Grassmann formula for vector space dimensions

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Theorem 0.1. *Let U and W be two finite-dimensional subspaces of the vector space V. Denoting by $U+W$ the sum subspace of U and W given by:*

$$U + W = \{\vec{w} + \vec{v} \mid \vec{w} \in W, \vec{u} \in U\} \tag{a}$$

and by $U \cap W$ the intersection subspace, then $U \cap W$ and $U + W$ have finite dimension and

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W).$$

In particular, $U + W$ is a direct sum of U and W if and only if

$$\dim(U + W) = \dim(U) + \dim(W).$$

Proof. $U \cap W$ is a subspace of U of finite dimension, so $U \cap W$ also has finite dimension.

Let

$$B_{U\cap W} = \{\vec{z}_1, \ldots, \vec{z}_q\}$$

be a basis of $U \cap W$. Then there exist $\vec{u}_1, \ldots, \vec{u}_t \in U$ and $\vec{w}_1, \ldots, \vec{w}_s \in W$ such that

$$B_U = \{\vec{z}_1, \ldots, \vec{z}_q, \vec{u}_1, \ldots, \vec{u}_t\}$$

is a basis of U and

$$B_W = \{\vec{z}_1, \ldots, \vec{z}_q, \vec{w}_1, \ldots, \vec{w}_s\}$$

is a basis of W.

Therefore, $\dim(V) = q + t$, $\dim(W) = q + s$, and $\dim(U \cap W) = q$. Thus,

$$\dim(U) + \dim(W) - \dim(U \cap W) = (q + t) + (q + s) - q = q + t + s.$$

To prove (a), it suffices to prove that

$$\{\vec{z}_1, \ldots, \vec{z}_q, \vec{u}_1, \ldots, \vec{u}_t, \vec{w}_1, \ldots, \vec{w}_s\}$$

is a basis of $U + W$.

Let

$$\vec{u} + \vec{w} \in U + W,$$

then there exist

$$a_1, \ldots, a_q, a'_1, \ldots, a'_q, b_1, \ldots, b_t, c_1, \ldots, c_s \in K$$

such that

$$\begin{aligned} \vec{u} &= a_1\vec{z}_1 + \ldots + a_q\vec{z}_q + b_1\vec{u}_1 + \ldots + b_t\vec{u}_t, \\ \vec{w} &= a'_1\vec{z}_1 + \ldots + a'_q\vec{z}_q + c_1\vec{w}_1 + \ldots + c_s\vec{w}_s. \end{aligned}$$

Therefore,

$$\begin{aligned} \vec{u} + \vec{w} = {} & (a_1\vec{z}_1 + \ldots + a_q\vec{z}_q + b_1\vec{u}_1 + \ldots + b_t\vec{u}_t) + \\ & (a'_1\vec{z}_1 + \ldots + a'_q\vec{z}_q + c_1\vec{w}_1 + \ldots + c_s\vec{w}_s) \\ = {} & (a_1 + a'_1)\vec{z}_1 + \ldots + (a_q + a'_q)\vec{z}_q + b_1\vec{u}_1 + \ldots + b_t\vec{u}_t + c_1\vec{w}_1 + \ldots + c_s\vec{w}_s. \end{aligned}$$

Thus,

$$\vec{u} + \vec{w} \in \langle \vec{z}_1, \ldots, \vec{z}_q, \vec{u}_1, \ldots, \vec{u}_t, \vec{w}_1, \ldots, \vec{w}_s \rangle.$$

Since this holds for every $\vec{u} + \vec{w}$, we have

$$\langle \vec{z}_1, \ldots, \vec{z}_q, \vec{u}_1, \ldots, \vec{u}_t, \vec{w}_1, \ldots, \vec{w}_s \rangle = U + W.$$

Let $a_1, \ldots, a_q, b_1, \ldots, b_t, c_1, \ldots, c_s \in K$ such that

$$a_1\vec{z}_1 + \ldots + a_q\vec{z}_q + b_1\vec{u}_1 + \ldots + b_t\vec{u}_t + c_1\vec{w}_1 + \ldots + c_s\vec{w}_s = \vec{0}.$$

We define

$$\begin{aligned} \vec{z} &= a_1\vec{z}_1 + \ldots + a_q\vec{z}_q, \\ \vec{u} &= b_1\vec{u}_1 + \ldots + b_t\vec{u}_t, \\ \vec{w} &= c_1\vec{w}_1 + \ldots + c_s\vec{w}_s. \end{aligned}$$

Thus,

$$\vec{z} + \vec{u} + \vec{w} = \vec{0} \ \rightarrow \ \vec{w} = -\vec{z} - \vec{u}.$$

Since

$$\vec{z} = a_1\vec{z}_1 + \ldots + a_q\vec{z}_q = a_1\vec{z}_1 + \ldots + a_q\vec{z}_q + 0\vec{u}_1 + \ldots + 0\vec{u}_t \in U$$
$$\vec{u} = b_1\vec{u}_1 + \ldots + b_t\vec{u}_t = 0\vec{z}_1 + \ldots + 0\vec{z}_q + b_1\vec{u}_1 + \ldots + b_t\vec{u}_t \in U$$

we can say that $\vec{w} \in U$, which implies $\vec{w} \in U \cap W$.

Since $\{\vec{z}_1, \ldots, \vec{z}_q\}$ is a basis of $U \cap W$, there exist

$$e_1\vec{z}_1 + \ldots + e_q\vec{z}_q \in K$$

such that

$$a_1\vec{z}_1 + \ldots + a_q\vec{z}_q + b_1\vec{u}_1 + \ldots + b_t\vec{u}_t = e_1\vec{z}_1 + \ldots + e_q\vec{z}_q,$$

which implies

$$(a_1 - e_1)\vec{z}_1 + \ldots + (a_q - e_q)\vec{z}_q + b_1\vec{u}_1 + \ldots + b_t\vec{u}_t = \vec{0}$$

by linear independence of $\vec{z}_1, \ldots, \vec{z}_q, \vec{u}_1, \ldots, \vec{u}_t$, all coefficients are 0, and in particular $b_1 = \ldots = b_t = 0$. Thus,

$$a_1\vec{z}_1 + \ldots + a_q\vec{z}_q + c_1\vec{w}_1 + \ldots + c_s\vec{w}_s = \vec{0}$$

By linear independence of $\vec{z}_1, \ldots, \vec{z}_q, \vec{w}_1, \ldots, \vec{w}_s$ we also have

$$a_1 = \ldots = a_q = c_1 = \ldots = c_s = 0.$$

Thus, $\vec{z}_1, \ldots, \vec{z}_q, \vec{u}_1, \ldots, \vec{u}_t, \vec{w}_1, \ldots, \vec{w}_s$ are linearly independent.

The last assertion of the theorem immediately follows from (0.1) and the definition of direct sum. $\square$