IS4303 Week 3

Linear Algebra Review

Announcement

- If you have registered the Monday's or Thursday morning's tutorial session, but you want to switch to the **Thursday night's** session (20:30 to 21:30, right after the class, same classroom), please send email to Yunkun (diszhao@nus.edu.sg) as soon as possible.

- Yunkun can help to add you into the **Thursday night's** session.

Agenda

- Review of Linear Algebra
  - http://polisci2.ucsd.edu/dhughes/204bLecture4LinearAlgebra.pdf
  - https://youtu.be/4Pm-htIGVMQ

- Linear Algebra and Data Cleaning in Python
  - Numpy and Pandas

What is a matrix?

- A typical matrix is a rectangular array of numbers arranged in rows and columns.
- Matrix is "sized" using the number of rows (m) by number of columns (n). The order of matrix is (m, n).

$$\underset{3x4}{A} = \begin{bmatrix} 13 & 26 & 30 & 31 \\ 49 & 77 & 61 & 72 \\ 52 & 36 & 44 & 93 \end{bmatrix} \qquad \underset{4x2}{B} = \begin{bmatrix} 17 & 42 \\ 62 & 70 \\ 21 & 43 \\ 57 & 95 \end{bmatrix}$$

Matrices

Mathematically, a matrix is denoted as:

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = {}_m\mathbf{A}^n \quad \begin{bmatrix} a_{11} & a_{12}\cdots & a_{ij} & a_{in} \\ a_{21} & a_{22}\cdots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

Row i: 1 to m

Column j: 1 to n

Special Matrices

- Square matrix: a square matrix is an (m x n) matrix where m = n.

$$\underset{3x3}{S} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Vector: a vector is an (m x n) matrix where m=1 or n=1.

$$\underset{4x1}{\text{ColumnVector}} = \begin{bmatrix} -2 \\ 92 \\ 0 \\ 5 \end{bmatrix} \qquad \underset{1x3}{\text{RowVector}} = \begin{bmatrix} -17 & -44 & 97 \end{bmatrix}$$

Special Matrices

- Scalar: a scalar is an (m x n) matrix where m=1 and n=1.

$$\underset{1x1}{\text{Scalar}} = \begin{bmatrix} 5 \end{bmatrix}$$

- Identity Matrix: a square (m x n) matrix with 1s on the diagonal and 0s on the off-diagonal places.

$$\underset{3x3}{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrices - Equality

Equality: All corresponding elements are equal

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad A = B\ ?$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad A = B\ ?$$

Matrices – Addition/Subtraction

The sum or difference of two matrices, **A** and **B** of the same size/order yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes/orders **cannot** be added or subtracted

Matrices - Operations

Commutative Law:
A + **B** = **B** + **A**

Associative Law:
A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A
2x3

B
2x3

C
2x3

Matrices – Scalar Multiplication

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then **kA = Ak**

Example: If k=4

$$4\times\begin{bmatrix}3 & -1\\2 & 1\\2 & -3\\4 & 1\end{bmatrix}=\begin{bmatrix}3 & -1\\2 & 1\\2 & -3\\4 & 1\end{bmatrix}\times 4=\begin{bmatrix}12 & -4\\8 & 4\\8 & -12\\16 & 4\end{bmatrix}$$

Matrices – Scalar Multiplication

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then **k$\mathbf{A}$ = $\mathbf{A}$k**

Properties:

- k ($\mathbf{A}$ + $\mathbf{B}$) = k$\mathbf{A}$ + k$\mathbf{B}$
- (k + g)$\mathbf{A}$ = k$\mathbf{A}$ + g$\mathbf{A}$
- k($\mathbf{AB}$) = (k$\mathbf{A}$)$\mathbf{B}$ = $\mathbf{A}$(k)$\mathbf{B}$
- k(g$\mathbf{A}$) = (kg)$\mathbf{A}$

Matrices – Matrix Multiplication

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

$$(1\text{x}3) \quad (3\text{x}1) \quad (1\text{x}1)$$

Matrices – Matrix Multiplication

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

B x **A** = Wrong !
(2x1) (4x2)

A x **B** = Wrong !
(6x2) (6x3)

A x **B** = **C**
(2x3) (3x2) (2x2)

Matrices – Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

$$C_{m\times n} = A_{m\times l} B_{l\times n}$$
$$c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}$$
$$i = 1, \ldots, m; \; j = 1, \ldots, n$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

Matrices – Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

IA = **A** and **AI** = **A**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Matrices - Operations

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. AI = IA = A
2. A(BC) = (AB)C = ABC - (associative law)
3. A(B+C) = AB + AC - (first distributive law)
4. (A+B)C = AC + BC - (second distributive law)

Caution!

1. AB not generally equal to BA, BA may not be conformable
2. If AB = 0, neither A nor B necessarily = 0
3. If AB = AC, B not necessarily = C

Matrices - Transposing a Matrix

- **Matrix Transpose** is the (m x n) matrix obtained by interchanging the rows and columns of a matrix (i.e., converting it to an n x m matrix)

$$a_{ij} \rightarrow a_{ji}$$

$$\underset{4x1}{X} = \begin{bmatrix} 12 \\ 9 \\ -4 \\ 0 \end{bmatrix} \qquad \underset{1x4}{X}' = \begin{bmatrix} 12 & 9 & -4 & 0 \end{bmatrix}$$

$$\underset{3x4}{A} = \begin{bmatrix} 21 & 62 & 33 & 93 \\ 44 & 95 & 66 & 13 \\ 77 & 38 & 79 & 33 \end{bmatrix} \qquad \underset{4x3}{A}' = \begin{bmatrix} 21 & 44 & 77 \\ 62 & 95 & 38 \\ 33 & 66 & 79 \\ 93 & 13 & 33 \end{bmatrix}$$

Matrices - Matrix Inverse

- Matrix Inverse: Need to perform the “division” of 2 square matrices
  - In scalar terms A/B is the same as A * 1/B
  - When we want to divide matrix A by matrix B we simply multiply by A by the inverse of B
  - An inverse matrix is defined as

$$\underset{nxn}{A^{-1}} \xrightarrow{Defined} \underset{nxn}{A}\,\underset{nxn}{A^{-1}} = \underset{nxn}{I}$$

$$\text{and } \underset{nxn}{A^{-1}}\,\underset{nxn}{A} = \underset{nxn}{I}$$

Matrices - Matrix Inverse

- For a 2x2 matrix the inverse is relatively simple

$$\underset{2x2}{C} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \qquad \underset{2x2}{C^{-1}} = \frac{1}{|C|}\begin{bmatrix} a_1 & -b_1 \\ -a_2 & b_2 \end{bmatrix}$$

$$\underset{2x2}{C} = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} \qquad |C| = -7$$

$$\underset{2x2}{C^{-1}} = \frac{1}{-7}\begin{bmatrix} 3 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -3/7 & 2/7 \\ 5/7 & -1/7 \end{bmatrix}$$

- Gaussian Elimination (But it is computational costly for high-order matrix)
- Suggestion: Use a computer…

Matrices – More…

- Orthogonal Matrix
- Eigenvalue and Eigenvector
- Linear Transformation
- Matrix Decomposition/Factorization (e.g., SVD for dimension reduction)
- And more…

Thank you!