Systems of Linear Equations

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[1, §1.1]

Hello world!

Motivation

Consider the system of m equations with n variables.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{$*$}$$

Q1. Does a solution exist?

Q2. How many solutions exist?

Q3. How do we find <u>all</u> solutions?

Ex. The system of 2 equations and 3 variables

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x & & & - & 2z & = & 3 \end{array}$$

is solved by $(x, y, z) = (5, -4, 1)$ and $(x, y, z) = (1, 2, -1)$. The system has <u>more than one</u> solution.

Ex. The system of 3 equations and 2 variables

$$\begin{array}{rcrcrl} x & + & 3y & = & 2 & (1) \\ -2x & + & 7y & = & 8 & (2) \\ 4x & - & 14y & = & 0 & (3) \end{array}$$

has <u>no solutions</u> because $2 \cdot (2) + (3) = \begin{cases} 0 & \texttt{LHS} \\ 16 & \texttt{RHS} \end{cases}$

EX. Find all solutions to the system

$$\begin{array}{rcrcrl} x & + & 2y & = & 3 & (1) \\ 3x & + & 4y & = & 0 & (2) \end{array}$$

BAD IDEA. Start deriving new equations. Equation (2) is

$$y = -\frac{3}{4}x \tag{3}$$

Plug (3) into (1) to obtain

$$x + 2\left(-\frac{3}{4}x\right) = 3 \Rightarrow x = -6 \tag{4}$$

Plug (4) into (3) to obtain

$$y = \left(-\frac{3}{4}\right)(-6) = \frac{9}{2} \tag{5}$$

Equations (4) and (5) then give $(x, y) = (-6, 9/2)$.

Q. Why is this a bad idea?

1. Equations get messy in high dimensions.
2. The procedure is not guaranteed to terminate.

DEF. Two systems are <u>equivalent</u> if they have the same solutions.

GOOD IDEA. Replace "hard" systems with equivalent "easy" systems.

Augmented Matrices and Reduced Row-Echelon Form

Def. The <u>augmented matrix</u> of the system (*) is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Ex.

$$\begin{array}{rcrcrcr} 3x & + & 4y & - & 8z & = & 0 \\ & & y & + & 10z & = & 6 \\ 9x & + & 7y & - & 2z & = & -5 \end{array} \qquad \longleftrightarrow \qquad \left[\begin{array}{ccc|c} 3 & 4 & -8 & 0 \\ 0 & 1 & 10 & 6 \\ 9 & 7 & -2 & -5 \end{array}\right]$$

Def. A matrix is in <u>reduced row-echelon form</u> (rref) if

1. any zero-rows occur at the bottom
2. the first nonzero entry of a nonzero row is 1 (called a <u>pivot</u>)
3. every pivot occurs to the right of any previous pivots
4. all other entries of a column containing a pivot are zero

Ex. Each of the following is in rref

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Ex. Each of the following is <u>not</u> in rref

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Def. A rref system is

<u>CONSISTENT</u> if the augmented column contains no pivot

<u>INCONSISTENT</u> if the augmented column contains a pivot

Def. A column in a consistent rref matrix corresponds to a

<u>FREE VARIABLE</u> if it contains no pivot

<u>DEPENDENT VARIABLE</u> if it contains a pivot

Idea. Solving consistent rref systems is <u>easy</u>: write the dependent variables in terms of the free variables.

Ex. The rref system

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is consistent. The free variables are $\{x_2, x_4\}$ and the dependent variables are $\{x_1, x_3\}$. The solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ x_2 \\ -2\,x_4 - 1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Note that there are <u>infinitely many solutions</u>.

Ex. The rref system

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

is consistent with no free variables. The only solution is $(x_1, x_2, x_3) = (3, 4, 8)$.

Ex. The rref system

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

is inconsistent. The last equation is $0=1$, which is impossible! This system has <u>no solutions</u>.

Thm. *A consistent rref system has*

<u>A UNIQUE SOLUTION</u> *if it has no free variables*

<u>INFINITELY MANY SOLUTIONS</u> *if it has at least one free variable*

An inconsistent rref system has no solutions.

Elementary Row Operations

Def. There are three types of <u>elementary row operations</u>:

<u>row switching</u>: a row can be switched with another row $R_i \leftrightarrow R_j$

<u>row multiplication</u>: a row can be multiplied by any nonzero scalar
$c \cdot R_i \to R_i$

<u>row addition</u>: a row can be replaced by the sum of that row and a multiple of another row $R_i + c \cdot R_j \to R_i$

Ex (row switching).

$$\left[\begin{array}{rrrr|r} 1 & -8 & -4 & -1 & 1 \\ -6 & -1 & -3 & 3 & -1 \\ 0 & 1 & -4 & 1 & 2 \\ 2 & -5 & -29 & 3 & 3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{rrrr|r} 1 & -8 & -4 & -1 & 1 \\ 2 & -5 & -29 & 3 & 3 \\ 0 & 1 & -4 & 1 & 2 \\ -6 & -1 & -3 & 3 & -1 \end{array}\right]$$

Ex (row multiplication).

$$\left[\begin{array}{rrrrr|r} -1 & -12 & 16 & -3 & -3 & 1 \\ -53 & -1 & -1 & 4 & 0 & 0 \\ 0 & -9 & -2 & -2 & 2 & 1 \\ -1 & -1 & 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{3 \cdot R_3 \to R_3} \left[\begin{array}{rrrrr|r} -1 & -12 & 16 & -3 & -3 & 1 \\ -53 & -1 & -1 & 4 & 0 & 0 \\ 0 & -27 & -6 & -6 & 6 & 3 \\ -1 & -1 & 0 & 1 & -1 & 3 \end{array}\right]$$

EX (row addition).

$$\left[\begin{array}{rrrrr|r} 3 & 8 & 1 & -1 & 0 & 1 \\ 0 & 1 & -16 & -5 & -2 & 0 \\ 1 & 2 & 0 & 4 & 0 & 1 \\ 4 & -1 & -1 & -1 & 1 & 3 \end{array}\right] \xrightarrow{R_3 - 3 \cdot R_2 \to R_3} \left[\begin{array}{rrrrr|r} 3 & 8 & 1 & -1 & 0 & 1 \\ 0 & 1 & -16 & -5 & -2 & 0 \\ 1 & -1 & 48 & 19 & 6 & 1 \\ 4 & -1 & -1 & -1 & 1 & 3 \end{array}\right]$$

THM. *Performing an elementary row operation on an augmented matrix yields an equivalent system.*

IDEA. To solve a given system, use elementary row operations to obtain a rref system.

THM (Gauß-Jordan Elimination). *Every augmented matrix is equivalent to a unique rref augmented matrix. Furthermore, its rref may be obtained via elementary row operations.*

Ex. Find all solutions to the system

$$\begin{array}{rcrcl} x & + & 2y & = & 3 \\ 3x & + & 4y & = & 0 \end{array}$$

Sol. Row reduce

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 0 \end{array}\right] \xrightarrow{R_2-3\cdot R_1\to R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -9 \end{array}\right]$$
$$\xrightarrow{-\frac{1}{2}\cdot R_2\to R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & \frac{9}{2} \end{array}\right]$$
$$\xrightarrow{R_1-2\cdot R_2\to R_1} \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & \frac{9}{2} \end{array}\right]$$

This shows that

$$\operatorname{rref}\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & \frac{9}{2} \end{array}\right]$$

The rref of the system is consistent and has no free variables.

Hence the unique solution is $(x,y)=(-6,9/2)$.

Ex. Find all solutions to

$$\begin{array}{rcrcrcr} & & 4x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 6x_2 & - & 2x_3 & = & 3 \\ 4x_1 & + & 8x_2 & - & 5x_3 & = & 4 \end{array}$$

Sol. Row reduce

$$\left[\begin{array}{rrr|r} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}\cdot R_1 \to R_1} \left[\begin{array}{rrr|r} 1 & 3 & -1 & \frac{3}{2} \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 - 4\cdot R_1 \to R_3} \left[\begin{array}{rrr|r} 1 & 3 & -1 & \frac{3}{2} \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}\cdot R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & 3 & -1 & \frac{3}{2} \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 - 3\cdot R_2 \to R_1} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + 4\cdot R_2 \to R_3} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This shows that

$$\operatorname{rref}\left[\begin{array}{rrr|r} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] = \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rref is consistent with free variable $\{x_3\}$ and dependent variables $\{x_1, x_2\}$. The system has infinitely many solutions given by

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{r} \frac{7}{4}\,x_3 \\ -\frac{1}{4}\,x_3 + \frac{1}{2} \\ x_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ \frac{1}{2} \\ 0 \end{array}\right] + \left[\begin{array}{r} \frac{7}{4} \\ -\frac{1}{4} \\ 1 \end{array}\right] x_3$$

Homogeneous Systems

Def. The system (*) is <u>homogeneous</u> if $b_1 = b_2 = \cdots = b_m = 0$. In a homogeneous system, the solution $x_1 = x_2 = \cdots = x_n = 0$ is the <u>trivial solution</u>. All other solutions are called <u>nontrivial solutions</u>.

Thm (Theorem 1.1 in [1]). *A homogeneous system of m equations and n variables has infinitely many solutions if $m < n$.*

Rank and Nullity

Def. The <u>rank</u> of a consistent system is the number of pivots in its rref. Equivalently, the rank is the number of dependent variables.

Thm. $\textbf{rank} \leq \#\textbf{variables}$ *and* $\textbf{rank} \leq \#\textbf{equations}$

Thm. *A consistent system has*

<u>a unique solution</u> *if* $\textbf{rank} = \#\textbf{variables}$

<u>infinitely many solutions</u> *if* $\textbf{rank} < \#\textbf{variables}$

Def. The <u>nullity</u> of a consistent system is the number of nonpivot columns in its rref. Equivalently, the rank is the number of free variables.

Thm. $\textbf{rank} + \textbf{nullity} = \#\textbf{variables}$

Thm. *A consistent system has*

<u>a unique solution</u> *if* $\textbf{nullity} = 0$

<u>infinitely many solutions</u> *if* $\textbf{nullity} > 0$

Ex. How many solutions will a system with coefficient matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & -1 & -2 \end{bmatrix}$$

have?

Sol. Row reduce

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & -1 & b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 2 & -1 & -1 & -2 & b_3 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 2 & -1 & -1 & -2 & b_3 \end{array}\right]$$

$$\xrightarrow{R_3 - 2\cdot R_1 \to R_3} \left[\begin{array}{cccc|r} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 0 & -1 & -2 & -1 & -b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{cccc|r} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 0 & 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

The system is consistent if and only if $-b_1 + b_2 + b_3 \neq 0$.

Suppose $-b_1 + b_2 + b_3 \neq 0 \neq 0$, so that the system is consistent.

Note that

$$\texttt{rank} = 2 < \#\texttt{variables} = 4$$

so the system has infinitely many solutions.

Alternatively,

$$\texttt{nullity} = 2 > 0$$

so the system has infinitely many solutions.

Thus, any system with coefficient matrix A will be inconsistent if $-b_1 + b_2 + b_3 \neq 0$ and consistent with infinitely many solutions if $-b_1 + b_2 + b_3 = 0$.

References

[1] G.L. Peterson and J.S. Sochacki. Linear Algebra and Differential Equations. Addison-Wesley, 2002.