SC4001 Exam 2018-2019 Sem 1

1. (a) Given two inputs x and y, you are to train a neuron to approximate the following function ϕ when $0 \leq x, y \leq 1.0$.

$$\phi(x, y) = x + 2y^3 + xy - 0.5$$

(i) Briefly state how you generate training data.

(3 marks)

Answer

- Generate random values for x and y within the interval [0, 1.0]
- Divide the range [0, 1.0] into N steps for x and y
- Generate M random pairs (x, y) where x and y are drawn uniformly from [0, 1.0]
- For each (x, y), compute the corresponding using the given function:

$$\phi(x, y) = x + 2y^3 + xy - 0.5$$

- Combine the input pairs and computed outputs into training dataset
- This dataset can then be used to train a neuron using supervised learning

(ii) State how you design the inputs to a linear neuron.

(3 marks)

Answer

$$\phi(x, y) = x + 2y^3 + xy - 0.5$$

- Linear neuron learns a linear function. The above equation can be written as a linear equation:

$$\phi(x, y) = x_1 + 2x_2 + x_3 - 0.5$$

- where the linear neuron receives 3 inputs: $x_1 = x,\ x_2 = y^3, and\ x_3 = xy.$

(iii) Write the activation function if a perceptron is used.

(4 marks)

Answer

- activation function: $Y = f(\boldsymbol{u}) = \dfrac{1}{1 + e^{-u}}$

1. (b) The softmax layer of has three neurons, receives 2-dimensional inputs $(x_1, x_2) \in \boldsymbol{R}^2$. The weight matrix $\boldsymbol{W}$ and bias vector $\boldsymbol{b}$ of the layer are given by

$$W = \begin{pmatrix} 2.0 & 0.0 & 1.0 \\ -1.0 & 1.0 & -2.0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0.5 \\ 1.0 \\ -0.5 \end{pmatrix}$$

1. (b) (i) Find the decision boundaries separating each pair of classes

(7 marks)

Answer

Synaptic inputs:

$$u_1 = 2x_1 - x_2 + 0.5$$
$$u_2 = x_2 + 1.0$$
$$u_3 = x_1 - 2x_2 - 0.5$$

Decision boundary separating Class 1 and Class 2:

$$u_1 = u_2$$
$$2x_1 - x_2 + 0.5 = x_2 + 1$$
$$2x_2 = 2x_1 - 0.5$$
$$x_2 = x_1 - 0.25$$

Decision boundary separating Class 1 and Class 3:

$$u_1 = u_3$$
$$2x_1 - x_2 + 0.5 = x_1 - 2x_2 - 0.5$$
$$x_2 = -x_1 - 1$$

Decision boundary separating Class 2 and Class 3:

$$u_2 = u_3$$
$$x_2 + 1 = x_1 - 2x_2 - 0.5$$
$$3x_2 = x_1 - 1.5$$
$$x_2 = \frac{1}{3}(x_1 - 1.5)$$

1. (b) (ii) Plot the decision boundaries separating the three classes, clearly indicating the regions belonging to each class.

(6 marks)

Answer

Decision boundary separating Class 1 and Class 2:

$x_2 = x_1 - 0.25$

plot

$when\ x_2 = 0, x_1 = \ \ 0.25$
$when\ x_1 = 0, x_2 = -0.25$

Decision boundary separating Class 1 and Class 3:

$x_2 = -x_1 - 1$

plot

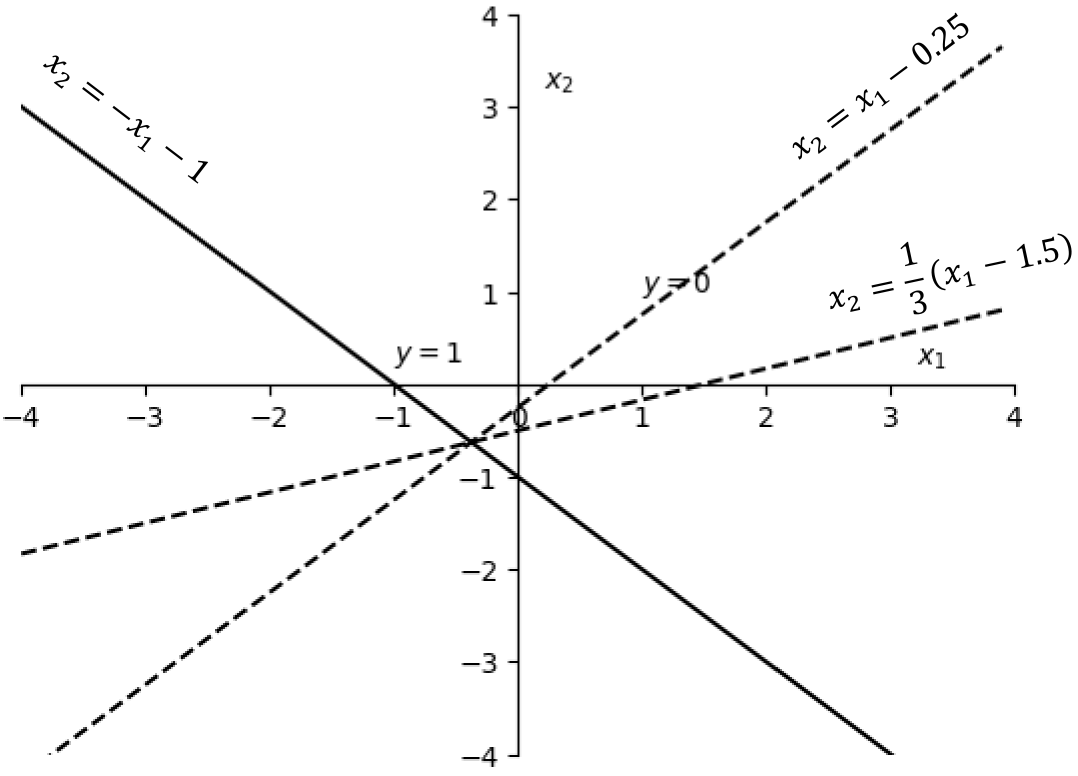
$when\ x_2 = 0, x_1 = -1$
$when\ x_1 = 0, x_2 = -1$

Decision boundary separating Class 2 and Class 3:

$x_2 = \frac{1}{3}(x_1 - 1.5)$

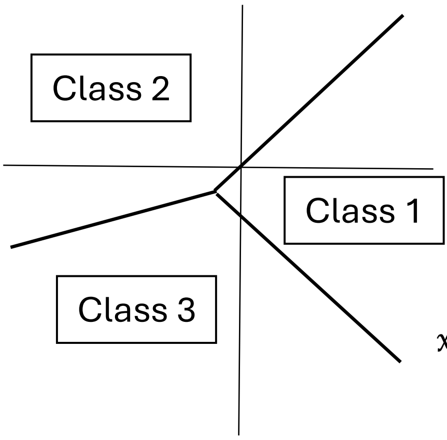
plot

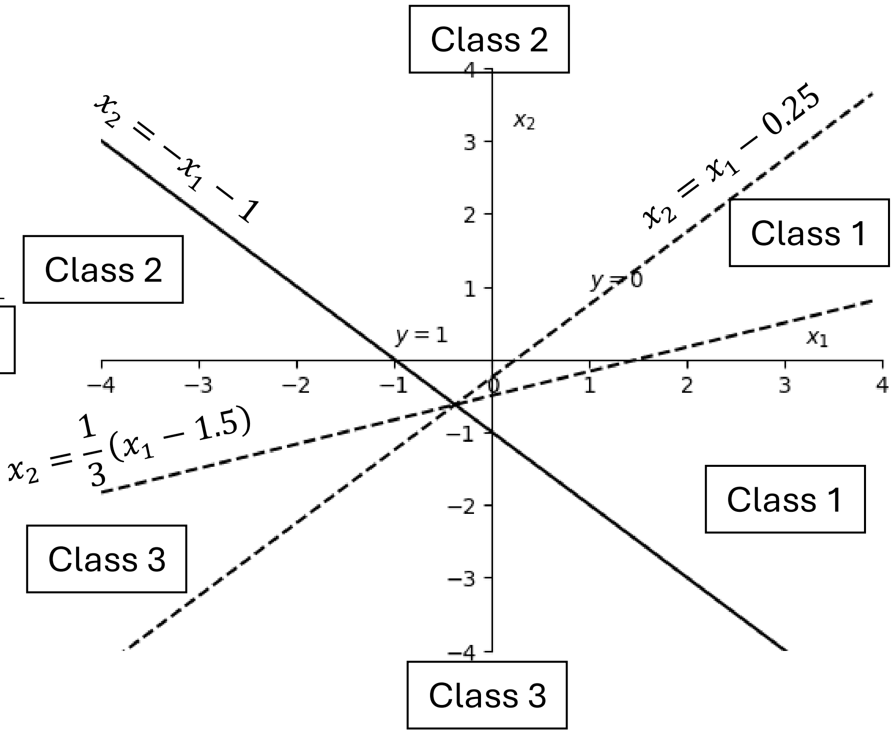
$when\ x_2 = 0, x_1 = \ \ 1.5$
$when\ x_1 = 0, x_2 = -0.5$



1. (b) (ii) cont

Answer





$$u_1 = 2x_1 - x_2 + 0.5$$
$$u_2 = x_2 + 1.0$$
$$u_3 = x_1 - 2x_2 - 0.5$$

Test point (x, y):

The u with highest value is the region

Test point (1, 3):

$$u_1 = 2(1) - (3) + 0.5 = -0.5$$
$$u_2 = (3) + 1.0 = 4.0$$
$$u_3 = (1) - 2(3) - 0.5 = -5.5$$

Region: Class 2

Test point (3, 1):

$$u_1 = 2(3) - (1) + 0.5 = 5.5$$
$$u_2 = (1) + 1.0 = 2.0$$
$$u_3 = (3) - 2(1) - 0.5 = 0.5$$

Region: Class 1

Test point (-3, -1):

$$u_1 = 2(-3) - (1) + 0.5 = -6.5$$
$$u_2 = (1) + 1.0 = 2.0$$
$$u_3 = (-3) - 2(1) - 0.5 = -5.5$$

Region: Class 2

Test point (3, -2):

$$u_1 = 2(3) - (-2) + 0.5 = 8.5$$
$$u_2 = (-2) + 1.0 = -1.0$$
$$u_3 = (3) - 2(-2) - 0.5 = 6.5$$

Region: Class 1

Test point (1, -3):

$$u_1 = 2(1) - (-3) + 0.5 = 5.5$$
$$u_2 = (-3) + 1.0 = -2.0$$
$$u_3 = (1) - 2(-3) - 0.5 = 6.5$$

Region: Class 3

Test point (-3, -2):

$$u_1 = 2(-3) - (-2) + 0.5 = -3.5$$
$$u_2 = (-2) + 1.0 = -1.0$$
$$u_3 = (-3) - 2(-2) - 0.5 = 0.5$$

Region: Class 3

1. (b) (iii) Find the output class label for an input pattern $\boldsymbol{x} = \begin{pmatrix} -0.5 \\ 1.0 \end{pmatrix}$.

(3 marks)

Answer

from the graph in 1(b)(i), it is Class 2.

Test point (-0.5, 1.0):

$$\begin{aligned} u_1 &= 2(-0.5) - (1) + 0.5 = -1.5 \\ u_2 &= (1) + 1.0 = 2.0 \\ u_3 &= (-0.5) - 2(1) - 0.5 = -3.0 \end{aligned}$$

Region: Class 2

output class label for $\boldsymbol{x} = \begin{pmatrix} -0.5 \\ 1.0 \end{pmatrix}$ is Class 2

2. The four-layer feedforward neural network shown in Figure Q2 receives 2-dimensional inputs $(x_1, x_2) \in \boldsymbol{R}^2$ and produces an output $y \in \boldsymbol{R}$. Each hidden- layer has two perceptrons and the output neuron is a linear neuron. The weights and biases of the networks are initialized as indicated in the figure.

The network is trained to produce a desired output $d = 1.0$ for an input $x = \begin{pmatrix} -1.0 \\ 0.5 \end{pmatrix}$, by using gradient decent learning. The learning factor $\alpha = 0.8$.

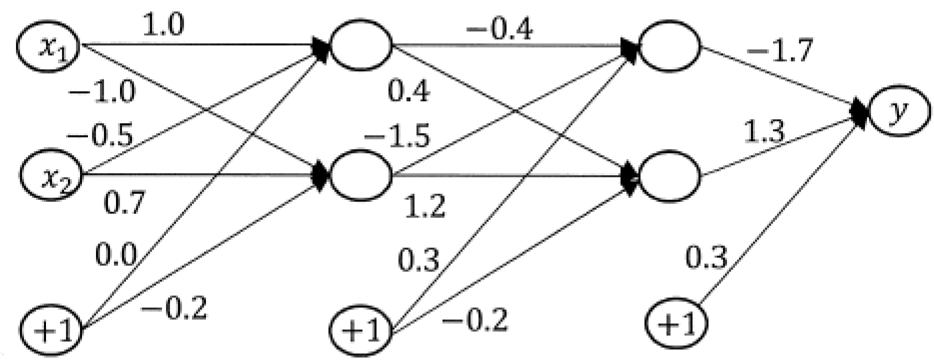


Figure Q2

For one iteration of stochastic gradient descent learning:

2. (a) Write the initial weight matrix $\boldsymbol{W}_1$ and bias vector $\boldsymbol{b}_1$ of the first hidden layer, the initial weight matrix $\boldsymbol{W}_2$ and bias vector $\boldsymbol{b}_2$ of the second hidden layer, and the initial weight vector $\boldsymbol{w}$ and bias b of the output neuron.

(3 marks)

Answer

$$W_1 = \begin{pmatrix} 1.0 & -1.0 \\ -0.5 & 0.7 \end{pmatrix} \qquad b_1 = \begin{pmatrix} 0.0 \\ -0.2 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix}$$

$$w = \begin{pmatrix} -1.7 \\ 1.3 \end{pmatrix} \qquad b = 0.3$$

2. (b) Find the synaptic input $\boldsymbol{u}_1$ and the activation $\boldsymbol{h}_1$ of the first hidden layer, the synaptic input $\boldsymbol{u}_2$ and the activation $\boldsymbol{h}_2$ of the second hidden layer, and the activation y of the output neuron.

(6 marks)

Answer

$$W_1 = \begin{pmatrix} 1.0 & -1.0 \\ -0.5 & 0.7 \end{pmatrix} \qquad b_1 = \begin{pmatrix} 0.0 \\ -0.2 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix}$$

Input $x = \begin{pmatrix} -1.0 \\ 0.5 \end{pmatrix}$

Synaptic input u_1,

$$u_1 = \boldsymbol{W}_1{}^T x + b_1 = \begin{pmatrix} 1.0 & -0.5 \\ -1.0 & 0.7 \end{pmatrix}\begin{pmatrix} -1.0 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.2 \end{pmatrix}$$

$$= \begin{pmatrix} (\ \ 1)(-1) + (-0.5)(0.5) \\ (-1)(-1) + (\ \ 0.7)(0.5) \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.2 \end{pmatrix}$$

$$= \begin{pmatrix} -1.25 \\ 1.35 \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.2 \end{pmatrix} = \begin{pmatrix} -1.25 \\ 1.15 \end{pmatrix}$$

activation h_1 (perceptron),

$$h_1 = sigmoid(u) = \frac{1}{1 + e^{-U}} = \begin{pmatrix} 0.22 \\ 0.76 \end{pmatrix}$$

Synaptic input to h_2,

$$u_2 = \boldsymbol{W}_2{}^T h_1 + b_2 = \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix}\begin{pmatrix} 0.22 \\ 0.76 \end{pmatrix} + \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix}$$

$$= \begin{pmatrix} (-0.4)(0.22) + (-1.5)(0.76) \\ (-1.5)(0.22) + (\ \ 1.2)(0.76) \end{pmatrix} + \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix}$$

$$= \begin{pmatrix} -1.23 \\ 0.58 \end{pmatrix} + \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix} = \begin{pmatrix} -0.93 \\ 0.38 \end{pmatrix}$$

activation h_2 (perceptron),

$$h_2 = sigmoid(u) = \frac{1}{1 + e^{-U}} = \begin{pmatrix} 0.28 \\ 0.59 \end{pmatrix}$$

2. (b) cont

Answer

$w = \begin{pmatrix} -1.7 \\ 1.3 \end{pmatrix}$ $b = 0.3$

activation y,

$$u_3 = \boldsymbol{W_3}^T h_2 + b_3 = (-1.7 \quad 1.3)\begin{pmatrix} 0.28 \\ 0.59 \end{pmatrix} + 0.3$$

$$= (-1.7)(0.28) + (1.3)(0.59) = -0.48 + 0.77 = 0.29$$

$$y = f(u_3) = u_3 = 0.29$$

<u>Summary</u>

synaptic input $u_1 = \begin{pmatrix} -1.25 \\ 1.15 \end{pmatrix}$

activation $h_1 = \begin{pmatrix} 0.22 \\ 0.76 \end{pmatrix}$

synaptic input $u_2 = \begin{pmatrix} -0.93 \\ 0.38 \end{pmatrix}$

activation $h_2 = \begin{pmatrix} 0.28 \\ 0.59 \end{pmatrix}$

activation $y = 0.29$

2. (c) Find the square error cost $J = \frac{1}{2}(d-y)^2$.

(1 mark)

Answer

$d = 1.0$ $y = 0.29$

square error cost $J = \dfrac{1}{2}(d-y)^2 = \dfrac{1}{2}(1-0.29)^2 = 0.25$

2. (d) Find the gradients $\nabla_{\boldsymbol{u_1}}J, \nabla_{\boldsymbol{u_2}}J$ and $\nabla_{\boldsymbol{y}}J$ of the cost J with respect to u_1, u_2 and y, respectively.

(8 marks)

Answer

Backpropagation for FFN:

Gradient $\nabla_{\boldsymbol{y}}J$,

$$d = 1.0 \qquad y = 0.29 \qquad w = \begin{pmatrix} -1.7 \\ 1.3 \end{pmatrix}$$

$$\nabla_{\boldsymbol{y}}J = -(d - y) = -(1 - 0.29) = -0.71$$

Gradient $\nabla_{\boldsymbol{u_2}}J$,

$$h_2 = \begin{pmatrix} 0.28 \\ 0.59 \end{pmatrix} \; w = \begin{pmatrix} -1.7 \\ 1.3 \end{pmatrix}$$

perceptron layer $\quad h_2 = f(\boldsymbol{u_2}) = \dfrac{1}{1 + e^{-U}}$

Derivative $\quad f'(\boldsymbol{u_2}) = h_2(1 - h_2) = \begin{pmatrix} 0.28(1 - 0.28) \\ 0.59(1 - 0.59) \end{pmatrix} = \begin{pmatrix} 0.20 \\ 0.24 \end{pmatrix}$

$$\nabla_{\boldsymbol{u_2}}J = w\left(\nabla_{\boldsymbol{y}}J\right) \cdot f'(\boldsymbol{u_2}) = \begin{pmatrix} -1.7 \\ 1.3 \end{pmatrix}(-0.71) \cdot \begin{pmatrix} 0.20 \\ 0.24 \end{pmatrix} = \begin{pmatrix} 1.21 \\ -0.92 \end{pmatrix} \cdot \begin{pmatrix} 0.20 \\ 0.24 \end{pmatrix} = \begin{pmatrix} 0.24 \\ -0.22 \end{pmatrix}$$

Gradient $\nabla_{\boldsymbol{u_1}}J$,

$$h_1 = \begin{pmatrix} 0.22 \\ 0.76 \end{pmatrix} \qquad W_2 = \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix}$$

perceptron layer $\quad h_1 = f(\boldsymbol{u_1}) = \dfrac{1}{1 + e^{-U}}$

Derivative $\quad f'(\boldsymbol{u_1}) = h_1(1 - h_1) = \begin{pmatrix} 0.22(1 - 0.22) \\ 0.76(1 - 0.76) \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.18 \end{pmatrix}$

$$\begin{aligned}
\nabla_{\boldsymbol{u_1}}J = W_2\left(\nabla_{\boldsymbol{u_2}}J\right) \cdot f'(\boldsymbol{u_1}) &= \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix}\begin{pmatrix} 0.24 \\ -0.22 \end{pmatrix} \cdot \begin{pmatrix} 0.17 \\ 0.18 \end{pmatrix} \\
&= \begin{pmatrix} (-0.4)(0.24) + (-1.5)(-0.22) \\ (-1.5)(0.24) + (\ 1.2)(-0.22) \end{pmatrix} \cdot \begin{pmatrix} 0.17 \\ 0.18 \end{pmatrix} \\
&= \begin{pmatrix} 0.23 \\ -0.62 \end{pmatrix} \cdot \begin{pmatrix} 0.17 \\ 0.18 \end{pmatrix} = \begin{pmatrix} 0.04 \\ -0.11 \end{pmatrix}
\end{aligned}$$

<u>Summary</u>

$$\nabla_{\boldsymbol{y}}J \ = -0.71$$

$$\nabla_{\boldsymbol{u_2}}J = \begin{pmatrix} 0.24 \\ -0.22 \end{pmatrix}$$

$$\nabla_{\boldsymbol{u_1}}J = \begin{pmatrix} 0.04 \\ -0.11 \end{pmatrix}$$

2. (e) Find the gradients $\nabla_{\boldsymbol{W}_2} J$, and $\nabla_{\boldsymbol{b}_2} J$ of cost J with respect to the weight matrix $\boldsymbol{W}_2$ and the bias vector $\boldsymbol{b}_2$ of the second hidden layer.

(4 marks)

Answer

Second hidden layer u_2,

$$h_1 = \begin{pmatrix} 0.22 \\ 0.76 \end{pmatrix} \qquad \nabla_{\boldsymbol{u}_2} J = \begin{pmatrix} 0.24 \\ -0.22 \end{pmatrix}$$

$$\nabla_{\boldsymbol{W}_2} J = h_1 \left(\nabla_{\boldsymbol{u}_2} J\right)^{\boldsymbol{T}} = \begin{pmatrix} 0.22 \\ 0.76 \end{pmatrix} (0.24 \quad -0.22) = \begin{pmatrix} (0.22)(0.24) & (0.22)(-0.22) \\ (0.76)(0.24) & (0.76)(-0.22) \end{pmatrix}$$

$$= \begin{pmatrix} 0.05 & -0.05 \\ 0.18 & -0.17 \end{pmatrix}$$

$$\nabla_{\boldsymbol{b}_2} J = \nabla_{\boldsymbol{u}_2} J = \begin{pmatrix} 0.24 \\ -0.22 \end{pmatrix}$$

Summary

$$\nabla_{\boldsymbol{W}_2} J = \begin{pmatrix} 0.05 & -0.05 \\ 0.18 & -0.17 \end{pmatrix}$$

$$\nabla_{\boldsymbol{b}_2} J = \begin{pmatrix} 0.24 \\ -0.22 \end{pmatrix}$$

2. (f) Find the updated weight matrix $\boldsymbol{W}_2$, and bias vector $\boldsymbol{b}_2$ of the second hidden layer.

(4 marks)

Answer

$$W_2 = \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix} \qquad \alpha = 0.8$$

$$W_2 = W_2 - \alpha \nabla_{\boldsymbol{W}_2} J = \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix} - 0.8 \begin{pmatrix} 0.05 & -0.05 \\ 0.18 & -0.17 \end{pmatrix}$$

$$= \begin{pmatrix} -0.4 & -1.5 \\ -1.5 & 1.2 \end{pmatrix} - \begin{pmatrix} 0.04 & -0.04 \\ 0.14 & -0.14 \end{pmatrix} = \begin{pmatrix} -0.44 & -1.46 \\ -1.64 & 1.34 \end{pmatrix}$$

$$b_2 = b_2 - \alpha \nabla_{\boldsymbol{b}_2} J = \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix} - 0.8 \begin{pmatrix} 0.24 \\ -0.22 \end{pmatrix} = \begin{pmatrix} 0.3 \\ -0.2 \end{pmatrix} - \begin{pmatrix} 0.19 \\ -0.18 \end{pmatrix} = \begin{pmatrix} 0.11 \\ -0.02 \end{pmatrix}$$

Summary

Updated $W_2 = \begin{pmatrix} -0.44 & -1.46 \\ -1.64 & 1.34 \end{pmatrix}$

Updated $b_2 = \begin{pmatrix} 0.11 \\ -0.02 \end{pmatrix}$

3. (a) An input image $\boldsymbol{X}$ is processed by a convolution layer and thereafter by a pooling layer. The convolution layer has filters with weights $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and a bias $b = 0.2$. and consists of neurons with *sigmoid* activation function. The convolution is performed at strides = [1,1] and with 'VALID' padding. The pooling layer performs max pooling and uses a 2 x 2 pooling window at strides = [2,2] and with 'SAME' padding.

Given an input image $\boldsymbol{X} = \begin{pmatrix} 0.4 & -0.1 & 0.2 & -0.3 \\ 0.7 & 0.1 & -0.3 & 0.4 \\ -1.5 & 0.2 & 0.0 & -0.3 \end{pmatrix}$, find the feature maps at

3. (a) (i) the convolution layer and

(7 marks)

Answer

$$\boldsymbol{X} = \begin{pmatrix} 0.4 & -0.1 & 0.2 & -0.3 \\ 0.7 & 0.1 & -0.3 & 0.4 \\ -1.5 & 0.2 & 0.0 & -0.3 \end{pmatrix} \qquad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- For 'VALID' padding, padding P = 0
- stride = [1, 1]
- bias = 0.2
- filter size = 2 x 2

output size:

$$\text{row} = \frac{R - F + 2P}{S} + 1 = \frac{3 - 2 + 2(0)}{1} + 1 = 2$$

$$\text{col} = \frac{C - F + 2P}{S} + 1 = \frac{4 - 2 + 2(0)}{1} + 1 = 3$$

Synaptic inputs to the feature map with filter w: $u = Conv(\boldsymbol{X}, w) + b$

$$u(1,1) = (0.4)(0) + (-0.1)(1) + (0.7)(1) + (0.1)(0) + 0.2$$
$$= 0 - 0.1 + 0.7 + 0 + 0.2 = 0.8$$
$$u(1,2) = (-0.1)(0) + (0.2)(1) + (0.1)(1) + (-0.3)(0) + 0.2$$
$$= 0 + 0.2 + 0.1 + 0 + 0.2 = 0.5$$
$$u(1,3) = (0.2)(0) + (-0.3)(1) + (-0.3)(1) + (0.4)(0) + 0.2$$
$$= 0 - 0.3 - 0.3 + 0 + 0.2 = -0.4$$
$$u(2,1) = (0.7)(0) + (0.1)(1) + (-1.5)(1) + (0.2)(0) + 0.2$$
$$= 0 + 0.1 - 1.5 + 0 + 0.2 = -1.2$$
$$u(2,2) = (0.1)(0) + (-0.3)(1) + (0.2)(1) + (0)(0) + 0.2$$
$$= 0 - 0.3 + 0.2 + 0 + 0.2 = 0.1$$
$$u(2,3) = (-0.3)(0) + (0.4)(1) + (0)(1) + (-0.3)(0) + 0.2$$
$$= 0 + 0.4 + 0 + 0 + 0.2 = 0.6$$

3. (a) cont

Answer

$$u = \begin{pmatrix} 0.8 & 0.5 & -0.4 \\ -1.2 & 0.1 & 0.6 \end{pmatrix}$$

Feature maps at the convolution layer:

$$\sigma(u) = \mathrm{sigmoid}(u) = \frac{1}{1+e^{-u}} = \begin{pmatrix} 0.69 & 0.62 & 0.40 \\ 0.23 & 0.52 & 0.65 \end{pmatrix}$$

3. (a) (ii) the pooling layer.

(4 marks)

Answer

- stride = [2, 2]
- pooling size = 2 x 2

pooling size:

$$\mathrm{row} = \frac{R-F}{S} + 1 = \frac{2-2}{2} + 1 = 1$$

$$\mathrm{col} = \frac{C-F}{S} + 1 = \frac{3-2}{2} + 1 = 1.5 = 1$$

Feature maps at the pooling layer:

max pooling: $P_{max} = 0.69$

3. (b) A recurrent neural network (RNN) with top-down recurrence receives 2- dimensional inputs and produces 2-dimensional hidden layer activations and 1-dimensional outputs. The hidden layer neurons have *tanh* activation functions and the output layer neurons have *sigmoid* activation functions.

The weight matrix $\boldsymbol{U}$ from the input layer to the hidden layer, the weight matrix $\boldsymbol{V}$ to the output layer, and the top-down recurrence weight matrix $\boldsymbol{W}$ are given by

$$\boldsymbol{U} = \begin{pmatrix} -1.0 & 0.5 \\ 0.5 & 0.3 \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} 2.0 \\ -1.5 \end{pmatrix} \text{ and } \boldsymbol{W} = (-2.0 \quad 1.5)$$

The hidden layer bias vector $\boldsymbol{b}$ and the output layer bias c are given by

$$\boldsymbol{b} = \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix} \text{ and } c = 0.4$$

The output layer is initialized to an output of 1.0.

Determine the output sequence of the RNN for an input sequence of $(\boldsymbol{x}(1), \boldsymbol{x}(2), \boldsymbol{x}(3))$ when

$$\boldsymbol{x}(1) = \begin{pmatrix} -1.0 \\ 2.0 \end{pmatrix}, \boldsymbol{x}(2) = \begin{pmatrix} 1.0 \\ -1.0 \end{pmatrix} \text{ and } \boldsymbol{x}(3) = \begin{pmatrix} 0.0 \\ 3.0 \end{pmatrix}$$

(14 marks)

Answer

$$\boldsymbol{U} = \begin{pmatrix} -1.0 & 0.5 \\ 0.5 & 0.3 \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} 2.0 \\ -1.5 \end{pmatrix} \text{ and } \boldsymbol{W} = (-2.0 \quad 1.5)$$

$$\boldsymbol{b} = \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix} \text{ and } c = 0.4$$

$$y = 1.0$$

$$\boldsymbol{x}(1) = \begin{pmatrix} -1.0 \\ 2.0 \end{pmatrix}, \boldsymbol{x}(2) = \begin{pmatrix} 2.0 \\ -1.5 \end{pmatrix} \text{ and } \boldsymbol{x}(3) = \begin{pmatrix} 0.0 \\ 3.0 \end{pmatrix}$$

$$\phi(u) = \tanh(u) \quad = \frac{e^u - e^{-u}}{e^u + e^{-u}}$$

$$\sigma(u) = \text{sigmoid}(u) = \frac{1}{1 + e^{-u}}$$

3. (b) cont

Answer

$$\phi(u) = \tanh(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}$$

$$\sigma(u) = \text{sigmoid}(u) = \frac{1}{1 + e^{-u}}$$

At $t = 1$, $\boldsymbol{x}(1) = \begin{pmatrix} -1.0 \\ 2.0 \end{pmatrix}$,

$$\boldsymbol{h}(t) = \phi(\boldsymbol{U}^T\boldsymbol{x}(t) + \boldsymbol{W}^T\boldsymbol{y}(t-1) + \boldsymbol{b})$$

$$\boldsymbol{h}(1) = tanh(\boldsymbol{U}^T\boldsymbol{x}(1) + \boldsymbol{W}^T\boldsymbol{y}(0) + \boldsymbol{b})$$

$$= \tanh\left(\begin{pmatrix} -1.0 & 0.5 \\ 0.5 & 0.3 \end{pmatrix}\begin{pmatrix} -1.0 \\ 2.0 \end{pmatrix} + \begin{pmatrix} -2.0 \\ 1.5 \end{pmatrix} 1.0 + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right)$$

$$= \tanh\left(\begin{pmatrix} (-1.0)(-1) + (0.5)(2) \\ (\ 0.5)(-1) + (0.3)(2) \end{pmatrix} + \begin{pmatrix} (-2.0)(1) \\ (\ 1.5)(1) \end{pmatrix} + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right)$$

$$= \tanh\left(\begin{pmatrix} 0 \\ 0.1 \end{pmatrix} + \begin{pmatrix} -2.0 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right) = \tanh\begin{pmatrix} 2.0 \\ 1.9 \end{pmatrix} = \begin{pmatrix} 0.964 \\ 0.956 \end{pmatrix}$$

$$\boldsymbol{y}(t) = \sigma(\boldsymbol{V}^T\boldsymbol{h}(t) + c)$$

$$\boldsymbol{y}(1) = sigmoid(\boldsymbol{V}^T\boldsymbol{h}(1) + c)$$

$$= \text{sigmoid}\left((2.0 \quad -1.5)\begin{pmatrix} 0.964 \\ 0.956 \end{pmatrix} + 0.4\right)$$

$$= \text{sigmoid}((2)(0.964) + (-1.5)(0.956) + 0.4)$$

$$= \text{sigmoid}(0.894) = 0.7097$$

At $t = 2$, $\boldsymbol{x}(2) = \begin{pmatrix} 1.0 \\ -1.0 \end{pmatrix}$,

$$\boldsymbol{h}(t) = \phi(\boldsymbol{U}^T\boldsymbol{x}(t) + \boldsymbol{W}^T\boldsymbol{y}(t-1) + \boldsymbol{b})$$

$$\boldsymbol{h}(2) = tanh(\boldsymbol{U}^T\boldsymbol{x}(2) + \boldsymbol{W}^T\boldsymbol{y}(1) + \boldsymbol{b})$$

$$= \tanh\left(\begin{pmatrix} -1.0 & 0.5 \\ 0.5 & 0.3 \end{pmatrix}\begin{pmatrix} 1.0 \\ -1.0 \end{pmatrix} + \begin{pmatrix} -2.0 \\ 1.5 \end{pmatrix} 0.7097 + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right)$$

$$= \tanh\left(\begin{pmatrix} (-1.0)(1) + (0.5)(-1) \\ (\ 0.5)(1) + (0.3)(-1) \end{pmatrix} + \begin{pmatrix} (-2.0)(0.7097) \\ (\ 1.5)(0.7097) \end{pmatrix} + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right)$$

$$= \tanh\left(\begin{pmatrix} -1.5 \\ 0.2 \end{pmatrix} + \begin{pmatrix} -1.419 \\ 1.065 \end{pmatrix} + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right) = \tanh\begin{pmatrix} -0.919 \\ 1.565 \end{pmatrix} = \begin{pmatrix} -0.725 \\ 0.916 \end{pmatrix}$$

$$\boldsymbol{y}(t) = \sigma(\boldsymbol{V}^T\boldsymbol{h}(t) + c)$$

$$\boldsymbol{y}(2) = sigmoid(\boldsymbol{V}^T\boldsymbol{h}(2) + c)$$

$$= \text{sigmoid}\left((2.0 \quad -1.5)\begin{pmatrix} -0.725 \\ 0.916 \end{pmatrix} + 0.4\right)$$

$$= \text{sigmoid}((2)(-0.725) + (-1.5)(0.916) + 0.4)$$

$$= \text{sigmoid}(-2.424) = 0.081$$

3. (b) cont

Answer

$$\phi(u) = \tanh(u) \quad = \frac{e^u - e^{-u}}{e^u + e^{-u}}$$

$$\sigma(u) = \text{sigmoid}(u) = \frac{1}{1 + e^{-u}}$$

At $t = 3$, $\boldsymbol{x}(3) = \begin{pmatrix} 0.0 \\ 3.0 \end{pmatrix}$,

$$\boldsymbol{h}(t) = \quad \phi(\boldsymbol{U}^T \boldsymbol{x}(t) + \boldsymbol{W}^T \boldsymbol{y}(t-1) + \boldsymbol{b})$$

$$\boldsymbol{h}(3) = tanh(\boldsymbol{U}^T \boldsymbol{x}(2) + \boldsymbol{W}^T \boldsymbol{y}(2) + \boldsymbol{b})$$

$$= \tanh\left(\begin{pmatrix} -1.0 & 0.5 \\ 0.5 & 0.3 \end{pmatrix}\begin{pmatrix} 0.0 \\ 3.0 \end{pmatrix} + \begin{pmatrix} -2.0 \\ 1.5 \end{pmatrix} 0.081 + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right)$$

$$= \tanh\left(\begin{pmatrix} (-1.0)(0) + (0.5)(3) \\ (\ \ 0.5)(0) + (0.3)(3) \end{pmatrix} + \begin{pmatrix} (-2.0)(0.081) \\ (\ \ 1.5)(0.081) \end{pmatrix} + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right)$$

$$= \tanh\left(\begin{pmatrix} 1.5 \\ 0.9 \end{pmatrix} + \begin{pmatrix} -0.162 \\ 0.122 \end{pmatrix} + \begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}\right) = \tanh\begin{pmatrix} 3.338 \\ 1.322 \end{pmatrix} = \begin{pmatrix} 0.997 \\ 0.867 \end{pmatrix}$$

$$\boldsymbol{y}(t) = \qquad \sigma(\boldsymbol{V}^T \boldsymbol{h}(t) + c)$$

$$\boldsymbol{y}(3) = sigmoid(\boldsymbol{V}^T \boldsymbol{h}(3) + c)$$

$$= \text{sigmoid}\left((2.0 \quad -1.5)\begin{pmatrix} 0.997 \\ 0.867 \end{pmatrix} + 0.4\right)$$

$$= \text{sigmoid}((2)(0.997) + (-1.5)(0.867) + 0.4)$$

$$= \text{sigmoid}(1.093) = 0.749$$

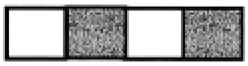
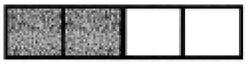
<u>Summary of output</u>

- $\boldsymbol{y}(1) = 0.7097$
- $\boldsymbol{y}(2) = 0.081$
- $\boldsymbol{y}(3) = 0.749$

4. (a) An autoencoder has four neurons at the input layer and two neurons at the hidden layer. All the neurons have *sigmoid* activation functions. The weight matrix $\boldsymbol{W}$ of the hidden layer, the bias vector $\boldsymbol{b}$ of the hidden layer and the bias vector $\boldsymbol{c}$ of the output layer are given by

$$W = \begin{pmatrix} 0.8 & 0.4 \\ -0.4 & -0.8 \\ 0.2 & 0.4 \\ -0.7 & 0.2 \end{pmatrix}, \; b = \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix}, \text{ and } \; c = \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix}$$

Consider the following two input patterns applied to the autoencoder:

$\boldsymbol{s}_1$  $\boldsymbol{s}_2$ 

4. (a) (i) Convert each input pattern to their respective vector representations by using the following notation: shaded box = 0 and white box = 1.

(2 marks)

Answer

$$\boldsymbol{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

4. (a) (ii) Find the hidden layer activations and the outputs of the autoencoder.

(7 marks)

Answer

$$W = \begin{pmatrix} 0.8 & 0.4 \\ -0.4 & -0.8 \\ 0.2 & 0.4 \\ -0.7 & 0.2 \end{pmatrix},\ b = \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix},\text{ and } c = \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix} \qquad \boldsymbol{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \boldsymbol{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

For input $\boldsymbol{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$,

$$u_1 = W^T s_1 + b = \begin{pmatrix} 0.8 & -0.4 & 0.2 & -0.7 \\ 0.4 & -0.8 & 0.4 & 0.2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} (0.8)(1) + (-0.4)(0) + (0.2)(1) + (-0.7)(0) \\ (0.4)(1) + (-0.8)(0) + (0.4)(1) + (\ 0.2)(0) \end{pmatrix} + \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 1.0 \\ 0.8 \end{pmatrix} + \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}$$

$$h_1 = \phi(u_1) = \text{sigmoid}(u) = \frac{1}{1 + e^{-u}} = \begin{pmatrix} 0.73 \\ 0.73 \end{pmatrix}$$

$$u_2 = W h_1 + c = \begin{pmatrix} 0.8 & 0.4 \\ -0.4 & -0.8 \\ 0.2 & 0.4 \\ -0.7 & 0.2 \end{pmatrix} \begin{pmatrix} 0.73 \\ 0.73 \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} (\ 0.8)(0.73) + (\ 0.4)(0.73) \\ (-0.4)(0.73) + (-0.8)(0.73) \\ (\ 0.2)(0.73) + (\ 0.4)(0.73) \\ (-0.7)(0.73) + (\ 0.2)(0.73) \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.876 \\ -0.876 \\ 0.438 \\ -0.365 \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.876 \\ -1.476 \\ 1.238 \\ -0.265 \end{pmatrix}$$

$$y_1 = \phi(u_2) = \text{sigmoid}(u_2) = \frac{1}{1 + e^{-u}} = \begin{pmatrix} 0.706 \\ 0.186 \\ 0.775 \\ 0.434 \end{pmatrix}$$

4. (a) (ii) cont

Answer

For input $\boldsymbol{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$,

$$u_2 = W^T s_2 + b = \begin{pmatrix} 0.8 & -0.4 & 0.2 & -0.7 \\ 0.4 & -0.8 & 0.4 & 0.2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} (0.8)(0) + (-0.4)(0) + (0.2)(1) + (-0.7)(1) \\ (0.4)(0) + (-0.8)(0) + (0.4)(1) + (\ \ 0.2)(1) \end{pmatrix} + \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} -0.5 \\ 0.6 \end{pmatrix} + \begin{pmatrix} 0.0 \\ 0.2 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.8 \end{pmatrix}$$

$$h_2 = \phi(u_2) = \text{sigmoid}(u_2) = \frac{1}{1 + e^{-u}} = \begin{pmatrix} 0.38 \\ 0.69 \end{pmatrix}$$

$$u_2 = W h_2 + c = \begin{pmatrix} 0.8 & 0.4 \\ -0.4 & -0.8 \\ 0.2 & 0.4 \\ -0.7 & 0.2 \end{pmatrix} \begin{pmatrix} 0.38 \\ 0.69 \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} (\ \ 0.8)(0.38) + (\ \ 0.4)(0.69) \\ (-0.4)(0.38) + (-0.8)(0.69) \\ (\ \ 0.2)(0.38) + (\ \ 0.4)(0.69) \\ (-0.7)(0.38) + (\ \ 0.2)(0.69) \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.58 \\ -0.70 \\ 0.35 \\ -0.13 \end{pmatrix} + \begin{pmatrix} 0.0 \\ -0.6 \\ 0.8 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.58 \\ -1.30 \\ 1.15 \\ -0.03 \end{pmatrix}$$

$$y_2 = \phi(u_2) = \text{sigmoid}(u_2) = \frac{1}{1 + e^{-u}} = \begin{pmatrix} 0.641 \\ 0.214 \\ 0.760 \\ 0.493 \end{pmatrix}$$

<u>Summary</u>

- $h_1 = \begin{pmatrix} 0.73 \\ 0.73 \end{pmatrix}$
- $y_1 = \begin{pmatrix} 0.706 \\ 0.186 \\ 0.775 \\ 0.434 \end{pmatrix}$
- $h_2 = \begin{pmatrix} 0.38 \\ 0.69 \end{pmatrix}$
- $y_2 = \begin{pmatrix} 0.641 \\ 0.214 \\ 0.760 \\ 0.493 \end{pmatrix}$

4. (a) (iii) Find the entropy at the output layer.

(4 marks)

Answer

$$\boldsymbol{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{y}_1 = \begin{pmatrix} 0.706 \\ 0.186 \\ 0.775 \\ 0.434 \end{pmatrix}, \qquad \boldsymbol{y}_2 = \begin{pmatrix} 0.641 \\ 0.214 \\ 0.760 \\ 0.493 \end{pmatrix}$$

For input $\boldsymbol{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$,

$$J_{cross-entropy} = -\sum_{p=1}^{P} \left(\boldsymbol{s}_p \log \boldsymbol{y}_p + \left(1 - \boldsymbol{s}_p\right) \log\left(1 - \boldsymbol{y}_p\right) \right)$$

$$L_1 = -[(1)\log(0.706) + (1-1)\log(1-0.706)] = 0.348$$

$$L_2 = -[(0)\log(0.186) + (1-0)\log(1-0.186)] = 0.205$$

$$L_3 = -[(1)\log(0.775) + (1-1)\log(1-0.775)] = 0.254$$

$$L_4 = -[(0)\log(0.434) + (1-0)\log(1-0.434)] = 0.569$$

$$J_{cross-entropy} = 0.348 + 0.205 + 0.254 + 0.569 = 1.376$$

For input $\boldsymbol{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$,

$$J_{cross-entropy} = -\sum_{p=1}^{P} \left(\boldsymbol{s}_p \log \boldsymbol{y}_p + \left(1 - \boldsymbol{s}_p\right) \log\left(1 - \boldsymbol{y}_p\right) \right)$$

$$L_1 = -[(0)\log(0.641) + (1-0)\log(1-0.641)] = 1.024$$

$$L_2 = -[(0)\log(0.214) + (1-0)\log(1-0.214)] = 0.241$$

$$L_3 = -[(1)\log(0.760) + (1-1)\log(1-0.760)] = 0.274$$

$$L_4 = -[(1)\log(0.493) + (1-1)\log(1-0.493)] = 0.707$$

$$J_{cross-entropy} = 1.024 + 0.241 + 0.274 + 0.707 = 2.246$$

<u>Summary</u>

$$J_{cross-entropy}(\boldsymbol{s}_1) = 1.376$$

$$J_{cross-entropy}(\boldsymbol{s}_2) = 2.246$$

4. (a) (iv) Find the Kullback-Leibler (KL) divergence of hidden layer activations with respect to a constant neuron activation ρ = 0.1.

(4 marks)

Answer

$$h_1 = \begin{pmatrix} 0.73 \\ 0.73 \end{pmatrix} \qquad h_2 = \begin{pmatrix} 0.38 \\ 0.69 \end{pmatrix}$$

$$\rho_j = \frac{1}{P}\sum_{p=1}^{P} h_{pj} = \frac{1}{P}\sum_{p=1}^{P} f\left(\boldsymbol{x}_p^T \boldsymbol{w}_j + b_j\right)$$

$$\rho_1 = \frac{1}{2}(h_{11} + h_{21}) = \frac{1}{2}(0.73 + 0.38) = \frac{1}{2}(1.11) = 0.555$$

$$\rho_2 = \frac{1}{2}(h_{12} + h_{22}) = \frac{1}{2}(0.73 + 0.69) = \frac{1}{2}(1.42) = 0.71$$

$$D(\boldsymbol{h}) = \sum_{j=1}^{M} \rho \log\frac{\rho}{\rho_j} + (1-\rho)\log\frac{1-\rho}{1-\rho_j}$$

$$D(h_1) = \left[0.1\log\frac{0.1}{0.555} + (1-0.1)\log\frac{1-0.1}{1-0.555}\right] = 0.463$$

$$D(h_2) = \left[0.1\log\frac{0.1}{0.71} + (1-0.1)\log\frac{1-0.1}{1-0.71}\right] = 0.823$$

$$D(\boldsymbol{h}) = 0.463 + 0.823 = 1.286$$

4. (b) Describe how the adversarial process between the generator and discriminator networks is implemented in the training of Generative Adversarial Networks (GAN).

(8 marks)

Answer

- In Generative Adversarial Networks (GANs), the adversarial process is implemented as a two-player minimax game between two networks.

- The two players are:
  1. Discriminator
  2. Generator

- Discriminator:
  - The discriminator's role is to distinguish between real data and synthetic data generated by the generator.

  - Objective is to correctly identify and classify the real data and the generated data.

- Generator:
  - The generator's role is to create synthetic data that resembles the real data.

  - Objective is to fool the Discriminator into classifying its fake outputs as real.

- During training:
  - The generator updates its parameters to improve the quality of its fake data, trying to make it harder for the discriminator to tell the difference.

  - The discriminator updates its parameters to become better at distinguishing real data from fake data generated by the generator.