Singular Value Decomposition Example

Math 218D-2

Problem. Consider the matrix A given by

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 5 & -3 \end{bmatrix}$$

Find a singular value decomposition $A = U\Sigma V^*$. Use your SVD to calculate the low-rank approximations of A.

Solution. First, note that

$$\overset{A^*A}{\begin{bmatrix} 5 & -10 & 10 \\ -10 & 27 & -13 \\ 10 & -13 & 27 \end{bmatrix}} = \overset{A^\intercal}{\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ 4 & 1 & 1 & -3 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 5 & -3 \end{bmatrix}}$$

The characteristic polynomial of A^*A is

$$\begin{aligned}
\chi_{A^*A}(t) &= \begin{vmatrix} t-5 & 10 & -10 \\ 10 & t-27 & 13 \\ -10 & 13 & t-27 \end{vmatrix} \\
&\xlongequal{\boldsymbol{r}_3+\boldsymbol{r}_2\to\boldsymbol{r}_3} \begin{vmatrix} t-5 & 10 & -10 \\ 10 & t-27 & 13 \\ 0 & t-14 & t-14 \end{vmatrix} \\
&= (t-14)\cdot\begin{vmatrix} t-5 & 10 & -10 \\ 10 & t-27 & 13 \\ 0 & 1 & 1 \end{vmatrix} \\
&\xlongequal[\boldsymbol{r}_2 - 13\cdot\boldsymbol{r}_3 \to \boldsymbol{r}_2]{\boldsymbol{r}_1 + 10\cdot\boldsymbol{r}_3 \to \boldsymbol{r}_1} (t-14)\cdot\begin{vmatrix} t-5 & 20 & 0 \\ 10 & t-40 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&= (t-14)\cdot\begin{vmatrix} t-5 & 20 \\ 10 & t-40 \end{vmatrix} \\
&= (t-14)\cdot\{(t-5)(t-40)-200\} \\
&= (t-14)\cdot\{t^2-45\,t\} \\
&= (t-14)\cdot(t-45)t
\end{aligned}$$

The eigenspaces of the positive eigenvalues of A^*A are

$$\mathcal{E}_{A^*A}(45) = \operatorname{Null}\begin{bmatrix} 40 & 10 & -10 \\ 10 & 18 & 13 \\ -10 & 13 & 18 \end{bmatrix} = \operatorname{Span}\left\{\frac{1}{3}\cdot\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right\}$$

$$\mathcal{E}_{A^*A}(14) = \operatorname{Null}\begin{bmatrix} 9 & 10 & -10 \\ 10 & -13 & 13 \\ -10 & 13 & -13 \end{bmatrix} = \operatorname{Span}\left\{\frac{1}{\sqrt{2}}\cdot\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

This gives the factorization

$$\overset{A^*A}{\begin{bmatrix} 5 & -10 & 10 \\ -10 & 27 & -13 \\ 10 & -13 & 27 \end{bmatrix}} = \overset{V}{\begin{bmatrix} 1/3 & 0 \\ -2/3 & 1/\sqrt{2} \\ 2/3 & 1/\sqrt{2} \end{bmatrix}} \overset{D}{\begin{bmatrix} 45 & 0 \\ 0 & 14 \end{bmatrix}} \overset{V^*}{\begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

Now, the formulas $\Sigma = \sqrt{D}$ and $U = AV\Sigma^{-1}$ give

$$\Sigma = \begin{bmatrix} \sqrt{45} & \\ & \sqrt{14} \end{bmatrix}$$

$$\overset{U}{\begin{bmatrix} 3/\sqrt{45} & 2/\sqrt{7} \\ 0 & 1/\sqrt{7} \\ 0 & 1/\sqrt{7} \\ -6/\sqrt{45} & 1/\sqrt{7} \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 5 & -3 \end{bmatrix}} \overset{V}{\begin{bmatrix} 1/3 & 0 \\ -2/3 & 1/\sqrt{2} \\ 2/3 & 1/\sqrt{2} \end{bmatrix}} \overset{\Sigma^{-1}}{\begin{bmatrix} 1/\sqrt{45} & \\ & 1/\sqrt{14} \end{bmatrix}}$$

This gives the singular value decomposition

$$\overset{A}{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 5 & -3 \end{bmatrix}} = \overset{U}{\begin{bmatrix} 3/\sqrt{45} & 2/\sqrt{7} \\ 0 & 1/\sqrt{7} \\ 0 & 1/\sqrt{7} \\ -6/\sqrt{45} & 1/\sqrt{7} \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sqrt{45} & \\ & \sqrt{14} \end{bmatrix}} \overset{V^*}{\begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

Our "compressed" sum is

$$\overset{A}{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 5 & -3 \end{bmatrix}} = \sqrt{45}\cdot \begin{bmatrix} 3/\sqrt{45} \\ 0 \\ 0 \\ -6/\sqrt{45} \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \end{bmatrix} + \sqrt{14}\cdot \begin{bmatrix} 2/\sqrt{7} \\ 1/\sqrt{7} \\ 1/\sqrt{7} \\ 1/\sqrt{7} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The "rank one" approximation is

$$\overset{A}{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 5 & -3 \end{bmatrix}} \approx \sqrt{45} \cdot \begin{bmatrix} 3/\sqrt{45} \\ 0 \\ 0 \\ -6/\sqrt{45} \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 4 & -4 \end{bmatrix} \quad \square$$