Inverses of Matrices

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[1, §1.3]

DEF. The <u>inverse</u> of $A \in M_{n\times n}(\mathbb{R})$ is a matrix $B \in M_{n\times n}(\mathbb{R})$ such that $AB = BA = I$.

Ex. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$
$$BA = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence B is an inverse of A.

Ex. The matrix $0_{n\times n}$ does not have an inverse since $0_{n\times n}B = 0_{n\times n} \neq I$ whenever $B \in M_{n\times n}(\mathbb{R})$.

Ex. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has no inverse since

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{bmatrix}$$

whenever $B \in M_{2\times 2}(\mathbb{R})$.

DEF. A matrix is <u>invertible</u> or <u>nonsingular</u> if it has an inverse.
A matrix is <u>noninvertible</u> or <u>singular</u> if it does not have an inverse.

DEF. The collection of all $n \times n$ invertible matrices is denoted by $\mathtt{GL}_n(\mathbb{R})$.

THM (Theorem 1.4 in [1]). *Inverses are unique. That is, let* $A \in \mathtt{GL}_n(\mathbb{R})$ *have inverses* B_1 *and* B_2. *Then* $B_1 = B_2$.

Proof. $B_1 = B_1 I = B_1(AB_2) = (B_1 A)B_2 = IB_2 = B_2$ □

NOTE. Since inverses are unique, we denote the inverse of A by A^{-1}.

Q. How do we find A^{-1}?

ANSWER.

1. Form the augmented matrix: $[A \mid I]$.
2. Use elementary row operations to reduce A into rref: $[A \mid I] \rightsquigarrow [\mathbf{rref}(A) \mid B]$.
3. If $\mathbf{rref}(A) = I$, then $B = A^{-1}$. Otherwise, A is not invertible.

Ex. Find the inverse of $A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$.

Sol. Row reduce

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - 2\cdot R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 4\cdot R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}\cdot R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - \frac{1}{2}\cdot R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{3} & \frac{5}{6} & 0 & -\frac{1}{6} \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}\cdot R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{3} & \frac{5}{6} & 0 & -\frac{1}{6} \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_1-\frac{7}{3}\cdot R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{7}{6} & -\frac{1}{6} \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_2+\frac{5}{3}\cdot R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{7}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

This gives $A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$.

Def. A $\underline{n \times n}$ elementary matrix is a matrix obtained by performing exactly one elementary row operation on I_n.

Ex.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{3\cdot R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 1/2\cdot R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Idea. Multiplication by elementary matrices on the left results in performing the corresponding elementary row operation.

Ex. Performing $3 \cdot R_2 \to R_2$ on $A = \begin{bmatrix} 1 & 7 \\ 0 & \frac{1}{3} \end{bmatrix}$ gives

$$\begin{bmatrix} 1 & 7 \\ 0 & \frac{1}{3} \end{bmatrix} \xrightarrow{3 \cdot R_2 \to R_2} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$$

The row-operation $3 \cdot R_2 \to R_2$ corresponds to the elementary matrix $E = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Computing EA gives

$$EA = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$$

Thm. *Elementary matrices are invertible and their inverses are*

$$[R_i \leftrightarrow R_j]^{-1} = [R_i \leftrightarrow R_j]$$

$$[\lambda \cdot R_i \to R_i]^{-1} = [{}^1/\lambda \cdot R_i \to R_i]$$

$$[R_i + \lambda \cdot R_j \to R_i]^{-1} = [R_i - \lambda \cdot R_j \to R_i]$$

Ex. Write A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

Sol. Row reduce

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3\cdot R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{-1\cdot R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2\cdot R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

This gives $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

These row reductions correspond to elementary matrices

$$E_1 = [R_2 - 3 \cdot R_1 \to R_2] = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$E_2 = [-R_2 \to R_2] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$E_3 = [R_1 - 2 \cdot R_2 \to R_1] = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

This gives $\underbrace{E_3 E_2 E_1}_{=A^{-1}} A = \mathtt{rref}(A) = I$.

THM. *Given* $A, B \in \mathtt{GL}_n(\mathbb{R})$, *let* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 0$. *Then*

1. $A^{-1} \in \mathtt{GL}_n(\mathbb{R})$ *and* $(A^{-1})^{-1} = A)$
2. $\lambda \cdot A \in \mathtt{GL}_n(\mathbb{R})$ *and* $(\lambda A)^{-1} = {}^1/_\lambda \cdot A^{-1}$
3. $AB \in \mathtt{GL}_n(\mathbb{R})$ *and* $(AB)^{-1} = B^{-1}A^{-1}$
4. $A^k \in \mathtt{GL}_n(\mathbb{R})$ *and* $(A^k)^{-1} = (A^{-1})^k$

THM (Fundamental Theorem of Invertible Matrices). *Let* $A \in M_{n\times n}(\mathbb{R})$. *Then the following are equivalent.*

1. *A is invertible*
2. *A is nonsingular*
3. $A \in \mathtt{GL}_n(\mathbb{R})$
4. $A\vec{x} = \vec{b}$ *has a unique solution for every* $\vec{b} \in \mathbb{R}^n$
5. $A\vec{x} = \vec{0}$ *has only the trivial solution* $\vec{x} = \vec{0}$
6. $\mathbf{rref}(A) = I_n$
7. $\mathbf{rank}(A) = n$
8. *A is a product of elementary matrices*

THM. *Let* $A, B \in M_{n\times n}(\mathbb{R})$ *such that* $AB = I_n$ *or* $BA = I_n$. *Then* $B = A^{-1}$.

Proof. Suppose $BA = I_n$ and consider the system $A\vec{x} = \vec{0}$. Then

$$\vec{0} = B\vec{0} = B(A\vec{x}) = (BA)\vec{x} = I_n\vec{x} = \vec{x}$$

So, the system $A\vec{x} = \vec{0}$ only has the trivial solution. By the fundamental theorem, $A \in \mathtt{GL}_n(\mathbb{R})$ and

$$B = BI_n = B(AA^{-1}) = (BA)A^{-1} = I_nA^{-1} = A^{-1}$$

The argument when $AB = I_n$ is similar. □

REFERENCES

[1] G.L. Peterson and J.S. Sochacki. Linear Algebra and Differential Equations. Addison-Wesley, 2002.