Transformations of Distributions

The binomial distribution

- Sequence of n independent trials of an experiment.

- Two possible outcomes for each trial:
  "Success" or "Failure"

- $p = P(\text{Success on any one trial})$

Let X = # successes in n trials.

$\boxed{X \sim \text{bin}(n, p)}$ pmf?

Suppose n = 4.

$$P(X = 3) = P(SSSF \text{ or } SSFS \text{ or } SFSS \text{ or } FSSS)$$

disjoint

$$= P(SSSF) + P(SSFS) + P(SFSS) + P(FSSS)$$

independent

$$= p \cdot p \cdot p \cdot (1 - p) + p \cdot p \cdot (1 - p) \cdot p + \cdots$$

counting

$$= 4p^3(1 - p)$$

n trials, x successes

There are $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ different ways to arrange the x S's in the n positions.

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} I_{\{0,1,\ldots,n\}}(x)$$

$$\boxed{X \sim \text{bin}(n, p)}$$

Transformations

Suppose that $X \sim \text{bin}(n, p)$.

What is the distribution of Y = n-X?

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} I_{\{0,1,\ldots,n\}}(x)$$

Just do it:

$$P(Y = y) = P(n - X = y)$$

$$= P(X = n - y)$$

$$= \binom{n}{n-y} p^x (1-p)^{n-(n-y)} I_{\{0,1,\ldots,n\}}(n-y)$$

Transformations

$$P(Y=y)=$$

$$\binom{n}{n-y} p^x (1-p)^{n-(n-y)} I_{\{0,1,\ldots,n\}}(n-y)$$

$$\boxed{n-y=0,1,\ldots,n} \Rightarrow \quad y=0,1,\ldots,n$$

$$\binom{5}{3}=\frac{5!}{3!2!}=\frac{5!}{2!3!}=\binom{5}{2}$$

$$P(Y=y)=\binom{n}{y}(1-p)^y p^{n-y} I_{\{0,1,\ldots,n\}}(y)$$

$$\Rightarrow \qquad \boxed{Y \sim \text{bin}(n, 1-p)}$$

Continuous Transformations

For X discrete or continuous,
the cumulative distribution function (cdf)
Is denoted by F(x) and is defined by

$$\boxed{F(x) = P(X \leq x)}$$

- **X discrete:**

$$F(x) = \sum_{u \leq x} P(X = u) = \sum_{u \leq x} f(u)$$

- **X continuous:**

$$F(x) = \int_{-\infty}^{x} f(u)\, du$$

X continuous:

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$\Downarrow$$

$$\boxed{\text{Fundamental Theorem of Calculus}}$$

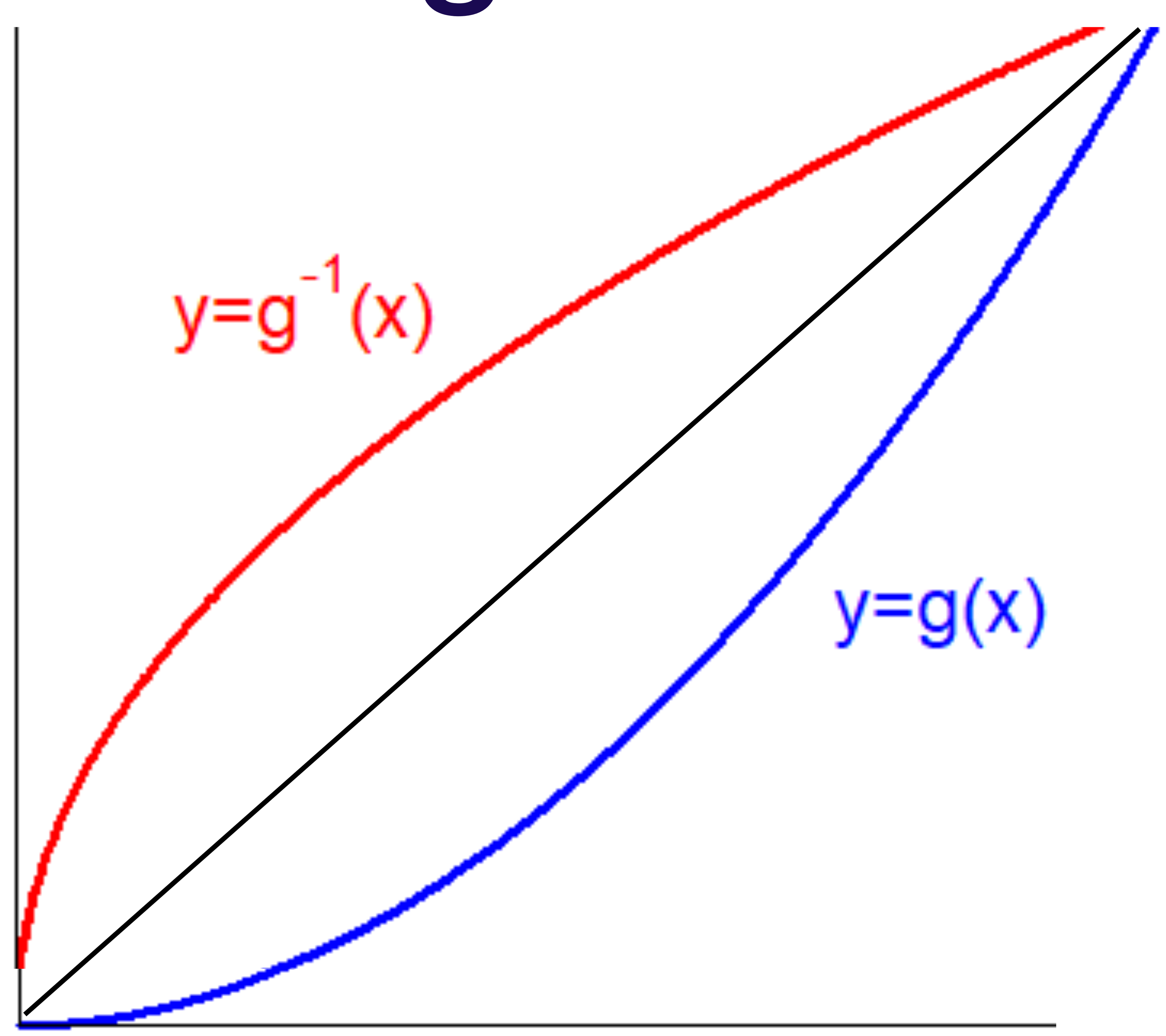
$$\Downarrow$$

$$f(x) = \frac{d}{dx}F(x)$$

- Suppose that X is continuous with pdf $f_X(x)$.
- Let g(x) be a continuous, differential, and invertible function.
- The g is either strictly increasing or strictly decreasing.

The inverse of an increasing function is increasing.

same for decreasing



X with pdf $f_X(x)$, Y=g(X) invertible

Case: g increasing

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(g(X) \leq y) \\
&= P(g^{-1}(g(X)) \leq g^{-1}(y)) \\
&= P(X \leq g^{-1}(y)) \\
&= F_X(g^{-1}(y))
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) \\
&= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)
\end{aligned}
$$

X with pdf $f_X(x)$, $Y=g(X)$ invertible

Case: g decreasing

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$
$$= P(g^{-1}(g(X)) \ge g^{-1}(y))$$
$$= P(X \ge g^{-1}(y)) = 1 - P(X < g^{-1}(y))$$
$$= 1 - P(X \le g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right)$$
$$= -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

negative

- X with pdf $f_X(x)$, Y=g(X) invertible
- g increasing <u>or</u> decreasing

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|}$$

Example: Suppose that X has a gamma distribution with shape parameter α and inverse scale parameter β.

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} I_{(0,\infty)}(x)$$

$$\boxed{X \sim \Gamma(\alpha, \beta)}$$

$$X \sim \Gamma(\alpha, \beta)$$

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} I_{(0,\infty)}(x)$$

What is the distribution of Y=cX for constant c>0 a constant?

$$y = g(x) = cx \quad \Rightarrow \quad x = g^{-1}(y)$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{1}{\Gamma(\alpha)} \beta^\alpha (x/c)^{\alpha-1} e^{-\beta x/c} I_{(0,\infty)}(x) \cdot |1/c|$$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} (y/c)^{\alpha-1} e^{-\beta y/c} I_{(0,\infty)(y/c)} \cdot |1/c|$$

$$0 < y/c < \infty \overset{c>0}{\Rightarrow} 0 < y < \infty$$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)} (\beta/c)^{\alpha} x^{\alpha-1} e^{-\beta x/c} I_{(0,\infty)(y)}$$

$$\Rightarrow \boxed{Y \sim \Gamma(\alpha, \beta/c)}$$