Exploration to Modern Geometry

Donghyun Park

February 15, 2026

This note is based on SNU 2025 Fall course: "Exploration to Modern Geometry".

1 Topology from the Differential Viewpoint

Topology from the Differentiable Viewpoint by John W. Milnor analyzes **topological aspects driven from purly differentiable objects of smooth manifolds**.

1.1 Proof of Sard's Theorem

One of my favorite part in this book is the proof of Sard's Theorem. It illustrates how Sard's Theorem can be proved essentially by the **'smoothness'** condition. There are three steps:
Define C a set of critical points of $f : U \subset \mathbb{R}^n \to \mathbb{R}^p$ and C_i the set of points whose all partial derivatives of order less or equal than i vanishing.
Step 1. $f(C - C_1)$ is measure zero
Step 2. $f(C_i - C_{i+1})$ is measure zero
Step 3. $f(C_k)$ is measure zero for large k
Roughly speaking the idea, Step 1, 2 uses change of coordinate charts. Step 3 uses Taylor's theorem so that for large amount of differentiation, the partitioning codomain becomes much larger than possible deviation of functions.

1.2 Brouwer degree

Chapter 4 and 5 are devoted to the **Brouwer degree**. This concept is the first one that appears by the geometric (local) definition respects topological feature.
Degree of the map $f : M \to N$ is defined by the sum of the sign of differential of f at regular value's preimage.

$$\deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{sgn} df_x$$

But the degree is in fact, does not depend on the choice of regular value and moreover, **invariant under homotopies!**

1.3 The Poincare Hopf Theorem

Next geometric object is **vector fields**. **Index** defined on zeros of vector fields which demonstrates local behavior around the zero, also respects the topology of manifold. The beautiful statement is;

$$\sum \iota = \chi(M)$$

2 Seifert Manifold

2.1 Principal Bundle

Let E, M, F be smooth manifolds. We call them total space, base space, fiber respectively. A fiber bundle with fiber F is a surjection $\pi : E \xrightarrow{C^\infty} M$ that have a local trivialization with fiber F. This means, there is a chart $\{(U_\alpha, \phi_\alpha)\}$ such that following diagram commutes.

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\text{proj}"] \\
& U_\alpha &
\end{tikzcd}$$

The most common example is when $F = \mathbb{R}^k$, we call it a vector bundle. Another example is Tautological bundle,

$$\mathcal{O}(-1) = \{([z], w) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | w \in \mathbb{C}z\}$$

which is in diagram,

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & \mathcal{O}(-1) \arrow[d] \\
& \mathbb{CP}^n
\end{tikzcd}$$

We call the fiber bundle is a **principal G-bundle** if for G a Lie group acts on P freely and local trivialization is G-equivariant.
- G acts on P freely : $g.x = x$ for some $x \in P$ then $g = e$
- $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is G-equivariant: $\phi_\alpha(g.p) = g.\phi_\alpha(p)$.

$$\begin{tikzcd}
G \arrow[r] & P \arrow[d, "\pi"] \\
& M
\end{tikzcd}$$

Most important example is the **Hopf bundle**. S^1 acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by $e^{i\theta}.(z_0, \cdots, z_n) = (e^{i\theta}z_1, \cdots, e^{i\theta}z_n)$

$$\begin{tikzcd}
S^1 \arrow[r] & S^{2n+1} \arrow[d] \\
& \mathbb{CP}^n
\end{tikzcd}$$

Or, more generally in Homogeneous space, such as G be a Lie group and $H \subset G$ a closed subgroup,

$$\begin{tikzcd}
H \arrow[r] & G \arrow[d] \\
& G/H
\end{tikzcd}$$

2.2 Euler Number

Let P be a principal S^1 bundle over a closed orientable surface Σ.

$$\begin{tikzcd}
S^1 \arrow[r] & P \arrow[d] \\
& \Sigma
\end{tikzcd}$$

Split $\Sigma = \Sigma_1 \cup \Sigma_2$, then it is known that every principal S^1 bundle over a surface with boundary is trivial. Thus,

$$\begin{tikzcd}
P|_{\Sigma_1} \arrow[r, "\cong"] \arrow[d] & \Sigma_1 \times S^1 \arrow[d] \\
\Sigma_1 \arrow[r, "id"] & \Sigma_1
\end{tikzcd}
\qquad
\begin{tikzcd}
P|_{\Sigma_2} \arrow[r, "\cong"] \arrow[d] & \Sigma_2 \times S^1 \arrow[d] \\
\Sigma_2 \arrow[r, "id"] & \Sigma_2
\end{tikzcd}$$

And the clutching map between $\partial\Sigma_1 = \partial\Sigma_2 \cong S^1$, $h : \partial\Sigma_1 \times S^1 \to \partial\Sigma_2 \times S^1$ sending (q, θ) to $(q, f(q)\theta)$ for some $f : \partial\Sigma_1 \to S^1$ exists. Now, define the Euler number $e(P \to \Sigma) = \deg f$.

2.2.1 Hopf fiber bundle has Euler number -1

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

We parametrize upper half hemisphere and lower half hemisphere by $[1, \xi]$ and $[\xi' : 1]$ respectively. Then each trivialized hemisphere with boundary is parametrized as

$$\left(\frac{t}{1+|\xi|^2}, \frac{t\xi}{1+|\xi|^2}\right), \left(\frac{s\xi'}{1+|\xi'|^2}, \frac{s}{1+|\xi'|^2}\right)$$

respectively.
Now, in clutching $([1:\xi], t)$ of $\Sigma_1 \times S^1$ with $([\xi' : 1], s)$, we attach $\xi = \xi'^{-1}$ so

$$\frac{t}{1+|\xi|^2} = \frac{s\xi^{-1}}{1+|\xi^{-1}|^2}$$

solving this, $t = s\frac{|\xi|^2}{\xi}$ so the mapping

$$S^1 \to S^1 : \xi \mapsto \frac{|\xi|^2}{\xi} = \xi^{-1}$$

has degree -1, which demonstrates why the Euler number is -1.

2.3 Correspondence of Principal S^1 bundle and Complex line bundle

Complex line bundle mean the complex vector bundle of rank 1.

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & E \\ & & \downarrow \\ & & \Sigma \end{array}$$

The correspondence between complex line bundle and principal S^1 bundle is,

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & E \\ & & \downarrow \\ & & \Sigma \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & P \\ & & \downarrow \\ & & \Sigma \end{array}$$

letting $P = \{(x, z) \in E|\ |z| = 1\}$ and S^1 action by $t.(x, e) = (x, et)$ and in other direction, $E = P \times_{S^1} \mathbb{C} = \{(p, z) \in P \times \mathbb{C}\}/(t.p, z) \sim (p, zt)$.

2.3.1 Hopf bundle and Tautological line bundle

Recall that Tautological line bundle is defined by

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathcal{O}(-1) \\ & & \downarrow \\ & & \mathbb{CP}^1 \end{array}$$

On the other hand, Hopf fiber bundle was

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

those two are in correspondence.

2.4 First Chern Number

First Chern number is defined in complex line bundle, $\Sigma = \Sigma_1 \cup \Sigma_2$ as before and in the complex line bundle,

$$\begin{array}{ccc} E|_{\Sigma_1} & \xrightarrow{\cong} & \Sigma_1 \times \mathbb{C} \\ \downarrow & & \downarrow \\ \Sigma_1 & \xrightarrow{id} & \Sigma_1 \end{array} \qquad \begin{array}{ccc} E|_{\Sigma_2} & \xrightarrow{\cong} & \Sigma_2 \times \mathbb{C} \\ \downarrow & & \downarrow \\ \Sigma_2 & \xrightarrow{id} & \Sigma_2 \end{array}$$

the clutching map $h : \partial\Sigma_1 \times \mathbb{C} \to \partial\Sigma_2\mathbb{C}$ mapping $(q, z) \mapsto (q, f(q) \cdot z)$ associates with $f : S^1 \to U(1)$. The first Chern number is $c_1(E \to \Sigma) = \deg f$.
In the correspondence of Principal S^1 bundle and Complex line bundle, Euler number and First Chern number is the same.

3 Seifert Manifold

Seifert Manifold is obtained by attatching manifold into another. We first establish theorem for pasting two manifolds. For M_1, M_2 smooth manifolds, assume there exists connected components $N_1 \subset \partial M_1$, $N_2 \subset \partial M_2$ and diffeomorphism $h : N_1 \xrightarrow{\cong} N_2$.
We can obtain $M = M_1 \cup_h M_2 = M_1 \cup M_2 / x \sim h(x)$.

Theorem 1. *Following holds.*
(a) M can be given a smooth structure that M_1 and M_2 are smooth submanifolds.
(b) Such a smooth structure on M is unique up to small isotopies.
(c) Let $h, h' : N_1 \to N_2$ be diffeomorphism. If $h'^{-1} \circ h : N_1 \to N_1$ can be extended to a diffeomorphism $H : M_1 \to M_1$ then $M_1 \cup_h M_2 \cong M_1 \cup_{h'} M_2$.
(d) Let $F_1 : M_1 \to M_1, F_2 : M_2 \to M_2$ be diffoemorphisms such that $f_1 = F_1|_{N_1} : N_1 \to N_1$ and $f_2 = F_2|_{N_2} : N_2 \to N_2$. Let $h' = f_2 \circ h \circ f_1 : N_1 \xrightarrow{\cong} N_2$ then $M_1 \cup_h M_2 \cong M_1 \cup_{h'} M_2$.
(e) If $h, h' : N_1 \xrightarrow{\cong} N_2$ are isotopic then $M_1 \cup_h M_2 \cong M_1 \cup_{h'} M_2$.

Gluing two manifolds are defined by: $f_1 : D^n \hookrightarrow \mathrm{int}M_1$ and $f_2 : D^n \hookrightarrow \mathrm{int}M_2$ are embeddings,

$$M_1 \# M_2 = (M_1 - \mathrm{int}f_1(D^n)) \cup_{f_2 \circ f_1^{-1}|_{f_1(\partial D^n)}} (M_2 - \mathrm{int}f_2(D^n))$$

If M_1 and M_2 are oriented and f_1, f_2 are orientation preserving, reversing respectively then orientation of M is well determined.
We are interested in attatching solid torus in 3-manifolds. Thus, it is important to determine **how many different ways to attach torus**.
Mapping Class Group of manifold M is isotopy classes of diffeomorphisms of M. We notate by $\mathrm{MCG}(M)$. The diffeomorphism preserving orientation is defined by $\mathrm{MCG}^+(M)$. Then,

$$\mathrm{MCG}(\mathbb{T}^2) \cong GL(2, \mathbb{Z})$$

$$\mathrm{MCG}^+(\mathbb{T}^2) \cong SL(2, \mathbb{Z})$$

Now, we define **Dehn Surgery**. For 3-dimensional smooth manifold M^3 such that $\mathbb{T}^2 \subset \partial M^3$, let c be a simple closed curve in $\mathbb{T}^2$. $h : \mathbb{T}^2 \to \partial D^2 \times S^1 \cong \mathbb{T}^2$ a diffeomrophism sending c to a merdian of $D^2 \times S^1$, which is $\partial D^2 \times \{pt\}$.
Define $M(c) = M \cup_h (D^2 \times S^1)$. This operation is called the Dehn surgery.

By our theorem and knowledge about Mapping Class Group,

Lemma 1. *Let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomrophism. Then following statements are equivalent.*
(a) h can be extended to a diffeomorphism $\bar{h} : D^2 \times S^1 \to D^2 \times S^1$.
(b) h sends a meridian to a meridian.
(c) $h_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ is expressed as

$$\begin{pmatrix} \pm 1 & c \\ 0 & \pm 1 \end{pmatrix}$$

.

Thus, $M(c)$ is well defined up to diffeomorphism. Let $h, h' : \mathbb{T}^2 \to \partial D^2 \times S^1$ sending c to meridian, $h' \circ h^{-1} : \partial D^2 \times S^1 \to \partial D^2 \times S^1$ sending meridian to meridian, then by lemma above, $h' \circ h^{-1}$ extends to $h : D^2 \times S^1 \to D^2 \times S^1$. By Theorem (c), $M \cup_h (D^2 \times S^1) \cong M \cup_{id} (D^2 \times S^1)$.

3.1 Seifert Fibration

Let M be an oriented 3-manifold, F a compact suface, $\pi : M \xrightarrow{C^\infty} F$ a surjection such that for every $x \in F$, some neighborhood $U \cong D^2$ of x such that
- $\pi^{-1}(D^2) \cong D^2 \times S^1$
- $\pi : D^2 \times S^1 \to D^2$, $(re^{it_1}, e^{it_2} \mapsto re^{i(pt_1+qt_2)}$ for some $p \neq 0, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$.

This triple (M, F, π) is called a **Seifert fibration**. The center of the disk D^2 and others behave differently. We call them exceptional fiber and typical fiber respectively. A typical fiber $\pi^{-1}(re^{is})$ can be parametrized as $(re^{i(s/p+qt)}, e^{-ipt})$ which winds longitude $-q$ times and meridian p times.

[figure]

Two Seifert fibrations (M, F, π) and (M', F', π') are isomorphic iff there exists orientation preserving diffeomorphism $f, \tilde{f}$ such that

$$\begin{tikzcd} M \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & M' \arrow[d, "\pi"] \\ F \arrow[r, "f"] & F' \end{tikzcd}$$

3.2 Seifert Manifold

Seifert manifold is parametrized by $(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n))$. $g, \alpha_i, \beta_i \in \mathbb{Z}$, $g \geq 0$ and $\alpha_i > 0$, $\gcd(\alpha_i, \beta_i) = 1$.
Take F a closed, oriented, connected surface of genus g. Next, remove n disks from F, $F_0 = F - (D_1^2 \cup \cdots \cup D_n^2)$. $M_0 = F_0 \times S^1$ then $\partial M_0 = \cup_{i=1}^n S_i^1 \times S^1$. Let $R = F_0 \times \{1\}$, the union of meridians.
Now, we attach solid torus into each holes. Define $Q_i = S_i^1 \times \{1\}$, $H_i = \{1\} \times S^1$, $T_i = D^2 \times S^1$, $M_i = \partial D^2 \times \{1\}$, $L_i = \{1\} \times S^1$.
Then, by definition

$$H_1(S_i^1 \times S^1) \cong \mathbb{Z}\langle Q_i, H_i \rangle,\ H_1(\partial T_i) \cong \mathbb{Z}\langle M_i, L_i \rangle$$

Glue with orientation preserving diffeomorphism, $h_i : \partial T_i \to S_i^1 \times S^1$ by $M_i \mapsto \alpha_i Q_i + \beta_i H_i$. Resulting manifold is Seifert manifold.

$$M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)) = M_0 \cup_h \big(\bigcup_{i=1}^n T_i\big)$$

[figure]

Now, we illustrate the fundamental theorem connecting Seifert Fibration and Seifert Manifold.

Theorem 2. *Following statements hold.*
(a) For any Seifert fibration (M, F, π) realizes as $M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n))$.
(b) $M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)) \cong M(g'; (\alpha'_1, \beta'_1), \cdots, (\alpha'_n, \beta'_n))$ if and only if $g = g'$, $\beta_i/\alpha_i = \beta'_i/\alpha'_i \pmod 1$ after permuting, and $\sum_{i=1}^n \beta_i/\alpha_i = \sum_{i=1}^n \beta'_i/\alpha'_i$.
(c) $M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)) \cong M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n), (1, 0))$

Thus, the operation : add or delete $(\alpha, \beta) = (1, 0)$ or replaceing (α_i, β_i) by $(\alpha_i, \beta_i + K_i\alpha_i)$ with $\sum K_i = 0$ does not change Seifert fibration.

Proof. First, we show (a). Let $P_1, \cdots, P_n \in F$ such that $\pi^{-1}(P_1), \cdots, \pi^{-1}(P_n)$ are all exceptional fibers. $D_1, \cdots, D_n \subset F$ is disjoint disk neighborhoods of P_i. Let $T_i = \pi^{-1}(D_i)$, $M_0 = M - \cup_{i=1}^n \mathrm{int} T_i$, $F_0 = F - \cup_{i=1}^n \mathrm{int} D_i$. Then $(M_0, F_0, \pi|_{M_0})$ is trivial S^1 bundle.
Choose one section $s : F_0 \to M_0$ of $\pi|_{M_0}$. Define $Q_i = s(F_0) \cap \partial T_i$, H_i be any fiber of $\partial T_i \xrightarrow{\pi} \partial D_i$. If l_i generates $H_1(T_i)$ then for α_i, β_i that $H_i = \alpha_i l_i, Q_i = -\beta_i l_i$,

$$(M, F, \pi) \cong M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)).$$

In (b), the equivalence is equipped by choosing different section. $s : F_0 \to M_0$ sends $x \in F_0$ to $(x, \phi(x)) \in F_0 \times S^1 \cong M_0$. Assume two sections s, s' exists and associated $\phi, \phi' : F_0 \to S^1$ exists. They can differ in $\partial F_0 = \sqcup \partial D_i$ by $(q_1, \cdots, q_n) \in \mathbb{Z}^n$ where $q_1 + \cdots + q_n = 0$. So $Q'_i = Q_i + q_i H_i$, $H'_i = H_i$ so $\alpha'_i = \alpha_i$ and $\beta'_i = \beta_i - q_i\alpha_i$.
Finally, $(1, 0)$ corresponds to a trivial bundle which demonstrates (c). ∎

The euler number of the Seifert fibration (M, F, π) is

$$e(M \xrightarrow{\pi} F) = -\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}.$$

Also, by the construction of Seifert manfold, we can easily compute fundamental group. First, we have

$$\pi_1(F - \bigcup_{i=1}^{n} D_i^2) = \langle a_1, \cdots, a_g, b_1, \cdots, b_g, q_1, \cdots, q_n \mid q_1 \cdots q_n [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle.$$

Thus

$$\pi_1\Big((F - \bigcup_{i=1}^{n} D_i^2) \times S^1\Big) = \langle a_1, \cdots, a_g, b_1, \cdots, b_g, q_1, \cdots, q_n, h \mid \prod q_j \prod [a_i, b_i] = 1, [h, a_i] = [h, b_i] = [h, q_j] = 1 \rangle.$$

Finally, Pasting T_i gives additional equality: $q_j^{\alpha_j} h^{\beta_j} = 1$.

$$\pi_1(M) = \langle a_i, b_i, q_j, h | \prod q_j \prod [a_i, b_i] = 1, [h, a_i] = [h, b_i] = [h, q_j] = 1, q_j^{\alpha_j} h^{\beta_j} = 1 \rangle$$

3.3 Len's Space as Seifert Manifold

$\gcd(p, q) = 1$ and let $p, q \in \mathbb{N}$. The $\mathbb{Z}_p$ action on S^3 is given by

$$(e^{2\pi i r/p}, (z_1, z_2)) \mapsto (e^{2\pi i r/p} z_1, e^{2\pi i r q/p} z_2).$$

Resulting quotient space is lens space. $L(p, q) = S^3/\mathbb{Z}_p$.

Theorem 3. *$L(p, q) \cong (D^2 \times S^1) \cup_h (D^2 \times S^1)$ where $h_* = \begin{pmatrix} -q & r \\ p & s \end{pmatrix}$, which $r, s \in \mathbb{Z}$ satisfies $\det h_* = -1$.*

Proof. Let $S^3 = \{|z_1|^2 + |z_2|^2 = 2\} = \{(z_1, z_2) \mid |z_2| \geq 1\} \cup \{(z_1, z_2) \mid |z_2| \leq 1\}$. The attatching map is $h_* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
The $\mathbb{Z}_p$ action acts on each $(D^2 \times S^1)_1, (D^2 \times S^1)_2$.
$(D^2 \times S^1)_1/\mathbb{Z}_p \cong D^2 \times S^1$ by $[z_1, z_2] \mapsto (z_2^{-s} z_1, z_2^p)$, $qs \equiv 1 (mod\, p)$ and similarly, $(D^2 \times S^1)_2/\mathbb{Z}_p \cong D^2 \times S^1$ by $[z_1, z_2] \mapsto (z_1^{-q} z_2, z_1^p)$.
Now attatching is $h : S^1 \times S^1 \to S^1 \times S^1$ where $(z_2^{-s} z_1, z_2^p) \mapsto (z_1^{-q} z_2, z_1^p)$. Resulting h is expressed as $\begin{pmatrix} -q & r \\ p & s \end{pmatrix}$. ☐

Theorem 4. *$L(p, q) \cong M(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$ where α_i, β_i satisfies*

$$p = \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\beta_1 & \beta_2 \end{pmatrix}, \; q = \det \begin{pmatrix} \alpha_1 & \alpha_2' \\ -\beta_1 & \beta_2' \end{pmatrix}, \; \det \begin{pmatrix} \alpha_2 & \alpha_2' \\ \beta_2 & \beta_2' \end{pmatrix} = 1.$$

Proof. By proceeding theorem and think as two $D^2 \times S^1$ attatched in intermediate cylinder. ☐

3.4 Other example of Seifert manifold

The following space is also an example of Seifert manifold. First, let

$$\Sigma(a_1, a_2, a_3) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\} \cap S^5$$

for $a_1, a_2, a_3 \geq 2$. There exists S^1 action on $\Sigma(a_1, a_2, a_3)$ which is for $a = \operatorname{lcm}(a_1, a_2, a_3)$, $t.(z_1, z_2, z_3) = (t^{a/a_1} z_1, t^{a/a_2} z_2, t^{a/a_3} z_3)$. The quotient space $(\Sigma(a_1, a_2, a_3), \Sigma(a_1, a_2, a_3)/S^1)$ has Seifert fibration structure.

Theorem 5. *Assume $\gcd(a_1, a_2) = \gcd(a_2, a_3) = \gcd(a_3, a_1) = 1$ then*

$$\Sigma(a_1, a_2, a_3)/S^1 \cong M(0; (a_1, b_1), (a_2, b_2), (a_3, b_3))$$

where b_1, b_2, b_3 satisfies

$$e(\Sigma(a_1, a_2, a_3) \to \Sigma(a_1, a_2, a_3)/S^1) = -\frac{1}{a_1 a_2 a_3}$$