Limits

Donato Martinelli
February 1, 2024

Contents

Part I
Limits

1 Neighborhoods

Let's present some fundamental notions of the topology of the set $\mathbb{R}$ of real numbers concerning their particular subsets.

Since there is a one-to-one correspondence between $\mathbb{R}$ and the points of an oriented line r, called the **real line**, we can identify every subset of $\mathbb{R}$ with a *subset of points* of the line r, and thus also talk about the topology of the line.

1.1 Neighborhoods of a Point

Complete Interior

Definition 1.1. Given a real number x_0, a **complete neighborhood** of x_0 is any open interval $I(x_0)$ containing x_0:

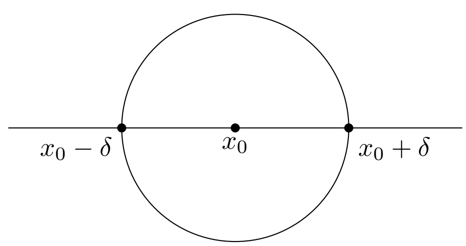
$$I(x_0) =]x_0 - \delta_1; x_0 + \delta_2[\quad \text{with } \delta_1, \delta_2 \in \mathbb{R}^+.$$

When $\delta_1 = \delta_2$, the point x_0 is the midpoint of the interval. In this case, we talk about the *circular neighborhood* of x_0.

Circular Neighborhood

Definition 1.2. Given a real number x_0 and a positive real number δ, the **circular neighborhood of x_0 with radius δ** is the open interval $I_\delta(x_0)$ centered at x_0 with radius δ:

$$I_\delta(x_0) =]x_0 - \delta; x_0 + \delta[= \{x \in \mathbb{R} \mid |x - x_0| < \delta\}$$



The set

$$J_{x_0} = \{I_\delta(x_0) | \delta > 0\}$$

is called the **collection of neighborhoods of x_0**.

Since the circular neighborhood of x_0 with radius δ is the set of points $x \in \mathbb{R}$ such that

$$x_0 - \delta < x < x_0 + \delta,$$

or equivalently

$$-\delta < x - x_0 < \delta,$$

we can also write:

$$I_\delta(x_0) = \{x \in \mathbb{R} \ : \ |x - x_0| < \delta\}$$

For complete and circular neighborhoods of a point x_0, the following property holds:

Proposition 1.1. *The intersection and union of two or more neighborhoods of x_0 are still neighborhoods of x_0.*

Right and Left Neighborhoods of a Point Given a neighborhood of a point x_0, sometimes we are interested in considering only the part of the neighborhood that is to the right of x_0 or the part that is to the left. In general, given a number $\delta \in \mathbb{R}^+$, we call:

- **right neighborhood** of x_0 the interval

$$I_\delta^+(x_0) =]x_0; x_0 + \delta[$$

- **left neighborhood** of x_0 the interval

$$I_\delta^-(x_0) =]x_0 - \delta; x_0[$$

1.2 Neighborhoods of Infinity

Given $a, b \in \mathbb{R}$, with $a < b$, we call:

- **neighborhood of negative infinity** any open interval unlimited below:

$$I(-\infty) =]-\infty; a[= \{x \in \mathbb{R} | x < a\};$$

- **neighborhood of positive infinity** any open interval unlimited above:

$$I(+\infty) =]b; +\infty[= \{x \in \mathbb{R} | x > b\};$$

We also define **neighborhood of infinity** as the union of a neighborhood of $-\infty$ and a neighborhood of $+\infty$, i.e.:

$$I(\infty) = I(-\infty) \cup I(+\infty) = \{x \in \mathbb{R} \mid x < a \vee x > b\}$$

Similar to the case of a real point x_0, we can talk about a **circular neighborhood of infinity**:

$$I_c(\infty) =]-\infty; -c[\cup]c; +\infty[\quad \text{with } c \in \mathbb{R}$$

1.3 Accumulation Points

Accumulation Point

Definition 1.3. The real number x_0 is called an accumulation point of A, a subset of $\mathbb{R}$, if every complete neighborhood of x_0 contains infinitely many points of A.

Note 1. The term *accumulation* indicates that the points of A gather around x_0.

Every point in an interval is an accumulation point for the interval itself. The endpoints of the interval are also its accumulation points. Alternatively, we can say that x_0 is an accumulation point of A if every complete neighborhood of x_0 contains at least one element of A distinct from x_0.

2 Definition of $\lim_{x\to x_0} f(x) = l$

Finite Limit as x Approaches x_0

Definition 2.1. We say that the function $f(x)$ has the real number l as its limit, as x approaches x_0, and we write

$$\lim_{x\to x_0} f(x) = l,$$

when, for every positive real number ϵ, it is possible to determine a complete neighborhood I of x_0 such that

$$l - \epsilon < f(x) < l + \epsilon, \quad \text{or } |f(x) - l| < \epsilon,$$

for every x belonging to I, different (at most) from x_0.

Note 2. The validity of the condition $|f(x)-l| < \epsilon$ assumes that $f(x)$ is defined in I (excluding at most x_0). The point x_0 is an accumulation point for the domain of the function. We are not concerned with the value that the function $f(x)$ may assume at x_0.

In symbols, the definition of $\lim_{x\to x_0} f(x) = l$ can be formulated as follows:

$$\forall \epsilon > 0, \exists I(x_0) : \forall x(\neq x_0) \in I(x_0) : |f(x) - l| < \epsilon.$$

In the definition just given, considering ϵ, we think of values that become increasingly smaller. We will say that ϵ is taken *small at will.* Furthermore, if we explicitly state the absolute value in the expression $|f(x) - l| < \epsilon$, we get

$$-\epsilon < f(x) - l < \epsilon \quad \to \quad l - \epsilon < f(x) < l + \epsilon,$$

i.e., $f(x)$ belongs to the interval $]l - \epsilon; l + \epsilon[$.

The definition tells us that l is the limit of $f(x)$ if, for any ϵ, even *very small*, we can always find a neighborhood of x_0 such that, for every $x \neq x_0$ in that neighborhood, $f(x)$ belongs to $]l - \epsilon; l + \epsilon[$, meaning $f(x)$ is *very close* to l.

In general, the existence of the limit of a function at a point x_0 is independent of the behavior of the function at x_0. The following cases are possible:

- $\exists \lim_{x\to x_0} f(x) = l \ \wedge \ l = f(x_0)$;
- $\exists \lim_{x\to x_0} f(x) = l \ \wedge \ l \neq f(x_0)$;
- $\exists \lim_{x\to x_0} f(x) = l \ \wedge \ \nexists f(x_0)$.

2.1 Continuous Functions

If for a function $f(x)$ it holds that, for a point x_0 belonging to the domain of f, the limit of $f(x)$ as $x \to x_0$ exists and

$$\lim_{x\to x_0} f(x) = f(x_0)$$

then f is called **continuous** at x_0. We then say that f is continuous in its domain D when it is continuous at every point in D. Functions whose graphs are uninterrupted curves are continuous in their domain; this includes, for example, a line or a parabola.

If a function is continuous at a point, the calculation of the limit at that point is simple because it suffices to calculate the value of the function at that point.

Here is a list of commonly used functions that are continuous in $\mathbb{R}$ (or in intervals of $\mathbb{R}$).

The Polynomial Function Every polynomial function, i.e., every function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n,$$

is continuous throughout $\mathbb{R}$.

The Square Root Function The function defined in $\mathbb{R}^+ \cup 0$,

$$y = \sqrt{x},$$

is continuous for every real positive or zero x. More generally, power functions with real exponents defined in $\mathbb{R}^+ : y = x^\alpha$ $(\alpha \in \mathbb{R})$ are continuous.

Note 3. The square root function is a special case of the power function with a real exponent. In fact,

$$f(x) = x^{\frac{1}{2}} = \sqrt{x}.$$

Trigonometric Functions The functions $\sin(x)$ and $\cos(x)$ are continuous in $\mathbb{R}$. The tangent function is also continuous in $\mathbb{R} - \{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$, and the cotangent function is continuous in $\mathbb{R} - \{k\pi,\ k \in \mathbb{Z}\}$. Finally, it can be shown that the secant, cosecant, arcsine, arccosine, arctangent, and arccotangent functions are continuous in their domains.

Note 4. The function $\tan(x)$ is not defined for $x = \frac{\pi}{2} + k\pi$.
The function $\cot(x)$ is not defined for $x = k\pi$.

Exponential and Logarithmic Functions The exponential function $y = a^x$, with $a > 0$, is continuous in $\mathbb{R}$. The logarithmic function $y = \log_a x$, with $a > 0$, $a \neq 1$, is continuous in $\mathbb{R}^+$.

2.2 Right and Left Limits

Right Limit The right limit of a function is indicated by the symbol:

$$\lim_{x \to x_0^+} f(x) = l.$$

This notation means that x approaches x_0 but always remains greater than x_0. The definition of the right limit is analogous to the previously given limit definition, with the only difference that the inequality $|f(x) - l| < \epsilon$ must be satisfied for every x belonging to a *right neighborhood* of x_0, i.e., a neighborhood of the form

$$]x_0; x_0 + \delta[$$

Left Limit The left limit of a function is indicated by the symbol:

$$\lim_{x \to x_0^-} f(x) = l.$$

This notation means that x approaches x_0 but always remains less than x_0. The same considerations made for the right limit also apply to the left limit, with the only difference that $|f(x) - l| < \epsilon$ must be satisfied for every x belonging to a *left neighborhood* of x_0, i.e., a neighborhood of the form

$$]x_0 - \delta; x_0[$$

Note that $\lim_{x \to x_0} f(x) = l$ exists if and only if both the right and left limits exist and coincide:

$$\lim_{x \to x_0} f(x) = l \Leftrightarrow \lim_{x \to x_0^+} f(x) = l \wedge \lim_{x \to x_0^-} f(x) = l$$

Indeed, given $\epsilon > 0$, the inequality $|f(x) - l| < \epsilon$ is verified in a complete neighborhood I of x_0, with at most $x \neq x_0$, if and only if it is verified both in a right neighborhood of x_0 and in a left neighborhood of x_0.

3 Definition of $\lim_{x\to x_0} f(x) = \infty$

3.1 $\lim_{x\to x_0} f(x) = +\infty$

Limit $+\infty$ as x approaches x_0

Definition 3.1. Let $f(x)$ be a function not defined at x_0. It is said that $f(x)$ tends to $+\infty$ as x approaches x_0, and it is written

$$\lim_{x\to x_0} f(x) = +\infty$$

when, for every positive real number M, a complete neighborhood I of x_0 can be determined such that

$$f(x) > M$$

for every x belonging to I and different from x_0.

In summary, we can say that $\lim_{x\to x_0} f(x) = +\infty$ if:

$$\forall M > 0, \exists I(x_0) : \forall x \in I(x_0) - \{x_0\}, f(x) > M$$

If $\lim_{x\to x_0} f(x) = +\infty$, it is also said that the function f **diverges positively**.

Note 5. In the definition, when we say "*for every positive real number M*," we think of values of M that become increasingly large. We will say that M is taken arbitrarily large.

3.2 $\lim_{x\to x_0} f(x) = -\infty$

Limit $-\infty$ as x approaches x_0

Definition 3.2. Let $f(x)$ be a function not defined at x_0. It is said that $f(x)$ tends to $-\infty$ as x approaches x_0, and it is written

$$\lim_{x\to x_0} f(x) = -\infty$$

when, for every positive real number M, a complete neighborhood I of x_0 can be determined such that

$$f(x) < -M$$

for every x belonging to I and different from x_0.

In symbols, we say that $\lim_{x\to x_0} f(x) = -\infty$ if:

$$\forall M > 0, \exists I(x_0) : \forall x \in I(x_0) - \{x_0\}, f(x) < -M$$

If $\lim_{x\to x_0} f(x) = -\infty$, it is also said that the function f **diverges negatively**.

If...	the inequality...	is satisfied for $x \neq x_0$, in a...
$\lim_{x \to x_0^+} f(x) = +\infty$	$f(x) > M$	right neighborhood of x_0
$\lim_{x \to x_0^-} f(x) = +\infty$	$f(x) > M$	left neighborhood of x_0
$\lim_{x \to x_0^+} f(x) = -\infty$	$f(x) < -M$	right neighborhood of x_0
$\lim_{x \to x_0^-} f(x) = -\infty$	$f(x) < -M$	left neighborhood of x_0

3.3 Right and Left Infinite Limits

Infinite limits can also be distinguished for right and left limits.

The definition of $\lim_{x \to x_0} f(x) = \infty$ is analogous to the previous ones, but with the following variation: **for every** $M > 0$**, it is possible to find a neighborhood** I **of** x_0 **such that, for every** $x (\neq x_0) \in I$ **in the domain of** f**,**$|f(x)| > M$. In symbols:

$$\forall M > 0, \exists I(x_0) : \forall x(\neq x_0) \in I, |f(x)| > M.$$

The inequality $|f(x)| > M$ can be written equivalently as $f(x) > M \vee f(x) < -M$, and therefore its solutions are the union of the solutions of the individual inequalities.

3.4 Vertical Asymptotes

Asymptote

Definition 3.3. A line is called an asymptote of the graph of a function if the distance from a generic point on the graph to that line tends to 0 as the abscissa or ordinate of the point tends to ∞.

Now let's study *vertical asymptotes.*

Vertical Asymptote

Definition 3.4. Given the function $y = f(x)$, if $\lim_{x \to c} f(x) = \infty$, it is said that the line $x = c$ is a vertical asymptote for the graph of the function.

The distance from a generic point on the graph of a function to its vertical asymptote, with equation $x = c$, tends to 0 as $x \to c$. Indeed, with $P(x; y)$ being the generic point on the graph, we have:

$$\lim_{x \to c} \overline{PH} = \lim_{x \to c} |x - c| = 0$$

The definition of a vertical asymptote is still valid if we consider the right limit $(x \to x_0^+)$ or the left limit $(x \to x_0^-)$, and both limits are infinite, but with opposite signs, or if only one of the two limits is infinite.

4 The Definition of $\lim_{x\to\infty} f(x) = l$

4.1 $x \to +\infty$

Finite Limit of a Function as x Approaches $+\infty$

Definition 4.1. A function $f(x)$ is said to tend to the real number l as x approaches $+\infty$, and it is written

$$\lim_{x\to+\infty} f(x) = l$$

when, no matter the positive real number ϵ chosen, we can determine a neighborhood I of $+\infty$ such that:

$$|f(x) - l| < \epsilon \quad \text{for every } x \in I.$$

Given that a neighborhood of $+\infty$ consists of all x greater than a number c, we can say that $\lim_{x\to+\infty} f(x) = l$ if:

$$\forall\epsilon > 0, \exists c > 0 : \forall x > c, |f(x) - l| < \epsilon$$

4.2 $x \to -\infty$

Finite Limit of a Function as x Approaches $-\infty$

Definition 4.2. A function $f(x)$ is said to have a real limit l as x approaches $-\infty$, and it is written

$$\lim_{x\to-\infty} f(x) = l$$

if, for every fixed positive real number ϵ, we can find a neighborhood I of $-\infty$ such that:

$$|f(x) - l| < \epsilon \quad \text{for every } x \in I$$

In symbols, $\lim_{x\to-\infty} f(x) = l$ if:

$$\forall\epsilon > 0, \exists c > 0 : \forall x < -c, |f(x) - l| < \epsilon$$

4.3 $x \to \infty$

The previous two cases can be summarized in one if we consider a neighborhood of ∞ determined by the x for which

$$|x| > c, \text{ i.e., } x < -c \lor x > c,$$

or also

$$x \in]-\infty; -x[\cup]c : +\infty[,$$

where c is a positive real number chosen arbitrarily large. We then say that x **tends to** ∞ omitting the sign $+$ or $-$. We say that $\lim_{x\to\infty} f(x) = l$ **when for every** $\epsilon > 0$ **we can find a neighborhood** I **of** ∞ **such that** $|f(x) - l| < \epsilon$ **for every** $x \in I$. In symbols:

$$\forall\epsilon > 0, \exists I(\infty) : \forall x \in I, |f(x) - l| < \epsilon$$

4.4 Horizontal Asymptotes

Horizontal Asymptote

Definition 4.3. Given the function $y = f(x)$, if one of the inequalities holds:

$$\lim_{x\to+\infty} f(x) = q \quad \text{or} \quad \lim_{x\to-\infty} f(x) = q \quad \text{or} \quad \lim_{x\to\infty} f(x) = q,$$

then the line $y = q$ is called a horizontal asymptote for the graph of the function.

The distance from a generic point P on the graph of a function to its horizontal asymptote, with equation $y = q$, tends to 0 as x tends to $+\infty$. Let $P(x; f(x))$ be the point, we have:

$$\lim_{x\to+\infty} \overline{PH} = \lim_{x\to+\infty} |f(x) - q| = 0.$$

Similar considerations hold for $x \to \infty$ or $x \to -\infty$.

5 The Definition of $\lim_{x\to\infty} f(x) = \infty$

5.1 The Limit is $+\infty$ as x Approaches $+\infty$ or $-\infty$

In this case, we can also say that **the function diverges positively**. Let's study the two cases:

$$\lim_{x\to+\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = +\infty$$

Limit $+\infty$ of a function as x approaches $+\infty$

Definition 5.1. The function $f(x)$ is said to have a limit of $+\infty$ as x approaches $+\infty$, and it is written

$$\lim_{x\to+\infty} f(x) = +\infty$$

when, for every positive real number M, we can determine a neighborhood I of $+\infty$ such that:

$$f(x) > M \quad \text{for every } x \in I$$

In symbols, $\lim_{x\to+\infty} f(x) = +\infty$ if:

$$\forall M > 0, \exists c > 0 : \forall x > c, f(x) > M.$$

Limit $+\infty$ of a function as x approaches $-\infty$

Definition 5.2. The function $f(x)$ is said to have a limit of $+\infty$ as x approaches $-\infty$, and it is written

$$\lim_{x\to-\infty} f(x) = -\infty$$

when, for every positive real number M, we can determine a neighborhood I of $-\infty$ such that:

$$f(x) > M \quad \text{for every } x \in I.$$

In symbols, $\lim_{x\to-\infty} f(x) = +\infty$ if:

$$\forall M > 0, \exists c > 0 : \forall x < -c, f(x) > M$$

5.2 The Limit is $-\infty$ as x Approaches $+\infty$ or $-\infty$

In this case, we can also say that **the function diverges negatively**. Let's study the cases:

$$\lim_{x\to+\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty$$

Limit $-\infty$ of a function as x approaches $+\infty$

Definition 5.3. The function $f(x)$ is said to have a limit of $-\infty$ as x approaches $+\infty$, and it is written $\lim_{x\to+\infty} f(x) = -\infty$ when, for every positive real number M, we can determine a neighborhood I of $+\infty$ such that $f(x) < -M$ for every $x \in I$.

In symbols:

$$\forall M > 0, \exists I(+\infty) : \forall x \in I, f(x) < -M$$

Limit $-\infty$ of a function as x approaches $-\infty$

Definition 5.4. The function $f(x)$ is said to have a limit of $-\infty$ as x approaches $-\infty$, and it is written $\lim_{x\to-\infty} f(x) = -\infty$ when, for every positive real number M, we can determine a neighborhood I of $-\infty$ such that $f(x) < -M$ for every $x \in I$.

In symbols:

$$\forall M > 0, \exists I(-\infty) : \forall x \in I, f(x) < -M$$

6 First Theorems on Limits

The following theorems and properties are valid for functions defined in any domain $D \subseteq \mathbb{R}$ and for points x_0 (where we calculate the limit) that are accumulation points of the domain D. They also hold for $x \to +\infty$ or $x \to -\infty$. However, we will consider specific domains D, i.e., intervals of $\mathbb{R}$ or unions of intervals, and x_0 as a point in D or as an endpoint of one of the intervals that make up D.

The theorems also hold if, instead of l, we have $+\infty$, $-\infty$, or ∞. They are also valid in the cases of the right limit or the left limit.

6.1 Uniqueness of Limit Theorem

Uniqueness of Limit Theorem

Theorem 6.1. *If the function $f(x)$ has the real number l as its limit as x approaches x_0, then this limit is unique.*

Proof. Let's prove the thesis by contradiction. Suppose that the thesis is false, i.e., that l is not unique. In that case, there should exist a real number l' different from l such that:

$$\lim_{x \to x_0} f(x) = l',\ l' \neq l$$

We can assume $l < l'$ and, since in the definition of limit we can choose ϵ arbitrarily as long as it is positive, consider:

$$\epsilon < \frac{l' - l}{2}.$$

Apply the definition of limit in both cases. There should exist two neighborhoods I and I' of x_0 such that:

$$\begin{aligned} |f(x) - l| < \epsilon \quad & \text{for every } x \in I \\ |f(x) - l'| < \epsilon \quad & \text{for every } x \in I' \end{aligned}$$

Notice that $I \cap I'$ is also a neighborhood of x_0. In $I \cap I'$, the two inequalities must hold simultaneously, i.e.:

$$\begin{cases} |f(x) - l| < \epsilon \\ |f(x) - l'| < \epsilon \quad \forall x \in I \cap I' \end{cases}$$

We can also write:

$$\begin{cases} l - \epsilon < f(x) < l + \epsilon \\ l' - \epsilon < f(x) < l' + \epsilon \end{cases}$$

Comparing the inequalities, remembering that $l < l'$, it follows that

$$l' - \epsilon < f(x) < l + \epsilon$$

which implies:

$$l' - \epsilon < l + \epsilon$$

Solving for ϵ, we get

$$-\epsilon - \epsilon < l - l' \quad \rightarrow \quad -2\epsilon < l - l' \quad \rightarrow \quad 2\epsilon > l' - l,$$

which contradicts the assumption of $\epsilon < \frac{l'-l}{2}$. The assumption that there are two limits is false. Therefore, if $\lim_{x\to x_0} f(x) = l$, the limit l is unique. □

6.2 Sign Preservation Theorem

Theorem 6.2. *If the limit of a function as x approaches x_0 is a number l different from 0, then there exists an interval I around x_0 (excluding at most x_0) in which both $f(x)$ and l are either both positive or both negative.*

Proof. By hypothesis,

$$\lim_{x\to x_0} f(x) = l \neq 0.$$

- If $l > 0$, for the arbitrariness of ϵ, we choose $\epsilon = l$. Then, there exists an interval I around x_0 such that

  $$|f(x) - l| < l,$$

  which implies

  $$-l < f(x) - l < l \quad \rightarrow \quad 0 < f(x) < 2l,$$

  and therefore

  $$f(x) > 0, \quad \forall x \in I;$$

  hence l and $f(x)$ are both positive.

- If $l < 0$, we choose $\epsilon = -l$. Then, there exists an interval I around x_0 such that

  $$|f(x) - l| < -l,$$

  which implies

  $$+l < f(x) - l < -l \quad \rightarrow \quad 2l < f(x) < 0,$$

  and therefore $f(x) < 0, \quad \forall x \in I$; hence l and $f(x)$ are both negative.

□

6.3 Comparison Theorem

Theorem 6.3. *Let $h(x)$, $f(x)$, and $g(x)$ be three functions defined in the same domain $D \subseteq \mathbb{R}$, excluding at most one point x_0. If, at every point different from x_0 in the domain, it holds that*

$$h(x) \leq f(x) \leq g(x)$$

and the limit of the two functions $h(x)$ and $g(x)$ as x approaches x_0 is the same number l, then the limit of $f(x)$ as x approaches x_0 is also equal to l.

Proof. Let $\epsilon > 0$ be arbitrary. It is true that:

$$\begin{aligned} |h(x) - l| < \epsilon, &\text{ for every } x \in I_1 \cap D, \text{ because } h(x) \to l \text{ as } x \to x_0; \\ |g(x) - l| < \epsilon, &\text{ for every } x \in I_2 \cap D, \text{ because } g(x) \to l \text{ as } x \to x_0. \end{aligned}$$

Both inequalities hold for every x in the domain belonging to the interval $I = I \cap I_1 \cap I_2$, excluding at most x_0. Therefore, for every $x \in I$, we have:

$$l - \epsilon < h(x) < l + \epsilon, \quad l - \epsilon < g(x) < l + \epsilon.$$

Taking into account the relationship between the functions, we have

$$l - \epsilon < h(x) \leq f(x) \leq g(x) < l + \epsilon,$$

for every $x \in I$, which implies

$$l - \epsilon < f(x) < l + \epsilon$$

for every $x \in I$, i.e.,

$$|f(x) - l| < \epsilon, \ \forall x \in I.$$

This last relation precisely means that $\lim_{x \to x_0} f(x) = l$. □

7 Limit of a Sequence

The concept of the limit of a sequence is similar to that of the limit of a function. However, in the case of sequences, we observe that the domain is the set of natural numbers $\mathbb{N}$ and not an interval.

Note 6. Remember that a sequence is a particular function from $\mathbb{N}$ to $\mathbb{R}$.

In particular, this implies that the independent variable n cannot tend to a finite value but only to $+\infty$.

7.1 $\lim_{n\to+\infty} a_n = +\infty$

Definition 7.1. Given the sequence with general term a_n, it is said that, as n tends to $+\infty$, the sequence has a limit of $+\infty$ when, for any arbitrarily chosen positive real number M, it is possible to determine a corresponding positive real number p_M such that:

$$a_n > M \quad \text{for every } n > p_M$$

Saying that M is a positive number arbitrarily chosen is equivalent to saying that every statement holds for every $M > 0$. This means that, for any arbitrarily chosen $M > 0$, from a certain index onwards, all the following terms are greater than M. In this case, the sequence is called **divergent positively**. Similarly, we give the definition of a sequence that tends to $-\infty$.

7.2 $\lim_{n\to+\infty} a_n = -\infty$

Definition 7.2. Given the sequence with general term a_n, it is said that, as n tends to $+\infty$, the sequence has a limit of $-\infty$ when, for any arbitrarily chosen positive real number M, it is possible to determine a corresponding positive real number p_M such that:

$$a_n < -M \quad \text{for every } n > p_M$$

Therefore, for any arbitrarily chosen number $M > 0$, from a certain index onwards, all terms of the sequence are less than $-M$. In this case, the sequence is called **divergent negatively**.

7.3 $\lim_{n\to+\infty} a_n = l$

Definition 7.3. Given the sequence with general term a_n, it is said that, as n tends to $+\infty$, the sequence has a limit equal to the number l when, for any arbitrarily chosen positive real number ϵ, it is possible to determine a corresponding positive real number p_ϵ such that:

$$|a_n - l| < \epsilon \quad \text{for every } n > p_\epsilon.$$

A sequence of this type is called **convergent**.

7.4 $\lim_{n\to+\infty} a_n$ does not exist

It may happen that a sequence is neither divergent nor convergent: in these cases, it is said that **the limit does not exist**, or the sequence is **indeterminate**.

7.5 Theorems on Limits of Sequences

The theorems we have proved for the limits of functions are valid, as special cases, also for sequences. In particular, let's recall the **comparison theorem**:

- given the sequences a_n, b_n, c_n such that $a_n \leq b_n \leq c_n$, $\forall n \in \mathbb{N}$, if $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = l$, then the limit of b_n also exists as n approaches $+\infty$ and is equal to l;
- given the sequences a_n, b_n such that $a_n \leq b_n, \forall n \in \mathbb{N}$, if $\lim_{n\to+\infty} a_n = +\infty$, then b_n also tends to $+\infty$ as n approaches $+\infty$, and similarly, if $\lim_{n\to+\infty} b_n = -\infty$, then a_n also tends to $-\infty$ as n approaches $+\infty$.

7.6 Subsequences

Subsequence (or Extracted Sequence)

Definition 7.4. A *subsequence* (or *extracted sequence*) of a sequence a_n is a new sequence a_{n_k} obtained by choosing an infinite subset of indices n_k from the original sequence.

From a sequence, we can derive infinitely many subsequences.

Limit of Subsequences

Theorem 7.1. *If a sequence a_n has a limit $l \in \mathbb{R}$, or $+\infty$ or $-\infty$, as n approaches $+\infty$, then every extracted sequence has the same limit as n approaches $+\infty$.*

If a sequence is indeterminate, it does not necessarily mean that its subsequences are also indeterminate. Moreover, if from a sequence we can extract a convergent subsequence, we cannot deduce that the sequence itself is convergent.

7.7 Limits of Monotonic Sequences

The following theorem holds for monotonic sequences.

Limit of a Monotonic Sequence

Theorem 7.2.
- *If an increasing sequence is upper-bounded, then it is convergent; if it is not upper-bounded, then it diverges positively.*
- *If a decreasing sequence is lower-bounded, then it is convergent; if it is not lower-bounded, then it diverges negatively.*

From the theorem, it follows that *a monotonic sequence is never indeterminate.*

7.8 Operations with Sequences

It is also possible to define the four operations with sequences. Given the sequences

$$a_0, a_1, a_2, ..., a_n, ... \quad \text{and} \quad b_0, b_1, b_2, ..., b_n, ...$$

let's define the following operations.

Addition The sum of the two sequences is called the sequence:

$$a_0 + b_0, a_1 + b_1, a_2 + b_2, ..., a_n + b_n, ...$$

Subtraction The difference of the two sequences is called the sequence:

$$a_0 - b_0, a_1 - b_1, a_2 - b_2, ..., a_n - b_n, ...$$

Multiplication The product of the two sequences is called the sequence:

$$a_0 \cdot b_0, a_1 \cdot b_1, a_2 \cdot b_2, ..., a_n \cdot b_n, ...$$

Division If $b_n \neq 0$ for all $n \in \mathbb{N}$, the quotient of the two sequences is called the sequence:

$$\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_n}{b_n}, ...$$

Part II
Limit Calculus

8 Operations on Limits

There are various theorems related to operations on limits. The following theorems are valid both in the case of a limit as x approaches a finite value and in the case of a limit as x approaches $+\infty$ or $-\infty$. Therefore, when it is not important to distinguish, we will use

$$x \to \alpha$$

to denote any of the following notations:

$$x \to x_0 \quad x \to x_0^+ \quad x \to x_0^- \quad x \to +\infty \quad x \to -\infty$$

8.1 The Limit of the Algebraic Sum of Two Functions

Functions Have Finite Limits

Theorem 8.1. *If* $\lim_{x\to\alpha} f(x) = l$ *and* $\lim_{x\to\alpha} g(x) = m$*, where* $l, m \in \mathbb{R}$*, then:*

$$\lim_{x\to\alpha}[f(x) + g(x)] = \lim_{x\to\alpha} f(x) + \lim_{x\to\alpha} g(x) = l + m.$$

Note 7. The limit of the sum of two functions is equal to the sum of their limits.

Functions Do Not Both Have Finite Limits With the symbols $+\infty$ and $-\infty$, operations cannot be performed as if dealing with real numbers. The various cases that may arise in the calculation of the limit of the sum of two functions are summarized in the table below.

$f(x) + g(x)$	l	$+\infty$	$-\infty$
m	$m + l$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?
$-\infty$	$-\infty$	?	$-\infty$

In the table, it can be observed that cases where $+\infty$ and $-\infty$ are added do not result in 0, as one might erroneously expect. This is a **form of indecision** or **indeterminate form**.

8.2 The Limit of the Product of Two Functions

Functions Have Finite Limits

Theorem 8.2. *If* $\lim_{x\to\alpha} f(x) = l$ *and* $\lim_{x\to\alpha} g(x) = m$*, with* $l, m \in \mathbb{R}$*, then:*

$$\lim_{x\to\alpha}[f(x) \cdot g(x)] = \lim_{x\to\alpha} f(x) \cdot \lim_{x\to\alpha} g(x) = l \cdot m.$$

Note 8. The limit of the product of two functions is equal to the product of their limits.

Case 1. If $f(x)$ is a constant function k, we have:

$$\lim_{x\to\alpha} f(x) \cdot g(x) = \lim_{x\to\alpha} k \cdot \lim_{x\to\alpha} g(x) = k \cdot m$$

Functions Do Not Both Have Finite Limits If the functions do not both have finite limits, various cases may arise for the limit of the product, summarized in the table. Note that even when using the symbols $+\infty$ and $-\infty$, the sign rule still applies.

$f(x) \cdot g(x)$	$l > 0$	$l < 0$	0	$+\infty$	$-\infty$
$m > 0$	$m \cdot l$	$m \cdot l$	0	$+\infty$	$-\infty$
$m < 0$	$m \cdot l$	$m \cdot l$	0	$-\infty$	$+\infty$
0	0	0	0	?	?
$+\infty$	$+\infty$	$-\infty$	?	$+\infty$	$-\infty$
$-\infty$	$-\infty$	$+\infty$	?	$-\infty$	$+\infty$

8.3 The Limit of a Power

Theorem 8.3. *If $n \in \mathbb{N} - \{0\}$ and $\lim_{x\to\alpha} f(x) = l$, then:*

$$\lim_{x\to\alpha} [f(x)]^n = [\lim_{x\to\alpha} f(x)]^n = l^n$$

This theorem can also be extended to the case of a real exponent a different from 0. When a is a positive irrational number, $[f(x)]^a$ exists only if $f(x) \geq 0$, so $f(x)$ can only tend to a number > 0.

The Function Has a Limit of $+\infty$ We have the following table.

$f(x)$	a	$[f(x)]^a$
$+\infty$	$a > 0$	$(+\infty)^a = +\infty$
$+\infty$	$a < 0$	$(+\infty)^a = 0$

The Exponent is a Function The power rule can be extended to the case $[f(x)]^{g(x)}$, considering that the power $[f(x)]^{g(x)}$ exists only if $f(x)$ is > 0.

$[f(x)]^{g(x)}$	0	$+\infty$	$-\infty$
$+\infty$	?	$+\infty$	0
0	?	0	$+\infty$
1	1	?	?
$0 < l < 1$	1	0	$+\infty$
$l > 1$	1	$+\infty$	0

In the table, we find three indeterminate forms:

$$\infty^0 \quad 0^0 \quad 1^\infty$$

8.4 The Limit of the Reciprocal Function

Theorem 8.4. *Consider a function $f(x)$ and its reciprocal $\dfrac{1}{f(x)}$:*

- *If* $\lim_{x\to\alpha} f(x) = l \in \mathbb{R}, l \neq 0$, *then*

$$\lim_{x\to\alpha} \frac{1}{f(x)} = \frac{1}{\lim_{x\to\alpha} f(x)} = \frac{1}{l};$$

- *If* $\lim_{x\to\alpha} f(x) = +\infty$ *or* $\lim_{x\to\alpha} f(x) = -\infty$, *then*

$$\lim_{x\to\alpha} \frac{1}{f(x)} = 0;$$

- *If* $\lim_{x\to\alpha} f(x) = 0$, *then*

$$\lim_{x\to\alpha} \frac{1}{f(x)} = \infty.$$

8.5 The Limit of the Quotient of Two Functions

Functions Have Finite Limits, One of Which is Nonzero

Theorem 8.5. *If* $\lim_{x\to\alpha} f(x) = l$ *and* $\lim_{x\to\alpha} g(x) = m$, *where* $m \neq 0$, *then:*

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\alpha} f(x)}{\lim_{x\to\alpha} g(x)} = \frac{l}{m}$$

Proof. Since we can write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$, by the theorem of the limit of the reciprocal function and the limit of the product of two functions, we have:

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{x\to\alpha} f(x) \cdot \lim_{x\to\alpha} \frac{1}{g(x)} = \lim_{x\to\alpha} f(x) \cdot \frac{1}{\lim_{x\to\alpha} g(x)} = \frac{l}{m}.$$

□

Note 9. Note that, by the sign preservation theorem, if $m \neq 0$, then $g(x) \neq 0$ in an entire neighborhood of α.

Functions Do Not Both Have Finite Limits Various cases can arise as summarized in the following table.

$\frac{f(x)}{g(x)}$	$m \neq 0$	0	$+\infty$	$-\infty$
$l \neq 0$	$\frac{l}{m}$	∞	0	0
0	0	?	0	0
$+\infty$	∞	∞	?	?
$-\infty$	∞	∞	?	?

We encounter the **indeterminate forms**:

$$\frac{0}{0} \quad \frac{\infty}{\infty}$$

9 Indeterminate Forms

As we have seen, the indeterminate forms encountered in the calculation of limits are seven:

$$+\infty - \infty \quad \infty \cdot 0 \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^\infty \quad 0^0 \quad \infty^0$$

Let's now examine, through some examples, how to calculate limits that appear in indeterminate form.

9.1 The indeterminate form $+\infty - \infty$

Example 9.1. The limit $\lim_{x\to+\infty}(x - \sqrt{x^2+1})$ appears in the indeterminate form $+\infty - \infty$ because:

$$\lim_{x\to+\infty} x = +\infty \quad \text{and} \quad \lim_{x\to+\infty}(-\sqrt{x^2+1}) = -\infty$$

To calculate this limit, we can rewrite the given function such that in the limit's argument, the difference $x - \sqrt{x^2+1}$ disappears, and instead, the sum $x + \sqrt{x^2+1}$ appears. To do this, multiply and divide the function by $x + \sqrt{x^2+1}$:

$$x - \sqrt{x^2+1} = (x - \sqrt{x^2+1}) \cdot \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2+1}} = \frac{x^2 - (x^2+1)}{x + \sqrt{x^2+1}} = \frac{-1}{x + \sqrt{x^2+1}}$$

As x approaches $+\infty$, the denominator of the fraction $x + \sqrt{x^2+1}$ tends to $+\infty$. Therefore, by the reciprocal function limit theorem, the fraction tends to 0, i.e.:

$$\lim_{x\to+\infty}(x - \sqrt{x^2+1}) = \lim_{x\to+\infty} \frac{-1}{x + \sqrt{x^2+1}} = 0$$

Example 9.2. Let's calculate the following limit:

$$\lim_{x\to+\infty}(x^4 - 3x^2 + 1)$$

It appears in the indeterminate form $+\infty - \infty$. Factoring out the x^4 term, the limit becomes:

$$\lim_{x\to+\infty} x^4(1 - \frac{3}{x^2} + \frac{1}{x^4}).$$

Since $\lim_{x\to+\infty}(-\frac{3}{x^2}) = 0$ and $\lim_{x\to+\infty} \frac{1}{x^4} = 0$, we have

$$\lim_{x\to+\infty}(1 - \frac{3}{x^2} + \frac{1}{x^4}) = 1.$$

Also, we know that $\lim_{x\to+\infty} x^4 = +\infty$, therefore, by the product limit theorem in the case of a finite limit (different from 0) and an infinite limit, we get:

$$\lim_{x\to+\infty} x^4(1 - \frac{3}{x^2} + \frac{1}{x^4}) = +\infty$$

The procedure used in Example 2 generalizes as follows.

The limit of a polynomial function In general, to calculate the limit of a polynomial function as x approaches $+\infty$ (or x approaches $-\infty$),

$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + ... + a_n),$$

we proceed as follows:

- Factor out x^n:
$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + ... + a_n)$$

- Since, for x approaching $+\infty$ or $-\infty$, the limit of $\frac{a_1}{x}, \frac{a_2}{x^2}, ..., \frac{a_n}{x^n}$ is 0, we have
$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + ... + a_n) = \lim_{x\to\pm\infty} a_0x^n.$$
This limit is $+\infty$ or $-\infty$. The sign is determined by applying the sign rule to the product a_0x^n.

9.2 The indeterminate form $\infty \cdot 0$

Example 9.3. Let's calculate the following limit:

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x)\cdot\tan x.$$

Through direct calculation, we obtain the indeterminate form $0\cdot\infty$ because:

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x) = 0 \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}^-}\tan x = +\infty.$$

Recall that $\tan x = \frac{\sin x}{\cos x}$ and multiply and divide the given function by $(1 + \sin x)$:

$$\begin{aligned}
&(1-\sin x)\cdot\tan x\cdot\frac{1+\sin x}{1+\sin x} \\
&= \frac{(1-\sin x)(1+\sin x)}{1+\sin x}\cdot\tan x \\
&= \frac{1-\sin^2 x}{1+\sin x}\cdot\frac{\sin x}{\cos x} \\
&= \frac{\cos^2 x}{1+\sin x}\cdot\frac{\sin x}{\cos x} \\
&= \frac{\sin x\cdot\cos x}{1+\sin x}
\end{aligned}$$

As x approaches $\frac{\pi}{2}^-$, the numerator $\sin x\cdot\cos x$ tends to 0, while the denominator $1+\sin x$ tends to 2. Therefore, by the limit quotient rule, the fraction tends to $\frac{0}{2}$, i.e., 0:

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x)\cdot\tan x = \lim_{x\to\frac{\pi}{2}^-}\frac{\sin x\cdot\cos x}{1+\sin x} = 0.$$

9.3 The indeterminate form $\frac{0}{0}$

Example 9.4. Let's calculate the limit

$$\lim_{x\to 3}\frac{x^2-2x-3}{2x^2-9x+9}$$

which is in the indeterminate form $\frac{0}{0}$ because:

$$\lim_{x\to 3}(x^2-2x-3)=0 \quad\text{and}\quad \lim_{x\to 3}(2x^2-9x+9)=0.$$

Since the value 3 makes both the numerator and the denominator zero, we factorize both:

$$x^2-2x-3\to(x-3)(x+1)$$

$$2x^2-9x+9\to(x-3)(2x-3)$$

$$\lim_{x\to 3}\frac{x^2-2x-3}{2x^2-9x+9}=\lim_{x\to 3}\frac{(x-3)(x+1)}{(x-3)(2x-3)}=\lim_{x\to 3}\frac{(x+1)}{(2x-3)}=\frac{4}{3}$$

9.4 The indeterminate form $\frac{\infty}{\infty}$

The limit of a fractional rational function as $x\to\infty$ Given the limit

$$\lim_{x\to\pm\infty}\frac{a_0x^n+a_1x^{n-1}+...+a_n}{b_0x^m+b_1x^{m-1}+...+b_m}$$

when at least one coefficient of the powers of x is nonzero both in the numerator and denominator, this limit presents itself in the form $\frac{\infty}{\infty}$ because both the numerator and denominator tend to ∞ as x approaches ∞. Here are three examples of limit calculations with $n>m,\ n=m,\ n<m$.

The degree of the numerator is greater than the degree of the denominator

Example 9.5. Let's calculate the limit

$$\lim_{x\to+\infty}\frac{x^5-2x^2+1}{3x^2-2x+6}.$$

Factor out x^5 in the numerator and x^2 in the denominator:

$$\lim_{x\to+\infty}\frac{x^5\cdot(1-\frac{2}{x^3}+\frac{1}{x^5})}{x^2\cdot(3-\frac{2}{x}+\frac{6}{x^2})}=\lim_{x\to+\infty}x^3\frac{(1-\frac{2}{x^3}+\frac{1}{x^5})}{(3-\frac{2}{x}+\frac{6}{x^2})}$$

We have $\lim_{x\to+\infty}x^3=+\infty$, $\lim_{x\to+\infty}(1-\frac{2}{x^3}+\frac{1}{x^5})=1$, $\lim_{x\to+\infty}(3-\frac{2}{x}+\frac{6}{x^2})=3$. Therefore, $\lim_{x\to+\infty}\frac{x^5-2x^2+1}{3x^2-2x+6}=+\infty$

The degree of the numerator is equal to the degree of the denominator

Example 9.6. Let's calculate the limit $\lim_{x\to\pm\infty} \frac{1-2x^2}{3x^2+2x-5}$. Factor out x^2 both in the numerator and the denominator:

$$\lim_{x\to\pm\infty} \frac{x^2 \cdot (\frac{1}{x^2} - 2)}{x^2 \cdot (3 + \frac{2}{x} - \frac{5}{x^2})} = \lim_{x\to\pm\infty} \frac{(\frac{1}{x^2} - 2)}{(3 + \frac{2}{x} - \frac{5}{x^2})}$$

By the limit quotient rule, the fraction tends to $-\frac{2}{3}$, therefore:

$$\lim_{x\to\pm\infty} \frac{1-2x^2}{3x^2+2x-5} = -\frac{2}{3}.$$

Note that $-\frac{2}{3}$ is the ratio of the coefficients of the highest degree term, i.e., x^2, in the numerator and denominator.

The degree of the numerator is less than the degree of the denominator

Example 9.7. Let's calculate the limit

$$\lim_{x\to-\infty} \frac{2x-1}{x^3+2x}.$$

Factor out x in the numerator and x^3 in the denominator:

$$\lim_{x\to-\infty} \frac{x \cdot (2 - \frac{1}{x})}{x^3 \cdot (1 + \frac{2}{x^2})} = \lim_{x\to-\infty} \frac{1}{x^2} \cdot \frac{(2 - \frac{1}{x})}{(1 + \frac{2}{x^2})}$$

We have $\lim_{x\to-\infty} \frac{1}{x^2} = 0$, $\lim_{x\to-\infty}(2 - \frac{1}{x}) = 2$, $\lim_{x\to-\infty}(1 + \frac{2}{x^2}) = 1$ Therefore,

$$\lim_{x\to-\infty} \frac{2x-1}{x^3+2x} = 0$$

In general, given a fractional rational function

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m},$$

with the numerator of degree n and the denominator of degree m, we have:

$$\lim_{x\to\pm\infty} \frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m} = \begin{cases} \pm\infty & \text{if } n > m \\ \frac{a^0}{b^0} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

The sign of ∞ in the case $n > m$ is given by the product of the signs of:

$$\lim_{x\to\pm\infty} x^{n-m} \quad \text{and} \quad \frac{a^0}{b^0}$$

10 Noteworthy Limits

Let's illustrate two particular limits, called noteworthy because they are fundamental in the applications of analysis.

A first noteworthy limit Consider

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

Since $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} x = 0$, we are dealing with the indeterminate form $\frac{0}{0}$. We prove that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Proof. Note that the function $\frac{\sin x}{x}$ is even since

$$\frac{\sin(-x)}{x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x},$$

making it symmetric with respect to the y-axis. Thus, we conclude that

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x}$$

and we can limit ourselves to proving the case

$$\lim_{x\to 0^+} \frac{\sin x}{x}.$$

Consider the unit circle and a positive angle of measure x. If x is in radians, its measure coincides with that of $\widehat{AP}$, while the measure of PQ is $\sin x$, and that of TA is $\tan x$. Since

$$\overline{PQ} < \widehat{AP} < \overline{TA},$$

we have

$$\sin x < x < \tan x.$$

Dividing the terms of the inequality by $\sin x$ yields,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

and taking reciprocals gives $\cos x < \frac{\sin x}{x} < 1$. The function $\frac{\sin x}{x}$ is sandwiched between the functions $\cos x$ and the constant function 1. We can apply the comparison theorem: since $\lim_{x\to 0} \cos x = 1$, the function $\frac{\sin x}{x}$ is sandwiched between two functions that tend to 1 as $x \to 0$, so it also tends to 1. □

From this noteworthy limit, we deduce the following limits, which are also in the indeterminate form $\frac{0}{0}$.

1. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Proof. Multiplying the numerator and denominator of $\frac{1-\cos x}{x}$ by $1+\cos x$, we obtain

$$\frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x} \cdot \sin x \cdot \frac{1}{1+\cos x},$$

and thus, by the limit product theorem:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \sin x \cdot \frac{1}{1+\cos x} = 1 \cdot 0 \cdot \frac{1}{2} = 0$$

□

2. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

   Proof. Applying the previous reasoning, we can write:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{1+\cos x} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$$

□

Another Remarkable Limit

$$\lim_{x\to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

From this remarkable limit, we can deduce others, which are in the indeterminate form $\frac{0}{0}$.

1. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

   Proof. Applying logarithm properties, we can write

$$\frac{\ln(1+x)}{x} = \frac{1}{x} \ln(1+x) = \ln(1+x)^{\frac{1}{x}}$$

   and thus, due to the continuity of the logarithmic function:

$$\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}] = \ln(\lim_{x\to 0} (1+x)^{\frac{1}{x}}).$$

   Now, let $y = \frac{1}{x}$, then $x = \frac{1}{y}$, and for $x \to 0$, we have $y \to \pm\infty$. Making the variable substitution in the previous limit, we get:

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \ln(\lim_{y\to \pm\infty} \left(1 + \frac{1}{y}\right)^y) = \ln e = 1.$$

□

2. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

Proof. Let $y = e^x - 1$, then $e^x = 1 + y$, and $x = \ln(1+y)$. Also, for $x \to 0$, we have $y \to 0$, so substituting the variable x yields:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{y}{\ln(1+y)} = \lim_{y\to 0} \frac{1}{\frac{\ln(1+y)}{y}} = \frac{1}{1} = 1,$$

by the reciprocal function limit theorem. □

Note 10. More generally, if $a > 0$:

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a.$$

Other Remarkable Limits

- Trigonometric functions
  - $$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$$
  - $$\lim_{x\to 0} \frac{\arcsin(x)}{x} = 1$$
  - $$\lim_{x\to 0} \frac{\arctan(x)}{x} = 1$$
- Exponential and logarithmic functions
  - $$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$
  - $$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$$
  - $$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$
  - $$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$$
  - $$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$
  - $$\lim_{x\to 0^+} x^\alpha \ln(x) = 0 \qquad \lim_{x\to +\infty} \frac{\ln(x)}{x^\alpha} = 0 (\alpha > 0)$$
  - $$\lim_{x\to +\infty} \frac{x^\alpha}{\alpha^x} = 0 \qquad \lim_{x\to +\infty} \frac{\ln(x)}{a^x} = 0 (a > 1)$$

10.1 Fundamental limits

$$\lim_{x\to+\infty} c = c \qquad \lim_{x\to-\infty} c = c$$

$$\lim_{x\to+\infty} x = +\infty \qquad \lim_{x\to-\infty} x = -\infty$$

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to+\infty} e^x = +\infty$$

$$\lim_{x\to(\frac{\pi}{2}+ka)} \tan x = -\infty \qquad \lim_{x\to(\frac{\pi}{2}+ka)} \tan x = +\infty$$

$$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2} \qquad \lim_{x\to+\infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x\to 0} \log x = -\infty \qquad \lim_{x\to+\infty} \log x = +\infty$$

11 Infinitesimals, Infinities, and Their Comparison

11.1 Infinitesimals

Infinitesimal as $x \to \alpha$

Definition 11.1. A function $f(x)$ is called an infinitesimal as $x \to \alpha$ when the limit of $f(x)$ as $x \to \alpha$ is equal to 0.

Remark 1. α can be finite, $+\infty$, or $-\infty$.

If $f(x)$ and $g(x)$ are both infinitesimals as $x \to \alpha$, they are called **simultaneous infinitesimals**. In this case, it is interesting to see which of the two infinitesimals tends to 0 *more rapidly*; we can establish this by determining the limit (if it exists) of their ratio as $x \to \alpha$. Let $f(x)$ and $g(x)$ be two simultaneous infinitesimals as $x \to \alpha$, and assume that there exists an interval I around α such that $g(x) \neq 0$ for every $x \in I$, with $x \neq \alpha$.

- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = l \neq 0$ (l finite), $f(x)$ and $g(x)$ are said to be **of the same order** (essentially meaning they tend to 0 at the same rate).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 0$, $f(x)$ is called a **higher-order infinitesimal** compared to $g(x)$ (i.e., f tends to 0 more rapidly than g).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \pm\infty$, $f(x)$ is called a **lower-order infinitesimal** compared to $g(x)$ (i.e., f tends to 0 less rapidly than g).
- If *the limit* $\lim_{x\to\alpha} \frac{f(x)}{g(x)}$ *does not exist*, the **infinitesimals** $f(x)$ and $g(x)$ **are not comparable**.

11.2 Infinity

Infinity as $x \to \alpha$

Definition 11.2. A function $f(x)$ is called infinity as $x \to \alpha$ when the limit of $f(x)$ as $x \to \alpha$ is $+\infty$, $-\infty$, or ∞.

For infinities, we can introduce concepts analogous to those seen for infinitesimals. If both $f(x)$ and $g(x)$ are infinities as $x \to \alpha$, they are called **simultaneous infinities**. Let $f(x)$ and $g(x)$ be simultaneous infinities as $x \to \alpha$.

- If $\lim_{x\to\alpha} = l \neq 0$ (l finite), $f(x)$ and $g(x)$ are said to be **of the same order** (essentially meaning they tend to ∞ at the same rate).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 0$, $f(x)$ is called a **lower-order infinity** compared to $g(x)$ (i.e., f tends to ∞ less rapidly than g).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \pm\infty$, $f(x)$ is called a **higher-order infinity** compared to $g(x)$ (i.e., f tends to ∞ more rapidly than g).

- If *the limit* $\lim_{x\to\alpha} \frac{f(x)}{g(x)}$ *does not exist*, the **infinities** $f(x)$ and $g(x)$ **are not comparable**.

12 Limits of Sequences

For sequences, which are special functions, all the theorems of limit calculus apply. Thus, given sequences a_n and b_n, if $\lim_{n\to+\infty} = l$ and $\lim_{n\to+\infty} b_n = l'$, the following theorems hold.

- **Sum of Limits Theorem**: $\lim_{n\to+\infty}(a_n + b_n) = l + l'$;
- **Difference of Limits Theorem**: $\lim_{n\to+\infty}(a_n - b_n) = l - l'$;
- **Product of Limits Theorem**: $\lim_{n\to+\infty}(a_n \cdot b_n) = l \cdot l'$;
- **Quotient of Limits Theorem**: if $b \neq 0, \forall n \in \mathbb{N}$ and $l' \neq 0$, then $\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{l}{l'}$;

These theorems are analogous to those studied for functions as $x \to +\infty$. Similar theorems are also valid when one or more sequences are divergent.

13 Continuous Functions

13.1 Definition of Continuous Function

Continuous Function at a Point

Definition 13.1. Let $f(x)$ be a function defined in an interval $[a; b]$, and x_0 be a point within the interval. The function $f(x)$ is said to be continuous at the point x_0 when the limit of $f(x)$ as x approaches x_0 exists and is equal to the value $f(x_0)$ of the function evaluated at x_0:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

.

Note 11. Applying the definition of limit, $f(x)$ is continuous at x_0 if, for all $\epsilon > 0$, there exists a complete neighborhood I of x_0 such that

$$|f(x) - f(x_0)| < \epsilon, \ \forall x \in I$$

A function $f(x)$ is thus continuous at x_0 if:

- it is defined at x_0, meaning $f(x_0)$ exists;
- the limit $\lim_{x \to x_0} f(x)$ is finite;
- the value of the limit equals $f(x_0)$.

If we consider only the right or left limit of a function $f(x)$, we can give the following definitions:

- $f(x)$ is **right-continuous** at x_0 if $f(x_0)$ coincides with the right limit of $f(x)$ as x approaches x_0:

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

- $f(x)$ is **left-continuous** at x_0 if $f(x_0)$ coincides with the left limit of $f(x)$ as x approaches x_0:

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

Continuity can also be discussed for points that are endpoints of the interval $[a; b]$ where the function is defined; at point a, we talk about right-continuity, while at point b, we talk about left-continuity.

Continuous Function in an Interval

Definition 13.2. A function defined in $[a;b]$ is said to be continuous in the interval $[a;b]$ if it is continuous at every point within the interval.

Functions that are rational, irrational (integer and fractional), exponential, logarithmic, and trigonometric are continuous in every interval of their domain. Moreover, if $f(x)$ and $g(x)$ are functions continuous at a point or in an interval, then the following functions are also continuous at the same point or interval:

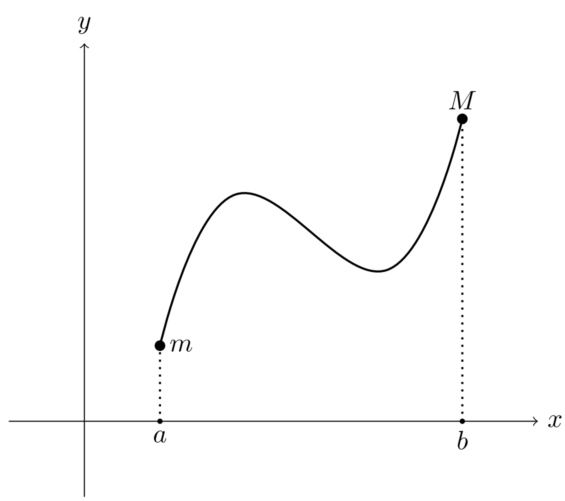
$$f(x) \pm g(x), \quad kf(x), \quad f(x) \cdot g(x), \quad [f(x)]^n, \quad \frac{f(x)}{g(x)}, \quad \text{with } g(x) \neq 0, k \in \mathbb{R}, n \in \mathbb{N} - \{0\}.$$

13.2 Theorems on Continuous Functions

Let's state some theorems that express important properties enjoyed by continuous functions and illustrate their graphical consequences.

Weierstrass' Theorem

Theorem 13.1. *If f is a continuous function in a bounded and closed interval $[a;b]$, then it takes on both the absolute maximum and the absolute minimum within that interval.*



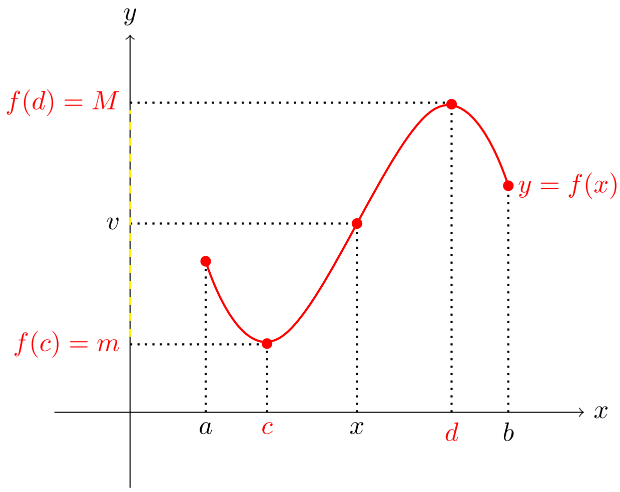
Note 12. Given the function $y = f(x)$ defined in the interval I, we call:

- the **absolute maximum** of $f(x)$, if it exists, the maximum M of the values assumed by the function in I;
- the **absolute minimum** of $f(x)$, if it exists, the minimum m of the values assumed by the function in I.

Intermediate Value Theorem

Theorem 13.2. *If f is a continuous function in a bounded and closed interval $[a;b]$, then it takes on every value at least once between the maximum and minimum. In symbols:*

$$\text{Let } f \text{ be continuous in } [a;b] \Rightarrow \forall v : m \leq v \leq M, \exists x \in [a;b] : f(x) = v$$

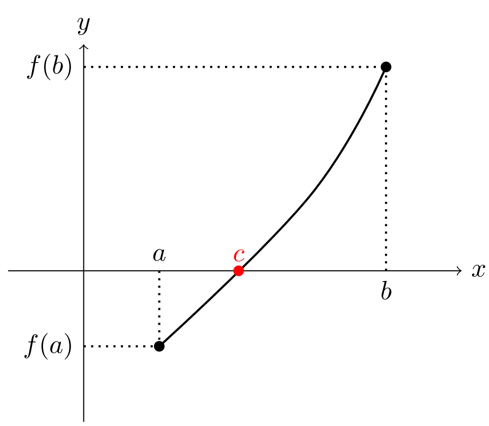


Existence of Zeros Theorem

Theorem 13.3. *If f is a continuous function in a bounded and closed interval $[a;b]$ and assumes values of opposite signs at the endpoints of this interval, then there exists at least one point c, within the interval, where f equals zero.*

In symbols:

$$f \text{ continuous in } [a;b], \quad f(a) < 0, f(b) > 0 \Rightarrow \exists c \in]a;b[: f(c) = 0$$



14 Points of Discontinuity of a Function

A point x_0 in an interval $[a; b]$ is called a **point of discontinuity** for a function $f(x)$ if the function is not continuous at x_0.

Note 13. A point of discontinuity is also called a **singular point**.

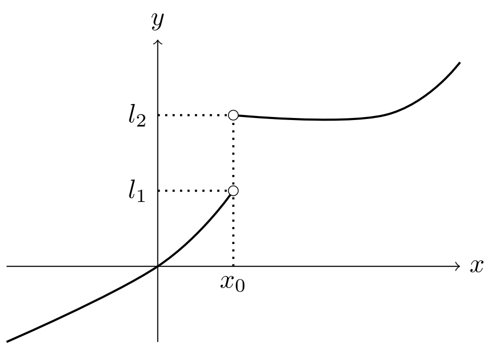
14.1 Points of First Kind Discontinuity

Point of First Kind Discontinuity

Definition 14.1. A point x_0 is called a point of first kind discontinuity for the function $f(x)$ when, as $x \to x_0$, both the right and left limits of $f(x)$ exist but are different from each other.

$$\lim_{x \to x_0^-} f(x) = l_1 \neq \lim_{x \to x_0^+} f(x) = l_2.$$

The difference $|l_2 - l_1|$ is called the **jump** of the function.



14.2 Points of Second Kind Discontinuity

Point of Second Kind Discontinuity

Definition 14.2. A point x_0 is called a point of second kind discontinuity for the function $f(x)$ when, as $x \to x_0$, at least one of the two limits, right or left, of $f(x)$ is infinite, or one of them does not exist.

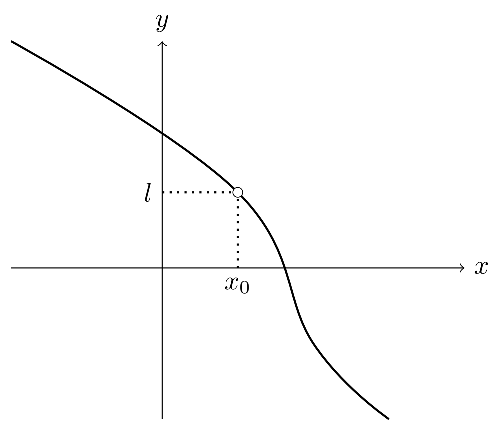
14.3 Points of Third Kind Discontinuity (or removable)

Point of Third Kind Discontinuity (or removable)

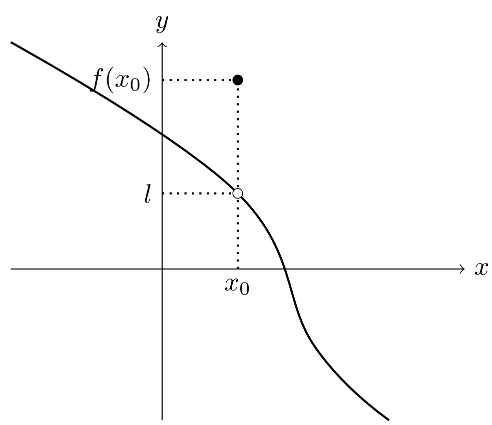
Definition 14.3. A point x_0 is called a point of third kind discontinuity for the function $f(x)$ when:

1. the limit of $f(x)$ as $x \to x_0$ exists and is finite, i.e., $\lim_{x \to x_0} f(x) = l$;
2. f is either undefined at x_0, or if defined, $f(x_0) \neq l$.

- f is undefined at x_0



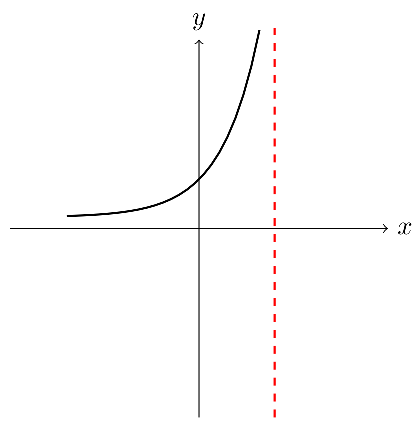
- f is defined at x_0, but $f(x_0) \neq l$



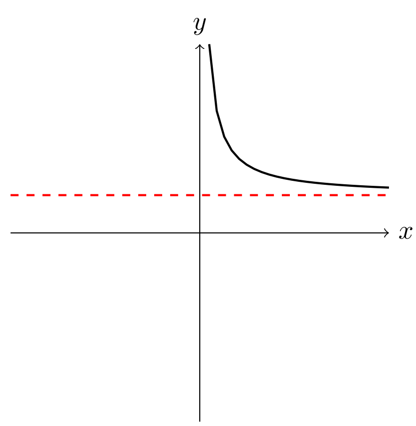
15 Asymptotes

An asymptote of a function $f(x)$ is a line whose distance from the graph of $f(x)$ tends to 0 as a generic point P on the graph moves towards infinity.

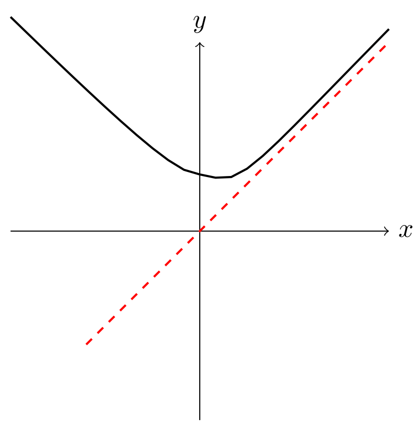
- **Vertical asymptotes**



- **Horizontal asymptotes**



- **Oblique Asymptotes**



15.1 Search for Horizontal and Vertical Asymptotes

In general, horizontal asymptotes are determined by calculating $\lim_{x\to\infty} f(x)$, while vertical asymptotes are determined by calculating $\lim_{x\to x_0} f(x)$, where x_0 does not belong to the domain.

Note 14. A horizontal asymptote with equation $y = c$ occurs when:

$$\lim_{x\to\infty} f(x) = c.$$

Note 15. A vertical asymptote with equation $x = x_0$ occurs when:

$$\lim_{x\to x_0} f(x) = \infty.$$

15.2 Oblique Asymptotes

Oblique Asymptote

Definition 15.1. Given the function $y = f(x)$, if it is verified that

$$\lim_{x\to\infty} [f(x) - (mx + q)] = 0,$$

then the line with equation $y = mx + q$ is called an oblique asymptote for the graph of the function.

Note 16. From

$$\lim_{x\to\infty} [f(x) - (mx + q)] = 0,$$

we deduce

$$\lim_{x\to\infty} = \lim_{x\to\infty} (mx + q),$$

leading to:

$$\lim_{x\to\infty} f(x) = \infty,$$

a necessary (but not sufficient) condition for the existence of the oblique asymptote.

The same definition applies by replacing ∞ with $+\infty$ or $-\infty$. For $x \to +\infty$, it is called a right oblique asymptote, and for $x \to -\infty$, it is called a left oblique asymptote.

15.3 Search for Oblique Asymptotes

Theorem 15.1. *If the graph of the function $y = f(x)$ has an oblique asymptote with equation $y = mx + q$, where $m \neq 0$, then m and q are given by the following limits:*

$$m = \lim_{x\to\infty} \frac{f(x)}{x}; \quad q = \lim_{x\to\infty} [f(x) - mx]$$

Proof. If an oblique asymptote exists, it is true that

$$\lim_{x\to\infty} [f(x) - (mx + q)] = 0,$$

and therefore, dividing by $x \neq 0$.

$$\lim_{x\to\infty} \frac{f(x) - (mx + q)}{x} = 0 \to \lim_{x\to\infty} [\frac{f(x)}{x} - m - \frac{q}{x}] = 0,$$

and, since $\lim_{x\to\infty} m = m$ and $\lim_{x\to\infty} \frac{q}{x} = 0$, it must be:

$$m = \lim_{x\to\infty} \frac{f(x)}{x}.$$

If m is not zero, to calculate q, consider again:

$$\begin{aligned}\lim_{x\to\infty} [f(x) - (mx + q)] = 0 &\to \lim_{x\to\infty} [(f(x) - mx) - q] = 0 \\ &\to \lim_{x\to\infty} [f(x) - mx] - q = 0 \\ &\to q = \lim_{x\to\infty} [f(x) - mx].\end{aligned}$$

□

Conversely, it can be shown that if $\lim_{x\to\infty} f(x) = \infty$ and the limits $m = \lim_{x\to\infty} \frac{f(x)}{x}$ and $q = \lim_{x\to\infty}[f(x) - mx]$ exist and are finite, with $m \neq 0$, then the graph of the function $y = f(x)$ has an oblique asymptote with equation $y = mx + q$.

Note 17. A function can have an oblique asymptote only if

$$\lim_{x\to\infty} f(x) = \infty,$$

or one of the analogous limits with $+\infty$ or $-\infty$.

Moreover, the theorem is valid even if we replace ∞ with $+\infty$ or $-\infty$.