RIOT
INVESTMENT SOCIETY

Essentials Calculus I Concepts

Limits, Derivatives, Function Analysis, Integrals

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February 7, 2024

Contents

Part I
Limits

1 Neighborhoods

Let's present some fundamental notions of the topology of the set $\mathbb{R}$ of real numbers concerning their particular subsets.

Since there is a one-to-one correspondence between $\mathbb{R}$ and the points of an oriented line r, called the **real line**, we can identify every subset of $\mathbb{R}$ with a *subset of points* of the line r, and thus also talk about the topology of the line.

1.1 Neighborhoods of a Point

Complete Interior

Definition 1.1. Given a real number x_0, a **complete neighborhood** of x_0 is any open interval $I(x_0)$ containing x_0:

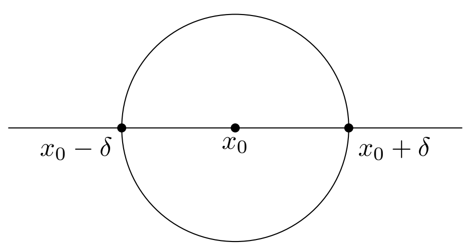
$$I(x_0) =]x_0 - \delta_1; x_0 + \delta_2[\quad \text{with } \delta_1, \delta_2 \in \mathbb{R}^+.$$

When $\delta_1 = \delta_2$, the point x_0 is the midpoint of the interval. In this case, we talk about the *circular neighborhood* of x_0.

Circular Neighborhood

Definition 1.2. Given a real number x_0 and a positive real number δ, the **circular neighborhood of x_0 with radius δ** is the open interval $I_\delta(x_0)$ centered at x_0 with radius δ:

$$I_\delta(x_0) =]x_0 - \delta; x_0 + \delta[= \{x \in \mathbb{R} \mid |x - x_0| < \delta\}$$



The set

$$J_{x_0} = \{I_\delta(x_0) | \delta > 0\}$$

is called the **collection of neighborhoods of x_0**.

Since the circular neighborhood of x_0 with radius δ is the set of points $x \in \mathbb{R}$ such that

$$x_0 - \delta < x < x_0 + \delta,$$

or equivalently

$$-\delta < x - x_0 < \delta,$$

we can also write:

$$I_\delta(x_0) = \{x \in \mathbb{R} \; : \; |x - x_0| < \delta\}$$

For complete and circular neighborhoods of a point x_0, the following property holds:

Proposition 1.1. *The intersection and union of two or more neighborhoods of x_0 are still neighborhoods of x_0.*

Right and Left Neighborhoods of a Point Given a neighborhood of a point x_0, sometimes we are interested in considering only the part of the neighborhood that is to the right of x_0 or the part that is to the left. In general, given a number $\delta \in \mathbb{R}^+$, we call:

- **right neighborhood** of x_0 the interval

$$I_\delta^+(x_0) =]x_0; x_0 + \delta[$$

- **left neighborhood** of x_0 the interval

$$I_\delta^-(x_0) =]x_0 - \delta; x_0[$$

1.2 Neighborhoods of Infinity

Given $a, b \in \mathbb{R}$, with $a < b$, we call:

- **neighborhood of negative infinity** any open interval unlimited below:

$$I(-\infty) =]-\infty; a[= \{x \in \mathbb{R} | x < a\};$$

- **neighborhood of positive infinity** any open interval unlimited above:

$$I(+\infty) =]b; +\infty[= \{x \in \mathbb{R} | x > b\};$$

We also define **neighborhood of infinity** as the union of a neighborhood of $-\infty$ and a neighborhood of $+\infty$, i.e.:

$$I(\infty) = I(-\infty) \cup I(+\infty) = \{x \in \mathbb{R} \mid x < a \vee x > b\}$$

Similar to the case of a real point x_0, we can talk about a **circular neighborhood of infinity**:

$$I_c(\infty) =]-\infty; -c[\cup]c; +\infty[\quad \text{with } c \in \mathbb{R}$$

1.3 Accumulation Points

Accumulation Point

Definition 1.3. The real number x_0 is called an accumulation point of A, a subset of $\mathbb{R}$, if every complete neighborhood of x_0 contains infinitely many points of A.

Note 1. The term *accumulation* indicates that the points of A gather around x_0.

Every point in an interval is an accumulation point for the interval itself. The endpoints of the interval are also its accumulation points. Alternatively, we can say that x_0 is an accumulation point of A if every complete neighborhood of x_0 contains at least one element of A distinct from x_0.

2 Definition of $\lim_{x \to x_0} f(x) = l$

Finite Limit as x Approaches x_0

Definition 2.1. We say that the function $f(x)$ has the real number l as its limit, as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = l,$$

when, for every positive real number ϵ, it is possible to determine a complete neighborhood I of x_0 such that

$$l - \epsilon < f(x) < l + \epsilon, \quad \text{or } |f(x) - l| < \epsilon,$$

for every x belonging to I, different (at most) from x_0.

Note 2. The validity of the condition $|f(x) - l| < \epsilon$ assumes that $f(x)$ is defined in I (excluding at most x_0). The point x_0 is an accumulation point for the domain of the function. We are not concerned with the value that the function $f(x)$ may assume at x_0.

In symbols, the definition of $\lim_{x \to x_0} f(x) = l$ can be formulated as follows:

$$\forall \epsilon > 0, \exists I(x_0) : \forall x (\neq x_0) \in I(x_0) : |f(x) - l| < \epsilon.$$

In the definition just given, considering ϵ, we think of values that become increasingly smaller. We will say that ϵ is taken *small at will*. Furthermore, if we explicitly state the absolute value in the expression $|f(x) - l| < \epsilon$, we get

$$-\epsilon < f(x) - l < \epsilon \quad \to \quad l - \epsilon < f(x) < l + \epsilon,$$

i.e., $f(x)$ belongs to the interval $]l - \epsilon; l + \epsilon[$.

The definition tells us that l is the limit of $f(x)$ if, for any ϵ, even *very small*, we can always find a neighborhood of x_0 such that, for every $x \neq x_0$ in that neighborhood, $f(x)$ belongs to $]l - \epsilon; l + \epsilon[$, meaning $f(x)$ is *very close* to l.

In general, the existence of the limit of a function at a point x_0 is independent of the behavior of the function at x_0. The following cases are possible:

- $\exists \lim_{x \to x_0} f(x) = l \ \wedge \ l = f(x_0)$;
- $\exists \lim_{x \to x_0} f(x) = l \ \wedge \ l \neq f(x_0)$;
- $\exists \lim_{x \to x_0} f(x) = l \ \wedge \ \nexists f(x_0)$.

2.1 Continuous Functions

If for a function $f(x)$ it holds that, for a point x_0 belonging to the domain of f, the limit of $f(x)$ as $x \to x_0$ exists and

$$\lim_{x \to x_0} f(x) = f(x_0)$$

then f is called **continuous** at x_0. We then say that f is continuous in its domain D when it is continuous at every point in D. Functions whose graphs are uninterrupted curves are continuous in their domain; this includes, for example, a line or a parabola.

If a function is continuous at a point, the calculation of the limit at that point is simple because it suffices to calculate the value of the function at that point.

Here is a list of commonly used functions that are continuous in $\mathbb{R}$ (or in intervals of $\mathbb{R}$).

The Polynomial Function Every polynomial function, i.e., every function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n,$$

is continuous throughout $\mathbb{R}$.

The Square Root Function The function defined in $\mathbb{R}^+ \cup 0$,

$$y = \sqrt{x},$$

is continuous for every real positive or zero x. More generally, power functions with real exponents defined in $\mathbb{R}^+ : y = x^\alpha$ $(\alpha \in \mathbb{R})$ are continuous.

Note 3. The square root function is a special case of the power function with a real exponent. In fact,

$$f(x) = x^{\frac{1}{2}} = \sqrt{x}.$$

Trigonometric Functions The functions $\sin(x)$ and $\cos(x)$ are continuous in $\mathbb{R}$. The tangent function is also continuous in $\mathbb{R} - \{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$, and the cotangent function is continuous in $\mathbb{R} - \{k\pi,\ k \in \mathbb{Z}\}$. Finally, it can be shown that the secant, cosecant, arcsine, arccosine, arctangent, and arccotangent functions are continuous in their domains.

Note 4. The function $\tan(x)$ is not defined for $x = \frac{\pi}{2} + k\pi$.
The function $\cot(x)$ is not defined for $x = k\pi$.

Exponential and Logarithmic Functions The exponential function $y = a^x$, with $a > 0$, is continuous in $\mathbb{R}$. The logarithmic function $y = \log_a x$, with $a > 0$, $a \neq 1$, is continuous in $\mathbb{R}^+$.

2.2 Right and Left Limits

Right Limit The right limit of a function is indicated by the symbol:

$$\lim_{x \to x_0^+} f(x) = l.$$

This notation means that x approaches x_0 but always remains greater than x_0. The definition of the right limit is analogous to the previously given limit definition, with the only difference that the inequality $|f(x) - l| < \epsilon$ must be satisfied for every x belonging to a *right neighborhood* of x_0, i.e., a neighborhood of the form

$$]x_0; x_0 + \delta[$$

Left Limit The left limit of a function is indicated by the symbol:

$$\lim_{x \to x_0^-} f(x) = l.$$

This notation means that x approaches x_0 but always remains less than x_0. The same considerations made for the right limit also apply to the left limit, with the only difference that $|f(x) - l| < \epsilon$ must be satisfied for every x belonging to a *left neighborhood* of x_0, i.e., a neighborhood of the form

$$]x_0 - \delta; x_0[$$

Note that $\lim_{x \to x_0} f(x) = l$ exists if and only if both the right and left limits exist and coincide:

$$\lim_{x \to x_0} f(x) = l \iff \lim_{x \to x_0^+} f(x) = l \wedge \lim_{x \to x_0^-} f(x) = l$$

Indeed, given $\epsilon > 0$, the inequality $|f(x) - l| < \epsilon$ is verified in a complete neighborhood I of x_0, with at most $x \neq x_0$, if and only if it is verified both in a right neighborhood of x_0 and in a left neighborhood of x_0.

3 Definition of $\lim_{x \to x_0} f(x) = \infty$

3.1 $\lim_{x \to x_0} f(x) = +\infty$

Limit $+\infty$ as x approaches x_0

Definition 3.1. Let $f(x)$ be a function not defined at x_0. It is said that $f(x)$ tends to $+\infty$ as x approaches x_0, and it is written

$$\lim_{x \to x_0} f(x) = +\infty$$

when, for every positive real number M, a complete neighborhood I of x_0 can be determined such that

$$f(x) > M$$

for every x belonging to I and different from x_0.

In summary, we can say that $\lim_{x \to x_0} f(x) = +\infty$ if:

$$\forall M > 0, \exists I(x_0) : \forall x \in I(x_0) - \{x_0\}, f(x) > M$$

If $\lim_{x \to x_0} f(x) = +\infty$, it is also said that the function f **diverges positively**.

Note 5. In the definition, when we say "*for every positive real number M*," we think of values of M that become increasingly large. We will say that M is taken arbitrarily large.

3.2 $\lim_{x \to x_0} f(x) = -\infty$

Limit $-\infty$ as x approaches x_0

Definition 3.2. Let $f(x)$ be a function not defined at x_0. It is said that $f(x)$ tends to $-\infty$ as x approaches x_0, and it is written

$$\lim_{x \to x_0} f(x) = -\infty$$

when, for every positive real number M, a complete neighborhood I of x_0 can be determined such that

$$f(x) < -M$$

for every x belonging to I and different from x_0.

In symbols, we say that $\lim_{x \to x_0} f(x) = -\infty$ if:

$$\forall M > 0, \exists I(x_0) : \forall x \in I(x_0) - \{x_0\}, f(x) < -M$$

If $\lim_{x \to x_0} f(x) = -\infty$, it is also said that the function f **diverges negatively**.

If...	the inequality...	is satisfied for $x \neq x_0$, in a...
$\lim_{x \to x_0^+} f(x) = +\infty$	$f(x) > M$	right neighborhood of x_0
$\lim_{x \to x_0^-} f(x) = +\infty$	$f(x) > M$	left neighborhood of x_0
$\lim_{x \to x_0^+} f(x) = -\infty$	$f(x) < -M$	right neighborhood of x_0
$\lim_{x \to x_0^-} f(x) = -\infty$	$f(x) < -M$	left neighborhood of x_0

3.3 Right and Left Infinite Limits

Infinite limits can also be distinguished for right and left limits.

The definition of $\lim_{x \to x_0} f(x) = \infty$ is analogous to the previous ones, but with the following variation: **for every $M > 0$, it is possible to find a neighborhood I of x_0 such that, for every $x(\neq x_0) \in I$ in the domain of f,** $|f(x)| > M$. In symbols:

$$\forall M > 0, \exists I(x_0) : \forall x(\neq x_0) \in I, |f(x)| > M.$$

The inequality $|f(x)| > M$ can be written equivalently as $f(x) > M \vee f(x) < -M$, and therefore its solutions are the union of the solutions of the individual inequalities.

3.4 Vertical Asymptotes

Asymptote

Definition 3.3. A line is called an asymptote of the graph of a function if the distance from a generic point on the graph to that line tends to 0 as the abscissa or ordinate of the point tends to ∞.

Now let's study *vertical asymptotes.*

Vertical Asymptote

Definition 3.4. Given the function $y = f(x)$, if $\lim_{x \to c} f(x) = \infty$, it is said that the line $x = c$ is a vertical asymptote for the graph of the function.

The distance from a generic point on the graph of a function to its vertical asymptote, with equation $x = c$, tends to 0 as $x \to c$. Indeed, with $P(x; y)$ being the generic point on the graph, we have:

$$\lim_{x \to c} \overline{PH} = \lim_{x \to c} |x - c| = 0$$

The definition of a vertical asymptote is still valid if we consider the right limit $(x \to x_0^+)$ or the left limit $(x \to x_0^-)$, and both limits are infinite, but with opposite signs, or if only one of the two limits is infinite.

4 The Definition of $\lim_{x\to\infty} f(x) = l$

4.1 $x \to +\infty$

Finite Limit of a Function as x Approaches $+\infty$

Definition 4.1. A function $f(x)$ is said to tend to the real number l as x approaches $+\infty$, and it is written

$$\lim_{x\to+\infty} f(x) = l$$

when, no matter the positive real number ϵ chosen, we can determine a neighborhood I of $+\infty$ such that:

$$|f(x) - l| < \epsilon \quad \text{for every } x \in I.$$

Given that a neighborhood of $+\infty$ consists of all x greater than a number c, we can say that $\lim_{x\to+\infty} f(x) = l$ if:

$$\forall\epsilon > 0, \exists c > 0 : \forall x > c, |f(x) - l| < \epsilon$$

4.2 $x \to -\infty$

Finite Limit of a Function as x Approaches $-\infty$

Definition 4.2. A function $f(x)$ is said to have a real limit l as x approaches $-\infty$, and it is written

$$\lim_{x\to-\infty} f(x) = l$$

if, for every fixed positive real number ϵ, we can find a neighborhood I of $-\infty$ such that:

$$|f(x) - l| < \epsilon \quad \text{for every } x \in I$$

In symbols, $\lim_{x\to-\infty} f(x) = l$ if:

$$\forall\epsilon > 0, \exists c > 0 : \forall x < -c, |f(x) - l| < \epsilon$$

4.3 $x \to \infty$

The previous two cases can be summarized in one if we consider a neighborhood of ∞ determined by the x for which

$$|x| > c, \text{ i.e., } x < -c \vee x > c,$$

or also

$$x \in] - \infty; -x[\cup]c : +\infty[,$$

where c is a positive real number chosen arbitrarily large. We then say that x **tends to** ∞ omitting the sign $+$ or $-$. We say that $\lim_{x\to\infty} f(x) = l$ **when for every** $\epsilon > 0$ **we can find a neighborhood** I **of** ∞ **such that** $|f(x) - l| < \epsilon$ **for every** $x \in I$. In symbols:

$$\forall\epsilon > 0, \exists I(\infty) : \forall x \in I, |f(x) - l| < \epsilon$$

4.4 Horizontal Asymptotes

Horizontal Asymptote

Definition 4.3. Given the function $y = f(x)$, if one of the inequalities holds:

$$\lim_{x \to +\infty} f(x) = q \quad \text{or} \quad \lim_{x \to -\infty} f(x) = q \quad \text{or} \quad \lim_{x \to \infty} f(x) = q,$$

then the line $y = q$ is called a horizontal asymptote for the graph of the function.

The distance from a generic point P on the graph of a function to its horizontal asymptote, with equation $y = q$, tends to 0 as x tends to $+\infty$. Let $P(x; f(x))$ be the point, we have:

$$\lim_{x \to +\infty} \overline{PH} = \lim_{x \to +\infty} |f(x) - q| = 0.$$

Similar considerations hold for $x \to \infty$ or $x \to -\infty$.

5 The Definition of $\lim_{x\to\infty} f(x) = \infty$

5.1 The Limit is $+\infty$ as x Approaches $+\infty$ or $-\infty$

In this case, we can also say that **the function diverges positively**. Let's study the two cases:

$$\lim_{x\to+\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = +\infty$$

Limit $+\infty$ of a function as x approaches $+\infty$

Definition 5.1. The function $f(x)$ is said to have a limit of $+\infty$ as x approaches $+\infty$, and it is written

$$\lim_{x\to+\infty} f(x) = +\infty$$

when, for every positive real number M, we can determine a neighborhood I of $+\infty$ such that:

$$f(x) > M \quad \text{for every } x \in I$$

In symbols, $\lim_{x\to+\infty} f(x) = +\infty$ if:

$$\forall M > 0, \exists c > 0 : \forall x > c, f(x) > M.$$

Limit $+\infty$ of a function as x approaches $-\infty$

Definition 5.2. The function $f(x)$ is said to have a limit of $+\infty$ as x approaches $-\infty$, and it is written

$$\lim_{x\to-\infty} f(x) = -\infty$$

when, for every positive real number M, we can determine a neighborhood I of $-\infty$ such that:

$$f(x) > M \quad \text{for every } x \in I.$$

In symbols, $\lim_{x\to-\infty} f(x) = +\infty$ if:

$$\forall M > 0, \exists c > 0 : \forall x < -c, f(x) > M$$

5.2 The Limit is $-\infty$ as x Approaches $+\infty$ or $-\infty$

In this case, we can also say that **the function diverges negatively**. Let's study the cases:

$$\lim_{x\to+\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty$$

Limit $-\infty$ of a function as x approaches $+\infty$

Definition 5.3. The function $f(x)$ is said to have a limit of $-\infty$ as x approaches $+\infty$, and it is written $\lim_{x\to+\infty} f(x) = -\infty$ when, for every positive real number M, we can determine a neighborhood I of $+\infty$ such that $f(x) < -M$ for every $x \in I$.

In symbols:

$$\forall M > 0, \exists I(+\infty) : \forall x \in I, f(x) < -M$$

Limit $-\infty$ of a function as x approaches $-\infty$

Definition 5.4. The function $f(x)$ is said to have a limit of $-\infty$ as x approaches $-\infty$, and it is written $\lim_{x\to-\infty} f(x) = -\infty$ when, for every positive real number M, we can determine a neighborhood I of $-\infty$ such that $f(x) < -M$ for every $x \in I$.

In symbols:

$$\forall M > 0, \exists I(-\infty) : \forall x \in I, f(x) < -M$$

6 First Theorems on Limits

The following theorems and properties are valid for functions defined in any domain $D \subseteq \mathbb{R}$ and for points x_0 (where we calculate the limit) that are accumulation points of the domain D. They also hold for $x \to +\infty$ or $x \to -\infty$. However, we will consider specific domains D, i.e., intervals of $\mathbb{R}$ or unions of intervals, and x_0 as a point in D or as an endpoint of one of the intervals that make up D.

The theorems also hold if, instead of l, we have $+\infty$, $-\infty$, or ∞. They are also valid in the cases of the right limit or the left limit.

6.1 Uniqueness of Limit Theorem

Uniqueness of Limit Theorem

Theorem 6.1. *If the function $f(x)$ has the real number l as its limit as x approaches x_0, then this limit is unique.*

Proof. Let's prove the thesis by contradiction. Suppose that the thesis is false, i.e., that l is not unique. In that case, there should exist a real number l' different from l such that:

$$\lim_{x \to x_0} f(x) = l', \ l' \neq l$$

We can assume $l < l'$ and, since in the definition of limit we can choose ϵ arbitrarily as long as it is positive, consider:

$$\epsilon < \frac{l' - l}{2}.$$

Apply the definition of limit in both cases. There should exist two neighborhoods I and I' of x_0 such that:

$$|f(x) - l| < \epsilon \quad \text{for every } x \in I$$
$$|f(x) - l'| < \epsilon \quad \text{for every } x \in I'$$

Notice that $I \cap I'$ is also a neighborhood of x_0. In $I \cap I'$, the two inequalities must hold simultaneously, i.e.:

$$\begin{cases} |f(x) - l| < \epsilon \\ |f(x) - l'| < \epsilon \quad \forall x \in I \cap I' \end{cases}$$

We can also write:

$$\begin{cases} l - \epsilon < f(x) < l + \epsilon \\ l' - \epsilon < f(x) < l' + \epsilon \end{cases}$$

Comparing the inequalities, remembering that $l < l'$, it follows that

$$l' - \epsilon < f(x) < l + \epsilon$$

which implies:

$$l' - \epsilon < l + \epsilon$$

Solving for ϵ, we get

$$-\epsilon - \epsilon < l - l' \quad \rightarrow \quad -2\epsilon < l - l' \quad \rightarrow \quad 2\epsilon > l' - l,$$

which contradicts the assumption of $\epsilon < \frac{l'-l}{2}$. The assumption that there are two limits is false. Therefore, if $\lim_{x\to x_0} f(x) = l$, the limit l is unique. □

6.2 Sign Preservation Theorem

Theorem 6.2. *If the limit of a function as x approaches x_0 is a number l different from 0, then there exists an interval I around x_0 (excluding at most x_0) in which both $f(x)$ and l are either both positive or both negative.*

Proof. By hypothesis,

$$\lim_{x\to x_0} f(x) = l \neq 0.$$

- If $l > 0$, for the arbitrariness of ϵ, we choose $\epsilon = l$. Then, there exists an interval I around x_0 such that

  $$|f(x) - l| < l,$$

  which implies

  $$-l < f(x) - l < l \quad \rightarrow \quad 0 < f(x) < 2l,$$

  and therefore

  $$f(x) > 0, \quad \forall x \in I;$$

  hence l and $f(x)$ are both positive.

- If $l < 0$, we choose $\epsilon = -l$. Then, there exists an interval I around x_0 such that

  $$|f(x) - l| < -l,$$

  which implies

  $$+l < f(x) - l < -l \quad \rightarrow \quad 2l < f(x) < 0,$$

  and therefore $f(x) < 0, \quad \forall x \in I$; hence l and $f(x)$ are both negative.

□

6.3 Comparison Theorem

Theorem 6.3. *Let $h(x)$, $f(x)$, and $g(x)$ be three functions defined in the same domain $D \subseteq \mathbb{R}$, excluding at most one point x_0. If, at every point different from x_0 in the domain, it holds that*

$$h(x) \leq f(x) \leq g(x)$$

and the limit of the two functions $h(x)$ and $g(x)$ as x approaches x_0 is the same number l, then the limit of $f(x)$ as x approaches x_0 is also equal to l.

Proof. Let $\epsilon > 0$ be arbitrary. It is true that:

$$\begin{aligned} |h(x) - l| < \epsilon, &\ \text{ for every } x \in I_1 \cap D, \ \text{ because } h(x) \to l \text{ as } x \to x_0; \\ |g(x) - l| < \epsilon, &\ \text{ for every } x \in I_2 \cap D, \ \text{ because } g(x) \to l \text{ as } x \to x_0. \end{aligned}$$

Both inequalities hold for every x in the domain belonging to the interval $I = I \cap I_1 \cap I_2$, excluding at most x_0. Therefore, for every $x \in I$, we have:

$$l - \epsilon < h(x) < l + \epsilon, \quad l - \epsilon < g(x) < l + \epsilon.$$

Taking into account the relationship between the functions, we have

$$l - \epsilon < h(x) \leq f(x) \leq g(x) < l + \epsilon,$$

for every $x \in I$, which implies

$$l - \epsilon < f(x) < l + \epsilon$$

for every $x \in I$, i.e.,

$$|f(x) - l| < \epsilon, \ \forall x \in I.$$

This last relation precisely means that $\lim_{x \to x_0} f(x) = l$. $\square$

7 Limit of a Sequence

The concept of the limit of a sequence is similar to that of the limit of a function. However, in the case of sequences, we observe that the domain is the set of natural numbers $\mathbb{N}$ and not an interval.

Note 6. Remember that a sequence is a particular function from $\mathbb{N}$ to $\mathbb{R}$.

In particular, this implies that the independent variable n cannot tend to a finite value but only to $+\infty$.

7.1 $\lim_{n\to+\infty} a_n = +\infty$

Definition 7.1. Given the sequence with general term a_n, it is said that, as n tends to $+\infty$, the sequence has a limit of $+\infty$ when, for any arbitrarily chosen positive real number M, it is possible to determine a corresponding positive real number p_M such that:

$$a_n > M \quad \text{for every } n > p_M$$

Saying that M is a positive number arbitrarily chosen is equivalent to saying that every statement holds for every $M > 0$. This means that, for any arbitrarily chosen $M > 0$, from a certain index onwards, all the following terms are greater than M. In this case, the sequence is called **divergent positively**. Similarly, we give the definition of a sequence that tends to $-\infty$.

7.2 $\lim_{n\to+\infty} a_n = -\infty$

Definition 7.2. Given the sequence with general term a_n, it is said that, as n tends to $+\infty$, the sequence has a limit of $-\infty$ when, for any arbitrarily chosen positive real number M, it is possible to determine a corresponding positive real number p_M such that:

$$a_n < -M \quad \text{for every } n > p_M$$

Therefore, for any arbitrarily chosen number $M > 0$, from a certain index onwards, all terms of the sequence are less than $-M$. In this case, the sequence is called **divergent negatively**.

7.3 $\lim_{n\to+\infty} a_n = l$

Definition 7.3. Given the sequence with general term a_n, it is said that, as n tends to $+\infty$, the sequence has a limit equal to the number l when, for any arbitrarily chosen positive real number ϵ, it is possible to determine a corresponding positive real number p_ϵ such that:

$$|a_n - l| < \epsilon \quad \text{for every } n > p_\epsilon.$$

A sequence of this type is called **convergent**.

7.4 $\lim_{n\to+\infty} a_n$ does not exist

It may happen that a sequence is neither divergent nor convergent: in these cases, it is said that **the limit does not exist**, or the sequence is **indeterminate**.

7.5 Theorems on Limits of Sequences

The theorems we have proved for the limits of functions are valid, as special cases, also for sequences. In particular, let's recall the **comparison theorem**:

- given the sequences a_n, b_n, c_n such that $a_n \le b_n \le c_n$, $\forall n \in \mathbb{N}$, if $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = l$, then the limit of b_n also exists as n approaches $+\infty$ and is equal to l;

- given the sequences a_n, b_n such that $a_n \le b_n, \forall n \in \mathbb{N}$, if $\lim_{n\to+\infty} a_n = +\infty$, then b_n also tends to $+\infty$ as n approaches $+\infty$, and similarly, if $\lim_{n\to+\infty} b_n = -\infty$, then a_n also tends to $-\infty$ as n approaches $+\infty$.

7.6 Subsequences

Subsequence (or Extracted Sequence)

Definition 7.4. A *subsequence* (or *extracted sequence*) of a sequence a_n is a new sequence a_{n_k} obtained by choosing an infinite subset of indices n_k from the original sequence.

From a sequence, we can derive infinitely many subsequences.

Limit of Subsequences

Theorem 7.1. *If a sequence a_n has a limit $l \in \mathbb{R}$, or $+\infty$ or $-\infty$, as n approaches $+\infty$, then every extracted sequence has the same limit as n approaches $+\infty$.*

If a sequence is indeterminate, it does not necessarily mean that its subsequences are also indeterminate. Moreover, if from a sequence we can extract a convergent subsequence, we cannot deduce that the sequence itself is convergent.

7.7 Limits of Monotonic Sequences

The following theorem holds for monotonic sequences.

Limit of a Monotonic Sequence

Theorem 7.2.

- *If an increasing sequence is upper-bounded, then it is convergent; if it is not upper-bounded, then it diverges positively.*

- *If a decreasing sequence is lower-bounded, then it is convergent; if it is not lower-bounded, then it diverges negatively.*

From the theorem, it follows that *a monotonic sequence is never indeterminate.*

7.8 Operations with Sequences

It is also possible to define the four operations with sequences. Given the sequences

$$a_0, a_1, a_2, ..., a_n, ... \quad \text{and} \quad b_0, b_1, b_2, ..., b_n, ...$$

let's define the following operations.

Addition The sum of the two sequences is called the sequence:

$$a_0 + b_0, a_1 + b_1, a_2 + b_2, ..., a_n + b_n, ...$$

Subtraction The difference of the two sequences is called the sequence:

$$a_0 - b_0, a_1 - b_1, a_2 - b_2, ..., a_n - b_n, ...$$

Multiplication The product of the two sequences is called the sequence:

$$a_0 \cdot b_0, a_1 \cdot b_1, a_2 \cdot b_2, ..., a_n \cdot b_n, ...$$

Division If $b_n \neq 0$ for all $n \in \mathbb{N}$, the quotient of the two sequences is called the sequence:

$$\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_n}{b_n}, ...$$

Part II
Limit Calculus

8 Operations on Limits

There are various theorems related to operations on limits. The following theorems are valid both in the case of a limit as x approaches a finite value and in the case of a limit as x approaches $+\infty$ or $-\infty$. Therefore, when it is not important to distinguish, we will use

$$x \to \alpha$$

to denote any of the following notations:

$$x \to x_0 \quad x \to x_0^+ \quad x \to x_0^- \quad x \to +\infty \quad x \to -\infty$$

8.1 The Limit of the Algebraic Sum of Two Functions

Functions Have Finite Limits

Theorem 8.1. *If* $\lim_{x\to\alpha} f(x) = l$ *and* $\lim_{x\to\alpha} g(x) = m$, *where* $l, m \in \mathbb{R}$, *then:*

$$\lim_{x\to\alpha} [f(x) + g(x)] = \lim_{x\to\alpha} f(x) + \lim_{x\to\alpha} g(x) = l + m.$$

Note 7. The limit of the sum of two functions is equal to the sum of their limits.

Functions Do Not Both Have Finite Limits With the symbols $+\infty$ and $-\infty$, operations cannot be performed as if dealing with real numbers. The various cases that may arise in the calculation of the limit of the sum of two functions are summarized in the table below.

$f(x) + g(x)$	l	$+\infty$	$-\infty$
m	$m + l$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?
$-\infty$	$-\infty$	?	$-\infty$

In the table, it can be observed that cases where $+\infty$ and $-\infty$ are added do not result in 0, as one might erroneously expect. This is a **form of indecision** or **indeterminate form**.

8.2 The Limit of the Product of Two Functions

Functions Have Finite Limits

Theorem 8.2. *If* $\lim_{x\to\alpha} f(x) = l$ *and* $\lim_{x\to\alpha} g(x) = m$, *with* $l, m \in \mathbb{R}$, *then:*

$$\lim_{x\to\alpha} [f(x) \cdot g(x)] = \lim_{x\to\alpha} f(x) \cdot \lim_{x\to\alpha} g(x) = l \cdot m.$$

Note 8. The limit of the product of two functions is equal to the product of their limits.

Case 1. If $f(x)$ is a constant function k, we have:

$$\lim_{x\to\alpha} f(x) \cdot g(x) = \lim_{x\to\alpha} k \cdot \lim_{x\to\alpha} g(x) = k \cdot m$$

Functions Do Not Both Have Finite Limits If the functions do not both have finite limits, various cases may arise for the limit of the product, summarized in the table. Note that even when using the symbols $+\infty$ and $-\infty$, the sign rule still applies.

$f(x)\cdot g(x)$	$l>0$	$l<0$	0	$+\infty$	$-\infty$
$m>0$	$m\cdot l$	$m\cdot l$	0	$+\infty$	$-\infty$
$m<0$	$m\cdot l$	$m\cdot l$	0	$-\infty$	$+\infty$
0	0	0	0	?	?
$+\infty$	$+\infty$	$-\infty$	?	$+\infty$	$-\infty$
$-\infty$	$-\infty$	$+\infty$	?	$-\infty$	$+\infty$

8.3 The Limit of a Power

Theorem 8.3. *If $n \in \mathbb{N} - \{0\}$ and $\lim_{x\to\alpha} f(x) = l$, then:*

$$\lim_{x\to\alpha}[f(x)]^n = [\lim_{x\to\alpha} f(x)]^n = l^n$$

This theorem can also be extended to the case of a real exponent a different from 0. When a is a positive irrational number, $[f(x)]^a$ exists only if $f(x) \geq 0$, so $f(x)$ can only tend to a number > 0.

The Function Has a Limit of $+\infty$ We have the following table.

$f(x)$	a	$[f(x)]^a$
$+\infty$	$a>0$	$(+\infty)^a = +\infty$
$+\infty$	$a<0$	$(+\infty)^a = 0$

The Exponent is a Function The power rule can be extended to the case $[f(x)]^{g(x)}$, considering that the power $[f(x)]^{g(x)}$ exists only if $f(x)$ is > 0.

$[f(x)]^{g(x)}$	0	$+\infty$	$-\infty$
$+\infty$	?	$+\infty$	0
0	?	0	$+\infty$
1	1	?	?
$0<l<1$	1	0	$+\infty$
$l>1$	1	$+\infty$	0

In the table, we find three indeterminate forms:

$$\infty^0 \quad 0^0 \quad 1^\infty$$

8.4 The Limit of the Reciprocal Function

Theorem 8.4. *Consider a function $f(x)$ and its reciprocal $\dfrac{1}{f(x)}$:*

- *If* $\lim_{x\to\alpha} f(x) = l \in \mathbb{R}, l \neq 0$, *then*

$$\lim_{x\to\alpha} \frac{1}{f(x)} = \frac{1}{\lim_{x\to\alpha} f(x)} = \frac{1}{l};$$

- *If* $\lim_{x\to\alpha} f(x) = +\infty$ *or* $\lim_{x\to\alpha} f(x) = -\infty$, *then*

$$\lim_{x\to\alpha} \frac{1}{f(x)} = 0;$$

- *If* $\lim_{x\to\alpha} f(x) = 0$, *then*

$$\lim_{x\to\alpha} \frac{1}{f(x)} = \infty.$$

8.5 The Limit of the Quotient of Two Functions

Functions Have Finite Limits, One of Which is Nonzero

Theorem 8.5. *If* $\lim_{x\to\alpha} f(x) = l$ *and* $\lim_{x\to\alpha} g(x) = m$, *where* $m \neq 0$, *then:*

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\alpha} f(x)}{\lim_{x\to\alpha} g(x)} = \frac{l}{m}$$

Proof. Since we can write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$, by the theorem of the limit of the reciprocal function and the limit of the product of two functions, we have:

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{x\to\alpha} f(x) \cdot \lim_{x\to\alpha} \frac{1}{g(x)} = \lim_{x\to\alpha} f(x) \cdot \frac{1}{\lim_{x\to\alpha} g(x)} = \frac{l}{m}.$$

□

Note 9. Note that, by the sign preservation theorem, if $m \neq 0$, then $g(x) \neq 0$ in an entire neighborhood of α.

Functions Do Not Both Have Finite Limits Various cases can arise as summarized in the following table.

$\frac{f(x)}{g(x)}$	$m \neq 0$	0	$+\infty$	$-\infty$
$l \neq 0$	$\frac{l}{m}$	∞	0	0
0	0	?	0	0
$+\infty$	∞	∞	?	?
$-\infty$	∞	∞	?	?

We encounter the **indeterminate forms**:

$$\frac{0}{0} \quad \frac{\infty}{\infty}$$

9 Indeterminate Forms

As we have seen, the indeterminate forms encountered in the calculation of limits are seven:

$$+\infty - \infty \quad \infty \cdot 0 \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^{\infty} \quad 0^0 \quad \infty^0$$

Let's now examine, through some examples, how to calculate limits that appear in indeterminate form.

9.1 The indeterminate form $+\infty - \infty$

Example 9.1. The limit $\lim_{x\to+\infty}(x - \sqrt{x^2+1})$ appears in the indeterminate form $+\infty - \infty$ because:

$$\lim_{x\to+\infty} x = +\infty \quad \text{and} \quad \lim_{x\to+\infty}(-\sqrt{x^2+1}) = -\infty$$

To calculate this limit, we can rewrite the given function such that in the limit's argument, the difference $x - \sqrt{x^2+1}$ disappears, and instead, the sum $x + \sqrt{x^2+1}$ appears. To do this, multiply and divide the function by $x + \sqrt{x^2+1}$:

$$x - \sqrt{x^2+1} = (x - \sqrt{x^2+1}) \cdot \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}} = \frac{x^2-(x^2+1)}{x+\sqrt{x^2+1}} = \frac{-1}{x+\sqrt{x^2+1}}$$

As x approaches $+\infty$, the denominator of the fraction $x + \sqrt{x^2+1}$ tends to $+\infty$. Therefore, by the reciprocal function limit theorem, the fraction tends to 0, i.e.:

$$\lim_{x\to+\infty}(x - \sqrt{x^2+1}) = \lim_{x\to+\infty} \frac{-1}{x+\sqrt{x^2+1}} = 0$$

Example 9.2. Let's calculate the following limit:

$$\lim_{x\to+\infty}(x^4 - 3x^2 + 1)$$

It appears in the indeterminate form $+\infty - \infty$. Factoring out the x^4 term, the limit becomes:

$$\lim_{x\to+\infty} x^4(1 - \frac{3}{x^2} + \frac{1}{x^4}).$$

Since $\lim_{x\to+\infty}(-\frac{3}{x^2}) = 0$ and $\lim_{x\to+\infty}\frac{1}{x^4} = 0$, we have

$$\lim_{x\to+\infty}(1 - \frac{3}{x^2} + \frac{1}{x^4}) = 1.$$

Also, we know that $\lim_{x\to+\infty} x^4 = +\infty$, therefore, by the product limit theorem in the case of a finite limit (different from 0) and an infinite limit, we get:

$$\lim_{x\to+\infty} x^4(1 - \frac{3}{x^2} + \frac{1}{x^4}) = +\infty$$

The procedure used in Example 2 generalizes as follows.

The limit of a polynomial function In general, to calculate the limit of a polynomial function as x approaches $+\infty$ (or x approaches $-\infty$),

$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + ... + a_n),$$

we proceed as follows:

- Factor out x^n:
$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + ... + a_n)$$
- Since, for x approaching $+\infty$ or $-\infty$, the limit of $\frac{a_1}{x}, \frac{a_2}{x^2}, ..., \frac{a_n}{x^n}$ is 0, we have
$$\lim_{x\to\pm\infty}(a_0x^n + a_1x^{n-1} + ... + a_n) = \lim_{x\to\pm\infty} a_0x^n.$$
This limit is $+\infty$ or $-\infty$. The sign is determined by applying the sign rule to the product a_0x^n.

9.2 The indeterminate form $\infty \cdot 0$

Example 9.3. Let's calculate the following limit:

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x)\cdot\tan x.$$

Through direct calculation, we obtain the indeterminate form $0\cdot\infty$ because:

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x) = 0 \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}^-}\tan x = +\infty.$$

Recall that $\tan x = \frac{\sin x}{\cos x}$ and multiply and divide the given function by $(1+\sin x)$:

$$\begin{aligned}
&(1-\sin x)\cdot\tan x\cdot\frac{1+\sin x}{1+\sin x}\\
&=\frac{(1-\sin x)(1+\sin x)}{1+\sin x}\cdot\tan x\\
&=\frac{1-\sin^2 x}{1+\sin x}\cdot\frac{\sin x}{\cos x}\\
&=\frac{\cos^2 x}{1+\sin x}\cdot\frac{\sin x}{\cos x}\\
&=\frac{\sin x\cdot\cos x}{1+\sin x}
\end{aligned}$$

As x approaches $\frac{\pi}{2}^-$, the numerator $\sin x\cdot\cos x$ tends to 0, while the denominator $1+\sin x$ tends to 2. Therefore, by the limit quotient rule, the fraction tends to $\frac{0}{2}$, i.e., 0:

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x)\cdot\tan x = \lim_{x\to\frac{\pi}{2}^-}\frac{\sin x\cdot\cos x}{1+\sin x} = 0.$$

9.3 The indeterminate form $\frac{0}{0}$

Example 9.4. Let's calculate the limit

$$\lim_{x\to 3}\frac{x^2-2x-3}{2x^2-9x+9}$$

which is in the indeterminate form $\frac{0}{0}$ because:

$$\lim_{x\to 3}(x^2-2x-3)=0 \quad \text{and} \quad \lim_{x\to 3}(2x^2-9x+9)=0.$$

Since the value 3 makes both the numerator and the denominator zero, we factorize both:

$$x^2-2x-3 \to (x-3)(x+1)$$

$$2x^2-9x+9 \to (x-3)(2x-3)$$

$$\lim_{x\to 3}\frac{x^2-2x-3}{2x^2-9x+9}=\lim_{x\to 3}\frac{(x-3)(x+1)}{(x-3)(2x-3)}=\lim_{x\to 3}\frac{(x+1)}{(2x-3)}=\frac{4}{3}$$

9.4 The indeterminate form $\frac{\infty}{\infty}$

The limit of a fractional rational function as $x\to\infty$ Given the limit

$$\lim_{x\to\pm\infty}\frac{a_0x^n+a_1x^{n-1}+...+a_n}{b_0x^m+b_1x^{m-1}+...+b_m}$$

when at least one coefficient of the powers of x is nonzero both in the numerator and denominator, this limit presents itself in the form $\frac{\infty}{\infty}$ because both the numerator and denominator tend to ∞ as x approaches ∞. Here are three examples of limit calculations with $n>m$, $n=m$, $n<m$.

The degree of the numerator is greater than the degree of the denominator

Example 9.5. Let's calculate the limit

$$\lim_{x\to+\infty}\frac{x^5-2x^2+1}{3x^2-2x+6}.$$

Factor out x^5 in the numerator and x^2 in the denominator:

$$\lim_{x\to+\infty}\frac{x^5\cdot(1-\frac{2}{x^3}+\frac{1}{x^5})}{x^2\cdot(3-\frac{2}{x}+\frac{6}{x^2})}=\lim_{x\to+\infty}x^3\frac{(1-\frac{2}{x^3}+\frac{1}{x^5})}{(3-\frac{2}{x}+\frac{6}{x^2})}$$

We have $\lim_{x\to+\infty}x^3=+\infty$, $\lim_{x\to+\infty}(1-\frac{2}{x^3}+\frac{1}{x^5})=1$, $\lim_{x\to+\infty}(3-\frac{2}{x}+\frac{6}{x^2})=3$. Therefore, $\lim_{x\to+\infty}\frac{x^5-2x^2+1}{3x^2-2x+6}=+\infty$

The degree of the numerator is equal to the degree of the denominator

Example 9.6. Let's calculate the limit $\lim_{x\to\pm\infty} \frac{1-2x^2}{3x^2+2x-5}$. Factor out x^2 both in the numerator and the denominator:

$$\lim_{x\to\pm\infty} \frac{x^2 \cdot (\frac{1}{x^2} - 2)}{x^2 \cdot (3 + \frac{2}{x} - \frac{5}{x^2})} = \lim_{x\to\pm\infty} \frac{(\frac{1}{x^2} - 2)}{(3 + \frac{2}{x} - \frac{5}{x^2})}$$

By the limit quotient rule, the fraction tends to $-\frac{2}{3}$, therefore:

$$\lim_{x\to\pm\infty} \frac{1-2x^2}{3x^2+2x-5} = -\frac{2}{3}.$$

Note that $-\frac{2}{3}$ is the ratio of the coefficients of the highest degree term, i.e., x^2, in the numerator and denominator.

The degree of the numerator is less than the degree of the denominator

Example 9.7. Let's calculate the limit

$$\lim_{x\to-\infty} \frac{2x-1}{x^3+2x}.$$

Factor out x in the numerator and x^3 in the denominator:

$$\lim_{x\to-\infty} \frac{x \cdot (2 - \frac{1}{x})}{x^3 \cdot (1 + \frac{2}{x^2})} = \lim_{x\to-\infty} \frac{1}{x^2} \cdot \frac{(2 - \frac{1}{x})}{(1 + \frac{2}{x^2})}$$

We have $\lim_{x\to-\infty} \frac{1}{x^2} = 0$, $\lim_{x\to-\infty}(2 - \frac{1}{x}) = 2$, $\lim_{x\to-\infty}(1 + \frac{2}{x^2}) = 1$ Therefore,

$$\lim_{x\to-\infty} \frac{2x-1}{x^3+2x} = 0$$

In general, given a fractional rational function

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m},$$

with the numerator of degree n and the denominator of degree m, we have:

$$\lim_{x\to\pm\infty} \frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m} = \begin{cases} \pm\infty & \text{if } n > m \\ \frac{a^0}{b^0} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

The sign of ∞ in the case $n > m$ is given by the product of the signs of:

$$\lim_{x\to\pm\infty} x^{n-m} \quad \text{and} \quad \frac{a^0}{b^0}$$

10 Noteworthy Limits

Let's illustrate two particular limits, called noteworthy because they are fundamental in the applications of analysis.

A first noteworthy limit Consider

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Since $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} x = 0$, we are dealing with the indeterminate form $\frac{0}{0}$. We prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Proof. Note that the function $\frac{\sin x}{x}$ is even since

$$\frac{\sin(-x)}{x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x},$$

making it symmetric with respect to the y-axis. Thus, we conclude that

$$\lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{x \to 0^+} \frac{\sin x}{x}$$

and we can limit ourselves to proving the case

$$\lim_{x \to 0^+} \frac{\sin x}{x}.$$

Consider the unit circle and a positive angle of measure x. If x is in radians, its measure coincides with that of $\widehat{AP}$, while the measure of PQ is $\sin x$, and that of TA is $\tan x$. Since

$$\overline{PQ} < \widehat{AP} < \overline{TA},$$

we have

$$\sin x < x < \tan x.$$

Dividing the terms of the inequality by $\sin x$ yields,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

and taking reciprocals gives $\cos x < \frac{\sin x}{x} < 1$. The function $\frac{\sin x}{x}$ is sandwiched between the functions $\cos x$ and the constant function 1. We can apply the comparison theorem: since $\lim_{x\to 0} \cos x = 1$, the function $\frac{\sin x}{x}$ is sandwiched between two functions that tend to 1 as $x \to 0$, so it also tends to 1. □

From this noteworthy limit, we deduce the following limits, which are also in the indeterminate form $\frac{0}{0}$.

1. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Proof. Multiplying the numerator and denominator of $\frac{1-\cos x}{x}$ by $1+\cos x$, we obtain

$$\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x}\cdot\sin x\cdot\frac{1}{1+\cos x},$$

and thus, by the limit product theorem:

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x\cdot\frac{1}{1+\cos x} = 1\cdot 0\cdot\frac{1}{2} = 0$$

□

2. $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$

   Proof. Applying the previous reasoning, we can write:

   $$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{1+\cos x} = 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2}$$

   □

Another Remarkable Limit

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$$

From this remarkable limit, we can deduce others, which are in the indeterminate form $\frac{0}{0}$.

1. $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

   Proof. Applying logarithm properties, we can write

   $$\frac{\ln(1+x)}{x} = \frac{1}{x}\ln(1+x) = \ln(1+x)^{\frac{1}{x}}$$

   and thus, due to the continuity of the logarithmic function:

   $$\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}] = \ln(\lim_{x\to 0}(1+x)^{\frac{1}{x}}).$$

   Now, let $y=\frac{1}{x}$, then $x=\frac{1}{y}$, and for $x\to 0$, we have $y\to\pm\infty$. Making the variable substitution in the previous limit, we get:

   $$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \ln(\lim_{y\to\pm\infty}\left(1+\frac{1}{y}\right)^y) = \ln e = 1.$$

   □

2. $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

Proof. Let $y = e^x - 1$, then $e^x = 1 + y$, and $x = \ln(1 + y)$. Also, for $x \to 0$, we have $y \to 0$, so substituting the variable x yields:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{y}{\ln(1+y)} = \lim_{y\to 0} \frac{1}{\frac{\ln(1+y)}{y}} = \frac{1}{1} = 1,$$

by the reciprocal function limit theorem. $\square$

Note 10. More generally, if $a > 0$:

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a.$$

Other Remarkable Limits

- Trigonometric functions
  - $$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$$
  - $$\lim_{x\to 0} \frac{\arcsin(x)}{x} = 1$$
  - $$\lim_{x\to 0} \frac{\arctan(x)}{x} = 1$$
- Exponential and logarithmic functions
  - $$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$
  - $$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$$
  - $$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$
  - $$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$$
  - $$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$
  - $$\lim_{x\to 0^+} x^\alpha \ln(x) = 0 \qquad \lim_{x\to +\infty} \frac{\ln(x)}{x^\alpha} = 0 (\alpha > 0)$$
  - $$\lim_{x\to +\infty} \frac{x^\alpha}{\alpha^x} = 0 \qquad \lim_{x\to +\infty} \frac{\ln(x)}{a^x} = 0 (a > 1)$$

10.1 Fundamental limits

$$\lim_{x\to+\infty} c = c \qquad \lim_{x\to-\infty} c = c$$

$$\lim_{x\to+\infty} x = +\infty \qquad \lim_{x\to-\infty} x = -\infty$$

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

$$\lim_{x\to0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to+\infty} e^x = +\infty$$

$$\lim_{x\to(\frac{\pi}{2}+ka)} \tan x = -\infty \qquad \lim_{x\to(\frac{\pi}{2}+ka)} \tan x = +\infty$$

$$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2} \qquad \lim_{x\to+\infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x\to0} \log x = -\infty \qquad \lim_{x\to+\infty} \log x = +\infty$$

11 Infinitesimals, Infinities, and Their Comparison

11.1 Infinitesimals

Infinitesimal as $x \to \alpha$

Definition 11.1. A function $f(x)$ is called an infinitesimal as $x \to \alpha$ when the limit of $f(x)$ as $x \to \alpha$ is equal to 0.

Remark 1. α can be finite, $+\infty$, or $-\infty$.

If $f(x)$ and $g(x)$ are both infinitesimals as $x \to \alpha$, they are called **simultaneous infinitesimals**. In this case, it is interesting to see which of the two infinitesimals tends to 0 *more rapidly*; we can establish this by determining the limit (if it exists) of their ratio as $x \to \alpha$. Let $f(x)$ and $g(x)$ be two simultaneous infinitesimals as $x \to \alpha$, and assume that there exists an interval I around α such that $g(x) \neq 0$ for every $x \in I$, with $x \neq \alpha$.

- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = l \neq 0$ (l finite), $f(x)$ and $g(x)$ are said to be **of the same order** (essentially meaning they tend to 0 at the same rate).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 0$, $f(x)$ is called a **higher-order infinitesimal** compared to $g(x)$ (i.e., f tends to 0 more rapidly than g).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \pm\infty$, $f(x)$ is called a **lower-order infinitesimal** compared to $g(x)$ (i.e., f tends to 0 less rapidly than g).
- If *the limit* $\lim_{x\to\alpha} \frac{f(x)}{g(x)}$ *does not exist*, the **infinitesimals** $f(x)$ and $g(x)$ **are not comparable**.

11.2 Infinity

Infinity as $x \to \alpha$

Definition 11.2. A function $f(x)$ is called infinity as $x \to \alpha$ when the limit of $f(x)$ as $x \to \alpha$ is $+\infty$, $-\infty$, or ∞.

For infinities, we can introduce concepts analogous to those seen for infinitesimals. If both $f(x)$ and $g(x)$ are infinities as $x \to \alpha$, they are called **simultaneous infinities**. Let $f(x)$ and $g(x)$ be simultaneous infinities as $x \to \alpha$.

- If $\lim_{x\to\alpha} = l \neq 0$ (l finite), $f(x)$ and $g(x)$ are said to be **of the same order** (essentially meaning they tend to ∞ at the same rate).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 0$, $f(x)$ is called a **lower-order infinity** compared to $g(x)$ (i.e., f tends to ∞ less rapidly than g).
- If $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \pm\infty$, $f(x)$ is called a **higher-order infinity** compared to $g(x)$ (i.e., f tends to ∞ more rapidly than g).

- If *the limit* $\lim_{x\to\alpha} \frac{f(x)}{g(x)}$ *does not exist*, the **infinities** $f(x)$ and $g(x)$ **are not comparable**.

12 Limits of Sequences

For sequences, which are special functions, all the theorems of limit calculus apply. Thus, given sequences a_n and b_n, if $\lim_{n\to+\infty} = l$ and $\lim_{n\to+\infty} b_n = l'$, the following theorems hold.

- **Sum of Limits Theorem**: $\lim_{n\to+\infty}(a_n + b_n) = l + l'$;
- **Difference of Limits Theorem**: $\lim_{n\to+\infty}(a_n - b_n) = l - l'$;
- **Product of Limits Theorem**: $\lim_{n\to+\infty}(a_n \cdot b_n) = l \cdot l'$;
- **Quotient of Limits Theorem**: if $b \neq 0, \forall n \in \mathbb{N}$ and $l' \neq 0$, then $\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{l}{l'}$;

These theorems are analogous to those studied for functions as $x \to +\infty$. Similar theorems are also valid when one or more sequences are divergent.

13 Continuous Functions

13.1 Definition of Continuous Function

Continuous Function at a Point

Definition 13.1. Let $f(x)$ be a function defined in an interval $[a;b]$, and x_0 be a point within the interval. The function $f(x)$ is said to be continuous at the point x_0 when the limit of $f(x)$ as x approaches x_0 exists and is equal to the value $f(x_0)$ of the function evaluated at x_0:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

.

Note 11. Applying the definition of limit, $f(x)$ is continuous at x_0 if, for all $\epsilon > 0$, there exists a complete neighborhood I of x_0 such that

$$|f(x) - f(x_0)| < \epsilon, \ \forall x \in I$$

A function $f(x)$ is thus continuous at x_0 if:

- it is defined at x_0, meaning $f(x_0)$ exists;
- the limit $\lim_{x \to x_0} f(x)$ is finite;
- the value of the limit equals $f(x_0)$.

If we consider only the right or left limit of a function $f(x)$, we can give the following definitions:

- $f(x)$ is **right-continuous** at x_0 if $f(x_0)$ coincides with the right limit of $f(x)$ as x approaches x_0:

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

- $f(x)$ is **left-continuous** at x_0 if $f(x_0)$ coincides with the left limit of $f(x)$ as x approaches x_0:

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

Continuity can also be discussed for points that are endpoints of the interval $[a;b]$ where the function is defined; at point a, we talk about right-continuity, while at point b, we talk about left-continuity.

Continuous Function in an Interval

Definition 13.2. A function defined in $[a;b]$ is said to be continuous in the interval $[a;b]$ if it is continuous at every point within the interval.

Functions that are rational, irrational (integer and fractional), exponential, logarithmic, and trigonometric are continuous in every interval of their domain. Moreover, if $f(x)$ and $g(x)$ are functions continuous at a point or in an interval, then the following functions are also continuous at the same point or interval:

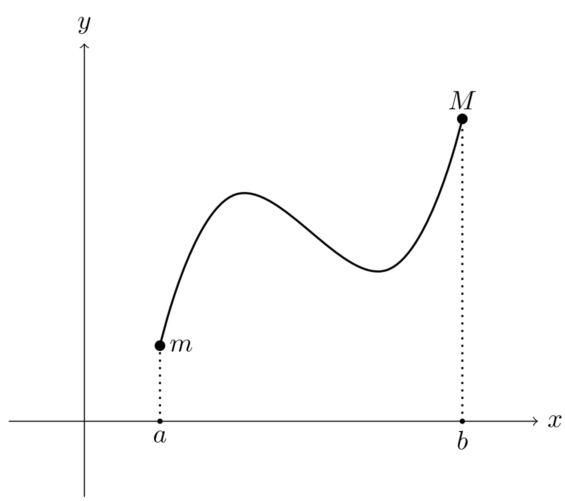
$$f(x) \pm g(x), \quad kf(x), \quad f(x) \cdot g(x), \quad [f(x)]^n, \quad \frac{f(x)}{g(x)}, \quad \text{with } g(x) \neq 0, k \in \mathbb{R}, n \in \mathbb{N} - \{0\}.$$

13.2 Theorems on Continuous Functions

Let's state some theorems that express important properties enjoyed by continuous functions and illustrate their graphical consequences.

Weierstrass' Theorem

Theorem 13.1. *If f is a continuous function in a bounded and closed interval $[a;b]$, then it takes on both the absolute maximum and the absolute minimum within that interval.*



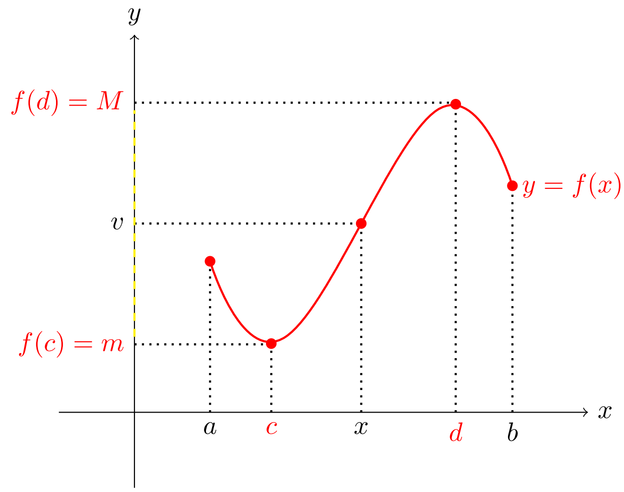
Note 12. Given the function $y = f(x)$ defined in the interval I, we call:

- the **absolute maximum** of $f(x)$, if it exists, the maximum M of the values assumed by the function in I;
- the **absolute minimum** of $f(x)$, if it exists, the minimum m of the values assumed by the function in I.

Intermediate Value Theorem

Theorem 13.2. *If f is a continuous function in a bounded and closed interval $[a;b]$, then it takes on every value at least once between the maximum and minimum. In symbols:*

$$\text{Let } f \text{ be continuous in } [a;b] \Rightarrow \forall v : m \leq v \leq M, \exists x \in [a;b] : f(x) = v$$

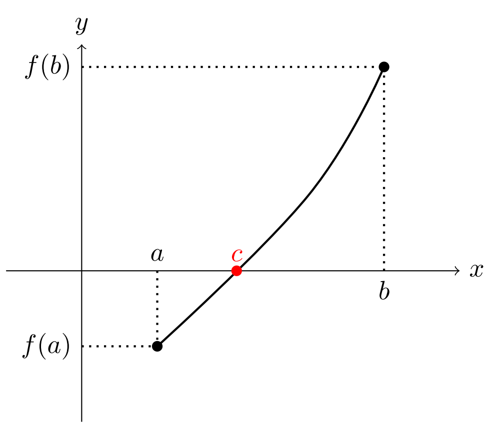


Existence of Zeros Theorem

Theorem 13.3. *If f is a continuous function in a bounded and closed interval $[a;b]$ and assumes values of opposite signs at the endpoints of this interval, then there exists at least one point c, within the interval, where f equals zero.*

In symbols:

$$f \text{ continuous in } [a;b], \quad f(a) < 0, f(b) > 0 \Rightarrow \exists c \in]a;b[: f(c) = 0$$



14 Points of Discontinuity of a Function

A point x_0 in an interval $[a; b]$ is called a **point of discontinuity** for a function $f(x)$ if the function is not continuous at x_0.

Note 13. A point of discontinuity is also called a **singular point**.

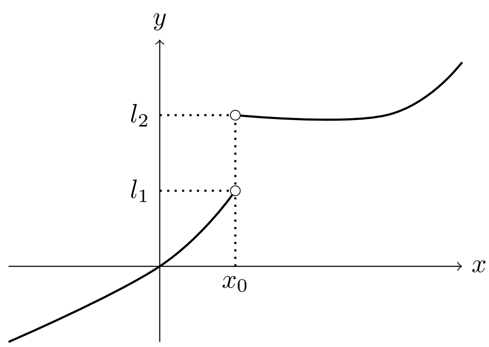
14.1 Points of First Kind Discontinuity

Point of First Kind Discontinuity

Definition 14.1. A point x_0 is called a point of first kind discontinuity for the function $f(x)$ when, as $x \to x_0$, both the right and left limits of $f(x)$ exist but are different from each other.

$$\lim_{x \to x_0^-} f(x) = l_1 \neq \lim_{x \to x_0^+} f(x) = l_2.$$

The difference $|l_2 - l_1|$ is called the **jump** of the function.



14.2 Points of Second Kind Discontinuity

Point of Second Kind Discontinuity

Definition 14.2. A point x_0 is called a point of second kind discontinuity for the function $f(x)$ when, as $x \to x_0$, at least one of the two limits, right or left, of $f(x)$ is infinite, or one of them does not exist.

14.3 Points of Third Kind Discontinuity (or removable)

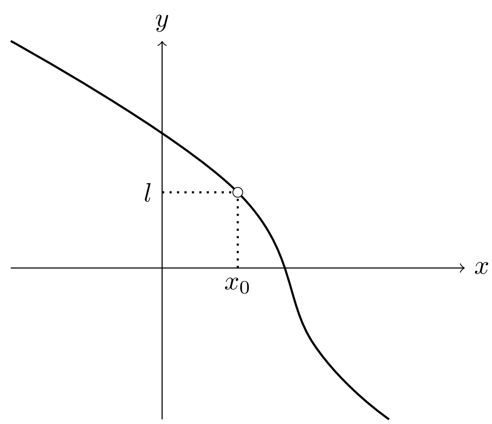
Point of Third Kind Discontinuity (or removable)

Definition 14.3. A point x_0 is called a point of third kind discontinuity for the function $f(x)$ when:

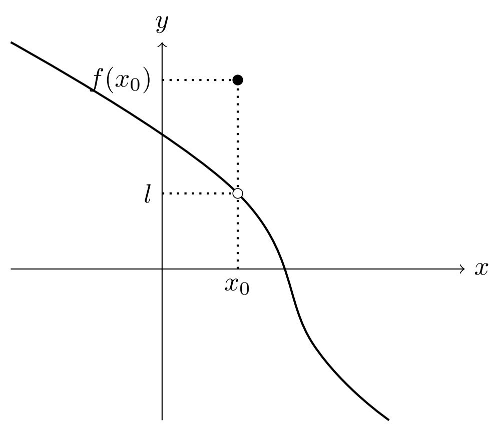
1. the limit of $f(x)$ as $x \to x_0$ exists and is finite, i.e., $\lim_{x \to x_0} f(x) = l$;

2. f is either undefined at x_0, or if defined, $f(x_0) \neq l$.

- f is undefined at x_0



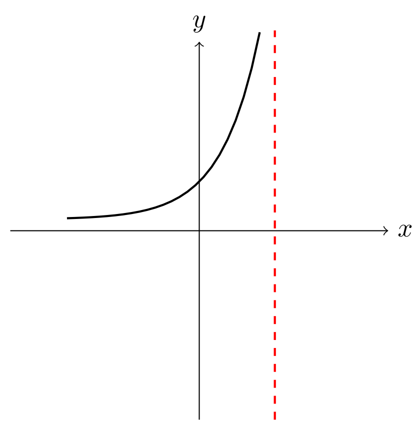
- f is defined at x_0, but $f(x_0) \neq l$



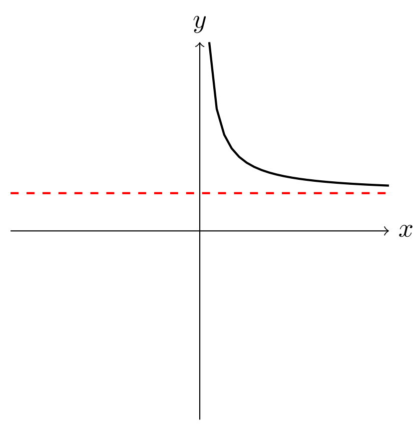
15 Asymptotes

An asymptote of a function $f(x)$ is a line whose distance from the graph of $f(x)$ tends to 0 as a generic point P on the graph moves towards infinity.

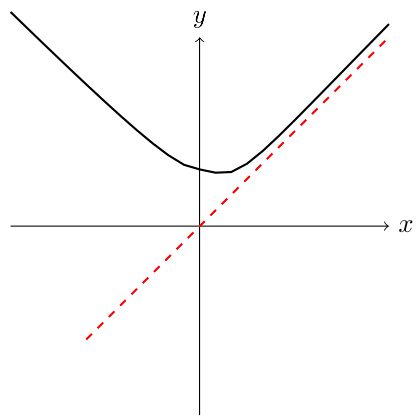
- **Vertical asymptotes**



- **Horizontal asymptotes**



- **Oblique Asymptotes**



15.1 Search for Horizontal and Vertical Asymptotes

In general, horizontal asymptotes are determined by calculating $\lim_{x\to\infty} f(x)$, while vertical asymptotes are determined by calculating $\lim_{x\to x_0} f(x)$, where x_0 does not belong to the domain.

Note 14. A horizontal asymptote with equation $y = c$ occurs when:

$$\lim_{x\to\infty} f(x) = c.$$

Note 15. A vertical asymptote with equation $x = x_0$ occurs when:

$$\lim_{x\to x_0} f(x) = \infty.$$

15.2 Oblique Asymptotes

Oblique Asymptote

Definition 15.1. Given the function $y = f(x)$, if it is verified that

$$\lim_{x\to\infty} [f(x) - (mx + q)] = 0,$$

then the line with equation $y = mx + q$ is called an oblique asymptote for the graph of the function.

Note 16. From

$$\lim_{x\to\infty} [f(x) - (mx + q)] = 0,$$

we deduce

$$\lim_{x\to\infty} = \lim_{x\to\infty} (mx + q),$$

leading to:

$$\lim_{x\to\infty} f(x) = \infty,$$

a necessary (but not sufficient) condition for the existence of the oblique asymptote.

The same definition applies by replacing ∞ with $+\infty$ or $-\infty$. For $x \to +\infty$, it is called a right oblique asymptote, and for $x \to -\infty$, it is called a left oblique asymptote.

15.3 Search for Oblique Asymptotes

Theorem 15.1. *If the graph of the function $y = f(x)$ has an oblique asymptote with equation $y = mx + q$, where $m \neq 0$, then m and q are given by the following limits:*

$$m = \lim_{x\to\infty} \frac{f(x)}{x}; \quad q = \lim_{x\to\infty} [f(x) - mx]$$

Proof. If an oblique asymptote exists, it is true that

$$\lim_{x\to\infty} [f(x) - (mx + q)] = 0,$$

and therefore, dividing by $x \neq 0$.

$$\lim_{x\to\infty} \frac{f(x) - (mx + q)}{x} = 0 \to \lim_{x\to\infty} [\frac{f(x)}{x} - m - \frac{q}{x}] = 0,$$

and, since $\lim_{x\to\infty} m = m$ and $\lim_{x\to\infty} \frac{q}{x} = 0$, it must be:

$$m = \lim_{x\to\infty} \frac{f(x)}{x}.$$

If m is not zero, to calculate q, consider again:

$$\begin{aligned} \lim_{x\to\infty} [f(x) - (mx + q)] = 0 &\to \lim_{x\to\infty} [(f(x) - mx) - q] = 0 \\ &\to \lim_{x\to\infty} [f(x) - mx] - q = 0 \\ &\to q = \lim_{x\to\infty} [f(x) - mx]. \end{aligned}$$

□

Conversely, it can be shown that if $\lim_{x\to\infty} f(x) = \infty$ and the limits $m = \lim_{x\to\infty} \frac{f(x)}{x}$ and $q = \lim_{x\to\infty}[f(x) - mx]$ exist and are finite, with $m \neq 0$, then the graph of the function $y = f(x)$ has an oblique asymptote with equation $y = mx + q$.

Note 17. A function can have an oblique asymptote only if

$$\lim_{x\to\infty} f(x) = \infty,$$

or one of the analogous limits with $+\infty$ or $-\infty$.

Moreover, the theorem is valid even if we replace ∞ with $+\infty$ or $-\infty$.

Part III
Derivatives

Fundamental Derivatives

Powers of x

$$\mathrm{D}\, k = 0$$
$$\mathrm{D}\, x^a = a x^{a-1}, \quad a \in \mathbb{R}$$
$$\mathrm{D}\, x = 1$$
$$\mathrm{D}\, \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$
$$\mathrm{D}\, \sqrt[n]{x} = \frac{1}{n\sqrt[n]{x^{n-1}}}, \quad x > 0,\ n \in \mathbb{N}$$
$$\mathrm{D}\, \frac{1}{x} = -\frac{1}{x^2}$$

Logarithmic and Exponential Functions

$$\mathrm{D}\, a^x = a^x \ln a, \quad a > 0$$
$$\mathrm{D}\, e^x = e^x$$
$$\mathrm{D}\, \log_a x = \frac{1}{x} \log_a e, \quad x > 0$$
$$\mathrm{D}\, \ln x = \frac{1}{x}, \quad x > 0$$

Trigonometric Functions

$$\mathrm{D}\, \sin x = \cos x$$
$$\mathrm{D}\, \sin x^\circ = \frac{\pi}{180^\circ} \cos x^\circ$$
$$\mathrm{D}\, \cos x = -\sin x$$
$$\mathrm{D}\, \cos x^\circ = -\frac{\pi}{180^\circ} \sin x^\circ$$
$$\mathrm{D}\, \tan x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$
$$\mathrm{D}\, \cot x = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x)$$

Inverse Trigonometric Functions

$$\mathrm{D}\ \arctan x = \frac{1}{1+x^2}$$
$$\mathrm{D}\,\mathrm{arccot}\ x = -\frac{1}{1+x^2}$$
$$\mathrm{D}\ \arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\mathrm{D}\ \arccos x = -\frac{1}{1-x^2}$$

Derivative Rules

$$\mathrm{D}\,[k \cdot f(x)] = k \cdot f'(x)$$
$$\mathrm{D}\,[f(x) + g(x)] = f'(x) + g'(x)$$
$$\mathrm{D}\,[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
$$\mathrm{D}\,[f(x) \cdot g(x) \cdot z(x)] = f'(x) \cdot g(x) \cdot z(x) + f(x) \cdot g'(x) \cdot z(x) + f(x) \cdot g(x) \cdot z'(x)$$
$$\mathrm{D}\,[f(x)]^a = a[f(x)]^{a-1} \cdot f'(x), \quad a \in \mathbb{R}$$
$$\mathrm{D}\ \left[\frac{1}{f(x)}\right] = -\frac{f'(x)}{f^2(x)}$$
$$\mathrm{D}\ \left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$
$$\mathrm{D}\,[f(g(x))] = f'(z) \cdot g'(x), \quad z = g(x)$$
$$\mathrm{D}\,[f(g(z(x)))] = f'(u) \cdot g'(t) \cdot z'(x), \quad t = z(x),\ u = g(t)$$
$$\mathrm{D}\,[f(x)]^{g(x)} = [f(x)]^{g(x)} \left[g'(x) \ln f(x) + \frac{g(x) \cdot f'(x)}{f(x)}\right]$$
$$\mathrm{D}\,[f^{-1}(y)] = \frac{1}{f'(x)}, \quad x = f^{-1}(y)$$

16 Derivative of a Function

16.1 Incremental Ratio

Given a function $y = f(x)$ defined in an interval $[a; b]$ and a point on its graph $A(c; f(c))$, let's increase the abscissa of A by an amount h to obtain the point B with coordinates:

$$x_B = c + h; \quad y_B = f(x_B) = f(c + h)$$

meaning,

$$B(c + h; f(c + h))$$

Let's consider the increments:

$$\Delta x = x_B - x_A = h \quad \text{and} \quad \Delta y = y_B - y_A = f(c + h) - f(c)$$

The ratio of the two increments is $\dfrac{\Delta y}{\Delta x}$.

Note 18. Generally, the notation Δt is called the **increment of the variable** t and indicates the difference between two values t_2 and t_1 of a quantity t:

$$\Delta t = t_2 - t_1.$$

Incremental Ratio

Definition 16.1. Given a function $y = f(x)$ defined in an interval $[a; b]$ and two real numbers c and $c + h$ within the interval, the incremental ratio of f (relative to c) is defined as:

$$\frac{\Delta y}{\Delta x} = \frac{f(c + h) - f(c)}{h}$$

Considering the points $A(c; f(c))$ and $B(c + h; f(c + h))$ on the graph of f, the *incremental ratio of f relative* to c is the **slope of the line passing through A and B.**

16.2 Derivative of a Function

Derivative of a Function

Definition 16.2. For a function $y = f(x)$ defined in an interval $[a; b]$, the derivative of the function at the point c within the interval, denoted by $f'(c)$, is the limit, if it exists and is *finite*, as h tends to 0, of the incremental ratio of f relative to c:

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

Remark 2. The derivative of a function at a point c represents the **slope of the tangent line** to the graph of the function at its abscissa c.

A function is said to be **differentiable** at a point c if the derivative $f'(c)$ exists. For a function to be differentiable at c, the following conditions must be satisfied:

1. the function is defined in a neighborhood of point c;
2. the limit of the incremental ratio relative to c exists for h tending to 0, i.e., the right-hand limit and left-hand limit of this ratio exist, and they coincide;
3. this limit is a finite number.

The derivative of a function $y = f(x)$ at a generic point x is denoted by one of the following symbols:

$$f'(x); \quad \mathrm{D}f(x); \quad y'.$$

If the limit as h tends to 0 of the incremental ratio of a function at a point *does not exist* or is *infinite*, the function is said to be **non-differentiable** at that point.

16.3 Calculating the Derivative

Example 16.1. Let's calculate the derivative of the function

$$y = (x-1)^2$$

at $c = 3$. Let f be the function, applying the definition:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}.$$

Calculate the values the function takes at the points with abscissa 3 and $3 + h$:

$$f(3) = (3-1)^2 = 2^2 = 4; \quad f(3+h) = (3+h-1)^2 = (2+h)^2 = h^2 + 4h + 4$$

Substitute these values into the incremental ratio and simplify:

$$f'(3) = \lim_{h \to 0} \frac{h^2 + 4h + 4 - 4}{h} = \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0} h + 4 = 4.$$

Therefore, $f'(3) = 4$.

Note 19. The derivative $f'(3)$ is a real number and is the slope of the tangent line to the graph of $f(x)$ at the point $(3; f(3))$.

We can also calculate the derivative of a function at a generic point. In this case, the value $f'(x)$ we obtain is a function of x, and for this reason, we also talk about the derivative function. The derivative function, as x varies, provides the slope of all tangent lines to the given function.

Example 16.2. Let's calculate the derivative of the function $f(x) = x^2 - x$ at a generic point x:

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 2xh - h}{h} \\ &= \lim_{h \to 0} \frac{h(h + 2x - 1)}{h} \\ &= \lim_{h \to 0} h + 2x - 1 \Rightarrow f'(x) = 2x - 1 \end{aligned}$$

16.4 Left Derivative and Right Derivative

Left Derivative and Right Derivative

Definition 16.3. The **left derivative** of a function at a point c is:

$$f'_{-}(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}.$$

The **right derivative** of a function at a point c is:

$$f'_{+}(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}.$$

A function is **differentiable** at a point c if the left derivative and the right derivative exist, are *finite*, and are *equal*.

Function Defined on an Interval

Definition 16.4. A function $y = f(x)$ is differentiable on a closed interval $[a; b]$ if it is differentiable at all internal points of $[a; b]$, and if the right derivative at a and the left derivative at b exist and are finite.

17 Tangent Line to the Graph of a Function

In general, given the function $y = f(x)$, the equation of the tangent line to the graph of f at the point $(x_0; y_0)$, if such a line exists and is not parallel to the y-axis, is:

$$y - y_0 = f'(x_0) \cdot (x - x_0)$$

17.1 Stationary Points

Stationary Point

Definition 17.1. For the function $y = f(x)$ and its point $x = c$, if $f'(c) = 0$, then $x = c$ is called a **stationary point** or a *point of horizontal tangency.*

18 Continuity and Differentiability

Theorem 18.1. *If a function $f(x)$ is differentiable at the point x_0, then the function is also continuous at that point.*

Proof.
Hypothesis:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Thesis:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

We write the relation

$$f(x_0 + h) = f(x_0) + \frac{f(x_0 + h) - f(x_0)}{h} \cdot h$$

which, after calculations, turns out to be an identity. Calculating the limit for $h \to 0$ on both sides, remembering that the limit of a sum is equal to the sum of the limits:

$$\lim_{h \to 0} f(x_0 + h) = \lim_{h \to 0} f(x_0) + \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot h.$$

In the second term, since the limit of a constant is the constant itself, we have:

$$\lim_{h \to 0} f(x_0) = f(x_0).$$

Moreover, since the limit of a product is equal to the product of the limits and recalling the hypothesis:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot 0 = f'(x_0) \cdot 0 = 0.$$

Therefore, substituting in the second term, the limit becomes:

$$\lim_{h \to 0} f(x_0 + h) = f(x_0) + f'(x_0) \cdot 0$$

$$\lim_{h \to 0} f(x_0 + h) = f(x_0.)$$

Setting $x_0 + h = x$, if $h \to 0$, it implies $x \to x_0$. Substituting into the previous relation, we conclude that the function $f(x)$ is continuous at x_0, as:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

□

Note 20. In the proof of the theorem, we have seen that the expression $\lim_{x \to 0} f(x_0 + h) = f(x_0)$ is equivalent to $\lim_{x \to x_0} f(x) = f(x_0)$. We can therefore assume it as the definition of a continuous function: a function is continuous if $\lim_{h \to 0} f(x_0 + h) = f(x_0)$.

From what we have said, we can assert that the set of differentiable functions is a subset of the set of continuous functions.

19 Fundamental Derivatives

Now let's determine the differentiation formulas for the most commonly used functions.

Theorem 19.1. *The derivative of a constant function is* 0*:*

$$\mathrm{D}k = 0$$

Proof. Remembering that if $f(x) = k$, then $f(x+h) = k$, calculate:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{k-k}{h} = 0.$$

□

Theorem 19.2. *The derivative of the function $f(x) = x$ is $f'(x) = 1$;*

$$\mathrm{D}x = 1.$$

Proof. If $f(x) = x$, then $f(x+h) = x+h$. Calculate $f'(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

□

Theorem 19.3. *The derivative of the function $f(x) = \sin(x)$, with x expressed in radians, is $f'(x) = \cos(x)$:*

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h} \\ &= \lim_{h\to 0} [\sin(x) \cdot \frac{(\cos(h)-1)}{h} + \cos(x) \cdot \frac{\sin(h)}{h}] \end{aligned}$$

Using known limits,

$$\lim_{h\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{1-\cos(x)}{x} = 0,$$

we have:

$$f'(x) = \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)$$

$\square$

Note 21. If x is measured in degrees:

$$\frac{d}{dx}\sin(x^\circ) = \frac{\pi}{180^\circ}\cdot\cos(x)$$

Theorem 19.4. *The derivative of the function $f(x) = \cos(x)$, with x expressed in radians, is $f'(x) = -\sin(x)$:*

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Note 22. If x is measured in degrees:

$$\frac{d}{dx}\cos(x^\circ) = -\frac{\pi}{180^\circ}\cdot\sin(x)$$

Theorem 19.5. *The derivative of the function $f(x) = a^x (a \in \mathbb{R}^+)$ is $f'(x) = a^x \ln(a)$:*

$$\frac{d}{dx}a^x = a^x \ln a.$$

Proof. Applying the definition of derivative to the function $f(x) = a^x$, we get:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x(a^h-1)}{h} \\ &= \lim_{h\to 0} a^x \cdot \lim_{h\to 0}\frac{a^h-1}{h} \end{aligned}$$

Using the well-known limit

$$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a,$$

we get

$$a^x \ln a.$$

$\square$

In particular
$$\frac{d}{dx}e^x = e^x.$$

Theorem 19.6. *The derivative of the function* $f(x) = \log_a x \quad (a \in \mathbb{R}^+ - \{1\}, x \in \mathbb{R}^+)$ *is* $f'(x) = \frac{1}{x} \cdot \log_a e$*:*

$$\frac{d}{dx}\log_a x = \frac{1}{x} \cdot \log_a e.$$

Proof.

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}.$$

Using the logarithmic property $\log_a x - \log_a y = \log_a \frac{x}{y}$, we write:

$$\log_a(x+h) - \log_a x = \log_a \frac{x+h}{h} = \log_a\left(1+\frac{h}{x}\right),$$

$$f'(x) = \lim_{h\to 0}\frac{\log_a\left(1+\frac{h}{x}\right)}{h}.$$

Multiplying and dividing the denominator h by x, we have

$$\lim_{h\to 0}\frac{\log_a\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \cdot \frac{1}{x} = \log_a e \cdot \frac{1}{x},$$

and therefore:
$$f'(x) = \frac{1}{x} \cdot \log_a e.$$

$\square$

In particular, for $a = e$ we have:

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

20 Theorems on the Calculation of Derivatives

20.1 Derivative of the Product of a Constant and a Function

Theorem 20.1. *The derivative of the product of a constant k and a differentiable function $f(x)$ is equal to the product of the constant and the derivative of the function:*

$$\mathrm{D}[k \cdot f(x)] = k \cdot f'(x).$$

Proof.

$$y' = \lim_{h\to 0} \frac{k \cdot f(x+h) - k \cdot f(x)}{h} = \lim_{h\to 0} \frac{k \cdot [f(x+h) - f(x)]}{h}.$$

Since k is a constant and recalling the definition of the derivative, we can write:

$$y' = k \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = k \cdot f'(x).$$

□

20.2 Derivative of the Sum of Functions

Theorem 20.2. *The derivative of the algebraic sum of two or more differentiable functions is equal to the algebraic sum of the derivatives of the individual functions:*

$$\mathrm{D}[f(x) + g(x)] = f'(x) + g'(x).$$

Proof. Let's calculate the limit of the incremental ratio of $f(x) + g(x)$:

$$\begin{aligned} y' &= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} = \\ &= \lim_{h\to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} = \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \\ &= f'(x) + g'(x) \end{aligned}$$

□

20.3 Derivative of the Product of Functions

Theorem 20.3. *The derivative of the product of two differentiable functions is equal to the sum of the derivative of the first function multiplied by the second non-derivative and the derivative of the second function multiplied by the first non-derivative:*

$$\mathrm{D}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Proof.

$$y' = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}.$$

In the numerator, let's add and subtract the product $g(x+h) \cdot f(x)$ to the first term:

$$y' = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - g(x+h) \cdot f(x) + g(x+h) \cdot f(x) - f(x) \cdot g(x)}{h}.$$

Factor out $g(x+h)$ from the first two terms and $f(x)$ from the last two terms:

$$y' = \lim_{h \to 0} \frac{g(x+h) \cdot [f(x+h) - f(x)] + f(x) \cdot [g(x+h) - g(x)]}{h}.$$

The limit of a sum is equal to the sum of the limits, so:

$$y' = \lim_{h \to 0} \left[g(x+h) \cdot \frac{f(x+h) - f(x)}{h} \right] + \lim_{h \to 0} \left[f(x) \cdot \frac{g(x+h) - g(x)}{h} \right].$$

The limit of a product is equal to the product of the limits:

$$y' = \lim_{h \to 0} g(x+h) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

Since $f(x)$ and $g(x)$ are differentiable and hence continuous by assumption, we have:

$$y' = g(x) \cdot f'(x) + f(x) \cdot g'(x).$$

□

Extending the theorem to the product of more functions, it can be shown that, for example, given the function $y = f(x) \cdot g(x) \cdot z(x)$, its first derivative is:

$$y' = f'(x) \cdot g(x) \cdot z(x) + f(x) \cdot g'(x) \cdot z(x) + f(x) \cdot g(x) \cdot z'(x).$$

In general, the derivative of the product of multiple differentiable functions is the sum of the products of the derivative of each function by the other non-derivative functions.

20.4 Derivative of the Power of a Function

Theorem 20.4. *The derivative of the n-th power of a differentiable function (with exponent $n \in \mathbb{N}$ and $n > 1$) is equal to the product of the exponent n and the function raised to the power of $n - 1$, multiplied by the derivative of the function itself:*

$$\mathrm{D}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x).$$

Proof.

$$y = \underbrace{f(x) \cdot f(x) \cdot f(x) \cdot \ldots \cdot f(x)}_{n \text{ factors}}$$

Using the theorem of the derivative of the product of multiple functions, we have:

$$y' = f'(x) \cdot \underbrace{f(x) \cdot \ldots \cdot f(x)}_{(n-1) \text{ factors}} + f(x) \cdot f'(x) \cdot \ldots \cdot f(x) + \ldots + \underbrace{f(x) \cdot f(x) \cdot \ldots}_{(n-1) \text{ factors}} \cdot f'(x).$$

In each of the n terms, the factor $f(x)$ appears $(n-1)$ times. Therefore:

$$y' = n \cdot [f(x)]^{n-1} \cdot f'(x).$$

□

It can be shown that the above theorem is also valid when the exponent of the power is any rational number.

$$\mathrm{D}[f(x)]^a = a \cdot [f(x)]^{a-1} \cdot f'(x), \quad a \in \mathbb{Q}.$$

Derivative of a Power of x If $f(x) = x$, since the derivative $f'(x) = 1$, we have:

$$\mathrm{D}x^a = a \cdot x^{a-1}, \quad a \in \mathbb{Q}.$$

20.5 Derivative of the Reciprocal of a Function

Theorem 20.5. *The derivative of the reciprocal of a non-zero differentiable function is equal to a fraction where:*

- *the numerator is the opposite of the derivative of the function;*
- *the denominator is the square of the function.*

$$\mathrm{D}\frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}, \quad \text{with } f(x) \neq 0.$$

Proof.

$$\begin{aligned}
y' &= \lim_{h \to 0} \frac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h} = \lim_{h \to 0} \frac{\dfrac{f(x) - f(x+h)}{f(x)f(x+h)}}{h} = \\
&= \lim_{h \to 0} \left[-\frac{f(x+h) - f(x)}{h} \cdot \frac{1}{f(x)f(x+h)} \right] = \\
&= -\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} \frac{1}{f(x)f(x+h)}.
\end{aligned}$$

Since $f(x)$ is differentiable, we have:

$$y' = -\frac{f'(x)}{f^2(x)}.$$

□

Note 23. The function value must be different from 0 at the points where we calculate the derivative.

20.6 Derivative of the Quotient of Two Functions

Theorem 20.6. *The derivative of the quotient of two differentiable functions (with a non-zero divisor function) is equal to a fraction with:*

- *the numerator as the difference between the derivative of the dividend multiplied by the non-derivative divisor and the non-derivative dividend multiplied by the derivative of the divisor;*
- *the denominator as the square of the divisor.*

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}, \quad g(x) \neq 0$$

Proof. Consider the quotient function as the product of two functions:

$$y = f(x)\cdot\frac{1}{g(x)}.$$

Apply the product rule of differentiation:

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = \mathrm{D}\left[f(x)\cdot\frac{1}{g(x)}\right] = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\mathrm{D}\left[\frac{1}{g(x)}\right].$$

Apply the rule of differentiation of the reciprocal of a function:

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left[\frac{-g'(x)}{g^2(x)}\right].$$

Combine with a common denominator and conclude:

$$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}.$$

□

From the above theorem, we can derive the derivatives of the tangent and cotangent functions as special cases.

Tangent Function Derivative Express $y = \tan(x)$ as $y = \dfrac{\sin(x)}{\cos(x)}$ and, applying the quotient rule, we have:

$$y' = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)} = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)}.$$

This result can also be written in the following ways:

$$y' = \frac{1}{\cos^2(x)} \quad \text{or} \quad y' = 1 + \tan^2(x).$$

Cotangent Function Derivative Similarly, since we can rewrite $y = \cot(x)$ as $y = \dfrac{\cos(x)}{\sin(x)}$, we can derive that:

$$y' = -\frac{1}{\sin^2(x)} \quad \text{or} \quad y' = -(1 + \cot^2(x)).$$

21 Derivative of a Composite Function

Let $z = g(x)$ be a function of the variable x, from domain A to codomain B, and let $y = f(x)$ be a function of the variable z, from domain B to codomain C. The composite function $y = f(g(x))$ is a composite function (or function of a function) because y is a function of z, which in turn is a function of x. The two functions $z = g(x)$ and $y = f(x)$ are called the *components* of the composite function.

Theorem 21.1. *If the function g is differentiable at the point x and the function f is differentiable at the point $z = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x and its derivative is the product of the derivatives of f with respect to z and g with respect to x:*

$$\mathrm{D}[f(g(x))] = f'(z) \cdot g'(x), \quad z = g(x).$$

Generalizing for a polynomial $P(x)$:

$$\mathrm{D}[P(x)]^n = n[P(x)]^{n-1} P'(x).$$

The derivative of a composite function can also be calculated directly without making substitutions. The above theorem can be extended to the derivative of a function y dependent on the variable x through any number of component functions.

For example, in the case of three functions, being

$$y = f(g(z(x))),$$

setting

$$t = z(x), \quad u = g(t), \quad y = f(u),$$

the formula for the derivative of the composite function can be written as:

$$\mathrm{D}f(g(z(x))) = f'(u) \cdot g'(t) \cdot z'(x).$$

22 The derivative of $[f(x)]^{g(x)}$

Using the formulas related to the derivative of a composite function and the derivative of a product, we can study a method for calculating the derivative of the function

$$y = [f(x)]^{g(x)}$$

where $f(x) > 0$ and $f(x)$ and $g(x)$ are differentiable functions.

Given the function

$$y = [f(x)]^{g(x)},$$

since $f(x) > 0$, it is also $f(x)^{g(x)} > 0$, so we can calculate the logarithms of both sides:

$$\ln y = \ln[f(x)]^{g(x)}.$$

Applying the property of the logarithm of a power, we have:

$$\ln y = g(x) \cdot \ln[f(x)]$$

If we now apply the theorems for the derivative of composite functions and the product of two functions to both sides of the equation, we obtain

$$\frac{1}{y} \cdot y' = g'(x) \cdot \ln[f(x)] + g(x) \cdot \frac{1}{f(x)} \cdot f'(x),$$

from which, considering y different from 0:

$$y' = y \cdot \left[g'(x) \cdot \ln[f(x)] + \frac{g(x) \cdot f'(x)}{f(x)} \right].$$

Since $y = [f(x)]^{g(x)}$, we can write:

$$y' = [f(x)]^{g(x)} \cdot \left[g'(x) \cdot \ln[f(x)] + \frac{g(x) \cdot f'(x)}{f(x)} \right].$$

In conclusion, we can highlight the following formula for the derivative of the function $y = [f(x)]^{g(x)}$:

$$\mathrm{D}[f(x)]^{g(x)} = [f(x)]^{g(x)} \cdot \left[g'(x) \cdot \ln[f(x)] + \frac{g(x) \cdot f'(x)}{f(x)} \right].$$

If in the function $[f(x)]^{g(x)}$ we take $g(x) = a$ ($a \in \mathbb{R}$), and apply the previous rule, we get:

$$\mathrm{D}[f(x)]^a = [f(x)]^a \cdot \frac{a \cdot f'(x)}{f(x)} = a \cdot [f(x)]^{a-1} \cdot f'(x).$$

So the derivative rule for powers of a function is true for every $a \in \mathbb{R}$.

In particular, if $x > 0$ and $a \in \mathbb{R}$, we have:

$$\mathrm{D}x^a = a \cdot x^{a-1}.$$

23 The derivative of an inverse function

Theorem 23.1. *Consider the function $y = f(x)$ defined and invertible in the interval I, and its inverse function $x = f^{-1}(y)$. If $f(x)$ is differentiable with a derivative different from 0 at every point in I, then $f^{-1}(y)$ is also differentiable, and the following relationship holds:*

$$\mathrm{D}[f^{-1}(y)] = \frac{1}{f'(x)}, \quad \text{with } x = f^{-1}(y).$$

Assuming that both derivatives exist, to justify the relationship between them, we recall that

$$f^{-1}[f(x)] = x.$$

Differentiating both sides of this equation, we have

$$\mathrm{D}[f^{-1}(y)] \cdot f'(x) = 1,$$

from which we obtain:

$$\mathrm{D}[f^{-1}(y)] = \frac{1}{f'(x)}.$$

Of particular interest is the application of the theorem in calculating the derivatives of inverse trigonometric functions. The function $y = \arcsin x$, defined for $x \in [-1, 1]$, is the inverse of $x = \sin y$, with $y \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

Furthermore, the sine function is differentiable in $]-\frac{\pi}{2}; \frac{\pi}{2}[$ with a non-zero derivative. By the previous theorem, the function $\arcsin x$ is differentiable in $]-1; 1[$ and we have:

$$\mathrm{D}\arcsin x = \frac{1}{\mathrm{D}\sin y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

$$\mathrm{D}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$

Similarly, the following formulas can be obtained:

$$\mathrm{D}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\arctan x = \frac{1}{1+x^2}$$

$$\mathrm{D}\ \mathrm{arccot} x = -\frac{1}{1+x^2}$$

24 Derivatives of order higher than the first

Consider the function:

$$y = f(x) = x^3 + 2x^2 - x - 1, \quad x \in \mathbb{R}.$$

Its derivative,

$$y' = 3x^2 + 4x - 1,$$

is, in turn, a function of the variable x, defined for $x \in \mathbb{R}$. We can also calculate the derivative of this function:

$$\mathrm{D}y' = 6x + 4.$$

This derivative is called the **second derivative** of the function $f(x)$ and is denoted by the symbol:

$$y'' \quad \text{or} \quad f''(x).$$

The obtained second derivative is also a function that we can differentiate; differentiating it, we get the **third derivative**:

$$y''' = 6.$$

In general, given a function $y = f(x)$, with the examined procedure, we can obtain the second, third, fourth derivatives, and so on. These are called **higher-order derivatives** of the given function.

Note 24. From the fourth derivative onwards, we use the number in parentheses:

$$y^{(4)}, y^{(5)}, y^{(6)}, \ldots$$

25 The differential of a function

Let $f(x)$ be a differentiable and therefore continuous function in an interval, and let x and $(x + \Delta x)$ be two points in that interval.

Differential

Definition 25.1. The differential of a function $f(x)$, relative to the point x and the increment Δx, is the product of the derivative of the function calculated at x and the increment Δx. The differential is indicated by $df(x)$ or dy:

$$dy = f'(x) \cdot \Delta x.$$

Note 25. Note that the differential depends on two elements: the point x where we calculate the differential and the increment Δx we consider.

Example 25.1. The differential of the function

$$y = 2x^3 + 3$$

is

$$dy = 6x^2 \cdot \Delta x,$$

which for $x = 1$ and $\Delta x = 0.3$ is

$$dy = 6 \cdot (1)^2 \cdot 0.3 = 1.8,$$

while for $x = 2$ and $\Delta x = 0.2$ is

$$dy = 6 \cdot (2)^2 \cdot 0.2 = 4.8.$$

Consider the function

$$y = x$$

and calculate its differential:

$$dy = 1 \cdot \Delta x.$$

Therefore:

$$dx = \Delta x.$$

This means that the *differential of the independent variable x is equal to the increment of the variable itself.*

Substituting into the definition of the differential, we can write

$$dy = f'(x) \cdot dx,$$

which means that the *differential of a function is equal to the product of its derivative and the differential of the independent variable.*

From this last relation, by solving for $f'(x)$, we have:

$$f'(x) = \frac{dy}{dx}.$$

The first derivative of a function is thus the ratio of the differential of the function to that of the independent variable.

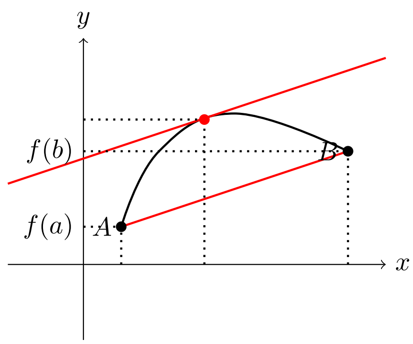
26 Theorems on differentiable functions

26.1 Lagrange's theorem

Lagrange's theorem

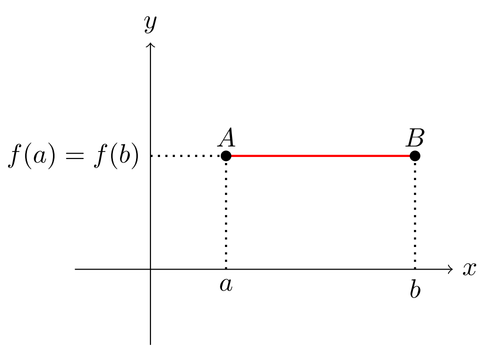
Theorem 26.1. *If a function $f(x)$ is continuous on a closed interval $[a;b]$ and is differentiable at every interior point, there exists at least one point c in the interval $[a;b]$ such that the relation holds:*

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$



From Lagrange's theorem, the following theorems follow.

Theorem 26.2. *If a function $f(x)$ is continuous on the interval $[a;b]$, differentiable on $]a;b[$, and such that $f'(x)$ is zero at every interior point of the interval, then $f(x)$ is constant throughout $[a;b]$.*



Proof. **Hypotheses:**

1. $f(x)$ is continuous in $[a;b]$;
2. $f'(x)=0$ in $]a;b[$.

Thesis:

$$f(x) = k \text{ in } [a; b].$$

Applying the Lagrange theorem in the interval $[a; x]$, where x is any point in $[a; b]$ different from a, we can write, with $c \in]a; x[$:

$$\frac{f(x) - f(a)}{x - a} = f'(c) = 0 \to f(x) - f(a) = 0 \to f(x) = f(a).$$

Therefore, f is constant throughout $[a; b]$. □

Theorem 26.3. *If $f(x)$ and $g(x)$ are two functions continuous in the interval $[a; b]$, differentiable in $]a; b[$, and such that $f'(x) = g'(x)$ for every $x \in]a; b[$, then they differ by a constant.*

Proof. **Hypotheses:**

1. $f(x)$ and $g(x)$ are continuous in $[a; b]$;
2. $f'(x) = g'(x)$ in $]a; b[$.

Thesis:

$$f(x) - g(x) = k \text{ in } [a; b].$$

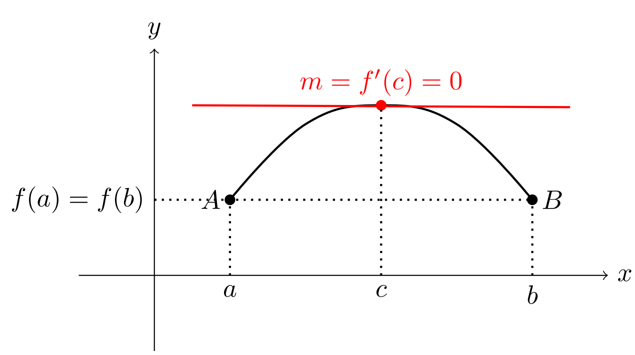
Calling $z(x)$ the difference between the given functions, i.e., $z(x) = f(x) - g(x)$, we have $z'(x) = f'(x) - g'(x)$. By hypothesis $f'(x) = g'(x)$, so $z'(x) = 0$, for every x in $]a; b[$. According to the previous theorem, $z(x) = k$ throughout $[a; b]$, and thus $f(x) - g(x) = k$. □

26.2 The Rolle's Theorem

If in the Lagrange theorem we add the hypothesis $f(a) = f(b)$, then $f'(c) = 0$. The following theorem is obtained.

Rolle's Theorem

Theorem 26.4. *If, for a function $f(x)$ continuous in the interval $[a; b]$ and differentiable at the interior points of this interval, the condition $f(a) = f(b)$ holds, then there exists at least one point c inside the interval such that $f'(c) = 0$.*



26.3 Cauchy's Theorem

Cauchy's Theorem

Theorem 26.5. *If the functions $f(x)$ and $g(x)$ are continuous on the interval $[a;b]$, differentiable at every interior point of this interval, and furthermore, $g'(x) \neq 0$ for all x in $]a;b[$, then there exists at least one point c in the interval $[a;b]$ such that:*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

In other words, the ratio of the increments of the functions $f(x)$ and $g(x)$ in the interval $[a;b]$ is equal to the ratio of their respective derivatives calculated at a particular point c within the interval.

26.4 L'Hôpital's Rule

The calculation of derivatives and the theorems studied so far are also useful for computing certain limits that appear in an indeterminate form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. This is possible thanks to the following theorem.

L'Hôpital's Rule

Theorem 26.6. *Given a neighborhood I of a point c and two functions $f(x)$ and $g(x)$ defined in I (excluding possibly c), if:*

- *$f(x)$ and $g(x)$ are differentiable in I with $g'(x) \neq 0$,*
- *both functions tend to either 0 or ∞ as $x \to c$,*
- *the limit of the ratio $\frac{f'(x)}{g'(x)}$ of their derivatives exists as $x \to c$,*

then the limit of the ratio of the functions exists and is:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

The theorem also extends to the limit as $x \to +\infty$ (or $-\infty$). In this case, the conditions of the theorem do not have to be true for a neighborhood of a point; instead, there must exist a value $M > 0$ such that these conditions are satisfied for all $x > M$ (or $x < -M$). The relation is then:

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)},$$

and a similar relationship holds for $x \to -\infty$.

In cases where the limit of the ratio of derivatives itself appears as an indeterminate form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and the functions $f'(x)$ and $g'(x)$ satisfy the assumptions of the theorem, one can proceed to the limit of the ratio of second derivatives, and so on for successive derivatives.

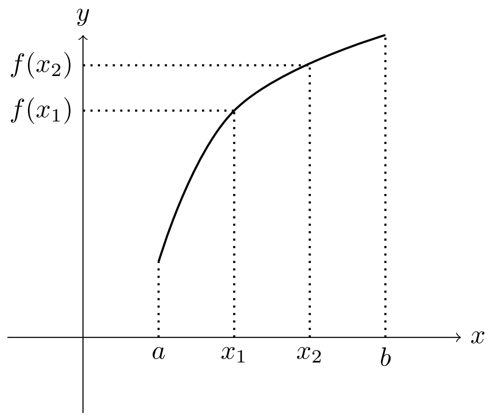
Part IV
Function Analysis

27 Increasing and Decreasing Functions and Their Derivatives

We provide two examples to recall the definitions related to increasing and decreasing functions.

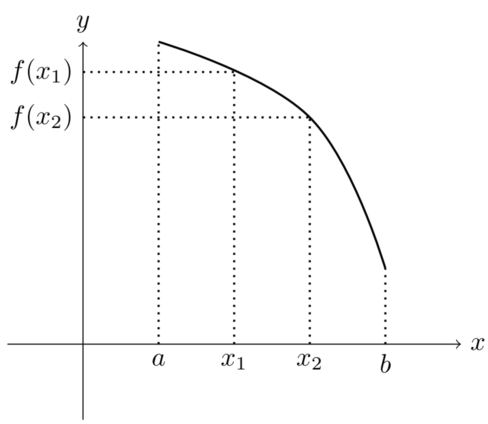
Example 27.1. If a function $y = f(x)$ is increasing in an interval, increasing the variable x results in an increase of y as well.

$$\forall x_1, x_2 \in [a; b], x_2 > x_1 \Rightarrow f(x_2) > f(x_1).$$



Example 27.2. If a function $y = f(x)$ is decreasing in an interval, increasing the variable x results in a decrease of y.

$$\forall x_1, x_2 \in [a; b], x_2 > x_1 \Rightarrow f(x_2) < f(x_1).$$



For increasing and decreasing functions, the following theorem holds.

Theorem 27.1. *Given a function $y = f(x)$, continuous in an interval I (both bounded and unbounded) and differentiable at the interior points of I, it is:*

1. *increasing in I, if its derivative is positive at every interior point of I;*

2. *decreasing in I, if its derivative is negative at every interior point of I.*

Proof. 1. Let x_1 and $x_2 \in I$, with $x_1 < x_2$. By the Mean Value Theorem applied to $f(x)$ in the interval $[x_1; x_2]$, we have:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad c \in]x_1; x_2[.$$

Since $x_2 - x_1 > 0$ and by hypothesis $f'(c) > 0$, we also have $f(x_2) - f(x_1) > 0$, hence $f(x_2) > f(x_1)$. Since x_1 and x_2 are arbitrary points of I, the function is increasing in I.

2. Proceeding similarly to the previous case, we obtain:

$$f(x_2) - f(x_1) < 0.$$

Indeed, $x_2 - x_1 > 0$ and by hypothesis $f'(c) < 0$, so $f(x_2) < f(x_1)$. Thus, the function is decreasing in I.

□

Note 26. This theorem is a **sufficient condition** to assert that a function is increasing or decreasing in an interval.

Note 27. The Mean Value Theorem holds because the function is continuous in $[x_1; x_2]$ and differentiable in $]x_1; x_2[$.

We can apply this theorem to determine the intervals in which a function is increasing or decreasing by studying the sign of its first derivative.

28 Maxima, Minima, and Inflection Points

28.1 Absolute Maxima and Minima

Absolute Maximum, Absolute Minimum

Definition 28.1. Given the function $y = f(x)$, defined in the interval I, we call:

- absolute maximum of $f(x)$, if it exists, the maximum M of the values assumed by the function in I, that is,

$$M = f(x_0), x_0 \in I \wedge M \geq f(x), \forall x \in I;$$

- absolute minimum of $f(x)$, if it exists, the minimum m of the values assumed by the function in I, that is,

$$m = f(x_1), x_1 \in I \wedge m \leq f(x), \forall x \in I.$$

M and m, if they exist, are unique.

- A point x_0 in I such that $f(x_0) = M$ is called an **absolute maximum point**.
- A point x_0 in I such that $f(x_0) = m$ is called an **absolute minimum point**.

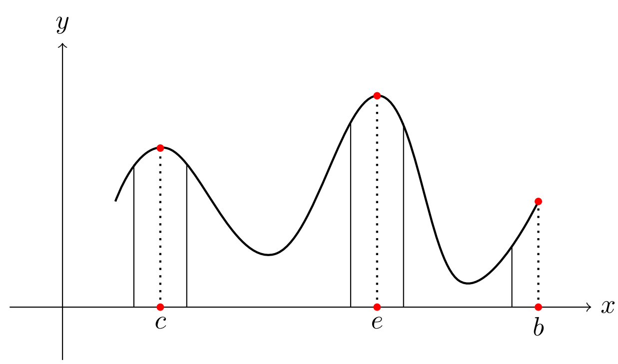
28.2 Relative Maxima and Minima

Relative Maximum

Definition 28.2. Given a function $y = f(x)$, defined in an interval I, the point x_0 of I is said to be a relative maximum if there exists a neighborhood I_{x_0} of x_0 such that $f(x_0)$ is greater than or equal to the value of the function for every x in the neighborhood I_{x_0}. $f(x_0)$ is called the relative maximum of the function in I.
In summary, let $y = f(x)$ be defined in I, x_0 is a relative maximum point if

$$\exists I_{x_0} : \forall x \in I_{x_0}, f(x_0) \geq f(x) \Rightarrow f(x_0) \text{ relative maximum.}$$

Example 28.1.

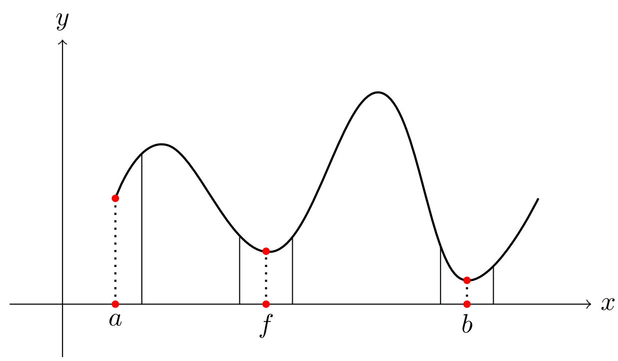


Relative Minimum

Definition 28.3. Given a function $y = f(x)$ defined on an interval I, the point x_0 in I is said to be a relative minimum if there exists a neighborhood I_{x_0} such that $f(x_0)$ is less than or equal to the value of the function for every x in the neighborhood I_{x_0}. $f(x_0)$ is called the relative minimum of the function in I.

In summary, let $y = f(x)$ be defined in I, then x_0 is a point of relative minimum if

$$\exists I_{x_0} : \forall x \in I_{x_0}, f(x_0) \leq f(x) \Rightarrow f(x_0) \text{ is a relative minimum.}$$



Example 28.2.

A point of an interval that is a point of relative maximum is also called a **maximizing point**; a point of relative minimum is called a **minimizing point**. A point of an interval is called an **extremum point** if it is a maximizing or minimizing point. The corresponding value of the function is called a **relative extremum**.

28.3 Concavity

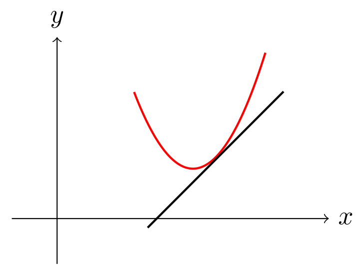
Let $y = f(x)$ be a function defined and differentiable in the interval I, and let the line with equation $y = t(x)$ be tangent to the graph of $f(x)$ at its abscissa point x_0, which is within the interval I.

Note 28. Since $f(x)$ is differentiable in I, the tangent line exists at every point.

Concave Upward

Definition 28.4. It is said that at x_0, the graph of the function $f(x)$ has concavity directed towards the positive y-axis (upward) if there exists a complete neighborhood I_{x_0} of x_0 such that, for every x belonging to the neighborhood and different from x_0, the ordinate of the point with abscissa x belonging to the graph is greater than that of the point belonging to the tangent t and having the same abscissa, i.e.:

$$f(x) > t(x) \quad \forall x \in I_{x_0} \wedge x \neq x_0.$$

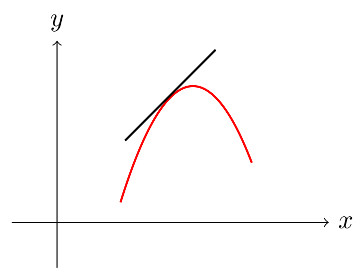


A function whose graph has concavity directed upward is said to be **convex**.

Concave Downward

Definition 28.5. It is said that at x_0, the graph of the function $f(x)$ has concavity directed towards the negative y-axis (downward) if there exists a complete neighborhood I_{x_0} of x_0 such that, for every x belonging to the neighborhood and different from x_0, the ordinate of the point with abscissa x belonging to the graph is less than that of the point belonging to the tangent t and having the same abscissa, i.e.:

$$f(x) < t(x) \quad \forall x \in I_{x_0} \wedge x \neq x_0.$$

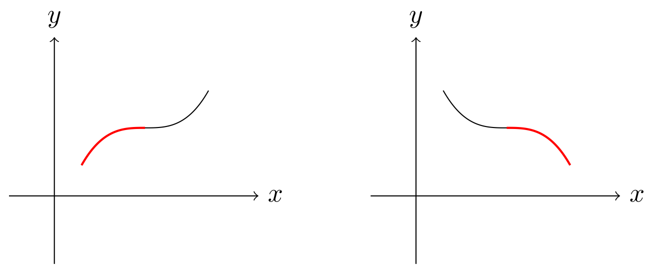


A function whose graph turns its concavity downward is called **concave**.

Given an interval I, we say that the graph has concavity upward (or downward) **within the interval** if it has concavity upward (or downward) at every interior point of the interval.

28.4 Inflection Points

Definition 28.6. Given the function $y = f(x)$ defined and continuous on the interval I, we say that it has an inflection point at x_0 within I if at that point the graph of $f(x)$ changes concavity.



If the function is differentiable at the inflection point, there exists a tangent to the curve at that point, and it is either oblique or parallel to the x-axis; if the derivative is infinite, the tangent is parallel to the y-axis. The tangent line has the characteristic of intersecting the curve.

Note 29. The tangent at an inflection point is also called an **inflectional tangent**.

If a tangent line exists at an inflection point, the inflection is termed:

- **horizontal** if the tangent at the inflection point is parallel to the x-axis;
- **vertical** if the tangent is parallel to the y-axis;
- **oblique** if the tangent is not parallel to either axis.

Note 30. If, in a neighborhood of the inflection point, the graph has concavity downwards to the left of the inflection and upwards to the right of the inflection, the inflection is **ascending**. If the concavity is upwards to the left of the inflection and downwards to the right, the inflection is **descending**.

29 Maxima, minima, horizontal inflection points, and first derivative

29.1 Stationary points

Stationary point

Definition 29.1. Given a differentiable function $y = f(x)$ and one of its points $x = c$, if $f'(c) = 0$, then $x = c$ is called a stationary point.

If $f'(c) = 0$, then the tangent at the point on the graph of the function where $x = c$ is parallel to the x-axis.

29.2 Relative maximum or minimum points

Theorem 29.1. *For a function $y = f(x)$ defined on an interval $[a; b]$ and differentiable in $]a; b[$, if $f(x)$ has a relative maximum or minimum at the point x_0 inside $[a; b]$, the derivative of the function at that point is zero, that is, $f'(x_0) = 0$.*

Note 31. The theorem states that the relative maximum and minimum points of a differentiable function, internal to the interval of definition, are stationary points.

From the geometric meaning of the derivative, the previous theorem implies that the tangent at a point of relative maximum or minimum (which is not an endpoint of the interval) is parallel to the x-axis.

29.3 Finding relative maxima and minima with the first derivative

Theorem 29.2. *Let $y = f(x)$ be defined and continuous in a complete neighborhood I_{x_0} of the point x_0 and differentiable in the same neighborhood for every $x \neq x_0$.*

1. *If for every x in the neighborhood $f'(x) > 0$ when $x < x_0$ and $f'(x) < 0$ when $x > x_0$, then x_0 is a relative maximum point.*
2. *If for every x in the neighborhood $f'(x) < 0$ when $x < x_0$ and $f'(x) > 0$ when $x > x_0$, then x_0 is a relative minimum point.*
3. *If the sign of the first derivative is the same for every $x \neq x_0$ in the neighborhood, then x_0 is not an extremum point.*

Proof. 1. For $x < x_0$, $f'(x) > 0$, hence $f(x)$ is increasing (by the theorem of increasing and decreasing functions); therefore, if $x < x_0$, $f(x) < f(x_0)$. For every $x \neq x_0$ in the neighborhood $f(x) < f(x_0)$, hence x_0 is a relative maximum point.

2. Similarly to the previous case: for $x < x_0$, $f'(x) < 0$, hence $f(x)$ is decreasing, i.e., if $x < x_0$, $f(x) > f(x_0)$; for $x > x_0$, $f'(x) > 0$, hence $f(x)$ is increasing, i.e., if $x > x_0$, $f(x) > f(x_0)$. For every $x \neq x_0$ in the neighborhood $f(x) > f(x_0)$, hence x_0 is a relative minimum point.

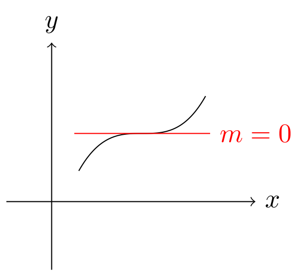
3. Suppose that for every $x \neq x_0$ in the neighborhood $f'(x) < 0$ (similar argument if $f'(x) > 0$). The function is decreasing both for $x < x_0$ and for $x > x_0$. Therefore, if $x < x_0$, $f(x) > f(x_0)$, while if $x > x_0$, $f(x) < f(x_0)$. We conclude that x_0 is neither a maximum nor a minimum point.

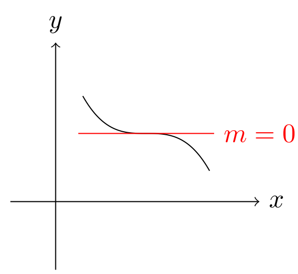
□

29.4 Stationary points of horizontal inflection

Theorem 29.3. *Given the function $y = f(x)$ defined and continuous in a complete neighborhood I_{x_0} of the point x_0 and differentiable in the same neighborhood, x_0 is a horizontal inflection point if the following conditions are satisfied:*

- $f'(x_0) = 0$;
- *the sign of the first derivative is the same for every $x \neq x_0$ in the neighborhood I_{x_0}.*





Note 32. In summary, for a continuous function $f(x)$, studying the sign of the first derivative is essential for **finding relative maximums and minimums, and horizontal inflection points**. The procedure is as follows:

- calculate the first derivative $f'(x)$ and determine its domain to find any points where the function is not differentiable (cusps, vertical tangents, corner points);
- solve the equation $f'(x) = 0$ to find stationary points;
- study the sign of $f'(x)$ to find relative maximums and minimums (including non-stationary ones) and horizontal tangent inflections.

The theorems stated are valid for points within the intervals of definition of the function, so it is necessary to also examine the values that the function takes at the endpoints of these intervals. Furthermore, if we need to find the **absolute maximum and minimum**:

- if the function $f(x)$ is continuous and the interval of definition of the function is closed and bounded, the Weierstrass theorem ensures the existence of absolute maximum and minimum; to determine them, compare the ordinates of the points of relative maximum and minimum with each other and with the values that $f(x)$ takes at the endpoints of the interval: the greater value corresponds to the absolute maximum point and the lesser one corresponds to the absolute minimum point;
- if the interval is not closed and bounded, absolute maximum and minimum may not exist.

30 Inflection Points and Second Derivative

30.1 Concavity and the Sign of the Second Derivative

A Criterion for Concavity A criterion to establish the concavity of the graph of a function at a point x_0 is given by the following theorem.

Theorem 30.1. *Let $y = f(x)$ be a function defined and continuous on an interval I, along with its first and second derivatives, and let x_0 be a point interior to this interval. If $f''(x_0) \neq 0$, then the graph of the function changes concavity at x_0:*

- *it is concave upwards if $f''(x_0) > 0$;*
- *it is concave downwards if $f''(x_0) < 0$.*

A Necessary Condition for Inflection Points For the identification of inflection points, the following theorem is useful, of which we provide only the statement.

Theorem 30.2. *Let $y = f(x)$ be a function defined on an interval $[a; b]$, and let its first and second derivatives exist in this interval. If $f(x)$ has an inflection point at x_0, inside $[a; b]$, then the second derivative of the function at that point vanishes, that is: $f''(x_0) = 0$.*

30.2 Inflection Points and Study of the Sign of the Second Derivative

To find inflection points, we can study the sign of the second derivative. The following theorem holds true.

Theorem 30.3. *Let $y = f(x)$ be a function defined and continuous in a complete neighborhood I_{x_0} of the point x_0, and let its first and second derivatives exist in this neighborhood for every $x \neq x_0$. If for every $x \neq x_0$ in the neighborhood,*

- *$f''(x) > 0$ for $x < x_0$ and $f''(x) < 0$ for $x > x_0$, or*
- *$f''(x) < 0$ for $x < x_0$ and $f''(x) > 0$ for $x > x_0$,*

then x_0 is an inflection point.

If, in addition to the hypotheses of the previous theorem, it is true that the second derivative is continuous at x_0, then necessarily $f''(x_0) = 0$. Therefore, inflection points of functions that have continuous first and second derivatives should be sought among the solutions of the equation $f''(x) = 0$. Furthermore, at the inflection point x_0, if $f'(x_0) \neq 0$ the inflection is oblique, if $f'(x_0) = 0$ the inflection is horizontal.

Note 33. Summarizing, given a function $f(x)$, continuous and differentiable, to find inflection points proceed as follows:

1. compute the second derivative $f''(x)$ and determine its domain;
2. study the sign of $f''(x)$ and look for the points where concavity changes, namely the inflection points;
3. if x_0 is an inflection point and:
   - $f'(x_0) = 0$, the inflection is **horizontal**;
   - $f'(x_0) \neq 0$, the inflection is **oblique**;

If the function $f(x)$ is not differentiable at a point x_0 where $f''(x)$ changes sign, then, when $\lim_{x \to x_0} f'(x) = +\infty$ or $\lim_{x \to x_0} f'(x) = -\infty$, at x_0 there is a **vertical** inflection.

31 Studying a Function

To study the main properties and plot the graph of a function $y = f(x)$, we can proceed by examining the following points.

1. The *domain* of the function.
2. Any *symmetries* and *periodicity*:
   - if the function is *even*, the graph is symmetric with respect to the y-axis;
     $$y = f(x) \text{ is even in } D, D \subseteq \mathbb{R}, \text{ if } f(-x) = f(x), \forall x \in D$$
   - if it is *odd*, it is symmetric with respect to the origin;
     $$y = f(x) \text{ is odd in } D, \text{ if } f(-x) = -f(x), \forall x \in D$$
   - if it is *periodic* with period T, we can limit ourselves to studying the function in a single interval of width T.
     $$y = f(x) \text{ is periodic with period } T\ (T > 0), \text{ if } f(x) = f(x+kT), \forall k \in \mathbb{Z}$$
3. The possible *intersection points* of the graph *with the Cartesian axes.*
4. The *sign of the function*: establish the intervals where it is positive, by setting $f(x) > 0$, and consequently find where it is negative.
5. The *behavior* of the function *at the extremes of the domain*: compute the respective *limits* and then look for any *asymptotes* of the function.

   Vertical asymptote: $x = x_0$ if $\lim_{x \to x_0} f(x) = \infty$.

   Horizontal asymptote: $y = y_0$ if $\lim_{x \to \infty} f(x) = y_0$.

   Oblique asymptote: $y = mx + q$, $\quad m = \lim_{x \to \infty} \dfrac{f(x)}{x}$ and $q = \lim_{x \to \infty} [f(x) - m \cdot x]$.

   Also classify any points of *discontinuity*, specifying whether they are of the first, second, or third kind.
6. The *first derivative* and its domain. From the *study of the sign of the first derivative*, determine the intervals where the function is *increasing* ($f'(x) > 0$) and consequently, those where it is *decreasing* ($f'(x) < 0$); look for any *local maxima* or *minima*, and horizontal inflection points, as well as points of non-differentiability for $f(x)$ (*vertical inflections, cusps, and corner points*).
7. The *second derivative* and its domain. From the *study of the sign of the second derivative*, determine the intervals where the graph changes concavity upwards ($f''(x) > 0$) or downwards ($f''(x) < 0$). Also look for *inflection points* with oblique tangent and possibly the inflection tangent.

Part V
Integrals

32 Indefinite Integral

32.1 Primitives

Primitive of a function

Definition 32.1. A function $F(x)$ is said to be a primitive of the function $f(x)$ defined in the interval $[a;b]$ if $F(x)$ is differentiable in the entire interval $[a;b]$ and its derivative is $f(x)$.

The primitive of a function is not unique.

In general, if a function $f(x)$ has a primitive $F(x)$, then it has infinitely many primitives of the form $F(x)+c$, where c is any real number. Indeed, since the derivative of a constant is zero:

$$\mathrm{D}\,[F(x)+c] = F'(x) = f(x), \quad \forall x \in \mathbb{R}.$$

Conversely, if two functions $F(x)$ and $G(x)$ are primitives of the same function $f(x)$, then the two functions differ by a constant,

$$\mathrm{D}\,[F(x)-G(x)] = F'(x) - G'(x) = f(x) - f(x) = 0,$$

and therefore

$$F(x) - G(x) = c.$$

We conclude that if $F(x)$ is a primitive of $f(x)$, then the functions $F(x)+c$, where c is any real number, are **all** and **only** the primitives of $f(x)$.

32.2 Indefinite Integral

Indefinite Integral

Definition 32.2. The indefinite integral of the function $f(x)$, denoted by

$$\int f(x)\; dx,$$

is the set of all primitives $F(x)+c$ of $f(x)$, where $c \in \mathbb{R}$.

$$\int f(x)\; dx = F(x) + c \Leftrightarrow \mathrm{D}\,[F(x)+c] = f(x)$$

Note 34. The symbol $\int f(x)\,dx$ is read as *indefinite integral of $f(x)$ with respect to dx.*

The primitive $F(x)$ obtained for $c = 0$ is called the **fundamental primitive**.

In the expression $\int f(x)\,dx$, the function $f(x)$ is called the **integrand function**, and the variable x is the **integration variable**.

From the previous definition, since

$$\frac{d}{dx}F(x) = f(x),$$

it follows that

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

This means that indefinite integration acts as the inverse operation of differentiation.

A function that admits a primitive (and hence infinitely many primitives) is said to be **integrable**.

Sufficient Condition for Integrability

Theorem 32.1. *If a function is continuous on* $[a;b]$*, then it admits primitives on the same interval.*

However, determining primitives even for fairly simple continuous functions is not always straightforward.

Properties of Indefinite Integration

First Linearity Property

Definition 32.3. The indefinite integral of a sum of integrable functions is equal to the sum of the indefinite integrals of the individual functions:

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Indeed, if we differentiate both sides, we respectively obtain:

$$\frac{d}{dx}\left[\int [f(x)+g(x)]\,dx\right] = f(x)+g(x);$$

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\left[\int f(x)\,dx\right] + \frac{d}{dx}\left[\int g(x)\,dx\right] = f(x)+g(x).$$

The two sides have the same derivative, thus representing primitives of the same function.

Second Linearity Property

Definition 32.4. The integral of the product of a constant by an integrable function is equal to the constant multiplied by the integral of the function:

$$\int k\cdot f(x)\,dx = k\cdot\int f(x)\,dx.$$

Again, if we differentiate both sides, we respectively obtain:

$$\frac{d}{dx}\left[\int k\cdot f(x)\,dx\right] = k\cdot f(x);$$

$$\frac{d}{dx}\left[k \cdot \int f(x)\,dx\right] = k \cdot \frac{d}{dx}\left[\int f(x)\,dx\right] = k \cdot f(x).$$

The two sides have the same derivative, thus representing primitives of the same function.

These linearity properties can be expressed in a single formula:

$$\int [c_1 f(x) + c_2 g(x)]\,dx = c_1 \int f(x)\,dx + c_2 \int g(x)\,dx.$$

The integral is also called a **linear operator**.

Note 35. It is always important to remember that:

1. $\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$,
2. $\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$.

33 Basic Indefinite Integrals

From the rules of differentiation of elementary functions, we derive fundamental indefinite integrals.

$$
\begin{aligned}
&\int x^a \, dx = \frac{x^{a+1}}{a+1} + c, \quad a \neq -1 \\
&\int \frac{1}{x} \, dx = \ln|x| + c \\
&\int e^x \, dx = e^x + c \\
&\int a^x \, dx = \frac{a^x}{\ln a} + c \\
&\int \sin x \, dx = -\cos x + c \\
&\int \cos x \, dx = \sin x + c \\
&\int \frac{1}{\cos^2 x} \, dx = \tan x + c \\
&\int \frac{1}{\sin^2 x} \, dx = -\cot x + c \\
&\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c \\
&\int \frac{1}{1+x^2} \, dx = \text{arccot } x + c
\end{aligned}
$$

Integration of Composite Functions

To integrate the power of a (composite) function using the power rule, it is necessary for the integrand function to be multiplied by the derivative of the more *internal* function in the composition. The same approach applies to calculate integrals of other composite functions reducible to different integration rules.

$$\int [f(x)]^a \cdot f'(x)\,dx = \frac{[f(x)]^{a+1}}{a+1} + c, \quad a \neq -1$$
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$
$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$
$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$
$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$
$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$
$$\int \frac{f'(x)}{\cos^2 f(x)}\,dx = \tan f(x) + c$$
$$\int \frac{f'(x)}{\sin^2 f(x)}\,dx = -\cot f(x) + c$$
$$\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx = \arcsin f(x) + c$$
$$\int \frac{f'(x)}{1+[f(x)]^2}\,dx = \arctan f(x) + c$$
$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \arcsin \frac{f(x)}{|a|} + c, \quad a \neq 0$$
$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\arctan \frac{f(x)}{a} + c, \quad a \neq 0$$
$$\int \tan x\,dx = -\ln|\cos x| + c$$
$$\int \cot x\,dx = \ln|\sin x| + c$$

34 The Definite Integral

The introduction of definite integrals arises from the necessity to determine the areas of plane figures with curvilinear boundaries.

34.1 The Trapezoid

Given a function $y = f(x)$ and a closed and bounded interval $[a;b]$ where the function is continuous and positive (or zero), a **trapezoid** is defined as the plane figure bounded by the x-axis, the lines parallel to the y-axis passing through the endpoints of the interval $[a;b]$, and the graph of the function f over this interval. Essentially, it is a quadrilateral with vertices $A(a;0)$, $B(b;0)$, $C(b;f(b))$, and $D(a;f(a))$.

The area S of a trapezoid cannot be calculated directly, but it can be approximated using the following procedure:

- Divide the interval $[a;b]$ into n equal parts of width $h = \dfrac{b-a}{n}$.
- Consider n rectangles, each having a base segment of the subdivision and a height equal to the minimum m the function assumes in that interval.
- Let s_n be the sum of the areas of all these n rectangles:

$$s_n = m_1 h + m_2 h + ... + m_n h.$$

The area of the trapezoid is approximated from below by s_n.

Similarly, we can approximate the area of the trapezoid from above, by summing the areas of rectangles associated with a partition of the interval $[a;b]$ into n equal parts and having heights equal to the maximum M_i of the function in the corresponding interval. Let's denote this sum as S_n:

$$S_n = M_1 h + M_2 h + ... + M_n h.$$

Thus, we obtain two sequences of areas s_n and S_n such that, for each n, the area S of the trapezoid lies between the underestimate and the overestimate, i.e., we can write:

$$s_n \leq S \leq S_n.$$

34.2 Definite Integral of a Non-Negative Function

The approximation of the areas s_n and S_n becomes better as the intervals of division of $[a;b]$ become smaller.

Theorem 34.1. *If a function $f(x)$ is continuous and non-negative (or zero) on the interval $[a;b]$, the limits as $n \to +\infty$ of the sequences s_n and S_n exist and are finite and coincident.*

Definite Integral ($f(x) \geq 0$)

Definition 34.1. Given a function $f(x)$ continuous and non-negative or zero on $[a;b]$, the definite integral over the interval $[a;b]$ is defined as the common value of the limit as $n \to +\infty$ of the two sequences s_n (underestimate) and S_n (overestimate). This value is denoted by:

$$\int_a^b f(x)\, dx.$$

Note 36. The symbol $\int$ represents an elongated S to remind that, in graphical representation, an integral corresponds to a sum of areas of rectangles with height $f(x)$ and base dx.

The definite integral, since $f(x) \geq 0$, provides the measure of the area of the trapezoid related to $f(x)$ with endpoints a and b. a and b are called the integration limits; a is the **lower limit**, b is the **upper limit**. The function $f(x)$ is called the **integrand function**.

Unlike the indefinite integral, which is a set of functions, the definite integral is a number and does not depend on the variable x.

34.3 General Definition of Definite Integral

Consider a function $y = f(x)$ continuous on $[a; b]$ and divide the interval into n closed intervals using i points $x_0, x_1, x_2, x_3, ..., x_n$, with:

$$x_0 < x_1 < x_2 < x_3 < ... < x_n.$$

x_0 coincides with a; x_n coincides with b.

The widths of the intervals may vary and are given by:

$$\begin{aligned}
\Delta x_1 &= x_1 - a, \\
\Delta x_1 &= x_2 - x_1, \\
\Delta x_1 &= x_3 - x_2, \\
&\vdots \\
\Delta x_n &= b - x_{n-1}.
\end{aligned}$$

For each of the intervals, select any point within the interval:

$$c_1, c_2, c_3, ..., c_n.$$

Consider the corresponding values of the function:

$$f(c_1), f(c_2), f(c_3), ..., f(c_n).$$

Then, write the sum $\bar{S}$ given by:

$$\bar{S} = f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2 + f(c_3) \cdot \Delta x_3 + ... + f(c_n) \cdot \Delta x_n.$$

The sum $\bar{S}$ depends on:

- the number of divisions;
- the widths Δx_n of the intervals;
- the points c_n chosen within the different intervals.

Among the widths of the intervals, denote the maximum as Δx_{max}: if $\Delta x_{max} \to 0$, all other widths tend to 0 as well.

It can be shown that if Δx_{max} tends to 0, all sums $\bar{S}$, obtained by choosing the division of the interval and the points within the different intervals in any way, tend to the same value S.

We then give the following definition.

Definite Integral

Definition 34.2. Given a function $f(x)$, continuous on $[a; b]$, the definite integral over the interval $[a; b]$ is defined as the value of the limit as Δx_{max} tends to 0 of the sum $\bar{S}$:

$$\int_a^b f(x)\, dx = \lim_{\Delta x_{max} \to 0} \bar{S}.$$

The previous definition, given for $f(x) \geq 0$, is a particular case of this.

According to this definition, **the definite integral can also be a negative or zero number and therefore, in general, does not correspond to the area enclosed between the graph of the function and the x-axis**. By convention, the following are set.

Definition 34.3.

- $$\int_a^a f(x)\, dx = 0;$$

- $$\int_a^b f(x)\, dx = -\int_a^b f(x)\, dx \quad \text{if} a > b.$$

If a function has a definite integral in an interval $[a; b]$, the function is said to be **integrable on** $[a; b]$.

35 Properties of the Definite Integral

35.1 Additivity of the Integral with Respect to the Integration Interval

Definition 35.1. If $f(x)$ is integrable over $[a; c]$ and $a < b < c$, then it is also integrable over $[a; b]$ and $[b; c]$; conversely, if $a < b < c$ and $f(x)$ is integrable over $[a; b]$ and $[b; c]$, then it is also integrable over $[a; c]$. This leads to:

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

35.2 Integral of the Sum of Functions

Definition 35.2. If $f(x)$ and $g(x)$ are integrable functions over $[a;b]$, then their sum $f(x) + g(x)$ is also integrable over $[a;b]$, and it holds that:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

35.3 Integral of a Constant Times a Function

Definition 35.3. If $f(x)$ is an integrable function over $[a;b]$, then $k \cdot f(x)$ is also integrable over $[a;b]$ for $k \in \mathbb{R}$, and it holds that:

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx.$$

35.4 Comparison between Integrals of Two Functions

Definition 35.4. If $f(x)$ and $g(x)$ are two continuous functions such that $f(x) \leq g(x)$ for every point in the interval $[a;b]$, then the integral from a to b of $f(x)$ is less than or equal to the integral of $g(x)$:

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

35.5 Integral of the Absolute Value of a Function

Definition 35.5. If $f(x)$ is a continuous function over the interval $[a;b]$, then the absolute value of the integral from a to b of $f(x)$ is less than or equal to the integral of the absolute value of $f(x)$:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

35.6 Integral of a Constant Function

Definition 35.6. If a function $f(x)$ is constant over the interval $[a;b]$, i.e., $f(x) = k$, then the integral from a to b of $f(x)$ is equal to the product of k and $(b - a)$:

$$\int_a^b k\,dx = k(b - a).$$

36 Fundamental Theorem of Calculus

36.1 Mean Value Theorem

We pose the following questions:

1. Is there a relationship between the indefinite integral $\int f(x)\,dx$ and the definite integral $\int_a^b f(x)\,dx$?

2. Is it possible to compute a definite integral $\int_a^b f(x)\,dx$?

To answer the first question, we introduce a theorem that relates the indefinite and definite integrals. This theorem is known as the *Fundamental Theorem of Calculus*. To prove it, we need to introduce another theorem, the mean value theorem.

Mean Value Theorem

Theorem 36.1. *If $f(x)$ is a continuous function over an interval $[a;b]$, there exists at least one point z in the interval such that:*

$$\int_a^b f(x)\,dx = (b-a)\cdot f(z), \quad z \in [a;b].$$

Proof. Since the function $f(x)$ is continuous on the interval $[a;b]$, by the Weierstrass theorem, the function attains its maximum value M and its minimum value m over $[a;b]$. Therefore, for every x belonging to $[a;b]$, the inequality holds:

$$m \leq f(x) \leq M.$$

By the properties of integrals, we also have:

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx.$$

Applying the property of the integral of a constant function, we can write:

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Dividing all terms of the inequality by $(b-a)$, we get:

$$m \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq M.$$

By the intermediate value theorem, the function must assume all values between its maximum and minimum, so there must exist a point z belonging to $[a;b]$ such that:

$$f(z) = \frac{\int_a^b f(x)\,dx}{b-a}.$$

Hence, there exists at least one point z belonging to $[a; b]$ such that:

$$\int_a^b f(x)\ dx = f(z)(b - a).$$

□

Note 37. Geometrically, if the function is positive on $[a; b]$, the mean value theorem expresses the equivalence between a trapezoid, whose area measures $\int_a^b f(x)\ dx$, and a rectangle, having equal base $b-a$. The height of the rectangle is given by the value of f at a particular point z in the interval $[a; b]$:

$$f(z) = \frac{\int_a^b f(x)\ dx}{b - a}.$$

36.2 The Integral Function

Let f be a continuous function on the interval $[a; b]$. Consider any point x in $[a; b]$. We define the **integral function** of f on $[a; b]$ as the function:

$$F(x) = \int_a^x f(t)\ dt,$$

which associates to each $x \in [a; b]$ the real number $\int_a^x f(t)\ dt$, where the independent variable x coincides with the upper limit of integration.

37 Fundamental Theorem of Calculus

37.1 Fundamental Theorem of Integral Calculus

Fundamental Theorem of Integral Calculus

Note 38. This theorem is also known as **Torricelli-Barrow**.

Theorem 37.1. *If a function $f(x)$ is continuous on $[a; b]$, then the derivative of its integral function exists*

$$F(x) = \int_a^x f(t)\ dt$$

for every point x in the interval $[a; b]$ and is equal to $f(x)$, that is:

$$F'(x) = f(x).$$

Thus $F(x)$ is an antiderivative of $f(x)$.

Proof. **Hypotheses**

1. $y = f(x)$ is continuous on $[a; b]$;

2. $F(x) = \int_a^x f(t)\ dt$.

Thesis

1. $F'(x)$ exists;
2. $F'(x) = f(x)$.

We prove the existence of the derivative of $F(x)$ and calculate it by applying the definition. Let's increase the variable x by a value $h \neq 0$ such that $a < x+h < b$ and calculate the difference $F(x+h) - F(x)$ using the expression of the integral function:

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt.$$

We apply the additivity property of the integral:

$$F(x+h) - F(x) = \int_a^x f(t)\ dt + \int_x^{x+h} f(t)\ dt - \int_x^a f(t)\ dt = \int_x^{x+h} f(t)\ dt.$$

By the mean value theorem, the value of the integral is equal to the product of the width h of the integration interval by the value $f(z)$, where z is a particular point in the interval $[x; x+h]$, if $h > 0$, or in the interval $[x+h; x]$, if $h < 0$; hence we can write:

$$F(x+h) - F(x) = h \cdot f(z).$$

We divide both sides by h:

$$\frac{F(x+h) - F(x)}{h} = f(z).$$

We analyze the behavior of $f(z)$ as h tends to 0. Let $h > 0$; since z is between x and $x+h$, if h tends to 0 (from the right), then z tends to x (from the right) and $\lim_{h \to 0^+} f(z) = \lim_{z \to x^+} f(z) = f(x)$ because f is continuous by hypothesis. With a similar reasoning, if $h < 0$, it follows that $\lim_{h \to 0^-} f(z) = \lim_{z \to x^-} f(z) = f(x)$. Therefore:

$$\lim_{h \to 0} f(z) = \lim_{z \to x} f(z) = f(x).$$

We can conclude that the limit also exists, as h tends to 0, of the expression on the left-hand side, i.e., the incremental ratio of F at point x, and:

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(z) = f(x).$$

The function F is therefore differentiable and we have

$$F'(x) = f(x).$$

$\square$

According to the theorem just proved, a function f continuous on $[a;b]$ has as a fundamental primitive the integral function $F(x)$, with x ranging in the interval $[a;b]$. Therefore, the indefinite integral of f, understood as the totality of its primitives, is expressed as:

$$\int f(x)\,dx = \int_a^x f(t)\,dt + c,$$

where c is any real constant.

37.2 Calculation of the Definite Integral

From the fundamental theorem of calculus, we can obtain the formula for calculating the definite integral. Let $\varphi(x)$ be any primitive of $f(x)$. From the fundamental theorem of calculus, we know that the integral function $F(x)$ is a particular primitive of the function f. Thus $\varphi(x)$ is of the form:

$$\varphi(x) = F(x) + c = \int_a^x f(t)\,dt + c,$$

where c is an arbitrary real constant.

- Let's calculate $\varphi(a)$ (we substitute the value a for the integration endpoint x);

$$\varphi(a) = \int_a^a f(t)\,dt + c = 0 + c = c.$$

- Let's calculate $\varphi(b)$ (we substitute the value b for the integration endpoint x);

$$\varphi(b) = \int_a^b f(t)\,dt + c. \quad \text{Since } \varphi(a) = c \text{, we obtain:}$$
$$\varphi(b) = \int_a^b f(t)\,dt + \varphi(a).$$

We bring $\varphi(a)$ to the left-hand side,

$$\varphi(b) - \varphi(a) = \int_a^b f(t)\,dt,$$

and write the equality from right to left:

$$\int_a^b f(t)\,dt = \varphi(b) - \varphi(a).$$

Since there are no more variable ambiguities, we can reuse the variable x and write:

$$\int_a^b f(x)\,dx = \varphi(b) - \varphi(a).$$

It is customary to indicate the difference $\varphi(b) - \varphi(a)$ as $[\varphi(x)]_a^b$. The formula found allows us to reduce the calculation of a definite integral to that of an indefinite integral. This overcomes the difficulty of calculating the limit of the sequence s_n, which, in general, is not easy to determine.

38 The Mean Value of a Function

38.1 The Mean Value Theorem

The mean value theorem states the existence of a point z where the property expressed by the theorem holds, but it does not provide indications on how to calculate it. However, $f(z)$ can be derived from the equality

$$\int_a^b f(x)\,dx = (b-a) \cdot f(z)$$

by dividing both sides by $b - a$ and rewriting the equality from left to right:

$$f(z) = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx.$$

The value $f(z)$ is defined as the **mean value** of the function $f(x)$ in the interval $[a; b]$.

Part VI
Integration

39 Integration Methods

39.1 Integration by Substitution

When the integral is not immediately solvable, it can be useful to apply the **substitution method**, which involves making a variable change that allows rewriting the given integral in a form that we know how to solve. The substitution method can also be used to compute integrals whose primitive is a composite function.

In general, to compute $\int f(x)\,dx$ using the substitution method:

- Let $x = g(t)$, or $t = g^{-1}(x)$, where $g(t)$ is invertible with continuous and nonzero derivative $g'(t)$.
- Calculate the differential dx, or dt.
- Substitute into the given integral to obtain an integral in the variable t, and compute, if possible, the integral with respect to t.
- Use the initial position to write the result in terms of x.

39.2 Integration by Parts

Given two differentiable functions $f(x)$ and $g(x)$ with continuous derivatives on an interval $[a; b]$, consider the derivative of their product:

$$\mathrm{D}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Integrating both sides:

$$\int \mathrm{D}\left[f(x) \cdot g(x)\right] dx = \int \left[f'(x) \cdot g(x) + f(x) \cdot g'(x)\right] dx,$$

$$f(x) \cdot g(x) = \int f'(x) \cdot g(x)\,dx + \int f(x) \cdot g'(x)\,dx.$$

Isolating $\int f(x) \cdot g'(x)\,dx$, we get:

$$\int f(x) \cdot g'(x)\,dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx,$$

which is called the **integration by parts** formula. The formula is useful when the integrand can be thought of as a *product of two factors*. $f(x)$ is called the **finite factor**, and $g'(x)\,dx$ is called the **differential factor**.

In applying the formula, one of the functions, the finite factor, is only differentiated, while the other, the differential factor, is only integrated. Thus, it is necessary to choose the two factors appropriately. The formula involves another integral on the right-hand side, making this integration method useful for transforming a difficult integral into a more manageable one.

In general, in integrals of the form

$$\int x^n \sin x\, dx, \qquad \int x^n \cos x\, dx, \qquad \int x^n e^x\, dx,$$

x^n is considered as the finite factor, while in integrals of the form

$$\int x^n \ln x\, dx, \qquad \int x^n \arctan x\, dx, \qquad \int x^n \arcsin x\, dx,$$

$x^n\, dx$ is considered as the differential factor. In particular, in integrals like

$$\int \ln x\, dx, \qquad \int \arctan x\, dx, \qquad \int \arcsin x\, dx,$$

the differential factor is considered as $x^0\, dx$, that is, $1 \cdot dx$.

39.3 Integration of Rational Functions

We now address the specific problem of calculating integrals of rational functions:

$$\int \frac{N(x)}{D(x)}\, dx,$$

where the numerator $N(x)$ and denominator $D(x)$ are polynomials. In our considerations, we assume that the degree of the denominator is less than the degree of the numerator. If this is not the case, we can always perform polynomial division of $N(x)$ by $D(x)$, obtaining a quotient polynomial $Q(x)$ and a remainder polynomial $R(x)$ of degree less than that of $D(x)$:

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

which yields:

$$\int \frac{N(x)}{D(x)}\, dx = \int \left[Q(x) + \frac{R(x)}{D(x)}\right]\, dx = \int Q(x)\, dx + \int \frac{R(x)}{D(x)}\, dx.$$

In the addition of the two integrals, the first one is calculable as it is the integral of a polynomial, while the second one is the integral of a rational function with a numerator of lesser degree than the denominator. Hence, we study integrals of the form $\int \frac{R(x)}{D(x)}\, dx$, where $R(x)$ is a polynomial of lesser degree than $D(x)$.

Numerator is the Derivative of the Denominator We have already seen that

$$\int \frac{f'(x)}{f(x)}\, dx = \ln |f(x)| + c,$$

which means the indefinite integral of a fraction where the numerator is the derivative of the denominator is equal to the natural logarithm of the absolute value of the denominator.

Denominator is of First Degree: $\int \frac{1}{ax+b}\,dx$ Also, the integral $\int \frac{1}{ax+b}\,dx, \quad a \neq 0$, where the algebraic fraction has a first-degree denominator, can be reduced to the case where the numerator is the derivative of the denominator.

Indeed, by multiplying the fraction by $\frac{a}{a}$ and applying the second property of linearity of the integral:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{a}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c.$$

Denominator is of Second Degree: $\int \frac{px+q}{ax^2+bx+c}\,dx$ To compute the integral $\int \frac{px+q}{ax^2+bx+c}\,dx, \quad a \neq 0$, different solving methods are used depending on the sign of the discriminant of the denominator $\Delta = b^2 - 4ac$.

Discriminant is Positive: $\Delta > 0$ **In general**, if $\Delta > 0$:

- Decompose the denominator: $ax^2 + bx + c = a(x - x_1)(x - x_2)$;
- Write the given fraction as the sum of fractions with first-degree denominators:
$$\frac{px+q}{ax^2+bx+c} = \frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)};$$
- Calculate the sum of the two fractions on the right-hand side;
- Determine the values of A and B by equating the coefficients of x and the constant terms;
- Solve the integral $\int \left[\frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)}\right] dx$.

Discriminant is Zero: $\Delta = 0$ **In general**, if $\Delta = 0$:

- Decompose the denominator: $ax^2 + bx + c = a(x - x_1)^2$;
- Write the given fraction as the sum of two fractions:
$$\frac{px+q}{ax^2+bx+c} = \frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)};$$
- Calculate the sum of the fractions on the right-hand side;
- Determine the values of A and B by equating the coefficients of x and the constant terms;
- Solve the integral $\int \left[\frac{A}{a(x-x_1)} + \frac{B}{(x-x_2)}\right] dx$.

Discriminant is Negative: $\Delta < 0$ We examine two cases.

1. The numerator is of degree zero, meaning the integral is of the form:

$$\int \frac{1}{ax^2 + bx + c}\, dx, \quad a \neq 0$$

   In general, to compute $\int \frac{1}{ax^2 + bx + c}\, dx$ if $\Delta < 0$:

   - Factor out the coefficient of x^2:

$$\frac{1}{a} \int \frac{1}{x^2 + nx + n}\, dx;$$

   - Write the denominator in the form: $(x+h)^2 + k^2$;
   - Calculate the integral

$$\frac{1}{a} \int \frac{1}{(x+h)^2 + k^2}\, dx = \frac{1}{ak} \arctan \frac{x+h}{k} + c.$$

2. The numerator is a first-degree polynomial, i.e., the integral is of the form:

$$\int \frac{px + q}{ax^2 + bx + c}\, dx, \quad a, p \neq 0.$$

   In general, to compute $\int \frac{px + q}{ax^2 + bx + c}\, dx$, with $a, p \neq 0$ and $\Delta < 0$:

   - Manipulate the numerator to make it the derivative of the denominator;
   - Write the integral as the sum of two integrals:

$$r \int \frac{2ax + b}{ax^2 + bx + c}\, dx + s \int \frac{1}{ax^2 + bx + c}\, dx;$$

   - Calculate the first integral using the fact that $\int \frac{f'(x)}{f(x)}\, dx = \ln |f(x)| + c$, thus:

$$\int \frac{2ax + b}{ax^2 + bx + c}\, dx = \ln |ax^2 + bx + c| + c_1;$$

   - Calculate the second integral using the method already seen;
   - Sum the obtained results.

40 Definite Integral

40.1 Calculation of Volumes of Solid of Revolution

Consider the function $y = f(x)$, continuous over the interval $[a;b]$ and non-negative, and the trapezoid extended to the interval $[a;b]$. If we rotate the trapezoid around the x-axis by a full revolution, we obtain a solid of revolution. Let's calculate the volume of this solid.

We take the trapezoid and divide the interval $[a;b]$ into n equal parts. Each of these parts has a length $h = \frac{b-a}{n}$. In each interval of the subdivision, we consider the minimum m_i and maximum M_i of $f(x)$ and draw the inscribed rectangles in the trapezoid with heights m_i, which approximate the area of the trapezoid defectively, and the circumscribed rectangles with heights M_i, which approximate the area of the trapezoid excessively.

In the complete rotation around the x-axis, each rectangle describes a circular cylinder with height h and base radius m_i or M_i. The sum of the volumes of the n cylinders with the base as the circle of radius m_i approximates the volume of the initial solid of revolution defectively, and the sum of the volumes of the n cylinders with the base as the circle of radius M_i approximates the volume of the same solid excessively.

Since the formula for the volume of a circular cylinder with radius r and height h is $\pi r^2 h$, the volume v_n of the cylinders approximating the solid defectively and the volume V_n of the cylinders approximating excessively are:

$$v_n = \pi m_1^2 h + \pi m_2^2 h + \pi m_3^2 h + \ldots + \pi m_n^2 h;$$

$$V_n = \pi M_1^2 h + \pi M_2^2 h + \pi M_3^2 h + \ldots + \pi M_n^2 h.$$

One can demonstrate that as $n \to \infty$, both sequences tend to the same limit, and this limit is equal to the product of π and the definite integral from a to b of the square of $f(x)$, namely:

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} V_n = \pi \cdot \int_a^b f^2(x)\, dx.$$

Volume of a Solid of Revolution

Definition 40.1. Given the trapezoid $ABCD$ extended to the interval $[a;b]$, bounded by the graph of the function $y = f(x)$ (non-negative), the x-axis, and the lines $x = a$ and $x = b$, the volume of the solid obtained by rotating the trapezoid around the x-axis by a full revolution is expressed by the following integral:

$$V = \pi \cdot \int_a^b f^2(x)\, dx.$$

40.2 Arc Length of a Curve

Consider a function $f(x)$ differentiable with continuous derivative over the interval $[a;b]$. We divide the interval $[a;b]$ into n parts and consider the polygonal with vertices at points $P_0, P_1, P_2, \ldots, P_n$. The length l_n of the inscribed polygonal in the curve can be calculated by summing the distances between the points $P_{()}, P_1, P_2, \ldots, P_n$.

It depends on the number n of subdivisions and the points chosen for the subdivision. The smaller the width of the intervals $[x_{i-1}; x_i]$, the better the polygonal approximates the curve. It can be demonstrated that as n tends to infinity, l_n has a finite limit given by:

$$\lim_{n\to\infty} l_n = \int_a^b \sqrt{1+[f'(x)]^2}\, dx.$$

Arc Length of a Curve

Definition 40.2. Given the function $y = f(x)$ differentiable over the interval $[a;b]$, the length of the curve represented by the graph of the function, bounded by the lines $x = a$ and $x = b$, is the number expressed by the following integral:

$$l = \int_a^b \sqrt{1+[f'(x)]^2}\, dx.$$

40.3 Surface Area of a Solid of Revolution

If the previously considered curve is rotated with a full rotation around the x-axis, a surface of revolution is obtained. We divide the interval $[a;b]$ into n parts and consider the polygonal inscribed in the curve, which has vertices at points $P_0, P_1, \ldots, P_n$. In the complete rotation around the x-axis, each segment P_iP_{i-1} of the polygonal describes a frustum of a cone with P_iP_{i-1} as apothem and two circles as bases with radii $f(x_{i-1})$ and $f(x_i)$ respectively.

The sum of the areas of the lateral surfaces of these frustums of cones approximates the area of the surface of revolution. By a similar procedure to that used for the arc length of a curve, we demonstrate the following formula:

$$S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1+[f'(x)]^2}\, dx$$

Surface Area of a Solid of Revolution

Definition 40.3. Given the function $y = f(x)$ differentiable over the interval $[a;b]$, the area of the surface obtained by rotating the graph of the function in a full rotation, bounded by the lines $x = a$ and $x = b$, is the number expressed by the following integral:

$$S = 2\pi \int_a^b f(x) \cdot \sqrt{1+[f'(x)]^2}\, dx$$

41 Improper Integrals

41.1 Integral of a Function with a Finite Number of Discontinuities in $[a; b]$

Consider the case where the function $f(x)$ is continuous at all points of the interval $[a; b[$ but not at b. Consider a point z inside the interval $[a; b[$: the function $f(x)$ is continuous in the interval $[a; z]$, so the integral $\int_a^z f(x)\ dx$ exists, whose value is a real number. This holds for all points z in the interval $[a; b[$, so we can construct the integral function

$$F(z) = \int_a^z f(x)\ dx,$$

defined in $[a; b[$. If the limit of $F(z)$ exists finitely as z approaches b from the left, that is, if

$$\lim_{x \to b^-} F(z),$$

exists, then we say that the function $f(x)$ is **improperly integrable in** $[a; b]$ and define:

$$\int_a^b f(x)\ dx = \lim_{x \to b^- f(x)\ dx}.$$

Note 39. It is also called that the function is **integrable in a generalized sense**.

The integral $\int_a^b f(x)\ dx$ is called the **improper integral** of the function $f(x)$ in $[a; b]$ and is also said to be **convergent**. If the limit considered does not exist or is infinite, we say that the function is not improperly integrable in $[a; b]$ or also that the integral is respectively **undetermined** or **divergent**.

If the function $f(x)$ is continuous at all points of the interval $]a; b]$, we can define the integral $\int_a^b f(x)\ dx$ in a similar way. Considering $z \in]a; b]$, if the limit of the function $F(z) = \int_a^b f(x)\ dx$ exists finitely as z approaches a from the right, that is, if

$$\lim_{x \to a^+} F(z),$$

exists, then we say that the function $f(x)$ is **improperly integrable in** $[a; b]$ and define:

$$\int_a^b f(x)\ dx = \lim_{x \to a^+} \int_z^b f(x)\ dx.$$

If the function has a discontinuity point at a point c inside the interval $[a; b]$, the integral $\int_a^b f(x)\ dx$ can be defined, in an improper sense, as the sum of the integrals $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$, if they exist. Such integrals are calculated using the previous definitions:

$$\int_a^b f(x)\ dx = \lim_{t \to c^-} \int_a^t f(x)\ dx + \lim_{x \to c^+} \int_z^b f(x)\ dx.$$

41.2 Integral of a Function over an Infinite Interval

Consider a function $f(x)$ continuous at all points of $[a;+\infty[$. For any point z inside the interval $[a;+\infty[$, the integral $\int_a^x f(x)\ dx$ exists and yields a real number, so we can also construct the integral function in this case:

$$F(z) = \int_a^z f(x)\ dx,$$

defined in $[a;+\infty[$. If the limit of the function $F(z)$ exists finitely as z tends to $+\infty$, that is, if

$$\lim_{x\to+\infty} F(z),$$

exists, then we say that the function $f(x)$ is **improperly integrable in** $[a;+\infty[$ and define:

$$\int_a^{+\infty} f(x)\ dx = \lim_{x\to+\infty} \int_a^x f(x)\ dx.$$

In this case, we also say that the integral $\int_a^{+\infty} f(x)\ dx$ is **convergent**.

Note 40. If the considered limit is infinite, we say that the integral $\int_a^{+\infty} f(x)\ dx$ is **divergent**. If the limit does not exist, the integral $\int_a^{+\infty} f(x)\ dx$ is **undetermined**. In both cases, we say that the function $f(x)$ is not improperly integrable in $[a;+\infty[$.

Similarly, if a function is continuous in $]-\infty;a]$ and if the limit $\lim_{x\to-\infty} \int_a^z f(x)\ dx$ exists finitely, we say that the function $f(x)$ is integrable in an improper sense in $]-\infty;a]$ and define:

$$\int_{-\infty}^a f(x)\ dx = \lim_{x\to-\infty} \int_z^a f(x)\ dx.$$