SNU Fall 2025, Lie Group

Donghyun Park

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1 Introduction

1.1 Motivation

In this course, the study of Lie group and Lie algebra came from Homogeneous dynamics. The Iwasawa decomposition and KAK decomposition gives rich view of homogeneous dynamics. The prototype is perhaps $SL_2\mathbb{Z}\backslash SL_2\mathbb{R}$.

1.2 Haar Measure

The existence of Haar measure, as we will accept as 'fact'. The statement is:

Theorem 1 (Haar, 1933)**.** *Every locally compact Hausdorff topological group has left Haar measure, unique up to constant. Left Haar measure is nonzero positive linear functional over $C_c(G)$, $m : C_c(G) \to \mathbb{C}$ which is invariant under left-translation $\lambda^*(g)m = m$*

1.3 Basics of Lie group and Lie algebra

Starting from the smooth manifold, Lie group is defined by smooth manifold with group structure. Lie algebra is defined by vector space of Left invariant vector field over Lie group with bracket operation.
Analyzing with the tool of smooth manifolds, we can analyze some Lie group and Lie algebras. One classical example is $O(n, \mathbb{R})$ and $\mathfrak{o}(n, \mathbb{R})$. We can analyze that the Lie algebra is isomorphic to skew-symmetric Matrix space with bracket operation $[A, B] = AB - BA$
Now we define 'abstract' version of Lie algebra. Lie algebra over field $\mathbb{k}$ is $\mathbb{k}$-vector space with bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies : anti-symmetric, Jacobi Identity:
- $[T_1, T_2] = -[T_2, T_1]$
- $[T_3, [T_1, T_2]] + [T_2, [T_3, T_1]] + [T_1, [T_2, T_3]] = 0$

Examples are smooth vector field over smooth manifold $\mathrm{Vect}^\infty(M)$, Left invariant smooth vector fields $\mathrm{Vect}^\infty(M)^G$, Associative $\mathbb{k}$ algebra with $[a, b] = ab - ba$ so for example, space of endomorphisms or derivatives.

1.4 Lie Group and Lie Algebra, some facts

- Hilbert's 5th problem, solved by Gleason, Montgomery, Zippin, Yamabe (1953) : A topological group which is also a topological manifold can be turned into a Lie group in a unique way.
- However, given topological manifold with no smooth structure exists (Kervaire, 1960) and given topological manifold, there can exist non diffeomorphic smooth structure (Milnor, 1956)
- Every Finite dimensional Lie algebra comes from Lie group
- Lie group is not uniquely determined by its Lie algebra. However, simply connected Lie group is uniquely determined by its Lie algebra.
- Closed subgroup of Lie group is Lie group (Cartan Theorem)

2 Exponential Map

2.1 Matrix Lie group and Matrix Lie algebra

Matrix Lie group and Matrix Lie algebra has powerful tool : the Exponential Map. Moreover, Matrix Lie algebra is extremely important because the **Lie algebra of adjoint map** $\mathrm{ad}(\mathfrak{g})$ **can be considered as Matrix Lie algebra**. When proving the Global Cartan Decomposition theorem and Iwasawa Decomposition, this fact is used.

Define $X \in \mathcal{M}_{n,n}(F)$. $F = \mathbb{R}, \mathbb{C}$.

$$\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

The statement itself is not powerful as expected but important theorem is Baker-Campbell-Hausdorff formula.

$$\exp(X)\exp(Y) = \exp(X + \int_0^1 g(e^{\mathrm{ad}X} e^{\mathrm{ad}(tY)})(Y)dt)$$

We can express $\exp(X)\exp(Y)$ as one exponential. Using Baker-Campbell-Hausdorff formula Critical Theorem holds.

Theorem 2. *For G, H matrix Lie group,* $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(H) = \mathfrak{h}$*. Assume G simply connected, connected. For $\phi : \mathfrak{g} \to \mathfrak{h}$ given, there exists unique $\Phi : G \to H$ such that $\Phi \circ \exp = \exp \circ \phi$.*

This theorem is important that we know the converse direction very well : For $\Phi : G \to H$, we can define $\phi : \mathfrak{g} \to \mathfrak{h}$ that

$$\begin{tikzcd} G \arrow[r, "\Phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\phi"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

2.2 Exponential Map in Lie group, Lie algebra

General exponential map is $\exp_G : \mathfrak{g} \to G$ for $\mathrm{Lie}(G) = \mathfrak{g}$ sending v to $\varphi_v(1)$. The flow with initial tangent vector v.

Surjectiveness of exponential map is often important. Because every element in Lie group can be expressed as exponential, and is much easier to analyze. There are some situations when the exponential map is surjective : (1) if G compact connected, (2) G is compact abelian or (3) $G = GL_n(\mathbb{C})$ then exponential map becomes surjective. Counterexample of exponential map being non surjective even connected group is $SL_2\mathbb{R}$.

2.3 Adjoint Map

Adjoint Map is core object to understand. I like the view from Fulton-Harris Representation Theory. Viewpoint starts from $\rho : G \to H$ homomorphism.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "m_g"'] & H \arrow[d, "m_{\rho(g)}"] \\ G \arrow[r, "\rho"] & H \end{tikzcd}$$

To look at the tangent space, it is convenient if base point is fixed, so consider conjugate

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "c_g"'] & H \arrow[d, "c_{\rho(g)}"] \\ G \arrow[r, "\rho"] & H \end{tikzcd}$$

Adjoint representation of Lie group is $\mathrm{Ad} : G \to GL(\mathfrak{g})$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "(d\rho)_e"] \arrow[d, "Ad(g)"'] & \mathfrak{h} \arrow[d, "Ad(\rho(g))"] \\ \mathfrak{g} \arrow[r, "(d\rho)_e"] & \mathfrak{h} \end{tikzcd}$$

Now taking differential again, we get Adjoint Representation of Lie algebra : $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "(d\rho)_e"] \arrow[d, "ad(g)"'] & \mathfrak{h} \arrow[d, "ad(\rho(g))"] \\ \mathfrak{g} \arrow[r, "(d\rho)_e"] & \mathfrak{h} \end{tikzcd}$$

which now, respects bracket operation

3 Prototype: Representation of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$

Representation of $\mathfrak{sl}_2\mathbb{C}$ is basic of basic and Representation of $\mathfrak{sl}_3\mathbb{C}$ will become our blueprint for analyzing general Lie algebras.
Every irreducible finite dimensional representation of $\mathfrak{sl}_2\mathbb{C}$ is

$$V^{(n)} = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Acts on the representation by H acts on itself, with the eigenvalue in the subindex. (So $V^{(n)}$ as direct sum is actually eigenspace decomposition. Remark : each eigenspace is 1 dimensional). X acts by translating V_j to V_{j+2}, Y acts by translating V_j to V_{j-2}.
Representation of $\mathfrak{sl}_3\mathbb{C}$ is more complex. First define

$$\mathfrak{h} = \{X \in \mathfrak{sl}_3\mathbb{C} : X \text{ diagonal}\}$$

We first do the **Root space decomposition**. Which is the special case of representation : $\mathfrak{g} \curvearrowright \mathfrak{g}$ by adjoint action. Then surprisingly, $\mathfrak{g}$ decompose into

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha} \mathfrak{g}_\alpha \right)$$

Is eigenspace decomposition of $\mathfrak{h}$ action, where now eigenvalue is not mattering since $\mathfrak{h}$ is two dimensional, but

$$\alpha \in \mathfrak{h}^*$$

Adjoint action of $\mathfrak{h}$ acts $\mathfrak{g}_\alpha$ to itself, and $\mathfrak{g}_\alpha$ acting on $\mathfrak{g}$ is similar to X, Y in $\mathfrak{sl}_2\mathbb{C}$ which translates $\mathrm{ad}(\mathfrak{g}_\alpha) : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$. We call α's **root**.
Now back to the irreducible representation $\pi : \mathfrak{g} \curvearrowright V$, eigenspace decomposition of $\mathfrak{h}$ is still valid. $V = \bigoplus_\alpha V_\alpha$. Also $\pi(\mathfrak{g}_\alpha) : V_\beta \to V_{\alpha+\beta}$. Here α is **root** but β is **weight**.
Choose the highest weight vector of V : kernel of $E_{1,2}, E_{1,3}, E_{2,3}$. Then we can identify all irreducible representation of $\mathfrak{sl}_3\mathbb{C}$. See the figure below.

4 Structure Theory : Introduction

Here is now a new Chapter of Lie group, Lie algebra theory. First some definitions

4.1 Simple, Semisimple, Nilpotent

$\mathfrak{g}$ is solvable if $\mathfrak{g}^j = 0$ for some j.

$$\mathfrak{g}^j = [\mathfrak{g}^{j-1}, \mathfrak{g}^{j-1}]$$

$\mathfrak{g}$ is nilpotent if $\mathfrak{g}_j = 0$ for some j.

$$\mathfrak{g}_j = [\mathfrak{g}_{j-1}, \mathfrak{g}]$$

$\mathfrak{g}$ is simple if $\mathfrak{g}$ is non abelian and has no nontrivial ideal.
$\mathfrak{g}$ is semisimple if it is non abelian and has no nontrivial solvable ideal.

4.2 Three Big Theorems

Theorem 3 ([Kna02] Theorem 1.25 (Lie's Theorem))**.** *$\mathfrak{g}$ be solvable. For finite dimensional representation $\pi : \mathfrak{g} \to End_{\mathbb{k}}V$, if $\mathbb{k}$ is algebraically closed or more generally, eigenvalues of $\pi(X)$, $X \in \mathfrak{g}$ lie in $\mathbb{k}$ then there is a simultaneous eigenvector $v \neq 0$ for all members $\pi(\mathfrak{g})$*

So inductively applying Lie's Theorem, representation $\pi : \mathfrak{g} \to End_{\mathbb{k}}V$ can be seen as flag.

$$V = V_0 \supset V_1 \supset \cdots \supset V_m = 0$$

Such that each V_i are stable under $\pi(\mathfrak{g})$.

Theorem 4 ([Kna02] Theorem 1.35 (Engel's Theorem))**.** *V a finite dimensional vector space over $\Bbbk$. $\mathfrak{g}$ is a Lie algebra of nilpotent endomorphisms of V. Then*
- $\mathfrak{g}$ is a nilpotent Lie algebra
- There exists $v \in V$ that $X(v) = 0$ for all $X \in \mathfrak{g}$
- With suitable basis of V, $\pi(\mathfrak{g})$ is expressed as strictly upper triangular matrix.

As a corollary, the Lie algebra whose ad$X$ is nilpotent, then Lie algebra itself is nilpotent.

Theorem 5 ([Kna02] Theorem 1.45 (Cartan's Criterion for semisimplicity))**.** *Lie algebra $\mathfrak{g}$ is semisimple if and only if Killing form B is nondegenerate.*

These three theorem will give us the starting point of analyzing the structure of Lie group and Lie algebras. Proof technique is also important, techniques handling Killing form and Bracket operation. For example

$$B((\mathrm{ad}(X))Y, Z) = -B(Y, (\mathrm{ad}(X)Z)$$

is oftenly used with the nondegeneracy of Killing form in semisimple Lie algebra. These techniques play significant role in analyzing structure.
For instance, by these three fundamental theorems we can see:

Theorem 6 ([Kna02] Theorem 1.54)**.** *Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ each $\mathfrak{g}_j$ are simple Lie algebra and an ideal.*

To go more from semisimple Lie algebras, we look at reductive Lie algebra: The Lie algebra is reductive if $\forall \mathfrak{a} \subseteq \mathfrak{g}$ an ideal then $\exists \mathfrak{b} \subseteq \mathfrak{g}$ that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Reductive Lie algebra decomposes into

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z_{\mathfrak{g}}$$

$[\mathfrak{g}, \mathfrak{g}]$ semisimple part, $Z_{\mathfrak{g}}$ abelian.

4.3 Analytic Subgroup

Analytic group means connected Lie group. One deep result is that for H an analytic subgroup of G and its Lie algebra $\mathfrak{h}$, the correspondence of analytic subgroup and Lie subalgebra is one-to-one onto.

$$\textbf{(Analytic Subgroup)} \longleftrightarrow \textbf{(Lie subalgebra)}$$

This fact will appear quite often when we analyze Lie algebra precisely, and then lift to the Lie group.

4.4 Root Space Decomposition

We focus on complex semisimple Lie algebras. At the end, we will find out the beautiful classification of complex simple Lie algebras.

Theorem 7. *Complex simple Lie algebra is isomorphic to one of the followings:*

$$\mathfrak{sl}_n\mathbb{C},\ \mathfrak{so}_n\mathbb{C},\ \mathfrak{sp}_{2n}\mathbb{C},\ G_2, F_4, E_6, E_7, E_8$$

Motivation of root space decomposition comes from $\mathfrak{sl}_3\mathbb{C}$ or more generally, $\mathfrak{sl}_n\mathbb{C}$. We found on $\mathfrak{sl}_3\mathbb{C}$, adjoint representation gives decomposition of $\mathfrak{sl}_3\mathbb{R}$ by the eigenspace of 'roots' which lie in $\mathfrak{h}^*$. This is indeed true for general semisimple Lie algebra.

Proposition 1 ([Kna02] Proposition 2.5)**.** *If $\mathfrak{g}$ is any finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ a nilpotent Lie subalgebra then*
- $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ the generalized weight space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\mathrm{ad}H - \alpha(H)1)^n X = 0\ \forall H \in \mathfrak{h}\}$$

- $\mathfrak{h} \subseteq \mathfrak{g}_0$
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

We call if $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$ then $\mathfrak{h}$ is **Cartan Subalgebra**. For complex Lie algebra, there exists Cartan subalgebra ([Kna02], Theorem 2.9) and is unique upto Int$\mathfrak{g}$. (Int is analytic subgroup of Aut$\mathfrak{g}$ whose Lie subalgebra is ad$\mathfrak{g}$, [Kna02] Theorem 2.15). Moreover, $\mathfrak{h}$ can be characterized by the maximal abelian subalgebra that $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ are simultaneoulsy diagonalizable.
In semisimple Lie algebra, using the Cartan's semisimplicity criteria, the Cartan subalgebra is abelian ([Kna02], Proposition 2.10). So $\mathfrak{g}$ semisimple can be written as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

We call Δ roots. Note that if Cartan subalgebra $\mathfrak{h}$ changes, roots will change. However, Cartan subalgebra is unique upto $\mathrm{Int}\mathfrak{g}$ so change of roots will be restricted also.

Example 1. $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$.
Then the Cartan subalgebra is

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} \mid h_1 h_2 \cdots h_n = 1,\ h_i \in \mathbb{C} \right\}$$

The roots are $\Delta = \{e_i - e_j \mid i \neq j\}$ and root spaces are each one dimensional, E_{ij} for the root $e_i - e_j$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}$$

Example 2. $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$
The Cartan subalgebra is

$$\mathfrak{h} = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix} \mid h_i \in \mathbb{C} \right\}$$

The roots are $\Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_k\}$ and root spaces are : for $\alpha = \pm e_i \pm e_j$, $\mathfrak{g}_\alpha$ is generated by E_α

$$E_\alpha = \begin{pmatrix} 0 & X_\alpha \\ -X_\alpha^t & 0 \end{pmatrix}$$

for i, j row and columns where

$$X_{e_i - e_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, X_{e_i + e_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, X_{-e_i + e_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, X_{-e_i - e_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

for $\alpha = \pm e_k$,

$$E_\alpha = \begin{pmatrix} 0 & X_\alpha \\ -X_\alpha^t & 0 \end{pmatrix}$$

each in k and $2n+1$ row and columns,

$$X_{e_k} = \begin{pmatrix} 1 \\ -i \end{pmatrix}, X_{-e_k} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Example 3. $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$
The Cartan subalgebra is

$$\mathfrak{h} = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} \end{pmatrix} \mid h_i \in \mathbb{C} \right\}$$

The roots are $\Delta = \{\pm e_i \pm e_j \mid i \neq j\}$ and root space as Example 2.

Example 4. $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$
The Cartan subalgebra is

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & & & & \\ & \ddots & & & & \\ & & h_n & & & \\ & & & -h_1 & & \\ & & & & \ddots & \\ & & & & & -h_n \end{pmatrix} \mid h_i \in \mathbb{C} \right\}$$

The roots are $\Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_k\}$ and root spaces are
$E_{e_i - e_j} = E_{ij} - E_{j+n,i+n}$, $E_{e_i+e_j} = E_{i,j+n} + E_{j,i+n}$, $E_{-e_i-e_j} = E_{i+n,j} + E_{j+n,i}$, $E_{2e_k} = E_{k,k+n}$, $E_{-2e_k} = E_{k+n,k}$

By Cartan's semismplicity criteria, we can analyze further with the Killing form B.
We can know that $B \mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate and $B|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ must be nondegenerate for $\alpha \in \Delta$. Thus we can guarantee $\alpha \in \Delta$ then $-\alpha \in \Delta$, moreover Δ spans $\mathfrak{h}^*$. $\mathfrak{g}_\alpha$ is each one dimensional and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ holds... Read [Kna02] Chapter 2 for more properties.
Moreover, for $\alpha \in \Delta$ we can let H_α that $\alpha(H) = B(H, H_\alpha)$ hold for all $H \in \mathfrak{h}$ (By non-degeneracy of $B|_{\mathfrak{h}\times\mathfrak{h}}$) and $E_\alpha \in \mathfrak{g}_\alpha$ that $[H, E_\alpha] = \alpha(H)E_\alpha$ (By Lie's theorem). Then $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ looks like (after some normalization) $\mathfrak{sl}_2\mathbb{C}$. $H'_\alpha = \frac{2}{\alpha(H_\alpha)}H_\alpha$, $E'_\alpha = \frac{2}{\alpha(H_\alpha)}E_\alpha$, $E'_{-\alpha} = E_{-\alpha}$
This introduces the terminology 'root string' and 'root reflection' E_α. Especially, $\mathfrak{g}_{\beta+n\alpha}$ string satisfies $-p \leq n \leq q$ then $p - q = \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$

4.5 Abstract Root System

From the motivation of studying root system of complex semisimple Lie algebra, we define an abstract root system for general finite dimensional real inner product space V,
- Δ spans V.
- $s_\alpha(\varphi) = \varphi - \frac{2\langle\varphi,\alpha\rangle}{|\alpha|^2}\alpha$ sends Δ to itself
- $\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$ is an integer

Connection of abstract root system to Lie algebra is that: $\mathfrak{g}$ a complex semisimple Lie algebra has abstract reduced root system if and only if $\mathfrak{g}$ is simple. So finding all simple Lie algebras (complex) is now become the problem finding all abstract reduced root system.
This problem is combinatoric. First define 'positivity' of roots and define atomic positive element : 'simple root'. Now abstract root system correspond to 'Cartan Matrix' and expressing as graph, the 'Dynkin diagram'. Thus, classification of complex simple Lie algebra is now established.

4.6 Weyl Group

On going from abstract reduced root system to abstract Cartan matrix, we need an ordering of roots (or positivity). The uniqueness matter of ordering is expressed as the Weyl group, defined $W(\Delta)$ a group generated by reflections of roots s_α.
One big theorem is that two simple systems are equivalent.

Theorem 8 ([Kna02] Theorem 2.63)**.** *Two simple systems for* Δ, Π, Π', *there exists unique element in* $W(\Delta)$ *that* $s\Pi = \Pi'$

For the root system with given positivity, we define the length. For $w \in W(\Delta)$, $l(w)$ is defined by number of positive roots $\alpha \in \Delta^+$ that $w\alpha < 0$. Then, this length equals to the number of reflection by simple roots to represent w.

5 Compact Lie Group, Compact Lie Algebra

To go on the deep structure theories, we need 'Compact Lie Algebras'. The fact that analytic subgroup is one to one onto correspondence of Lie subalgebra; when does the analytic subgroup become compact?
$\mathfrak{g}$ is compact **real** Lie algebra if Int$\mathfrak{g}$ is compact. (Recall Int$\mathfrak{g}$ is analytic subgroup of $Aut_{\mathbb{R}}\mathfrak{g}$ whose Lie algebra is ad$\mathfrak{g}$)

Theorem 9 ([Kna02] Corollary 4.25)**.** *G a compact Lie group then its Lie algebra is reductive*

Proposition 2 ([Kna02] Proposition 4.27)**.** *If the Killing form of a real Lie algebra $\mathfrak{g}$ is negative definite then $\mathfrak{g}$ is compact Lie algebra.*

By the analysis of compact Lie algebras, we can find the structure of Compact Lie group.

Theorem 10 ([Kna02] Theorem 4.29)**.** *G a compact, connected Lie group. Center be Z_G and G_{ss} an analytic subgroup of Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ then G_{ss} has finite center and*

$$G = (Z_G)_0 G_{ss}$$

as a commuting product of closed subgroups $(Z_G)_0$ and G_{ss}.

We define **Maximal tori** of compact connected Lie group by maximal torus subgroup. This definition correspond to: analytic subgroup correspond to maximal abelian subalgebra of $\mathrm{Lie}(G) = \mathfrak{g}_0$. (Knapp, Proposition 4.30). Complexification of $\mathfrak{t}_0, \mathfrak{g}_0$, $\mathfrak{t} = \mathfrak{t}_0^{\mathbb{C}}$ gives analogy of root space decomposition.
(Technically, we only know well about root space decomposition for complex semisimple Lie algebras. So we decompose $\mathfrak{t}$ into terms on centralizer $Z_{\mathfrak{g}}$ and semisimple part $[\mathfrak{g}, \mathfrak{g}]$ and combine result)

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha$$

Each roots are extended to be zero at $Z_{\mathfrak{g}}$ so roots $\Delta(\mathfrak{g}, \mathfrak{t})$ are almost the same comming from $[\mathfrak{g}, \mathfrak{g}]$ semisimple part.
From the root space decomposition,

Theorem 11 ([Kna02] Theorem 4.34)**.** *For compact connected Lie group G, two maximal abelian subalgebras of $\mathfrak{g}_0 = \mathrm{Lie}(G)$ are conjugate via $\mathrm{Ad}(G)$.*

Moreover, every element belongs to the maximal torus;

Theorem 12 ([Kna02] Theorem 4.36)**.** *For compact connected Lie group G, maximal torus T, each element of G is conjugate to element of T.*

This simple looking theorem is hard to proof: the hardest part is while lifting the Lie algebra analysis to Lie group level.

5.1 Weyl group

We defined Weyl group in terms of Lie algebra : $W(\Delta(\mathfrak{g}, \mathfrak{h}))$ was the group generated by root reflections.
For compact connected Lie group G and maximal torus T, define analytic Weyl group:

$$W(G, T) = N_G(T)/Z_G(T)$$

This version of Weyl group can also act on Lie algebra $\mathfrak{t}_{\mathbb{R}}^*$ by Ad action. Indeed, in the root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha$$

$\alpha|_{\mathfrak{t}_0}$ are purely imaginary since $\mathrm{Ad}(t)$ action to $\mathfrak{g}_\alpha$ is multiplicate by a complex unit, thus its differential at $t = 0$ is purely imaginary. So $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_0$ is real form of $\mathfrak{t} = \mathfrak{t}_0^{\mathbb{C}}$ that all roots become real. Negative definite Ad invariant bilinear form B on $\mathfrak{g}_0$ extends to Hermitian form on $\mathfrak{g}$. Restricted to $\mathfrak{t}_{\mathbb{R}}$, B gives an inner product here.
We can regard $\mathfrak{t}_{\mathbb{R}} = iZ_{\mathfrak{g}_0} \oplus \Delta(\mathfrak{g}, \mathfrak{t})$ so $W(\Delta(\mathfrak{g}, \mathfrak{h}))$ **and** $W(G, T)$ **acts on** $\mathfrak{t}_{\mathbb{R}}^*$. $W(\Delta(\mathfrak{g}, \mathfrak{h}))$ acts by root reflection on $\Delta(\mathfrak{g}, \mathfrak{h})$ then extends trivially, $W(G, T)$ acts on T, hence differentiating acts on $\mathfrak{t}_0$ hence $\mathfrak{t}$ and $\mathfrak{t}_{\mathbb{R}}$ finally to its dual.

Theorem 13 ([Kna02] Theorem 4.54)**.** *For a compact connected Lie group G with maximal torus T, the analytically defined Weyl Group $W(G, T)$ acting on $\mathfrak{t}_{\mathbb{R}}^*$ coincides with Weyl Group $W(\Delta(\mathfrak{g}, \mathfrak{t}))$*

Example. $G = SU(2)$.

$$G = SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

$$\mathfrak{g}_0 = \mathfrak{su}(2) = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\rangle$$

$$\mathfrak{g} = \mathfrak{su}(2)^{\mathbb{C}} = \mathfrak{sl}_2\mathbb{C}$$

The $\mathfrak{t} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$, $\mathfrak{t}_{\mathbb{R}}$ is real multiplication of the matrix.
The $W(G,T) = N_G(T)/T = \left\{ id, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts by Adjoint action,

$$\sigma \curvearrowright \left\langle \begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix} \right\rangle \text{ by } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \\ & \alpha \end{pmatrix}$$
$$\sigma \curvearrowright \mathfrak{t}_0 \text{ by } \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \to_\sigma \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\sigma \curvearrowright \mathfrak{t}_{\mathbb{R}} \text{ by } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to_\sigma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma \curvearrowright \mathfrak{t}_{\mathbb{R}}^* \text{ by } e_1 - e_2 \to_\sigma e_2 - e_1$$

6 Cartan Decomposition, Iwasawa Decomposition

6.1 Split Real Form and Compact Real Form

For the complex semisimple Lie Algebra $\mathfrak{g}$ we can define two kinds of real form.
Define

$$\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \alpha(H) \in \mathbb{R} \ \alpha \in \Delta\}$$

and

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R} X_\alpha$$

gives **split real form**. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as vector space.
On the other hand the **compact real form** of $\mathfrak{g}$ exists for complex semisimple Lie algebra.

$$\mathfrak{u}_0 = \sum_{\alpha \in \Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha \in \Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Delta} \mathbb{R} i(X_\alpha + X_{-\alpha})$$

is closed under bracket but the Killing form restricted to $\mathfrak{u}_0$ is negative definte hence compact.

6.2 Cartan Decomposition

Cartan Decomposition comes from the motivation from Matrix groups. In complex matrix X, the map $\theta(X) = -X^*$ is an involution. $[\theta(X), \theta(Y)] = \theta([X,Y])$. Moreover, the Killing form B, defining $B_\theta(X,Y) = -B(X, \theta Y)$ is symmetric positive definite.
We define for general real semisimple Lie algebra $\mathfrak{g}_0$, define **Cartan involution** an involution θ that is Lie algebra map (preserves bracket) that makes $B_\theta(X,Y) = -B(X,\theta Y)$ be symmetric positive definite.
For existence, it is easy for real semisimple Lie algebra that is also a complex Lie algebra. It has compact real from $\mathfrak{u}_0$, the $\mathfrak{g} = \mathfrak{u}_0 + i\mathfrak{u}_0$ conjugation $u + iu' \to u - iu'$ is Cartan involution.
Indeed, for general real semisimple Lie algebra, the Cartan involution exists and for uniqueness, they are conjugate.

Theorem 14 ([Kna02] Corollary 6.18, 6.19)**.** *If $\mathfrak{g}_0$ is a real semisimple Lie algebra then $\mathfrak{g}_0$ has a Cartan involution and two Cartan involutions are conjugate by* $\mathrm{Int}\mathfrak{g}_0$

Now eigenspace decomposition by θ of $\mathfrak{g}_0$ gives $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ as vector space. It satisfies:

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$$

$$\mathfrak{k}_0, \mathfrak{p}_0 \text{ are orthogonal under } B_\theta \text{ and } B_{\mathfrak{g}_0}$$

The decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ that satisfies above condition is the **Cartan Decomposition**. Cartan decomposition and Cartan involution have one-to-one correspondence so Cartan decomposition will also unique under some 'conjugacy'. Actually we can regard any Cartan decomposition as Matrix Lie algebra.

Proposition 3 ([Kna02] Proposition 6.28)**.** *$\mathfrak{g}_0$ is a real semisimple Lie algebra then $\mathfrak{g}_0$ is isomorphic to a Lie algebra of real matrices that is closed under transpose. The Cartan involution θ of $\mathfrak{g}_0$ can be correspond into negative transpose.*

Proof. Since $\mathfrak{g}_0$ is semisimple, $\mathfrak{g}_0 \cong \mathrm{ad}\mathfrak{g}_0$ identification. □

6.3 Cartan Decomposition of Lie Group

We will lift the Cartan decomposition of Lie algebra onto Lie group decomposition. The proof technique will be also used in the proof of Iwasawa decomposition: **first prove on** $\mathrm{Ad}(G)$ **then lift to** G. The analysis of center will be the remaining task.

Theorem 15 ([Kna02] Theorem 6.31)**.** *G a semisimple (real) Lie group and θ a Cartan involution of Lie algebra $\mathfrak{g}_0$ and Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. K be the analytic subgroup with Lie algebra $\mathfrak{k}_0$.*
(a) There exists a Lie group automorphism Θ with differential θ that $\Theta^2 = 1$.
(b) The subgroup fixed by Θ is K.
(c) $K \times \mathfrak{p}_0 \to G$ by $(k, X) \mapsto k \exp X$ is diffeomorphism onto.
(d) K is closed.
(e) $Z \subseteq K$
(f) K is compact if and only if Z is finite.
(g) If Z is finite, K is maximal compact subgroup of G.

6.4 Iwasawa Decomposition

Motivation is QR decomposition. Here are statements:

Theorem 16 ([Kna02] Proposition 6.43 (Iwasawa decomposition of Lie algebra))**.** *$\mathfrak{g}_0$ is real semisimple Lie algebra. Then $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ as vector space decomposition. $\mathfrak{a}_0$ is abelian, $\mathfrak{n}_0$ is nilpotent.*

This decomposition can be gained by **restricted root space decomposition**. First, in the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, the maximal abelian subspace $\mathfrak{a}_0$ gives a decomposition of $\mathfrak{g}_0$

$$\mathfrak{g}_{0\lambda} = \{X \in \mathfrak{g}_0 \mid (\mathrm{ad} H)X = \lambda(H)X \quad \forall H \in \mathfrak{a}_0\}$$

is restricted root space. Similar to root space decomposition,

Proposition 4 ([Kna02] Proposition 6.40)**.** *Restricted roots satisfy:*
(a) $\mathfrak{g}_0$ decompose into $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_{0\lambda}$
(b) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$
(c) $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$
(d) $\mathfrak{g}_{00} = \mathfrak{a}_0 \oplus \mathfrak{m}_o$ orthogonally in Killing form. $\mathfrak{m}_0 = Z_{\mathfrak{k}_o}(\mathfrak{a}_0)$

The restricted root system is actually, an abstract root system in $\mathfrak{a}_0^*$
So Iwasawa decomposition is under this Restricted root space decomposition,

$$\begin{aligned} X &= H + X_0 + \sum_{\lambda \in \Sigma} X_\lambda \\ &= \left(X_0 + \sum_{\lambda \in \Sigma} (X_{-\lambda} + \theta X_{-\lambda}) \right) + H + \left(\sum_{\lambda \in \Sigma} (X_\lambda - \theta X_{-\lambda}) \right) \end{aligned}$$

is $\mathfrak{k}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$ respectively.

Example 1. $G = SL_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$$\mathfrak{g}_0 = \mathfrak{sl}_n \mathbb{K} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

The $\mathfrak{k}_0$ is skew-Hermitian part, $\mathfrak{p}_0$ is Hermitian. Now, one maximal abelian subspace is real diagonals, restricted roots are $\Sigma = \{f_i - f_j \mid i \neq j\}$. Each $\mathfrak{g}_{f_i - f_j} = \mathbb{K} E_{ij}$

$$\mathfrak{g}_0 = \mathfrak{sl}_n \mathbb{K} = \mathfrak{g}_{00} \oplus \bigoplus_{f_i - f_j \in \Sigma} \mathbb{K} E_{ij}$$

each restricted root space are dimension 1, 2, 4 respectively.
Decomposing $\mathfrak{g}_{00} = \mathfrak{a}_0 + \mathfrak{m}_0$, $\mathfrak{m}_0$ is 0 in $\mathbb{K} = \mathbb{R}$, purely imaginary in $\mathbb{K} = \mathbb{C}, \mathbb{H}$

Example 2. $G = SU(p, q)$
The Lie algebra is

$$\mathfrak{g}_0 = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

$\mathfrak{a}, \mathfrak{d}$ are skew-Hermitian with $Tr\ a + Tr\ d = 0$. $\mathfrak{k}_0 = \{b = 0\}$, $\mathfrak{p}_0 = \{a = d = 0\}$. The maximal abelian subspace of $\mathfrak{p}_0$ is

$$\mathfrak{a}_0 = \left\{ b = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & a_q \\ \vdots & \ddots & \vdots \\ a_1 & \cdots & 0 \end{pmatrix} \right\}$$

so restricted roots are $\Sigma = \{\pm f_i \pm f_j \mid i \neq j\} \cup \{\pm 2f_i\}$ for $p \neq q$ and $\Sigma = \{\pm f_i \pm f_j \mid i \neq j\} \cup \{\pm 2f_i\} \cup \{\pm f_i\}$ for $p = q$

The Lie algebra decomposition lifts as it was at the Cartan decomposition onto Lie group.

Theorem 17 ([Kna02] Theorem 6.46 (Iwasawa decomposition for Lie group))**.** *G a semisimple (real) Lie group and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ a Iwasawa decomposition of Lie algebra. Let A, N analytic subgroups of G with Lie algebras $\mathfrak{a}_0, \mathfrak{n}_0$, then the multiplication map $K \times A \times N \to G$ by multiplication is diffeomorphism onto. A, N are simply connected*

We know from the Cartan decomposition that the Cartan involution so $\mathfrak{k}_0$ is unique by $\mathrm{Int}\mathfrak{g}_0$. We now determine uniqueness of $\mathfrak{a}_0$ and $\mathfrak{n}_0$. First, fixing $\mathfrak{k}_0$ and $\mathfrak{p}_0$, maximal abelian subspace $\mathfrak{p}_0$ is unique upto $\mathrm{Ad}K$.

Theorem 18 ([Kna02] Theorem 6.51)**.** *$\mathfrak{a}_0, \mathfrak{a}_0'$ are two maximal abelian subspaces of $\mathfrak{p}_0$ then there exists an element in K that $\mathrm{Ad}(k)\mathfrak{a}_0' = \mathfrak{a}_0$. Thus $\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}_0$*

Restricted roots Σ is an abstract root system in $\mathfrak{a}_0^*$ and $\mathrm{Ad}(k)$ acts like root reflection. The uniqueness of $\mathfrak{n}_0$ is now determined by analogous of Weyl group. In fact, $W(G, A) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0) = W(\Sigma)$.

Theorem 19 ([Kna02] Corollary 6.55)**.** *Any two choices of $\mathfrak{n}_0$ are conjugate by Ad of $N_K(\mathfrak{a}_0)$*

7 KAK decomposition, Bruhat decomposition

7.1 Reductive Lie group

Reductive Lie group is 4-tuple (G, K, θ, B) which are Lie group, a compact subgroup, Lie algebra involution, nondegenerate $Ad(G), \theta$ invariant bilinear form on $\mathfrak{g}_0$. They need to satisfy
- $\mathfrak{g}_0$ is reductive Lie algebra
- θ decomposes $\mathfrak{g}_0$ into eigenspaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ corresponding eigenvalue $+1, -1$ respectively. $\mathrm{Lie}(K) = \mathfrak{k}_0$
- $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under B. B is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$.
- $K \times \exp \mathfrak{p}_0 \to G$ is diffeomorphism onto
- Every automorphism $\mathrm{Ad}(g)$ on $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ is inner. $\mathrm{Ad}(g) \in \mathrm{Int}(\mathfrak{g})$

G_{ss} be a semisimpmle analytic subgroup of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. If G_{ss} has finite center, we call this Lie group **Harish-Chandra class**.
The Iwasawa decomposition continues on the reductive Lie group.

Proposition 5 ([Kna02] Proposition 7.29)**.** *G be a reductive Lie group. If $\mathfrak{a}_0, \mathfrak{a}_0'$ are two maximal abelian subspaces of $\mathfrak{p}_0$ then there exists $k \in K \cap G_{ss}$ that $\mathrm{Ad}(k)\mathfrak{a}_0' = \mathfrak{a}_0$.*

$$\mathfrak{p}_0 = \bigcup_{k \in K_{ss}} \mathrm{Ad}(k)\mathfrak{a}_0$$

Form the restricted root of $(\mathfrak{g}_0, \mathfrak{a}_0)$, $\lambda \in \mathfrak{a}_0^*$ is restricted root, and restricted roots comes from the semisimple part $[\mathfrak{g}_0, \mathfrak{g}_0]$. It extends to $\mathfrak{a}_0$ by assigning 0 to $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$. Defining Σ a restricted roots, $\mathfrak{n}_0 = \bigoplus_{\lambda \in \Sigma^+} (\mathfrak{g}_0)_\lambda$
Iwasawa decomposition of reductive Lie algebra is now

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$$

Proposition 6 ([Kna02] Proposition 7.31)**.** *G a reductive Lie group. Let A, N be the analytic subgroups with Lie algebras $\mathfrak{a}_0$ and $\mathfrak{n}_0$. The map $K \times A \times N \to G$ is diffeomorphism onto. A, N are simply connected.*

As in semisimple case, we can define Weyl group in two way. $M = Z_K(\mathfrak{a}_0)$ then $W(G, A) = N_K(\mathfrak{a}_0)/M$

Proposition 7. *If G is reductive Lie group, then $W(\Sigma) = W(G, A)$*

We call the Lie subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ Cartan subalgebra if $\mathfrak{h} = \mathfrak{h}_0^{\mathbb{C}}$ is Cartan subalgebra of $\mathfrak{g}$. It will be the form of

$$\mathfrak{h}_0 = Z_{\mathfrak{g}_0} \oplus (\mathfrak{h}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$$

7.2 KAK decomposition

Theorem 20 ([Kna02] Theorem 7.39 (KAK decomposition))**.** *For reductive Lie group G, every element decomposes as k_1ak_2 for $k_1, k_2 \in K$ and $a \in A$. a is uniquely determined up to conjugation by $W(G, A)$. If a is fixed satisfying $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$ then k_1 is unique up to right multiplication of $M = Z_K(\mathfrak{a}_0)$.*

7.3 Bruhat Decomposition

G be a reductive Lie group. $M = Z_K(\mathfrak{a}_0)$ is compact and $B = MAN$ be a Borel subgroup. Iwasawa decomposition gives $M \times A \times N \to B$ is diffeomorphism onto. We want to decompose G into double coset of B.

Theorem 21 ([Kna02] Theorem 7.40 (Bruhat decomoposition))**.** *A reductive Lie group G can be parametrized by*

$$G = \bigcup_{w \in W(G,A)} B\tilde{w}B$$

Example 1. $G = SL_2\mathbb{R}$. $B = MAN = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R} \right\}$
$N_K(\mathfrak{a}_0) = \left\{ \pm I, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$, $Z_K(\mathfrak{a}_0) = \{\pm I\}$
$W(G, A) = \{id, w\}$ and $\tilde{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- $c = 0$ then $g \in B$
- $c \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}$$

Example 2. $G = SL_3\mathbb{R}$. $B = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} : a_{11}a_{22}a_{33} = 1 \right\}$

The Weyl group $W \cong S_3$. Let $E_1 = \langle \vec{e}_1 \rangle, E_2 = \langle \vec{e}_1, \vec{e}_2 \rangle$ be a standard flag. Then the double coset $B\tilde{w}B$ is corresponds to the relative position of two flags, $\mathcal{F}_{std}$ and $\mathcal{F}$ which is $0 \subseteq V_1 \subseteq V_2 \subseteq V$.

$w \in W$	S_3	Relative Position of Flags	$\dim(BwB)$	Matrix form
$w = e$	$(1,2,3) \to (1,2,3)$	$V_1 = E_1, V_2 = E_2$	$\dim(BwB) = 5$	$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$
$w = s_1$	$(1,2,3) \to (2,1,3)$	$V_1 \neq E_1, V_2 = E_2$	$\dim(BwB) = 6$	$\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$
$w = s_2$	$(1,2,3) \to (1,3,2)$	$V_1 = E_1, V_2 \neq E_2$	$\dim(BwB) = 6$	$\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}$
$w = s_2s_1$	$(1,2,3) \to (3,1,2)$	$V_1 \nsubseteq E_2, E_1 \subseteq V_2$	$\dim(BwB) = 7$	$\begin{pmatrix} * & * & * \\ * & * & * \\ & * & * \end{pmatrix}$
$w = s_1s_2$	$(1,2,3) \to (2,3,1)$	$V_1 \subseteq E_2, E_1 \nsubseteq V_2$	$\dim(BwB) = 7$	$\begin{pmatrix} * & * & * \\ & * & * \\ * & * & * \end{pmatrix}$
$w = s_1s_2s_1$	$(1,2,3) \to (3,2,1)$	$V_1 \nsubseteq E_2, E_1 \nsubseteq V_2$	$\dim(BwB) = 8$	$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

8 Theorem of the Highest Weight

Though in this lecture, Theorem of the Highest Weight was not in important subject I want to state the theorem which is very powerful theorem on irreducible representation of Lie algebra.

Theorem 22 ([Kna02] Theorem 5.5 (Theorem of the Hightest Weight))**.** *The equivalence class of irreducible finite dimensional representations of* $\mathfrak{g}$ *stand in one-to-one correspondence with the* ***dominant algebraically integral linear functionals*** $\lambda \in \mathfrak{h}^*$*, the correspondence being that* λ *is the highest weight of* φ_λ*. The highest weight* λ *of* φ_λ *has following additional properties: (a)* λ *only depends on the simple system* Π *not on ordering*

(b) V_λ *is one dimensional*
(c) For each root vector E_α *for* $\alpha \in \Delta^+$ *annihilates the members of* V_λ*. This characterizes* V_λ
(d) Every weight of φ_λ *is of the form* $\lambda - \sum_{i=1}^{l} n_i \alpha_i$ *with* $n_i \in \mathbb{N} \cup \{0\}$ *and* $\alpha_i \in \Pi$
(e) $\dim(V_{w\mu}) = \dim(V_\mu)$ *for every weight space* V_μ *and Weyl group member* $w \in W(\Delta)$*. Each weight has* $|\mu| \leq |\lambda|$*. Equality holds for the* $W(\Delta)$ *orbit of* λ

References

[Kna02] Anthony W. Knapp. *Lie Groups Beyond an Introduction*. Vol. 140. Progress in Mathematics. Boston: Birkhäuser, 2002.