COMP90084 Assignment 1 (Part 1)

Chenyang Dong
1074314

August 2022

Question 1

	Real part	Imaginary part
3i	0	3
$\sqrt{7}$	$\sqrt{7}$	0
$6+\pi i$	6	π

Question 2

	Modulus	Complex conjugate
3i	3	-3i
$\sqrt{7}$	$\sqrt{7}$	$\sqrt{7}$
$6+\pi i$	$\sqrt{36+\pi^2}$	$6-\pi i$

Question 3

$$z = \rho e^{i\phi} = \rho[\cos(\phi) + i\sin(\phi)]; \rho = |z|$$

	Trigonometric form	Exponential form
3i	$3(\cos\arctan\frac{3}{0} + i\sin\arctan\frac{3}{0})$	$3e^{i\arctan\frac{3}{0}}$
$\sqrt{7}$	$\sqrt{7}(\cos\arctan\frac{0}{\sqrt{7}} + i\sin\arctan\frac{0}{\sqrt{7}})$	$\sqrt{7}e^{i\arctan\frac{0}{\sqrt{7}}}$
$6+\pi i$	$\sqrt{36+\pi^2}(\cos\arctan\frac{\pi}{6} + i\sin\arctan\frac{\pi}{6})$	$\sqrt{36+\pi^2}e^{i\arctan\frac{\pi}{6}}$

Question 4

(1)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$
$$XX^\dagger = \begin{bmatrix} 0 \times 0 + 1 \times 1 & 0 \times 1 + 1 \times 0 \\ 1 \times 0 + 0 \times 0 & 1 \times 1 + 0 \times 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = X^\dagger X = I_2$$

Therefore, X is Hermitian and unitary.

(2)

$$Y = \begin{bmatrix} 0 & \text{-i} \\ \text{i} & 0 \end{bmatrix}$$

$$Y^\dagger = \begin{bmatrix} 0 & \text{-i} \\ \text{i} & 0 \end{bmatrix} = Y$$
$$YY^\dagger = \begin{bmatrix} 0 \times 0 + -i \times i & 0 \times -i + -i \times 0 \\ i \times 0 + 0 \times i & i \times -i + 0 \times 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Y^\dagger Y = I_2$$

Therefore, Y is Hermitian and unitary.

(3)

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$
$$ZZ^\dagger = \begin{bmatrix} 1 \times 1 + 0 \times 0 & 1 \times 0 + 0 \times -1 \\ 0 \times 1 + -1 \times 0 & 0 \times 0 + -1 \times -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Z^\dagger Z = I_2$$

Therefore, Z is Hermitian and unitary.

Question 5

(1)

$$\begin{aligned}\sum_i P_i &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ &= \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\end{aligned}$$

(2)

$$\begin{aligned}\langle\psi|P_1|\psi\rangle &= (a^\star\langle 0| + b^\star\langle 1|)\,(|0\rangle\langle 0|)\,(a|0\rangle + b|1\rangle) \\ &= (a^\star\langle 0||0\rangle\langle 0| + b^\star\langle 1||0\rangle\langle 1|)\,(a|0\rangle + b|1\rangle) \\ &= a^\star\langle 0|\,(a|0\rangle + b|1\rangle) \\ &= a^\star a\langle 0||0\rangle + a^\star b\langle 0||1\rangle \\ &= a^\star a\end{aligned}$$

$$\begin{aligned}\langle\psi|P_2|\psi\rangle &= (a^\star\langle 0| + b^\star\langle 1|)\,(|1\rangle\langle 1|)\,(a|0\rangle + b|1\rangle) \\ &= (a^\star\langle 0||1\rangle\langle 1| + b^\star\langle 1||1\rangle\langle 1|)\,(a|0\rangle + b|1\rangle) \\ &= b^\star\langle 1|\,(a|0\rangle + b|1\rangle) \\ &= b^\star a\langle 1||0\rangle + b^\star b\langle 1||1\rangle \\ &= b^\star b\end{aligned}$$

(3)

$$\begin{aligned}\langle\psi|(\sum_i P_i)|\psi\rangle &= (a^\star\langle 0| + b^\star\langle 1|)\,(|0\rangle\langle 0| + |1\rangle\langle 1|)\,(a|0\rangle + b|1\rangle) \\ &= (a^\star\langle 0||0\rangle\langle 0| + a^\star\langle 0||1\rangle\langle 1| + b^\star\langle 1||0\rangle\langle 0| + b^\star\langle 1||1\rangle\langle 1|)\,(a|0\rangle + b|1\rangle) \\ &= (a^\star\langle 0| + b^\star\langle 1|)\,(a|0\rangle + b|1\rangle) \\ &= a^\star a\langle 0||0\rangle + a^\star b\langle 0||1\rangle + b^\star a\langle 1||0\rangle + b^\star b\langle 1||1\rangle \\ &= a^\star a + b^\star b \\ &= 1\end{aligned}$$

Question 6

If the state is not entangled, it is able to transform from the form of

$$ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

to the tensor product form of

$$(a|0\rangle + b|1\rangle)\ (c|0\rangle + d|1\rangle)$$

In this case, the state $|\psi\rangle = (\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + 0|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$.
If $|\psi\rangle$ is not entangled, all of coefficients can be solved, which

$$ac = \frac{1}{\sqrt{2}} \qquad ad = 0 \qquad bc = 0 \qquad bd = \frac{1}{\sqrt{2}}$$

and clearly it cannot be solved. Therefore, $|\psi\rangle$ is an entangled state.

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho_1 = tr_2\rho = \sum_{i=0}^{1}(I \otimes \langle i|)\ \rho\ (I \otimes |i\rangle) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$$

To prove the equation: $tr(\rho_1 T) = tr(\rho(T \otimes I))$

$$\begin{aligned} LHS &= tr(\rho_1 T) \\ &= tr(\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}) \\ &= tr(\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}) \end{aligned}$$

$$
\begin{aligned}
RHS &= tr(\rho(T \otimes I)) \\
&= tr(\frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}(\begin{bmatrix}1 & 0\\ 0 & e^{\frac{i\pi}{4}}\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix})) \\
&= tr(\frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & e^{\frac{i\pi}{4}} & 0\\ 0 & 0 & 0 & e^{\frac{i\pi}{4}}\end{bmatrix}) \\
&= tr(\frac{1}{2}\begin{bmatrix}1 & 0 & 0 & e^{\frac{i\pi}{4}}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & e^{\frac{i\pi}{4}}\end{bmatrix}) \\
&= tr(\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & e^{\frac{i\pi}{4}}\end{bmatrix}) \\
&= LHS
\end{aligned}
$$

Therefore, $tr(\rho_1 T) = tr(\rho(T \otimes I))$.

Question 7

(1)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To calculate the eigenvalues:

$$\begin{aligned} det(X - \lambda I) &= det(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \\ &= det(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}) \\ &= \lambda^2 - 1 = 0 \\ &\longrightarrow \lambda_1 = -1 \qquad \lambda_2 = 1 \end{aligned}$$

Let $|\lambda_1\rangle = (x, y)^T$ be an eigenvector corresponding to λ.

$$X|\lambda_1\rangle = |\lambda_1\rangle \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{cases} y = -x \\ x = -y \end{cases}$$

These relations are satified with x = -1, y = 1. Thus the normalized eigenvector is:

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Same as $|\lambda_1\rangle$, let $|\lambda_2\rangle = (x, y)^T$ be an eigenvector corresponding to λ.

$$X|\lambda_2\rangle = |\lambda_1\rangle \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{cases} y = x \\ x = y \end{cases}$$

These relations are satified with x = 1, y = 1. Thus the normalized eigenvector is:

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(2)

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

To calculate the eigenvalues:

$$\begin{aligned} det(Z - \lambda I) &= det(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \\ &= det(\begin{bmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{bmatrix}) \\ &= (1 - \lambda)(-1 - \lambda) = 0 \\ &\longrightarrow \lambda_1 = -1 \qquad \lambda_2 = 1 \end{aligned}$$

Let $|\lambda_1\rangle = (x, y)^T$ be an eigenvector corresponding to λ.

$$Z|\lambda_1\rangle = |\lambda_1\rangle \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{cases} x = -x \\ -y = -y \end{cases}$$

These relations are satified with x = 0, y = 1. Thus the normalized eigenvector is:

$$|\lambda_1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Same as $|\lambda_1\rangle$, let $|\lambda_2\rangle = (x, y)^T$ be an eigenvector corresponding to λ.

$$Z|\lambda_1\rangle = |\lambda_1\rangle \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{cases} x = x \\ -y = y \end{cases}$$

These relations are satified with x = 1, y = 0. Thus the normalized eigenvector is:

$$|\lambda_2\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$