Special Matrices and Additional Properties of Matrices

Brian D. Fitzpatrick

[1, §1.4]

DEF. A square matrix A is <u>diagonal</u> if $a_{ij} = 0$ for $i \neq j$.

EX. The matrices

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_n \qquad 0_{n\times n}$$

are diagonal.

NOTE. We write $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

THM (Theorem 1.11 in [1]). *Let* $A = \mathrm{diag}(a_1, \ldots, a_n)$ *and* $B = \mathrm{diag}(b_1, \ldots, b_n)$. *Then*

1. $A + B = \mathrm{diag}(a_1 + b_1, \ldots, a_n + b_n)$
2. $AB = \mathrm{diag}(a_1 b_2, \ldots, a_n b_n)$
3. $A \in \mathrm{GL}_n(\mathbb{R})$ *if and only if each* $a_i \neq 0$. *Furthermore, in this case,* $A^{-1} = \mathrm{diag}(1/a_1, \ldots, 1/a_n)$

DEF. A square matrix A is

<u>UPPER TRIANGULAR</u> if $a_{ij} = 0$ for $i > j$

<u>LOWER TRIANGULAR</u> if $a_{ij} = 0$ for $j > i$

EX. We have

$$\underbrace{\begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{upper triangular}} \qquad \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 9 & 7 \end{bmatrix}}_{\text{lower triangular}}$$

THM (Theorem 1.12 in [1]). *Suppose* A *and* B *are upper triangular. Then*

1. $A + B$ *is upper triangular.*
2. AB *is upper triangular.*
3. A *is invertible if and only if each diagonal entry is nonzero.*

The corresponding theorem holds for lower triangular matrices.

DEF. The <u>transpose</u> of a $m \times n$ matrix A is the $n \times m$ matrix $A^\top$ whose entries are $[A^\top]_{ij} = a_{ji}$.

EX. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \implies A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

THM (Theorem 1.13 in [1]).

1. $(A^\top)^\top = A$
2. $(A+B)^\top = A^\top + B^\top$
3. $(\lambda A)^\top = \lambda A^\top$
4. $(AB)^\top = B^\top A^\top$
5. $(A^\top)^{-1} = (A^{-1})^\top$

Proof. To prove that $(AB)^\top = B^\top A^\top$, note that

$$\begin{aligned}
[(AB)^\top]_{ij} &= [AB]_{ji} \\
&= \sum_{k=1}^{n} [A]_{jk}[B]_{ki} \\
&= \sum_{k=1}^{n} [A^\top]_{kj}[B^\top]_{ik} \\
&= \sum_{k=1}^{n} [B^\top]_{ik}[A^\top]_{kj} \\
&= [B^\top A^\top]_{ij}
\end{aligned}$$

Hence $(AB)^\top = B^\top A^\top$. $\square$

DEF. A matrix A is <u>symmetric</u> if $A^\top = A$.

IDEA. rows of $A =$ columns of $A^\top$

DEF. We denote the collection of all $n \times n$ matrices by $\mathrm{Sym}_n(\mathbb{R})$.

EX. We have

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}}_{\text{symmetric}} \qquad \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{\text{not symmetric}}$$

THM (Theorem 1.14 in [1]).

1. *If A and B are symmetric, then $A+B$ is symmetric.*
2. *If A is symmetric, then λA is symmetric.*
3. *$A^\top A$ and $AA^\top$ is symmetric.*
4. *If A is invertible and symmetric, then A^{-1} is symmetric.*

REFERENCES

[1] G.L. Peterson and J.S. Sochacki. <u>Linear Algebra and Differential Equations</u>. Addison-Wesley, 2002.