Function analysis

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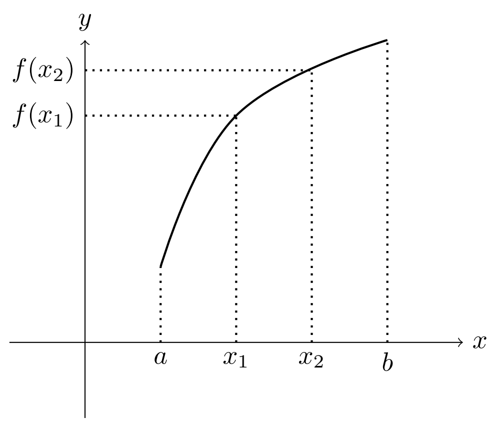
1 Increasing and Decreasing Functions and Their Derivatives

We provide two examples to recall the definitions related to increasing and decreasing functions.

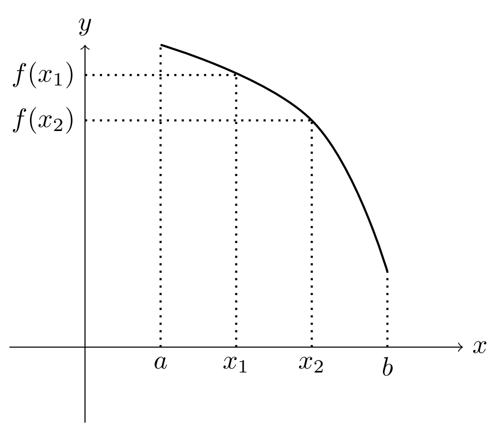
Example 1.1. If a function $y = f(x)$ is increasing in an interval, increasing the variable x results in an increase of y as well.

$$\forall x_1, x_2 \in [a; b], x_2 > x_1 \Rightarrow f(x_2) > f(x_1).$$



Example 1.2. If a function $y = f(x)$ is decreasing in an interval, increasing the variable x results in a decrease of y.

$$\forall x_1, x_2 \in [a; b], x_2 > x_1 \Rightarrow f(x_2) < f(x_1).$$



For increasing and decreasing functions, the following theorem holds.

Theorem 1.1. *Given a function $y = f(x)$, continuous in an interval I (both bounded and unbounded) and differentiable at the interior points of I, it is:*

1. *increasing in I, if its derivative is positive at every interior point of I;*
2. *decreasing in I, if its derivative is negative at every interior point of I.*

Proof. 1. Let x_1 and $x_2 \in I$, with $x_1 < x_2$. By the Mean Value Theorem applied to $f(x)$ in the interval $[x_1; x_2]$, we have:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad c \in]x_1; x_2[.$$

Since $x_2 - x_1 > 0$ and by hypothesis $f'(c) > 0$, we also have $f(x_2) - f(x_1) > 0$, hence $f(x_2) > f(x_1)$. Since x_1 and x_2 are arbitrary points of I, the function is increasing in I.

2. Proceeding similarly to the previous case, we obtain:

$$f(x_2) - f(x_1) < 0.$$

Indeed, $x_2 - x_1 > 0$ and by hypothesis $f'(c) < 0$, so $f(x_2) < f(x_1)$. Thus, the function is decreasing in I.

□

Note 1. This theorem is a **sufficient condition** to assert that a function is increasing or decreasing in an interval.

Note 2. The Mean Value Theorem holds because the function is continuous in $[x_1; x_2]$ and differentiable in $]x_1; x_2[$.

We can apply this theorem to determine the intervals in which a function is increasing or decreasing by studying the sign of its first derivative.

2 Maxima, Minima, and Inflection Points

2.1 Absolute Maxima and Minima

Absolute Maximum, Absolute Minimum

Definition 2.1. Given the function $y = f(x)$, defined in the interval I, we call:

- absolute maximum of $f(x)$, if it exists, the maximum M of the values assumed by the function in I, that is,

$$M = f(x_0), x_0 \in I \wedge M \geq f(x), \forall x \in I;$$

- absolute minimum of $f(x)$, if it exists, the minimum m of the values assumed by the function in I, that is,

$$m = f(x_1), x_1 \in I \wedge m \leq f(x), \forall x \in I.$$

M and m, if they exist, are unique.

- A point x_0 in I such that $f(x_0) = M$ is called an **absolute maximum point**.
- A point x_0 in I such that $f(x_0) = m$ is called an **absolute minimum point**.

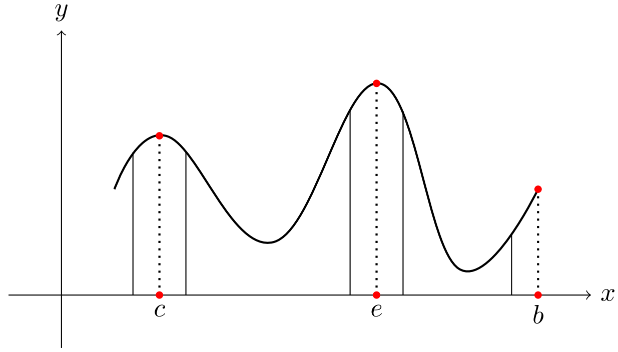
2.2 Relative Maxima and Minima

Relative Maximum

Definition 2.2. Given a function $y = f(x)$, defined in an interval I, the point x_0 of I is said to be a relative maximum if there exists a neighborhood I_{x_0} of x_0 such that $f(x_0)$ is greater than or equal to the value of the function for every x in the neighborhood I_{x_0}. $f(x_0)$ is called the relative maximum of the function in I.
In summary, let $y = f(x)$ be defined in I, x_0 is a relative maximum point if

$$\exists I_{x_0} : \forall x \in I_{x_0}, f(x_0) \geq f(x) \Rightarrow f(x_0) \text{ relative maximum.}$$

Example 2.1.



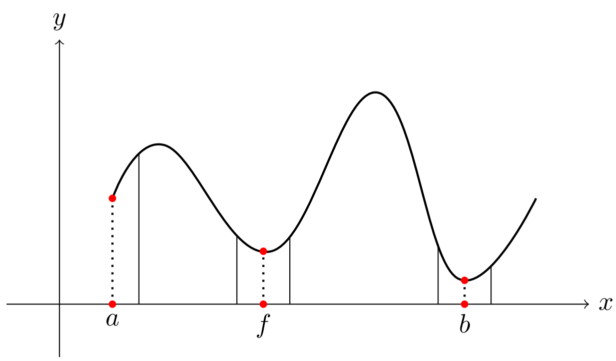
Relative Minimum

Definition 2.3. Given a function $y = f(x)$ defined on an interval I, the point x_0 in I is said to be a relative minimum if there exists a neighborhood I_{x_0} such that $f(x_0)$ is less than or equal to the value of the function for every x in the neighborhood I_{x_0}. $f(x_0)$ is called the relative minimum of the function in I.

In summary, let $y = f(x)$ be defined in I, then x_0 is a point of relative minimum if

$$\exists I_{x_0} : \forall x \in I_{x_0}, f(x_0) \leq f(x) \Rightarrow f(x_0) \text{ is a relative minimum.}$$

Example 2.2.



A point of an interval that is a point of relative maximum is also called a **maximizing point**; a point of relative minimum is called a **minimizing point**. A point of an interval is called an **extremum point** if it is a maximizing or minimizing point. The corresponding value of the function is called a **relative extremum**.

2.3 Concavity

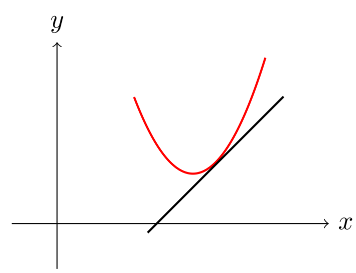
Let $y = f(x)$ be a function defined and differentiable in the interval I, and let the line with equation $y = t(x)$ be tangent to the graph of $f(x)$ at its abscissa point x_0, which is within the interval I.

Note 3. Since $f(x)$ is differentiable in I, the tangent line exists at every point.

Concave Upward

Definition 2.4. It is said that at x_0, the graph of the function $f(x)$ has concavity directed towards the positive y-axis (upward) if there exists a complete neighborhood I_{x_0} of x_0 such that, for every x belonging to the neighborhood and different from x_0, the ordinate of the point with abscissa x belonging to the graph is greater than that of the point belonging to the tangent t and having the same abscissa, i.e.:

$$f(x) > t(x) \quad \forall x \in I_{x_0} \wedge x \neq x_0.$$

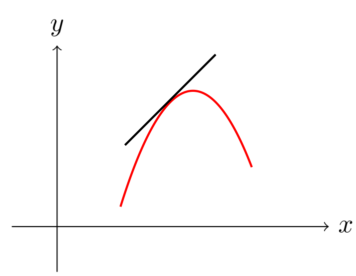


A function whose graph has concavity directed upward is said to be **convex**.

Concave Downward

Definition 2.5. It is said that at x_0, the graph of the function $f(x)$ has concavity directed towards the negative y-axis (downward) if there exists a complete neighborhood I_{x_0} of x_0 such that, for every x belonging to the neighborhood and different from x_0, the ordinate of the point with abscissa x belonging to the graph is less than that of the point belonging to the tangent t and having the same abscissa, i.e.:

$$f(x) < t(x) \quad \forall x \in I_{x_0} \wedge x \neq x_0.$$

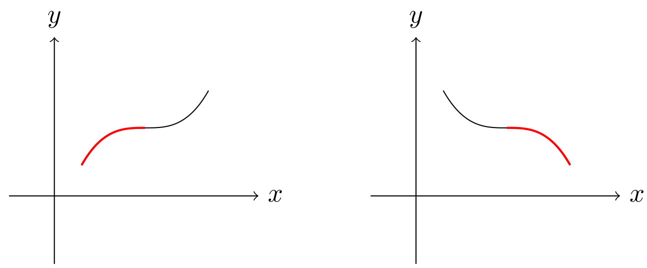


A function whose graph turns its concavity downward is called **concave**.

Given an interval I, we say that the graph has concavity upward (or downward) **within the interval** if it has concavity upward (or downward) at every interior point of the interval.

2.4 Inflection Points

Definition 2.6. Given the function $y = f(x)$ defined and continuous on the interval I, we say that it has an inflection point at x_0 within I if at that point the graph of $f(x)$ changes concavity.



If the function is differentiable at the inflection point, there exists a tangent to the curve at that point, and it is either oblique or parallel to the x-axis; if the derivative is infinite, the tangent is parallel to the y-axis. The tangent line has the characteristic of intersecting the curve.

Note 4. The tangent at an inflection point is also called an **inflectional tangent**.

If a tangent line exists at an inflection point, the inflection is termed:

- **horizontal** if the tangent at the inflection point is parallel to the x-axis;
- **vertical** if the tangent is parallel to the y-axis;
- **oblique** if the tangent is not parallel to either axis.

Note 5. If, in a neighborhood of the inflection point, the graph has concavity downwards to the left of the inflection and upwards to the right of the inflection, the inflection is **ascending**. If the concavity is upwards to the left of the inflection and downwards to the right, the inflection is **descending**.

3 Maxima, minima, horizontal inflection points, and first derivative

3.1 Stationary points

Stationary point

Definition 3.1. Given a differentiable function $y = f(x)$ and one of its points $x = c$, if $f'(c) = 0$, then $x = c$ is called a stationary point.

If $f'(c) = 0$, then the tangent at the point on the graph of the function where $x = c$ is parallel to the x-axis.

3.2 Relative maximum or minimum points

Theorem 3.1. *For a function $y = f(x)$ defined on an interval $[a; b]$ and differentiable in $]a; b[$, if $f(x)$ has a relative maximum or minimum at the point x_0 inside $[a; b]$, the derivative of the function at that point is zero, that is, $f'(x_0) = 0$.*

Note 6. The theorem states that the relative maximum and minimum points of a differentiable function, internal to the interval of definition, are stationary points.

From the geometric meaning of the derivative, the previous theorem implies that the tangent at a point of relative maximum or minimum (which is not an endpoint of the interval) is parallel to the x-axis.

3.3 Finding relative maxima and minima with the first derivative

Theorem 3.2. *Let $y = f(x)$ be defined and continuous in a complete neighborhood I_{x_0} of the point x_0 and differentiable in the same neighborhood for every $x \neq x_0$.*

1. *If for every x in the neighborhood $f'(x) > 0$ when $x < x_0$ and $f'(x) < 0$ when $x > x_0$, then x_0 is a relative maximum point.*
2. *If for every x in the neighborhood $f'(x) < 0$ when $x < x_0$ and $f'(x) > 0$ when $x > x_0$, then x_0 is a relative minimum point.*
3. *If the sign of the first derivative is the same for every $x \neq x_0$ in the neighborhood, then x_0 is not an extremum point.*

Proof. 1. For $x < x_0$, $f'(x) > 0$, hence $f(x)$ is increasing (by the theorem of increasing and decreasing functions); therefore, if $x < x_0$, $f(x) < f(x_0)$. For every $x \neq x_0$ in the neighborhood $f(x) < f(x_0)$, hence x_0 is a relative maximum point.

2. Similarly to the previous case: for $x < x_0$, $f'(x) < 0$, hence $f(x)$ is decreasing, i.e., if $x < x_0$, $f(x) > f(x_0)$; for $x > x_0$, $f'(x) > 0$, hence $f(x)$ is increasing, i.e., if $x > x_0$, $f(x) > f(x_0)$. For every $x \neq x_0$ in the neighborhood $f(x) > f(x_0)$, hence x_0 is a relative minimum point.

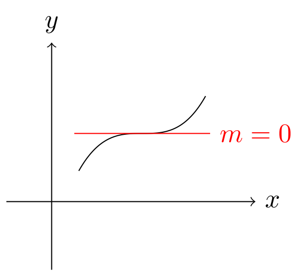
3. Suppose that for every $x \neq x_0$ in the neighborhood $f'(x) < 0$ (similar argument if $f'(x) > 0$). The function is decreasing both for $x < x_0$ and for $x > x_0$. Therefore, if $x < x_0$, $f(x) > f(x_0)$, while if $x > x_0$, $f(x) < f(x_0)$. We conclude that x_0 is neither a maximum nor a minimum point.

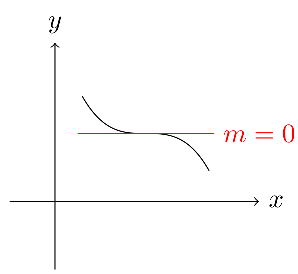
□

3.4 Stationary points of horizontal inflection

Theorem 3.3. *Given the function $y = f(x)$ defined and continuous in a complete neighborhood I_{x_0} of the point x_0 and differentiable in the same neighborhood, x_0 is a horizontal inflection point if the following conditions are satisfied:*

- $f'(x_0) = 0$;
- *the sign of the first derivative is the same for every $x \neq x_0$ in the neighborhood I_{x_0}.*





Note 7. In summary, for a continuous function $f(x)$, studying the sign of the first derivative is essential for **finding relative maximums and minimums, and horizontal inflection points**. The procedure is as follows:

- calculate the first derivative $f'(x)$ and determine its domain to find any points where the function is not differentiable (cusps, vertical tangents, corner points);
- solve the equation $f'(x) = 0$ to find stationary points;
- study the sign of $f'(x)$ to find relative maximums and minimums (including non-stationary ones) and horizontal tangent inflections.

The theorems stated are valid for points within the intervals of definition of the function, so it is necessary to also examine the values that the function takes at the endpoints of these intervals. Furthermore, if we need to find the **absolute maximum and minimum**:

- if the function $f(x)$ is continuous and the interval of definition of the function is closed and bounded, the Weierstrass theorem ensures the existence of absolute maximum and minimum; to determine them, compare the ordinates of the points of relative maximum and minimum with each other and with the values that $f(x)$ takes at the endpoints of the interval: the greater value corresponds to the absolute maximum point and the lesser one corresponds to the absolute minimum point;
- if the interval is not closed and bounded, absolute maximum and minimum may not exist.

4 Inflection Points and Second Derivative

4.1 Concavity and the Sign of the Second Derivative

A Criterion for Concavity A criterion to establish the concavity of the graph of a function at a point x_0 is given by the following theorem.

Theorem 4.1. *Let $y = f(x)$ be a function defined and continuous on an interval I, along with its first and second derivatives, and let x_0 be a point interior to this interval. If $f''(x_0) \neq 0$, then the graph of the function changes concavity at x_0:*

- *it is concave upwards if $f''(x_0) > 0$;*
- *it is concave downwards if $f''(x_0) < 0$.*

A Necessary Condition for Inflection Points For the identification of inflection points, the following theorem is useful, of which we provide only the statement.

Theorem 4.2. *Let $y = f(x)$ be a function defined on an interval $[a; b]$, and let its first and second derivatives exist in this interval. If $f(x)$ has an inflection point at x_0, inside $[a; b]$, then the second derivative of the function at that point vanishes, that is: $f''(x_0) = 0$.*

4.2 Inflection Points and Study of the Sign of the Second Derivative

To find inflection points, we can study the sign of the second derivative. The following theorem holds true.

Theorem 4.3. *Let $y = f(x)$ be a function defined and continuous in a complete neighborhood I_{x_0} of the point x_0, and let its first and second derivatives exist in this neighborhood for every $x \neq x_0$. If for every $x \neq x_0$ in the neighborhood,*

- *$f''(x) > 0$ for $x < x_0$ and $f''(x) < 0$ for $x > x_0$, or*
- *$f''(x) < 0$ for $x < x_0$ and $f''(x) > 0$ for $x > x_0$,*

then x_0 is an inflection point.

If, in addition to the hypotheses of the previous theorem, it is true that the second derivative is continuous at x_0, then necessarily $f''(x_0) = 0$. Therefore, inflection points of functions that have continuous first and second derivatives should be sought among the solutions of the equation $f''(x) = 0$. Furthermore, at the inflection point x_0, if $f'(x_0) \neq 0$ the inflection is oblique, if $f'(x_0) = 0$ the inflection is horizontal.

Note 8. Summarizing, given a function $f(x)$, continuous and differentiable, to find inflection points proceed as follows:

1. compute the second derivative $f''(x)$ and determine its domain;
2. study the sign of $f''(x)$ and look for the points where concavity changes, namely the inflection points;
3. if x_0 is an inflection point and:
   - $f'(x_0) = 0$, the inflection is **horizontal**;
   - $f'(x_0) \neq 0$, the inflection is **oblique**;

If the function $f(x)$ is not differentiable at a point x_0 where $f''(x)$ changes sign, then, when $\lim_{x \to x_0} f'(x) = +\infty$ or $\lim_{x \to x_0} f'(x) = -\infty$, at x_0 there is a **vertical** inflection.

5 Studying a Function

To study the main properties and plot the graph of a function $y = f(x)$, we can proceed by examining the following points.

1. The *domain* of the function.
2. Any *symmetries* and *periodicity*:
   - if the function is *even*, the graph is symmetric with respect to the y-axis;
     $$y = f(x) \text{ is even in } D, D \subseteq \mathbb{R},\ \text{if } f(-x) = f(x), \forall x \in D$$
   - if it is *odd*, it is symmetric with respect to the origin;
     $$y = f(x) \text{ is odd in } D,\ \text{if } f(-x) = -f(x), \forall x \in D$$
   - if it is *periodic* with period T, we can limit ourselves to studying the function in a single interval of width T.
     $$y = f(x) \text{ is periodic with period } T\ (T > 0),\ \text{if } f(x) = f(x+kT), \forall k \in \mathbb{Z}$$
3. The possible *intersection points* of the graph *with the Cartesian axes.*
4. The *sign of the function*: establish the intervals where it is positive, by setting $f(x) > 0$, and consequently find where it is negative.
5. The *behavior* of the function *at the extremes of the domain*: compute the respective *limits* and then look for any *asymptotes* of the function.

   Vertical asymptote: $x = x_0$ if $\lim\limits_{x \to x_0} f(x) = \infty$.

   Horizontal asymptote: $y = y_0$ if $\lim\limits_{x \to \infty} f(x) = y_0$.

   Oblique asymptote: $y = mx + q$, $\quad m = \lim\limits_{x \to \infty} \dfrac{f(x)}{x}$ and $q = \lim\limits_{x \to \infty} [f(x) - m \cdot x]$.

   Also classify any points of *discontinuity*, specifying whether they are of the first, second, or third kind.
6. The *first derivative* and its domain. From the *study of the sign of the first derivative*, determine the intervals where the function is *increasing* ($f'(x) > 0$) and consequently, those where it is *decreasing* ($f'(x) < 0$); look for any *local maxima* or *minima*, and horizontal inflection points, as well as points of non-differentiability for $f(x)$ (*vertical inflections, cusps, and corner points*).
7. The *second derivative* and its domain. From the *study of the sign of the second derivative*, determine the intervals where the graph changes concavity upwards ($f''(x) > 0$) or downwards ($f''(x) < 0$). Also look for *inflection points* with oblique tangent and possibly the inflection tangent.