

Systems of Linear Equations

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[1, §1.1]

MOTIVATION

Consider the system of m equations with n variables.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \quad (*)$$

Q1. Does a solution exist?

Q2. How many solutions exist?

Q3. How do we find all solutions?

Ex. The system of 2 equations and 3 variables

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x & & & & - & 2z & = & 3 \end{array}$$

is solved by $(x, y, z) = (5, -4, 1)$ and $(x, y, z) = (1, 2, -1)$. The system has more than one solution.

Ex. The system of 3 equations and 2 variables

$$x + 3y = 2 \quad (1)$$

$$-2x + 7y = 8 \quad (2)$$

$$4x - 14y = 0 \quad (3)$$

has no solutions because $2 \cdot (2) + (3) = \begin{cases} 0 & \text{LHS} \\ 16 & \text{RHS} \end{cases}$

Ex. Find all solutions to the system

$$x + 2y = 3 \quad (1)$$

$$3x + 4y = 0 \quad (2)$$

BAD IDEA. Start deriving new equations. Equation (2) is

$$y = -\frac{3}{4}x \quad (3)$$

Plug (3) into (1) to obtain

$$x + 2\left(-\frac{3}{4}x\right) = 3 \Rightarrow x = -6 \quad (4)$$

Plug (4) into (3) to obtain

$$y = \left(-\frac{3}{4}\right)(-6) = \frac{9}{2} \quad (5)$$

Equations (4) and (5) then give $(x, y) = (-6, 9/2)$.

Q. Why is this a bad idea?

1. Equations get messy in high dimensions.
2. The procedure is not guaranteed to terminate.

DEF. Two systems are equivalent if they have the same solutions.

GOOD IDEA. Replace "hard" systems with equivalent "easy" systems.

AUGMENTED MATRICES AND REDUCED ROW-ECHELON FORM

DEF. The augmented matrix of the system (*) is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

Ex.

$$\begin{array}{rcl} 3x + 4y - 8z = 0 \\ y + 10z = 6 \\ 9x + 7y - 2z = -5 \end{array} \longleftrightarrow \left[\begin{array}{ccc|c} 3 & 4 & -8 & 0 \\ 0 & 1 & 10 & 6 \\ 9 & 7 & -2 & -5 \end{array} \right]$$

DEF. A matrix is in reduced row-echelon form (rref) if

1. any zero-rows occur at the bottom
2. the first nonzero entry of a nonzero row is 1 (called a pivot)
3. every pivot occurs to the right of any previous pivots
4. all other entries of a column containing a pivot are zero

Ex. Each of the following is in rref

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array} \right] \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Ex. Each of the following is not in rref

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

DEF. A rref system is

CONSISTENT if the augmented column contains no pivot

INCONSISTENT if the augmented column contains a pivot

DEF. A column in a consistent rref matrix corresponds to a

FREE VARIABLE if it contains no pivot

DEPENDENT VARIABLE if it contains a pivot

IDEA. Solving consistent rref systems is easy: write the dependent variables in terms of the free variables.

Ex. The rref system

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

is consistent. The free variables are $\{x_2, x_4\}$ and the dependent variables are $\{x_1, x_3\}$. The solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ x_2 \\ -2x_4 - 1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Note that there are infinitely many solutions.

Ex. The rref system

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 8 \end{array} \right]$$

is consistent with no free variables. The only solution is $(x_1, x_2, x_3) = (3, 4, 8)$.

Ex. The rref system

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

is inconsistent. The last equation is $0=1$, which is impossible!

This system has no solutions.

THM. A consistent rref system has

A UNIQUE SOLUTION if it has no free variables

INFINITELY MANY SOLUTIONS if it has at least one free variable

An inconsistent rref system has no solutions.

ELEMENTARY ROW OPERATIONS

DEF. There are three types of elementary row operations:

ROW SWITCHING: a row can be switched with another row $R_i \leftrightarrow R_j$

ROW MULTIPLICATION: a row can be multiplied by any nonzero scalar
 $c \cdot R_i \rightarrow R_i$

ROW ADDITION: a row can be replaced by the sum of that row and a multiple of another row $R_i + c \cdot R_j \rightarrow R_i$

Ex (row switching).

$$\left[\begin{array}{cccc|c} 1 & -8 & -4 & -1 & 1 \\ -6 & -1 & -3 & 3 & -1 \\ 0 & 1 & -4 & 1 & 2 \\ 2 & -5 & -29 & 3 & 3 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & -8 & -4 & -1 & 1 \\ 2 & -5 & -29 & 3 & 3 \\ 0 & 1 & -4 & 1 & 2 \\ -6 & -1 & -3 & 3 & -1 \end{array} \right]$$

Ex (row multiplication).

$$\left[\begin{array}{ccccc|c} -1 & -12 & 16 & -3 & -3 & 1 \\ -53 & -1 & -1 & 4 & 0 & 0 \\ 0 & -9 & -2 & -2 & 2 & 1 \\ -1 & -1 & 0 & 1 & -1 & 3 \end{array} \right] \xrightarrow{3 \cdot R_3 \rightarrow R_3} \left[\begin{array}{ccccc|c} -1 & -12 & 16 & -3 & -3 & 1 \\ -53 & -1 & -1 & 4 & 0 & 0 \\ 0 & -27 & -6 & -6 & 6 & 3 \\ -1 & -1 & 0 & 1 & -1 & 3 \end{array} \right]$$

Ex (row addition).

$$\left[\begin{array}{ccccc|c} 3 & 8 & 1 & -1 & 0 & 1 \\ 0 & 1 & -16 & -5 & -2 & 0 \\ 1 & 2 & 0 & 4 & 0 & 1 \\ 4 & -1 & -1 & -1 & 1 & 3 \end{array} \right] \xrightarrow{R_3 - 3 \cdot R_2 \rightarrow R_3} \left[\begin{array}{ccccc|c} 3 & 8 & 1 & -1 & 0 & 1 \\ 0 & 1 & -16 & -5 & -2 & 0 \\ 1 & -1 & 48 & 19 & 6 & 1 \\ 4 & -1 & -1 & -1 & 1 & 3 \end{array} \right]$$

THM. *Performing an elementary row operation on an augmented matrix yields an equivalent system.*

IDEA. To solve a given system, use elementary row operations to obtain a rref system.

THM (Gauß-Jordan Elimination). *Every augmented matrix is equivalent to a unique rref augmented matrix. Furthermore, its rref may be obtained via elementary row operations.*

Ex. Find all solutions to the system

$$x + 2y = 3$$

$$3x + 4y = 0$$

SOL. Row reduce

$$\begin{aligned} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 0 \end{array} \right] & \xrightarrow{R_2 - 3 \cdot R_1 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -9 \end{array} \right] \\ & \xrightarrow{-\frac{1}{2} \cdot R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & \frac{9}{2} \end{array} \right] \\ & \xrightarrow{R_1 - 2 \cdot R_2 \rightarrow R_1} \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & \frac{9}{2} \end{array} \right] \end{aligned}$$

This shows that

$$\text{rref} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 0 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & \frac{9}{2} \end{array} \right]$$

The rref of the system is consistent and has no free variables.

Hence the unique solution is $(x, y) = (-6, 9/2)$.

Ex. Find all solutions to

$$\begin{aligned}4x_2 + x_3 &= 2 \\2x_1 + 6x_2 - 2x_3 &= 3 \\4x_1 + 8x_2 - 5x_3 &= 4\end{aligned}$$

SOL. Row reduce

$$\begin{aligned}\left[\begin{array}{ccc|c}0 & 4 & 1 & 2 \\2 & 6 & -2 & 3 \\4 & 8 & -5 & 4\end{array}\right] &\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c}2 & 6 & -2 & 3 \\0 & 4 & 1 & 2 \\4 & 8 & -5 & 4\end{array}\right] \\&\xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \left[\begin{array}{ccc|c}1 & 3 & -1 & \frac{3}{2} \\0 & 4 & 1 & 2 \\4 & 8 & -5 & 4\end{array}\right] \\&\xrightarrow{R_3 - 4R_1 \rightarrow R_3} \left[\begin{array}{ccc|c}1 & 3 & -1 & \frac{3}{2} \\0 & 4 & 1 & 2 \\0 & -4 & -1 & -2\end{array}\right] \\&\xrightarrow{\frac{1}{4}R_2 \rightarrow R_2} \left[\begin{array}{ccc|c}1 & 3 & -1 & \frac{3}{2} \\0 & 1 & \frac{1}{4} & \frac{1}{2} \\0 & -4 & -1 & -2\end{array}\right] \\&\xrightarrow{R_1 - 3R_2 \rightarrow R_1} \left[\begin{array}{ccc|c}1 & 0 & -\frac{7}{4} & 0 \\0 & 1 & \frac{1}{4} & \frac{1}{2} \\0 & -4 & -1 & -2\end{array}\right] \\&\xrightarrow{R_3 + 4R_2 \rightarrow R_3} \left[\begin{array}{ccc|c}1 & 0 & -\frac{7}{4} & 0 \\0 & 1 & \frac{1}{4} & \frac{1}{2} \\0 & 0 & 0 & 0\end{array}\right]\end{aligned}$$

This shows that

$$\text{rref} \left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The rref is consistent with free variable $\{x_3\}$ and dependent variables $\{x_1, x_2\}$. The system has infinitely many solutions given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{4}x_3 \\ -\frac{1}{4}x_3 + \frac{1}{2} \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix} x_3$$

HOMOGENEOUS SYSTEMS

DEF. The system (*) is homogeneous if $b_1 = b_2 = \dots = b_m = 0$. In a homogeneous system, the solution $x_1 = x_2 = \dots = x_n = 0$ is the trivial solution. All other solutions are called nontrivial solutions.

THM (Theorem 1.1 in [1]). *A homogeneous system of m equations and n variables has infinitely many solutions if $m < n$.*

RANK AND NULLITY

DEF. The rank of a consistent system is the number of pivots in its rref. Equivalently, the rank is the number of dependent variables.

THM. $\text{rank} \leq \# \text{variables}$ and $\text{rank} \leq \# \text{equations}$

THM. *A consistent system has*

A UNIQUE SOLUTION *if* $\text{rank} = \# \text{variables}$

INFINITELY MANY SOLUTIONS *if* $\text{rank} < \# \text{variables}$

DEF. The nullity of a consistent system is the number of nonpivot columns in its rref. Equivalently, the rank is the number of free variables.

THM. $\text{rank} + \text{nullity} = \# \text{variables}$

THM. *A consistent system has*

A UNIQUE SOLUTION *if* $\text{nullity} = 0$

INFINITELY MANY SOLUTIONS *if* $\text{nullity} > 0$

Ex. How many solutions will a system with coefficient matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & -1 & -2 \end{bmatrix}$$

have?

SOL. Row reduce

$$\begin{aligned} \left[\begin{array}{cccc|c} 2 & 0 & 1 & -1 & b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 2 & -1 & -1 & -2 & b_3 \end{array} \right] & \xrightarrow{\frac{1}{2} \cdot R_1 \rightarrow R_1} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 2 & -1 & -1 & -2 & b_3 \end{array} \right] \\ & \xrightarrow{R_3 - 2 \cdot R_1 \rightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 0 & -1 & -2 & -1 & -b_1 + b_3 \end{array} \right] \\ & \xrightarrow{R_3 + R_2 \rightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 0 & 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array} \right] \end{aligned}$$

The system is consistent if and only if $-b_1 + b_2 + b_3 \neq 0$.

Suppose $-b_1 + b_2 + b_3 \neq 0$, so that the system is consistent.

Note that

$$\text{rank} = 2 < \# \text{variables} = 4$$

so the system has infinitely many solutions.

Alternatively,

$$\text{nullity} = 2 > 0$$

so the system has infinitely many solutions.

Thus, any system with coefficient matrix A will be inconsistent if $-b_1 + b_2 + b_3 \neq 0$ and consistent with infinitely many solutions if $-b_1 + b_2 + b_3 = 0$.

REFERENCES

- [1] G.L. Peterson and J.S. Sochacki. Linear Algebra and Differential Equations. Addison-Wesley, 2002.