Numerical simulations of particles inside a Penning Trap

Fuad Dadvar — Duncan Wilkins — Erlin Nupen — Don.P Bullecer
Department of Physics, University of Oslo, Norway
Github address: Link here

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Abstract

The goal of this project was to simulate particles moving inside a penning trap. First only a single particle was simulated inside the penning trap using both Euler-Cromer and Rung-Kutte 4. These simulations was compared to the analytical solution over a time period over 100 μs. After simulating one particle, we simulated two particles with the Coulomb-interaction between the two particles. The simulations were somewhat expected, but there were several deviations in the simulations which was thoroughly discussed in the discussions section.

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1 Introduction

To capture particles and study their properties under severe conditions can be revolutionizing. It's a popular topic and many laboratories worldwide including CERN use these penning traps to store antimatter such as antiprotons.

In this article we are going to study the effects of this device exactly, a penning trap. A penning trap is a device that stores particles within a confinement. We will introduce laws of electrodynamic and classical mechanics for this case of issue. To fully trap these particles we need to utilize a static configuration of electric and magnetic fields to confine them. Once these particles are confined we can do various of measurements and analysis.

2 Theory

2.1 Classical Mechanics

Isac Newton formulated three laws of motion, for this article we will have a look at the second law of motion. Which is central in classical mechanics. It is given by:

$$m\ddot{\mathbf{r}} = \sum_i \mathbf{F}_i, \tag{1}$$

where m is given as the mass of the particle, $\ddot{\mathbf{r}} \equiv \frac{d^2\mathbf{r}}{dt^2}$ and $\sum_i \mathbf{F}_i$ is all forces acting on the particle.

2.2 The Electrodynamics of a Penning Trap

2.2.1 Electric field

An electric field surrounds charged particles, and depending if the charge is positive or negative will either repel or attract other charged particles. The electrostatic field can be expressed as the gradient of the electrostatic potential.

$$\mathbf{E}(\mathbf{r}) = -\nabla \mathbf{V}(\mathbf{r}), \tag{2}$$

Where ∇ is the differential operator with respects to all the spatial coordinates, and $\mathbf{V}$ is the electric potential at a point $\mathbf{r}$. In our case the electric field can be expressed as the gradient of the electric potential. Since our magnetic field is constant in time, and from Faradays law of induction the curl has to be zero:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0. \tag{3}$$

With a magnetic field **B** (5), constant in time, the electric field is conservative. And we can express the the electric field as the gradient of **V** as in eq (2)

In our Penning trap, gravitational forces are small in comparison to the Lorentz-force, (6) and is therefore negligible. Quantum effects are also negligible, since we are looking at a classical Penning Trap. Continuing, we will consider our pennings trap to be ideal, where the electric potential is defined as:

$$V(x, y, z) = \frac{V_0}{2d^2}(2z^2 - x^2 - y^2), \tag{4}$$

Where V_0 is the potential applied to the electrodes, d is the length between our electrodes and is called the characteristic dimension, and is defined as $d = \sqrt{z_0^2 + r_0^2/2}$. Here z_0 is the distance from the center of one of the end caps, and r_0 is the distance from the center of the ring.

2.2.2 Magnetic field

If we were to send in a particle into our penning trap with only our electric field turned on, the particle could escape in the radial direction, or xy-plane. To prohibit this, an homogeneous magnetic field is imposed in the z-direction of the apparatus:

$$\mathbf{B} = B_0 \hat{e}_z = (0, 0, B_0), \tag{5}$$

Where B_0 is the field strength. If the field is strong enough, it will counter act the radial outwards force acting on the electron, and instead force it into an orbital motion around the center of our penning trap.

2.2.3 Lorentz Force

The relation between an electric field **E** and a magnetic field **B** acting on a particle **q** is given by the Lorentz force which is given as:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}, \tag{6}$$

where **v** is the velocity of the particle. When the particle is inside the penning trap, then the Lorentz Force is the only force acting on the particle. We can see that if the velocity **v**, is parallel with the magnetic field **B**, then the cross product between the velocity of the particle and the magnetic field, is zero. The force acting on the particle is then equal to:

$$\mathbf{F} = q\mathbf{E}, \tag{7}$$

2.2.4 Coloumb's Law

Charles de Coulomb found that a force between two objects as a function of their charges q_1 and q_2 and the distance $\mathbf{r}$, between the objects is given by Coulomb's law:

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}_{12}}{|\mathbf{r}_{12}|^2}, \tag{8}$$

where $\mathbf{r}_{12}$ denotes the vector distance between two arbitrary charges at $\mathbf{r_1}$ and $\mathbf{r_2}$ and $\hat{\mathbf{r}}_{12}$ is unit vector in the direction of the force. If the charges q_1 and q_2 have opposite charge, the negative sign indicates the direction of the force is opposite that of the direction of the unit vector. ϵ_0 is the electric constant.

2.3 General motion

2.3.1 Motion of a particle inside a Penning Trap

Now that we have established the electro-dynamic laws governing our penning trap, we can derive the equations of motion for our particle. Writing out Newtons' second law of motion for a particle which is only affected by the Lorentz Force:

$$m\ddot{\mathbf{r}} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}, \tag{9}$$

Which gives the following equations of motion for our particle:

$$\ddot{x} - \omega_0 \dot{y} - \frac{1}{2}\omega_z^2 x = 0, \tag{10}$$

$$\ddot{y} + \omega_0 \dot{x} - \frac{1}{2}\omega_z^2 y = 0, \tag{11}$$

$$\ddot{z} + \omega_z^2 z = 0. \tag{12}$$

See Appendix [A.2] for a more detailed explanation and calculations. Coupled differential equations are difficult to work with, if not impossible. Introducing a complex function, $f(t) = x(t) + iy(t)$, the differential equations for x and y can be decoupled into a single differential equation of one variable:

$$\ddot{f} + i\omega_0 \dot{f} - \frac{1}{2}\omega_z^2 f = 0, \tag{13}$$

where $\omega_0 = \frac{qB_0}{m}$ and $\omega_z^2 =$ A.3.

2.3.2 Position of the Particle

With the equations of motion decoupled, we can begin to map how the particle moves around inside the Penning Trap. The goal of the Penning Trap is to confine the particle inside the apparatus. With such a confinement comes certain restrictions on our equations of motion. The general solution to our decoupled equation of motion $f(t)$, is given as:

$$f(t) = A_+ e^{-i\omega_+ t} + A_- e^{-i\omega_- t}, \tag{14}$$

With the negative and positive frequency solution given as:

$$\omega_\pm = \frac{\omega_0 \pm \sqrt{\omega_0^2 - 2, \omega_z^2}}{2}. \tag{15}$$

The physical coordinates of the particle in the xy-plane is then given as $\Re f(t)$ and $\Im f(t)$. Where $x(t)$ and $y(t)$ is equal to:

$$x(t) = \Re f(t) = A_+ \cos(\omega_+ t) + A_- \cos(\omega_- t), \tag{16}$$

$$y(t) = \Im f(t) = -A_+ \sin(\omega_+ t) - A_- \sin(\omega_- t), \tag{17}$$

Where $A_\pm = \pm \frac{v_0 + \omega_\mp x_0}{\omega_- - \omega_+}$. Along with the position of the particle in z-direction, which is given by the differential equation **??**, we can construct the full position vector to the particle:

$$\mathbf{R} = [\Re f(t), \Im f(t), z] \tag{18}$$

Where z is given by:

$$z(t) = z_0 \cos(\sqrt{\omega_0} t) \tag{19}$$

If we were to let the Penning Trap stay on for a long time, equivalent to $t \to \infty$ then our solution would blow up, ie the electron would escape our Penning trap and would be lost forever. Therefor we need to apply certain restrictions on ω_0 and ω_z to make sure the exponentials in our solution don't blow up. Such a solution is fittingly called a bounded solution of the particle, and from our solution it is clear that our exponentials blow up if the positive or negative frequancy solutions are complex. This constraint on our system can be written as:

$$\omega_0^2 - 2\omega_z^2 > 0 \tag{20}$$

$$\omega_0 > \sqrt{2}\omega_z \tag{21}$$

With this condition fulfilled, the particle will never be able to escape in the xy-plane no matter how long we let the Penning Trap stay on. This restriction can also be expressed in terms of the different Penning Trap parameters:

$$\left(\frac{qB_0}{m}\right)^2 > 2\left(\frac{2qV_0}{md^2}\right) \tag{22}$$

$$\frac{d^2B_0^2}{V_0} > 4\frac{m}{q} \tag{23}$$

With these boundaries set in place, the upper and lower bound on the particle's distance from the origin in the xy-plane is given by:

$$R_+ = A_+ + A_- \tag{24}$$

$$R_- = |A_+ - A_-| \tag{25}$$

Where R_+ is the upper bound, and R_- is the lower bound. See appendixC for derivations.

2.4 Euler's Method

Up until now we have been able to solve a small bunch of differential equations analytically, in reality the vast majority of differential equations can not be solved. This leads us to a numerical method which allows us to make approximated solutions of differential equations, implementing an old and easy method introduced by Euler. We are then able to solve them efficiently with a small margin of error. This is called Euler's method.

The Euler method can be applied to a general first order differential equation which is given by:

$$\frac{dy}{dt} = f(t, y), \tag{26}$$

here f is a given function and y is either a scalar or vector. In general the Euler method produces an approximate solution $y_n + 1$ for a given t_n at a point y_n, using the following equation:

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n), \tag{27}$$

defining:

$$f_n = f(t_n, y_n), \tag{28}$$

and:

$$(t_{n+1} - t_n) = h, \tag{29}$$

We are able to get a simplified equation given by:

$$y_{n+1} = y_n + f_n h, \tag{30}$$

Euler method can simply be used by starting of with a uniform step-size.

2.5 Euler-Cromer method

In mathematics we have that the Euler-Cromer method is a modification of the Euler method solving Hamilton's equations, a system of ordinary differential equations that arises in classical mechanics. And for our issue of matter that is the correct method.

The Euler-Cromer method can be applied to a pair of differential equations of form:

$$\frac{dx}{dt} = f(t, v), \tag{31}$$

$$\frac{dv}{dt} = g(t, x) \tag{32}$$

where f and g are given functions. Here, x and v may either be scalars or vectors. We have that the Euler-cromer method produces an approximate discrete solution by itering

$$v_{n+1} = v_n + g(t_v, x_n)\Delta t, \tag{33}$$

$$x_{n+1} = x_n + f(t_n, v_{n+1})\Delta t, \tag{34}$$

where Δt is the time step and $t_n = t_0 + n\Delta t$ is the time after n steps.

The difference with standard Euler method is that the Euler-cromer method uses v_{n+1} in the equation for x_{n+1}, while the Euler method uses v_n.

By applying the method with a negative time step to the computation of (x_n, v_n) from (x_{n+1}, v_{n+1}), and rearranging leads tot the second variant of the Euler-cromer method

2.6 Predictor-Corrector method

One of the many modifications to Euler's method designed to integrate ordinary differential equation is the Predictor-Corrector method.

While the Euler method uses a gradient of a single point (f_i) to predict the next point, the Predictor-Corrector is an improvement as it uses an average gradient between the two points t_i and t_{i+1}.

Although f_{i+1} is unknown, we can predict it using a simple forward Euler step. This algorithm has 2 steps, namely the "prediction" step and the "corrector" step.

1. Prediction step: extrapolates the state of the system or a function's value at a subsequent point.

2. Corrector step: polishes the initial approximation by including the current value of the function to interpolate the unknown value of the function at the same subsequent point.

One advantage of using this method is that it produces a local error of $O(h^3)$ and a global error of $O(h^2)$ which is one order better than Forward Euler, but with the drawback that we are required to do one extra evaluation of f.

Through the general proof for this method which is shown in Appendix x, we get this equation for Predictor-Corrector method,

$$y(t+h) = y(t) + h \cdot \frac{f(t+h) + f(t)}{2} + O(h^3)$$

where, in our case, $f_i = f(t_i, y_i)$, and thus it's discretized form is written as

$$y_{i+1} = y_i + h \cdot \frac{f_{i+1}^* + f_i}{2}$$

However, this method is usually written in an alternate notation where it's essentially the same but expressed in terms of the gradients k_1 and k_2.

Alternate notation:

1.Prediction:

$$k_1 = h \cdot f_i = h \cdot f(t_i, y_i)$$
$$k_2 = h \cdot f_{i+1}^* = h \cdot f(t_i, y_{i+1}^*)$$

2. Correction:

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$$

2.7 Runge-Kutta method

This then ties with the main method we will use in the simulation for the Penning Trap, the Runge-Kutta method of 4th degree (RK4).

Essentially a more "sophisticated" version of the Predictor-Corrector method, a Runge-Kutta method of m-th order uses m estimates of the gradient on intervals $t_i \leq t \leq t_{i+1}$ to determine y_{i+1}.

And based on the calculations and proof for the general Runge-Kutta of m-th order explained in Appendix x, this method gives a local error of degree $m + 1$, and a global error of order m.

In RK4, $m = 4$, thus requiring 4 evaluations of f. This means a local error of degree 5 and global error of 4. Thus it

Additionally, the RK4 equation for 1 step is given by

$$y_{n+1} = y_n + \frac{1}{6}h \cdot [k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4] + O(h^5)$$
$$t_{n+1} = t_n + h$$

for n = 0, 1, 2, ..., n, using the gradients

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$
$$k_3 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$$
$$k_4 = f(t_n + h, y_n + k_3)$$

such that

- k_1 is the slope at the start of the interval, using y (predicted via Euler's method).
- k_2 the slope at the midpoint of the interval between y and k_1.
- k_3 is the same as k_2 but is instead between y and k_2.
- k_4 is the slope at the end of the interval, using y and k_3.

The descretized equations thus become Discretized Gradients:

$$k_1 = h \cdot f(t_i, y_i)$$
$$k_2 = hf(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1)$$
$$k_3 = hf(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2)$$
$$k_4 = hf(t_i + h, y_i + k_3)$$

RK4 Discretized Equation for 1 step:

$$y_{i+1} = y_i + \frac{1}{6}[k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4]$$

algorithm

3 Method

3.1 Implementation of Euler-Cromer method

[H] **for** $i = 0, 1, ...n$ **do**

for $j = 0, 1, ...P_{tot}$ **do**

Calculate *Accelration*.

Calculate v_{i+1}. $\longleftarrow$ $\mathrm{v}_i + h \cdot Acceleration$

Calculate r_{i+1}. $\longleftarrow \mathrm{r}_i + h \cdot v_{i+1}$

3.2 Implementation of Runge Kutta method

[H] **for** $i = 0, 1, ...n$ **do**

for $j = 0, 1, ...P_{tot}$ **do**

$f(t_i, y_i) \longleftarrow$ Save original conditions of particles.

Calculate Acceleration.

Calculate k_n. $\longleftarrow$ for n = 1,2,3,4

Calculate y_{i+1}. $\longleftarrow \mathrm{y}_i + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

Save y_{i+1} for p_j in t_{i+1}.

4 Results

4.1 2D movement

4.1.1 A single particle

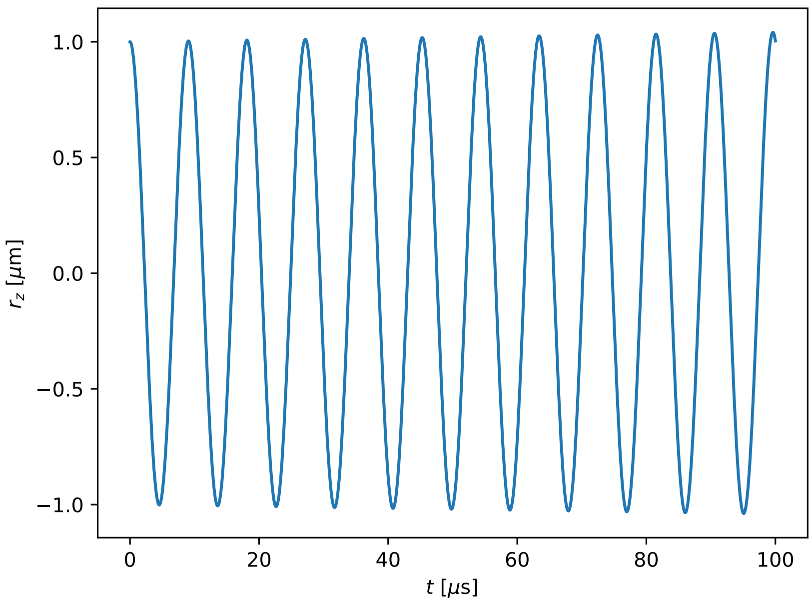

Figure 1: Time vs. Position of the particle in the z-direction

4.1.2 Two particles

4.1.2.1 With coulomb interaction

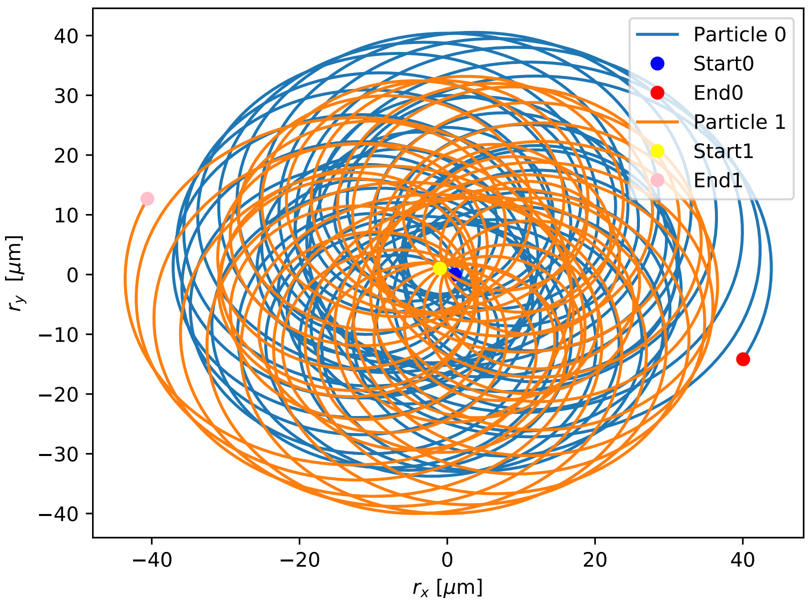

Figure 2: XY-plan of two particles with Coloumb interaction

4.1.2.2 Without coulomb interaction

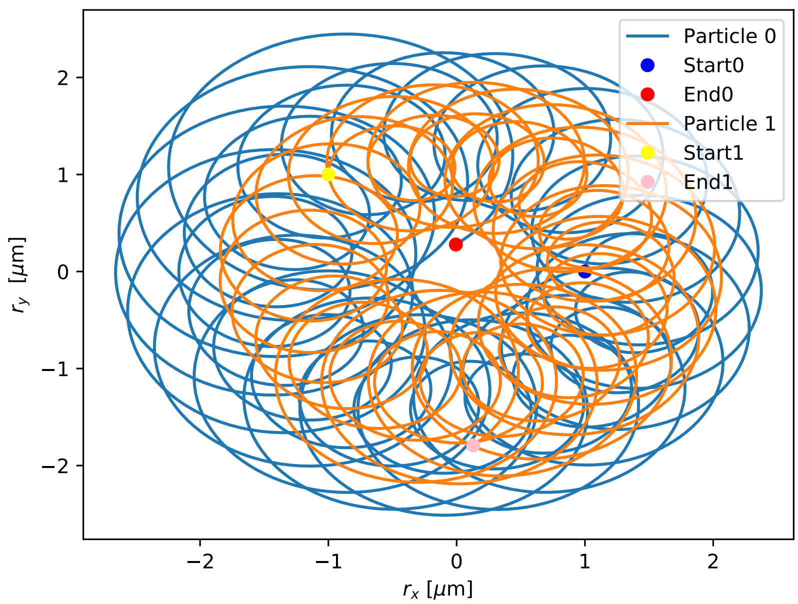

Figure 3: XY-plan of two particles without Coloumb interaction

4.1.3 Two particles in Phase Space

4.1.3.1 Without Coulomb's interaction

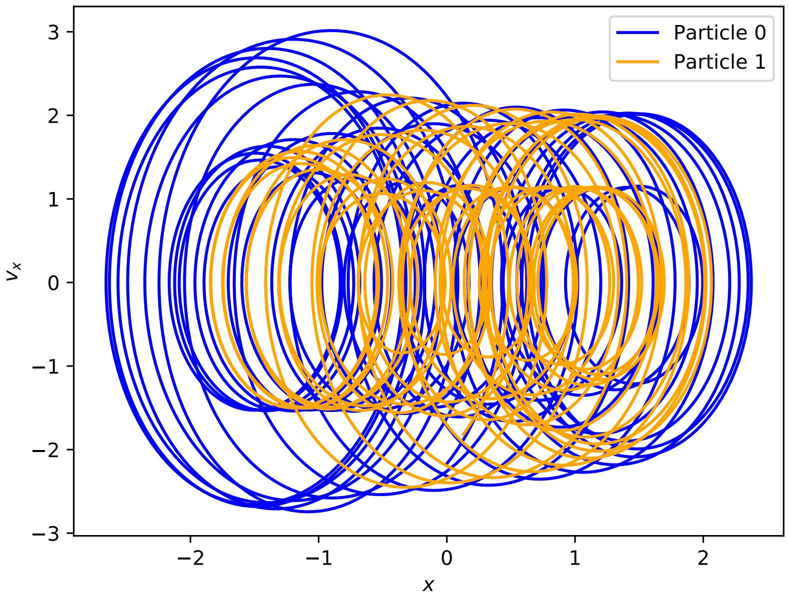

Figure 4: Phase space x plane

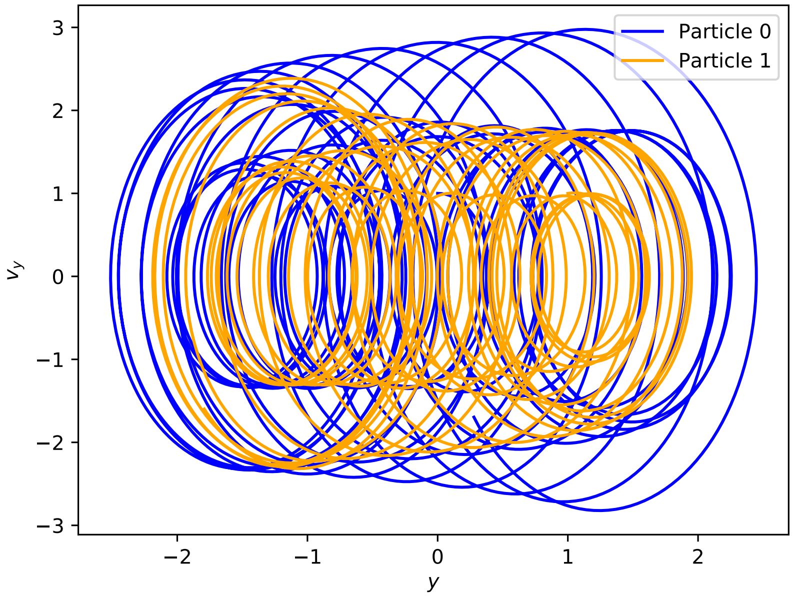

Figure 5: Phase space y plane

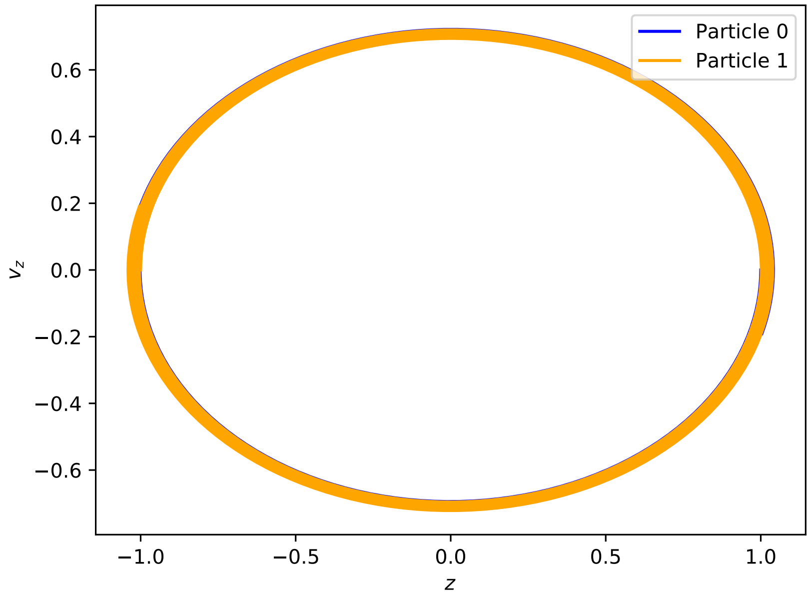

Figure 6: Phase space z plane

4.1.3.2 With coulomb's interaction

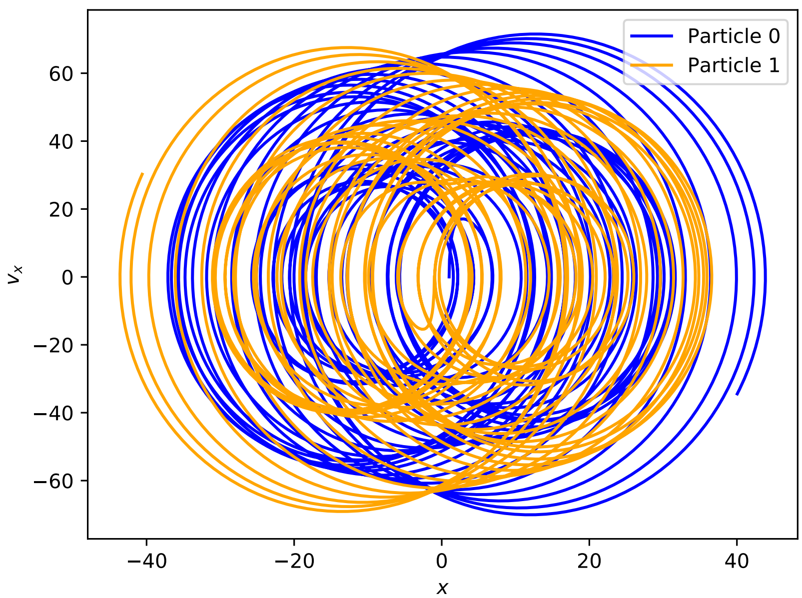

Figure 7: Phase space x plane

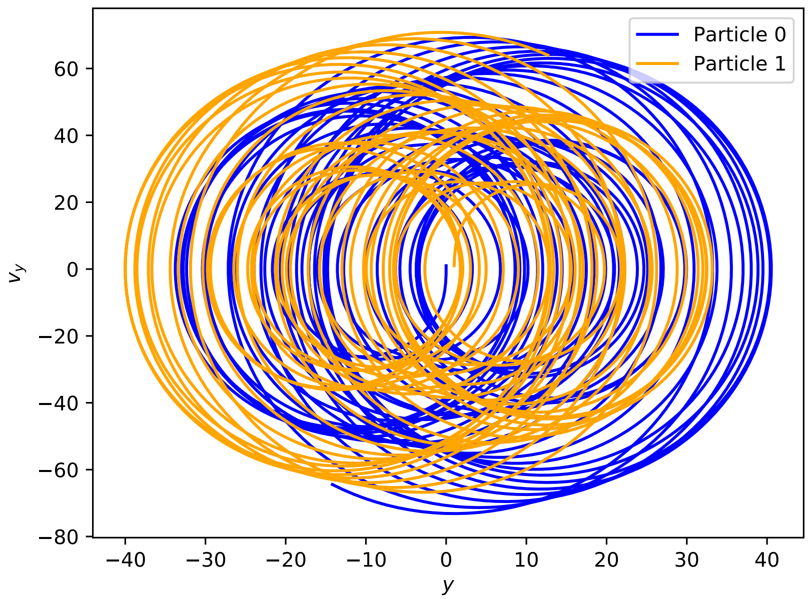

Figure 8: Phase space y plane

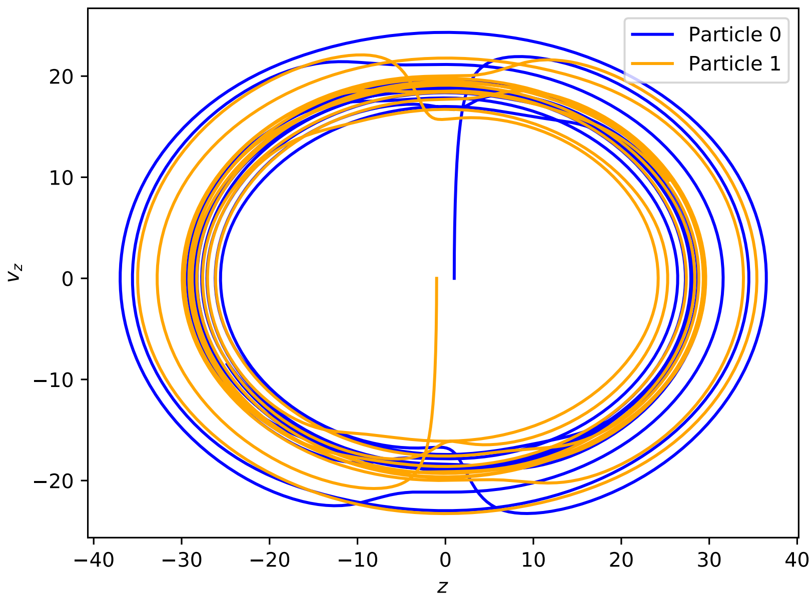

Figure 9: Phase space z plane

4.2 3D movement

4.2.1 Two particles with coulomb's interaction

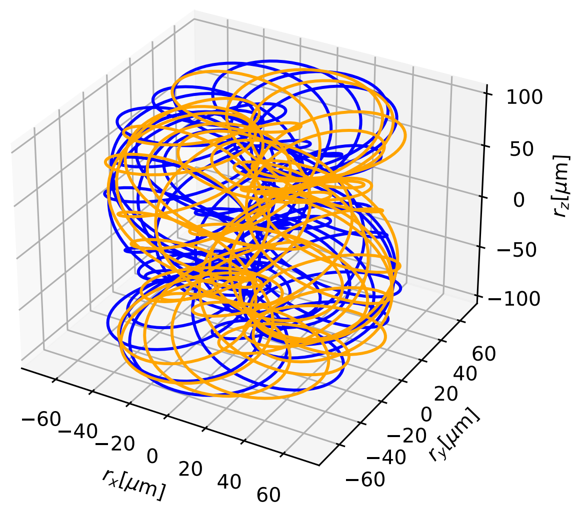

Figure 10: 3D trajectory with

4.2.2 Two particles without coulomb's interaction

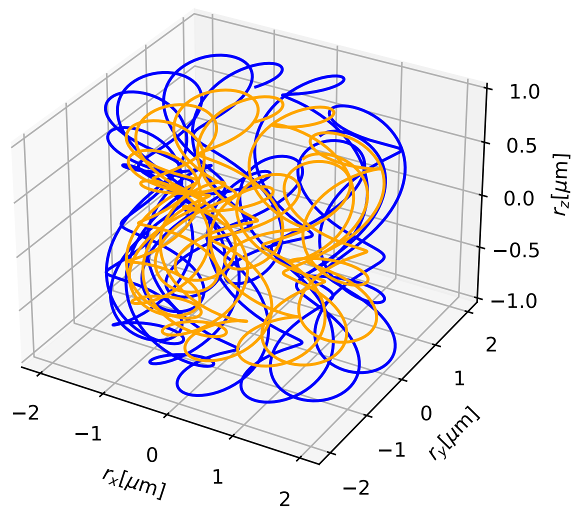

Figure 11: 3D trajectory without

4.3 Difference in choice of timesteps, h

4.3.1 Stepsize 1

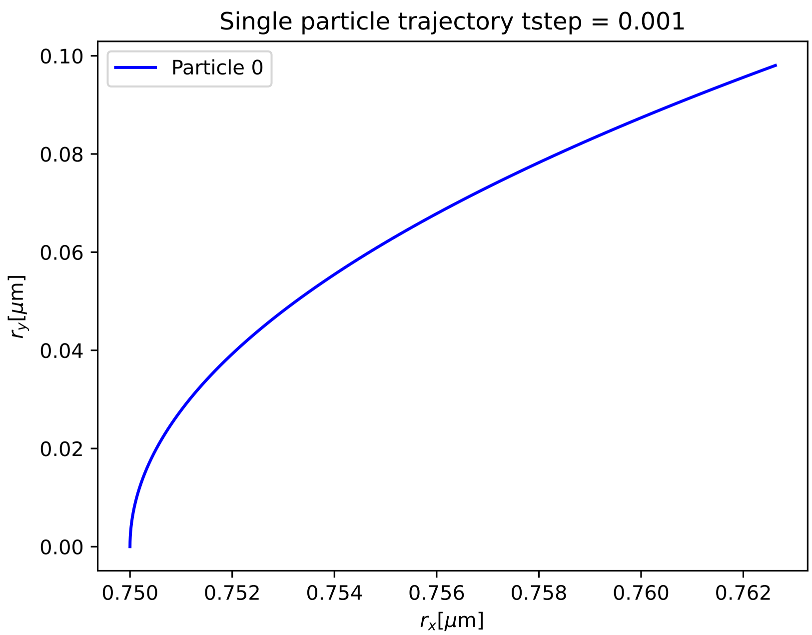

Figure 12: Single Particle Trajectory $h = 0.001$

4.3.2 Stepsize 2

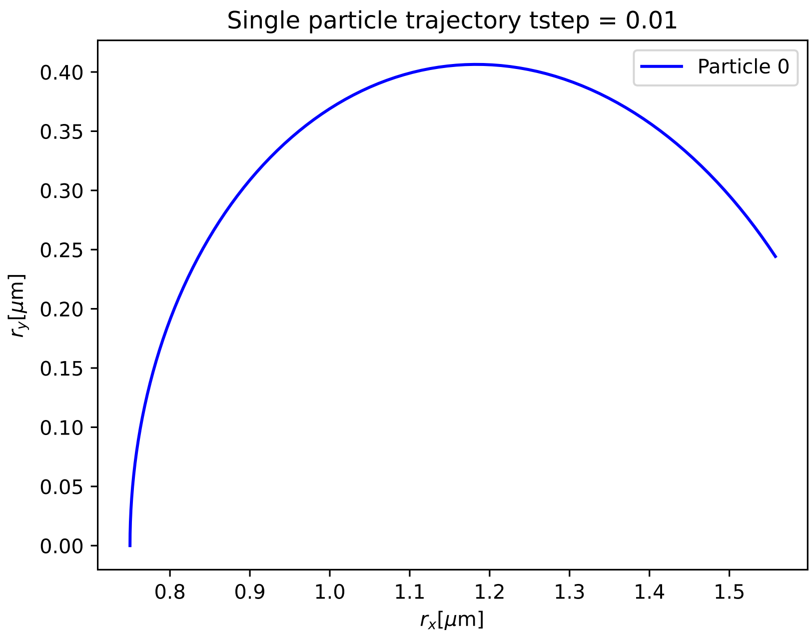

Figure 13: Single Particle Trajectory $h = 0.01$

4.3.3 Stepsize 3

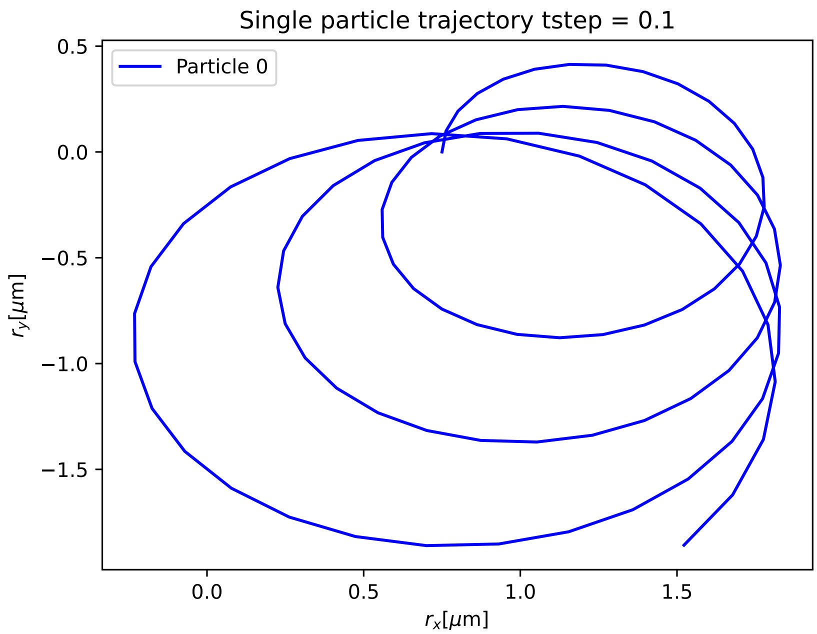

Figure 14: Single Particle Trajectory $h = 0.1$

4.3.4 Stepsize 4

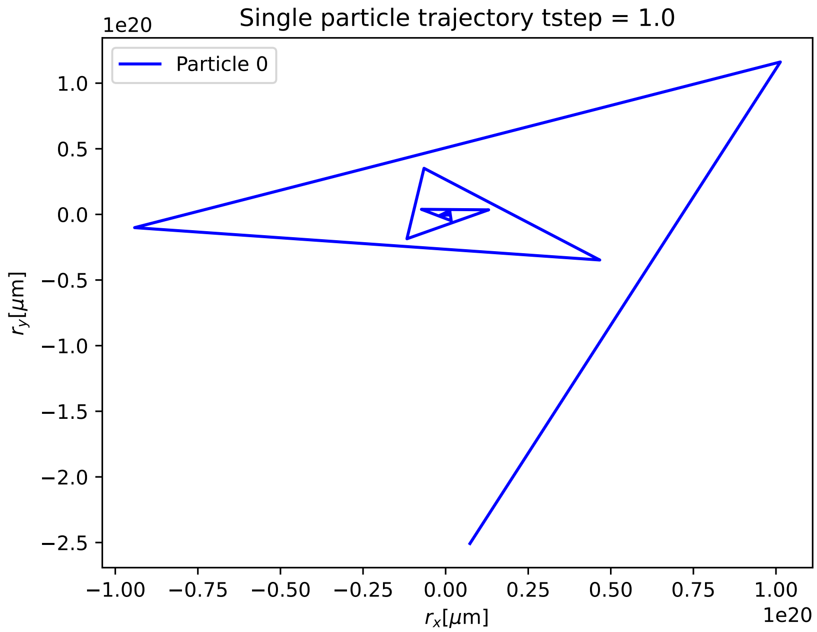

Figure 15: Single Particle Trajectory $h = 1.0$

4.3.5 Stepsize 5

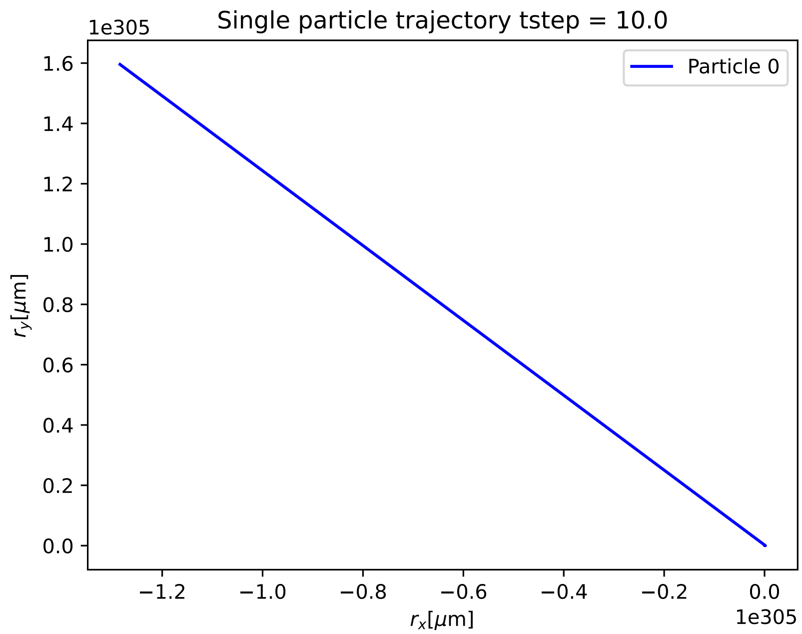

Figure 16: Single Particle Trajectory $h = 10.0$

5 Discussion

5.1 2D movement

5.1.1 A single particle

From figure 1 we can see that the particle oscillates with a constant amplitude in the z-direction as expected for a particle trapped in a penning trap.

5.1.2 Two particles

4.1.2.1 With coulomb interactions From figure 2 we can see that the particle start close together, but the force from the Coulomb-interaction, causes the particles to radially repel each-other. This is displayed in figure 2, as the amplitudes in the x and y plane that get bigger over time.

4.1.2.2 Without coulomb interactions If we now turn the Coulomb interaction off, we can see from figure 3 that the particles do not interact with each-other and overlap more than in figure 2. Furthermore we can see that the axis are considerably smaller in figure 3 than than in figure 2

5.1.3 Two particles in phase space

4.1.3.1 Without coulomb's interaction For the phase space plots for the x and y direction, figure 4 and figure 5, we can see that the velocity in xy direction slowly converges over time towards 0. Since our penning trap accounts for the particles interacting with one another, it is reasonable to assume that the loss of energy over time would result in the particles being stuck in the middle of the penning trap with zero velocity

4.1.3.2 With coulomb's interaction Unlike the previous graphs for when the particle did not have any Coulomb interaction, we clearly see here that when there is an interaction in figures 7, 8, and 9. The particles' trajectory is affected by each other resulting in a sort of symmetric pattern, where one of the particle's trajectory is displaced to one side.

5.2 3D movement

In our 3D plots, wee see the difference of the plots when Coloumb's interaction is in play. There is more chaotic patterns when the interaction is on. However, we expected a clearer difference when the interaction is off, wherein the graph would show a periodic patter for both particles where they do not react to each others' presence.

5.3 Difference in choice of timesteps, h

We observered that the higher the value of h-steps results in a higher truncation error and therefore a larger error when we simulate our particles in our penning trap.

6 Conclusion

In conclusion we discovered our code had mistakes that ultimately was the cause of the multiple deviations in the plots. These mistakes were derived from the magnetic field and electric field being slightly out of bound enabling the particles to escape from the trap. These deviations lead to a not so ideal penning trap which did not caught the particles as expected.

We are able to take with us the knowledge of how a penning trap works, and which elements that play a role in catching the electrons in place, which include the 3 electrodes that generate the electric field and the b field that keeps the electrons from escaping.

A Appendix

A.1 Source code

All the source code is located in this GitHub Repository

A.2 Derivation of motion equation and finding the general solution of z

Inserting the relation we found for the electric potential (4), and the relation for the magnetic field (5), we get:

$$\ddot{\mathbf{r}} = \frac{q}{m}[(-\frac{V_0}{2d^2}((-2x\hat{\imath} - 2y\hat{\jmath} + 4z\hat{k}) + (\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}) \times (B_0\hat{k}))] \tag{35}$$

$$\ddot{\mathbf{r}} = \frac{q}{m}[(-\frac{V_0}{2d^2}((-2x\hat{\imath} - 2y\hat{\jmath} + 4z\hat{k}) + \dot{y}B_0\hat{\imath} - \dot{x}B_0\hat{\jmath} + 0\hat{k})] \tag{36}$$

$$\ddot{\mathbf{r}} = (\frac{qV_0x}{md^2}\hat{\imath} + \frac{qV_0y}{md^2}\hat{\jmath} - \frac{qV_02z}{md^2}\hat{k}) + \frac{q\dot{y}B_0}{m}\hat{\imath} - \frac{q\dot{x}_0}{m}\hat{\jmath} + 0\hat{k}) \tag{37}$$

Rearranging eq 3, setting $\frac{qB_0}{m} = \omega_0$ and $\omega_z^2 = \frac{2qV_0}{md^2}$.

$$\ddot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x = 0 \tag{38}$$

$$\ddot{y} + \omega_0\dot{x} - \frac{1}{2}\omega_z^2 y = 0 \tag{39}$$

$$\ddot{z} + \omega_z^2 z = 0 \tag{40}$$

We will set an ansatz:

$$z = Ce^{at},$$

this gives us:

$$(a^2 + \omega_0)Ce^{at} = 0 \tag{41}$$

This can only be equal to zero if $a = \sqrt{-\omega_0}$, meaning:

$$z(t) = C_1 e^{i\sqrt{\omega_0 t}} + C_2 e^{-i\sqrt{\omega_0 t}} \tag{42}$$

A.3 From coupled differential equations to a single differential equations depending on one variable

$$\ddot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x + i(\ddot{y} + \omega_0\dot{x} - \frac{1}{2}\omega_z^2 y) \tag{43}$$

$$\ddot{x} + i\ddot{y} - \omega_0\dot{y} + \omega_0\dot{x} - (\frac{1}{2}\omega_z^2 x + \frac{1}{2}\omega_z^2 y) \tag{44}$$

Using the trick $i\omega_0(x+iy)$ I can write my new differential equation only dependent on f.

$$\ddot{f} + i\omega_0\dot{f} - \frac{1}{2}\omega_z^2 f = 0 \tag{45}$$

A.4 Calculation of upper and lower bounds

The upper and lower bound on the particles distance from the origin, is governed the maximum and minimum of the value of $|f(t)|$.

$$f(t) = A_+(\cos\omega_+ t - i\sin\omega_+ t)) + A_-(\cos\omega_- t - i\sin\omega_- t) \tag{46}$$

$$= (A_+\cos\omega_+ t + A_-\omega_- t) + i(-A_+\sin\omega_+ t - A_-\sin\omega_- t) \tag{47}$$

Now that we have identified real part as:

$$x(t) = (A_+\cos\omega_+ t + A_-\omega_- t) \tag{48}$$

, and imagin $y(t) = (-A_+\sin\omega_+ t - A_-\sin\omega_- t)$, I can put them back into the relation eq(relation) to find the upper and lower bounds:

$$|x+iy| = \sqrt{x^2+y^2} = \sqrt{(A_+\cos\omega_+ t + A_-\omega_- t)^2 + (-A_+\sin\omega_+ t - A_-\sin\omega_- t)^2} \tag{49}$$

$$= \sqrt{A_+^2\cos^2\omega_+ t + 2A_+A_-\cos\omega_+ t\cdot\cos\omega_- t A_-^2\cos\omega_t}$$
$$+ \sqrt{A_+^2\sin^2\omega_+ t + 2A_+A_-\sin\omega_+ t\cdot sin\omega_- t + A_-^2\sin^2\omega_- t}$$

Here I can make use of the identity $\cos^2 + \sin^2 = 1$ aswell as the trignomertric identity:

$$[(\alpha-\beta)x] = \cos(\alpha t)\cos(\beta t) + \sin(\alpha t)\sin(\beta t) \tag{50}$$

Applying these trigonometrical identities I get the following function for the distance from origo, $|f(t)| = |x + iy|$:

$$|x + iy| = \sqrt{A_+^2 + A_-^2 + 2A_+A_- \cos[(\omega_+\omega_-)t]} \tag{51}$$

Now, it is easy to see that if I want the upper bound, the argument of my cosine must be $n \cdot 2\pi$ for n = 0,1,2.. , resulting in a max value of 1 for my cosine term. This gives the upper bound:

$$R_+ = \sqrt{A_+^2 + A_-^2 + 2A_+A_-} = \sqrt{(A_+ + A_-)^2} = A_+ + A \tag{52}$$

Now for the lower bound the cosine must be minus one, which it is for the arguments $(\omega_+\omega_-)t = n \cdot \pi$ for n = 0,1,2,3 ... This results in a radius equal to:

$$R_- = \sqrt{(A_+ - A_-)^2} = A_+ - A_- \tag{53}$$

But since we can not have an negative radius, the we must take the absolute value of R_-. Since it is not given that A_+ is bigger than A_-. If it were, absolut value would not be needed. And we get the results we wanted:

$$R_+ = A_+ + A_- \tag{54}$$

$$R_- = |A_+ - A_-| \tag{55}$$

A.5 Initial condition calculation

Using the initial conditions:

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= 0 \\ y(0) &= 0 \\ \dot{y}(0) &= v_0 \end{aligned}$$

Solving $f(t)$ for these initial conditions to find A_+ and A_-, gives two equations:

$$x_0 = A_+ + A_- \tag{56}$$

$$v_0 = -\omega_+ A_+ - A_-\omega_- \tag{57}$$

Multiplying equation (23) with ω_+ we can use the relation below to expand the equation (22).

$$\omega_+ A_+ = -\omega_- A_- - v_0 \tag{58}$$

Expanding equation (22) to solve with respects to A_-:

$$\begin{aligned} x_0\omega_+ &= -v_0 + (\omega_+ - \omega_-)A_- \\ A_- &= -\frac{v_0 + \omega_+ x_0}{\omega_- - \omega_+} \end{aligned}$$

With A_- found, A_+ can be found in the same manner. Z is found in the same manner by using the initial conditions $z(0) = z_0$ and $\dot{z} = 0$, which yields two equations:

$$\begin{aligned} z_0 &= C_1 + C_2 \\ 0 &= C_1 i\sqrt{\omega_0} - C_2 i\sqrt{\omega_0} \end{aligned}$$

From which it is clear that $C_1 = C_2 = \frac{z_0}{2}$

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