Linear Combinations

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Let $\vec{v}_1, \ldots, \vec{v}_n \in V$ and $a_1, ..., a_n \in \mathrm{K}$. The vector

$$a_1\vec{v}_1 + ... + a_n\vec{v}_n \tag{a}$$

is called a *linear combination of the vectors* $\vec{v}_1, \ldots, \vec{v}_n$, and $a_1, ..., a_n$ are called *coefficients of the linear combination.* If all coefficients are equal to 0, then (a) equals the zero vector $\vec{0}$ and is called a *trivial linear combination of* $\vec{v}_1, \ldots, \vec{v}_n$. Every linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ where the coefficients are not all zero is called *non-trivial.*

Proposition 0.1. *Let $\{\vec{v}_1, ..., \vec{v}_n\}$ be a finite subset of vectors in V. The set*

$$\langle \vec{v}_1, ..., \vec{v}_n \rangle$$

consisting of all linear combinations of $\vec{v}_1, ..., \vec{v}_n$ is a subspace of V. It is equal to the intersection of all subspaces of V containing $\{\vec{v}_1, ..., \vec{v}_n\}$.

We call $\langle \vec{v}_1, ..., \vec{v}_n \rangle$ the subspace generated by $\vec{v}_1, ..., \vec{v}_n$. Note that if $1 \leq m < n$, the subspace $\langle \vec{v}_1, ..., \vec{v}_m \rangle$ is contained in $\langle \vec{v}_1, ..., \vec{v}_n \rangle$, because every linear combination of $\vec{v}_1, ..., \vec{v}_m$ is also a linear combination of $\vec{v}_1, ..., \vec{v}_n$.

$$a_1\vec{v}_1 + ... + a_m\vec{v}_m = a_1\vec{v}_1 + ... + a_m\vec{v}_m + 0\vec{v}_{m+1} + ... + 0\vec{v}_n.$$

We say that $\vec{v}_1, ..., \vec{v}_n$ generate V, or that $\{\vec{v}_1, ..., \vec{v}_n\}$ is a *set of generators* of V, if $\langle \vec{v}_1, ..., \vec{v}_n \rangle = V$.
In symbols,

$$\vec{v}_1, ..., \vec{v}_n \text{ generate } V \iff \forall\, \vec{v} \in V,\ \exists\, a_1, ..., a_n \in K \,:\, \vec{v} = a_1\vec{v}_1 + ... + a_n\vec{v}_n.$$

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Linearly Dependent and Independent Vectors

Definition 0.1. Vectors $\vec{v}_1, ..., \vec{v}_n \in V$ are called linearly dependent if there exist scalars $a_1, ..., a_n \in K$, not all zero, such that

$$a_1\vec{v}_1 + ... + a_n\vec{v}_n = \vec{0},$$

or equivalently, if the vector $\vec{0}$ can be expressed as a nontrivial linear combination of them.

Otherwise, $\vec{v}_1, ..., \vec{v}_n$ are called linearly independent.

Fundamental Properties of Linearly Dependent and Independent Vectors

Proposition 0.2. *A vector $\vec{v}$ is linearly dependent if and only if $\vec{v} = 0$.*

Proof. The proof is obvious. □

Proposition 0.3. *If $\vec{v}_1$ and $\vec{v}_2$ are two vectors such that $\vec{v}_2$ is proportional to $\vec{v}_1$, i.e., such that $\vec{v}_2 = a\vec{v}_1$ for some $a \in K$, then $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent.*

Proof. Indeed, $a\vec{v}_1 - \vec{v}_2 = \vec{0}$ is a linear combination of them with coefficients a and -1, and therefore not both zero.

Conversely, if $\vec{v}_1$ and $\vec{v}_2$ are two linearly dependent vectors, then one of them is a multiple of the other. Specifically, $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{0}$, i.e., $a_2\vec{v}_2 = -a_1\vec{v}_1$, with $(a_1, a_2) \neq (0, 0)$, implies, assuming $a_2 \neq 0$, that $\vec{v}_2 = a\vec{v}_1$, where $a = -a_1a_2^{-1}$. □

Proposition 0.4. *$\vec{v}_1, ..., \vec{v}_n \in V$ $n \leq 2$ are linearly dependent if and only if at least one of them can be expressed as a linear combination of the others.*

Proof. If $\vec{v}_1, ..., \vec{v}_n$ are linearly dependent, then $\vec{0} = a_1\vec{v}_1 + ... + a_n\vec{v}_n$, with $a_i \neq 0$ for some i; hence

$$a_i\vec{v}_i = -(a_1\vec{v}_1 + ... + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + ... + a_n\vec{v}_n)$$

that is,

$$\vec{v}_i = -a_i^{-1}(a_1\vec{v}_1 + ... + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + ... + a_n\vec{v}_n)$$

$$\vec{v}_i = -a_i^{-1}a_1\vec{v}_1 - ... - a_i^{-1}a_{i-1}\vec{v}_{i-1} - a_i^{-1}a_{i+1}\vec{v}_{i+1} - ... - a_i^{-1}a_n\vec{v}_n$$

Conversely, if for some i

$$\vec{v}_i = b_1\vec{v}_1 + ... + b_{i-1}\vec{v}_{i-1} + b_{i+1}\vec{v}_{i+1} + ... + b_n\vec{v}_n$$

then $\vec{v}_1, ..., \vec{v}_n$ are linearly dependent. □

Proposition 0.5. *If the set* $\{\vec{v}_1, ..., \vec{v}_n\}$ *contains the vector* $\vec{0}$*, then* $\vec{v}_1, ..., \vec{v}_n$ *are linearly dependent.*

Proof. Suppose we have $\vec{v}_i = \vec{0}$ for $1 \leq i \leq n$. Then we have

$$0\vec{v}_1 + ... + 0\vec{v}_{i-1} + 1\vec{v}_i + 0\vec{v}_{i+1} + ... + 0\vec{v}_n = \vec{v}_i = \vec{0},$$

and $\vec{0}$ is a nontrivial linear combination of $\vec{v}_1, ..., \vec{v}_n$. □

Proposition 0.6. *If* $\vec{v}_1, ..., \vec{v}_n$ *are linearly independent, and* $a_1, ..., a_n, b_1, ..., b_n \in K$ *are such that*

$$a_1\vec{v}_1 + .. + a_n\vec{v}_n = b_1\vec{v}_1 + .. + b_n\vec{v}_n,$$

then $a_1 = b_1,\ a_2 = b_2,\ ...,\ a_n = b_n$.

Proof. Since

$$\vec{0} = a_1\vec{v}_1 + .. + a_n\vec{v}_n - (b_1\vec{v}_1 + .. + b_n\vec{v}_n) = (a_1 - b_1)\vec{v}_1 + ... + (a_n - b_n)\vec{v}_n$$

we must have

$$a_1 - b_1 = ... = a_n - b_n = 0.$$

□