RIOT
INVESTMENT SOCIETY

Integrals

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Part I
Integrals

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1.1 Primitives

Primitive of a function

Definition 1.1. A function $F(x)$ is said to be a primitive of the function $f(x)$ defined in the interval $[a; b]$ if $F(x)$ is differentiable in the entire interval $[a; b]$ and its derivative is $f(x)$.

The primitive of a function is not unique.

In general, if a function $f(x)$ has a primitive $F(x)$, then it has infinitely many primitives of the form $F(x) + c$, where c is any real number. Indeed, since the derivative of a constant is zero:

$$\mathrm{D}\left[F(x) + c\right] = F'(x) = f(x), \quad \forall x \in \mathbb{R}.$$

Conversely, if two functions $F(x)$ and $G(x)$ are primitives of the same function $f(x)$, then the two functions differ by a constant,

$$\mathrm{D}\left[F(x) - G(x)\right] = F'(x) - G'(x) = f(x) - f(x) = 0,$$

and therefore

$$F(x) - G(x) = c.$$

We conclude that if $F(x)$ is a primitive of $f(x)$, then the functions $F(x) + c$, where c is any real number, are **all** and **only** the primitives of $f(x)$.

1.2 Indefinite Integral

Indefinite Integral

Definition 1.2. The indefinite integral of the function $f(x)$, denoted by

$$\int f(x)\, dx,$$

is the set of all primitives $F(x) + c$ of $f(x)$, where $c \in \mathbb{R}$.

$$\int f(x)\, dx = F(x) + c \Leftrightarrow \mathrm{D}\left[F(x) + c\right] = f(x)$$

Note 1. The symbol $\int f(x)\, dx$ is read as *indefinite integral of $f(x)$ with respect to dx.*

The primitive $F(x)$ obtained for $c = 0$ is called the **fundamental primitive**.

In the expression $\int f(x)\, dx$, the function $f(x)$ is called the **integrand function**, and the variable x is the **integration variable**.

From the previous definition, since

$$\frac{d}{dx} F(x) = f(x),$$

it follows that

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

This means that indefinite integration acts as the inverse operation of differentiation.

A function that admits a primitive (and hence infinitely many primitives) is said to be **integrable**.

Sufficient Condition for Integrability

Theorem 1.1. *If a function is continuous on* $[a;b]$*, then it admits primitives on the same interval.*

However, determining primitives even for fairly simple continuous functions is not always straightforward.

Properties of Indefinite Integration

First Linearity Property

Definition 1.3. The indefinite integral of a sum of integrable functions is equal to the sum of the indefinite integrals of the individual functions:

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Indeed, if we differentiate both sides, we respectively obtain:

$$\frac{d}{dx}\left[\int [f(x)+g(x)]\,dx\right] = f(x)+g(x);$$

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\left[\int f(x)\,dx\right] + \frac{d}{dx}\left[\int g(x)\,dx\right] = f(x)+g(x).$$

The two sides have the same derivative, thus representing primitives of the same function.

Second Linearity Property

Definition 1.4. The integral of the product of a constant by an integrable function is equal to the constant multiplied by the integral of the function:

$$\int k\cdot f(x)\,dx = k\cdot\int f(x)\,dx.$$

Again, if we differentiate both sides, we respectively obtain:

$$\frac{d}{dx}\left[\int k\cdot f(x)\,dx\right] = k\cdot f(x);$$

$$\frac{d}{dx}\left[k \cdot \int f(x)\,dx\right] = k \cdot \frac{d}{dx}\left[\int f(x)\,dx\right] = k \cdot f(x).$$

The two sides have the same derivative, thus representing primitives of the same function.

These linearity properties can be expressed in a single formula:

$$\int [c_1 f(x) + c_2 g(x)]\,dx = c_1 \int f(x)\,dx + c_2 \int g(x)\,dx.$$

The integral is also called a **linear operator**.

Note 2. It is always important to remember that:

1. $\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$,
2. $\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$.

2 Basic Indefinite Integrals

From the rules of differentiation of elementary functions, we derive fundamental indefinite integrals.

$$\int x^a \, dx = \frac{x^{a+1}}{a+1} + c, \quad a \neq -1$$
$$\int \frac{1}{x} \, dx = \ln|x| + c$$
$$\int e^x \, dx = e^x + c$$
$$\int a^x \, dx = \frac{a^x}{\ln a} + c$$
$$\int \sin x \, dx = -\cos x + c$$
$$\int \cos x \, dx = \sin x + c$$
$$\int \frac{1}{\cos^2 x} \, dx = \tan x + c$$
$$\int \frac{1}{\sin^2 x} \, dx = -\cot x + c$$
$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$$
$$\int \frac{1}{1+x^2} \, dx = \text{arccot } x + c$$

Integration of Composite Functions

To integrate the power of a (composite) function using the power rule, it is necessary for the integrand function to be multiplied by the derivative of the more *internal* function in the composition. The same approach applies to calculate integrals of other composite functions reducible to different integration rules.

$$\int [f(x)]^a \cdot f'(x)\,dx = \frac{[f(x)]^{a+1}}{a+1} + c, \quad a \neq -1$$
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$
$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$
$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$
$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$
$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$
$$\int \frac{f'(x)}{\cos^2 f(x)}\,dx = \tan f(x) + c$$
$$\int \frac{f'(x)}{\sin^2 f(x)}\,dx = -\cot f(x) + c$$
$$\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx = \arcsin f(x) + c$$
$$\int \frac{f'(x)}{1+[f(x)]^2}\,dx = \arctan f(x) + c$$
$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \arcsin \frac{f(x)}{|a|} + c, \quad a \neq 0$$
$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\arctan \frac{f(x)}{a} + c, \quad a \neq 0$$
$$\int \tan x\,dx = -\ln|\cos x| + c$$
$$\int \cot x\,dx = \ln|\sin x| + c$$

3 The Definite Integral

The introduction of definite integrals arises from the necessity to determine the areas of plane figures with curvilinear boundaries.

3.1 The Trapezoid

Given a function $y = f(x)$ and a closed and bounded interval $[a; b]$ where the function is continuous and positive (or zero), a **trapezoid** is defined as the plane figure bounded by the x-axis, the lines parallel to the y-axis passing through the endpoints of the interval $[a;b]$, and the graph of the function f over this interval. Essentially, it is a quadrilateral with vertices $A(a;0)$, $B(b;0)$, $C(b;f(b))$, and $D(a;f(a))$.

The area S of a trapezoid cannot be calculated directly, but it can be approximated using the following procedure:

- Divide the interval $[a;b]$ into n equal parts of width $h = \dfrac{b-a}{n}$.
- Consider n rectangles, each having a base segment of the subdivision and a height equal to the minimum m the function assumes in that interval.
- Let s_n be the sum of the areas of all these n rectangles:

$$s_n = m_1 h + m_2 h + ... + m_n h.$$

The area of the trapezoid is approximated from below by s_n.

Similarly, we can approximate the area of the trapezoid from above, by summing the areas of rectangles associated with a partition of the interval $[a;b]$ into n equal parts and having heights equal to the maximum M_i of the function in the corresponding interval. Let's denote this sum as S_n:

$$S_n = M_1 h + M_2 h + ... + M_n h.$$

Thus, we obtain two sequences of areas s_n and S_n such that, for each n, the area S of the trapezoid lies between the underestimate and the overestimate, i.e., we can write:

$$s_n \leq S \leq S_n.$$

3.2 Definite Integral of a Non-Negative Function

The approximation of the areas s_n and S_n becomes better as the intervals of division of $[a;b]$ become smaller.

Theorem 3.1. *If a function $f(x)$ is continuous and non-negative (or zero) on the interval $[a;b]$, the limits as $n \to +\infty$ of the sequences s_n and S_n exist and are finite and coincident.*

Definite Integral ($f(x) \geq 0$)

Definition 3.1. Given a function $f(x)$ continuous and non-negative or zero on $[a;b]$, the definite integral over the interval $[a;b]$ is defined as the common value of the limit as $n \to +\infty$ of the two sequences s_n (underestimate) and S_n (overestimate). This value is denoted by:

$$\int_a^b f(x)\, dx.$$

Note 3. The symbol $\int$ represents an elongated S to remind that, in graphical representation, an integral corresponds to a sum of areas of rectangles with height $f(x)$ and base dx.

The definite integral, since $f(x) \geq 0$, provides the measure of the area of the trapezoid related to $f(x)$ with endpoints a and b. a and b are called the integration limits; a is the **lower limit**, b is the **upper limit**. The function $f(x)$ is called the **integrand function**.

Unlike the indefinite integral, which is a set of functions, the definite integral is a number and does not depend on the variable x.

3.3 General Definition of Definite Integral

Consider a function $y = f(x)$ continuous on $[a; b]$ and divide the interval into n closed intervals using i points $x_0, x_1, x_2, x_3, ..., x_n$, with:

$$x_0 < x_1 < x_2 < x_3 < ... < x_n.$$

x_0 coincides with a; x_n coincides with b.

The widths of the intervals may vary and are given by:

$$\begin{aligned}
\Delta x_1 &= x_1 - a, \\
\Delta x_1 &= x_2 - x_1, \\
\Delta x_1 &= x_3 - x_2, \\
&\vdots \\
\Delta x_n &= b - x_{n-1}.
\end{aligned}$$

For each of the intervals, select any point within the interval:

$$c_1, c_2, c_3, ..., c_n.$$

Consider the corresponding values of the function:

$$f(c_1), f(c_2), f(c_3), ..., f(c_n).$$

Then, write the sum $\bar{S}$ given by:

$$\bar{S} = f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2 + f(c_3) \cdot \Delta x_3 + ... + f(c_n) \cdot \Delta x_n.$$

The sum $\bar{S}$ depends on:

- the number of divisions;
- the widths Δx_n of the intervals;
- the points c_n chosen within the different intervals.

Among the widths of the intervals, denote the maximum as Δx_{max}: if $\Delta x_{max} \to 0$, all other widths tend to 0 as well.

It can be shown that if Δx_{max} tends to 0, all sums $\bar{S}$, obtained by choosing the division of the interval and the points within the different intervals in any way, tend to the same value S.

We then give the following definition.

Definite Integral

Definition 3.2. Given a function $f(x)$, continuous on $[a; b]$, the definite integral over the interval $[a; b]$ is defined as the value of the limit as Δx_{max} tends to 0 of the sum $\bar{S}$:

$$\int_a^b f(x)\,dx = \lim_{\Delta x_{max} \to 0} \bar{S}.$$

The previous definition, given for $f(x) \geq 0$, is a particular case of this.

According to this definition, **the definite integral can also be a negative or zero number and therefore, in general, does not correspond to the area enclosed between the graph of the function and the x-axis**. By convention, the following are set.

Definition 3.3.

- $$\int_a^a f(x)\,dx = 0;$$
- $$\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx \quad \text{if} a > b.$$

If a function has a definite integral in an interval $[a; b]$, the function is said to be **integrable on** $[a; b]$.

4 Properties of the Definite Integral

4.1 Additivity of the Integral with Respect to the Integration Interval

Definition 4.1. If $f(x)$ is integrable over $[a; c]$ and $a < b < c$, then it is also integrable over $[a; b]$ and $[b; c]$; conversely, if $a < b < c$ and $f(x)$ is integrable over $[a; b]$ and $[b; c]$, then it is also integrable over $[a; c]$. This leads to:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

4.2 Integral of the Sum of Functions

Definition 4.2. If $f(x)$ and $g(x)$ are integrable functions over $[a;b]$, then their sum $f(x)+g(x)$ is also integrable over $[a;b]$, and it holds that:

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

4.3 Integral of a Constant Times a Function

Definition 4.3. If $f(x)$ is an integrable function over $[a;b]$, then $k\cdot f(x)$ is also integrable over $[a;b]$ for $k\in\mathbb{R}$, and it holds that:

$$\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx.$$

4.4 Comparison between Integrals of Two Functions

Definition 4.4. If $f(x)$ and $g(x)$ are two continuous functions such that $f(x)\le g(x)$ for every point in the interval $[a;b]$, then the integral from a to b of $f(x)$ is less than or equal to the integral of $g(x)$:

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

4.5 Integral of the Absolute Value of a Function

Definition 4.5. If $f(x)$ is a continuous function over the interval $[a;b]$, then the absolute value of the integral from a to b of $f(x)$ is less than or equal to the integral of the absolute value of $f(x)$:

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

4.6 Integral of a Constant Function

Definition 4.6. If a function $f(x)$ is constant over the interval $[a;b]$, i.e., $f(x)=k$, then the integral from a to b of $f(x)$ is equal to the product of k and $(b-a)$:

$$\int_a^b k\,dx = k(b-a).$$

5 Fundamental Theorem of Calculus

5.1 Mean Value Theorem

We pose the following questions:

1. Is there a relationship between the indefinite integral $\int f(x)\ dx$ and the definite integral $\int_a^b f(x)\ dx$?

2. Is it possible to compute a definite integral $\int_a^b f(x)\ dx$?

To answer the first question, we introduce a theorem that relates the indefinite and definite integrals. This theorem is known as the *Fundamental Theorem of Calculus*. To prove it, we need to introduce another theorem, the mean value theorem.

Mean Value Theorem

Theorem 5.1. *If $f(x)$ is a continuous function over an interval $[a;b]$, there exists at least one point z in the interval such that:*

$$\int_a^b f(x)\ dx = (b-a)\cdot f(z), \quad z \in [a;b].$$

Proof. Since the function $f(x)$ is continuous on the interval $[a;b]$, by the Weierstrass theorem, the function attains its maximum value M and its minimum value m over $[a;b]$. Therefore, for every x belonging to $[a;b]$, the inequality holds:

$$m \leq f(x) \leq M.$$

By the properties of integrals, we also have:

$$\int_a^b m\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b M\ dx.$$

Applying the property of the integral of a constant function, we can write:

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$

Dividing all terms of the inequality by $(b-a)$, we get:

$$m \leq \frac{\int_a^b f(x)\ dx}{b-a} \leq M.$$

By the intermediate value theorem, the function must assume all values between its maximum and minimum, so there must exist a point z belonging to $[a;b]$ such that:

$$f(z) = \frac{\int_a^b f(x)\ dx}{b-a}.$$

Hence, there exists at least one point z belonging to $[a;b]$ such that:

$$\int_a^b f(x)\,dx = f(z)(b-a).$$

□

Note 4. Geometrically, if the function is positive on $[a;b]$, the mean value theorem expresses the equivalence between a trapezoid, whose area measures $\int_a^b f(x)\,dx$, and a rectangle, having equal base $b-a$. The height of the rectangle is given by the value of f at a particular point z in the interval $[a;b]$:

$$f(z) = \frac{\int_a^b f(x)\,dx}{b-a}.$$

5.2 The Integral Function

Let f be a continuous function on the interval $[a;b]$. Consider any point x in $[a;b]$. We define the **integral function** of f on $[a;b]$ as the function:

$$F(x) = \int_a^x f(t)\,dt,$$

which associates to each $x \in [a;b]$ the real number $\int_a^x f(t)\,dt$, where the independent variable x coincides with the upper limit of integration.

6 Fundamental Theorem of Calculus

6.1 Fundamental Theorem of Integral Calculus

Fundamental Theorem of Integral Calculus

Note 5. This theorem is also known as **Torricelli-Barrow**.

Theorem 6.1. *If a function $f(x)$ is continuous on $[a;b]$, then the derivative of its integral function exists*

$$F(x) = \int_a^x f(t)\,dt$$

for every point x in the interval $[a;b]$ and is equal to $f(x)$, that is:

$$F'(x) = f(x).$$

Thus $F(x)$ is an antiderivative of $f(x)$.

Proof. **Hypotheses**

1. $y = f(x)$ is continuous on $[a;b]$;

2. $F(x) = \int_a^x f(t)\,dt$.

Thesis

1. $F'(x)$ exists;

2. $F'(x) = f(x)$.

We prove the existence of the derivative of $F(x)$ and calculate it by applying the definition. Let's increase the variable x by a value $h \neq 0$ such that $a < x+h < b$ and calculate the difference $F(x+h) - F(x)$ using the expression of the integral function:

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

We apply the additivity property of the integral:

$$F(x+h) - F(x) = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_x^a f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

By the mean value theorem, the value of the integral is equal to the product of the width h of the integration interval by the value $f(z)$, where z is a particular point in the interval $[x; x+h]$, if $h > 0$, or in the interval $[x+h; x]$, if $h < 0$; hence we can write:

$$F(x+h) - F(x) = h \cdot f(z).$$

We divide both sides by h:

$$\frac{F(x+h) - F(x)}{h} = f(z).$$

We analyze the behavior of $f(z)$ as h tends to 0. Let $h > 0$; since z is between x and $x+h$, if h tends to 0 (from the right), then z tends to x (from the right) and $\lim_{h\to 0^+} f(z) = \lim_{z\to x^+} f(z) = f(x)$ because f is continuous by hypothesis. With a similar reasoning, if $h < 0$, it follows that $\lim_{h\to 0^-} f(z) = \lim_{z\to x^-} f(z) = f(x)$. Therefore:

$$\lim_{h\to 0} f(z) = \lim_{z\to x} f(z) = f(x).$$

We can conclude that the limit also exists, as h tends to 0, of the expression on the left-hand side, i.e., the incremental ratio of F at point x, and:

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} f(z) = f(x).$$

The function F is therefore differentiable and we have

$$F'(x) = f(x).$$

□

According to the theorem just proved, a function f continuous on $[a;b]$ has as a fundamental primitive the integral function $F(x)$, with x ranging in the interval $[a;b]$. Therefore, the indefinite integral of f, understood as the totality of its primitives, is expressed as:

$$\int f(x)\,dx = \int_a^x f(t)\,dt + c,$$

where c is any real constant.

6.2 Calculation of the Definite Integral

From the fundamental theorem of calculus, we can obtain the formula for calculating the definite integral. Let $\varphi(x)$ be any primitive of $f(x)$. From the fundamental theorem of calculus, we know that the integral function $F(x)$ is a particular primitive of the function f. Thus $\varphi(x)$ is of the form:

$$\varphi(x) = F(x) + c = \int_a^x f(t)\,dt + c,$$

where c is an arbitrary real constant.

- Let's calculate $\varphi(a)$ (we substitute the value a for the integration endpoint x);
$$\varphi(a) = \int_a^a f(t)\,dt + c = 0 + c = c.$$
- Let's calculate $\varphi(b)$ (we substitute the value b for the integration endpoint x);

$$\varphi(b) = \int_a^b f(t)\,dt + c. \quad \text{Since } \varphi(a) = c, \text{ we obtain:}$$
$$\varphi(b) = \int_a^b f(t)\,dt + \varphi(a).$$

We bring $\varphi(a)$ to the left-hand side,

$$\varphi(b) - \varphi(a) = \int_a^b f(t)\,dt,$$

and write the equality from right to left:

$$\int_a^b f(t)\,dt = \varphi(b) - \varphi(a).$$

Since there are no more variable ambiguities, we can reuse the variable x and write:

$$\int_a^b f(x)\,dx = \varphi(b) - \varphi(a).$$

It is customary to indicate the difference $\varphi(b) - \varphi(a)$ as $[\varphi(x)]_a^b$. The formula found allows us to reduce the calculation of a definite integral to that of an indefinite integral. This overcomes the difficulty of calculating the limit of the sequence s_n, which, in general, is not easy to determine.

7 The Mean Value of a Function

7.1 The Mean Value Theorem

The mean value theorem states the existence of a point z where the property expressed by the theorem holds, but it does not provide indications on how to calculate it. However, $f(z)$ can be derived from the equality

$$\int_a^b f(x)\,dx = (b-a)\cdot f(z)$$

by dividing both sides by $b - a$ and rewriting the equality from left to right:

$$f(z) = \frac{1}{b-a}\cdot\int_a^b f(x)\,dx.$$

The value $f(z)$ is defined as the **mean value** of the function $f(x)$ in the interval $[a; b]$.

Part II
Integration

8 Integration Methods

8.1 Integration by Substitution

When the integral is not immediately solvable, it can be useful to apply the **substitution method**, which involves making a variable change that allows rewriting the given integral in a form that we know how to solve. The substitution method can also be used to compute integrals whose primitive is a composite function.

In general, to compute $\int f(x)\,dx$ using the substitution method:

- Let $x = g(t)$, or $t = g^{-1}(x)$, where $g(t)$ is invertible with continuous and nonzero derivative $g'(t)$.
- Calculate the differential dx, or dt.
- Substitute into the given integral to obtain an integral in the variable t, and compute, if possible, the integral with respect to t.
- Use the initial position to write the result in terms of x.

8.2 Integration by Parts

Given two differentiable functions $f(x)$ and $g(x)$ with continuous derivatives on an interval $[a; b]$, consider the derivative of their product:

$$\mathrm{D}\,[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

Integrating both sides:

$$\int \mathrm{D}\,[f(x)\cdot g(x)]\,dx = \int [f'(x)\cdot g(x) + f(x)\cdot g'(x)]\,dx,$$

$$f(x)\cdot g(x) = \int f'(x)\cdot g(x)\,dx + \int f(x)\cdot g'(x)\,dx.$$

Isolating $\int f(x)\cdot g'(x)\,dx$, we get:

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx,$$

which is called the **integration by parts** formula. The formula is useful when the integrand can be thought of as a *product of two factors*. $f(x)$ is called the **finite factor**, and $g'(x)\,dx$ is called the **differential factor**.

In applying the formula, one of the functions, the finite factor, is only differentiated, while the other, the differential factor, is only integrated. Thus, it is necessary to choose the two factors appropriately. The formula involves another integral on the right-hand side, making this integration method useful for transforming a difficult integral into a more manageable one.

In general, in integrals of the form

$$\int x^n \sin x \, dx, \quad \int x^n \cos x \, dx, \quad \int x^n e^x \, dx,$$

x^n is considered as the finite factor, while in integrals of the form

$$\int x^n \ln x \, dx, \quad \int x^n \arctan x \, dx, \quad \int x^n \arcsin x \, dx,$$

$x^n \, dx$ is considered as the differential factor. In particular, in integrals like

$$\int \ln x \, dx, \quad \int \arctan x \, dx, \quad \int \arcsin x \, dx,$$

the differential factor is considered as $x^0 \, dx$, that is, $1 \cdot dx$.

8.3 Integration of Rational Functions

We now address the specific problem of calculating integrals of rational functions:

$$\int \frac{N(x)}{D(x)} \, dx,$$

where the numerator $N(x)$ and denominator $D(x)$ are polynomials. In our considerations, we assume that the degree of the denominator is less than the degree of the numerator. If this is not the case, we can always perform polynomial division of $N(x)$ by $D(x)$, obtaining a quotient polynomial $Q(x)$ and a remainder polynomial $R(x)$ of degree less than that of $D(x)$:

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

which yields:

$$\int \frac{N(x)}{D(x)} \, dx = \int \left[Q(x) + \frac{R(x)}{D(x)} \right] \, dx = \int Q(x) \, dx + \int \frac{R(x)}{D(x)} \, dx.$$

In the addition of the two integrals, the first one is calculable as it is the integral of a polynomial, while the second one is the integral of a rational function with a numerator of lesser degree than the denominator. Hence, we study integrals of the form $\int \frac{R(x)}{D(x)} \, dx$, where $R(x)$ is a polynomial of lesser degree than $D(x)$.

Numerator is the Derivative of the Denominator We have already seen that

$$\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + c,$$

which means the indefinite integral of a fraction where the numerator is the derivative of the denominator is equal to the natural logarithm of the absolute value of the denominator.

Denominator is of First Degree: $\int \frac{1}{ax+b}\,dx$ Also, the integral $\int \frac{1}{ax+b}\,dx, \quad a \neq 0$, where the algebraic fraction has a first-degree denominator, can be reduced to the case where the numerator is the derivative of the denominator.

Indeed, by multiplying the fraction by $\frac{a}{a}$ and applying the second property of linearity of the integral:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{a}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c.$$

Denominator is of Second Degree: $\int \frac{px+q}{ax^2+bx+c}\,dx$ To compute the integral $\int \frac{px+q}{ax^2+bx+c}\,dx, \quad a \neq 0$, different solving methods are used depending on the sign of the discriminant of the denominator $\Delta = b^2 - 4ac$.

Discriminant is Positive: $\Delta > 0$ **In general**, if $\Delta > 0$:

- Decompose the denominator: $ax^2+bx+c = a(x-x_1)(x-x_2)$;
- Write the given fraction as the sum of fractions with first-degree denominators:
$$\frac{px+q}{ax^2+bx+c} = \frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)};$$
- Calculate the sum of the two fractions on the right-hand side;
- Determine the values of A and B by equating the coefficients of x and the constant terms;
- Solve the integral $\int \left[\frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)}\right]\,dx$.

Discriminant is Zero: $\Delta = 0$ **In general**, if $\Delta = 0$:

- Decompose the denominator: $ax^2+bx+c = a(x-x_1)^2$;
- Write the given fraction as the sum of two fractions:
$$\frac{px+q}{ax^2+bx+c} = \frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)};$$
- Calculate the sum of the fractions on the right-hand side;
- Determine the values of A and B by equating the coefficients of x and the constant terms;
- Solve the integral $\int \left[\frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)}\right]\,dx$.

Discriminant is Negative: $\Delta < 0$ We examine two cases.

1. The numerator is of degree zero, meaning the integral is of the form:

$$\int \frac{1}{ax^2 + bx + c}\, dx, \quad a \neq 0$$

In general, to compute $\int \frac{1}{ax^2 + bx + c}\, dx$ if $\Delta < 0$:

- Factor out the coefficient of x^2:

$$\frac{1}{a} \int \frac{1}{x^2 + nx + n}\, dx;$$

- Write the denominator in the form: $(x + h)^2 + k^2$;
- Calculate the integral

$$\frac{1}{a} \int \frac{1}{(x + h)^2 + k^2}\, dx = \frac{1}{ak} \arctan \frac{x + h}{k} + c.$$

2. The numerator is a first-degree polynomial, i.e., the integral is of the form:

$$\int \frac{px + q}{ax^2 + bx + c}\, dx, \quad a, p \neq 0.$$

In general, to compute $\int \frac{px + q}{ax^2 + bx + c}\, dx$, with $a, p \neq 0$ and $\Delta < 0$:

- Manipulate the numerator to make it the derivative of the denominator;
- Write the integral as the sum of two integrals:

$$r \int \frac{2ax + b}{ax^2 + bx + c}\, dx + s \int \frac{1}{ax^2 + bx + c}\, dx;$$

- Calculate the first integral using the fact that $\int \frac{f'(x)}{f(x)}\, dx = \ln |f(x)| + c$, thus:

$$\int \frac{2ax + b}{ax^2 + bx + c}\, dx = \ln |ax^2 + bx + c| + c_1;$$

- Calculate the second integral using the method already seen;
- Sum the obtained results.

9 Definite Integral

9.1 Calculation of Volumes of Solid of Revolution

Consider the function $y = f(x)$, continuous over the interval $[a; b]$ and non-negative, and the trapezoid extended to the interval $[a; b]$. If we rotate the trapezoid around the x-axis by a full revolution, we obtain a solid of revolution. Let's calculate the volume of this solid.

We take the trapezoid and divide the interval $[a; b]$ into n equal parts. Each of these parts has a length $h = \frac{b-a}{n}$. In each interval of the subdivision, we consider the minimum m_i and maximum M_i of $f(x)$ and draw the inscribed rectangles in the trapezoid with heights m_i, which approximate the area of the trapezoid defectively, and the circumscribed rectangles with heights M_i, which approximate the area of the trapezoid excessively.

In the complete rotation around the x-axis, each rectangle describes a circular cylinder with height h and base radius m_i or M_i. The sum of the volumes of the n cylinders with the base as the circle of radius m_i approximates the volume of the initial solid of revolution defectively, and the sum of the volumes of the n cylinders with the base as the circle of radius M_i approximates the volume of the same solid excessively.

Since the formula for the volume of a circular cylinder with radius r and height h is $\pi r^2 h$, the volume v_n of the cylinders approximating the solid defectively and the volume V_n of the cylinders approximating excessively are:

$$v_n = \pi m_1^2 h + \pi m_2^2 h + \pi m_3^2 h + \ldots + \pi m_n^2 h;$$

$$V_n = \pi M_1^2 h + \pi M_2^2 h + \pi M_3^2 h + \ldots + \pi M_n^2 h.$$

One can demonstrate that as $n \to \infty$, both sequences tend to the same limit, and this limit is equal to the product of π and the definite integral from a to b of the square of $f(x)$, namely:

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} V_n = \pi \cdot \int_a^b f^2(x)\, dx.$$

Volume of a Solid of Revolution

Definition 9.1. Given the trapezoid $ABCD$ extended to the interval $[a; b]$, bounded by the graph of the function $y = f(x)$ (non-negative), the x-axis, and the lines $x = a$ and $x = b$, the volume of the solid obtained by rotating the trapezoid around the x-axis by a full revolution is expressed by the following integral:

$$V = \pi \cdot \int_a^b f^2(x)\, dx.$$

9.2 Arc Length of a Curve

Consider a function $f(x)$ differentiable with continuous derivative over the interval $[a; b]$. We divide the interval $[a; b]$ into n parts and consider the polygonal with vertices at points $P_0, P_1, P_2, \ldots, P_n$. The length l_n of the inscribed polygonal in the curve can be calculated by summing the distances between the points $P_{()}, P_1, P_2, \ldots, P_n$.

It depends on the number n of subdivisions and the points chosen for the subdivision. The smaller the width of the intervals $[x_{i-1}; x_i]$, the better the polygonal approximates the curve. It can be demonstrated that as n tends to infinity, l_n has a finite limit given by:

$$\lim_{n \to \infty} l_n = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Arc Length of a Curve

Definition 9.2. Given the function $y = f(x)$ differentiable over the interval $[a; b]$, the length of the curve represented by the graph of the function, bounded by the lines $x = a$ and $x = b$, is the number expressed by the following integral:

$$l = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

9.3 Surface Area of a Solid of Revolution

If the previously considered curve is rotated with a full rotation around the x-axis, a surface of revolution is obtained. We divide the interval $[a; b]$ into n parts and consider the polygonal inscribed in the curve, which has vertices at points $P_0, P_1, \ldots, P_n$. In the complete rotation around the x-axis, each segment $P_i P_{i-1}$ of the polygonal describes a frustum of a cone with $P_i P_{i-1}$ as apothem and two circles as bases with radii $f(x_{i-1})$ and $f(x_i)$ respectively.

The sum of the areas of the lateral surfaces of these frustums of cones approximates the area of the surface of revolution. By a similar procedure to that used for the arc length of a curve, we demonstrate the following formula:

$$S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$$

Surface Area of a Solid of Revolution

Definition 9.3. Given the function $y = f(x)$ differentiable over the interval $[a; b]$, the area of the surface obtained by rotating the graph of the function in a full rotation, bounded by the lines $x = a$ and $x = b$, is the number expressed by the following integral:

$$S = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$$

10 Improper Integrals

10.1 Integral of a Function with a Finite Number of Discontinuities in $[a; b]$

Consider the case where the function $f(x)$ is continuous at all points of the interval $[a; b[$ but not at b. Consider a point z inside the interval $[a; b[$: the function $f(x)$ is continuous in the interval $[a; z]$, so the integral $\int_a^z f(x)\ dx$ exists, whose value is a real number. This holds for all points z in the interval $[a; b[$, so we can construct the integral function

$$F(z) = \int_a^z f(x)\ dx,$$

defined in $[a; b[$. If the limit of $F(z)$ exists finitely as z approaches b from the left, that is, if

$$\lim_{x \to b^-} F(z),$$

exists, then we say that the function $f(x)$ is **improperly integrable in** $[a; b]$ and define:

$$\int_a^b f(x)\ dx = \lim_{x \to b^- f(x)\ dx}.$$

Note 6. It is also called that the function is **integrable in a generalized sense**.

The integral $\int_a^b f(x)\ dx$ is called the **improper integral** of the function $f(x)$ in $[a; b]$ and is also said to be **convergent**. If the limit considered does not exist or is infinite, we say that the function is not improperly integrable in $[a; b]$ or also that the integral is respectively **undetermined** or **divergent**.

If the function $f(x)$ is continuous at all points of the interval $]a; b]$, we can define the integral $\int_a^b f(x)\ dx$ in a similar way. Considering $z \in]a; b]$, if the limit of the function $F(z) = \int_a^b f(x)\ dx$ exists finitely as z approaches a from the right, that is, if

$$\lim_{x \to a^+} F(z),$$

exists, then we say that the function $f(x)$ is **improperly integrable in** $[a; b]$ and define:

$$\int_a^b f(x)\ dx = \lim_{x \to a^+} \int_z^b f(x)\ dx.$$

If the function has a discontinuity point at a point c inside the interval $[a; b]$, the integral $\int_a^b f(x)\ dx$ can be defined, in an improper sense, as the sum of the integrals $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$, if they exist. Such integrals are calculated using the previous definitions:

$$\int_a^b f(x)\ dx = \lim_{t \to c^-} \int_a^t f(x)\ dx + \lim_{x \to c^+} \int_z^b f(x)\ dx.$$

10.2 Integral of a Function over an Infinite Interval

Consider a function $f(x)$ continuous at all points of $[a;+\infty[$. For any point z inside the interval $[a;+\infty[$, the integral $\int_a^x f(x)\ dx$ exists and yields a real number, so we can also construct the integral function in this case:

$$F(z) = \int_a^z f(x)\ dx,$$

defined in $[a;+\infty[$. If the limit of the function $F(z)$ exists finitely as z tends to $+\infty$, that is, if

$$\lim_{x\to+\infty} F(z),$$

exists, then we say that the function $f(x)$ is **improperly integrable in** $[a;+\infty[$ and define:

$$\int_a^{+\infty} f(x)\ dx = \lim_{x\to+\infty} \int_a^x f(x)\ dx.$$

In this case, we also say that the integral $\int_a^{+\infty} f(x)\ dx$ is **convergent**.

Note 7. If the considered limit is infinite, we say that the integral $\int_a^{+\infty} f(x)\ dx$ is **divergent**. If the limit does not exist, the integral $\int_a^{+\infty} f(x)\ dx$ is **undetermined**. In both cases, we say that the function $f(x)$ is not improperly integrable in $[a;+\infty[$.

Similarly, if a function is continuous in $]-\infty;a]$ and if the limit $\lim_{x\to-\infty} \int_a^z f(x)\ dx$ exists finitely, we say that the function $f(x)$ is integrable in an improper sense in $]-\infty;a]$ and define:

$$\int_{-\infty}^a f(x)\ dx = \lim_{x\to-\infty} \int_z^a f(x)\ dx.$$