Howe-Moore Theorem and Moore's Ergodicity Theorem

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Dynamics on Hyperbolic Plane

- We have identified $PSL_2\mathbb{R}$ to $T^1\mathbb{H}^2$
  - Fix the reference vector $(z_0, v_0) = (i, i)$ on $T^1\mathbb{H}$
  - The correspondence

$$PSL_2\mathbb{R} \longleftrightarrow T^1\mathbb{H}^2$$

$$g \longleftrightarrow Dg(z_0, v_0)$$

- The Geodesic Flow on $T^1\mathbb{H}^2$ corresponds to right multiplication $g \mapsto R_{a_t}(g) = ga_t^{-1}$ on $PSL_2\mathbb{R}$

Geodesic Flow on $\Gamma\backslash G$

- Γ is Lattice of Lie Group G if the quotient $X = \Gamma\backslash G$ has finite Haar measure.
- For $G = SL_2\mathbb{R}$, $\Gamma = SL_2\mathbb{Z}$, the geodesic flow on $\Gamma\backslash G$ has a connection to the dynamic of Gauss Map $y \mapsto \left\{\frac{1}{y}\right\}$

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Main Question

- Does a geodesic flow on $\Gamma\backslash G$ is ergodic? Furthermore, does a geodesic flow is Mixing? Mixing of all orders?

Associated Unitary Operators

- For a measure preserving transform of (X, μ), $T : X \to X$, the associated operator $U_T : L^2_\mu \to L^2_\mu$ is unitary

$$U_T(f) = f \circ T$$

  - T is ergodic if and only if 1 is a simple eigenvalue of U_T
  - T is mixing if and only if $\lim_{n\to\infty} \langle U_T^n f, g\rangle = \langle f, 1\rangle \cdot \langle 1, g\rangle$
- Lie group G, lattice Γ and $X = \Gamma\backslash G$ we'll consider

$$\pi : G \curvearrowright L^2_{m_X}(X),\ \pi(g)(f)(x) = f(xg^{-1})$$

Mautner's Phenomenon

- G be a locally compact group. Define for $\alpha = \{a_n\}$

$$S(\alpha) = \left\{ g \in G \mid e \in \overline{\{a_n^{-1} g a_n \mid n \in \mathbb{N}\}} \right\}$$

Theorem (Mautner's Phenomenon)

G be a locally compact group and $(\pi, \mathcal{H}_\pi)$ a strongly continuous unitary representation of G. $\{a_n\}$ be a sequence in G and let $\xi, \xi_0 \in \mathcal{H}_\pi$ such that

$$\lim_{n\to\infty} \pi(a_n)\xi = \xi_0$$

Then

$$\pi(g)\xi_0 = \xi_0$$

for g in the closure of the subgroup generated by $S(\alpha)$; N_α^+

Mautner's Phenomenon

Proof.

$\lim_{n\to\infty} \pi(a_n)\xi = \xi_0$ need to be fixed by $g \in S(\alpha)$
Take a subsequence $\{a_{n_k}\}$ that $a_{n_k}^{-1} g a_{n_k} \to e$. Then by unitarity of π,

$$\begin{aligned} |\ \langle \pi(g)\xi_0, \eta\rangle - \langle \xi_0, \eta\rangle\ | &= \lim_{k\to\infty} |\ \langle \pi(g a_{n_k})\xi, \eta\rangle - \langle \pi(a_{n_k})\xi, \eta\rangle\ | \\ &= \lim_{k\to\infty} |\ \langle \pi(a_{n_k}^{-1} g a_{n_k})\xi, \pi(a_{n_k}^{-1})\eta\rangle - \langle \xi, \pi(a_{n_k}^{-1})\eta\rangle\ | \\ &\leq \lim_{k\to\infty} \|(\pi(a_{n_k}^{-1} g a_{n_k}) - \mathrm{id}_{\mathcal{H}_\pi})\,\xi\| \|\eta\| = 0 \end{aligned}$$

Application : Ergodicity of Geodesic Flow on $\Gamma\backslash SL_2\mathbb{R}$

- Apply Mautner's Phenomenon to

$$\alpha = \{(a_t)^n\}_n = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}^n \right\}_n$$

$$\begin{pmatrix} e^{-nt/2} & 0 \\ 0 & e^{nt/2} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{nt/2} & 0 \\ 0 & e^{-nt/2} \end{pmatrix} = \begin{pmatrix} 1 & se^{-nt} \\ 0 & 1 \end{pmatrix} \to 1$$

- $U^- \subset S(\alpha)$. U^-, U^+ generates $SL_2(\mathbb{R})$.

Ergodicity of Horocycle Flow on $\Gamma\backslash SL_2\mathbb{R}$

- Mautner's Phenomenon for $\alpha = \{(u_s^-)^n\}_n = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}^n \right\}$ gives nothing, because $S(\alpha) = \{\pm I\}$.
- However similar argument can be applied. Consider the function $\varphi(g) = \langle \pi(g)\xi_0, \xi_0 \rangle$. ξ_0 invariant by u_s^-.
- $\varphi(g) = \varphi(u_s^- g u_t^-)$

$$\begin{pmatrix} 1 & \alpha\lambda_n^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\lambda_n^{-1} \\ \lambda_n & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\lambda_n^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \lambda_n & \alpha^{-1} \end{pmatrix}$$

- $\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right) = \lim_{n\to\infty} \varphi\left(\begin{pmatrix} 0 & -\lambda_n^{-1} \\ \lambda_n & 0 \end{pmatrix} \right)$

Howe and Moore's Theorem

Theorem (Howe-Moore Theorem for $SL_2\mathbb{R}$)

Let $\mathcal{H}_\pi$ be a Hilbert space carrying a unitary representation of $SL_2\mathbb{R}$ without any invariant vectors. Then for any $v, w \in \mathcal{H}_\pi$ the matrix coefficients vanish at ∞

$$\langle g_n v, w \rangle \to 0$$

Theorem

Γ be a lattice in $G = SL_2\mathbb{R}$. The action of $SL_2\mathbb{R}$ on $X = \Gamma \backslash G$ is mixing.

Proof

Lemma

$\alpha = \{g_n\}$ be a sequence in $SL_2\mathbb{R}$ converging to ∞ then $S(\alpha)$ contains nontrivial unipotent element.

- $\alpha = \{g_n\}$ that $g_n \to \infty$. For $f \in L^2_{m_X}(X)$, $g_n(f)$ will have convergent subsequence in a weak* sense. $g_{n_k}(f) \to f_0$
- $u \in S(\alpha)$, $u(f_0) = f_0$. By ergodicity f_0 is constant $f_0 = \int_X f dm_X$

$$\int_X f(xg_n^{-1})\bar{h}(x)dm_X = \langle g_n(f), h\rangle \to \langle \int_X f dm_X, h\rangle = \int_X f dm_X \int_X \bar{h} dm_X$$

In general Hilbert space $\mathcal{H}_\pi$, that does not admit any invariant vectors,

$$\langle g_n v, w\rangle \to 0$$

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Structure of Semisimple Lie Groups

- G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$$

  each of $\mathfrak{g}_i$'s are ideals and also are simple Lie algebras.

- Subgroup of G with Lie algebra $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$ be $S_1, \cdots, S_n$ then

$$G = S_1 \cdots S_n$$

  each subgroups are simple, normal, closed and commutes with every elements of other subgroup.

Howe-Moore Vanishing Theorem

Theorem (Howe-Moore Vanishing Theorem)

Let G a connected semisimple Lie group with finite center. $(\pi, \mathcal{H}_\pi)$ a strongly continuous unitary representation of G. Assume that the restriction of π to any non-compact simple factor S_i of G has no non-trivial invariant vector. Then all the matrix coefficients of π vanish at infinity.

Proof

- In $G = KAK$ decomposition, element in A is unique upto Weyl Group
- **Lemma.** Let $(\pi, \mathcal{H}_\pi)$ a strongly continuous unitarty representation. Suppose that all matrix coefficients $\varphi_{\xi\eta}(g) = \langle \pi(g)\xi, \eta \rangle$ vanish at infinity for all sequences $\{a_n\}_n$ in $\overline{A^+} = \overline{\{\exp H \mid H \in \mathfrak{a}^+\}}$ that $\lim_{n\to\infty} a_n = \infty$. Then all matrix coefficients vanish at infinity.
  - Decompose $g_n = k_n a_n h_n$. Then subsequence of $\pi(h_n)\xi$, $\pi(k_n^{-1})\eta$ converges in norm to $\overline{\xi}, \overline{\eta}$.

$$\langle \pi(g_n)\xi, \eta \rangle = \langle \pi(a_n)\pi(h_n)\xi, \pi(k_n^{-1})\eta \rangle \to 0$$

Proof

- Recall. for the sequence $\alpha = \{a_n\}_n$, define $S(\alpha) = \{g \in G \mid e \in \overline{\{a_n^{-1} g a_n \mid n \in \mathbb{N}\}}\}$ and call its closure N_α^+
- $\alpha = \{b^n\}_n$ write $S(b), N_b^+$
- **Lemma.** For every sequence α in $\overline{A^+}$ converging to infinity, there exists $b \in \overline{A^+} \backslash \{e\}$ with $N_\alpha^+ = N_b^+$
- For simple roots R and $\alpha = \{\exp_G(a_n)\}_n$ there exists $\lambda \in R$

$$\limsup_{n \to \infty} \lambda(a_n) = \infty$$

b is defined by $b = \exp_G H$, $H \in \overline{\mathfrak{a}^+}$

$$\lambda(H) = \begin{cases} 1 & \textit{if } \lambda \in R_\alpha \\ 0 & \textit{if } \lambda \notin R_\alpha \end{cases}$$

Proof

- $\xi_0 = \lim_{n\to\infty} \pi(a_n)\xi$ is invariant under $N_\alpha^+ = N_b^+$. (Mautner's Phenomenon)
- $\forall \lambda \in \Sigma^+$ that $\lambda(H) > 0$, $X \in \mathfrak{g}^\lambda \backslash \{0\}$ then $X \in N_b^+$.
- $\{X, \theta(X), H_\lambda\}$ generates the copy of $\mathfrak{sl}_2\mathbb{R}$, $\mathfrak{g}_X$. Corresponding subgroup G_X is locally isomorphic to $SL(2,\mathbb{R})$
- ξ_0 is invariant under G_X and hence $N_{b^{-1}}^+$
- **Lemma.** For $b \in A\backslash\{e\}$, let G_b be the closed subgroup generated by N_b^+ and $N_{b^{-1}}^+$. Then G_b is a non-discrete normal subgroup of G

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Moore's Ergodicity Theorem

Theorem (Moore's Ergodicity Theorem)

G be a semisimple Lie group with finite center. G act on a probability space (X, μ) and assume that the restriction of this action to any simple non-compact factor of G is ergodic. H be a subgroup of G with a non-compact closure. Then the action of H on X is strongly mixing.

- Application to Homogeneous space : If G a simple Lie group with finite center, Γ a lattice and H a non-compact closed subgroup then H acts ergodically on $\Gamma \backslash G$
- Also Moore's Duality theorem gives Γ acts ergodically on G/H
  - $G = SL_n(\mathbb{R})$, $H = Stab((1, 0, \cdots, 0)^t)$, $\Gamma = SL_n(\mathbb{Z})$. Γ acts ergodically on $G/H = \mathbb{R}^n \backslash \{0\}$

Applications and Extensions

- G/K be a Riemannian symmetric space of rank 1. Γ be a lattice in G then the geodesic flow of $\Gamma\backslash G/K$ is ergodic, strongly mixing.
  - Geodesic flow of any Riemannian manifold of constant negative curvature and finite volume is ergodic, strongly mixing.
- p-adic extensions:
  - Howe-Moore theorem on $G = SL_n(\mathbb{Q}_p)$
- **Mozes.** Let G be a connected semisimple Lie group with finite center and no compact factors. If G acts on Lebesgue probability measure space (X, μ) in a continuous, measure-preserving way and ergodic in each non-central normal subgroup, the action is mixing of all orders

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