\ \forall\, \vec{v} \in V, \quad (\lambda + \mu)\vec{v} = \lambda\vec{u} + \mu\vec{v}$;
- **VS7**: $\forall\, \lambda, \mu \in K,\ \forall\, \vec{v} \in V, \quad (\lambda\mu)\vec{v} = \lambda(\mu\vec{v})$;
- **VS8**: $\forall\, \vec{v} \in V, \quad 1\vec{v} = \vec{v}$.

Let U and W be two K-vector spaces. The Cartesian product of U and W, $U \times W$ is a K-vector space if, $\forall\, (u, w), (u', w') \in U \times W, \quad \forall a \in K$:

1. $(\vec{u}, \vec{w}) + (\vec{u'}, \vec{w'}) = (\vec{u} + \vec{u'}, \vec{w} + \vec{w'}) \in U \times W$
2. $a(\vec{u}, \vec{w}) = (a\vec{u}, a\vec{w}) \in U \times W$

The zero vector of $U \times W$ is $(\vec{0}, \vec{0})$.

- **SV1**: $\forall\,(\vec{u},\vec{w}),(\vec{u'},\vec{w'}),(\vec{u''},\vec{w''})\in U\times W,\quad ((\vec{u},\vec{w})+(\vec{u'},\vec{w'}))+(\vec{u''},\vec{w''}) = (\vec{u},\vec{w})+((\vec{u'},\vec{w'})+(\vec{u''},\vec{w''}))$;

  Proof.

$$\begin{aligned}((\vec{u},\vec{w})+(\vec{u'},\vec{w'}))+(\vec{u''},\vec{w''}) &= (\vec{u}+\vec{u'},\vec{w}+\vec{w'})+(\vec{u''},\vec{w''})\\ &= (\vec{u}+\vec{u'}+\vec{u''},\vec{w}+\vec{w'}+\vec{w''})\\ &= (\vec{u},\vec{w})+(\vec{u'}+\vec{u''},\vec{w'}+\vec{w''})\\ &= (\vec{u},\vec{w})+((\vec{u'},\vec{w'})+(\vec{u''}+\vec{w''}))\end{aligned}$$

  □

- **SV2**: $\forall\,(\vec{u},\vec{w})\in U\times W,\quad (\vec{u},\vec{w})+(\vec{0},\vec{0}) = (\vec{0},\vec{0})+(\vec{u},\vec{w}) = (\vec{u},\vec{w})$;

  Proof.

$$\begin{aligned}(\vec{u},\vec{w})+(\vec{0},\vec{0}) &= (\vec{u}+\vec{0},\vec{w}+\vec{0})\\ &= (\vec{0}+\vec{u},\vec{0}+\vec{w})\\ &= (\vec{u},\vec{w})\end{aligned}$$

  □

- **SV3**: $\forall\,(\vec{u},\vec{w})\in U\times W,\quad (\vec{u},\vec{w})+(-1)(\vec{u},\vec{w}) = (-1)(\vec{u},\vec{w})+(\vec{u},\vec{w}) = \vec{0}$;

  Proof.

$$\begin{aligned}(\vec{u},\vec{w})+(-1)(\vec{u},\vec{w}) &= (\vec{u},\vec{w})+((-1)\vec{u},(-1)\vec{w})\\ &= (\vec{u}+(-1)\vec{u},\vec{w}+(-1)\vec{w})\\ &= (\vec{u}-\vec{u},\vec{w}-\vec{w})\\ &= (\vec{0},\vec{0})\end{aligned}$$

  □

- **SV4**: $\forall\,(\vec{u},\vec{w}),(\vec{u'},\vec{w'})\in U\times W,\quad (\vec{u},\vec{w})+(\vec{u'},\vec{w'}) = (\vec{u'},\vec{w'})+(\vec{u},\vec{w})$;

  Proof.

$$\begin{aligned}(\vec{u},\vec{w})+(\vec{u'},\vec{w'}) &= (\vec{u}+\vec{u'},\vec{w}+\vec{w'})\\ &= (\vec{u'}+\vec{u},\vec{w'}+\vec{w})\\ &= (\vec{u'},\vec{w'})+(\vec{u},\vec{w})\end{aligned}$$

  □

- **SV5**: $\forall\, \lambda \in K,\ \forall\, (\vec{u},\vec{w}),(\vec{u'},\vec{w'}) \in U \times W, \quad \lambda((\vec{u},\vec{w})+(\vec{u'},\vec{w'})) = \lambda(\vec{u},\vec{w}) + \lambda(\vec{u'},\vec{w'})$;

  Proof. Proof is trivial. □

- **SV6**: $\forall\, \lambda,\mu \in K,\ \forall\, (\vec{u},\vec{w}) \in U \times W, \quad (\lambda+\mu)(\vec{u},\vec{w}) = \lambda(\vec{u},\vec{w}) + \mu(\vec{u},\vec{w})$;

  Proof. Proof is trivial. □

- **SV7**: $\forall\, \lambda,\mu \in K,\ \forall\, (\vec{u},\vec{w}) \in U \times W, \quad (\lambda\mu)(\vec{u},\vec{w}) = \lambda(\mu(\vec{u},\vec{w}))$;

  Proof.

$$
\begin{aligned}
(\lambda\mu)(\vec{u},\vec{w}) &= ((\lambda\mu)\vec{u},(\lambda\mu)\vec{w}) \\
&= (\lambda(\mu\vec{u}),\lambda(\mu\vec{w})) \\
&= \lambda(\mu\vec{u},\mu\vec{w}) \\
&= \lambda(\mu(\vec{u},\vec{w}))
\end{aligned}
$$

  □

- **SV8**: $\forall\, (\vec{u},\vec{w}) \in U \times W, \quad 1(\vec{u},\vec{w}) = (\vec{u},\vec{w})$.

  Proof.

$$
\begin{aligned}
1(\vec{u},\vec{w}) &= (1\vec{u},1\vec{w}) \\
&= (\vec{u},\vec{w})
\end{aligned}
$$

  □

Vector Subspace

Definition 0.2. Let V be a vector space over K. A non-empty subset W of V is called a *vector subspace* of V if:

1. for every $\vec{w}_1,\vec{w}_2 \in W$, the sum $\vec{w}_1+\vec{w}_2$ belongs to W;
2. for every $\vec{w} \in W$ and every $c \in K$, the product $c\vec{w}$ belongs to W.

The subsets

$$U' = \{(\vec{u}, \vec{0}) \,|\, \vec{u} \in U\}$$
$$W' = \{(\vec{0}, \vec{w}) \,|\, \vec{w} \in W\}$$

are two subspaces of $U \times W$.

Let $(\vec{u}, \vec{0}), (\vec{u'}, \vec{0}) \in U'$ and $c \in K$. Then

$$\begin{aligned}(\vec{u}, \vec{0}) + (\vec{u'}, \vec{0}) &= (\vec{u} + \vec{u'}, \vec{0} + \vec{0}) \\ &= (\vec{u} + \vec{u'}, \vec{0}) \in U'\end{aligned}$$

$$\begin{aligned}c(\vec{u}, \vec{0}) &= (c\vec{u}, c\vec{0}) \\ &= (c\vec{u}, \vec{0}) \in U'\end{aligned}$$

Let $(\vec{0}, \vec{w}), (\vec{0}, \vec{w'}) \in W'$ and $c \in K$. Then

$$\begin{aligned}(\vec{0}, \vec{w}) + (\vec{0}, \vec{w'}) &= (\vec{0} + \vec{0}, \vec{w} + \vec{w'}) \\ &= (\vec{0}, \vec{w} + \vec{w'}) \in W'\end{aligned}$$

$$\begin{aligned}c(\vec{0}, \vec{w}) &= (c\vec{0}, c\vec{w}) \\ &= (\vec{0}, c\vec{w}) \in W'\end{aligned}$$

We have

$$U' \cap W' = (\vec{0}, \vec{0});$$

furthermore,

$$(\vec{u}, \vec{w}) = (\vec{u}, \vec{0}) + (\vec{0}, \vec{w})$$

for every $(u, w) \in U \times W$. Hence we have

$$U \times W = U' \oplus W'.$$