COMP90084 Quantum Software Fundamentals

Semester 2, 2022, Assignment 1, Part 1

This is Part 1 of the first assignment and consists of **7** questions. The total marks of this part is **10**. The goal of this assignment is to familiarize yourself with some basic calculations in quantum computing.

I've uploaded a file called "The Matrix Cookbook" to the Resources section in Subject Modules. It could be a good reference for matrix-related stuff.

The Pauli matrices are

$$\begin{aligned} \sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{1}$$

Question 1 (0.5 mark) Find the real and imaginary parts of the following complex numbers:

$$3\mathrm{i};$$
$$\sqrt{7};$$
$$6 + \pi\mathrm{i};$$

Question 2 (1 mark) Find the modulus and complex conjugate of the complex numbers in the previous question.

Question 3 (1 mark) Convert the complex numbers in the first question into trigonometric and exponential forms. You can have the $\arctan$ function in your answers.

Question 4 (1.5 marks) Check whether the Pauli matrices X, Y and Z are unitary/Hermitian. Show your work.

Question 5 (2 marks) **(Projective measurements.)** Let $P_1 = |0\rangle \langle 0|, P_2 = |1\rangle \langle 1|$, and $|\psi\rangle = a|0\rangle + b|1\rangle$ with normalization condition $a^*a + b^*b = aa^* + bb^* = 1$. If we sandwich P_i between a bra and a ket of $|\psi\rangle$ to get $\langle\psi| P_i |\psi\rangle$ (can also be written as $\langle P_i\rangle$), we are measuring the state $|\psi\rangle$ with observable P_i. This can also be regarded as projecting the state vector $|\psi\rangle$ onto the basis of P_i. Calculate the followings:

$$\sum_i P_i \tag{2}$$

$$\langle\psi| P_1 |\psi\rangle, \langle\psi| P_2 |\psi\rangle \tag{3}$$

$$\langle\psi| (\sum_i P_i) |\psi\rangle \tag{4}$$

Question 6 (2 marks) Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{5}$$

$$\rho = |\psi\rangle \langle\psi| \tag{6}$$

$$\rho_1 = \operatorname{tr}_2 \rho = \sum_{i=0}^{1} (I \otimes \langle i|)\rho(I \otimes |i\rangle) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \tag{8}$$

T acts on the first qubit (ρ_1). First, check whether the state $|\psi\rangle$ is entangled or not, and show your work. Then show that

$$\operatorname{tr}(\rho_1 T) = \operatorname{tr}(\rho(T \otimes I)) \tag{9}$$

show your work.

Eigenvalue Problems [NO08] Suppose we apply a matrix A on a vector $|v\rangle \in \mathbb{C}^n$, where $|v\rangle \neq \mathbf{0}$. If we have $\lambda \in \mathbb{C}$, such that

$$A|v\rangle = \lambda |v\rangle \tag{10}$$

Then λ is called an **eigenvalue** of A, while $|v\rangle$ is called the corresponding **eigenvector**. We often use the symbol $|\lambda\rangle$ for an eigenvector corresponding to an eigenvalue $|\lambda\rangle$. Let $\{|e_k\rangle\}$ be an orthonormal basis in $\mathbb{C}^n$ and let $\langle e_i| A |e_j\rangle = A_{ij}$ and $v_i = \langle e_i|v\rangle$ be the components of A and $|v\rangle$ with respect to the basis. Then the component expression for Eqn. 10 is obtained from

$$A|v\rangle = \sum_{i,j} |e_i\rangle \langle e_i| A |e_j\rangle \langle e_j|v\rangle = \sum_{i,j} A_{i,j} v_j |e_i\rangle$$

as

$$\sum_j A_{ij} v_j = \lambda v_i \tag{11}$$

To find the eigenvalues, we rewrite the eigenvalue equation as

$$\sum_j (A - \lambda I)_{ij} v_j = 0 \tag{12}$$

This equation in v_j has nontrivial solutions if and only if the matrix $A - \lambda I$ has no inverse, namely:

$$D(\lambda) \equiv \det(A - \lambda I) = 0 \tag{13}$$

If it had the inverse, then $|v\rangle = (A - \lambda I)^{-1}\mathbf{0} = 0$, where **0** is the zero vector, would be the unique solution. This is called the **characteristic equation** or the **eigen equation** of A.

Let A be an n by n matrix. Then the characteristic equation has n solutions, including the multiplicity, which we write as $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Then function $D(\lambda)$ is also written as

$$\begin{aligned} D(\lambda) &= \Pi_{i=1}^n (\lambda_i - \lambda) \\ &= (-\lambda)^n + \sum_i \lambda_i (-\lambda)^{n-1} + \cdots + \Pi_{i=1}^n \lambda_i \\ &= (-\lambda)^n + \operatorname{tr} A (-\lambda)^{n-1} + \cdots + \det A \end{aligned} \tag{14}$$

where we used $\operatorname{tr} A = \sum_i \lambda_i$ and $\det A = \Pi_i \lambda_i$.

Question 7 (2 marks) Calculate the eigenvalues and corresponding eigenvectors for Pauli matrices X and Z. An example has been done for the Pauli Y matrix in Example 1.3 of [NO08]. Show your work. Write the eigenvectors in Dirac's bra-ket notation.

References

[NO08] M. Nakahara and T. Ohmi. *Quantum Computing: From Linear Algebra to Physical Realizations*. Taylor & Francis, 2008.