University of Basilicata

# Solving Equations and Inequalities

*Donato Martinelli*

March 14, 2024

# Contents

## Part I

# Quadratic Equations and Inequalities

# 1 Quadratic Equations

Any polynomial equation in one variable $x$ can be written in the normal form $p^n(x) = 0$ where $p^n(x) = 0$ is a polynomial of degree $n$ in $x$.
If the polynomial $p(x)$ is of the second degree, then it is a quadratic equation.
The normal form of a quadratic equation is: $ax^2 + bx + c = 0$, where $a \neq 0$.
The letters $a$, $b$, and $c$ represent real numbers or literal expressions and are called the first, second, and third coefficient of the equation; $c$ is also known as the constant term.
If besides $a \neq 0$, we also have $b \neq 0$ and $c \neq 0$, the equation is said to be complete.
If the equation is incomplete, we have the following cases:

| Coefficients | Normal Form | Name |
|---|---|---|
| $b \neq 0,\ c = 0$ | $ax^2 + bx = 0$ | degenerate |
| $b = 0,\ c \neq 0$ | $ax^2 + c = 0$ | pure |
| $b = 0,\ c = 0$ | $ax^2 = 0$ | monomial |

A solution (or root) of the equation is a value that, when substituted for the variable, satisfies the equality between the two sides.

## 1.1 Solving a Quadratic Equation

### 1.1.1 Completing the Square Method

To find the roots of the complete equation

$$ax^2 + bx + c = 0 \quad \text{where } a, b, c \neq 0$$

we apply the completing the square method.

$$\begin{aligned}
ax^2 + bx &= -c \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + 2 \cdot x \cdot \frac{b}{2a} &= -\frac{c}{a} \\
x^2 + 2 \cdot x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}
\end{aligned}$$

The expression on the left-hand side is a square, so it is always positive or zero. For the equation to have real solutions, the fraction on the right-hand side must also be non-negative.
Since the denominator of the fraction is always positive, the numerator must be non-negative, i.e., $b^2 - 4ac \geq 0$.
If $b^2 - 4ac \geq 0$, there are two values, one the negative of the other, that satisfy the equation.
We obtain them by taking the square root:

$$\begin{aligned} x + \frac{b}{2a} &= \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

This is called the quadratic formula.

### 1.1.2 Discriminant and Solutions

We call discriminant the expression under the square root in the quadratic formula, i.e.,

$$\Delta = b^2 - 4ac$$

To determine if real solutions exist for a quadratic equation, it is sufficient to calculate the discriminant ($\Delta$):

- If $\Delta > 0$, the equation has two real and distinct solutions:
  - $S = \{x_1, x_2\}$, with $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$
  - $x_1 = \dfrac{-b + \sqrt{\Delta}}{2a}$, $x_2 = \dfrac{-b - \sqrt{\Delta}}{2a}$
- If $\Delta = 0$, the equation has two real and coincident solutions:
  - $S = \{x_1\}$, with $x_1 \in \mathbb{R}$ and $x_1 = x_2$
  - $x_1 = x_2 = -\dfrac{b}{2a}$
- If $\Delta < 0$, the equation has no real solutions:
  - $\nexists x \in \mathbb{R}$

An equation of the second degree has real solutions only if the discriminant is a positive number or zero because in the set $\mathbb{R}$, the square root of a negative number does not exist.

## 1.2 Reduced and Most Reduced Forms

There are variants of the solution formula for second-degree equations that can be applied when the normal form of the equations satisfies particular conditions on the coefficients:

**Reduced Formula (or Quarter Delta)** Given $ax^2+bx+c=0$ with $a,b,c\in\mathbb{R}_0$. When the coefficient $b$ of the $x$ term is an even number, we can apply a reduced formula to simplify calculations:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Setting $b=2k$, where $b$ is an even number, and substituting, we get:

$$\begin{aligned}
x&=\frac{-2k\pm\sqrt{4k^2-4ac}}{2a}\\
&=\frac{-2k\pm\sqrt{4(k^2-ac)}}{2a}\\
&=\frac{-2k\pm2\sqrt{k^2-ac}}{2a}\\
&=\frac{-2k\pm2\sqrt{k^2-ac}}{2a}\\
&=\frac{2(-k\pm\sqrt{k^2-ac})}{2a}\\
&=\frac{-k\pm\sqrt{k^2-ac}}{a}
\end{aligned}$$

Substituting $k=\frac{b}{2}$, we obtain:

$$x_{1,2}=\frac{-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-ac}}{a}$$

The discriminant of this formula is one-fourth of the discriminant of the standard formula, hence indicated by:

$$\frac{\Delta}{4}=\left(\frac{b}{2}\right)^2-ac$$

**Most Reduced Formula** If $a=1$, the formula becomes:

$$x_{1,2}=-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c}$$

With a discriminant:

$$\frac{\Delta}{4}=\left(\frac{b}{2}\right)^2-c$$

## 1.3 Special Cases

Let's revisit the scenarios of incomplete second-degree equations:

| Coefficients | Normal Form | Name |
|---|---|---|
| $b \neq 0,\ c = 0$ | $ax^2 + bx = 0$ | spurious |
| $b = 0,\ c \neq 0$ | $ax^2 + c = 0$ | pure |
| $b = 0,\ c = 0$ | $ax^2 = 0$ | monomial |

**Spurious Equations**

- $ax^2 + bx = 0$ with $a \neq 0$ and $a, b \in \mathbb{R}_0$
  - $x(ax + b) = 0 \quad \rightarrow \quad x = 0 \quad ax + b = 0$
  - Solutions:
    * $x_1 = 0$
    * $ax + b = 0 \quad \rightarrow \quad ax = -b \quad \rightarrow \quad x_2 = -\frac{b}{a}$
- A spurious second-degree equation always has two real solutions, one of which is zero.

**Pure Equations**

- $ax^2 + c = 0$ with $a \neq 0$ and $a, c \in \mathbb{R}_0$
  - $ax^2 = -c \quad \rightarrow \quad x^2 = -\frac{c}{a} \quad \rightarrow \quad x = \pm\sqrt{-\frac{c}{a}}$
  - Depending on the sign of $-\frac{c}{a}$, two cases can occur:
    * $-\frac{c}{a} > 0$ : $a$ and $c$ have opposite signs
    * $-\frac{c}{a} < 0$ : $a$ and $c$ have the same sign
- A pure second-degree equation can have two real opposite solutions or no real solutions at all.

**Monomial Equations**

- $ax^2 = 0$ with $a \neq 0$ and $a \in \mathbb{R}_0$
- A monomial second-degree equation has two coinciding solutions with $x = 0$ ($x_1 = x_2 = 0$).

## 1.4 Sum and Product of Roots

**Sum of Roots** Given the equation $ax^2 + bx + c = 0$, with $\Delta \geq 0$, we calculate:

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}$$

The sum $s$ of the roots of a quadratic equation with non-negative discriminant equals the negated ratio of the coefficient of $x$ to the coefficient of $x^2$.

$$s = -\frac{b}{a}$$

**Product of Roots**

$$x_1 \cdot x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{c}{a}$$

The product $p$ of the roots of a quadratic equation with non-negative discriminant equals the ratio of the constant term to the coefficient of $x^2$.

$$p = \frac{c}{a}$$

**Sum and Product of Roots and the Equation in Normal Form**

$$\begin{aligned} ax^2 + bx + c =& 0 \\ x^2 + \frac{b}{a}x + \frac{c}{a} =& 0 \\ x^2 - \left(-\frac{b}{a}\right) x + \frac{c}{a} =& 0 \\ x^2 - sx + p =& 0 \end{aligned}$$

In a quadratic equation reduced to normal form where the leading coefficient is 1, the second coefficient is the sum $s$ of the roots with the sign changed, and the constant term is the product $p$ of the roots.

$$x^2 - sx + p = 0$$

or:

$$x^2 - (x_1 + x_2)x + x_1 x_2 = 0$$

## 1.5 Factoring a Quadratic Trinomial

Given a quadratic trinomial: $ax^2 + bx + c$

- $\Delta > 0$: The trinomial has two distinct real zeros, thus:

  $$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

  **Proof:**

  $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

  Utilizing $-\frac{b}{a} = x_1 + x_2$ and $\frac{c}{a} = x_1 \cdot x_2$, we rewrite it as:

  $$\begin{aligned} & a\left[x^2 - \left(-\frac{b}{a}\right) x + \frac{c}{a}\right] \\ & a[x^2 - (x_1 + x_2)x + x_1 \cdot x_2] \\ & a[x^2 - x_1 x - x_2 x + x_1 x_2] \\ & a[x(x - x_1) - x_2(x - x_1)] \\ & a(x - x_1)(x - x_2) \end{aligned}$$

- $\Delta = 0$: The trinomial has only one zero, as $x_1 = x_2$; thus:

$$\begin{aligned}
&ax^2 + bx + c \\
&a(x - x_1)(x - x_1) \\
&a(x - x_1)^2
\end{aligned}$$

- $\Delta < 0$: The trinomial has no real zeros and cannot be factored into real factors, hence it's irreducible.

## 1.6 Descartes' Rule of Signs

If a quadratic equation has a positive or zero discriminant, and thus has real solutions, the signs of the solutions can be determined without solving the equation, simply by deducing them from the signs of its coefficients.
It is said that there is a sign permanence in the equation when two consecutive terms have the same sign, and there is a sign variation when two consecutive terms have different signs.
Assuming $a$ is always positive, in a quadratic equation, there can be four cases regarding sign permanences and sign variations:

| | a | b | c | |
|---|---|---|---|---|
| 1st case | + | + | + | 2 permanences |
| 2nd case | + | - | + | 2 variations |
| 3rd case | + | + | - | 1 permanence, 1 variation |
| 4th case | + | - | - | 1 variation, 1 permanence |

To determine the sign of each solution $x_1$ and $x_2$ of the equation, we reason about the sign of their sum and product, and then we also indicate the sign of the sum and product of the solutions, which are obtained from the signs of the coefficients.

| | a | b | c | $x_1 + x_2 = -\frac{b}{a}$ | $x_1 \cdot x_2 = \frac{c}{a}$ |
|---|---|---|---|---|---|
| 1st case | + | + | + | - | + |
| 2nd case | + | - | + | + | + |
| 3rd case | + | + | - | - | - |
| 4th case | + | - | - | + | - |

Possible scenarios:

- **Two permanences**
  - The product is positive, so $x_1$ and $x_2$ are of the same sign.
  - The sum is negative, thus both $x_1$ and $x_2$ are negative.

- **Two variations**
  - The product is positive, so $x_1$ and $x_2$ are of the same sign.
  - The sum is positive, hence both $x_1$ and $x_2$ are positive.
- **One permanence and one variation**
  - The product is negative, so $x_1$ and $x_2$ have opposite signs.
  - The sum is negative, therefore the negative solution $x_1$ has a greater absolute value than the positive solution $x_2$.
- **One variation and one permanence**
  - The product is negative, so $x_1$ and $x_2$ have opposite signs.
  - The sum is positive, thus the positive solution $x_1$ has a greater absolute value than the negative solution $x_1$.

In conclusion:

- Two permanences of sign correspond to two negative solutions.
- Two variations of sign correspond to two positive solutions.
- One variation and one permanence correspond to one positive solution and one negative solution.

**Cartesian Signs Rule**
Therefore, the following rule, called the **Cartesian Signs Rule**, holds:
In every quadratic equation with a positive or zero discriminant, for each *variation of signs* of the coefficients, there corresponds a *positive solution*, and for each permanence, there corresponds a *negative solution*; when the solutions are of opposite signs, if the variation precedes the permanence, the positive solution has a greater absolute value than the negative solution, and vice versa.

# 2 Second-Degree Inequalities

Second-degree inequalities are inequalities in which one side is a polynomial of degree 2, while the other side is a polynomial of degree at most 2.

To study the methods of solving quadratic inequalities, it is convenient to refer to a general form, called the **normal form of second-degree inequalities**

$$ax^2 + bx + c \gtreqless 0$$

where $a, b, c \in \mathbb{R}$ are real numbers, specifically:

- $a$ is the coefficient of the term of degree 2, or *leading coefficient*;
- $b$ is the coefficient of the linear term;
- $c$ is the coefficient of the constant term, or *constant term*.

For the normal form to be consistent, a very important condition must be imposed:

$$a \neq 0.$$

Otherwise, the inequality would not be of second degree and would reduce to a first-degree inequality. The terms $b$ and $c$, however, can potentially be zero. Regarding the number of solutions of quadratic inequalities, considerations entirely analogous to those of first degree apply. There are three possibilities:

- Determinate inequality, with a finite number of solutions;
- Indeterminate inequality, with an infinite number of solutions;
- Impossible inequality, with no solutions.

## 2.1 Algebraic Method for Quadratic Inequalities

1. Ensure that the leading coefficient $a$ is positive. If it is not, make it positive. If $a$ is positive, no action is needed. If it is negative, change its sign by multiplying both sides by $-1$ and remember to change the direction of the inequality.

2. Solve the associated quadratic equation:

   $$ax^2 + bx + c = 0 \ (a > 0)$$

   We know that the solutions of a quadratic equation are given by the discriminant formula:

   $$x_{1,2} = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

3. Under the condition $a > 0$, once we obtain the solutions of the equation, we can solve the original quadratic inequality using a simple chart.

   - Two real and distinct solutions $x_1 \neq x_2$ with $x_1 < x_2$

$$
\begin{aligned}
&ax^2 + bx + c > 0 \Rightarrow x < x_1 \vee x > x_2 \\
&ax^2 + bx + c \geq 0 \Rightarrow x \leq x_1 \vee x \geq x_2 \\
&ax^2 + bx + c < 0 \Rightarrow x_1 < x < x_2 \\
&ax^2 + bx + c \leq 0 \Rightarrow x_1 \leq x \leq x_2
\end{aligned}
$$

   - Two real and coincident solutions $x_1 = x_2$

$$
\begin{aligned}
&ax^2 + bx + c > 0 \Rightarrow \forall x,\ x \neq x_1 \\
&ax^2 + bx + c \geq 0 \Rightarrow \forall x \\
&ax^2 + bx + c < 0 \Rightarrow \nexists x \\
&ax^2 + bx + c \leq 0 \Rightarrow x = x_1
\end{aligned}
$$

   - No real solutions $\Delta < 0$

$$
\begin{aligned}
&ax^2 + bx + c > 0 \Rightarrow \forall x \\
&ax^2 + bx + c \geq 0 \Rightarrow \forall x \\
&ax^2 + bx + c < 0 \Rightarrow \nexists x \\
&ax^2 + bx + c \leq 0 \Rightarrow \nexists x
\end{aligned}
$$

**Part II**

# Equations and Inequalities of Degree Higher than Two

# 3 Equations of Degree Higher than Two

**Equations of Degree Higher than Two** Algebraic equations of degree higher than two in the variable $x$ are equations that can be reduced to the normal form:

$$p^n(x) = 0$$

where $p^n(x)$ is a polynomial with real coefficients of degree $n > 2$, ordered according to decreasing powers of $x$.
By solving first and second-degree equations, we have been able to verify that:

- every first-degree equation (which is not an identity) has at most one real solution;
- every second-degree equation has at most two real solutions, which are distinct if $\Delta > 0$ and coincident (with multiplicity 2) if $\Delta = 0$.

In general, it can be stated that every algebraic equation of degree $n$ in the set $\mathbb{R}$ of real numbers has at most $n$ solutions, each considered with its multiplicity.

**Fundamental Theorem of Algebra** Every algebraic equation of degree $n$ always has $n$ solutions, which can be real or non-real.

## 3.1 Equations Resolvable by Factorization

Consider an algebraic equation of degree higher than two, written in normal form:

$$p^n(x) = 0$$

and suppose that the polynomial $p^n(x)$ can be factored into the product of first and second-degree factors, for example:

$$a(x) \cdot b(x) \cdot c(x) = 0$$

By applying the zero-product property, the equation splits into three equations:

$$a(x) = 0 \quad b(x) = 0 \quad c(x) = 0$$

The solutions of the equation of degree higher than two are all the solutions of the three split equations.
The resolution of an equation $p^n(x) = 0$ of degree higher than two depends on the possibility and ability to factorize the polynomial $p^n(x)$ into the product of first or second-degree factors.

## 3.2 Binomial Equations

An equation is called binomial if its normal form is: $ax^n + b = 0$ with $a, b \in \mathbb{R}_0, n \in \mathbb{N}_0$:

$$x^n = -\frac{b}{a} \quad \rightarrow \quad x = \sqrt[n]{-\frac{b}{a}}$$

The existence of real solutions and their number depends on the values of $a$ and $b$, and the index of the root $n$; the following cases can be verified:

1. $n$ is even and $a$ and $b$ have the same signs: $-\frac{b}{a}$ is a negative number, and the even root of a negative number does not exist in the set $\mathbb{R}$. The binomial equation has no real solutions.

2. $n$ is even and $a$ and $b$ have opposite signs: $-\frac{b}{a}$ is a positive number, and thus, the binomial equation has two real solutions which are opposite to each other, namely: $x = -\sqrt[n]{-\frac{b}{a}}$ and $x = \sqrt[n]{-\frac{b}{a}}$

3. $n$ is odd: The binomial equation always has a single real solution, which is $x = \sqrt[n]{-\frac{b}{a}}$, regardless of the values of $a$ and $b$.

## 3.3 Trinomial Equations

An equation is called trinomial if its normal form is:

$$ax^{2n} + bx^n + c = 0 \quad \text{with} \quad a, b, c \in \mathbb{R}_0, n \in \mathbb{N}_0$$

The solutions are obtained by substitution:

$$x^n = t$$

with which the equation transforms into a second-degree equation:

$$t^2 + bt + c = 0$$

If the discriminant is greater than or equal to 0, the equation has two real solutions which, when substituted back with $x^n = t$, give the solutions of the trinomial equation.
In the particular case where $n = 2$, trinomial equations take the form: $ax^4 + bx^2 + c = 0$, and are called **biquadratic equations**.

# 4 Reciprocal Equations

## 4.1 Reciprocal Equations

A **reciprocal equation** is defined as any equation in which, in its standard form, the coefficients of terms equidistant from the ends are either equal or opposite numbers.
If the coefficients of terms equidistant from the ends are equal numbers, the equation is called of the **first kind**.
If the coefficients of terms equidistant from the ends are opposite numbers, the equation is called of the **second kind**.
It can be shown that if a reciprocal equation has the real number $k$ as a solution, then its reciprocal, $\frac{1}{k}$, is also a solution. This property is why such equations are called reciprocal.

## 4.2 Reciprocal Equations of the First Degree

The standard form of reciprocal equations of the first degree is:

$$ax^3 + bx^2 + bx + a = 0 \quad \text{where} \quad a, b \in \mathbb{R}_0$$

It is easy to verify that $-1$ is a root of the cubic polynomial:

$$p(-1) = -a + b - b + a = 0$$

hence it is always divisible by $x + 1$.
Therefore, these equations can be solved by factoring the polynomial using Ruffini's rule, into the product of $x+1$ and a quadratic factor, and then applying the zero-product property.
Thus, all reciprocal equations of the first degree have the solution $x = -1$.

## 4.3 Reciprocal Equations of the Second Degree

The standard form of reciprocal equations of the second degree is:

$$ax^3 + bx^2 - bx - a = 0 \quad \text{where} \quad a, b \in \mathbb{R}_0$$

It is easy to verify that 1 is a root of the cubic polynomial:

$$p(1) = a + b - b - a = 0$$

hence it is always divisible by $x - 1$.
Therefore, these equations can be solved by factoring the polynomial into the product of $x - 1$ and a quadratic factor.
Thus, all reciprocal equations of the second degree have the solution $x = 1$.

## 4.4 Reciprocal Equations of the Third Degree

The standard form of reciprocal equations of the third degree is:

$$ax^4 + bx^3 + cx^2 + bx + a = 0 \quad \text{where} \quad a, b, c \in \mathbb{R}_0$$

Dividing all terms by $x^2$, we get:

$$ax^2 + bx + c + \frac{b}{x} + \frac{a}{x^2} = 0$$

Grouping terms equidistant from the ends, we have:

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$$

Let:

$$x + \frac{1}{x} = t$$

Then, squaring both sides, we obtain:

$$x^2 + \frac{1}{x^2} + 2 = t^2 \quad \rightarrow \quad x^2 + \frac{1}{x^2} = t^2 - 2$$

Substituting into the equation yields a quadratic equation in $t$:

$$a(t^2 - 2) + bt + c = 0$$

Solving it, we find two solutions $t_1$ and $t_2$ (if they exist).
Returning to $x + \frac{1}{x} = t$, we get two equations in $x$: $x + \frac{1}{x} = t_1$ and $x + \frac{1}{x} = t_2$.
Solving these gives the solutions of the reciprocal equation.

## 4.5 Reciprocal Equations of the Fourth Degree

The standard form of reciprocal equations of the fourth degree is:

$$ax^4 + bx^3 + cx^2 + bx + a = 0 \quad \text{where} \quad a, b, c \in \mathbb{R}_0$$

where the central term in $x^2$ is missing. It should be equal to itself and its opposite, thus it must be 0.
It can be verified that $-1$ and 1 are roots of the quartic polynomial:

$$p(-1) = a - b + b - a = 0 \quad \text{and} \quad p(1) = a + b - b - a = 0$$

thus it is always divisible by both $(x + 1)$ and $(x - 1)$ and a quadratic factor.
Therefore, all reciprocal equations of the fourth degree have solutions $x = -1$ and $x = 1$.

### 4.5.1 Reciprocal Equations of Fifth Degree

All reciprocal equations of the fifth degree of the first kind have the solution $x = -1$, just like those of the third degree of the first kind.
They are then reduced in degree using the Ruffini's rule, resulting in a fourth-degree equation, which is still reciprocal and thus solvable.
Similarly, all reciprocal equations of the fifth degree of the second kind have the solution $x = 1$, just like those of the third degree of the second kind.
They are also reduced in degree using the Ruffini's rule, resulting in a fourth-degree equation that remains reciprocal.

# 5 Inequalities of Degree Higher than Second

An inequality is said to be of degree higher than the second if the variable appears at least once with an exponent greater than 2. To study inequalities of degree higher than the second, we refer to their standard form

$$P(x) \gtreqless 0 \quad \text{with } \deg(P) > 2$$

where $P(x)$ is a polynomial with real coefficients of degree $n > 2$. In explicit form:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \gtreqless 0$$

with $a_0, a_1, ..., a_n \in \mathbb{R}$. Even in this context, the usual observation holds. The inequalities may not necessarily appear in standard form; for example, they could be of the form

$$R(x) \gtreqless S(x) \quad \text{with } \deg(R) > 2,\ \deg(S) \in \mathbb{N}$$

and algebraic work may be required to simplify the expressions and reduce them to standard form.

## 5.1 Method for Solving Inequalities of Degree Higher than the Second

1. Factorize the polynomial $P(x)$. Suppose, for simplicity, that the polynomial $P(x)$ factors into $k$ factors $P_1(x), ..., P_k(x)$, and rewrite the inequality as

   $$P_1(x) \cdot P_2(x) \cdot \cdots \cdot P_k(x) \gtreqless 0.$$

   The idea is to obtain factors $P_1(x), ..., P_k(x)$ with degrees 1 or 2.

2. Study the sign of each individual factor $P_1(x), ..., P_k(x)$ separately.

   - If we have

     $$P_1(x) \cdot P_2(x) \cdot \cdots \cdot P_k(x) \geq 0 \quad \vee \quad P_1(x) \cdot P_2(x) \cdot \cdots \cdot P_k(x) \leq 0$$

     then study the sign of the individual factors by always setting them greater than or equal to zero:

     $$\begin{aligned} P_1(x) &\geq 0 \\ P_2(x) &\geq 0 \\ &\vdots \\ P_k(x) &\geq 0 \end{aligned}$$

- If we have

$$P_1(x) \cdot P_2(x) \cdot \dots \cdot P_k(x) > 0 \quad \vee \quad P_1(x) \cdot P_2(x) \cdot \dots \cdot P_k(x) < 0$$

  Then study the sign of the individual factors by always setting them greater than zero:

$$\begin{aligned} P_1(x) &> 0 \\ P_2(x) &> 0 \\ &\vdots \\ P_k(x) &> 0 \end{aligned}$$

  In both cases, we will end up solving first or second-degree inequalities.

3. The goal at this point is to apply the sign rule and use the information about the signs of the individual factors to determine the sign of the product, i.e., the sign of the polynomial $P(x)$. We resort to the so-called **graphical method of signs**. Draw an oriented half-line, on which we mark the endpoints of the solutions of all $k$ inequalities; below it, represent, on $k$ different lines, the solutions of each of the $k$ inequalities.

   - As we have set up the individual inequalities, on each line, the solutions correspond to the intervals where the respective factor is positive. Represent them with a solid line;
   - On each line, the intervals outside the solid lines are those in which the respective factor is negative. Represent them with a dashed line;
   - On each line, the endpoints correspond to the values that nullify the respective factor. If the inequality symbol is greater than or equal to, indicate the endpoints with filled dots (included), if the symbol is strictly greater, indicate the endpoints with empty dots (excluded).

4. Finally, conclude by applying the sign rule. Analyze the tabular representation by looking at the various intervals vertically, and count the number of dashed lines on each interval:

   - If there is an **even number of dashed lines**, or if there are none, we have an odd number of negative factors on that interval. The solid lines correspond to positive factors, so the overall product is positive;
   - If there is an **odd number of dashed lines**, we have an even number of negative factors on that interval. The solid lines correspond to positive factors, so the overall product is negative.

5. Look at the inequality symbol in the last step of the decomposition because in performing the decomposition, we may have collected and simplified a negative term. Always refer to the inequality symbol at the end of the algebraic steps:

   - If it is greater than or equal to, the solutions of the inequality are the intervals that have an overall positive sign (included endpoints);
   - If it is strictly greater, the solutions of the inequality are the intervals that have an overall positive sign (excluded endpoints);
   - If it is less than or equal to, the solutions of the inequality are the intervals that have an overall negative sign (included endpoints);
   - If it is strictly less than, the solutions of the inequality are the intervals that have an overall negative sign (excluded endpoints).

**Part III**

# Fractional Equations and Inequalities

## 6 Fractional Inequalities

**Fractional inequalities** are inequalities in which the variable appears at least once in a denominator; to solve them, values of the variable that nullify the denominators are excluded from the solution set, algebraic expressions are reduced to a single ratio, and the signs of the numerator and denominator are studied separately. A fractional inequality is defined as such if at least one of the two members contains a ratio with the variable $x$ in the denominator. Regardless of the initial form in which a fractional inequality appears, in general, we can always apply the two principles of inequality equivalence and reduce it to one of the following cases:

$$\frac{N(x)}{D(x)} > 0 \quad \frac{N(x)}{D(x)} \geq 0$$

$$\frac{N(x)}{D(x)} < 0 \quad \frac{N(x)}{D(x)} \leq 0$$

where $N(x), D(x)$ are expressions containing the variable $x$.

## 7 Method for Solving Rational Inequalities

The general method involves the following steps:

1. Discuss the existence conditions and identify the solution's domain. Before any calculations, ensure that every denominator containing $x$ is nonzero. Keep in mind that the existence conditions must all hold simultaneously. If we have $n$ terms involving the variable $x$, we can express the existence conditions as:

$$CE: \begin{cases} \text{denominator}_1 \neq 0 \\ \vdots \\ \text{denominator}_n \neq 0 \end{cases}$$

   It's important to note that the $n$ conditions regarding the denominators translate into equations, contributing to a system of equations. If the inequality involves not only polynomials but also logarithms, even roots, and so on, we need to consider their respective existence conditions. These conditions must be added to the system for existence conditions:

$$CE: \begin{cases} \text{denominator}_1 \neq 0 \\ \qquad\vdots \\ \text{denominator}_n \neq 0 \\ \text{other relevant conditions} \end{cases}$$

2. After identifying the solution's domain, proceed with the calculations to reduce it to one of the following forms:

$$\frac{N(x)}{D(x)} > 0 \quad \frac{N(x)}{D(x)} \geq 0$$

$$\frac{N(x)}{D(x)} < 0 \quad \frac{N(x)}{D(x)} \leq 0$$

3. Analyze the sign of the numerator and denominator separately. Regardless of the direction of the inequality symbol, assume:

   - The numerator is greater than or equal to zero if the inequality symbol includes equality, and strictly greater than zero if the inequality symbol does not include equality.
   - The denominator is strictly greater than zero to exclude points where it equals zero, regardless of whether the inequality symbol includes equality or not.

   By doing this, we can determine for which values of the variable the numerator and denominator are respectively positive, negative, or zero:

$$\frac{N(x)}{D(x)} > 0 \quad \wedge \quad \frac{N(x)}{D(x)} < 0 \quad \rightarrow \quad N(x) > 0,\ D(x) > 0$$

$$\frac{N(x)}{D(x)} \geq 0 \quad \wedge \quad \frac{N(x)}{D(x)} \leq 0 \quad \rightarrow \quad N(x) \geq 0,\ D(x) > 0$$

   This helps in identifying the intervals where the numerator and denominator are respectively positive.

4. Analyze the sign of the ratio $\dfrac{N(x)}{D(x)}$.

5. Knowing the intervals where the ratio $\dfrac{N(x)}{D(x)}$ is positive $(+)$, negative $(-)$, or zero, and which values should be excluded from the solutions:

- $$\frac{N(x)}{D(x)} > 0$$

  The solutions are intervals with the + sign, excluding all points with open circles.

- $$\frac{N(x)}{D(x)} \geq 0$$

  The solutions are intervals with the + sign, including points with filled circles but excluding those with open circles. If any point is covered by both a filled circle and an open circle, it should be excluded.

- $$\frac{N(x)}{D(x)} < 0$$

  The solutions are intervals with the $-$ sign, excluding all points with open circles.

- $$\frac{N(x)}{D(x)} \leq 0$$

  The solutions are intervals with the $-$ sign, including points with filled circles but excluding those with open circles. If any point is covered by both a filled circle and an open circle, it should be excluded.

# 8 Equations and Irrational Inequalities

## 8.1 Irrational Equations

**Irrational equations** involve operations between polynomials, where at least one non-constant polynomial is raised to a fractional exponent.

According to radical theory, this means that the variable $x$ must appear in at least one polynomial under a radical.

We can define the standard form of an irrational equation as

$$\sqrt[n]{f(x)} = g(x)$$

where $f(x)$ and $g(x)$ are polynomials with real coefficients.

For this type of equation, we need to consider two different procedures depending on whether the root index $n$ is even or odd.

## 8.2 Irrational Equations with Even Root Index

Consider solving an **irrational equation with an even index**

$$\sqrt[n]{f(x)} = g(x) \quad n \text{ even.}$$

The first step is to eliminate the root by appropriately raising both sides to a power. Firstly, we impose existence conditions: since the root has an even index, it is well-defined only if the argument is non-negative (greater than or equal to zero). This translates to the inequality

$$f(x) \geq 0.$$

Additionally, since an even-index root only yields positive or zero values, we risk expanding the solution set by raising both sides to the $n$th power, which may result in unacceptable solutions because even exponents do not preserve the base's sign. For these reasons, we must also add the so-called **sign agreement condition**

$$g(x) \geq 0.$$

With these conditions in place, we can raise both sides of the equation to the $n$th power. The resolution scheme for irrational equations with an even index can be summarized in the following system

$$\begin{cases} f(x) \geq 0 & \text{existence condition} \\ g(x) \geq 0 & \text{sign agreement condition} \\ f(x) = [g(x)]^n \end{cases}$$

Once done, we only need to solve the equation and then compare the obtained solutions with the system of inequalities between the existence condition and the sign agreement condition. Solutions are acceptable if and only if they satisfy both the existence condition and the sign agreement condition.

## 8.3 Irrational Equations with Odd Root Index

Now let's move to solving an irrational equation with an odd index, such as

$$\sqrt[n]{f(x)} = g(x) \quad n \text{ odd.}$$

Since odd-index roots can have radicands with any sign, no existence condition is required. Additionally, since an odd-index root assumes a value with the same sign as the radicand, it can take values of any sign, and no sign agreement condition is required.

Therefore, we only need to raise both sides to the power of $n$ and solve the equation

$$f(x) = [g(x)]^n$$

and accept all solutions.

## 8.4 Irrational Equations with Multiple Roots

When dealing with an equation with two roots:

1. Having the **same index**
   We should impose existence conditions - one for each even index root. Then, we try to reduce the equation to standard form by isolating the two roots, such that one is on the left side alone and the other is on the right side. Before raising both sides to the common root index, it's crucial to have separated the two roots; otherwise, mixed products may introduce new roots. Furthermore, before raising both sides to the $n$th power, we need to impose sign agreement conditions:

   - If the index is odd, no sign agreement condition is required.
   - If the index is even, we isolate one of the roots on the left side and everything else on the right side; we impose the sign agreement condition on the left side, which may result in a separate irrational inequality to solve. We then square both sides and possibly repeat the process to return to the standard form of irrational equations.

2. Having **different indices**

   $$\sqrt[n]{f(x)} = \sqrt[m]{g(x)}$$

   The process and delicate aspects to consider are almost the same as in point (1). The only difference lies in raising both sides to the least common multiple of the root indices to ensure the elimination of both roots.

   For **equations with more than two roots**, we proceed with the same logic as with two roots.

# 9 Irrational Inequalities

An inequality is said to be irrational if the variable appears as an argument of at least one root. In general, the standard form of irrational inequalities is considered to be

$$\sqrt[n]{f(x)} \gtreqless g(x),$$

where $f(x), g(x)$ are any expressions containing the variable $x$, and $n$ is the index of the root.

## 9.1 Method of Solving Irrational Inequalities

The method of solving involves analyzing the standard form of the irrational inequality, paying attention to two aspects:

- The index of the root, which can be even or odd;
- The inequality symbol.

For each possible case, we can rely on a specific solving scheme.

### 9.1.1 Irrational Inequalities with Even Root

- Symbol *greater than*
  Solving
  $$\sqrt[n]{f(x)} > g(x)$$
  is equivalent to solving two systems and then considering the union of their solutions:
  $$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) > [g(x)]^n \end{cases} \quad \bigcup \quad \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases}$$
  Analyzing one by one the inequalities composing the two systems, which are distinguished based on the sign of the second term $g(x)$. The first system refers to the case where $g(x)$ is positive or zero:
  $$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) > [g(x)]^n \end{cases}$$
  1. The first condition concerns the existence condition of the even-index root.

2. The second condition is equivalent to imposing that $g(x)$ is positive or zero since we don't know beforehand if the second term is positive, negative, or zero.
3. The third condition is imposed as a consequence of the first two, indeed if the root exists (inequality 1) and if there is a positive or zero quantity on the right (inequality 2), then we can raise both sides to the $n$ to eliminate the root.

At the same time, we need to consider the second system, the one related to the case where $g(x)$ is negative:

$$\begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases}$$

1. The first condition is again the existence condition for the even-index root.
2. With the second condition, we're saying that we're discussing the case where $g(x)$ is negative.

We don't need any other inequalities in this system because if $f(x)$ is greater than or equal to zero (inequality 1) and if $g(x)$ is negative (inequality 2), then the inequality

$$\sqrt[n]{f(x)} > g(x)$$

is automatically verified. This is because an even-index root only assumes positive values or possibly zero.

Before continuing with the other cases, let's briefly revisit the first system, and observe that the third condition

$$f(x) > [g(x)]^n$$

implies that $f(x)$ is greater than a even power, obviously non-negative by definition, so the first of the three conditions

$$f(x) \geq 0$$

is implicitly included in the third and can be omitted. Recapitulating:

$$\sqrt[n]{f(x)} > g(x), \text{ with } n \text{ even } \rightarrow \begin{cases} g(x) \geq 0 \\ f(x) > [g(x)]^n \end{cases} \quad \bigcup \quad \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases}$$

- Symbol *greater than or equal to*
  The reasoning is identical to the previous case. Just adjust the inequality resulting from raising both sides to the $n$:

$$\sqrt[n]{f(x)} \geq g(x), \text{ with } n \text{ even } \rightarrow \begin{cases} g(x) \geq 0 \\ f(x) \geq [g(x)]^n \end{cases} \quad \bigcup \quad \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases}$$

- Symbol *less than*
  Solving
  $$\sqrt[n]{f(x)} < g(x)$$
  is equivalent to solving
  $$\begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) < [g(x)]^n \end{cases}$$
  Here too, let's try to understand the logic of the solving scheme.
  - The first condition concerns the existence condition of the even-index root. In particular, if it is satisfied then the first term of the inequality
    $$\sqrt[n]{f(x)} < g(x)$$
    is certainly positive or zero.
  - Consequently, the second term must be necessarily positive. If it were negative or zero, the inequality wouldn't be satisfied since, as we repeat, the first term is a positive quantity that is worth at least zero.
  - In the third condition, we can therefore raise both sides to the $n$ to remove the root.
- Symbol *less than or equal to*
  Just consider the symbol greater than or equal to instead of just greater than in the second condition of the system, and the symbol less than or equal to instead of just less than in the third condition:
  $$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) \leq [g(x)]^n \end{cases}$$

### 9.1.2 Irrational Inequalities with Odd Root

Odd-index roots do not require existence conditions, so the radicand can be any real number of any sign. To solve an irrational inequality with odd-index roots, it is sufficient to raise both sides of the inequality to that index. In practice,

$$\sqrt[n]{f(x)} \gtreqless g(x) \text{ with } n \text{ odd } \rightarrow f(x) \gtreqless [g(x)]^n.$$

This holds regardless of the inequality symbol. Odd-index roots are defined independently of the sign of the radicand and assume values with the same sign as the radicand, so we can raise both sides to $n$ and solve the resulting inequality.

# Part IV
# Systems of Equations and Inequalities

# 10 Systems of Equations

## 10.1 Second Degree Systems in Two Unknowns

Systems of equations are sets of two or more equations for which the common solutions are sought.
The **degree of a system** is the product of the degrees of the individual equations composing it.
Second-degree systems of two equations in two unknowns $x$ and $y$ are thus composed of one linear equation and one quadratic equation in $x$ and $y$.
The method of substitution is generally used:

1. Solve for one of the unknowns in the linear equation (e.g., $x$).
2. Substitute the expression found for $x$ into the quadratic equation, obtaining a quadratic equation in $y$ known as the *resolvent equation of the system*.

The existence and number of solutions of the system depend on the discriminant of the resolvent equation:

- If $\Delta \geq 0$: The resolvent equation has two real solutions (distinct or coincident), each corresponding to a value of $x$; therefore, the system has two real solutions (distinct or coincident), which are two ordered pairs of real numbers.
- If $\Delta < 0$: The resolvent equation has no real solutions, and thus the system also has no real solutions.

Some observations are appropriate:

- In the case where the resolvent equation turns out to be an identity, the system has infinite solutions and is called *indeterminate*;
- It may happen that the two equations of the system are incompatible: in this case, the resolvent equation is absurd, and the system is called *impossible*;
- In the case where the resolvent equation is a linear equation, the system is of first degree and therefore solved accordingly.

## 10.2 Second Degree Systems in Multiple Unknowns

Second-degree systems in $n$ unknowns consist of $n-1$ linear equations and one quadratic equation, and they are generally solved using the **method of substitution**. For instance, considering a second-degree system of three equations in three unknowns $x$, $y$, and $z$, composed of two linear equations and one quadratic equation, the resolution process is as follows:

- Initially solve the linear system formed by the two linear equations, obtaining, for example, the two unknowns $x$ and $y$ as functions of the third unknown $z$.
- Substitute the expressions thus obtained for $x$ and $y$ into the quadratic equation, obtaining a resolvent equation in the single variable $z$.
- Determine the solutions $z_1$ and $z_2$ of the resolvent equation (if they exist) and substitute them into the expressions for $x$ and $y$.

## 10.3 Symmetric Systems

Symmetric systems are special systems of second degree and higher degrees that, depending on their structure, can be solved straightforwardly. **Symmetric** systems are those consisting of two equations in two unknowns that remain unchanged if the two unknowns are interchanged.

### 10.3.1 Second Degree Symmetric Systems

The simplest second-degree symmetric system is called the *fundamental symmetric system* and is presented in the form:

$$\begin{cases} x + y = s \\ xy = p \end{cases}$$

with $p, s \in \mathbb{R}$. Since two numbers $x$ and $y$ must be determined knowing their sum $s$ and their product $p$, the two numbers are solutions of the equation:

$$t^2 - st + p = 0$$

referred to as the *equation associated with the system.* The existence of solutions of the system and their number depends on the discriminant of the associated equation:

- If $\Delta > 0$: The associated equation has two real and distinct solutions $t_1$ and $t_2$, and hence the system also has two real and distinct solutions, which are the ordered pairs $(t_1, t_2)$ and $(t_2, t_1)$.
- If $\Delta = 0$: The associated equation has two real and coincident solutions $t_1 = t_2$, and hence the system also has two real and coincident solutions $(t_1, t_1)$.
- If $\Delta < 0$: The associated equation has no real solutions, and hence the system also has no real solutions.

Another type of second-degree symmetric system is the one presented in the form:

$$\begin{cases} x + y = s \\ x^2 + y^2 = r^2 \end{cases}$$

which is solved by reducing it to the fundamental symmetric system. In fact, using the notable product $(x+y)^2 = x^2+y^2+2xy$ yields $x^2+y^2 = (x+y)^2-2xy$, known as the *first Waring formula*. Using the Waring formula in the second equation of the system, we get:

$$\begin{cases} x + y = x \\ (x + y)^2 - 2xy = r^2 \end{cases}$$

By substituting $x$ into the second equation as $x + y$, we obtain:

$$\begin{cases} x + y = s \\ s^2 - 2xy = r^2 \end{cases}$$

$$\begin{cases} x + y = s \\ -2xy = r^2 - s^2 \end{cases}$$

$$\begin{cases} x + y = s \\ xy = \dfrac{s^2 - r^2}{2} \end{cases}$$

which is the fundamental symmetric system.

### 10.3.2 Symmetric Systems of Degree Higher Than Second

Symmetric systems of degree higher than the second generally appear in the forms:

$$\begin{cases} x + y = s \\ x^m + y^m = a \end{cases}$$

with $m > 2$ or

$$\begin{cases} xy = p \\ x^m + y^m = a \end{cases}$$

with $m \geq 2$. These can be reduced to the fundamental symmetric system of second degree by transforming the second equation $x^m + y^m = a$ using the *Waring formulas*, which are obtained, with appropriate adjustments, from the notable products.

### 10.3.3 Systems Reducible to Symmetric Systems

Some systems can be transformed, with appropriate adjustments, into equivalent symmetric systems, and thus be solved as symmetric systems; however, the transformation is not always convenient.

## 10.4 Homogeneous Systems

A particular type of systems of degrees higher than the second is constituted by homogeneous systems.
A system is said to be **homogeneous** when it consists of homogeneous equations, i.e., equations whose terms, excluding the constant term, are all of the same degree.
We only consider **homogeneous systems of the fourth degree**, composed of two homogeneous second-degree equations in two unknowns, which are presented in the form:

$$\begin{cases} ax^2 + bxy + cy^2 = d \\ a'x^2 + b'xy + c'y^2 = d' \end{cases}$$

where all coefficients are real numbers.
These systems can be solved straightforwardly, i.e., they can be reduced to systems of first and second degrees.
The resolution procedure depends on the values of the constants $d$ and $d'$, which can both be different from zero, or both be zero, or one be different from zero and the other be zero.
In general, we perform the substitution $\frac{x}{y} = t$, after dividing one of the two equations by $y^2$ or by adding the two equations term by term.

## 11 Systems of Inequalities

Solving a system of inequalities means finding all and only the common solutions to the inequalities present in the system. In other words, the solutions of a system of inequalities are obtained from the intersection of the solution sets of the individual inequalities that compose it.

A system of inequalities is represented by curly braces enclosing and listing the involved inequalities.

$$\begin{cases} \text{inequality}_1 \\ \vdots \\ \text{inequality}_n \end{cases}$$

This notation denotes intersection: systematizing $n$ inequalities means considering the solutions common to all inequalities in the list, i.e., taking the intersection of the solution sets of all involved inequalities. In the language of logical connectives, the previous notation is equivalent to:

$$\text{inequality}_1 \wedge \cdots \wedge \text{inequality}_n$$

**Method for Solving Systems of Inequalities**

1. Given a system

$$\begin{cases} \text{inequality}_1 \\ \vdots \\ \text{inequality}_n \end{cases}$$

we will solve each inequality separately, as if it were an exercise on its own, and obtain their solutions. After solving all inequalities of the system, we rewrite it in the form

$$\begin{cases} \text{solutions}_1 \\ \vdots \\ \text{solutions}_n \end{cases}$$

2. How do we derive the solutions of the system, or rather how do we intersect the $n$ solution sets? To do this, we use a graphical comparison, through a suitable tabular representation.

   - Draw a half-line to indicate the real number line;
   - Below the half-line, fill in as many rows as there are inequalities in the system. On each row, represent the solutions of the corresponding inequality with a solid line, indicating inclusive endpoints with a filled dot and exclusive endpoints with an empty dot;
   - After representing the solutions of all inequalities, look at the table vertically. The solutions of the system are given by all and only the intervals corresponding to solid lines on each row. Regarding

endpoints, even a single empty dot on a row excludes the respective value from the solutions of the system.

## 11.1 Special Cases and Various Observations on Systems of Inequalities

Before moving on to examples, it is worth noting some observations:

- If even one of the inequalities that form the system is impossible, i.e., if it has no solutions ($\nexists x$), then the entire system will have no solutions;
- For each inequality in the system with solutions $\forall x$, we represent a solid line covering the entire real axis;
- The double inequalities

$$a \leq x \leq b$$
$$a < x \leq b$$
$$a \leq x < b$$
$$a < x < b$$

are nothing but synthetic forms representing small systems of inequalities

$$a \leq x \leq b \rightarrow \begin{cases} x \geq a \\ x \leq b \end{cases}$$

$$a < x \leq b \rightarrow \begin{cases} x > a \\ x \leq b \end{cases}$$

$$a \leq x < b \rightarrow \begin{cases} x \geq a \\ x < b \end{cases}$$

$$a < x < b \rightarrow \begin{cases} x > a \\ x < b \end{cases}$$

# Part V
# Absolute Value Equations and Inequalities

# 12 Equations with Absolute Value

**Equations with absolute value** are equations in which the unknown is present inside at least one absolute value. In general terms, such equations may have one or more absolute values, and they are solved by studying the signs of the arguments of the various absolute values, in order to eliminate them by explicating the signs of the arguments.

Let's consider the different standard forms for equations with one or more absolute values.

- Equations with one absolute value and constant term

$$|A(x)| = k$$

- Equations with two absolute values

$$|A(x)| = |B(x)|$$

- Equations with one absolute value and variable term

$$|A(x)| = B(x)$$

- Equations with two or more absolute values
- Equations with one absolute value inside another

## 12.1 Equations with One Absolute Value and Constant Term

In standard form, we have

$$|A(x)| = k.$$

Let $A(x)$ be a polynomial, and let $k \in \mathbb{R}$. We distinguish different cases.

1. If $k > 0$, solving the equation

$$|A(x)| = k$$

is equivalent to solving the following equations separately, and uniting their solution sets:

$$A(x) = k \ \lor \ A(x) = -k.$$

2. If $k = 0$, the equation

$$|A(x)| = 0$$

is equivalent to

$$A(x) = 0.$$

3. If $k < 0$, the equation
$$A(x) = k$$
has no solutions because it is equivalent to comparing a non-negative term with a negative term. We will say that the set of solutions is the empty set.

## 12.2 Equations with Two Absolute Values

The second type of equations with absolute value we consider is given by **equations with two absolute values**, which in standard form reduce to

$$|A(x)| = |B(x)|.$$

Let $A(x), B(x)$ be two polynomials depending on the variable $x$. We don't need to impose any sign agreement condition, because both sides are in absolute value and therefore are both greater than or equal to zero.

The set of solutions of the equation coincides with the union of the solution sets of the equations

$$A(x) = B(x) \ \vee \ A(x) = -B(x),$$

so we just need to separately solve the two equations and unite the obtained solutions.

## 12.3 Equations with One Absolute Value and Variable Term

Equations with one absolute value and variable term are presented in standard form as
$$|A(x)| = B(x)$$
where $A(x), B(x)$ are polynomials with real coefficients.

This type of equations is equivalent to the union of two mixed systems, obtained by specifying the sign of the argument of the absolute value and eliminating the absolute value.

$$\begin{cases} A(x) \geq 0 \\ A(x) = B(x) \end{cases} \quad \cup \quad \begin{cases} A(x) < 0 \\ -A(x) = B(x) \end{cases}$$

The reason for this equivalence comes from the definition of absolute value:

$$|A(x)| = \begin{cases} A(x) & \text{if } A(x) \geq 0 \\ -A(x) & \text{if } A(x) < 0 \end{cases}$$

## 12.4 Equations with Two or More Absolute Values

Dealing with equations containing two or more absolute values opens the door to numerous types of modulus equations. Fortunately, they can be solved using the same technique.

The basic idea of the solution method is analogous to that of equations with a single absolute value and variable constant term: we need to study the signs of the arguments and appropriately eliminate the absolute values by specifying the signs of the arguments. Since there will be two or more absolute values, we aim to accomplish this in as few logical steps as possible. To aid us, we utilize a sign chart that allows us to quickly determine how to specify the signs of the arguments and eliminate all absolute values simultaneously.

Furthermore, this method does not require any conditions for sign agreement.

## 12.5 Equations with an Absolute Value Nested Inside Another

Cases we've previously analyzed, where $A(x)$ and $B(x)$ were polynomials, are perfectly suited for solving equations with nested absolute values. In general:

- We reason from the outside in, first dealing with the outermost absolute values, then the innermost ones.
- It might happen that a condition on studying the signs of the arguments translates into an absolute value inequality. In such cases, we simply solve it separately.
- Our goal always involves reducing the equation to a standard form. The nesting of absolute values compels us to work on the outer ones, studying the conditions related to the signs of the arguments and eliminating them; once done, we iterate the process, addressing the innermost absolute values and their respective sub-cases.

# 13 Inequalities with Absolute Value

**Inequalities involving absolute values** are inequalities where the unknown appears in at least one absolute value; they are solved by decomposing them into unions of systems of inequalities, where conditions are imposed to eliminate the absolute values and attribute the correct sign to the arguments.

The notation $|x|$ has a precise meaning. According to the definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

as the possible values of $x$ vary, the absolute value $|x|$ is always a non-negative number, and in particular, it's zero only when $x = 0$.

Since we don't know the value of the unknown $x$ beforehand, what we'll do is:

- Impose the condition on the sign of the argument of the absolute value;
- Rewrite the inequality by eliminating the absolute value and attributing the correct sign to the argument;
- Systematize the condition and the inequality.

As there are two conditions on the sign of the argument of the absolute value

$$(\geq, <),$$

we'll have two systems of inequalities; since both conditions are valid, we'll need to consider the union of the solutions of the two systems.

The same reasoning applies even when dealing with absolute values where the argument is an expression $f(x)$ containing the unknown $x$. With these premises, let's see how to solve the three possible types of absolute value inequalities:

- Elementary inequalities with a single absolute value, in the form

$$|f(x)| \gtreqless c$$

- Non-elementary inequalities with a single absolute value, of the type

$$|f(x)| \gtreqless g(x)$$

- Inequalities with two or more absolute values.

| Symbol | Sign of $c$ | Solutions |
|---|---|---|
| $\geq$ | $c > 0$ | To be analyzed |
| $\geq$ | $c = 0$ | $\forall x$ |
| $\geq$ | $c < 0$ | $\forall x$ |
| $>$ | $c > 0$ | To be analyzed |
| $>$ | $c = 0$ | $\forall x \; - \; \{ \; x \mid f(x) = 0 \; \}$ |
| $>$ | $c < 0$ | $\forall x$ |
| $\leq$ | $c > 0$ | To be analyzed |
| $\leq$ | $c = 0$ | $\{ \; x \mid f(x) = 0 \; \}$ |
| $\leq$ | $c < 0$ | $\nexists x$ |
| $<$ | $c > 0$ | To be analyzed |
| $<$ | $c = 0$ | $\nexists x$ |
| $<$ | $c < 0$ | $\nexists x$ |

## 13.1 Elementary Inequalities with Absolute Value

Elementary inequalities with a single absolute value are those presented in the form

$$|f(x)| \gtreqless c$$

where $c$ is a real number.

The first step is to impose any existence conditions related to the expression $f(x)$ and consider them when deriving the solutions of the inequality.

Since the absolute value $|f(x)|$ is, by definition, a non-negative quantity if $f(x) \neq 0$, or zero if $f(x) = 0$, depending on the sign of $c$, the inequality might be trivial:

The only interesting case occurs when $c > 0$, because in all other cases we obtain either an immediate inequality or an equation. Let's focus on

$$|f(x)| \gtreqless c \text{ with } c > 0$$

. Regardless of the symbol, if $c$ is positive, the inequality translates into the union of two systems of inequalities:

$$\begin{cases} f(x) \geq 0 \\ f(x) \gtreqless c \end{cases} \quad \bigcup \quad \begin{cases} f(x) < 0 \\ -f(x) \gtreqless c \end{cases}$$

What we've done is use the definition of absolute value to eliminate it:

- by assigning the correct sign to the argument $f(x)$ (second set of inequalities);
- under the respective condition on the sign of the argument (first set of inequalities).

Since both possibilities are valid, we need to solve the two systems separately and consider the union of their solution sets.

## 13.2 Case $c > 0$ and symbol $<$ (or $\leq$)

$$|f(x)| < c \quad \text{with } c > 0$$

Rewriting the union of the two solving systems:

$$\begin{cases} f(x) \geq 0 \\ f(x) < c \end{cases} \quad \bigcup \quad \begin{cases} f(x) < 0 \\ -f(x) < c \end{cases}$$

The second inequality of the second system can be written as $f(x) > -c$:

$$\begin{cases} f(x) \geq 0 \\ f(x) < c \end{cases} \quad \bigcup \quad \begin{cases} f(x) < 0 \\ f(x) > -c \end{cases}$$

Since we're assuming $c > 0$, it's evident that $-c < 0$. Therefore, we can express the two systems of inequalities as double inequalities

$$0 \leq f(x) < c \text{ or } -c < f(x) < 0$$

, and rewrite everything in a more compact form

$$-c < f(x) < c$$

. Translating this double inequality into a system of inequalities, we obtain

$$\begin{cases} f(x) > -c \\ f(x) < c \end{cases}$$

Similar considerations hold for the $\leq$ symbol.

## 13.3 Case $c > 0$ and symbol $>$ (or $\geq$)

$$|f(x)| > c \quad \text{with } c > 0$$

Starting again from the union of the two solving systems:

$$\begin{cases} f(x) \geq 0 \\ f(x) > c \end{cases} \quad \bigcup \quad \begin{cases} f(x) < 0 \\ -f(x) > c \end{cases}$$

We express the second inequality of the second system as $f(x) < -c$. It's immediately seen that each system reduces to a single inequality

$$f(x) > c \text{ or } f(x) < -c$$

, or alternatively

$$f(x) < -c \text{ or } f(x) > c$$

. Therefore, we just need to separately solve these inequalities and consider the union of their respective solution sets.

## 13.4 Non-elementary Inequalities with Absolute Value

Now, let's move to the more general case of inequalities with an absolute value:

$$|f(x)| \gtreqless g(x)$$

where $f(x), g(x)$ are expressions containing the variable $x$. The procedure is essentially the same as we've seen in the elementary case with $c > 0$. After imposing any necessary existence conditions on $f(x), g(x)$, we need to:

1. Study the sign of the argument of the absolute value by setting $f(x) \geq 0$;
2. Draw the sign table, identifying intervals where $f(x)$ is positive or negative;
3. For each group of intervals where the sign is positive or negative, write a system consisting of two inequalities:
   - the first inequality is the condition that fixes the sign of $f(x)$;
   - the second is the rewrite of the original inequality, where we eliminate the absolute value and assign the correct sign to its argument.
4. Solve the two systems separately;
5. Find the solutions of the original inequality as the union of the solutions of the two systems.

In summary:

$$\begin{cases} f(x) \geq 0 \\ f(x) \gtreqless g(x) \end{cases} \quad \bigcup \quad \begin{cases} f(x) < 0 \\ -f(x) \gtreqless g(x) \end{cases}$$

## 13.5 Inequalities with Two or More Absolute Values

Finally, let's analyze the case of inequalities with more than one absolute value. The procedure is essentially the same, with the difference that we could have more than two conditions related to the signs of the arguments (and hence more than two systems). After imposing any necessary existence conditions...

1. Study the signs of the arguments of each absolute value;
2. Draw the sign table to identify intervals where the various arguments have positive or negative signs.
3. For each interval, set up a system combining the condition on the sign of the argument with the inequality; in the latter, eliminate the absolute values and assign the sign to each argument according to the sign table.
4. Solve each system separately.
5. Determine the solutions of the original inequality as the union of the solution sets of all systems.

## Part VI
# Exponential Equations and Inequalities

# 14 Exponential Equations

**Exponential equations** are equations in which exponential functions appear, with the variable appearing in at least one exponent.

To study the resolution methods, we will consider various **standard forms of exponential equations** and see how to exploit them to determine any solutions:

- Elementary exponential equations
$$a^{f(x)} = b$$
- Exponential equations with exponential term known and with variable
$$a^{f(x)} = b^{g(x)}$$
- Exponential equations by substitution

## 14.1 Elementary Exponential Equations

Elementary exponential equations are of the form

$$a^{f(x)} = b$$

where $a$ is a positive real number different from 1, and $f(x)$ is any mathematical expression containing the variable $x$.

Regardless of whether the base $a$ is between 0 and 1 or greater than 1, we know that the exponential term $a^{f(x)}$ is positive by definition. Thus, a simple preliminary analysis assures us that the exponential equation will be:

- Impossible if $b \leq 0$;
- Determined if $b > 0$.

In particular, note that exponential equations of the previous form cannot be indeterminate. Under the assumption $b > 0$, there are essentially two methods to solve elementary exponential equations. The choice depends on the relationship between the numbers $a$ and $b$, and in this regard, we have two possibilities:

1. $b$ can be expressed as a power of $a$, that is,
$$\exists\, k \in \mathbb{Q} \,:\, b = a^k;$$
2. $b$ cannot be expressed as a power of $a$, that is,
$$\nexists\, k \in \mathbb{Q} \,:\, b = a^k.$$

Case (1) leads to the method of solving elementary exponential equations with powers, while (2) requires the solution technique of elementary exponential equations with logarithms.

### 14.1.1 Elementary Exponential Equations Solvable with Powers

Considering case (1), bearing in mind the assumptions

$$a^{f(x)} = b \text{ with } a > 0,\, a \neq 1 \quad \wedge \quad b > 0 \quad \exists\, k \in \mathbb{Q} : b = a^k$$

It will be possible to find solutions simply by writing $b$ as a power of $a$, i.e., bringing it to the form

$$a^{f(x)} = a^k$$

and then equating the exponents

$$f(x) = k.$$

This way, we reduce it to a non-exponential equation, which we could solve with appropriate solution techniques.

### 14.1.2 Elementary Exponential Equations Solvable with Logarithms

Now, let's analyze case (2) of elementary exponential equations

$$a^{f(x)} = b \text{ with } a > 0,\, a \neq 1 \quad \wedge \quad b > 0 \quad \nexists\, k \in \mathbb{Q} : b = a^k$$

After bringing it to standard form, if we cannot find a rational exponent $k$ such that $b = a^k$, that is, if we cannot express $b$ as a power of $a$, the resolution method involves resorting to logarithms.
Recalling that

$$a^c = b \Leftrightarrow c = \log_a(b) \text{ with } a > 0,\, a \neq 1,\, b > 0$$

We apply the logarithm with base $a$ to both sides of the equation:

$$\log_a(a^{f(x)}) = \log_a(b)$$

and, using the definition of logarithm, we obtain

$$f(x) = \log_a(b)$$

where $\log_a(b)$ is simply a number, as it does not depend on the variable.

By doing this, we have eliminated the exponential component of the equation and reduced it to another type of equation, which depends on the specific case and which we will solve using known methods.

## 14.2 Exponential Equations with Known Exponential Term and Variable

Let's now consider exponential equations that can be reduced to the standard form

$$a^{f(x)} = b^{g(x)} \quad a, b > 0 \;\; a, b \neq 1$$

where $f(x), g(x)$ are mathematical expressions dependent on the variable $x$.

This is a case very similar to the previous one, as we need to try to write $b$ as a power of $a$ and then equate the exponents.

## 14.3 Exponential Equations by Substitution

The case of exponential equations by substitution is extremely general and does not include a specific standard form. If we encounter a complicated-looking exponential equation, where:

- There are sums and differences between multiple exponentials
- There is a repetition of an exponential term, eventually after reducing the exponential terms to the same base.

then we need to attempt applying the substitution method, which involves replacing the repeated exponential term with a new variable.

# 15 Exponential Inequalities

**Exponential inequalities** are inequalities in which the variable appears as an exponent of at least one exponential term, regardless of whether the base of that term is constant or variable.

For this type of inequalities, there is no single resolution method. What we can do is divide the procedure into a general part, valid in any case, and a specific part, which depends on the standard form to which we can reduce the initial inequality.

Regarding the general part of the procedure, the steps are as follows:

1. We observe the inequality as it is presented and establish the existence conditions. In this regard, it is good to remember a fundamental property of exponentials with constant base: they are positive wherever they are defined. Terms of the form

   $$a^{f(x)} \text{ with } a > 0 \ a \neq 1$$

   do not require existence conditions as exponentials, but may require conditions of existence related to the expression $f(x)$. Moreover, within the domain of $f(x)$, the term $a^{f(x)}$ is always positive. Consequently, it is not negative and especially it does not become zero:

   $$a^{f(x)} > 0 \quad \forall\, x, \exists\, f(x).$$

   When setting the existence conditions, it will be fundamental to remember this property, and naturally check that there are no expressions that require existence conditions by their nature (such as denominators, roots with even indices, logarithms,...)

2. After setting the existence conditions, we can proceed to calculations and simplifications, with the goal of reaching one of the following types ($a, b, c$ denote numbers and $f(x), g(x)$ expressions containing the variable):

   - Elementary exponential inequalities

     $$a^{f(x)} \gtreqless a^{g(x)} \ \text{ with } a > 0, \ a \neq 1$$

   - Exponential inequalities with logarithmic transition, of the first type

     $$a^{f(x)} \gtreqless c \ \text{ with } a > 0, \ a \neq 1$$

   - Exponential inequalities with logarithmic transition, of the second type

     $$a^{f(x)} \gtreqless b^{g(x)} \ \text{ with } a, b > 0, \ a, b \neq 1, \ a \neq b$$

- Exponential inequalities reducible to polynomial inequalities by substitution, of the type

$$C_n a^{nf(x)} + ... + C_2 a^{2f(x)} + C_1 a^{f(x)} + C_0 \gtreqless 0$$
$$\text{with } C_0, C_1, ..., C_n \in \mathbb{R},\ a > 0,\ a \neq 1$$

- Exponential inequalities with variable base, containing at least one term of the form

$$[f(x)]^{g(x)}$$

- Exponential inequalities of another type

3. Finally, we apply the specific method for the type of exponential inequality we have identified, determine the solutions, and compare them with the existence conditions set at the beginning.

## 15.1 Elementary Exponential Inequalities

We consider exponential inequalities of the form:

$$a^{f(x)} \gtreqless a^{g(x)} \text{ where } a > 0,\ a \neq 1$$

If we refer to the standard form, we must ensure to impose the existence conditions related to the expressions $f(x), g(x)$ in a system:

$$\begin{cases} \text{existence of } f(x) \\ \text{existence of } g(x) \\ a^{f(x)} \gtreqless a^{g(x)} \end{cases}$$

To clarify, let's reason with the symbol $\geq$, noting that similar reasoning applies to the other inequality symbols:

$$a^{f(x)} \geq a^{g(x)}$$

In our hypothesis, the bases of the two exponentials are equal. Here, it's straightforward because we can eliminate the bases and directly compare the exponents. In doing so, we only need to pay attention to the value of the base:

- $a > 1$
  If the base is greater than 1, we compare the exponents while preserving the inequality symbol.

$$\begin{cases} \text{existence of } f(x) \\ \text{existence of } g(x) \\ a^{f(x)} \geq a^{g(x)} \end{cases}$$

- $0 < a < 1$
  If the base is between 0 and 1, we compare the exponents by reversing the inequality symbol.
$$\begin{cases} \text{existence of } f(x) \\ \text{existence of } g(x) \\ a^{f(x)} \leq a^{g(x)} \end{cases}$$

*Note* 1. But why do we change the direction when the base is between 0 and 1? Consider the exponential inequality

$$a^{f(x)} \geq a^{g(x)} \text{ with } 0 < a < 1.$$

We use the definition of a power with a negative exponent to express both sides in an equivalent form:

$$\left(\frac{1}{a}\right)^{-f(x)} \geq \left(\frac{1}{a}\right)^{-g(x)}.$$

Since $0 < a < 1$, the reciprocal of $a$ is greater than 1:

$$0 < a < 1 \rightarrow \frac{1}{a} > 1$$

Therefore, we can compare the exponents in an inequality with the same direction as the one we just wrote:

$$-f(x) \geq -g(x)$$

Hence, by changing the signs, we get:

$$f(x) \leq g(x)$$

which is the inequality we expected for exponentials with bases between 0 and 1.

## 15.2 Exponential Inequalities with Logarithmic Transition, First Type

Another type of exponential inequalities involves applying an appropriate logarithm to both sides, taking the form:

$$a^{f(x)} \gtreqless c \quad \text{where } a > 0,\ a \neq 1$$

Referring to the standard form, we impose any Existence Conditions (EC) related to the expression $f(x)$:

$$\begin{cases} \text{existence of } f(x) \\ a^{f(x)} \gtreqless c \end{cases}$$

Considering that the exponential on the left is positive where $f(x)$ is defined, we have different cases depending on the inequality symbol and the sign of $c$, some of which are rather obvious.

| **Symbol** | **Sign of** $c$ | **Solutions** |
|---|---|---|
| $\geq$ | $c > 0$ | To be analyzed |
| $\geq$ | $c = 0$ | $\forall x$ |
| $\geq$ | $c < 0$ | $\forall x$ |
| $>$ | $c > 0$ | To be analyzed |
| $>$ | $c = 0$ | $\forall x$ |
| $>$ | $c < 0$ | $\forall x$ |
| $\leq$ | $c > 0$ | To be analyzed |
| $\leq$ | $c = 0$ | $\nexists x$ |
| $\leq$ | $c < 0$ | $\nexists x$ |
| $<$ | $c > 0$ | To be analyzed |
| $<$ | $c = 0$ | $\nexists x$ |
| $<$ | $c < 0$ | $\nexists x$ |

The only interesting scenario is when $c$ is positive because in all other cases, the positivity of the left member immediately leads to solutions. So, let's focus on

$$a^{f(x)} \gtreqless c \text{ where } c > 0$$

and for clarity, consider the greater-equal symbol (the process is similar for other cases):

$$a^{f(x)} \geq c \text{ with } c > 0$$

We apply the logarithm with base $a$ to both sides to eliminate the exponential on the left. We can do this because the logarithm requires the argument to be positive, and both members satisfy this condition.

When calculating the logarithm of both sides, we must check the value of the base $a$, following what we have seen for logarithmic inequalities:

- $a > 1$
  If the base is greater than 1, we're applying a logarithm with a base greater than 1, so we leave the inequality symbol unchanged.

  $$\log_a(a^{f(x)}) \geq \log_a(c)$$

  $$\begin{cases} \text{existence of } f(x) \\ f(x) \geq \log_a(c) \end{cases}$$

- $0 < a < 1$
  If the base is between 0 and 1, we're applying a logarithm with a base between 0 and 1, so we reverse the inequality symbol.

  $$\log_a(a^{f(x)}) \leq \log_a(c)$$

$$\begin{cases} \text{existence of } f(x) \\ f(x) \leq \log_a(c) \end{cases}$$

Depending on the type of inequality we obtain:

$$f(x) \gtreqless \log_a(c)$$

we'll proceed to solve it using the most suitable method.

## 15.3 Exponential Inequalities with Logarithmic Transition, Second Type

Another type extends basic exponential inequalities, involving a comparison between exponentials with different bases:

$$a^{f(x)} \gtreqless b^{g(x)} \text{ with } a, b > 0,\ a, b \neq 1,\ a \neq b$$

Beyond the usual considerations (checking the initial form of the inequality and calculations to reach the standard form), if the bases of the two exponential terms are different, unfortunately, the situation becomes more complicated.

Firstly, we impose any necessary conditions for expressions $f(x)$ and $g(x)$:

$$\begin{cases} \text{existence of } f(x) \\ \text{existence of } g(x) \\ a^{f(x)} \gtreqless b^{g(x)} \end{cases}$$

and consider the standard form with the $\geq$ symbol (the reasoning is similar for other cases):

$$a^{f(x)} \geq b^{g(x)} \quad \text{with } a \neq b$$

In general, it is convenient to apply the natural logarithm to both sides, which we can do because both exponentials are positive (where defined).

Since the base of the natural logarithm is Euler's number $e$, which is greater than 1, we keep the direction of the inequality symbol unchanged:

$$\ln(a^{f(x)}) \geq \ln(b^{g(x)})$$

Using the logarithmic property concerning the logarithm of a power:

$$f(x) \cdot \ln(a) \geq g(x) \cdot \ln(b)$$

Considering that $\ln(a)$ and $\ln(b)$ are numerical coefficients, at this point, we will do our best to solve the resulting inequality, which could be of any type.

## 15.4 Exponential Inequalities Reducible to Polynomials (by Substitution)

Now let's move on to another variant:

$$C_n a^{nf(x)} + ... + C_2 a^{2f(x)} + C_1 a^{f(x)} + C_0 \gtreqless 0$$
$$\text{with } C_0, C_1, ..., C_n \in \mathbb{R},\ a > 0,\ a \neq 1$$

The essence of the discussion refers to inequalities in which the same exponential term repeats with a different integer coefficient among the various exponents.

To understand this, we can rewrite the left-hand side using the property for powers of powers:

$$C_n \left[a^{f(x)}\right]^n + ... + C_2 \left[a^{f(x)}\right]^2 + C_1 \left[a^{f(x)}\right] + C_0 \gtreqless 0$$

To solve these types of inequalities, we first ensure that the expression $f(x)$ is defined:

$$\begin{cases} \text{existence of } f(x) \\ C_n a^{nf(x)} + ... + C_2 a^{2f(x)} + C_1 a^{f(x)} + C_0 \gtreqless 0 \end{cases}$$

Then we perform a simple substitution:

$$y = a^{f(x)}$$

which reduces the problem to a first or second-degree inequality, or even to an inequality of degree higher than the second:

$$C_n y^n + ... + C_2 y^2 + C_1 y + C_0 \gtreqless 0$$

Next, we solve the inequality and determine its solutions.

Afterward, we need to revert to the variable $x$. Assuming the solutions are:

$$y < y_1 \ \vee \ y_2 < y < y_3 \ \vee \ y > y_4$$

We perform the inverse substitution $y = a^{f(x)}$:

$$a^{f(x)} < y_1 \ \vee \ y_2 < a^{f(x)} < y_3 \ \vee \ a^{f(x)} > y_4$$

Each of the inequalities expressing the solution translates into an exponential inequality with logarithmic transition, of the first type. When solving them, we need to pay attention to some aspects:

- The logical connector $\vee$ means union, so we can separately solve the exponential inequalities and consider the union of the solutions.
- Double inequalities are to be understood as systems.
- After translating all solutions from the variable $y$ to the variable $x$, let's not forget about the existence conditions stated at the beginning.

## 15.5 Exponential Inequalities with Variable Base

The last type of exponential inequalities involves exponential terms with a variable base, that is, of the form

$$[f(x)]^{g(x)}.$$

Since expressions involving such terms are literally infinite, instead of a method, it's better to provide two suggestions:

- The first one concerns existence conditions. For an exponential with a variable base to make sense, it requires the base to be positive, in addition to the conditions concerning expressions $f(x)$ and $g(x)$:

$$\begin{cases} \text{existence of } f(x) \\ \text{existence of } g(x) \\ f(x) > 0 \end{cases}$$

- The second one is an algebraic trick that allows us to rewrite exponentials with a variable base into a more approachable form, using the **logarithm-exponential identity**:

$$[f(x)]^{g(x)} = e^{\ln([f(x)]^{g(x)})}$$

  Equivalently, due to a well-known property of logarithms:

$$[f(x)]^{g(x)} = e^{g(x)\ln(f(x))}$$

# 16 Logarithmic Equations

**Logarithmic equations** are equations in which the variable appears at least once in the argument of a logarithm, or possibly in the base of a logarithm.

There are various methods for solving logarithmic equations. Some of them involve reducing to a specific normal form, while others require a technique.

In general, we should:

1. Impose the existence conditions, always remembering that by definition a logarithm must have:
   - positive argument;
   - positive base, different from 1.
2. Reduce to a suitable normal form, or a form on which we can work, and apply an appropriate solving method;
3. Determine if the obtained solutions are acceptable.

The methods and types we will deal with are as follows:

- Elementary logarithmic equations

$$\log_a(f(x)) = \log_a(g(x))$$

- Logarithmic equations solvable by converting to exponential form

$$\log_a(f(x)) = b$$

**Existence Conditions of a Logarithmic Equation** Before proceeding with one of the solving methods, we must ensure that the arguments of the logarithms containing the variable are greater than zero. If the variable also appears in the bases, we will require that these bases be greater than 0 and different from 1.

$$\log_{\text{base}(x)}(\text{argument}(x)) \to \begin{cases} \text{base}(x) > 0 \\ \text{base}(x) \neq 1 \\ \text{argument}(x) > 0 \end{cases}$$

This way, we obtain an inequality or a system of inequalities to be solved. Additionally:

- if the inequality or the system of inequalities has no solutions, it is useless to continue; the logarithmic equation is impossible;
- otherwise, the solutions of the existence conditions identify the set of solutions of the logarithmic equation. We set them aside, remembering to keep them in mind at the end.

After finding the acceptability conditions of the equation's solutions, we perform the calculations to try to fall into one of the following cases.

## 16.1 Elementary Logarithmic Equations

If after imposing the existence conditions we fall into an elementary logarithmic equation, whose normal form is of the type:

$$\log_a[f(x)] = \log_a[g(x)], \quad \text{with } a > 0, a \neq 1$$

to find the solutions, it suffices to equate the arguments:

$$f(x) = g(x).$$

At this point, we need to find the solutions of the equation $f(x) = g(x)$, remembering, in the end, to verify whether the obtained solutions respect the acceptability conditions.

## 16.2 Logarithmic Equations Solvable by Converting to Exponential Form

Once we have found the acceptability conditions of the solutions, and after performing the calculations, if we fall into a logarithmic equation of the form

$$\log_a[f(x)] = b, \quad \text{with } a > 0,\ a \neq 1$$

we will find the solutions by converting to exponential form, i.e., using the definition of logarithm.
That is

$$\begin{aligned} &\log_a[f(x)] = b \\ \Rightarrow\ &\log_a[f(x)] = b \log_a a \\ \Rightarrow\ &\log_a[f(x)] = \log_a a^b \\ \Rightarrow\ &f(x) = a^b \end{aligned}$$

At this point, we just need to find the solutions of this last equation and then verify the acceptability of the solutions.

# 17 Logarithmic Inequalities

A **logarithmic inequality** is one where the variable appears as the argument or base of at least one logarithm.

There is no universal method for solving them; rather, we can talk about a *general procedure*, valid for any type of logarithmic inequality, which allows us to reduce the inequality to a specific normal form; each normal form then involves applying a specific method.

Let's see the steps of the general procedure, and then we'll separately analyze each of the normal forms and the respective methods:

1. Before carrying out any calculation, we impose the existence conditions. We remember that by definition, a logarithm requires that the argument be positive, and that the base be positive and different from 1. So, we will look at the left and right sides of the inequality, and for each logarithm containing the variable in the argument or in the base, we will impose a corresponding condition according to the following scheme:

$$\log_{\text{base}(x)}(\text{argument}(x)) \rightarrow \begin{cases} \text{base}(x) > 0 \\ \text{base}(x) \neq 1 \\ \text{argument}(x) > 0 \end{cases}$$

   Clearly, in addition to the existence conditions related to logarithms, we must also impose those related to other relevant aspects of the inequality. All existence conditions must be enclosed in a single system of inequalities.

2. Calculations and simplifications, in order to reduce to one of the following normal forms (we denote with $a, b, c$ numerical terms and with $f(x), g(x)$ generic expressions containing the variable $x$):

   - Elementary logarithmic inequalities, of the type

$$\log_a[f(x)] \gtreqless \log_b[g(x)] \text{ with } a, b > 0, \quad a, b \neq 1$$

   - Logarithmic inequalities with conversion to exponential form, of the type

$$\log_a[f(x)] \gtreqless c \text{ with } c \in \mathbb{R},\ a > 0\ a \neq 1$$

   - Logarithmic inequalities reducible to polynomial inequalities by substitution, of the type

$$C_n \log_a^n[f(x)] + ... + C_2 \log_a^2[f(x)] + C_1 \log_a[f(x)] + C_0 \gtreqless 0$$

     with $C_0, C_1, .., C_n \in \mathbb{R},\ a > 0,\ a \neq 1.$

- Logarithmic inequalities with the variable in the base, containing at least one term of the form

$$\log_{f(x)}(a) \text{ or } \log_{f(x)}(g(x))$$

3. Apply the specific method for the normal form we have identified, obtain the solutions, and establish whether they are acceptable by comparing them with the existence conditions.

## 17.1 Elementary Logarithmic Inequalities

Let's see how to tackle elementary logarithmic inequalities, namely those solvable by eliminating logarithms:

$$\log_a[f(x)] \gtreqless \log_b[g(x)] \quad \text{where} \quad a, b > 0, \quad a, b \neq 1$$

Before proceeding, algebraic steps might be necessary to arrive at this form, and even before performing them, we need to consider any possible domain restrictions on the original inequality.

Referring to the standard form, we must establish the existence conditions related to the arguments of the logarithms, which must be positive, and we write the system of inequalities:

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \\ \log_a[f(x)] \gtreqless \log_b[g(x)] \end{cases}$$

To clarify, let's assume that the inequality symbol is $\geq$, as similar considerations hold for other symbols.

$$\log_a[f(x)] \geq \log_b[g(x)]$$

If the bases of the two logarithms are **equal**, i.e., $a = b$, then we have a straightforward path. There are no issues with the first two inequalities of the system; concerning the last one, we can directly eliminate the logarithms by comparing the arguments. The critical aspect concerns the value of the base.

- $a > 1$
  If the base is greater than 1, we eliminate the logarithms while preserving the inequality symbol:

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x) \geq g(x) \text{ (same symbol)} \end{cases}$$

- $0 < a < 1$
  If the base is between 0 and 1, we eliminate the logarithms by reversing the inequality symbol:

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x) \leq g(x) \text{ (reversal)} \end{cases}$$

If the bases of the two logarithms are **different**, we need to appropriately change the base of one of the logarithms to bring it to the case of equal bases. Let's revisit the standard form with the $\geq$ symbol.

$$\log_a[f(x)] \geq \log_b[g(x)]$$

We use the logarithmic *change of base* formula and ensure that the right-hand logarithm has the same base as the left-hand one.

$$\log_b[g(x)] = \frac{\log_a[g(x)]}{\log_a(b)}$$

Substituting:

$$\log_a[f(x)] \geq \frac{\log_a[g(x)]}{\log_a(b)}$$

The term $\log_a(b)$ is purely numerical, so we can move it to the left-hand side. However, first, we establish whether it is positive or negative because in the latter case, we must invert the inequality symbol:

$$\log_a(b) > 0 \to \log_a(b) \cdot \log_a[f(x)] \geq \log_a[g(x)]$$
$$\log_a(b) < 0 \to \log_a(b) \cdot \log_a[f(x)] \leq \log_a[g(x)]$$

Now, we use the logarithmic property related to the exponent rule:

$$\log_a(b) > 0 \to \log_a[[f(x)]^{\log_a(b)}] \geq \log_a[g(x)]$$
$$\log_a(b) < 0 \to \log_a[[f(x)]^{\log_a(b)}] \leq \log_a[g(x)]$$

And we have brought ourselves back to the case of equal bases, which we already know how to handle.

## 17.2 Logarithmic Inequalities with Exponential Conversion

Logarithmic inequalities with exponential conversion are those presented in the form

$$\log_a[f(x)] \gtreqless c \text{ with } c \in \mathbb{R},\ a > 0,\ a \neq 1$$

The solution procedure for this standard form consists of the existence condition and the inequality itself

$$\begin{cases} f(x) > 0 \\ \log_a[f(x)] \gtreqless c \end{cases}$$

To illustrate, let's consider the inequality with the symbol $\geq$; the procedure remains the same for other inequality symbols:

$$\log_a[f(x)] \geq c$$

We notice that we can rewrite the second inequality by applying the definition of logarithm

$$c = \log_a(a^c)$$

thus the inequality becomes

$$\log_a[f(x)] \geq \log_a(a^c)$$

and we can stop here because we have reduced it to a basic logarithmic inequality that we already know how to solve.

## 17.3 Logarithmic Inequalities Reducible to Polynomials (by Substitution)

The third type of logarithmic inequalities we analyze is solved by substitution, reducing the expression containing logarithms to a polynomial form:

$$C_n \log_a^n[f(x)] + ... + C_2 \log_a^2[f(x)] + C_1 \log_a[f(x)] + C_0 \gtreqless 0$$

with $C_0, C_1, ..., C_n \in \mathbb{R},\ a > 0,\ a \neq 1$.

The above form identifies inequalities where the same logarithm appears in various powers.

Regarding the solution method, the standard form requires the existence conditions of the logarithm, hence the system

$$\begin{cases} f(x) > 0 \\ C_n \log_a^n[f(x)] + ... + C_2 \log_a^2[f(x)] + C_1 \log_a[f(x)] + C_0 \gtreqless 0 \end{cases}$$

Let's set

$$y = \log_a[f(x)]$$

and obtain

$$\begin{cases} f(x) > 0 \\ C_n y^n + ... + C_2 y^2 + C_1 y + C_0 \gtreqless 0 \end{cases}$$

Thus, we have translated the logarithmic inequality into a polynomial inequality of degree $n$. After identifying its solutions, we need to bring them back to the variable $x$. Suppose the solutions are

$$y < y_1 \ \vee \ y_2 < y < y_3 \ \vee \ y > y_4$$

To revert back to the variable $x$, we must substitute $y = \log_a[f(x)]$

$$\log_a[f(x)] < y_1 \ \vee \ y_2 < \log_a[f(x)] < y_3 \ \vee \ \log_a[f(x)] > y_4$$

Each of the inequalities expressing the solutions will translate into a logarithmic inequality with exponential conversion, which we can solve separately. There are some aspects to pay particular attention to:

- the logical connective $\vee$ signifies union, so we can solve the logarithmic inequalities separately and consider the union of solutions;
- double inequalities are to be understood as systems

$$y_2 < \log_a[f(x)] < y_3 \rightarrow \begin{cases} \log_a[f(x)] > y_2 \\ \log_a[f(x)] < y_3 \end{cases}$$

- after translating all solutions from the variable $y$ to the variable $x$, let's not forget the existence conditions set at the beginning.

## 17.4 Logarithmic Inequalities with Variable in the Base

For this type of inequality, we only need to observe that the presence of the variable in a base, say

$$\log_{f(x)}(a)$$

implies imposing the existence conditions derived from the definition of logarithm

$$\begin{cases} f(x) > 0 \\ f(x) \neq 1 \end{cases}$$

For the rest, the solution technique involves using the logarithmic change of base formula

$$\log_{f(x)}(a) = \frac{\log_b(a)}{\log_b(f(x))}$$

and from here, we can solve it as we have reduced it to a form we know how to work with.

Part VII

# Trigonometric Equations and Inequalities

# 18 Trigonometric Equations

| $\alpha°$ | $\alpha$ | $\cos\alpha$ | $\sin\alpha$ | $\tan\alpha$ | $\cot\alpha$ | $\sec\alpha$ | $\csc\alpha$ |
|---|---|---|---|---|---|---|---|
| $0°$ | $0$ | $1$ | $0$ | $0$ | $-$ | $1$ | $-$ |
| $30°$ | $\frac{\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ | $\frac{2\sqrt{3}}{3}$ | $2$ |
| $45°$ | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ | $1$ | $\sqrt{2}$ | $\sqrt{2}$ |
| $60°$ | $\frac{\pi}{3}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\sqrt{3}$ | $\frac{\sqrt{3}}{3}$ | $2$ | $\frac{2\sqrt{3}}{3}$ |
| $90°$ | $\frac{\pi}{2}$ | $0$ | $1$ | $-$ | $0$ | $-$ | $1$ |
| $120°$ | $\frac{2\pi}{3}$ | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $-\sqrt{3}$ | $-\frac{\sqrt{3}}{3}$ | $-2$ | $\frac{2\sqrt{3}}{3}$ |
| $135°$ | $\frac{3\pi}{4}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $-1$ | $-1$ | $-\sqrt{2}$ | $\sqrt{2}$ |
| $150°$ | $\frac{5\pi}{6}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{3}$ | $-\sqrt{3}$ | $-\frac{2\sqrt{3}}{3}$ | $2$ |
| $180°$ | $\pi$ | $-1$ | $0$ | $0$ | $-$ | $-1$ | $-$ |
| $210°$ | $\frac{7\pi}{6}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ | $\frac{2\sqrt{3}}{3}$ | $-2$ |
| $225°$ | $\frac{5\pi}{4}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $1$ | $1$ | $\sqrt{2}$ | $-\sqrt{2}$ |
| $240°$ | $\frac{4\pi}{3}$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $\sqrt{3}$ | $\frac{\sqrt{3}}{3}$ | $-2$ | $-\frac{2\sqrt{3}}{3}$ |
| $270°$ | $\frac{3\pi}{2}$ | $0$ | $-1$ | $-$ | $0$ | $-$ | $-1$ |
| $300°$ | $\frac{5\pi}{3}$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-\sqrt{3}$ | $-\frac{\sqrt{3}}{3}$ | $-2$ | $-\frac{2\sqrt{3}}{3}$ |
| $315°$ | $\frac{7\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-1$ | $-\sqrt{2}$ | $-\sqrt{2}$ |
| $330°$ | $\frac{11\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{3}$ | $-\sqrt{3}$ | $-\frac{2\sqrt{3}}{3}$ | $-2$ |

Trigonometric equations are those equations in which the unknown appears as an argument of a trigonometric function: sine, cosine, tangent, cotangent, secant, cosecant. The methods for solving trigonometric equations depend on the standard form to which they can be reduced.

There is a multitude of trigonometric equations, also known as **trigonometric equations**, the main ones are listed below.

1. Elementary Trigonometric Equations

$$\begin{aligned} \sin(x) = m \quad &\text{or} \quad \cos(x) &= n \\ \tan(x) = p \quad &\text{or} \quad \cot(x) &= q \\ \sec(x) = r \quad &\text{or} \quad \csc(x) &= s \end{aligned}$$

2. Trigonometric Equations Reducible to Elementary Ones

   - Trigonometric equations by substitution

$$\begin{aligned}
\sin[f(x)] = m \quad \text{or} \quad \cos[f(x)] &= n \\
\tan[f(x)] = p \quad \text{or} \quad \cot[f(x)] &= q \\
\sec[f(x)] = r \quad \text{or} \quad \csc[f(x)] &= s
\end{aligned}$$

   - Trigonometric equations by comparison

$$\begin{aligned}
\sin[f(x)] = \sin[g(x)] \quad \text{or} \quad \cos[f(x)] &= \cos[g(x)] \\
\tan[f(x)] = \tan[g(x)] \quad \text{or} \quad \cot[f(x)] &= \cot[g(x)] \\
\sec[f(x)] = \sec[g(x)] \quad \text{or} \quad \csc[f(x)] &= \csc[g(x)]
\end{aligned}$$

   - Trigonometric equations reducible to elementary ones using trigonometric formulas
     - Trigonometric equations solvable with definitions
     - Trigonometric equations solvable with the fundamental trigonometric relation
     - Trigonometric equations solvable with associated angle formulas
     - Trigonometric equations solvable with addition and subtraction formulas
     - Trigonometric equations solvable with duplication formulas
     - Trigonometric equations solvable with bisection formulas
     - Trigonometric equations solvable with Werner's formulas
     - Trigonometric equations solvable with prosthaphaeresis formulas

3. Linear Trigonometric Equations in Sine and Cosine

$$a\sin(x) + b\cos(x) = c$$

4. Second-Degree Trigonometric Equations in Sine and Cosine

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) = d$$

## 18.1 Elementary Trigonometric Equations

### 18.1.1 Elementary Trigonometric Equations of the Form $\sin(x) = m$

Recalling that the sine of an angle represents the *ordinate* of the point on the unit circle associated with that angle, solving the equation is equivalent to finding the points on the unit circle with ordinate equal to $m$ and determining the angles they represent.

We draw a unit circle centered at the origin, locate the value $m$ on the $y$-axis, and draw the line with equation $y = m$. The cases that can occur are as follows:

- If $m < -1$ or $m > 1$, the equation is impossible because the line will be outside the circle.
- If $m = 1$, the equation is satisfied for $x = \frac{\pi}{2} + 2k\pi$ for all integers $k$, where $k$ is essential to account for the periodicity of the sine function ($2\pi$).
- If $m = -1$, the equation is satisfied for $x = \frac{3\pi}{2} + 2k\pi$ for all integers $k$.
- If $m = 0$, the solutions of the equation are $x = 2k\pi$ or $x = \pi + 2k\pi$ for all integers $k$. In a more compact form, sine equals zero every $\pi$ radians, thus the solutions can be expressed as $x = k\pi$ for all integers $k$.
- If $0 < m < 1$, the line $y = m$ intersects the circle at two distinct points located in the first and second quadrants. Joining the origin with these points forms the two angles that satisfy the trigonometric equation. The next step is to determine their magnitude:
  - If $m$ is a known value from the trigonometric function table, then the solutions are obtained directly. Let $\alpha$ be the angle in the first quadrant such that $\sin(\alpha) = m$, then the solutions of the equation are $x = \alpha + 2k\pi$ or $x = (\pi - \alpha) + 2k\pi$ for all integers $k$.
  - If $m$ is not a known value, we resort to the arcsine function. In this case, the equation is satisfied for $x = \arcsin(m) + 2k\pi$ or $x = [\pi - \arcsin(m)] + 2k\pi$ for all integers $k$.
- If $-1 < m < 0$, the procedure is the same. The line $y = m$ intersects the unit circle in the third and fourth quadrants. By joining these points with the origin, we form the two angles satisfying the equation. To determine their magnitude, we again refer to the trigonometric function table or use the arcsine function. After finding the angle, the solutions are $x = (\pi + \alpha) + 2k\pi$ or $x = (2\pi - \alpha) + 2k\pi$ for all integers $k$, or potentially $x = [\pi + \arcsin(|m|)] + 2k\pi$ or $x = [2\pi - \arcsin(|m|)] + 2k\pi$ for all integers $k$.

### 18.1.2 Elementary Trigonometric Equations of the Form $\cos(x) = n$

Recalling that the cosine of an angle represents the x-coordinate of the point on the unit circle associated with that angle, we can repeat the same reasoning by considering the points of intersection of the circle with the line given by the equation

$$x = n.$$

Again, we draw a circle centered at the origin with a radius of 1 and, after identifying the value $n$ on the $x$-axis, we draw the line with the equation $x = n$. Let's analyze the possible cases:

- $$n < -1 \vee n > 1$$

  The equation has no solutions.

- $$n = 1$$

  The equation is satisfied for

  $$x = 2k\pi \quad \forall\, k \in \mathbb{Z}.$$

- $$n = -1$$

  The equation is satisfied for

  $$x = \pi + 2k\pi \quad \forall\, k \in \mathbb{Z}.$$

- $$n = 0$$

  The equation has solutions

  $$x = \frac{\pi}{2} + 2k\pi \;\vee\; x = \frac{3\pi}{2} + 2k\pi \quad \forall\, k \in \mathbb{Z}.$$

  In a more compact form, we can express the solutions as

  $$x = \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}.$$

- $$0 < n < 1$$

  The line with the equation $x = n$ intersects the circle at two distinct points, located in the first and fourth quadrants. Connecting the origin with these points gives us the two angles that satisfy the trigonometric equation.

  - If $n$ is a value present in the table of fundamental values of trigonometric functions, let $\beta$ be the angle in the first quadrant such that $\cos(\beta) = n$, then the solutions will be:

    $$x = \beta + 2k\pi \;\vee\; x = (2\pi - \beta) + 2k\pi \quad \forall\, k \in \mathbb{Z}.$$

  - If $n$ is not a known value, we will resort to the arccosine function, and the equation will be satisfied for

    $$x = \arccos(n) + 2k\pi \;\vee\; x = [2\pi - \arccos(n)] + 2k\pi \quad \forall\, k \in \mathbb{Z}.$$

- $$-1 < n < 0$$

  We draw the line with the equation $x = n$, which in this case intersects the unit circle at two points in the second and third quadrants. To find the two angles that satisfy the equation, we connect the two points with the origin, and to determine their magnitude, we will use the table of significant values or express them using the arccosine function. In particular, we look for the angle $\beta$ **in the first quadrant** such that

  $$\cos(\beta) = |n|,$$

  so the solutions will be

  $$x = (\pi - \beta) + 2k\pi \vee x = (\pi + \beta) + 2k\pi \quad \forall\, k \in \mathbb{Z}$$

  or, possibly,

  $$x = [\pi - \arccos(|n|)] + 2k\pi \ \vee \ x = [\pi + \arccos(|n|)] + 2k\pi \quad \forall\, k \in \mathbb{Z}.$$

### 18.1.3 Elementary Trigonometric Equations of the Form $\tan(x) = p$

Recall that tangent is defined as the ratio of sine to cosine, so we must impose the existence conditions to ensure that the denominator does not become zero:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \ \rightarrow \ x \neq \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}.$$

Let's draw the unit circle and draw the tangent line to the circle at the point $(1, 0)$. On the tangent line, consider the point $T(1, p)$ with ordinate $p$, and draw the line passing through the origin and the point $T$. This will form the two angles that satisfy the trigonometric equation.

Since tangent is a periodic function with period $\pi$, once we find the angle $\alpha$ in the first quadrant such that

$$\tan(\alpha) = |p|,$$

the solutions to the elementary trigonometric equation can be of two types:

- If $p \geq 0$, then
  $$x = \alpha + k\pi \quad \forall\, k \in \mathbb{Z};$$
- If $p < 0$, then
  $$x = (\pi - \alpha) + k\pi \quad \forall\, k \in \mathbb{Z}.$$

Of course, the angle $\alpha$ needs to be searched among the fundamental values of trigonometric functions, and if $|p|$ does not appear in the table, we resort to arctangent to obtain the solutions:

- If $p \geq 0$, then
$$x = \arctan(p) + k\pi \quad \forall\, k \in \mathbb{Z};$$

- If $p < 0$, then
$$x = [\pi - \arctan(|p|)] + k\pi \quad \forall\, k \in \mathbb{Z}.$$

### 18.1.4 Elementary Trigonometric Equations of the Form $\cot(x) = p$, $\sec(x) = r$, $\csc(x) = s$

$$\cot(x) = p, \ \sec(x) = r, \ \csc(x) = s$$

In these cases, we need to:

- Impose the existence conditions, according to the definitions:

$$\begin{aligned}
\cot(x) &= \frac{\cos(x)}{\sin(x)} \ \rightarrow \ x \neq k\pi \quad \forall\, k \in \mathbb{Z} \\
\sec(x) &= \frac{1}{\cos(x)} \ \rightarrow \ x \neq \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z} \\
\csc(x) &= \frac{1}{\sin(x)} \ \rightarrow \ x \neq k\pi \quad \forall\, k \in \mathbb{Z}
\end{aligned}$$

- Refer to the definitions of the involved trigonometric functions to identify the base angles that satisfy the equation, with the condition
$$0 \leq x \leq 2\pi;$$

- Determine all and only the solutions by extending the base solutions using the periodicity of the specific trigonometric function;

- Compare the solutions with the existence conditions.

## 18.2 Trigonometric Equations Reducible to Elementary Ones

### 18.2.1 Trigonometric Equations by Substitution

Trigonometric equations of the form

$$\begin{aligned} \sin[f(x)] &= m \qquad \cos[f(x)] = n \\ \tan[f(x)] &= p \qquad \cot[f(x)] = q \\ \sec[f(x)] &= r \qquad \csc[f(x)] = s \end{aligned}$$

require, first of all, a preliminary discussion of the existence conditions, according to the definitions. In addition to any existence conditions related to the specific expression of $f(x)$, we must consider that sine or cosine do not require any existence conditions by themselves; on the contrary, tangent and cotangent, and secant and cosecant can be rewritten as ratios, so we must prevent the denominator from becoming zero.

$$\tan[f(x)] = \frac{\sin[f(x)]}{\cos[f(x)]} \ \rightarrow \ f(x) \neq \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}$$

$$\cot[f(x)] = \frac{\cos[f(x)]}{\sin[f(x)]} \ \rightarrow \ f(x) \neq k\pi \quad \forall\, k \in \mathbb{Z}$$

$$\sec[f(x)] = \frac{1}{\cos[f(x)]} \ \rightarrow \ f(x) \neq \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}$$

$$\csc[f(x)] = \frac{1}{\sin[f(x)]} \ \rightarrow \ f(x) \neq k\pi \quad \forall\, k \in \mathbb{Z}$$

With these premises, we can reduce to an elementary trigonometric equation by making an obvious substitution:

$$f(x) = y$$

where $y$ is called the **auxiliary unknown**, so that we have:

$$\sin(y) = m \qquad \cos(y) = n$$

$$\tan(y) = p \qquad \cot(y) = q$$

$$\sec(y) = r \qquad \csc(y) = s$$

which we know how to solve.

After identifying the solutions related to the unknown $y$, we will perform the inverse substitution in order to express the solutions with respect to the unknown $x$.

### 18.2.2 Trigonometric Equations by Comparison

An important type of trigonometric equations involves comparing different arguments with the same trigonometric function. Let's see how to proceed in solving equations of the form:

$$\sin[f(x)] = \sin[g(x)] \quad \text{or} \quad \cos[f(x)] = \cos[g(x)]$$

$$\tan[f(x)] = \tan[g(x)] \quad \text{or} \quad \cot[f(x)] = \cot[g(x)]$$

$$\sec[f(x)] = \sec[g(x)] \quad \text{or} \quad \csc[f(x)] = \csc[g(x)]$$

The first step is to impose any existence conditions related to $f(x)$ and $g(x)$. In addition to these, we need to impose conditions of existence for tangent, cotangent, secant, and cosecant, following the scheme seen earlier. Once this is done, we can proceed with the resolution.

In such cases, it's easy to make mistakes by writing $f(x) = g(x)$. To avoid errors, it suffices to remember that two angles have the same sine or cosine, the same tangent or cotangent, or the same secant or cosecant when:

- Two angles have the same sine if they differ by an integer multiple of full circles, or if one of them differs by an integer multiple of full circles from the supplement of the other:

$$\begin{aligned}&\sin[f(x)] = \sin[g(x)] \;\Rightarrow\\ &f(x) = g(x) + 2k\pi \text{ or } f(x) = [\pi - g(x)] + 2k\pi \quad \forall\, k \in \mathbb{Z}\end{aligned}$$

- Two angles have the same cosine if they differ by an integer multiple of full circles, or if one of them differs by an integer multiple of full circles from the opposite of the other:

$$\begin{aligned}&\cos[f(x)] = \cos[g(x)] \;\Rightarrow\\ &f(x) = g(x) + 2k\pi \text{ or } f(x) = [2\pi - g(x)] + 2k\pi \quad \forall\, k \in \mathbb{Z}\end{aligned}$$

- Two angles have the same tangent if they differ by an integer multiple of straight angles and are both different from $\frac{\pi}{2} + k\pi \; \forall \, k \in \mathbb{Z}$:

$$\tan[f(x)] = \tan[g(x)] \;\Rightarrow$$
$$f(x) = g(x) + k\pi \quad \text{where } f(x), g(x) \neq \frac{\pi}{2} + k\pi \; \forall \, k \in \mathbb{Z}$$

- Two angles have the same cotangent if they differ by an integer multiple of straight angles and are both different from $k\pi$ as $k$ varies over $\mathbb{Z}$:

$$\cot[f(x)] = \cot[g(x)] \;\Rightarrow$$
$$f(x) = g(x) + k\pi \quad \text{where } f(x), g(x) \neq k\pi \quad \forall \, k \in \mathbb{Z}$$

- Two angles have the same secant if they differ by an integer multiple of full circles and are both different from $\frac{\pi}{2} + k\pi \; \forall \, k \in \mathbb{Z}$:

$$\sec[f(x)] = \sec[g(x)] \;\Rightarrow$$
$$f(x) = g(x) + 2k\pi \text{ or } f(x) = [2\pi - g(x)] + 2k\pi \quad \text{where } f(x), g(x) \neq \frac{\pi}{2} + k\pi \; \forall \, k \in \mathbb{Z}$$

- Two angles have the same cosecant if they differ by an integer multiple of full circles and are both different from $k\pi \; \forall \, k \in \mathbb{Z}$:

$$\csc[f(x)] = \csc[g(x)] \;\Rightarrow$$
$$f(x) = g(x) + 2k\pi \text{ or } f(x) = [\pi - g(x)] + 2k\pi \quad \text{where } f(x), g(x) \neq k\pi \; \forall \, k \in \mathbb{Z}$$

### 18.2.3 Trigonometric Equations Reduced to Elementary Forms using Trigonometric Identities

**Trigonometric Equations Solvable with Definitions** If in a trigonometric equation functions other than sine, cosine, or tangent appear, we can simply reduce them by recalling how cotangent, secant, and cosecant are defined.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

**Example 18.1.**

$$\tan(x) + 2\cot(x) = 3$$

CE: $\begin{cases} x \neq \frac{\pi}{2} + k\pi \\ x \neq k\pi \end{cases} \quad \rightarrow \; x \neq k\frac{\pi}{2}$

$\tan(x) + \frac{2}{\tan(x)} - 3 = 0$

$\frac{\tan^2(x) - 3\tan(x) + 2}{\tan(x)} = 0$

$\tan^2(x) - 3\tan(x) + 2 = 0 \quad$ let $y = \tan(x)$

$y^2 - 3y + 2 = 0 \quad \rightarrow \quad y_{1,2} = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2} = \frac{3 \pm \sqrt{1}}{2} \quad \rightarrow \quad y_1 = 1 \; \vee \; y_2 = 2$

$\tan(x) = 1 \; \rightarrow \; x = \frac{\pi}{4} + k\pi \quad \vee \quad \tan(x) = 2 \; \rightarrow \; x = \arctan(2) + k\pi \quad \forall\, k \in \mathbb{Z}$

**Trigonometric Equations Solvable with the Fundamental Trigonometric Relation** If we have (or have reduced ourselves to) an equation in which sine and cosine appear, one or both squared, we can use the fundamental trigonometric identity:

$$\sin^2(x) + \cos^2(x) = 1$$

to reduce it to a single function and proceed with the methods already seen.

**Fundamental Trigonometric Identity** The fundamental trigonometric identity allows us to rewrite sine in terms of cosine and vice versa.

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

Depending on the needs, we may need to use it in the forms:

- $\sin^2(\alpha) = 1 - \cos^2(\alpha)$
- $\cos^2(\alpha) = 1 - \sin^2(\alpha)$

**Example 18.2.**

$$5 - 2\cos^2(x) - 4\sin(x) = 2\cos^2(x)$$

$$\begin{aligned}
&5 - 2\cos^2(x) - 4\sin(x) - 2\cos^2(x) = 0\\
&-4\cos^2(x) - 4\sin(x) + 5 = 0\\
&-4[1 - \sin^2(x)] - 4\sin(x) + 5 = 0\\
&-4 + 4\sin^2(x) - 4\sin(x) + 5 = 0\\
&4\sin^2(x) - 4\sin(x) + 1 = 0 \quad \text{let } y = \sin(x)\\
&4y^2 - 4y + 1 = 0 \quad \Rightarrow \quad y_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 4 \cdot 1}}{8} = \frac{4}{8} = \frac{1}{2}\\
&y = \frac{1}{2} \quad \Rightarrow \quad \sin(x) = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6} + 2k\pi \ \vee \ x = \frac{5\pi}{6} + 2k\pi \quad \forall\, k \in \mathbb{Z}
\end{aligned}$$

**Trigonometric Equations Solvable with Associated Angle Formulas** The associated angle formulas are very useful for rewriting many equations into more manageable forms, especially trigonometric equations for comparison.

**Formulas for Associated Angles for Sine and Cosine** Thanks to the formulas for associated angles, we can derive the value of sine and cosine for particular angles, called associated angles.

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right)&=\cos(\alpha); & \cos\left(\frac{\pi}{2}-\alpha\right)&=\sin(\alpha)\\
\sin\left(\frac{\pi}{2}+\alpha\right)&=\cos(\alpha); & \cos\left(\frac{\pi}{2}+\alpha\right)&=-\sin(\alpha)\\
\sin(\pi-\alpha)&=\sin(\alpha); & \cos(\pi-\alpha)&=-\cos(\alpha)\\
\sin(\pi+\alpha)&=-\sin(\alpha); & \cos(\pi+\alpha)&=-\cos(\alpha)\\
\sin\left(\frac{3}{2}\pi-\alpha\right)&=-\cos(\alpha); & \cos\left(\frac{3}{2}\pi-\alpha\right)&=-\sin(\alpha)\\
\sin\left(\frac{3}{2}\pi+\alpha\right)&=-\cos(\alpha); & \cos\left(\frac{3}{2}\pi+\alpha\right)&=\sin(\alpha)\\
\sin(-\alpha)&=-\sin(\alpha); & \cos(-\alpha)&=\cos(\alpha)
\end{aligned}
$$

**Example 18.3.**

$$\sin(2x)=-\sin(3x)$$

$$\sin(2x)=\sin(-3x)$$

$$
\begin{aligned}
2x&=-3x+2k\pi \quad 2x=\pi-(-3x)+2k\pi\\
5x&=+2k\pi \quad 2x=\pi+3x+2k\pi\\
x&=\frac{2}{5}k\pi \quad \forall\, k\in\mathbb{Z} \quad -x=\pi+2k\pi\\
&x=-\pi-2k\pi \quad \forall\, k\in\mathbb{Z}
\end{aligned}
$$

**Trigonometric Equations Solvable Using Addition and Subtraction Formulas** If the argument of trigonometric functions is a sum or difference, one possible method to solve the equation involves using addition and subtraction formulas for angles.

**Summation Formulas for Sine, Cosine, Tangent** The angle addition formulas allow rewriting trigonometric functions applied to the sum (or difference) of two angles by decoupling the angles.

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \quad \forall\, \alpha, \beta, \alpha+\beta \neq \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}$$

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)} \quad \forall\, \alpha, \beta, \alpha-\beta \neq \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}$$

**Example 18.4.**

$$\sin\left(x+\frac{\pi}{4}\right) - \sin\left(x-\frac{\pi}{4}\right) = 1$$

$$\begin{aligned}
&\sin(x)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos(x) - \left[\sin(x)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos(x)\right] = 1\\
&\sin(x)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos(x) - \sin(x)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos(x) = 1\\
&\frac{\sqrt{2}}{2}\sin(x) + \frac{\sqrt{2}}{2}\cos(x) - \frac{\sqrt{2}}{2}\sin(x) + \frac{\sqrt{2}}{2}\cos(x) = 1\\
&\frac{\sqrt{2}}{2}\cos(x) + \frac{\sqrt{2}}{2}\cos(x) = 1\\
&\frac{\sqrt{2}\cos(x) + \sqrt{2}\cos(x)}{2} = 1\\
&\frac{\cancel{2}\sqrt{2}\cos(x)}{\cancel{2}} = 1\\
&\sqrt{2}\cos(x) = 1\\
&\cos(x) = \frac{1}{\sqrt{2}}\\
&\cos(x) = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad x = \frac{\pi}{4} + 2k\pi \ \vee\ x = \frac{7\pi}{4} + 2k\pi \quad \forall\, k \in \mathbb{Z}
\end{aligned}$$

**Trigonometric Equations Solvable Using Double Angle Formulas**

**Double Angle Formulas** Double angle formulas provide an alternative expression for a trigonometric function applied to twice an angle.

$$\begin{aligned}
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1 \\
\tan(2\alpha) &= \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}, \quad \alpha \neq \frac{\pi}{4} + k\frac{\pi}{2} \wedge \alpha \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}
\end{aligned}$$

**Example 18.5.**

$$\cos(2x) + \cos(x) = 0$$

$$\begin{aligned}
&2\cos^2(x) - 1 + \cos(x) = 0 \\
&2\cos^2(x) + \cos(x) - 1 = 0 \quad \text{Let } y = \cos(x) \\
&2y^2 + y - 1 = 0 \quad \Rightarrow \quad y_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot -1}}{4} = \frac{-1 \pm \sqrt{9}}{4} = \frac{-1 \pm 3}{4} \\
&y_1 = -1 \ \vee \ y_2 = \frac{1}{2} \quad \Rightarrow \quad \cos(x) = -1 \ \vee \ \cos(x) = \frac{1}{2} \\
&x = \pi + 2k\pi \ \vee \ x = \frac{\pi}{3} + 2k\pi \ \vee \ x = \frac{5\pi}{3} + 2k\pi \quad \forall k \in \mathbb{Z}
\end{aligned}$$

**Trigonometric Equations Solvable Using Half Angle Formulas**

**Half Angle Formulas** Half angle formulas are equalities through which we can rewrite trigonometric functions applied to half of an angle.

$$\begin{aligned}
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos(\alpha)}{2}} \\
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos(\alpha)}{2}} \\
\tan\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}, \quad \alpha \neq \pi + 2k\pi, \quad k \in \mathbb{Z}
\end{aligned}$$

**Example 18.6.**

$$\tan^2\left(\frac{x}{2}\right) + \cos(x) = 1$$

CE: $x \neq \pi + 2k\pi$

$$\frac{1-\cos(x)}{1+\cos(x)} + \cos(x) = 1$$

$$\frac{1-\cos(x)}{1+\cos(x)} + \cos(x) - 1 = 0$$

$$\frac{1-\cos(x)+\cos(x)(1+\cos(x))-1(1+\cos(x))}{1+\cos(x)} = 0$$

$$\frac{1-\cos(x)+\cos(x)+\cos^2(x)-1-\cos(x)}{1+\cos(x)} = 0$$

$$\frac{\cos^2(x)-\cos(x)}{1+\cos(x)} = 0$$

$$\cos^2(x) - \cos(x) = 0$$

$$\cos(x)[\cos(x)-1] = 0 \quad \Rightarrow \quad \cos(x) = 0 \ \vee \ \cos(x) = 1$$

$$x = \frac{\pi}{2} + k\pi \ \vee \ x = 2k\pi \quad \forall\, k \in \mathbb{Z}$$

**Trigonometric Equations Solvable Using Werner's Formulas**

**Werner's Formulas** Werner's formulas serve as a sort of inverse representation of the angle addition and subtraction formulas for the possible products between the sine and cosine of two distinct angles.

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

**Example 18.7.**

$$\sin(4x)\sin(3x) = \sin(2x)\sin(x)$$

$$
\begin{aligned}
&\frac{1}{2}[\cos(4x-3x)-\cos(4x+3x)] = \frac{1}{2}[\cos(2x-x)-\cos(2x+x)]\\
&\frac{1}{2}[\cos(x)-\cos(7x)] = \frac{1}{2}[\cos(x)-\cos(3x)]\\
&\cos(x)-\cos(7x) = \cos(x)-\cos(3x)\\
&-\cos(7x) = -\cos(3x)\\
&\cos(7x) = \cos(3x)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
7x &= 3x + 2k\pi\\
4x &= 2k\pi\\
x &= \frac{1}{2}k\pi \quad \forall\, k \in \mathbb{Z}
\end{aligned}
$$

and

$$
\begin{aligned}
7x &= -3x + 2k\pi\\
10x &= 2k\pi\\
x &= \frac{1}{5}k\pi \quad \forall\, k \in \mathbb{Z}.
\end{aligned}
$$

**Trigonometric Equations Solvable Using Prosthaphaeresis Formulas**

**Prosthaphaeresis Formulas** Prosthaphaeresis formulas are used to rewrite the sums and differences of sine and cosine applied to two angles as products of sine and cosine.

$$
\begin{aligned}
\sin(\alpha)+\sin(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\sin(\alpha)-\sin(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\\
\cos(\alpha)+\cos(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\cos(\alpha)-\cos(\beta) &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}
$$

**Example 18.8.**

$$\sin(4x) + \sin(3x) + \sin(2x) + \sin(x) = 0$$

$$\begin{aligned}
&2\sin\left(\frac{4x+3x}{2}\right)\cos\left(\frac{4x-3x}{2}\right) + 2\sin\left(\frac{2x+x}{2}\right)\cos\left(\frac{2x-x}{2}\right) = 0\\
&2\sin\left(\frac{7x}{2}\right)\cos\left(\frac{x}{2}\right) + 2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) = 0\\
&2\cos\left(\frac{x}{2}\right)\left[\sin\left(\frac{7x}{2}\right) + 2\sin\left(\frac{3x}{2}\right)\right] = 0\\
&2\cos\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad \sin\left(\frac{7x}{2}\right) + 2\sin\left(\frac{3x}{2}\right) = 0
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&2\cos\left(\frac{x}{2}\right) = 0\\
&\cos\left(\frac{x}{2}\right) = 0 \quad \to \quad \frac{x}{2} = \frac{\pi}{2} + k\pi \quad \to \quad x = \pi + 2k\pi \quad \forall\, k \in \mathbb{Z}
\end{aligned}$$

and

$$\begin{aligned}
\sin\left(\frac{7x}{2}\right) + \sin\left(\frac{3x}{2}\right) &= 0\\
\sin\left(\frac{7x}{2}\right) &= -\sin\left(\frac{3x}{2}\right)\\
\sin\left(\frac{7x}{2}\right) &= \sin\left(-\frac{3x}{2}\right)
\end{aligned}$$

$$\begin{array}{rl}
\frac{7x}{2} = -\frac{3x}{2} + 2k\pi & \frac{7x}{2} = \pi - (-\frac{3x}{2}) + 2k\pi\\
\frac{7x}{2} + \frac{3x}{2} = 2k\pi & \frac{7x}{2} = \pi + \frac{3x}{2} + 2k\pi\\
\frac{7x+3x}{2} = 2k\pi & \frac{7x}{2} - \frac{3x}{2} = \pi + 2k\pi\\
\frac{10x}{2} = 2k\pi & \frac{7x-3x}{2} = \pi + 2k\pi\\
5x = 2k\pi & \frac{4x}{2} = \pi + 2k\pi\\
x = \frac{2}{5}k\pi \quad \forall\, k \in \mathbb{Z} & 2x = \pi + 2k\pi\\
 & x = \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}
\end{array}$$

Hence, we have

$$x = \pi + 2k\pi \quad \text{or} \quad x = \frac{2}{5}k\pi \quad \text{or} \quad x = \frac{\pi}{2} + k\pi \quad \forall\, k \in \mathbb{Z}$$

## 18.3 Linear Equations in Sine and Cosine

Linear equations in sine and cosine derive their name from the structure that distinguishes them:

$$a\sin(x) + b\cos(x) = c$$

where $a$ and $b$ are constants different from zero. Although they are trigonometric equations in a single variable, the form in which they appear closely resembles that of classical linear equations with two variables.

There are three methods to determine the values of the unknown $x$ that satisfy the equality:

- Linear equations in sine and cosine using parametric formulas;
- Linear equations in sine and cosine using the system passage method;
- Linear equations in sine and cosine using the method of the added angle.

### 18.3.1 Trigonometric Linear Equations with Parametric Formulas

Suppose we want to solve the equation:

$$a\sin(x) + b\cos(x) = c \quad \text{with } a, b \neq 0$$

By using the parametric formulas for sine and cosine:

$$t = \tan\left(\frac{x}{2}\right) \quad \Rightarrow \quad \begin{aligned} \sin(x) &= \tfrac{2t}{1+t^2} \\ \cos(x) &= \tfrac{1-t^2}{1+t^2} \end{aligned}$$

Before proceeding with substitutions, we must consider that these formulas are subject to conditions of existence that would restrict the domain of solutions, and that, on the other hand, the original equation in its form is not subject to any domain restrictions. Since the conditions of existence that allow the application of parametric formulas are given by:

$$\frac{x}{2} \neq \frac{\pi}{2} + k\pi \to x \neq \pi + 2k\pi \quad \text{with } k \in \mathbb{Z}$$

we must manually check if $x = \pi$ is a solution of the linear equation in sine and cosine. If not, we proceed without delay; if affirmative, we note down the initial solutions of the equation extending them periodically $x = \pi + 2k\pi \quad \text{with } k \in \mathbb{Z}$.

We substitute the expressions of the parametric formulas into the linear equation in sine and cosine. We obtain a second-degree equation in $t$:

$$(b + c)t^2 - 2at - b + c = 0.$$

Based on the associated discriminant, we will have two real solutions (distinct or coincident) or no real solutions:

- If the associated discriminant is positive, we will have two solutions: $t = t_1$ or $t = t_2$. Since $t = \tan\left(\frac{x}{2}\right)$, we perform the inverse substitution, moving to trigonometric equations that we can solve:

$$t = t_1 \Rightarrow \tan\left(\frac{x}{2}\right) = t_1$$
$$t = t_2 \Rightarrow \tan\left(\frac{x}{2}\right) = t_2$$

To these solutions, if necessary, we will add those previously determined ($x = \pi + 2k\pi$).

- If the associated discriminant is zero, we will have one solution: $t = \bar{t}$, from which the trigonometric equation $t = \bar{t} \Rightarrow \tan\left(\frac{x}{2}\right) = \bar{t}$ arises, and, if necessary, we will add the previously determined solutions ($x = \pi + 2k\pi$).

- If the discriminant is negative, the second-degree equation is impossible. The linear equation in sine and cosine is impossible if not even $x = \pi + 2k\pi$ are solutions of the equation in the initial form; otherwise, they will be the only solutions.

### 18.3.2 Linear Equations in Sine and Cosine using the System Passage Method

Another technique that comes to our aid is the system passage method for linear equations in sine and cosine.

$$a\sin(x) + b\cos(x) = c$$

This procedure is suggested by the name itself, or rather by the structure of such equations; it involves considering two auxiliary variables and setting:

$$\begin{cases} Y = \sin(x) \\ X = \cos(x) \end{cases}$$

so that the equation translates into $aY + bX = c$. The idea is to reduce the original equation to a system of equations in the variables $X$ and $Y$.

To determine $X$ and $Y$, we need another condition, so that we have a system of two equations in two unknowns. We can use the fundamental trigonometric relation

$$\cos^2(x) + \sin^2(x) = 1$$

which translates to

$$X^2 + Y^2 = 1.$$

Now we are in a position to write a system of two equations in two unknowns

$$\begin{cases} aY + bX = c \\ X^2 + Y^2 = 1 \end{cases}$$

From this point, we have two alternatives:

1. We can **solve the system by substitution**: we will use the first equation to express one of the two unknowns (say $Y$) in terms of the other and substitute the obtained expression into the second equation. By doing so, the second equation will become a second-degree equation in one unknown (in this case $X$), and three cases may arise:

    - The equation admits two distinct real solutions $X_1, X_2$. We substitute the two values into the first equation and obtain two solutions for the other unknown $Y_1, Y_2$. The system admits two pairs of solutions $(X_1, Y_1)$, $(X_2, Y_2)$. We then perform the inverse substitutions

    $$\begin{cases} \sin(x) = Y_1 \\ \cos(x) = X_1 \end{cases} \qquad \text{or} \qquad \begin{cases} \sin(x) = Y_2 \\ \cos(x) = X_2 \end{cases}$$

    and identify, on one hand, the values of the unknown $x$ that satisfy the elementary trigonometric equations of the first system, and on the other hand, the values of $x$ that satisfy the elementary trigonometric equations of the second system. The solutions of the linear trigonometric equation are given by the union of such solutions.

    - The equation admits two real and coincident solutions $\overline{X}$. We substitute the value into the first equation and obtain one solution for the other unknown $\overline{Y}$. The system admits a single pair of solutions $(\overline{X}, \overline{Y})$. We then perform the inverse substitutions

    $$\begin{cases} \sin(x) = \overline{Y} \\ \cos(x) = \overline{X} \end{cases}$$

    and determine the values of the unknown $x$ that satisfy the elementary trigonometric equations of the system. These will be the solutions of the **linear trigonometric equation**.

- The equation has no real solutions. In this case, neither does the system have solutions, and we can conclude that the trigonometric equation is impossible.

2. An alternative technique to solve the system in $X$ and $Y$ involves resorting to graphical methods: solving the system is equivalent to determining the points of intersection $(X, Y)$ between the geometric loci associated with the two equations.

$$X^2 + Y^2 = 1$$

represents the unit circle, centered at the origin of the axes with radius 1.

$$aX + bY = c$$

is the equation of a line. Now, there are three possibilities:

(a) The line and the circle intersect at two distinct points (secant line).

(b) The line and the circle intersect at a single point (tangent line).

(c) The line and the circle do not intersect (external line).

### 18.3.3 Linear Equations in Sine and Cosine Using the Auxiliary Angle Method

The auxiliary angle method is essentially based on the addition formulas of sine. It allows expressing a linear trigonometric equation as an equivalent trigonometric equation solely in sine, thus reducing it to an elementary equation. To achieve this, we will use an angle sometimes called the *phase angle* or *auxiliary angle*. We aim to determine the solutions of the equation:

$$a\sin(x) + b\cos(x) = c,$$

we need to determine the auxiliary angle $\alpha$ that solves the system of elementary equations:

$$\begin{cases} \sin(\alpha) = \dfrac{b}{\sqrt{a^2+b^2}} \\ \cos(\alpha) = \dfrac{a}{\sqrt{a^2+b^2}} \end{cases}$$

Once the auxiliary angle is determined, we can construct the equation equivalent to the original one

$$\sin(x+\alpha) = \frac{c}{\sqrt{a^2+b^2}}$$

## 18.4 Second Degree Trigonometric Equations in Sine and Cosine

### 18.4.1 Homogeneous Second Degree Equations in Sine and Cosine

The homogeneous second degree equations in sine and cosine are trigonometric equations presented in the form

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) = 0$$

with $a$, $b$, $c$ as real coefficients not all equal to zero. The structure of such equations resembles, in the left-hand side, that of a homogeneous polynomial in the indeterminates $X$, $Y$

$$aX^2 + bXY + cY^2 = 0.$$

They are called homogeneous equations because, in the standard form, the left-hand side can be reduced to a homogeneous polynomial.

Let's now see how to proceed. Since eliminating a coefficient among $a$, $b$, and $c$ still leaves us with a homogeneous equation, let's start by analyzing **specific cases** for simplicity.

1. If $a = 0$, we reduce the equation to

   $$b\sin(x)\cos(x) + c\cos^2(x) = 0.$$

   Factoring out $\cos(x)$, we get

   $$\cos(x)(b\sin(x) + c\cos(x)) = 0.$$

   At this point, we can apply the zero-product property and split it into two equations:

   $$\cos(x) = 0,$$

   which is a basic trigonometric equation, and

   $$b\sin(x) + c\cos(x) = 0,$$

   which is a linear equation in sine and cosine.

2. If $c = 0$, we reduce the equation to

   $$a\sin^2(x) + b\sin(x)\cos(x) = 0.$$

   Simply factoring out $\sin(x)$ leads us back to the form

   $$\sin(x)(a\sin(x) + b\cos(x)) = 0,$$

   giving us

   $$\sin(x) = 0$$

   and

   $$a\sin(x) + b\cos(x) = 0.$$

   Once again, we are faced with a basic equation and a linear one.

3. If $b = 0$, we reduce the equation to

$$a \sin^2(x) + c \cos^2(x) = 0,$$

and proceed as in the following general case...

**Second-Degree Homogeneous Equations in Sine and Cosine** The resolution method for the general case refers to the possibility that $a$, $b$, $c$ are not null, but it also applies to the specific case where $a$, $c$ are not null and $b = 0$.

We proceed by dividing both sides by $\cos^2(x)$:

$$\frac{a \sin^2(x) + b \sin(x) \cos(x) + c \cos^2(x)}{\cos^2(x)} = 0.$$

Hence, dividing term by term,

$$a \frac{\sin^2(x)}{\cos^2(x)} + b \frac{\sin(x) \cos(x)}{\cos^2(x)} + c \frac{\cos^2(x)}{\cos^2(x)} = 0.$$

After simplifying and recalling the definition of tangent of an angle, we have

$$a \tan^2(x) + b \tan(x) + c = 0,$$

which is a trigonometric equation reducible to a basic one, where we can apply the substitution method.

We need to be careful with the division by $\cos^2(x)$, which would require appropriate existence conditions. These conditions are only legitimate if the values of $x$ for which $\cos(x) = 0$ are not solutions of the second-degree homogeneous equation. Otherwise, by doing so, we would erroneously restrict the set of solutions' existence.

On the other hand, if we assume

$$\cos(x) = 0,$$

by the fundamental trigonometric relation, it follows that

$$\sin^2(x) + \cos^2(x) = 1 \quad \Rightarrow \quad \sin^2(x) = 1.$$

Substituting into the original equation, we get:

$$a \cdot 1 + b \cdot 0 + c \cdot 0 = 0,$$

implying $a = 0$, which is absurd since we initially supposed $a$ to be non-zero.

**Non-Homogeneous Second-Degree Equations in Sine and Cosine** The most general case for second-degree equations in sine and cosine is the non-homogeneous one, where the polynomial on the left side loses its homogeneity due to the non-zero constant term.

If we have an equation of the form

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) = d$$

with $d \neq 0$, we can easily reduce it to homogeneous ones using the well-known fundamental identity:

$$\sin^2(x) + \cos^2(x) = 1$$

We can rewrite the original equation as

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) = d \cdot 1.$$

Utilizing the fundamental identity:

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) = d \cdot [\sin^2(x) + \cos^2(x)]$$

Expanding the product and moving everything to the left side, we get

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) - d\sin^2(x) - d\cos^2(x) = 0.$$

Now, we just need to perform some appropriate factoring:

$$(a - d)\sin^2(x) + b\sin(x)\cos(x) + (c - d)\cos^2(x) = 0.$$

Depending on the relationships between the coefficients $a$, $b$, $d$, we will fall into the case of homogeneous second-degree trigonometric equations, and possibly into one of the related special cases.

# 19 Trigonometric Inequalities

An inequality is called *trigonometric* if the unknown appears as an argument of at least one trigonometric function, namely one of sine, cosine, tangent, cotangent, secant, or cosecant.

### Solution Method

1. Establish the existence conditions. In this step, we not only need to pay attention to expressions that inherently require existence conditions such as denominators, even roots, and logarithms. The existence conditions must be included in a system of inequalities along with the inequality we want to solve.

2. After setting the existence conditions, we need to determine the type of inequality proposed. To do this, some simple algebraic calculations may be necessary, following the principles of inequality equivalence. The most common cases in exercises are:

   - Elementary trigonometric inequalities, such as

   $$\sin(x) \gtreqless m \qquad \cos(x) \gtreqless n$$

   $$\tan(x) \gtreqless p \qquad \cot(x) \gtreqless q$$

   $$\sec(x) \gtreqless r \qquad \csc(x) \gtreqless s$$

   - Elementary trigonometric inequalities by substitution, like

   $$\sin[f(x)] \gtreqless m \qquad \cos[f(x)] \gtreqless n$$

   $$\tan[f(x)] \gtreqless p \qquad \cot[f(x)] \gtreqless q$$

   $$\sec[f(x)] \gtreqless r \qquad \csc[f(x)] \gtreqless s$$

   - Trigonometric inequalities reducible to polynomials by substitution, where the same trigonometric function appears with different positive integer exponents and the same argument.
   - Trigonometric inequalities solvable by applying definitions and trigonometric formulas, thereby reducing the original inequality to a different type of trigonometric inequality.

- Linear trigonometric inequalities in sine and cosine, such as

$$a\sin(x) + b\cos(x) \gtreqless c$$

- Second-degree trigonometric inequalities in sine and cosine, such as

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) \gtreqless d$$

Each type of trigonometric inequality can appear in many different forms and may involve any of the six trigonometric functions. Therefore, it is necessary to:

- Understand thoroughly how to solve elementary trigonometric inequalities and extend solutions for periodicity.
- Keep in mind the definitions of trigonometric functions and trigonometric formulas, and if necessary, use them to rewrite the inequality in a more manageable form or one that we know how to solve.
- Fully comprehend trigonometric equations. They provide useful insights for solving corresponding inequalities, such as in linear sine and cosine equations and second-degree sine and cosine equations. The only significant difference concerns comparison trigonometric equations, like

$$\sin[f(x)] = \sin[g(x)] \qquad \cos[f(x)] = \cos[g(x)]$$

$$\tan[f(x)] = \tan[g(x)] \qquad \cot[f(x)] = \cot[g(x)]$$

$$\sec[f(x)] = \sec[g(x)] \qquad \csc[f(x)] = \csc[g(x)]$$

  In such cases, the method for solving the corresponding inequalities

$$\sin[f(x)] \gtrless \sin[g(x)] \qquad \cos[f(x)] \gtrless \cos[g(x)]$$

$$\tan[f(x)] \gtrless \tan[g(x)] \qquad \cot[f(x)] \gtrless \cot[g(x)]$$

$$\sec[f(x)] \gtrless \sec[g(x)] \qquad \csc[f(x)] \gtrless \csc[g(x)]$$

  differs from that of equations, and it is much more convenient to use formulas and definitions to reduce them to elementary trigonometric inequalities.

3. Finally, we will determine the solutions within the periodicity interval, extend them for periodicity to the entire real axis, and then compare them with the existence conditions set at the beginning.

## 19.1 Elementary Trigonometric Inequalities

Let's see how to solve elementary trigonometric inequalities, i.e., those that appear in the form:

$$\begin{aligned} \sin(x) &\gtreqless m & \cos(x) &\gtreqless n \\ \tan(x) &\gtreqless p & \cot(x) &\gtreqless q \\ \sec(x) &\gtreqless r & \csc(x) &\gtreqless s \end{aligned}$$

It is sufficient for us to know how to derive the solutions of:

$$\begin{aligned} \sin(x) &\gtreqless m \\ \cos(x) &\gtreqless n \\ \tan(x) &\gtreqless p \end{aligned}$$

because in the presence of a cotangent, a secant, or a cosecant, we can still refer to the aforementioned cases by applying the definitions:

$$\begin{aligned} \cot(x) &= \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} \\ \sec(x) &= \frac{1}{\cos(x)} \\ \csc(x) &= \frac{1}{\sin(x)} \end{aligned}$$

### 19.1.1 Elementary Trigonometric Inequalities with Sine

To determine the solutions of the inequality:

$$\sin(x) \gtreqless m$$

the simplest way to proceed is to reason in the Cartesian plane $OXY$, where we denote $X$ and $Y$ as abscissa and ordinate respectively. In particular, the abscissa $X$ should not be confused with the variable $x$, which represents an angle for us.

Regardless of the inequality symbol, we draw a unit circle and the horizontal line with equation $Y = m$.

The sine of an angle between 0 and $2\pi$, by definition, is the ordinate $Y$ of the point $(X, Y)$ on the unit circle associated with the angle $x$. Consequently:

- if the inequality symbol is $>$ or $\geq$, the set of solutions consists of angles corresponding to points on the unit circle located above the line;

| **Inequality** | **Value of $m$** | **Solutions in the Periodicity Interval** | **Solutions in $\mathbb{R}$** |
|---|---|---|---|
| $\sin(x) > m$ | $m > 1$ | $\nexists x$ | $\nexists x$ |
| $\sin(x) \geq m$ | $m > 1$ | $\nexists x$ | $\nexists x$ |
| $\sin(x) < m$ | $m > 1$ | $0 \leq x < 2\pi$ | $\forall x$ |
| $\sin(x) \leq m$ | $m > 1$ | $0 \leq x < 2\pi$ | $\nexists x$ |
| $\sin(x) \gtreqless m$ | $-1 \leq m \leq 1$ | To be analyzed | To be analyzed |
| $\sin(x) > m$ | $m < -1$ | $0 \leq x < 2\pi$ | $\forall x$ |
| $\sin(x) \geq m$ | $m < -1$ | $0 \leq x < 2\pi$ | $\forall x$ |
| $\sin(x) < m$ | $m < -1$ | $\nexists x$ | $\nexists x$ |
| $\sin(x) \leq m$ | $m < -1$ | $\nexists x$ | $\nexists x$ |

- if the inequality symbol is $<$ or $\leq$, the set of solutions consists of angles corresponding to points on the unit circle located below the line.

Once this is done, we only need to pay attention to two aspects

- **Periodicity Interval and Solution Extension:** The angles of the trigonometric circle vary in $0 \leq x < 2\pi$, which is also the periodicity interval of the sine function:

$$0 \leq x < 2\pi$$

  However, we must consider $x$ as a real number. We will solve the inequality considering $0 \leq x < 2\pi$ and extend the solutions to the entire real set $\mathbb{R}$ by adding $2k\pi$ to the endpoints, varying $k \in \mathbb{Z}$.

- **Limitation of Sine Values:** The sine $\sin(x)$ takes values between $-1$ and 1

$$-1 \leq \sin(x) \leq 1$$

  so we have different possibilities depending on the values of the right-hand side $m$:

  - if $-1 \leq m \leq 1$, that is, if the line intersects the circle, to find the angles corresponding to the intersection points, we must solve the associated elementary trigonometric equation

$$\sin(x) = m$$

  - if $m < -1$ or if $m > 1$, the inequality is immediate

**Alternative Method for Elementary Trigonometric Inequalities with Sine** Alternatively, to solve an elementary trigonometric inequality with sine, we can plot the graph of the sine function $y = \sin(x)$ in the Cartesian plane $Oxy$, limiting ourselves to the abscissas $0 \leq x \leq 2\pi$.

In addition to the graph, we draw the line with equation $y = m$, then:

- if the inequality symbol is $<$ or $\leq$, we consider the intervals of the $x$-axis where the graph lies below the line;
- if the inequality symbol is $>$ or $\geq$, we consider the intervals of the $x$-axis where the graph lies above the line.

### 19.1.2 Elementary Trigonometric Inequalities with Cosine

Moving on to trigonometric inequalities of the form

$$\cos(x) \gtreqless n.$$

Again, we reason in the Cartesian plane $OXY$, and draw the trigonometric circle and the vertical line $X = n$.

The cosine of an angle $0 \leq x < 2\pi$ is, by definition, the abscissa $X$ of the point $(X, Y)$ on the trigonometric circle associated with the angle $x$. Consequently:

- If the inequality symbol is $>$ or $\geq$, the solution set consists of the angles corresponding to the points of the circle located **to the right** of the line;
- If the inequality symbol is $<$ or $\leq$, the solution set consists of the angles corresponding to the points of the circle located **to the left** of the line.

Here too, we must take into account the two most delicate aspects of the procedure

- **Periodicity Interval and Solution Extension:** The angles of the trigonometric circle can vary in $0 \leq x < 2\pi$, which is also the periodicity interval of the cosine function.

  $$0 \leq x < 2\pi$$

  Just like in elementary trigonometric inequalities with sine, we will determine the solutions of the inequality in the interval $0 \leq x < 2\pi$, then extend the solutions to the entire real set $\mathbb{R}$ by adding $2k\pi$ to the endpoints, varying $k \in \mathbb{Z}$

- **Limitation of Cosine Values:** The cosine $\cos(x)$ takes values between $-1$ and $1$

  $$-1 \leq \cos(x) \leq 1$$

  so depending on the values of the right-hand side $n$, we can distinguish between different cases:

– if $-1 \leq n \leq 1$, that is, if the line intersects the circle, to find the angles corresponding to the intersection points, we must solve the associated elementary trigonometric equation

$$\cos(x) = n$$

for which it may be useful to consult the table of notable values of trigonometric functions.

– if $n < -1$ or $n > 1$, the inequality is trivial.

| **Inequality** | **Value of $n$** | **Solutions in the Periodicity Interval** | **Solutions in $\mathbb{R}$** |
|---|---|---|---|
| $\cos(x) > n$ | $n > 1$ | $\nexists x$ | $\nexists x$ |
| $\cos(x) \geq n$ | $n > 1$ | $\nexists x$ | $\nexists x$ |
| $\cos(x) < n$ | $n > 1$ | $0 \leq x < 2\pi$ | $\forall x$ |
| $\cos(x) \leq n$ | $n > 1$ | $0 \leq x < 2\pi$ | $\forall x$ |
| $\cos(x) \gtreqless n$ | $-1 \leq n \leq 1$ | To be analyzed | To be analyzed |
| $\cos(x) > n$ | $n < -1$ | $0 \leq x < 2\pi$ | $\forall x$ |
| $\cos(x) \geq n$ | $n < -1$ | $0 \leq x < 2\pi$ | $\forall x$ |
| $\cos(x) < n$ | $n < -1$ | $\nexists x$ | $\nexists x$ |
| $\cos(x) \leq n$ | $n < -1$ | $\nexists x$ | $\nexists x$ |

**Alternative Method for Elementary Trigonometric Inequalities with Cosine** In this case as well, we can resort to the graph of the cosine function $y = \cos(x)$, plotted in the $xy$-plane within the interval $0 \leq x < 2\pi$, and draw the line with equation $y = n$. To determine the solutions, we consider:

- the intervals on the $x$-axis where the graph lies below the line, if the symbol is $<$ or $\leq$;
- the intervals on the $x$-axis where the graph lies above the line, if the symbol is $>$ or $\geq$.

### 19.1.3 Elementary Trigonometric Inequalities with Tangent

As the third and final case, we have trigonometric inequalities of the form

$$\tan(x) \gtreqless p$$

For elementary trigonometric inequalities with tangent, it's not advisable to use the unit circle, as it's not as straightforward as with sine and cosine inequalities.

For elementary inequalities with tangent, it's more convenient to resort to the graph of the tangent function $y = \tan(x)$ in the $xy$-plane, limited to the interval $0 \leq x < \pi$, and draw the line with equation $y = p$.

At this point:

- If the symbol of the inequality is $>$ or $\geq$, we will consider the part of the graph above the line;
- if the symbol of the inequality is $<$ or $\leq$, we will consider the part of the graph below the line.

Remember that the tangent requires as conditions of existence

$$\tan(x) \to x \neq \frac{\pi}{2} + k\pi \quad \text{where } k \in \mathbb{Z}$$

Additionally, its periodicity interval is

$$0 \leq x < \frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{2} < x \leq \pi$$

or, equivalently,

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

The choice is indifferent, although the second option is more convenient to write. The important thing is to consider an interval with an amplitude of $\pi$ and choose consistently with the interval on which the graph is decided to be drawn. To extend the solutions, furthermore, we'll need to add multiples of $\pi$ to the extremes.

### 19.1.4 Other Elementary Trigonometric Inequalities

For other elementary trigonometric inequalities, namely cotangent, secant, or cosecant, we can resort to the graphical method just as in the case of tangent. In each case, we'll draw the graph in the $xy$-plane, identifying the solutions and extending them for periodicity. For convenience, let's recall the conditions of existence and the periodicity intervals:

- cotangent function

$$y = \cot(x) \text{ ; CE and periodicity: } 0 < x < \pi$$

- secant function

$$y = \sec(x) \text{ ; CE and periodicity: } -\frac{\pi}{2} < x < \frac{\pi}{2} \vee \frac{\pi}{2} < x < \frac{3\pi}{2}$$

- cosecant function

$$y = \csc(x) \ ; \ \text{CE and periodicity: } 0 < x < \pi \vee \pi < x < 2\pi$$

## 19.2 Trigonometric Inequalities Reducible to Elementary Ones by Substitution

The first case we consider is that of *quasi*-elementary trigonometric inequalities, namely of the form:

$$\begin{array}{rr} \sin[f(x)] \gtreqless m & \cos[f(x)] \gtreqless n \\ \tan[f(x)] \gtreqless p & \cot[f(x)] \gtreqless q \\ \sec[f(x)] \gtreqless r & \csc[f(x)] \gtreqless s \end{array}$$

To solve them, one must first impose the existence conditions related to the expression $f(x)$, and possibly for the involved trigonometric function (tangent and cotangent, secant and cosecant). Then, one proceeds with the substitution $y = f(x)$. At this point, solve the resulting elementary inequality, report the solutions to the variable $x$ on the periodicity interval of the trigonometric function, and finally extend the solutions to the real numbers.

**Alternative Method:** In certain cases, it might be more convenient to apply a suitable trigonometric formula, such as the formulas for associated angles or the addition and subtraction formulas for arcs.

## 19.3 Trigonometric Inequalities Reducible to Polynomials

This type of inequality is characterized by the same trigonometric function appearing multiple times, with the same argument, in the form of a power with integer exponents. The resolution method involves making a substitution to reduce it to a first-degree, second-degree, or higher-than-second-degree inequality. After determining its solutions, they need to be reported back to the original variable and extended for periodicity.

## 19.4 Linear Trigonometric Inequalities in Sine and Cosine

In this case, the procedure is inspired by linear trigonometric equations in sine and cosine. One needs to apply parametric formulas and express sine and cosine in an equivalent form by setting:

$$t = \tan\left(\frac{x}{2}\right) \quad \Rightarrow \quad \begin{array}{l} \sin(x) = \frac{2t}{1+t^2} \\ \\ \cos(x) = \frac{1-t^2}{1+t^2} \end{array}$$

If we set $t = \tan\left(\frac{x}{2}\right)$ and proceed to solve the inequality:

$$a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2} \gtreqless c$$

we must consider that the tangent imposes its own existence conditions:

$$\frac{x}{2} \neq \frac{\pi}{2} + k\pi \quad \rightarrow \quad x \neq \pi + 2k\pi.$$

These values are excluded in the transition to the new inequality but not necessarily in the original inequality. Before diving into calculations, we need to establish whether they satisfy the initial inequality, and to do so, we can limit ourselves to values within the periodicity interval of sine and cosine, which is $0 \leq x < 2\pi$.

$$\begin{array}{l} k = -1 \quad \rightarrow \quad x = \pi - 2\pi = -\pi < 0 \\ k = 0 \quad \rightarrow \quad x = \pi + 0 = \pi \quad \rightarrow \quad 0 \leq \pi < 2\pi \\ k = 1 \quad \rightarrow \quad x = \pi + 2\pi = 3\pi > 2\pi \end{array}$$

We will then check if $x = \pi$ satisfies the original inequality, by simple substitution:

$$a\sin(\pi) + b\cos(\pi) \gtreqless c \quad \Rightarrow \quad < b \gtreqless x.$$

If the inequality holds true, we will consider that $x = \pi$ is also a solution; otherwise, it is not.

After doing so, we will solve the inequality in $t$, report the solutions to the variable $x$ and to the periodicity interval $0 \leq x < 2\pi$, and finally extend them for periodicity to the real axis.

## 19.5 Second-degree Trigonometric Inequalities in Sine and Cosine

The last type of trigonometric inequalities we analyze is of the form

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) \gtreqless d$$

The procedure follows the same steps as for homogeneous second-degree equations in sine and cosine. If the right-hand side is non-zero, $d \neq 0$, we are in the non-homogeneous case.

We use the fundamental trigonometric identity

$$\sin^2(x) + \cos^2(x) = 1$$

to rewrite it as

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) \gtreqless d(\sin^2(x) + \cos^2(x))$$

and bringing everything to the left-hand side, we get

$$(a-d)\sin^2(x) + b\sin(x)\cos(x) + (c-d)\cos^2(x) \gtreqless 0$$

This effectively reduces the problem to the *homogeneous* case, where the right-hand side is zero, $d = 0$. Let's focus on this, as it's the most interesting case:

$$a\sin^2(x) + b\sin(x)\cos(x) + c\cos^2(x) \gtreqless 0$$

We divide both sides by $\cos^2(x)$ to leverage the definition of the tangent of an angle:

$$a\frac{\sin^2(x)}{\cos^2(x)} + b\frac{\sin(x)\cos(x)}{\cos^2(x)} + c\frac{\cos^2(x)}{\cos^2(x)} \gtreqless 0$$

Note that we can divide without worrying about sign issues, as $\cos^2(x)$ is non-negative. The only case we need to exclude is

$$\cos^2(x) = 0 \to \cos(x) = 0$$

so we must impose

$$x \neq \frac{\pi}{2} + k\pi$$

These values must be excluded for the algebraic procedure to be valid, but they might still satisfy the original inequality. Therefore, we'll check if the respective values within the periodic interval $0 \leq x < 2\pi$ satisfy the inequality:

$$\begin{aligned}
k = -1 \to x &= \frac{\pi}{2} - \pi = -\frac{\pi}{2} < 0 \\
k = 0 \to x &= \frac{\pi}{2} + 0 = \frac{\pi}{2} \to 0 \leq \frac{\pi}{2} < 2\pi \\
k = 1 \to x &= \frac{\pi}{2} + \pi = \frac{3\pi}{2} \to 0 \leq \frac{3\pi}{2} < 2\pi \\
k = 2 \to x &= \frac{\pi}{2} + 2\pi = \frac{5\pi}{2} > 2\pi
\end{aligned}$$

To understand this, let's substitute $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$

$$\begin{aligned}
x = \frac{\pi}{2} &\to a\sin^2\left(\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}\cos(\frac{\pi}{2})\right) + c\cos^2\left(\frac{\pi}{2}\right) \gtreqless 0 \to a \gtreqless 0 \\
x = \frac{3\pi}{2} &\to a\sin^2\left(\frac{3\pi}{2}\right) + b\sin\left(\frac{3\pi}{2}\cos(\frac{3\pi}{2})\right) + c\cos^2\left(\frac{\pi}{2}\right) \gtreqless 0 \to a \gtreqless 0
\end{aligned}$$

If we obtain a verified inequality, then we'll keep $x = \dfrac{\pi}{2} + k\pi$ as solutions of the original inequality. Returning to

$$a\tan^2(x) + b\tan(x) + c \gtreqless 0$$

We have a trigonometric inequality reducible to a polynomial, which we can solve by letting $z = \tan(x)$.