Determinants

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[1, §1.5]

DEF. The <u>(i, j)-submatrix</u> of a $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A. We denote the (i, j)-submatrix of A by A_{ij}.

EX. $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 0 & 10 \\ 0 & -1 & 6 \end{bmatrix} \implies A_{21} = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$

DEF. The <u>determinant</u> of a 1×1 matrix $A = [a_{11}]$ is $\det(A) = a_{11}$.

DEF. The <u>determinant</u> of a $n \times n$ matrix A is $\det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det(A_{1k})$

NOTE. We often write $\det(A) = |A|$.

EX. Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

SOL.

$$\begin{aligned}
\det(A) &= \sum_{k=1}^{2} (-1)^{1+k} a_{1k} \det(A_{1k}) \\
&= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) \\
&= a \det([d]) - b \det([c]) \\
&= ad - bc
\end{aligned}$$

Ex. Compute $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Sol.

$$\begin{aligned}
\det(A) &= \sum_{k=1}^{3}(-1)^{1+k}a_{1k}\det(A_{1k}) \\
&= (-1)^{1+1}a_{11}\det(A_{11}) + (-1)^{1+2}a_{12} + (-1)^{1+3}a_{13}\det(A_{13}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}$$

DEF. The <u>(i,j)-minor</u> of A is $M_{ij} = \det(A_{ij})$. The <u>(i,j)-cofactor</u> of A is $C_{ij} = (-1)^{i+j} M_{ij}$.

NOTE. We this notation, our formula for determinants is

$$\det(A) = \sum_{k=1}^{n} a_{1k} C_{1k}$$

EX. For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ we have

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

NOTE. The signs of the cofactors can be remembered by

$$\begin{matrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

THM (Laplace Cofactor Expansion Theorem, Theorem 1.16 in [1]).
For any $1 \leq i \leq n$ *we have*

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} \qquad (i\text{th row expansion})$$

$$\det(A) = \sum_{k=1}^{n} a_{ki} C_{ki} \qquad (i\text{th column expansion})$$

IDEA. The Laplace Cofactor Expansion Theorem allows us to compute determinants by "expanding" about any row or column. This is nice because some rows or columns may be easier to expand about than others.

Ex. Compute $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

Sol. Expansion about Row_2 gives

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (-4)\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + (5)\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-6)\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 0$$

Expansion about Col_3 gives

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (3)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} + (-6)\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + (9)\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 0$$

Ex. Compute $\begin{vmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{vmatrix}$.

Sol. Expansion about Col_3 gives

$$\begin{vmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{vmatrix} = (-2)\begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix}$$

Then expand about Col_1 to obtain

$$\begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix} = (7)\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 7(6-12) = -42$$

Hence

$$\begin{vmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{vmatrix} = (-2)(-42) = 84$$

THM (Corollary 1.17 in [1]). *If* A *has a row or column of zeros, then* $\det(A) = 0$.

THM (Corollary 1.18 in [1]). *The determinant of a triangular matrix is the product of its diagonal entries.*

THM (Theorem 1.19 in [1]). $\det(A^\top) = \det(A)$

THM (Theorem 1.20 in [1]).

1. $A \xrightarrow{R_i \leftrightarrow R_j} B \implies \det(B) = -\det(A)$
2. $A \xrightarrow{\lambda R_i \to R_i} B \implies \det(B) = \lambda \det(A)$
3. $A \xrightarrow{R_i + \lambda R_j \to R_i} B \implies \det(B) = \det(A)$

IDEA. Determinants play nicely with elementary row operations.

REFERENCES

[1] G.L. Peterson and J.S. Sochacki. Linear Algebra and Differential Equations. Addison-Wesley, 2002.