Induction Principle

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Induction Principle

Theorem 0.1. *Let A be a property, then $A(\varphi)$ holds for all $\varphi \in \mathrm{PROP}$ if*

1. $\forall i \in \mathbb{N}, A(p_i)$ and $A(\perp)$,
2. $A(\varphi), A(\psi) \Rightarrow A((\varphi \square \psi))$,
3. $A(\varphi) \Rightarrow A((\neg\varphi))$.

Note 1. p_i and $\perp$ are called *atoms* or *atomic propositions*, meaning they are indivisible. φ and ψ are called *meta-variables* and indicate both atomic and compound propositions.

We call an application of the previous theorem an induction proof on φ.

Example 0.1. Every proposition has an even number of parentheses.

Proof.

1. Every atom has 0 parentheses, and 0 is an even number.
2. Suppose φ and ψ have respectively $2n$ and $2m$ parentheses, then $(\varphi \square \psi)$ has $2n + 2m + 2 = 2(n + m + 1)$.
3. Suppose φ has $2n$ parentheses, then $(\neg\varphi)$ has $2n + 2 = 2(n + 1)$.

$\square$

Example 0.2. Every proposition has a formation sequence.

Proof.

1. Let φ be an atom, then the formation sequence of φ consists solely of φ.

2. Let $\varphi_0, ..., \varphi_n$ and $\psi_0, ..., \psi_m$ be formation sequences of φ and ψ, then it follows that $\varphi_0, ..., \varphi_n,\ \psi_0, ..., \psi_m,\ (\varphi_n \Box \psi_m)$ is a formation sequence of $(\varphi_n \Box \psi_m)$.

3. Let $\varphi_0, ..., \varphi_n$ be a formation sequence of φ, then $\bot, \varphi_0, ..., \varphi_n$ is a formation sequence of $(\neg\varphi)$.

$\square$