INVESTMENT SOCIETY

Derivatives

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February 4, 2024

Contents

Fundamental Derivatives

Powers of x

$$
\begin{aligned}
&\mathrm{D}\, k = 0 \\
&\mathrm{D}\, x^a = a x^{a-1}, \quad a \in \mathbb{R} \\
&\mathrm{D}\, x = 1 \\
&\mathrm{D}\, \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0 \\
&\mathrm{D}\, \sqrt[n]{x} = \frac{1}{n\sqrt[n]{x^{n-1}}}, \quad x > 0,\ n \in \mathbb{N} \\
&\mathrm{D}\, \frac{1}{x} = -\frac{1}{x^2}
\end{aligned}
$$

Logarithmic and Exponential Functions

$$
\begin{aligned}
&\mathrm{D}\, a^x = a^x \ln a, \quad a > 0 \\
&\mathrm{D}\, e^x = e^x \\
&\mathrm{D}\, \log_a x = \frac{1}{x} \log_a e, \quad x > 0 \\
&\mathrm{D}\, \ln x = \frac{1}{x}, \quad x > 0
\end{aligned}
$$

Trigonometric Functions

$$
\begin{aligned}
&\mathrm{D}\, \sin x = \cos x \\
&\mathrm{D}\, \sin x^\circ = \frac{\pi}{180^\circ} \cos x^\circ \\
&\mathrm{D}\, \cos x = -\sin x \\
&\mathrm{D}\, \cos x^\circ = -\frac{\pi}{180^\circ} \sin x^\circ \\
&\mathrm{D}\, \tan x = \frac{1}{\cos^2 x} = 1 + \tan^2 x \\
&\mathrm{D}\, \cot x = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x)
\end{aligned}
$$

Inverse Trigonometric Functions

$$\mathrm{D}\arctan x = \frac{1}{1+x^2}$$
$$\mathrm{D}\,\mathrm{arccot}\ x = -\frac{1}{1+x^2}$$
$$\mathrm{D}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\mathrm{D}\arccos x = -\frac{1}{1-x^2}$$

Derivative Rules

$$\mathrm{D}\,[k \cdot f(x)] = k \cdot f'(x)$$
$$\mathrm{D}\,[f(x) + g(x)] = f'(x) + g'(x)$$
$$\mathrm{D}\,[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
$$\mathrm{D}\,[f(x) \cdot g(x) \cdot z(x)] = f'(x) \cdot g(x) \cdot z(x) + f(x) \cdot g'(x) \cdot z(x) + f(x) \cdot g(x) \cdot z'(x)$$
$$\mathrm{D}\,[f(x)]^a = a[f(x)]^{a-1} \cdot f'(x), \quad a \in \mathbb{R}$$
$$\mathrm{D}\,\left[\frac{1}{f(x)}\right] = -\frac{f'(x)}{f^2(x)}$$
$$\mathrm{D}\,\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$
$$\mathrm{D}\,[f(g(x))] = f'(z) \cdot g'(x), \quad z = g(x)$$
$$\mathrm{D}\,[f(g(z(x)))] = f'(u) \cdot g'(t) \cdot z'(x), \quad t = z(x),\ u = g(t)$$
$$\mathrm{D}\,[f(x)]^{g(x)} = [f(x)]^{g(x)}\left[g'(x)\ln f(x) + \frac{g(x) \cdot f'(x)}{f(x)}\right]$$
$$\mathrm{D}\,[f^{-1}(y)] = \frac{1}{f'(x)}, \quad x = f^{-1}(y)$$

1 Derivative of a Function

1.1 Incremental Ratio

Given a function $y = f(x)$ defined in an interval $[a; b]$ and a point on its graph $A(c; f(c))$, let's increase the abscissa of A by an amount h to obtain the point B with coordinates:

$$x_B = c + h; \quad y_B = f(x_B) = f(c + h)$$

meaning,

$$B(c + h; f(c + h))$$

Let's consider the increments:

$$\Delta x = x_B - x_A = h \quad \text{and} \quad \Delta y = y_B - y_A = f(c + h) - f(c)$$

The ratio of the two increments is $\dfrac{\Delta y}{\Delta x}$.

Note 1. Generally, the notation Δt is called the **increment of the variable** t and indicates the difference between two values t_2 and t_1 of a quantity t:

$$\Delta t = t_2 - t_1.$$

Incremental Ratio

Definition 1.1. Given a function $y = f(x)$ defined in an interval $[a; b]$ and two real numbers c and $c+h$ within the interval, the incremental ratio of f (relative to c) is defined as:

$$\frac{\Delta y}{\Delta x} = \frac{f(c + h) - f(c)}{h}$$

Considering the points $A(c; f(c))$ and $B(c + h; f(c + h))$ on the graph of f, the *incremental ratio of f relative* to c is the **slope of the line passing through A and B.**

1.2 Derivative of a Function

Derivative of a Function

Definition 1.2. For a function $y = f(x)$ defined in an interval $[a; b]$, the derivative of the function at the point c within the interval, denoted by $f'(c)$, is the limit, if it exists and is *finite*, as h tends to 0, of the incremental ratio of f relative to c:

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

Remark 1. The derivative of a function at a point c represents the **slope of the tangent line** to the graph of the function at its abscissa c.

A function is said to be **differentiable** at a point c if the derivative $f'(c)$ exists. For a function to be differentiable at c, the following conditions must be satisfied:

1. the function is defined in a neighborhood of point c;
2. the limit of the incremental ratio relative to c exists for h tending to 0, i.e., the right-hand limit and left-hand limit of this ratio exist, and they coincide;
3. this limit is a finite number.

The derivative of a function $y = f(x)$ at a generic point x is denoted by one of the following symbols:

$$f'(x); \quad \mathrm{D}f(x); \quad y'.$$

If the limit as h tends to 0 of the incremental ratio of a function at a point *does not exist* or is *infinite*, the function is said to be **non-differentiable** at that point.

1.3 Calculating the Derivative

Example 1.1. Let's calculate the derivative of the function

$$y = (x-1)^2$$

at $c = 3$. Let f be the function, applying the definition:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}.$$

Calculate the values the function takes at the points with abscissa 3 and $3 + h$:

$$f(3) = (3-1)^2 = 2^2 = 4; \quad f(3+h) = (3+h-1)^2 = (2+h)^2 = h^2 + 4h + 4$$

Substitute these values into the incremental ratio and simplify:

$$f'(3) = \lim_{h \to 0} \frac{h^2 + 4h + 4 - 4}{h} = \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0} h + 4 = 4.$$

Therefore, $f'(3) = 4$.

Note 2. The derivative $f'(3)$ is a real number and is the slope of the tangent line to the graph of $f(x)$ at the point $(3; f(3))$.

We can also calculate the derivative of a function at a generic point. In this case, the value $f'(x)$ we obtain is a function of x, and for this reason, we also talk about the derivative function. The derivative function, as x varies, provides the slope of all tangent lines to the given function.

Example 1.2. Let's calculate the derivative of the function $f(x) = x^2 - x$ at a generic point x:

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 2xh - h}{h} \\ &= \lim_{h \to 0} \frac{h(h + 2x - 1)}{h} \\ &= \lim_{h \to 0} h + 2x - 1 \Rightarrow f'(x) = 2x - 1 \end{aligned}$$

1.4 Left Derivative and Right Derivative

Left Derivative and Right Derivative

Definition 1.3. The **left derivative** of a function at a point c is:

$$f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}.$$

The **right derivative** of a function at a point c is:

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}.$$

A function is **differentiable** at a point c if the left derivative and the right derivative exist, are *finite*, and are *equal*.

Function Defined on an Interval

Definition 1.4. A function $y = f(x)$ is differentiable on a closed interval $[a; b]$ if it is differentiable at all internal points of $[a; b]$, and if the right derivative at a and the left derivative at b exist and are finite.

2 Tangent Line to the Graph of a Function

In general, given the function $y = f(x)$, the equation of the tangent line to the graph of f at the point $(x_0; y_0)$, if such a line exists and is not parallel to the y-axis, is:

$$y - y_0 = f'(x_0) \cdot (x - x_0)$$

2.1 Stationary Points

Stationary Point

Definition 2.1. For the function $y = f(x)$ and its point $x = c$, if $f'(c) = 0$, then $x = c$ is called a **stationary point** or a *point of horizontal tangency.*

3 Continuity and Differentiability

Theorem 3.1. *If a function $f(x)$ is differentiable at the point x_0, then the function is also continuous at that point.*

Proof.
Hypothesis:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Thesis:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

We write the relation

$$f(x_0 + h) = f(x_0) + \frac{f(x_0 + h) - f(x_0)}{h} \cdot h$$

which, after calculations, turns out to be an identity. Calculating the limit for $h \to 0$ on both sides, remembering that the limit of a sum is equal to the sum of the limits:

$$\lim_{h \to 0} f(x_0 + h) = \lim_{h \to 0} f(x_0) + \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot h.$$

In the second term, since the limit of a constant is the constant itself, we have:

$$\lim_{h \to 0} f(x_0) = f(x_0).$$

Moreover, since the limit of a product is equal to the product of the limits and recalling the hypothesis:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot 0 = f'(x_0) \cdot 0 = 0.$$

Therefore, substituting in the second term, the limit becomes:

$$\lim_{h\to 0} f(x_0+h) = f(x_0) + f'(x_0)\cdot 0$$

$$\lim_{h\to 0} f(x_0+h) = f(x_0.)$$

Setting $x_0 + h = x$, if $h \to 0$, it implies $x \to x_0$. Substituting into the previous relation, we conclude that the function $f(x)$ is continuous at x_0, as:

$$\lim_{x\to x_0} f(x) = f(x_0)$$

$\square$

Note 3. In the proof of the theorem, we have seen that the expression $\lim_{x\to 0} f(x_0+h) = f(x_0)$ is equivalent to $\lim_{x\to x_0} f(x) = f(x_0)$. We can therefore assume it as the definition of a continuous function: a function is continuous if $\lim_{h\to 0} f(x_0+h) = f(x_0)$.

From what we have said, we can assert that the set of differentiable functions is a subset of the set of continuous functions.

4 Fundamental Derivatives

Now let's determine the differentiation formulas for the most commonly used functions.

Theorem 4.1. *The derivative of a constant function is* 0*:*

$$\mathrm{D}k = 0$$

Proof. Remembering that if $f(x) = k$, then $f(x+h) = k$, calculate:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{k-k}{h} = 0.$$

□

Theorem 4.2. *The derivative of the function $f(x) = x$ is $f'(x) = 1$;*

$$\mathrm{D}x = 1.$$

Proof. If $f(x) = x$, then $f(x+h) = x + h$. Calculate $f'(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

□

Theorem 4.3. *The derivative of the function $f(x) = \sin(x)$, with x expressed in radians, is $f'(x) = \cos(x)$:*

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} [\sin(x) \cdot \frac{(\cos(h)-1)}{h} + \cos(x) \cdot \frac{\sin(h)}{h}]
\end{aligned}$$

Using known limits,

$$\lim_{h\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{1-\cos(x)}{x} = 0,$$

we have:

$$f'(x) = \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)$$

$\square$

Note 4. If x is measured in degrees:

$$\frac{d}{dx} \sin(x^\circ) = \frac{\pi}{180^\circ} \cdot \cos(x)$$

Theorem 4.4. *The derivative of the function $f(x) = \cos(x)$, with x expressed in radians, is $f'(x) = -\sin(x)$:*

$$\frac{d}{dx} \cos(x) = -\sin(x).$$

Note 5. If x is measured in degrees:

$$\frac{d}{dx} \cos(x^\circ) = -\frac{\pi}{180^\circ} \cdot \sin(x)$$

Theorem 4.5. *The derivative of the function $f(x) = a^x (a \in \mathbb{R}^+)$ is $f'(x) = a^x \ln(a)$:*

$$\frac{d}{dx} a^x = a^x \ln a.$$

Proof. Applying the definition of derivative to the function $f(x) = a^x$, we get:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\ &= \lim_{h\to 0} a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h} \end{aligned}$$

Using the well-known limit

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a,$$

we get

$$a^x \ln a.$$

$\square$

In particular

$$\frac{d}{dx}e^x = e^x.$$

Theorem 4.6. *The derivative of the function* $f(x) = \log_a x \quad (a \in \mathbb{R}^+ - \{1\}, x \in \mathbb{R}^+)$ *is* $f'(x) = \frac{1}{x} \cdot \log_a e$*:*

$$\frac{d}{dx}\log_a x = \frac{1}{x} \cdot \log_a e.$$

Proof.

$$\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log_a(x+h) - \log_a x}{h}.$$

Using the logarithmic property $\log_a x - \log_a y = \log_a \frac{x}{y}$, we write:

$$\log_a(x+h) - \log_a x = \log_a \frac{x+h}{h} = \log_a \left(1 + \frac{h}{x}\right),$$

$$f'(x) = \lim_{h \to 0} \frac{\log_a\left(1 + \frac{h}{x}\right)}{h}.$$

Multiplying and dividing the denominator h by x, we have

$$\lim_{h \to 0} \frac{\log_a\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \cdot \frac{1}{x} = \log_a e \cdot \frac{1}{x},$$

and therefore:

$$f'(x) = \frac{1}{x} \cdot \log_a e.$$

□

In particular, for $a = e$ we have:

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

5 Theorems on the Calculation of Derivatives

5.1 Derivative of the Product of a Constant and a Function

Theorem 5.1. *The derivative of the product of a constant k and a differentiable function $f(x)$ is equal to the product of the constant and the derivative of the function:*

$$\mathrm{D}[k \cdot f(x)] = k \cdot f'(x).$$

Proof.

$$y' = \lim_{h \to 0} \frac{k \cdot f(x+h) - k \cdot f(x)}{h} = \lim_{h \to 0} \frac{k \cdot [f(x+h) - f(x)]}{h}.$$

Since k is a constant and recalling the definition of the derivative, we can write:

$$y' = k \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = k \cdot f'(x).$$

□

5.2 Derivative of the Sum of Functions

Theorem 5.2. *The derivative of the algebraic sum of two or more differentiable functions is equal to the algebraic sum of the derivatives of the individual functions:*

$$\mathrm{D}[f(x) + g(x)] = f'(x) + g'(x).$$

Proof. Let's calculate the limit of the incremental ratio of $f(x) + g(x)$:

$$\begin{aligned} y' &= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} = \\ &= \lim_{h \to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} = \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \\ &= f'(x) + g'(x) \end{aligned}$$

□

5.3 Derivative of the Product of Functions

Theorem 5.3. *The derivative of the product of two differentiable functions is equal to the sum of the derivative of the first function multiplied by the second non-derivative and the derivative of the second function multiplied by the first non-derivative:*

$$\mathrm{D}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Proof.

$$y' = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}.$$

In the numerator, let's add and subtract the product $g(x+h) \cdot f(x)$ to the first term:

$$y' = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - g(x+h) \cdot f(x) + g(x+h) \cdot f(x) - f(x) \cdot g(x)}{h}.$$

Factor out $g(x+h)$ from the first two terms and $f(x)$ from the last two terms:

$$y' = \lim_{h \to 0} \frac{g(x+h) \cdot [f(x+h) - f(x)] + f(x) \cdot [g(x+h) - g(x)]}{h}.$$

The limit of a sum is equal to the sum of the limits, so:

$$y' = \lim_{h \to 0} \left[g(x+h) \cdot \frac{f(x+h) - f(x)}{h} \right] + \lim_{h \to 0} \left[f(x) \cdot \frac{g(x+h) - g(x)}{h} \right].$$

The limit of a product is equal to the product of the limits:

$$y' = \lim_{h \to 0} g(x+h) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

Since $f(x)$ and $g(x)$ are differentiable and hence continuous by assumption, we have:

$$y' = g(x) \cdot f'(x) + f(x) \cdot g'(x).$$

□

Extending the theorem to the product of more functions, it can be shown that, for example, given the function $y = f(x) \cdot g(x) \cdot z(x)$, its first derivative is:

$$y' = f'(x) \cdot g(x) \cdot z(x) + f(x) \cdot g'(x) \cdot z(x) + f(x) \cdot g(x) \cdot z'(x).$$

In general, the derivative of the product of multiple differentiable functions is the sum of the products of the derivative of each function by the other non-derivative functions.

5.4 Derivative of the Power of a Function

Theorem 5.4. *The derivative of the n-th power of a differentiable function (with exponent $n \in \mathbb{N}$ and $n > 1$) is equal to the product of the exponent n and the function raised to the power of $n-1$, multiplied by the derivative of the function itself:*

$$\mathrm{D}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x).$$

Proof.

$$y = \underbrace{f(x) \cdot f(x) \cdot f(x) \cdot ... \cdot f(x)}_{n \text{ factors}}$$

Using the theorem of the derivative of the product of multiple functions, we have:

$$y' = f'(x) \cdot \underbrace{f(x) \cdot ... \cdot f(x)}_{(n-1) \text{ factors}} + f(x) \cdot f'(x) \cdot ... \cdot f(x) + ... + \underbrace{f(x) \cdot f(x) \cdot ...}_{(n-1) \text{ factors}} \cdot f'(x).$$

In each of the n terms, the factor $f(x)$ appears $(n-1)$ times. Therefore:

$$y' = n \cdot [f(x)]^{n-1} \cdot f'(x).$$

□

It can be shown that the above theorem is also valid when the exponent of the power is any rational number.

$$\mathrm{D}[f(x)]^a = a \cdot [f(x)]^{a-1} \cdot f'(x), \quad a \in \mathbb{Q}.$$

Derivative of a Power of x If $f(x) = x$, since the derivative $f'(x) = 1$, we have:

$$\mathrm{D}x^a = a \cdot x^{a-1}, \quad a \in \mathbb{Q}.$$

5.5 Derivative of the Reciprocal of a Function

Theorem 5.5. *The derivative of the reciprocal of a non-zero differentiable function is equal to a fraction where:*

- *the numerator is the opposite of the derivative of the function;*
- *the denominator is the square of the function.*

$$\mathrm{D}\frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}, \quad \text{with } f(x) \neq 0.$$

Proof.

$$\begin{aligned} y' &= \lim_{h \to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \lim_{h \to 0} \frac{\frac{f(x) - f(x+h)}{f(x)f(x+h)}}{h} = \\ &= \lim_{h \to 0} \left[-\frac{f(x+h) - f(x)}{h} \cdot \frac{1}{f(x)f(x+h)} \right] = \\ &= -\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} \frac{1}{f(x)f(x+h)}. \end{aligned}$$

Since $f(x)$ is differentiable, we have:

$$y' = -\frac{f'(x)}{f^2(x)}.$$

□

Note 6. The function value must be different from 0 at the points where we calculate the derivative.

5.6 Derivative of the Quotient of Two Functions

Theorem 5.6. *The derivative of the quotient of two differentiable functions (with a non-zero divisor function) is equal to a fraction with:*

- *the numerator as the difference between the derivative of the dividend multiplied by the non-derivative divisor and the non-derivative dividend multiplied by the derivative of the divisor;*
- *the denominator as the square of the divisor.*

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}, \quad g(x) \neq 0$$

Proof. Consider the quotient function as the product of two functions:

$$y = f(x)\cdot\frac{1}{g(x)}.$$

Apply the product rule of differentiation:

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = \mathrm{D}\left[f(x)\cdot\frac{1}{g(x)}\right] = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\mathrm{D}\left[\frac{1}{g(x)}\right].$$

Apply the rule of differentiation of the reciprocal of a function:

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left[\frac{-g'(x)}{g^2(x)}\right].$$

Combine with a common denominator and conclude:

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}.$$

□

From the above theorem, we can derive the derivatives of the tangent and cotangent functions as special cases.

Tangent Function Derivative Express $y = \tan(x)$ as $y = \dfrac{\sin(x)}{\cos(x)}$ and, applying the quotient rule, we have:

$$y' = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)} = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)}.$$

This result can also be written in the following ways:

$$y' = \frac{1}{\cos^2(x)} \quad \text{or} \quad y' = 1 + \tan^2(x).$$

Cotangent Function Derivative Similarly, since we can rewrite $y = \cot(x)$ as $y = \dfrac{\cos(x)}{\sin(x)}$, we can derive that:

$$y' = -\frac{1}{\sin^2(x)} \quad \text{or} \quad y' = -(1 + \cot^2(x)).$$

6 Derivative of a Composite Function

Let $z = g(x)$ be a function of the variable x, from domain A to codomain B, and let $y = f(x)$ be a function of the variable z, from domain B to codomain C. The composite function $y = f(g(x))$ is a composite function (or function of a function) because y is a function of z, which in turn is a function of x. The two functions $z = g(x)$ and $y = f(x)$ are called the *components* of the composite function.

Theorem 6.1. *If the function g is differentiable at the point x and the function f is differentiable at the point $z = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x and its derivative is the product of the derivatives of f with respect to z and g with respect to x:*

$$\mathrm{D}[f(g(x))] = f'(z) \cdot g'(x), \quad z = g(x).$$

Generalizing for a polynomial $P(x)$:

$$\mathrm{D}[P(x)]^n = n[P(x)]^{n-1} P'(x).$$

The derivative of a composite function can also be calculated directly without making substitutions. The above theorem can be extended to the derivative of a function y dependent on the variable x through any number of component functions.

For example, in the case of three functions, being

$$y = f(g(z(x))),$$

setting

$$t = z(x), \quad u = g(t), \quad y = f(u),$$

the formula for the derivative of the composite function can be written as:

$$\mathrm{D}f(g(z(x))) = f'(u) \cdot g'(t) \cdot z'(x).$$

7 The derivative of $[f(x)]^{g(x)}$

Using the formulas related to the derivative of a composite function and the derivative of a product, we can study a method for calculating the derivative of the function

$$y = [f(x)]^{g(x)}$$

where $f(x) > 0$ and $f(x)$ and $g(x)$ are differentiable functions.

Given the function

$$y = [f(x)]^{g(x)},$$

since $f(x) > 0$, it is also $f(x)^{g(x)} > 0$, so we can calculate the logarithms of both sides:

$$\ln y = \ln[f(x)]^{g(x)}.$$

Applying the property of the logarithm of a power, we have:

$$\ln y = g(x) \cdot \ln[f(x)]$$

If we now apply the theorems for the derivative of composite functions and the product of two functions to both sides of the equation, we obtain

$$\frac{1}{y} \cdot y' = g'(x) \cdot \ln[f(x)] + g(x) \cdot \frac{1}{f(x)} \cdot f'(x),$$

from which, considering y different from 0:

$$y' = y \cdot \left[g'(x) \cdot \ln[f(x)] + \frac{g(x) \cdot f'(x)}{f(x)} \right].$$

Since $y = [f(x)]^{g(x)}$, we can write:

$$y' = [f(x)]^{g(x)} \cdot \left[g'(x) \cdot \ln[f(x)] + \frac{g(x) \cdot f'(x)}{f(x)} \right].$$

In conclusion, we can highlight the following formula for the derivative of the function $y = [f(x)]^{g(x)}$:

$$\mathrm{D}[f(x)]^{g(x)} = [f(x)]^{g(x)} \cdot \left[g'(x) \cdot \ln[f(x)] + \frac{g(x) \cdot f'(x)}{f(x)} \right].$$

If in the function $[f(x)]^{g(x)}$ we take $g(x) = a$ ($a \in \mathbb{R}$), and apply the previous rule, we get:

$$\mathrm{D}[f(x)]^{a} = [f(x)]^{a} \cdot \frac{a \cdot f'(x)}{f(x)} = a \cdot [f(x)]^{a-1} \cdot f'(x).$$

So the derivative rule for powers of a function is true for every $a \in \mathbb{R}$.

In particular, if $x > 0$ and $a \in \mathbb{R}$, we have:

$$\mathrm{D}x^{a} = a \cdot x^{a-1}.$$

8 The derivative of an inverse function

Theorem 8.1. *Consider the function $y = f(x)$ defined and invertible in the interval I, and its inverse function $x = f^{-1}(y)$. If $f(x)$ is differentiable with a derivative different from 0 at every point in I, then $f^{-1}(y)$ is also differentiable, and the following relationship holds:*

$$\mathrm{D}[f^{-1}(y)] = \frac{1}{f'(x)}, \quad \text{with } x = f^{-1}(y).$$

Assuming that both derivatives exist, to justify the relationship between them, we recall that

$$f^{-1}[f(x)] = x.$$

Differentiating both sides of this equation, we have

$$\mathrm{D}[f^{-1}(y)] \cdot f'(x) = 1,$$

from which we obtain:

$$\mathrm{D}[f^{-1}(y)] = \frac{1}{f'(x)}.$$

Of particular interest is the application of the theorem in calculating the derivatives of inverse trigonometric functions. The function $y = \arcsin x$, defined for $x \in [-1, 1]$, is the inverse of $x = \sin y$, with $y \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

Furthermore, the sine function is differentiable in $]-\frac{\pi}{2}; \frac{\pi}{2}[$ with a non-zero derivative. By the previous theorem, the function $\arcsin x$ is differentiable in $]-1; 1[$ and we have:

$$\mathrm{D} \arcsin x = \frac{1}{\mathrm{D} \sin y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

$$\mathrm{D} \arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

Similarly, the following formulas can be obtained:

$$\mathrm{D} \arccos x = -\frac{1}{\sqrt{1 - x^2}}$$
$$\mathrm{D} \arctan x = \frac{1}{1 + x^2}$$
$$\mathrm{D}\ \mathrm{arccot} x = -\frac{1}{1 + x^2}$$

9 Derivatives of order higher than the first

Consider the function:

$$y = f(x) = x^3 + 2x^2 - x - 1, \quad x \in \mathbb{R}.$$

Its derivative,

$$y' = 3x^2 + 4x - 1,$$

is, in turn, a function of the variable x, defined for $x \in \mathbb{R}$. We can also calculate the derivative of this function:

$$\mathrm{D}y' = 6x + 4.$$

This derivative is called the **second derivative** of the function $f(x)$ and is denoted by the symbol:

$$y'' \quad \text{or} \quad f''(x).$$

The obtained second derivative is also a function that we can differentiate; differentiating it, we get the **third derivative**:

$$y''' = 6.$$

In general, given a function $y = f(x)$, with the examined procedure, we can obtain the second, third, fourth derivatives, and so on. These are called **higher-order derivatives** of the given function.

Note 7. From the fourth derivative onwards, we use the number in parentheses:

$$y^{(4)}, y^{(5)}, y^{(6)}, \ldots$$

10 The differential of a function

Let $f(x)$ be a differentiable and therefore continuous function in an interval, and let x and $(x + \Delta x)$ be two points in that interval.

Differential

Definition 10.1. The differential of a function $f(x)$, relative to the point x and the increment Δx, is the product of the derivative of the function calculated at x and the increment Δx. The differential is indicated by $df(x)$ or dy:

$$dy = f'(x) \cdot \Delta x.$$

Note 8. Note that the differential depends on two elements: the point x where we calculate the differential and the increment Δx we consider.

Example 10.1. The differential of the function

$$y = 2x^3 + 3$$

is

$$dy = 6x^2 \cdot \Delta x,$$

which for $x = 1$ and $\Delta x = 0.3$ is

$$dy = 6 \cdot (1)^2 \cdot 0.3 = 1.8,$$

while for $x = 2$ and $\Delta x = 0.2$ is

$$dy = 6 \cdot (2)^2 \cdot 0.2 = 4.8.$$

Consider the function

$$y = x$$

and calculate its differential:

$$dy = 1 \cdot \Delta x.$$

Therefore:

$$dx = \Delta x.$$

This means that the *differential of the independent variable x is equal to the increment of the variable itself.*

Substituting into the definition of the differential, we can write

$$dy = f'(x) \cdot dx,$$

which means that the *differential of a function is equal to the product of its derivative and the differential of the independent variable.*

From this last relation, by solving for $f'(x)$, we have:

$$f'(x) = \frac{dy}{dx}.$$

The first derivative of a function is thus the ratio of the differential of the function to that of the independent variable.

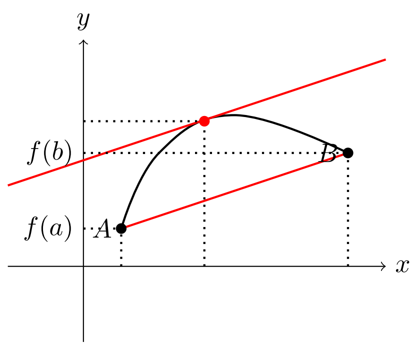
11 Theorems on differentiable functions

11.1 Lagrange's theorem

Lagrange's theorem

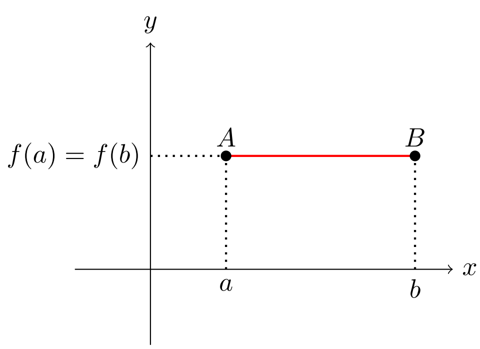
Theorem 11.1. *If a function $f(x)$ is continuous on a closed interval $[a;b]$ and is differentiable at every interior point, there exists at least one point c in the interval $[a;b]$ such that the relation holds:*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$



From Lagrange's theorem, the following theorems follow.

Theorem 11.2. *If a function $f(x)$ is continuous on the interval $[a;b]$, differentiable on $]a;b[$, and such that $f'(x)$ is zero at every interior point of the interval, then $f(x)$ is constant throughout $[a;b]$.*



Proof. **Hypotheses:**

1. $f(x)$ is continuous in $[a;b]$;
2. $f'(x) = 0$ in $]a;b[$.

Thesis:

$$f(x) = k \text{ in } [a; b].$$

Applying the Lagrange theorem in the interval $[a; x]$, where x is any point in $[a; b]$ different from a, we can write, with $c \in]a; x[$:

$$\frac{f(x) - f(a)}{x - a} = f'(c) = 0 \to f(x) - f(a) = 0 \to f(x) = f(a).$$

Therefore, f is constant throughout $[a; b]$. □

Theorem 11.3. *If $f(x)$ and $g(x)$ are two functions continuous in the interval $[a; b]$, differentiable in $]a; b[$, and such that $f'(x) = g'(x)$ for every $x \in]a; b[$, then they differ by a constant.*

Proof. **Hypotheses:**

1. $f(x)$ and $g(x)$ are continuous in $[a; b]$;
2. $f'(x) = g'(x)$ in $]a; b[$.

Thesis:

$$f(x) - g(x) = k \text{ in } [a; b].$$

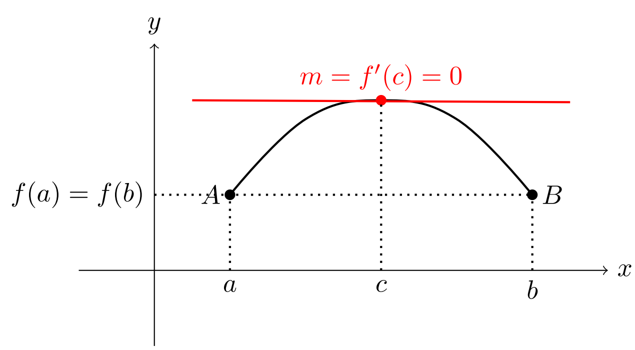
Calling $z(x)$ the difference between the given functions, i.e., $z(x) = f(x) - g(x)$, we have $z'(x) = f'(x) - g'(x)$. By hypothesis $f'(x) = g'(x)$, so $z'(x) = 0$, for every x in $]a; b[$. According to the previous theorem, $z(x) = k$ throughout $[a; b]$, and thus $f(x) - g(x) = k$. □

11.2 The Rolle's Theorem

If in the Lagrange theorem we add the hypothesis $f(a) = f(b)$, then $f'(c) = 0$. The following theorem is obtained.

Rolle's Theorem

Theorem 11.4. *If, for a function $f(x)$ continuous in the interval $[a; b]$ and differentiable at the interior points of this interval, the condition $f(a) = f(b)$ holds, then there exists at least one point c inside the interval such that $f'(c) = 0$.*



11.3 Cauchy's Theorem

Cauchy's Theorem

Theorem 11.5. *If the functions $f(x)$ and $g(x)$ are continuous on the interval $[a;b]$, differentiable at every interior point of this interval, and furthermore, $g'(x) \neq 0$ for all x in $]a;b[$, then there exists at least one point c in the interval $[a;b]$ such that:*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

In other words, the ratio of the increments of the functions $f(x)$ and $g(x)$ in the interval $[a;b]$ is equal to the ratio of their respective derivatives calculated at a particular point c within the interval.

11.4 L'Hôpital's Rule

The calculation of derivatives and the theorems studied so far are also useful for computing certain limits that appear in an indeterminate form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. This is possible thanks to the following theorem.

L'Hôpital's Rule

Theorem 11.6. *Given a neighborhood I of a point c and two functions $f(x)$ and $g(x)$ defined in I (excluding possibly c), if:*

- *$f(x)$ and $g(x)$ are differentiable in I with $g'(x) \neq 0$,*
- *both functions tend to either 0 or ∞ as $x \to c$,*
- *the limit of the ratio $\frac{f'(x)}{g'(x)}$ of their derivatives exists as $x \to c$,*

then the limit of the ratio of the functions exists and is:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

The theorem also extends to the limit as $x \to +\infty$ (or $-\infty$). In this case, the conditions of the theorem do not have to be true for a neighborhood of a point; instead, there must exist a value $M > 0$ such that these conditions are satisfied for all $x > M$ (or $x < -M$). The relation is then:

$$\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{x\to +\infty} \frac{f'(x)}{g'(x)},$$

and a similar relationship holds for $x \to -\infty$.

In cases where the limit of the ratio of derivatives itself appears as an indeterminate form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and the functions $f'(x)$ and $g'(x)$ satisfy the assumptions of the theorem, one can proceed to the limit of the ratio of second derivatives, and so on for successive derivatives.