p-adic Numbers 강의록

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January 12, 2026

1 4강. Analysis on p-adic numbers 1

1.1 Sequences and Series

Lemma 1 ([Gou20] Lemma 5.1.1). *A sequence (a_n) in $\mathbb{Q}_p$ is a Cauchy sequence (or Convergent) if and only if*

$$\lim_{n\to\infty} |a_{n+1} - a_n| = 0$$

Example. $a_1 = (1+p)$, $a_n = (a_{n-1})^p$ 로 정의된 수열
$|a_1 - 1| < 1$

$$(1+p)^p = 1 + p^2 + \binom{p}{2} p^2 + \cdots + p^p \equiv 1 (mod\ p^2)$$

$|a_2 - 1| < 1/p$
Now, repeating $|a_n - 1| \le p^{-n}$. a_n converges to 1 as $n \to \infty$
Example. $a_n = \{n!\}$ converges to 0
Example. $a_n = n$ nonconverges.

Corollary 1 ([Gou20] Corollary 5.1.2). *An infinite series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ is convergent if and only if $\lim_{n\to\infty} a_n = 0$. Also*

$$\left| \sum_{n=0}^{\infty} a_n \right| \le \max_n |a_n|$$

Remark 1. *실수에서 봤던 것과는 꽤 다르다... 실수에서는 그저 0으로 converge 하는 것은... 안됐었는데?*

Problem 1.1. *Sum of a convergent series in $\mathbb{Q}_p$ does not change when we reorder the terms.*

다음으로, double sequence를 보자. 실수에서는 꽤나 까다로운 문제였다.

Remark 2. *$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ 가 성립하는 두 가지 경우가 있었습니다.*
- $a_{ij} \ge 0$
- Absolutely converging: $\sum_i |a_{ij}|$ 로 구성된 sequence의 합이 수렴한다...

성립 안하는 예시로

$$a_{mn} = \begin{cases} 0 & m < n \\ -1 & m = n \\ 2^{n-m} & m > n \end{cases}$$

$$\sum_m \sum_n a_{mn} = \sum_m -\frac{1}{2^{m-1}} = -2$$

$$\sum_n \sum_m a_{mn} = \sum_n 0 = 0$$

다음을 정의하자. $\lim_{i\to\infty} b_{ij} = 0$ uniformly in j를 $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}$ that $i \ge N$ then $|b_{ij}| < \epsilon$

Lemma 2 ([Gou20] Lemma 5.1.3). *Let $b_{ij} \in \mathbb{Q}_p$, FSAE*
(a) For every i, $\lim_{j\to\infty} b_{ij} = 0$ and $\lim_{i\to\infty} b_{ij} = 0$ uniformly in j.
(b) Given any $\epsilon > 0$, $\exists N$ depending only on $\epsilon > 0$,

$$\max(i,j) \ge N \Rightarrow |b_{ij}| < \epsilon$$

(c) $\lim_{i\to\infty} b_{ij} = 0$ uniformly in j and $\lim_{j\to\infty} b_{ij} = 0$ uniformly in i.

Proof. (b) then (c), (c) then (a).
(a) then (b): uniformly in j allows us to choose N_0 depending on ϵ but not on j, $|b_{ij}| < \epsilon$ if $i \ge N_0$.
We can find $N_1(i)$ that if $j \ge N_1(i)$, $|b_{ij}| < \epsilon$

$$N = \max(N_0, N_1(0), \cdots, N_1(N_0 - 1))$$

if $\max(i,j) \ge N$ then either $i \ge N_0$ or $i < N_0$ and $j \ge N$. □

Proposition 1 ([Gou20] Proposition 5.1.4). *$b_{ij} \in \mathbb{Q}_p$ and suppose*
(a) For every i, $\lim_{j\to\infty} b_{ij} = 0$
(b) $\lim_{i\to\infty} b_{ij} = 0$ uniformly in j
Then both series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}, \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} b_{ij}$$

converges and their sums are equal.

Proof. From the lemma, choose N, if $\max(i,j) \ge N$ then $|b_{ij}| < \epsilon$. The $\sum_{j=0}^{\infty} b_{ij}$, $\sum_{i=0}^{\infty} b_{ij}$ converges for all i, j respectively.

$$\left| \sum_{j=0}^{\infty} b_{ij} \right| \le \max_j |b_{ij}| < \epsilon$$

for $i \ge N$ and similar result for j.
thus

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} b_{ij} = 0$$

$$\lim_{j\to\infty} \sum_{i=0}^{\infty} b_{ij} = 0$$

both double series converge.
Their sums are equal?

$$\left| \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} - \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} \right| = \left| \sum_{i=0}^{N} \sum_{j=N+1}^{\infty} b_{ij} + \sum_{i=N+1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \right|$$

By the ultrametric inequality, for all i, $|\sum_{j=N+1}^{\infty} b_{ij}| < \epsilon$ for $j \ge N+1$. Ultrametric inequality,

$$\left| \sum_{i=0}^{N} \sum_{j=N+1}^{\infty} b_{ij} \right| < \epsilon$$

On the other hand, also holds

$$\left| \sum_{i=N+1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \right| < \epsilon$$

So

$$\left| \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} - \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} \right| < \epsilon$$

reversing i, j gives

$$\left| \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} - \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} b_{ij} \right| < \epsilon$$

□

Problem 1.2 ([Gou20] Problem 150). *Show that if $a = \sum a_n$, $b = \sum b_n$ convergent then*

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

the series $\sum c_n$ is convergent with sum ab.
(The same condition in real number was false!)

1.2 Functions, Continuity, Derivatives

무엇이 같고 무엇이 다른가. Interval이라는 개념이 없다...(연결성에서 문제가 많다) 하지만 open set는 있다.
Definition. $U \subset \mathbb{Q}_p$, $f : U \to \mathbb{Q}_p$ is continuous at $a \in U$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x \in U$

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

A function is uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in U$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Problem 1.3. *If $f : U \to \mathbb{Q}_p$ is continuous and U compact, show that f is uniformly continuous.*

Definition. $U \subset \mathbb{Q}_p$ an open set, $f : U \to \mathbb{Q}_p$ a function. f is differentiable at $x \in U$ if

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.
Example([Gou20] Problem 153) Consider the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ that maps

$$x = a_0 + a_1 p + \cdots$$

to

$$f(x) = a_0 + a_1 p^2 + \cdots$$

f is clearly injective but $f'(x) = 0$ for all x.

Remark 3. *This is one of the reasons why the derivative seems to play such a minor role in p-adic analysis... [Gou20] p.117*

More precise analysis: in the real case, Mean Value Thoerem (MVT). f differentiable in (a, b) continuous in $[a, b]$ then there exists $\xi \in (a, b)$ that

$$f(b) - f(a) = f'(\xi)(b - a)$$

But p-adic case breaks down because no 'interval' exists....
그러면 비슷한 형식의 이런 정리가 성립할까?
$f(X)$ differentiable with derivative on $U \subset \mathbb{Q}_p$ and if $|f'(x)| \leq M$ for all $x \in U$ then for any two numbers $a, b \in U$

$$|f(b) - f(a)| \leq M|b - a|$$

Counterexample: $U = \mathbb{Z}_p$, $f(x) = x^p$, $a = 1$, $b = 0$. $f'(x) = px^{p-1}$...

MVT가 성립 안하니 derivative 가 0이라도 locally constant 가 안된다...

1.3 Power Series

$f(X) = \sum_{n=0}^{\infty} a_n X^n$
In the real setting, region of convergence...

Proposition 2 ([Gou20] Proposition 5.4.1). *Define*

$$\rho = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|a_n|}}$$

so that $0 \leq \rho \leq \infty$
(a) If $\rho = 0$ then $f(x)$ converges only when $x = 0$
(b) If $\rho = \infty$ then $f(x)$ converges for every $\mathbb{Q}_p$
(c) If $0 < \rho < \infty$ and $\lim_{n\to\infty} |a_n|\rho^n = 0$ then $f(x)$ converges if and only if $|x| \leq \rho$
(d) If $0 < \rho < \infty$ and $\lim_{n\to\infty} |a_n|\rho^n$ do not converges to zero then $f(x)$ converges if and only if $|x| < \rho$

Proof. If $|x| > \rho$, $|a_n||x|^n$ cannot tend to zero.
If $|x| < \rho$ then $|x| < \rho_1 < \rho$. All but finitely many n, $|a_n| < 1/\rho_1^n$ so $|a_n x^n| \leq |x|^n/\rho_1^n \to 0$
When $|x| = \rho$, Exactly Corollary 1,
An infinite series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ is convergent if and only if $\lim_{n\to\infty} a_n = 0$. Also

$$\left| \sum_{n=0}^{\infty} a_n \right| \leq \max_n |a_n|$$

□

Define $f(X) = \sum_{n=0}^{\infty} a_n X^n$ and $g(X) = \sum_{n=0}^{\infty} b_n X^n$ the

$$(f + g)(X) = \sum_{n=0}^{\infty} (a_n + b_n) X^n$$

$$(fg)(X) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n$$

Proposition 3 ([Gou20] Proposition 5.4.2). *Let $f(X), g(X)$ be a formal power series, and suppose $x \in \mathbb{Q}_p$. If both $f(X), g(X)$ converge, then*
(a) $(f + g)(x)$ converges and is equal to $f(x) + g(x)$
(b) $(fg)(x)$ converges and equal to $f(x)g(x)$
Thus, the radii of convergence of $f + g$ and fg are greater or equal then the smaller radii of convergence of f and g.

Proof. Problem 1.2에 의해 (b)가 성립... (a)는 더 쉽다! □

다음으로 합성함수를 해봅시다...
$f(X) = \sum_{n=0}^{\infty} a_n X^n$ and $g(X) = \sum_{n=0}^{\infty} b_n X^n$, suppose $b_0 = 0$. $h(X) = f(g(X))$를 어떻게 정의할까요? coefficient-wise! (가장 자연스러운 정의)

Generally, 이렇게 정의된 formal power series의 radii of convergence는 찾기 어려움.

Theorem 1 ([Gou20] Theorem 5.4.3)**.** *Let $f(X) = \sum_{n=0}^{\infty} a_n X^n$ and $g(X) = \sum_{n=0}^{\infty} b_n X^n$ with $g(0) = 0$ and $h(X)$ be their formal composition. Suppose that*
(a) $g(x)$ converges
(b) $f(g(x))$ converges (plugging $g(x)$ into $f(X)$ converges)
(c) For every n, we have $|b_n x^n| \le |g(x)|$
Then $h(x)$ also converges and $f(g(x)) = h(x)$

Proof. $g(X)^m = \sum_{n=m}^{\infty} d_{m,n} X^n$ where

$$d_{m,n} = \sum_{i_1+i_2+\cdots+i_m=n} b_{i_1} \cdots b_{i_m}$$

so write $h(X)$:

$$h(X) = a_0 + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} a_m d_{m,n} \right) X^n$$

Since $g(x)$ converges, plugging $X = x$ to $g(X)^m$ gives

$$g(x)^m = \sum_{n=m}^{\infty} d_{m,n} x^n$$

Also, for every n we have

$$|d_{m,n} x^n| \le |g(x)^m|$$

because ultrametric inequality (for $n \ge m$)

$$\begin{aligned} |d_{m,n} x^n| &= \left| \sum_{i_1+i_2+\cdots+i_m=n} b_{i_1} x^{i_1} \cdots b_{i_m} x^{i_m} \right| \\ &\le \max(|b_{i_1} x^{i_1}| \cdot |b_{i_2} x^{i_2}| \cdots |b_{i_m} x^{i_m}|) \le |g(x)^m| \end{aligned}$$

Now, we have

$$f(g(x)) = a_0 + \sum_{m=1}^{\infty} a_m g(x)^m = a_0 + \sum_{m=1}^{\infty} a_m \left(\sum_{n=m}^{\infty} d_{m,n} x^n \right)$$

합의 순서만 바꿀 수 있다면..! Double series에 대한 정리를 적용하자..
$|a_m d_{m,n} x^n| \le |a_m g(x)^m|$ 가 성립하므로... $|a_m g(x)^m| \to 0$ 인 사실을 이용하면..
$\lim_{m\to\infty} a_m d_{m,n} x^n = 0$ uniformly in n.
$g(x)^m$ 가 수렴한다는 사실로부터 $\lim_{n\to\infty} a_m d_{m,n} x^n = 0$. Proposition 1. □

Example. $\mathbb{Q}_2$에서,

$$f(X) = 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!}$$

$$g(X) = 2X^2 - 2X$$

(앞으로 보일 내용) $f(x)$ converges for $x \in 4\mathbb{Z}_2$. $g(x)$는 모든 x에서 converge... $g(1) = 0$이고, $f(g(1)) = 1$.
$h(X) = f(g(X))$를 powers series로 정의하자. $h(X) = \sum a_n X^n$.
(a) $v_2(a_n) \ge 1 + n/4$ for $n \ge 2$ (Fact : 쉬운 예제입니다)
(b) Theorem 1의 세 조건에서 (c)를 보면 $|2x| = 1/2 > |g(1)| = 0$으로 성립하지 않음.
(c) $h(X) = 1 - 2X + 4X^2 + \cdots$ 따라서 $h(1) \equiv 3 (mod\ 4)$
(d) $h(1) \ne f(g(1))$

1.4 Functions Defined by Power series

Power series로 정의된 함수의 성질?

Lemma 3 ([Gou20] Lemma 5.5.1)**.** *$f(X) = \sum a_n X^n$ be a power series with coefficients in $\mathbb{Q}_p$. If $f(x)$ converges when $|x| \le r$ then the function $f : \bar{B}(0, r) \to \mathbb{Q}_p$ is bounded and uniformly continuous.*

Proof. $M_r = \max_{n\ge0} |a_n| r^n$ 는 유한값.
Boundness

$$|f(x)| = \left| \sum_{n=0}^{\infty} a_n x^n \right| \le \max(|a_n x^n|) \le M_r$$

Uniform continuity

$$\begin{aligned} f(x) - f(y) &= \sum_{n=1}^{\infty} a_n (x - y)(x^{n-1} + \cdots + y^{n-1}) \\ &= (x - y) \sum_{n=1}^{\infty} a_n (x^{n-1} + \cdots + y^{n-1}) \end{aligned}$$

Ultrametric inequality에 의해 $|x^{n-1} + \cdots + y^{n-1}| \le r^{n-1}$
따라서 $|f(x) - f(y)| \le \frac{M_r}{r} |x - y|$ □

Remark 4. *사실... closed ball in $\mathbb{Q}_p$는 항상 compact이므로... 그렇지만 증명 과정 상에서 ultrametric inequality만을 활용하였기에 이 증명은 적용범위가 더 넓습니다. (그러면 $\mathbb{Q}_p$ 보다 더 큰 object를 생각하겠다는 말이겠지요)*

Corollary 2 ([Gou20] Corollary 5.5.2)**.** *$f(X) = \sum a_n X^n$ be a power series with coefficients in $\mathbb{Q}_p$. $\mathcal{D} \subset \mathbb{Q}_p$ be region of convergence.*
Then the function $f : \mathcal{D} \to \mathbb{Q}_p$ is continuous on $\mathcal{D}$

또 다른 문제. 주어진 power series 전개를 다른 곳에서도 할 수 있을까.

Proposition 4 ([Gou20] Proposition 5.5.3)**.** *$f(X) = \sum a_n X^n$ be a power series with coefficients in $\mathbb{Q}_p$. Let $0 \ne \alpha \in \mathbb{Q}_p$ where $f(\alpha)$ converges. Define*

$$b_m = \sum_{n\ge m} \binom{n}{m} a_n \alpha^{n-m}$$

and consider the power series

$$g(X) = \sum_{m=0}^{\infty} b_m (X - \alpha)^m$$

(a) b_m converges for every m.
(b) The power series $f(X)$ and $g(X)$ have the same region of convergence, $f(\lambda)$ converges if and only if $g(\lambda)$ converges
(c) λ in the region of convergence, we have $g(\lambda) = f(\lambda)$

Proof. (a) $|\binom{n}{m} a_n \alpha^{n-m}| \le |a_n \alpha^{n-m}| = |\alpha|^{-m} \cdot |a_n \alpha^n| \to 0$ for fixed m. Thus it converges
(b), (c): for λ in the region of convergence of $f(X)$,

$$f(\lambda) = \sum_n \sum_{m\le n} \binom{n}{m} a_n \alpha^{n-m} (\lambda - \alpha)^m$$

이제 또 다시 double series를 바꿀 수 있는지 체크하면 된다.

$$\beta_{nm} := \binom{n}{m} a_n \alpha^{n-m} (\lambda - \alpha)^m$$

for $m \le n$ and 0 otherwise.
$|\beta_{nm}| \le |a_n \alpha^{n-m} (\lambda - \alpha)^m|$, λ, α가 동시에 region of convergence에 있으므로, rho_1 such that $|\lambda| \le \rho_1$ and $|\alpha| \le \rho_1$, ρ_1 smaller or equal than region of the convergence.

$|\alpha|^{n-m} \le \rho_1^{n-m}$, $|\lambda-\alpha|^m \le \rho_1^m$. 따라서 $|\beta_{nm}| \le |a_n|\rho_1^n$. 따라서 double series 의 교환 가능. 이제 $f(\lambda) = g(\lambda)$이고, region of convergence는? □

The region of the convergence가 같은 것은 사실은 조금 안 좋은 일... function을 더 큰 domain으로 continue할 때 다른 점에서 analytic expansion을 한 것이 사용불가능하기 때문. Example.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Z}_p \\ 0 & x \notin \mathbb{Z}_p \end{cases}$$

이 함수는 어떤 점이든 open ball이 있어 formal power series와 값이 같은... analytic한 함수가 되는데..

References

[Gou20] Fernando Q. Gouvêa. *p-adic Numbers: An Introduction*. 3rd. Universitext. Springer, 2020. ISBN: 978-3-030-47295-5. DOI: 10.1007/978-3-030-47295-5.