Further Properties of Determinants

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[1, §1.6]

THM (Theorem 1.21 in [1]). *A square matrix A is invertible if and only if* $\det(A) \neq 0$*.*

Proof. By [1, Theorem 1.20], performing an elementary row operation $A \to B$ results in $\det(B) = \lambda \cdot \det(A)$ where $\lambda \neq 0$. Since $\operatorname{rref}(A)$ is obtained from A by elementary row operations, it follows that $\det(\operatorname{rref} A) = \lambda \cdot \det(A)$. Since A is invertible if and only if $\det(\operatorname{rref} A) \neq 0$, it follows that A is invertible if and only if $\det(A) \neq 0$. $\square$

Ex. Note that $\det\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} = (-3)\det\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = (-3)(2)(2) \neq 0$. Hence $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is invertible.

THM (Lemma 1.22 in [1]). *Elementary matrices have determinants*

$$\det[R_i \leftrightarrow R_j] = -1$$
$$\det[\lambda \cdot R_i \to R_i] = \lambda$$
$$\det[R_i + \lambda \cdot R_j \to R_i] = 1$$

EX. In the 3×3 case we have

$$\det[R_2 \leftrightarrow R_3] = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = -1$$
$$\det[-3 \cdot R_2 \to R_2] = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -3$$
$$\det[R_1 + 2 \cdot R_2 \to R_1] = \det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

THM (Theorem 1.24 in [1]). *If A and B are square matrices, then* $\det(AB) = \det(A)\det(B)$.

THM (Corollary 1.25 in [1]). *If A is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Proof. Note that

$$\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$$

so that $\det(A^{-1}) = \dfrac{1}{\det(A)}$. $\square$

DEF. The cofactor matrix of a square matrix A is the $n \times n$ matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

The adjoint of A is

$$\operatorname{adj}(A) = C^{\top} = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{11} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

EX. For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$C_{11} = d \qquad C_{12} = -c$$

$$C_{21} = -b \qquad C_{22} = a$$

Hence

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \qquad \operatorname{adj}(A) = C^{\top} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

THM (Theorem 1.26 in [1]). $A\,\mathrm{adj}(A) = \mathrm{adj}(A)A = \det(A)I$

Ex. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we found $\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Thus

$$A\,\mathrm{adj}(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -bc+ad & 0 \\ 0 & -bc+ad \end{bmatrix} = \det(A)I$$

THM (Corollary 1.27 in [1]). *If A is invertible, then*

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$$

Ex. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \frac{1}{-bc+ad} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided that $-bc+ad \neq 0$.

THM (Cramer's Rule). *Consider the system $A\vec{x}=\vec{b}$ where A is invertible. Let A_i be the matrix obtained by replacing the ith column of A by $\vec{b}$. Then $x_i = \dfrac{\det(A_i)}{\det(A)}$.*

EX. Consider $A\vec{x}=\vec{b}$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and suppose that A is invertible. Then

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} b_1 & b \\ b_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-bb_2 + b_1 d}{-bc + ad}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} a & b_1 \\ c & b_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ab_2 - b_1 c}{-bc + ad}$$

REFERENCES

[1] G.L. Peterson and J.S. Sochacki. Linear Algebra and Differential Equations. Addison-Wesley, 2002.