Matrices and Matrix Operations

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[1, §1.2]

DEF. A <u>matrix of size $m \times n$</u> is an object of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \underbrace{m}_{\texttt{\#rows}} \times \underbrace{n}_{\texttt{\#columns}}$$

where $a_{ij} \in \mathbb{R}$.

EX.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \\ 16 & -\frac{3}{4} \end{bmatrix} \qquad 3 \times 2 \qquad a_{21} = 4$$

$$B = \begin{bmatrix} 3 & 0 & 8 & 1 & 2 \\ 7 & -\frac{1}{2} & 3 & 2 & 10 \\ 32 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad 3 \times 5 \qquad b_{32} = 1$$

NOTATION. The entries of $A = [a_{ij}]$ are often denoted by

$$a_{ij} = [A]_{ij} = \texttt{ent}_{ij}(A)$$

DEF. The collection of all $m \times n$ matrices is denoted by $M_{m\times n}(\mathbb{R})$.

DEF. Matrices in $M_{1\times n}(\mathbb{R})$ are called <u>row vectors</u>. Matrices in $M_{m\times 1}(\mathbb{R})$ are called <u>column vectors</u>.

EX. We have the identifications

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \, a_{ij} \in \mathbb{R} \right\} \qquad \mathbb{R}^n = M_{n\times 1} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \, a_i \in \mathbb{R} \right\}$$

MATRIX ADDITION

DEF. The <u>sum</u> of two $m\times n$ matrices A and B is the $m\times n$ matrix $A+B$ whose entries are $[A+B]_{ij} = a_{ij} + b_{ij}$.

EX. $\begin{bmatrix} 3 & 1 & 8 \\ 2 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -3 & -4 \\ 1 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 4 \\ 3 & 6 & -8 \end{bmatrix}$

NOTE. $A+B$ is only defined when A and B <u>have the same size</u>

SCALAR MULTIPLICATION

DEF. The <u>scalar product</u> of $\lambda \in \mathbb{R}$ and $A \in M_{m\times n}(\mathbb{R})$ is the $m \times n$ matrix λA whose entries are $[\lambda A]_{ij} = \lambda a_{ij}$.

EX. $6\begin{bmatrix} 3 & 1 & 8 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 48 \\ 12 & 24 & -6 \end{bmatrix}$

THM (Theorem 1.2 in [1]). *Let* $A, B, C \in M_{m\times n}(\mathbb{R})$ *and let* $\alpha, \beta \in \mathbb{R}$. *Then*

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $\alpha(\beta A) = (\alpha\beta)A$
4. $\alpha(A + B) = \alpha A + \alpha B$
5. $(\alpha + \beta)A = \alpha A + \beta A$

DEF. The <u>$m \times n$ zero matrix</u> is the $m \times n$ matrix $0_{m\times n}$ whose entries are $[0_{m\times n}]_{ij} = 0$.

EX. $0_{3\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

THM. *Let* $A \in M_{m\times n}(\mathbb{R})$. *Then*

1. $A + 0_{m\times n} = A$
2. $0 \cdot A = 0_{m\times n}$
3. $A - A = 0_{m\times n}$

MATRIX MULTIPLICATION

DEF. Let $A \in M_{l\times m}(\mathbb{R})$ and $B \in M_{m\times n}(\mathbb{R})$. The <u>product</u> of A and B is the $l \times n$ matrix AB whose entries are

$$[AB]_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

NOTE. The product AB is only defined when

$$\# \text{ columns of } A = \# \text{ rows of } B$$

IDEA. The (i,j) entry of AB is given by the dot-product

$$[AB]_{ij} = (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B)$$

$$= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}$$

EX. Find AB and BA where

$$A = \begin{bmatrix} 3 & 4 \\ 1 & -8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

SOL. Note that A is 2×2 and B is 3×2. Since $2 \neq 3$, AB is not defined.

However, BA is defined and

$$BA = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & -8 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 10 & 4 \end{bmatrix}$$

THM (Theorem 1.3 in [1]). *Let A, B, and C be matrices and let $\lambda \in \mathbb{R}$. Then*

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$
3. $(A+B)C = AC + BC$
4. $\lambda(AB) = (\lambda A)B = A(\lambda B)$

DEF. The $n \times n$ identity matrix is the $n \times n$ matrix I_n whose entries are

$$[I_n]_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

EX.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

THM. *Let* $A \in M_{m\times n}(\mathbb{R})$. *Then*

1. $0_{l\times m}A = 0_{l\times n}$ *and* $A0_{n\times l} = 0_{m\times l}$
2. $I_m A = AI_n = A$

Proof. To prove that $I_m A = A$, note that

$$[I_m A]_{ij} = \sum_{k=1}^{m} [I_m]_{ik} a_{kj} = \sum_{k=1}^{m} \delta_{ik} a_{kj} = a_{ij}$$

Proving that $AI_n = A$ is similar. □

The previous two theorems show that matrix arithmetic is similar to "usual" arithmetic. However, there are some key differences.

EX. Find AB and BA where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

SOL. Compute

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So $AB \neq BA$. Furthermore, $AB = 0_{2\times 2}$ even though $A \neq 0_{w\times 2}$ and $B \neq 0_{2\times 2}$.

Motivation

The system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

may be written as $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Ex. The system

$$\begin{array}{rcrcrcr} 2x & - & y & + & 4z & = & 1 \\ x & - & 7y & + & z & = & 3 \\ -x & + & 2y & + & z & = & 2 \end{array}$$

may be written as

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & -7 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

References

[1] G.L. Peterson and J.S. Sochacki. Linear Algebra and Differential Equations. Addison-Wesley, 2002.