Smooth Mainfold by Lee

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This post illustrates important points (in my view) of the John H.Lee's Introduction to Smooth Manifold [Lee13]

Chapter 1. Smooth Manifold

"Smooth Manifold" is **"Topological manifold"** with some **"Smoothness"**. "Topological manifold" satisfies : Hausdorff, Second Countable, Locally Euclidean (which is homeomorphism)
Major properties of Topological Manifolds are :
- Have countable basis of precompact balls
- Countable components
- Countable fundamental group $\pi_1(M)$

"Smoothness" structure is given by transition of two coordinate charts. Since we only know "smoothness" on "Euclidean spaces", we define smoothness by smoothness of transition of coordinate charts. Manifold with "Smooth structure" is topological manifold given with maximal smooth atlas.
One useful thing is that every smooth atlas is contained in unique maximal smooth atlas. Thus, if we want to build some **'strange'** smooth structure, we should figure some atlas! and argue that this atlas expands to smooth structure.

Chapter 2. Smooth Maps

"Diffeomorphism" is $F : M \to N$ that is bijective, smooth and satisfying F^{-1} smooth.
"Lie group" is a smooth manifold G that has group structure, satisfying $m : G \times G \to G$, $i : G \to G$ are smooth. One must not confuse with the definition of **"Lie Algebra"**.
"Smooth Covering Map" is $\pi : \tilde{M} \to M$ that is smooth, surjective and similar to topological covering map, some neighborhood's inverse image is component-wisely "diffeomorphism".

If M is smooth manifold and there exists topological covering map, then we can give smooth structure to $\tilde{M}$ to make π smooth covering map. This type of **"Assigning Smooth Structure to satisfy smoothness of some map"** argument is Very Very important.
For Lie group G, there exists a Universal smooth covering map $\pi : \tilde{G} \to G$ (Theorem 2.13)

"Proper Map" is the map that satisfies : Any preimage of compact set is compact. It is useful that **proper continuous map is closed** and closed continuous map with (1) injectiveness, (2) surjectiveness, (3) bijectiveness satisfies (1) topological embedding, (2) quotient map, (3) homeomorphism respectively.

Partition of Unity : Since every topological manifolds are paracompact, any smooth manifold with arbitrary open cover gains smooth partition of unity subordinate to the open cover.

Chapter 3 : Tangent Vectors

We defined smooth manifold or just topological manifold as some "patches" of Euclidean spaces. This means the definition of manifold is independent to its embedding on larger Euclidean space. This is the bottleneck defining tangent vector : since our intuition considers some embedded manifolds.

Tangent Vector is defined as the Derivation. For $C^\infty(M)$ a smooth real valued function on M, tangent vectors are **Derivations**. $X(fg) = f(a)X(g) + g(a)X(f)$
Pushforward is defined by $F_*(X)(f) = X(f \circ F)$. Notice that pushforward in terms of derivation.
Representation of Tangent vector in coordinate system is interesting since coordinate system is local, but tangent vectors are defined on global smooth functions.
Idea is to pushforward by chart map. (U, φ) be the chart then $\varphi_* : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. Since Euclidean spaces, $\frac{\partial}{\partial x^i}\Big|_p$ are basis, its pushforward becomes basis.

In this coordinate, $F : M \to N$ pushforward F_* is represented by jacobian matrix of $\hat{F} = \psi \circ F \circ \varphi^{-1}$.
Calculation by Curve : If γ curve satisfies $\gamma(0) = p$, $\gamma'(0) = X$ then $F_*(X) = (F \circ \gamma)'(0)$.

Chapter 4 : Vector Fields

Tangent Bundle. Tangent bundle is originally set $TM = \coprod_{p \in M} T_pM$. However, this set can be given topology and smooth structure so that TM is smooth manifold and $\pi : TM \to M$ is smooth map. Its coordinate representation is useful that for $(U, (x^1, \cdots, x^n))$ chat for M, TM coordinate is represented with $(x^1(p), \cdots, x^n(p), v^1, \cdots, v^n)$.

Smooth Vector Field is **rough vector field** endowed with smoothness. Rough vector field is just a section of $\pi : TM \to M$. "Smoothness" is smoothness of the section as map of M to TM.
- If $Y_p = Y^i(p) \frac{\partial}{\partial x^i}\Big|_p$, then Y is smooth iff Y^i are smooth.
- Y is smooth iff $\forall f \in C^\infty(U)$, Yf is smooth.

Smooth Vector Field can be used in two different ways. - **Module over ring** $C^\infty(M)$: $(f \cdot Y)_p = f(p)Y_p$ which is multiplying tangent vectors with scalar function values. - **Derivation** and thus mapping $Y : C^\infty(M) \to C^\infty(M)$.

Pushforward of Vector Field : $F : M \to N$ generally does not induce pushforward of vector field since if F is not surjective or not injective, well-definedness might fail. One sufficient condition is diffeomorphism. **Pushforward of Vector Field, to be well-defined needs diffeomorphim**.

Lie Brackets

Lie bracket is operator from two vector fields to vector field. $[V, W] = VW - WV$ become 'derivation'.

Lie Algebra

For Lie group G, $L_g(h) = gh$ and $R_g(h) = hg$ are diffeomorphism. (As a composition $G \xrightarrow{i_g} G \times G \xrightarrow{m} G$)
So, Lie group can be **systematically map any point to any other by a global diffeomorphism**.
Lie Algebra is defined in algebraic sense : Real vector space $\mathfrak{g}$ with bracket operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying
- $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, $[Z, aX + bY] = a[Z, X] + b[Z, Y]$
- $[X, Y] = -[Y, X]$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

For Lie group G, left invariant vector fields are Lie algebra $Lie(G)$. For Lie group, $Lie(G)$ is completely determined by tangent vector at $e \in G$.

Theorem 1 (Theorem 4.20)**.** *G be Lie group. Then the evaluation map $\epsilon : Lie(G) \to T_e(G)$ is vector space isomorphism.*

Idea is using global diffeomorphism L_g and $Lie(G)$ is defined by left-invariant vector fields.
Moreover, from this evaluation map we can find that **Lie group homomorphism induces Lie algebra homomorphism**.
After several chapters, we will determine many Lie algebras of Lie groups. First one to figure out is Lie algebra of $GL(n, \mathbb{R})$

Proposition 1 (Proposition 4.23)**.** *$Lie(GL(n, \mathbb{R})) \to \mathfrak{gl}(n, \mathbb{R})$ is Lie algebra isomorphism. (Here, $\mathfrak{gl}(n, \mathbb{R})$ is the matrix algebra, vector space $\mathcal{M}(n, \mathbb{R})$ with $[A, B] = AB - BA$)*

Chapter 5 : Vector Bundles

Viewing manifolds as vector bundle from the base space is powerful in some situations.
Smooth Vector Bundle is M, E smooth manifolds with $\pi : E \to M$ a smooth map satisfying - $\forall p \in M$, $E_p = \pi^{-1}(p)$ is k dimensional vector space. - $\forall p \in M$, $\exists U \in M$, $\exists \Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a diffeomorphism and sends each fiber E_q into $\{q\} \times \mathbb{R}^k$ (Local trivialization)
One example is Tangent Bundle TM.

Vector Bundles admits **sections** and **frames**. One idea of understanding vector bundle is to understand **Local trivialization and Local Frame are identical**. Local frame induces Local trivialization and vice versa. In this viewpoint, **Tangent Bundle is unique vector bundle over M that all coordinate vector fields be smooth local sections**

- Lie Group is parallelizable. Thus TG is trivial bundle.
- $\mathbb{S}^3$ is parallelizable. We can find $i : \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$ related vector fields

$$X_1 = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4}$$

$$X_2 = -x^3 \frac{\partial}{\partial x^1} - x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^4}$$

$$X_3 = -x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}$$

Also in the above viewpoint, **Bundle Map of the same base space is mapping of smooth sections which is linear over** $C^\infty(M)$. $\pi : E \to M$, $\pi' : E' \to M$ then smooth bundle map $F : E \to E'$ over M associates with $\mathcal{F} : \mathcal{E}(M) \to \mathcal{E}'(M)$ by $\mathcal{F}(\sigma) = F \circ \sigma$

Final gist to point out is **Vector Bundle Construction Lemma**. It is highly technical lemma but useful to prove whether following space is vector bundle. This requires :
- E_p a real vector space for $p \in M$.
- $\{U_\alpha\}_{\alpha \in A}$ an open cover
- Local trivialization-like bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ which sends E_p to $\{p\} \times \mathbb{R}^k$
- A smooth transition map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ that $\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$.

Chapter 6 : The Cotangent Bundle

Cotangent Bundle is $T^*M = \coprod_{p \in M} T_p^*M$. Standard coordinate for cotangent bundle is dual of $\frac{\partial}{\partial x^i}$, λ^i thus (x^i, λ^i).
Smooth covector field is just a section of Cotangent bundle. One good thing is that **Pullback of smooth covector field always exist**. $(F^*\omega)(X) = \omega(F_*(X))$. This is different from vector fields, which such a diffeomorphism condition is needed.
One type of smooth covector field is **differential**. For $f : M \to \mathbb{R}$, df is defined by $df_p(X_p) = X_p f$. Or, in curve view, $(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t))$. Its pullback is easy to compute : $G^*(df) = d(f \circ G)$.

Calculation Formula of pushforward, pullback

It is useless to know sophisticated formulas but don't know the method calculating pushforward and pullback. Every instances requires calculations so we need to handle computations easily.
Pushforward

$$F_* \frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j}\bigg|_{F(p)}$$

e.g. $F(t) = (cos(t), sin(t))$ then

$$F_*\left(\frac{\partial}{\partial t}\right) = \frac{\partial(cos(t))}{\partial t}\frac{\partial}{\partial x} + \frac{\partial sin(t)}{\partial t}\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Change of basis

$$X = X^i \frac{\partial}{\partial x^i}\bigg|_p = \tilde{X^j} \frac{\partial}{\partial \tilde{x^j}}\bigg|_p$$

$$\tilde{X^j} = \frac{\partial \tilde{x^j}}{\partial x^i}(\hat{p}) X^i$$

Lie Bracket

$$[V, W] = (VW^j - WV^j)\frac{\partial}{\partial x^j}$$

e.g.

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$\begin{aligned}[X,Y] &= (X(z) - Y(0))\frac{\partial}{\partial x} + (X(0) - Y(-z))\frac{\partial}{\partial y} + (X(-x) - Y(y))\frac{\partial}{\partial z} \\ &= y\frac{\partial}{\partial x} + (-x)\frac{\partial}{\partial y} + (0-0)\frac{\partial}{\partial z} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\end{aligned}$$

Pullback

$$G^*\omega = (\omega_j \circ G)d(y^j \circ G)$$

e.g. $G(x,y,z) = (x^2y, ysin(z))$ and $\omega = udv + vdu$

$$\begin{aligned}G^*(\omega) &= (ysin(z))d(x^2y) + (x^2y)d(ysin(z)) \\ &= (ysin(z))(2xydx + x^2dy) + (x^2y)(sin(z)dy + ycos(z)dz) \\ &= 2xy^2sin(z)dx + 2x^2ysin(z)dy + x^2y^2cos(z)dz\end{aligned}$$

Line Integral

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt$$

e.g. $\omega = \frac{xdy - ydx}{x^2+y^2}$ and $\gamma(t) = (cos(t), sin(t))$

$$\int_\gamma \omega = \int_0^{2\pi} (-sin(t)) \cdot (-sin(t)) + (cos(t)) \cdot (cos(t))dt = 2\pi$$

Chapter 7. Submersions, Immersions, and Embeddings

For the map $F : M \to N$ the map is
- **Submersion** if F is surjective (So, rank$F$ = dim$N$)
- **Immersion** if F is injective (So, rank$F$ = dim$M$)
- **Smooth Embedding** if F is immersion and is a topological embedding, i.e. $F(M)$ and M are homeomorphic where $F(M)$ induce with **subspace topology**

Figure Eight is important example which is immersion but not smooth embedding. Figure Eight is defined by $\gamma : (-\pi/2, 3\pi/2) \to \mathbb{R}^2$, $\gamma(t) = (sin(2t), cos(t))$.

Proposition 2 (Proposition 7.4)**.** *$F : M \to N$ is injective immersion. If M is compact or F is proper map (i.e. every preimage of compact set is compact) then F is topological embedding*

Rank Theorem

By the inverse function theorem, we gain powerful theorem that represents submersion, immersion and generally, constant rank maps.

Theorem 2 (Theorem 7.13, Representation of Constant Rank Map)**.** *$F : M \to N$ is constant rank k smooth map. Then for every $p \in M$, there exist smooth coordinates $(x^1, \cdots, x^m)$ of M that F has the coordinate representation*

$$F(x^1, \cdots, x^k, x^{k+1}, \cdots, x^m) = (x^1, \cdots, x^k, 0, \cdots, 0)$$

Thus, from the Rank theorem, we can figure out if $F : M \to N$ is smooth constant rank map,
- If F is surjective then it is a submersion
- If F is injective then it is an immersion
- If F is bijective, then it is a diffeomorphism

Moreover, if $\pi : M \to N$ is submersion then
- π is an open map
- If π is surjective then it is quotient map

Quotient Map as in point set topology, can induce maps which are impossible at continuous maps.

Proposition 3 (Proposition 7.17). *If $\pi : M \to N$ is surjective submersion (so quotient map) and $F : N \to P$ be any map. Then F is smooth iff $F \circ \pi$ is smooth.*

Proposition 4 (Proposition 7.18). *If $\pi : M \to N$ is surjective submersion (so quotient map) and $F : M \to P$ be smooth map that is constant on the fibers of π then there exist a smooth map $\tilde{F} : N \to P$ s.t. $\tilde{F} \circ \pi = F$.*

Chapter 8. Submanifolds

With the knowledge of submersions, immersions we can define 'submanifolds'.
Embedded Submanifold is if $\forall p \in S$, there exists chart (U, φ) such that $U \cap S$ is k slice of U. Then S with subspace topology is topological manifold and have unique smooth strucutre on S such that $S \hookrightarrow M$ is a smooth embedding. Converse is also true. Therefore, **Embedded Submanifolds are exactly the image of Smooth Embedding**.
Immersed Submanifold is $S \subset M$ with k manifold topology (**Do not need to be subspace topology**) with smooth structure that makes $S \hookrightarrow M$ be smooth immersion. Similar to embedded submanifolds, **Immersed Submanifolds are exactly the image of Injective Immersions**. One significant example is **Figure Eight**. However, not always given subset cannot be an immersed submanifold, the example **Cusp** $x^3 = y^2$ is it.
Interestingly, immersed submanifold can have more than one topology and smooth structure to be immersed submanifold. However, if **Topology is given then there exists only one smooth structure**. In other words, (Top1, Smooth1), (Top2, Smooth2) ... can exist but if (Top, Smooth1) and (Top, Smooth2) are two structures then Smooth1 = Smooth2. (Exercise 8.12)

Level Set being embedded submanifolds

Theorem 3 (Theorem 8.8, Constant Rank Level Set Theorem). *If $\Phi : M \to N$ be smooth map with constant rank k. Then each level set of Φ is a **closed embedded submanifold** of codimension k*

Here, **constant rank** condition can be weakened in some situations

Corollary 1 (Corollary 8.9). *If Φ is a submersion then each level set is closed embedded submanifold.*

Corollary 2 (Corollary 8.10). ***(Regular Level Set Theorem)** Each regular level set ($\Phi : T_pM \to T_{\Phi(p)}N$ is surjective for every level set) is a **closed embedded submanifold** of codimension equal to* $\dim N$.

Moreover, Proposition 8.12 states **Every embedded submanifold is locally, level set of some submersion**. As a consequence, we can **characterize the tangent space of an embedded submanifold**.

Lemma 1 (Lemma 8.15, Characterization of tangent space). *If $S \subset M$ is an embedded submanifold and $\Phi : U \to N$ is any local defining map for S ($U \subset M$ and $U \cap S$ become regular level set of Φ) then*

$$T_pS = Ker\Phi_* : T_pM \to T_{\Phi(p)}N$$

Restricting to Submanifolds

Nice property of immersed submanifolds or embedded submanifolds are **these submanifolds enables restrict domain or codomain of smooth map**.
- If $F : M \to N$ is smooth map and $S \subset M$ is an (immersed or embedded) submanifold then $F|_S : S \to N$ is smooth.
- If $F : M \to N$ is smooth map and $F(M) \subset S$ which is immersed or embedded submanifold of N. If $F : M \to S$ is continuous, then $F : M \to S$ is smooth. (**Restricting codomain is more tricky**)

Also, we can restrict vector fields and covector fields
- If $S \subset M$ is an immersed submanifold, Y is tangent vector field of M is also tangent to S then there exists unique smooth vector field $Y|_S$ which is i-related to Y.
- Since every pullback of covector field is well-defined, $\omega|_S$ exists, which is $i\omega$.

Lie Subgroups

Lie subgroup is very important topic. We originally know Lie group $GL(n, \mathbb{R})$ and it has $\mathfrak{gl}(n, \mathbb{R})$ as a Lie algebra. Proposition 8.30 enables discussion of Lie group can be hung with embedded submanifold. G **be a Lie group and** $H \subset G$ **is subgroup and embedded submanifold. Then** H **is closed Lie subgroup of** G.
Thus, we can just check **(1) Subgroup condition** and **(2) Embedded submanifold (Level set or ...)** to figure various Lie groups.

Identification of Lie algebra of a Lie subgroup. Lie algebra of a Lie subgroup can be identified as : for $H \subset G$ is Lie subgroup,

$$\tilde{\mathfrak{h}} = \{X \in \text{Lie}(G) : X_e \in T_e H\}$$

is canonically isomorphic to Lie(H). This is because Lie algebra of Lie subgroup is totally defined by tangent vector at identity.

- $O(n)$
$O(n)$ is level set of $\Phi : GL(n, \mathbb{R}) \to S(n, \mathbb{R})$

$$\Phi(A) = A^T A$$

$\Phi : T_A GL(n, \mathbb{R}) \to T_{\Phi(A)} S(n, \mathbb{R})$ can be calculated by curve $\gamma(t) = A + tB$,

$$\Phi_* B = \frac{d}{dt}\bigg|_{t=0} \Phi(A + tB) = B^T A + A^T B$$

So Φ is surjective, which means $\Phi : GL(n, \mathbb{R}) \to S(n, \mathbb{R})$ is submersion. Thus $O(n) = \Phi^{-1}(I_n)$ is embedded submanifold, and also subgroup thus Lie subgroup of $GL(n, \mathbb{R})$.

$O(n)$ is thus, $\frac{n(n-1)}{2}$ dimensional Lie group.

Moreover, $T_{I_n} O(n) = \text{Ker}\Phi$ by the characterization of tangent space of embedded submanifold, so $T_{I_n} O(n) = \{B \in \mathfrak{gl}(n, \mathbb{R}) : B^T + B = 0\}$

$$\mathfrak{o}(n) = \text{Lie}(O(n)) \approx \{B \in \mathfrak{gl}(n, \mathbb{R}) : B^T + B = 0\}$$

- $SL(n, \mathbb{R})$
$SL(n, \mathbb{R})$ is level set of $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$.

$$d(det)_A(B) = (detA)tr(A^{-1}B)$$

so $d(det)_A$ is nonzero, thus det is submersion. This means $SL(n, \mathbb{R})$ is Lie subgroup of dimension $n^2 - 1$.

$$T_{I_n} SL(n, \mathbb{R}) = \text{Ker}(det_*) = \{B \in \mathfrak{gl}(n, \mathbb{R}) : tr(B) = 0\}$$

so we can find

$$\mathfrak{sl}(n, \mathbb{R}) = \text{Lie}(sl(n, \mathbb{R})) = \{B \in \mathfrak{gl}(n, \mathbb{R}) : tr(B) = 0\}$$

- $SO(n)$
$SO(n)$ is open subset of $O(n)$ thus Lie group. Also since $SO(n)$ is open in $O(n)$, Lie algebra are identical.

$$\mathfrak{so}(n) = \mathfrak{o}(n)$$

- $GL(n, \mathbb{C})$
By $\beta : GL(n, \mathbb{C}) \to GL(2n, \mathbb{R})$ that

$$\beta \begin{pmatrix} a_1^1 + ib_1^1 & \cdots & a_1^n + ib_1^n \\ \vdots & \ddots & \vdots \\ a_n^1 + ib_n^1 & \cdots & a_n^n + ib_n^n \end{pmatrix} = \begin{pmatrix} a_1^1 & -b_1^1 & \cdots & a_1^n & -b_1^n \\ b_1^1 & a_1^1 & \cdots & b_1^n & a_1^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n^1 & -b_n^1 & \cdots & a_n^n & -b_n^n \\ b_n^1 & a_n^1 & \cdots & b_n^n & a_n^n \end{pmatrix}$$

becomes injective Lie group homomorphism. So $GL(n, \mathbb{C})$ can be thought of embedded submanifold of $GL(2n, \mathbb{R})$. $\beta : \text{Lie}(GL(n, \mathbb{C})) \to \text{Lie}(GL(2n, \mathbb{R}))$ is Lie algebra homomorphism. Using β, one can find the map

$$\text{Lie}(GL(n, \mathbb{C})) \xrightarrow{\epsilon} T_{I_n} GL(n, \mathbb{C}) \xrightarrow{\varphi} \mathfrak{gl}(n, \mathbb{C})$$

which is originally vector space isomorphism becomes Lie algebra isomorphism (Here, we need to prove $\varphi \circ \epsilon$ preserves bracket operation).

- $\mathbb{S}^3$
Define $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ and bilinear product $(a,b)(c,d) = (ac - \bar{d}b, da + b\bar{c})$ for $a,b,c,d \in \mathbb{C}$. This is **quaternion algebra**.
Its basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ are $\mathbf{1} = (1,0)$, $\mathbf{i} = (i,0)$, $\mathbf{j} = (0,1)$, $\mathbf{k} = (0,i)$. It satisfies

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$$

Quaternion multiplication is associative but not commutative, and we can define norm and inner product on quaternions. $\langle p, q \rangle = \frac{1}{2}(pq + qp)$.
Then unit quaternions is Lie group. This is because first, $\mathbb{H}$ multiplication map and inverse map is smooth, by calculation formula via its basis. Unit quaternions are subgroup. Additionally, it is embedded submanifold as a view of level set $\|\cdot\| : \mathbb{H} \to \mathbb{R}$. Thus restricting domain and codomain of multiplication map and inverse map gives $m|_{\mathcal{S}}$ and $i|_{\mathcal{S}}$ are smooth. Its smooth structure is diffeomorphic to $\mathbb{S}^3$ if we identify $\mathbb{H}$ as $\mathbb{R}^4$.

In this point of view, the statement $\mathbb{S}^3$ can be parallelizable is easy to show. Since vector fields $X_1|_q = q\mathbf{i}$, $X_2|_q = q\mathbf{j}$, $X_3|_q = q\mathbf{k}$ are non vanishing tangent vector fields, $\mathbb{S}^3$ diffeormorphic to unit quoternions is parallelizable.
Same logic can be applied to **octonions**. However after **sedenions**, parallelizable argument does not applied becaus **division algebra** condition brokes down. ($X_i|_q = qe_i$ nonvanishing argument depends the fact that if $pq = 0$, $p = 0$ or $q = 0$, but after sedenions, this does not hold)

Chapter 9. Lie Group Actions

Lie group actions make us to generate a lot of useful maifolds. **Action** means $\theta : G \times M \to M$ with several conditions and action is continuous if θ is continuous, action is smooth if θ is smooth.
For smooth acion θ, each θ_g then playes diffeomorphism on M.

Theorem 4 (Theorem 9.7, Equivariant Rank Theorem)**.** *M, N are smooth manifolds and G a Lie group. If $F : M \to N$ is smooth and **equivariant** (which means, $F(g \cdot p) = g \cdot F(p)$ holds) w.r.t. **transitive** smooth action on M and smooth action on N. Then F is constant rank so each level sets are closed embedded submanifolds.*

Equivariant Rank Theorem is convinient tool of checking the map being constant rank. One should check **equivariantness** and **transitive** action on the domain.

Proposition 5 (Proposition 9.8)**.** *Lie group homomorphism $F : G \to H$ is constant rank.*

For the proof, define action G on H by $\theta_g(h) = F(g)h$.

- $U(n)$
Define $\Phi : GL(n,\mathbb{C}) \to \mathcal{M}(n,\mathbb{C})$ by $\Phi(A) = AA$. Then $U(n) = \Phi^{-1}(I_n)$.
If we want to prove $U(n)$ is submanifold, we had to prove I_n is regular value. However, with equivariant rank theorem, we can choose alternative way.
Define right action with Lie group $G = GL(n,\mathbb{C})$. This acts on $GL(n,\mathbb{C})$ by $A \cdot B = AB$ and $\mathcal{M}(n,\mathbb{C})$ by $X \cdot B = BXB$. Then Φ is equivariant map and G acts transitively on $GL(n,\mathbb{C})$. Thus Φ is equivariant map.
As our routine proof, Lie algebra of $U(n)$ can be calculated by

$$\Phi_* B = \left.\frac{d}{dt}\right|_{t=0} \Phi \circ \gamma(t) = \left.\frac{d}{dt}\right|_{t=0} (I + tB)^*(I + tB) = B^* + B$$

So $U(n)$ is n^2-dimensional Lie subgroup and

$$\mathfrak{u}(n) = \{B \in \mathfrak{gl}(n,\mathbb{C}) : B^* + B = 0\}$$

- $SL(n,\mathbb{C})$
$\det : GL(n,\mathbb{C}) \to \mathbb{C}^\times$ is Lie group homomorphism. Thus Proposition 9.8 states $SL(n,\mathbb{C})$ is an embedded Lie subgroup. It's dimension will be $(2n^2 - 2)$.
We have saw that $det = tr$ at the identity matrix. Thus,

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}) : \text{tr}(A) = 0\}$$

- $SU(n)$

$SU(n)$ is embedded submanifold of $U(n)$ by restricting domain of det to $\det|_{U(n)} : U(n) \to \mathbb{C}^\times$. Then this map can restrict codomain by $\det|_{U(n)} : U(n) \to \mathbb{S}^1$. Which is Lie group homomorphism and $SU(n)$ is level set.
As a closed submanifold of $U(n)$ and $SL(n, \mathbb{C})$

$$\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C})$$

- Further relationships of $O(n)$, $SO(n)$, $U(n)$, $SU(n)$

(1) $SO(2)$, $U(1)$, $\mathbb{S}^1$ are all isomorphic Lie groups
This is easy when we see these manifolds on algebraic view but more topological perspective exists (from the mathstackexchange answer)
$\phi : U(1) \to SO(2)$ by

$$a + bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then this map is injective and equivariant via $\mathbb{S}^1$ action. It is constant rank and since injective, this is immersion. $U(1)$ is compact so closedness of the map is gauranteed so is embedding. Since codimension is zero and $U(1)$, $SO(2)$ are connected, they are diffeomorphic. Under the map, group operation still holds so is Lie group isomorphism.

(2) $U(1) \times SU(n)$ is diffeomorphic to $U(n)$ but not isomorphic as a Lie group.
Define $U(n) \to SU(n) \times U(1)$ by

$$A \longmapsto \frac{1}{\sqrt[n]{\det(A)}} A \times \sqrt[n]{\det(A)}$$

Then it is smooth. First think this mapping on $GL(n, \mathbb{C}) \to GL(n, \mathbb{C}) \times \mathbb{C}^\times$. Then restricting to embedded submanifold solves it.
However, these are not isomorphic. It's because they do not have isomorphic centers, for $U(n)$, center is isomorphic to $\mathbb{S}^1$ but $U(1) \times SU(n)$ center is isomorphic to $\mathbb{S}^1 \times \mathbb{Z}_n$.

(3) $SU(2)$ is isomorphic to unit quaternions
Idea is thinking representing 'basis elements'. We can find basis

$$\left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, A_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$$

Then one can show that with $a^2 + b^2 + c^2 + d^2 = 1$, $a, b, c, d \in \mathbb{R}$, formula $aA_1 + bA_2 + cA_3 + dA_4$ is exactly the representation of $SU(2)$.

(4) $SO(3)$ is isomorphic to

$$SU(2)/\{\pm e\}$$

For unique quaternion q, $\rho(q)$ be the matrix representation of $v \mapsto qvq$ with respect to $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. Then this map preserves inner product so $\rho(q) \in SO(3)$ and this ρ is surjective with kernel $\{\pm 1\}$.

(5) $SL(n, \mathbb{R})$ is diffeomorphic to $SO(n) \times \mathbb{R}^{n(n+1)/2-1}$. and $SL(n, \mathbb{C})$ is diffeomorphic to $SU(n) \times \mathbb{R}^{n^2-1}$.
These arguments are by the QR decomposition. We can show the uniqueness of QR decomposition and its process is smooth (because Gram-Schmidt process) so diffeomorphism established.

Quotient Manifold

We will look at the quotient manifolds. This **Quotient Manifolds are very very important object on manifolds since it has a lot of applications**. Before that, we need to define **proper action**. **Proper action** has three equivalent definitions,

- $G \times M \to M \times M$ by $(g, p) \longmapsto (g \cdot p, p)$ is proper map
- $\forall K \subset M$ compact, $G_K = \{g \in G : (g \cdot K) \cap K \neq \phi\}$ compact
- $\{p_i\}$ convergent sequence in M and $\{g_i \cdot p_i\}$ converges. Then some subsequence of $\{g_i\}$ converges.

Theorem 5 (Theorem 9.16, Quotient Manifold Theorem). *Lie group G acts* ***smoothly, freely*** *($g \neq e$ then does not fix any point),* ***properly*** *on a smooth manifold M. Then M/G the orbit space is a topological manifold and has a unique smooth structure that makes $\pi : M \to M/G$ be a smooth submersion.*

Proof of Quotient Manifold Theorem is long but instructive. The art of the proof is showing **adapted coordinate chart**. In conclusion, M have an **adapted coordinate chart** for each point, $(U, (x^1, \cdots, x^k, y^1 \cdots, y^n))$ that
- $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^k \times \mathbb{R}^n$
- (Orbit) $\cap U$ is either empty or of form $\{y^1 = c^1, \cdots, y^n = c^n\}$

Properness and **freeness** really matters in this proof so these two conditions are **Essential** to make Quotient Manifold Theorem holds.
Quotient Manifold Theorem have a lot of applications.

Smooth covering map

For $\pi : \tilde{M} \to M$ a smooth covering map, $\mathcal{C}_\pi(\tilde{M})$ a covering group is discrete Lie group and its action is smooth, free, proper.
In addition, if $\mathcal{C}_\pi(\tilde{M})$ is normal, then M is diffeomorphic to $\tilde{M}/\mathcal{C}_\pi(\tilde{M})$ by following theorem.

Theorem 6 (Theorem 9.19). *$\tilde{M}$ a connected smooth manifold, Γ is a discrete group whose action is smooth, free, proper. Then $\tilde{M}/\Gamma$ has a unique smooth structure making $\pi : \tilde{M} \to \tilde{M}/\Gamma$ to be a smooth normal covering map.*

Homogeneous Space

If Lie group action is transitive, we call M a homogeneous G-space. Then **Homogeneous Spaces are exactly the Quotient of Lie group by closed Lie subgroup**.

Theorem 7 (Theorem 9.22). *If G a Lie group and H closed Lie subgroup, then G/H has a unique smooth structure that makes $\pi : G \to G/H$ be smooth submersion. Moreover, G/H is homogeneous G-space.*

Theorem 8 (Theorem 9.24). *M be a homogeneous G-space. Then for any $p \in M$, $F : G/G_p \to M$, $F(gG_p) = g \cdot p$ is an equivariant diffeomorphism.*

0.0.1 Making set to smooth manifold

Proposition 6 (Proposition 9.31). *Suppose X a set and transitive action of Lie group G acts on X. Suppose that isotropy group of $p \in X$ is a closed Lie subgroup G. Then X has a unique smooth manifold structure making the action smooth.*

E.g. **Grassmann manifold** $G_k(\mathbb{R}^n)$
Initially, defining Grassmann manifold is by following method :
For $P \in G_k(\mathbb{R}^n)$, write $Q = P^\perp$. Define the neighborhood of P by the k dimensional subspaces that

$$U_P = \{R \in G_k(\mathbb{R}^n) : R \cap Q = \phi\}$$

. Then the map $\varphi : U_P \to L(P, Q)$ by the inverse map $\psi : L(P, Q) \to U_P$,

$$\psi(A) = \{x + Ax : x \in P\}$$

. Then identifying $L(P, Q)$ as $k \times (n-k)$ matrix, Grassmann manifold is smooth manifold under these atlas

$$\{(U_P, \varphi_P)\}$$

Another method defining smooth structure on just the 'set' of k dimensional subspaces is using Proposition 9.31. Since the action $GL(n, \mathbb{R})$ acts transitively and its isotropy group is

$$H = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : A \in GL(k, \mathbb{R}), D \in GL(n-k, \mathbb{R}) \right\}$$

We can show that two smooth structures are the same by uniqueness argument on Proposition 9.31. We can show that under the original smooth structure, group action is smooth.
Let me describe this more precisely. We need to show the following map is finally smooth.

$$\begin{tikzcd}
\psi(A) = \{x + Ax : x \in P\} \in G_k(\mathbb{R}^n) \arrow[r, "B \in GL(n,\mathbb{R})"] \arrow[d, "\psi"] & B\psi(A) = \{Bx + BAx : x \in P\} \in G_k(\mathbb{R}^n) \arrow[d, "\varphi"] \\
A \in L(P, Q) \arrow[r] & A' \in L(P', Q')
\end{tikzcd}$$

Here, $\{Bx + BAx : x \in P\} = \{Bx + A'Bx : x \in P\}$ for some A'. We need coordinate of A' are smooth function of A.
$x' \in P$, then $Bx' + A'Bx' = Bx + BAx$ for some $x \in P$. Then $Bx + BAx - Bx' \in Q'$ so if we set $\pi_{P'} : \mathbb{R}^n \to P'$ a projection, $\pi_{P'}(Bx + BAx - Bx') = 0$ thus $(\pi_{P'} \circ (B + BA))^{-1}(x') = x$ and

$$A'(Bx') = (B + BA) \circ (\pi_{P'} \circ (B + BA))^{-1}(x') - Bx'$$

$$A' = (B + BA) \circ (\pi_{P'} \circ (B + BA))^{-1} \circ B^{-1} - Id$$

This is smooth function of both A and B so smoothness of action is proven.

e.g. Flag manifold $F_K(V)$
Similar to Grassmann manifolds, if V is real vector space and $K = (k_1, \cdots, k_m)$ is $0 < k_1 < \cdots < k_m < n$ integers, then flag manifold is built from the following set :

$$\{S_1 \subset S_2 \subset \cdots \subset S_m \subset V : \dim(S_i) = k_i\}$$

and give smoothness structure by $GL(V)$ acting transitively on this set.

Connectivity of Lie group

Proposition 7 (Proposition 9.34)**.** *Lie group G acts smoothly, freely, properly on a manifold M. If G and M/G are connected, then M is connected.*

Representation of Lie group, Lie algebra

A finite dimensional **representation of Lie group** G is Lie group homomorphism $\rho : G \to GL(V)$ for finite dimensional real or complex vector space V.
If ρ is injective (**faithful**) then we can identify G as Lie subgroup $\rho(G) \subset GL(V)$ or, Lie subgroup of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.
A finite dimensional **representation of Lie algebra** $\mathfrak{g}$ is Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$. Thus, if there exists a representation of Lie group then there exists a representation of Lie algebra of the Lie group. Lie algebra representation is more 'good' than Lie group representation.

Theorem 9 (Ado's Theorem)**.** *Every finite-dimensional Lie algebra admits a faithful finite-dimensional representation.*

One instance of representation method is **Adjoint Representation**. For Lie group G, the adjoint representation is

$$\mathrm{Ad}(g) = (C_g)_* : \mathfrak{g} \to \mathfrak{g}$$

where $C_g(h) = ghg^{-1}$ is conjugation. Then $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is smooth. (This is shown on Lee's textbook)
However, **Adjoint representation for Lie algebra** is more subtle but interesting. For Lie algebra (finite-dimensional) $\mathfrak{g}$, define

$$\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{ad}(X)Y = [X, Y]$$

So $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. However, we need to show that this representation is Lie algebra homomorphism... Interestingly, this holds because of **Jacobi Identity**. The problem is, we don't know if

$$\mathrm{ad}([X, Y]) = \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)$$

But, this equation applied to $Z \in \mathfrak{g}$ became

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]]$$

which is just a Jacobi identity!!

Chapter 10. Embedding and Approximation Theorems

All of this chapter is saying **Every manifold can be thought as a embedded submanifold of Euclidean space**.
This argument uses **Sard's Theorem** which is if $F : M \to N$ is any smooth map then the set of critical values of F is measure zero in N.

Theorem 10 (Theorem 10.9, Whitney Immersion Theorem)**.** *Every smooth n-manifold admits an immersion to* $\mathbb{R}^{2n}$*.*

This theorem uses argument that if we give a little **peturbation** to a smooth map $F : M \to \mathbb{R}^m$ then we can get smooth immersion $\tilde{F} : M \to \mathbb{R}^m$ if $m \geq 2n$ (This argument itself is very interesting, it uses inductive argument, finding little perturbation on little open set).
Then again inductively constructing 'injectiveness' by finding little perturbation gives the **Whitney Embedding Theorem**.

Theorem 11 (Theorem 10.11, Whitney Embedding Theorem)**.** *Every smooth n-manifold admits a proper smooth embedding into* $\mathbb{R}^{2n+1}$

Strong Whitney Embedding Theorem and Strong Whitney Immersion Theorem holds for more strict dimension! (Interestingly!) If I study the proof, then I will add the proof in other post.
Second topic is **Any continuous function** $F : M \to N$ **can be approximated to smooth function** $\tilde{F} : M \to N$. This is called **Whitney Approximation Theorem**.

Theorem 12 (Theorem 10.16, Whitney Approximation Theorem)**.** $F : M \to \mathbb{R}^k$ *a continuous function. Then given positive function* $\delta : M \to \mathbb{R}$*, there exists smooth function* $\tilde{F} : M \to \mathbb{R}^k$ *that is* δ*-close to* F*. This* $\tilde{F}$ *could be set relative to closed subset* $A \subset M$ *if* F *is smooth on* A*.*

Tubular Neighborhood

What is the advantage of the perspective : M an any smooth manifold can be thought as the embedded submanifold of $\mathbb{R}^k$?
Here, the answer is existence of **Tubular Neighborhood**. We can make M inflate slightly so we can imagine some **tube** wrapping the manifold.
First, we define **Normal Bundle**.

$$NM = \coprod_{x \in M} N_x M = \{(x, v) \in T\mathbb{R}^n : x \in M,\, v \in N_x M\}$$

So, Normal bundle is in set view, the subset of tangent bundle of $\mathbb{R}^n$ which is diffeomorphic to $\mathbb{R}^{2n}$. Every element has a representation (x, v).
There exists a smooth structure making $\pi_{NM} : NM \to M$ surjective submersion. It is **smooth vector bundle** of rank $n - m$ over M, **embedded submanifold** of $T\mathbb{R}^n$.
Tubular manifold is defined by neighborhood U of M in $\mathbb{R}^n$ which satisfies : for $E : NM \to \mathbb{R}^n$, $E(x, v) = x + v$

$$E : V \to U$$

$$V = \{(x, v) \in NM : |v| < \delta(x)\}$$

is a diffeomorphism.
Not surprisingly, **Every embedded submanifold of** $\mathbb{R}^n$ **has a tubular neighborhood**. This is very nice. Because tubular neighborhood looks like **tube**, **similar to original manifold** (Smooth retraction exists). Its application as follows :

Theorem 13 (Theorem 10.21, Whitney Approximation on Manifolds)**.** *If* $F : N \to M$ *a continuous map, then* F *is homotopic to smooth map* $\tilde{F} : N \to M$*.*

Proof idea : Embed M to euclidean space and construct its tubular neighborhood. Then thinking as F mapping to euclidean space, small perturbation makes smooth map's image be totally contained in tubular manifold. Using retraction concludes the proof.

Proposition 8 (Proposition 10.22)**.** *If* $F, G : M \to N$ *homotopic relative to* A *(closed subset) then they are smoothly homotopic relative to* A*.*

The same idea.

Problem 10.5 If M is smooth, compact manifold and admits a nowhere vanishing vector field, then there is a smooth map $F : M \to M$ that is homotopic to identity and has no fixed points.
Proof idea : Embed M to Euclidean space. Imagine tubular manifold. Now for each point $p \in M$, move p by smooth vector field in euclidean space. Now small move makes is still remain in the tubular manifold (compactness is used) and retract it.

Chapter 11. Tensors

For vector spaces V, W, the map

$$T : V \times \cdots \times V \to \mathbb{R}$$

is called covariant k-tensor if it is multilinear. Then we can define the tensor product which $S \in T^k(V), T \in T^l(V)$ then

$$S \otimes T(X_1, \cdots, X_{k+l}) = S(X_1, \cdots, X_k)T(X_{k+1}, \cdots, X_{k+l})$$

$T^k(V)$ has basis elements $\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$ which ϵ^i are dual basis to any basis of V.
However, there is a more abstract definition of tensor product of vector spaces. This uses quotient space. Main result of this concept is considering tensor product to satisfy

$$a(v \otimes w) = (av) \otimes w = v \otimes (aw)$$

$$v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$$

$$v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w$$

For f.d.v.s V, W then tensor product is **universal** that for any vector space X with bilinear map $A : V \times W \to X$, there exists a unique bilinear map $\tilde{A} : V \otimes W \to X$ that commutes with A.
Also one thing to note is that there exists **canonical (do not depend on basis representation)** isomorphism

$$V^* \otimes W^* \approx B(V, W)$$

$$(\omega, \eta) \longleftrightarrow \Big((v, w) \mapsto \omega(v)\eta(w)\Big)$$

thus $T^k(V)$ is canonically isomorphic to tensor product of k copies of V^*, so we can define contravariant and mixed tensors by

$$T^k(V) = V^* \otimes \cdots \otimes V^*$$

$$T_k(V) = V \otimes \cdots \otimes V$$

$$T^k_l(V) = V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V$$

Tensor on Manifolds

Now we can define tensor bundles and its smooth sections
- covariant k-tensor bundle

$$T^k M = \coprod_{p \in M} T^k(T_p M)$$

$\mathscr{T}^k(M)$ be its smooth sections. In **local coordinate view**

$$\sigma = \sigma_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

- covariant l-tensor bundle

$$T_l M = \coprod_{p \in M} T_l(T_pM)$$

$\mathscr{T}_l(M)$ be its smooth sections. In **local coordinate view**

$$\sigma = \sigma^{j_1 \cdots j_l} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_k}}$$

- mixed (k, l)-tensor bundle

$$T^k_l M = \coprod_{p \in M} T^k_l(T_pM)$$

$\mathscr{T}^k_l(M)$ be its smooth sections.
Then, we can define several operations on these tensor bundles and its sections.
- Tensor product
$\sigma \in \mathscr{T}^k(M)$, $\tau \in \mathscr{T}^l(M)$ then we can define $\sigma \otimes \tau$
- Pullbacks
$F : M \to N$ smooth then we can define $F^* : T^kN \to T^kM$. Since it is $C^\infty(M)$ **linear, F^* is a bundle map**.

Riemannian Metric

Riemannian metric is example of symmetric tensors. Symmetric tensors are covariant k tensor that satisfies

$$T(X_1, \cdots, X_i, \cdots, X_j, \cdots X_k) = T(X_1, \cdots, X_j, \cdots, X_i, \cdots X_k)$$

.
We denote these tensors by $\Sigma^k(V)$. Any tensor can be mapped to symmetric tensor by symmetrization.
Riemannian Metric is **smooth symmetric 2-tensor field that is positive definite at each point**. So each g_p determines inner product on T_pM. We can represent in coordinate system,

$$g = g_{ij} dx^i \otimes dx^j = g_{ij} dx^i dx^j$$

since it is symmetric. Each g_p determining inner product allows us to define norm, angle on T_pM. Moreover the length of curve can be defined and importantly, it is independent of parametrization:

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt$$

- Isometry
$F : (M, g) \to (\tilde{M}, \tilde{g})$ is an isometry if it is diffeomorphism and $F^*\tilde{g} = g$

- Flat
$\forall p \in M$, $\exists U \subset M$ that $(U, g|_U)$ isometric to an open subset of $(\mathbb{R}^n, \bar{g})$

Actually, (M, g) to be flat, it is equivalent to following statement: "Each point of M has a smooth coordinate neighborhood in which the coordinate frame is orthonormal"
Proof is easy : $(U, g|_U)$ isomorphic to some $(V, \bar{g}|_V)$ of $V \subset \mathbb{R}^n$ open
$\Leftrightarrow \exists \varphi : U \to V$ that $\varphi(v_1), \cdots, \varphi(v_n)$ are orthonormal or this coordinate frame φ is orthonormal.

Interesting properties of riemannian metrics are :
- There exists **Smooth orthonormal frame** locally. (Gram-Schmidt)

- Distance function d defined by

$$d(p, q) = \inf_\gamma \{\gamma : I \to M | \gamma(0) = p, \gamma(1) = q\}$$

induces the same topology with M.

- Like as the setting of Whiteney's embedding theorem, for $S \subset M$ a Riemannian submanifold, we can define normal bundle.

$$NS = \coprod_{p \in S} N_pS$$

We can do this by defining $\Phi : \pi^{-1}_{NS}(S \cap U) \to (S \cap U) \times \mathbb{R}^{n-m}$, $\Phi(a^i E_i|_x) = (x, (a^{m+1}, \cdots, a^n))$ where E_i are adapted orthonormal frame. Then its transition matrix is smooth so it is a bundle.

- Tangent-Cotangent Isomorphism
$\tilde{g} : TM \to T^*M$ bundle map can be defined from the Riemannian metric.

$$\tilde{g}(X_p)(Y_p) = g_p(X_p, Y_p)$$

since $\tilde{g}$ is $C^\infty(M)$ linear, $\tilde{g}$ is a bundle map. Moreover clearly $\tilde{g}$ is injective so for dimensionality reason $\tilde{g}$ is bijective so bundle isomorphism.
In coordinate representation, $\tilde{g}(X) = g_{ij}X^i dy^j$ thus it is natural to notate $X_j = g_{ij}X^i$ and say **flat** : $X \mapsto X^\flat$.
Similarly, for $\omega \in \mathscr{T}^*M$, $\tilde{g}^{-1}(\omega) = g^{ij}\omega_j \frac{\partial}{\partial x^i}$ which we notate **sharp** : $\omega \mapsto \omega^\sharp$.
This enables us to define **"gradient"** as in euclidean space.

$$\mathrm{grad} f = (df)^\sharp$$

Or equivalently,

$$\langle \mathrm{grad} f, X \rangle_g = Xf$$

More properties of Tangent-Cotangent Isomorphism can be checked on Riemannian Manifold book of Lee.

Also I want to note from **Problem 11.11** that generally TM and T^*M are isomorphic vector bundles but this isomorphism is not **canonical**. There does not exist a bundle isomorphism $\lambda_M : TM \to T^*M$ that is **canonical**. The reason is if the following diagram commutes :

$$\begin{tikzcd} TM \arrow[r, "F_*"] \arrow[d, "\lambda_M"] & TN \arrow[d, "\lambda_N"] \\ T^*M & F^*N \arrow[l, "F^*"'] \end{tikzcd}$$

then chasing the diagram, one finds $\lambda_N(F_*(v))(F_*(w)) = \lambda_M(v)(w)$. Fixing v, right hand side is linear over w so general relation $\lambda_N(F_*(v))(F_*(w_1 + w_2) - F_*(w_1) - F_*(w_2)) = 0$ holds, which cannot be true for all diffeomorphism F.

- Existence of Riemannian metric Argument
This is from the **Problem 11.22**. Existence proof of Riemannian metric argument can be generalized to general settings, which can be used to prove existence of orientation form, etc.

Argument : For smooth vector bundle E over M and $V \subset E$ an open set that $V \cap E_p$ is **convex**, nonempty. Then there exists a smooth global section whose image lies in V.

Proof : For local trivialization (U_α, Φ_α) there exist local section of V, v_α. Now, for partition of unity $\{\phi_\alpha\}$ subordinate to $\{U_\alpha\}$, $\sum \phi_\alpha v_\alpha$ is it. Here, convexity enables us to ensure summation be possible.

Notations

$\Lambda^k M$ is vector bundle of alternating k tensors.

$\mathscr{A}^k(M)$ is smooth section of $\Lambda^k M$. We call them **k-forms**.

Chapter 12. Differential Forms

Likewise the symmetric tensors, there exists alternating tensors. Alternating tensors are defined by if vectors permutes, the value changes sign depending on the sign of the permutation. Alternating tensors $\Lambda^k(V)$ have basis (ϵ^I) where

$$I = \{i_1 < i_2 < \cdots < i_k\}$$

Alternating tensors have the **Wedge product**, in Lee's textbook wedge product is defined:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta)$$

Then for any multi-indices I, J the formula $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$ holds. Thus $\epsilon^I = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$.
Alternating tensors with wedge product gives **exterior algebra**

$$\Lambda^*(V) = \bigoplus_{k=0}^{n} \Lambda^k V$$

As we did on the tensor to manifolds, we can define the vector bundle over manifold

$$\Lambda^k M = \coprod_{p\in M} \Lambda^k(T_pM)$$

Its section is called **differential form** and notate by $\mathscr{A}^k(M)$. Also, pullback is well defined.

Exterior Derivatives

Unique property of differential form is that there exists additional operation called **Exterior Derivatives**. Exterior derivative is an unique linear map $d : \mathscr{A}^k(M) \to \mathscr{A}^{k+1}(M)$ that $df(X) = Xf$ for 0-form f, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ and $d \circ d = 0$. Following properties holds.

Lemma 2 (Lemma 12.16)**.** *If $G : M \to N$ is a smooth map, then $G^* : \mathscr{A}^k(N) \to \mathscr{A}^k(M)$ commutes with d.*

$$G^*(d\omega) = d(G^*\omega)$$

Proposition 9 (Proposition 12.19)**.** *For $\omega \in \mathscr{A}^k(M)$, following formula holds:*

$$\begin{aligned} d\omega(X_1, \cdots, X_{k+1}) = &\sum_{1\le i\le k+1} (-1)^{i-1} X_i(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{k+1})) \\ &+ \sum_{1\le i<j\le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1}) \end{aligned}$$

Symplectic Forms

Symplectic tensor is a nondegenerate alternating 2-tensor on vector space. We can define symplectic complement for subspace $S \subset V$ by

$$S^\perp = \{X \in V : \omega(X, Y) = 0 \ \ \forall Y \in S\}$$

Then dimensional reason, $\mathrm{dim}S + \mathrm{dim}S^\perp = \mathrm{dim}V$.
We call $S \subset V$ subspace is
- symplectic if $S \cap S^\perp = \{0\}$ (or equivalently, $\omega|_S$ nondegenerate)
- isotropic if $S \subset S^\perp$ (or equivalently, $\omega|_S = 0$)
- coisotropic if $S \supset S^\perp$
- Lagrangian if $S = S^\perp$

One important proposition is that there exists a basis that symplectic tensor expressed in canonical form.

Proposition 10 (Proposition 12.22)**.** *If ω is a symplectic tensor on an m-dimensional vector space V, then m is even and there exists a basis for V, $A_1, \cdots A_n, B_1 \cdots B_n$ that*

$$\omega = \sum_{i=1}^n \alpha^i \wedge \beta^i$$

Using this canonical basis, we can represent subspace of symplectic manifolds as following :

- symplectic if there exists symplectic basis (A_i, B_i) for V that $S = \mathrm{span}(A_1, B_1, \cdots, A_k, B_k)$
- isotropic if there exists symplectic basis (A_i, B_i) for V that $S = \mathrm{span}(A_1, \cdots, A_k)$
- coisotropic if there exists symplectic basis (A_i, B_i) for V that $S = \mathrm{span}(A_1, \cdots A_n, B_1, \cdots, B_k)$
- Lagrangian if if there exists symplectic basis (A_i, B_i) for V that $S = \mathrm{span}(A_1, \cdots A_n)$

Proofs are easy, think of the basis of S first, then extend it to the basis of V to satisfy symplectic basis conditions.

Manifold equiped with symplectic form (smooth, closed, nondegenerate 2-form) is symplectic manifold. Symplectic manifold takes advantages of enabling some **geometric argument with differential forms**.

Tautological 1-form

Tautological 1-form is **canonical** form on cotangent bundle. $M = T^*Q$ then for the projection map $\pi : T^*Q \to Q$, $\tau \in \Lambda^1(T^*Q)$ is defined by

$$\tau_{(q,\varphi)} = \pi^*\varphi$$

This definition does not depend on the coordinate representation so is canonical.

Proposition 11 (Proposition 12.24)**.** *Tautological 1-form is smooth and $\omega = -d\tau$ is a symplectic form on T^*Q.*

This is by computing on coordinates. If (x^i) a coordinate on Q, then standard coordinate on T^*Q is (x^i, ξ_i). (Here, $\varphi = \xi_i dx^i$)

Then $\tau_{(x,\xi)} = \pi^*(\xi_i dx^i) = \xi_i dx^i$ so smooth and $\omega = \sum_i dx^i \wedge d\xi_i$ thus symplectic.

With this tautological 1 form, we can think smooth 1-form in M as an embedding to T^*M and interpret closedness of 1-form into geometry of embedded structure.

Proposition 12 (Proposition 12.25)**.** *σ be a smooth 1-form on M. Then as a smooth map from M to T^*M, σ is a smooth embedding, and σ is closed iff $\sigma(M)$ is a Lagrangian submanifold of T^*M.*

The argument closedness $\Leftrightarrow$ Lagrangian submanifold comes from the observation :

$$\sigma^*\tau = \sigma^*(\xi_i dx^i) = \sigma_i dx^i = \sigma$$

so $\sigma^*\omega = -\sigma^* d\tau = -d\sigma$.

Generally, let S be an embedded submanifold of T^*Q. Then S is the image of a smooth closed 1-form on Q if S is Lagrangian, transverse to the fibers and intersects each fiber in exactly one point.

To see S satisfying Lagrangian, transverse, intersects only one then S is an image of 1-form on Q, since S intersects fiber only once, S is a image of (possibly non-continuous at all) section σ. However we know S is an embedded submanifold and $T_{(p,\varphi)}S + T_{(p,\varphi)}\pi^{-1}(p)$ spans T_pM, so σ is smooth 1-form and Lagrangian property ensures closedness.

Geometric interpretation of Symplectomorphism

For symplectic manifolds (M, ω) and $(\tilde{M}, \tilde{\omega})$, 2-form $\Omega = \pi^*\omega - \tilde{\pi}^*\tilde{\omega}$ on $M \times \tilde{M}$ is symplectic form. This can be shown if $\Omega(X, Y) = 0$ of all Y, identifying $Y = Y_1 + Y_2$, each in $T_{\pi(p)}M$ and $T_{\tilde{\pi}(p)}\tilde{M}$, $\omega(X_1, Y_1) = \tilde{\omega}(X_2, Y_2)$ holds for arbitrary Y_1, Y_2 which means $X_1 = 0$, $X_2 = 0$

In this analogy, $F : M \to \tilde{M}$ diffeomorphism is **symplectomorphism if and only if the graph is Lagrangian submanifold of** $(M \times \tilde{M}, \Omega)$.

The graph $\Gamma(F)$ become Lagrangian submanifold if $\Omega|_{\Gamma(F)} = 0$ (since dimensional condition already satisfied). It is equivalent to

$$\Omega|_{\Gamma(F)} = 0 \Leftrightarrow i^*\pi^*\omega = i^*\tilde{\pi}^*\tilde{\omega} \Leftrightarrow i^*\pi^*(\omega - F^*\tilde{\omega}) = 0 \Leftrightarrow \omega = F^*\tilde{\omega}$$

Symplectic group

Symplectic group is $\mathrm{Sp}(n, \mathbb{R}) \subset GL(2n, \mathbb{R})$ that makes $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ invariant. $A^*\omega = \omega$.

(a) $A \in \mathrm{Sp}(n, \mathbb{R})$ iff it takes standard basis to a symplectic basis.
This is trivial by the definition of Symplectic group.

(b) $A \in \mathrm{Sp}(n, \mathbb{R})$ iff $A^T J A = J$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Proof is calculus. Say $A \in \mathrm{Sp}(n, \mathbb{R})$.

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

Then for the basis (x^i) with (y^i) direct calculation gives $\sum_{j=1}^n d(\tilde{x})^j \wedge d(\tilde{y})^j$ consist of

$$\sum_{j=1}^n (A_1)^j_i (A_3)^j_k - (A_1)^j_k (A_3)^j_i dx^i \wedge dx^k$$

and

$$\sum_{j=1}^n (A_2)^j_i (A_4)^j_k - (A_2)^j_k (A_4)^j_i dy^i \wedge dy^k$$

and

$$\sum_{j=1}^n (A_1)^j_i (A_4)^j_k - (A_2)^j_k (A_3)^j_i dx^i \wedge dy^k$$

.

So the equations being

$$\sum_{j=1}^n d(\tilde{x})^j \wedge d(\tilde{y})^j = \sum_{j=1}^n dx^j \wedge dy^j$$

is equivalent to $A_1^T A_3 = A_1 A_3^T$, $A_2^T A_4 = A_2 A_4^T$, $A_4^T A_1 - A_2^T A_3 = I_n$. This is exactly the same formula $A^T J A = J$

(c) $\mathrm{SP}(n, \mathbb{R})$ is embedded Lie subgroup of $\mathrm{GL}(2n, \mathbb{R})$ whose dimension is $2n^2 + n$

$\Phi(A) = A^T J A$ be a map from $\mathrm{GL}(2n, \mathbb{R})$ to itself, we claim J is the regular value. Calculation gives

$$(\Phi_*)_A(B) = B^T J A + A^T J B$$

(by the curve $\gamma(t) = A + tB$) so if $B = A$, $(\Phi_*)_A(A) = 2J \neq 0$. Thus $\mathrm{Sp}(n, \mathbb{R})$ is a regular level set, so embedded submanifold. Since $\mathrm{Sp}(n, \mathbb{R})$ is closed under multiplication, it is Lie subgroup.
Now,

$$\ker(\Phi_*)_I = \{B | B^T J + J B = 0\}$$

so dimension of $\mathrm{Sp}(n, \mathbb{R})$ is $2n^2 + n$.

(d) The Lie algebra of $\mathrm{Sp}(n, \mathbb{R})$ is by previous calculation

$$\mathfrak{sp}(n, \mathbb{R}) = \{B \in \mathfrak{gl}(2n, \mathbb{R}) | B^T J + J B = 0\}$$

(e) $\mathrm{Sp}(n, \mathbb{R})$ is non compact since following matrices are not bounded.

$$\begin{pmatrix} kI & I \\ I & \frac{2}{k} I \end{pmatrix}$$

Lagrangian subspaces

If we denote $\Lambda_n \subset G_n(\mathbb{R}^{2n})$ the set of Lagrangian subspaces, we want to explore manifold structure. Observation gives $\mathrm{Sp}(n, \mathbb{R})$ acts transitively on Λ_n, which is previous observation. Thus, by

Proposition 13 (Proposition 9.31)**.** *we can give Λ_n a unique smooth manifold structure that action of* $\mathrm{Sp}(n, \mathbb{R})$ *be smooth. Here, determining the isotropy group*

$$\begin{pmatrix} A & 0 \\ B & (A^{-1})^T \end{pmatrix}$$

$A \in \mathrm{GL}(n, \mathbb{R})$ and $B^T A = A^T B$. Isotropy group is closed Lie subgroup and has dimension $n^2 + \frac{1}{2} n(n+1)$ (A has n^2 dimension and B has $\frac{1}{2} n(n-1)$ constraints so $\frac{1}{2} n(n+1)$ dimension)

Thus $\dim \Lambda_n = \frac{1}{2} n(n+1)$

Finally Λ_n is compact. Since $G_n(\mathbb{R}^{2n})$ is compact, we only need to show closedness.
By coordinate calculation, $V \mapsto V^\perp$ the map from $G_n(\mathbb{R}^{2n})$ to itself is continuous. Thus $f : V \mapsto (V, V^\perp)$ is continuous, so its preimage of diagonal Δ, $f^{-1}(\Delta) = \Lambda_n$ is closed, compact.

Chapter 13. Orientations

Orientation of manifold is originated from the orientation of vector space. Orientation of vector space is equivalence class of ordered basis. $(E_1, \cdots, E_n)$ and $(\tilde{E}_1, \cdots, \tilde{E}_n)$ are equivalent if the determinant is positive. Also lemma gives vector space V orientation is determined by some nonzero element of $\Lambda^n(V)$. $\Omega \in \Lambda^n(V)$ determines orientation by $\Omega(E_1, \cdots, E_n) > 0$
Orientation of Manifold must be given 'continuity'. Since locally manifold is euclidean space, so is vector space. If two chart $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) overlapping area $U_\alpha \cap U_\beta$ the $\varphi_\beta \circ \varphi_\alpha^{-1}$ is positive Jacobian then we can say two charts are consistent. If this consistence can be globally defined, we say M is oriented. Analogue of n-form determination holds for manifold : M has orientation if and only if there exists nonvanishing n-form Ω on M. As a corollary, parallelizable manifolds are orientable. Thus every **Lie groups are orientable**.

Orientation Covering

Nonzero n covectors on smooth manifold M is denoted by $\Lambda^n_* M$. The action $\mathbb{R}^+$ on $\Lambda^n_* M$ is smooth, free, proper (Lemma 13.8) so we can define quotient space $\hat{M} = \Lambda^n_* M/\mathbb{R}^+$.

Theorem 14 (Theorem 13.9). *M is smooth connected manifold. $\hat{\pi} : \hat{M} \to M$ be orientation covering. Then following holds*
(2) $\hat{M}$ has a canonical orientation
(4) $\hat{M}$ is connected if and only if M is non-orientable.

Canonical orientation is by the map $\hat{\pi}_* : T_{\hat{q}}\hat{M} \to T_q M$. By lifting the orientation of $T_q M$ to the orientation of $T_{\hat{q}}\hat{M}$, it becomes continuous.

Orientation of Hypersurfaces

In ths section, we want to determine orientability of hypersurface $S \subset M$. To do this, the map $i_X : \Lambda^k(V) \to \Lambda^{k-1}(V)$ is important.

$$i_X\omega(Y_1, \cdots, Y_{k-1}) = \omega(X, Y_1, \cdots, Y_{k-1})$$

We notate $i_X\omega = X \lrcorner \omega$. The **interior multiplication map plays quite important role on proving crucial equalities on differential forms**

Lemma 3 (Lemma 13.11). *V be a f.d.v.s. $X \in V$*
(1) $i_X \circ i_X = 0$
(2) If ω is k-covector and η is l-covector

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X\eta)$$

To apply interior multiplication to smooth manifold, $X \in \mathscr{T}(M)$ and $\omega \in \mathscr{A}^k(M)$ then applying pointwise, $X \lrcorner \omega$ is $(k-1)$-form.

Lemma 4 (Lemma 13.11 (Manifold version)). *M be smooth manifold and $X \in \mathscr{T}(M)$*
(1) $i_X : \mathscr{A}^k(M) \to \mathscr{A}^{k-1}(M)$ is linear over $C^\infty(M)$ so is bundle map $i_X : \Lambda^k M \to \Lambda^{k-1} M$
(2) $i_X \circ i_X = 0$
(3) If $\omega \in \mathscr{A}^k(M)$, $\eta \in \mathscr{A}^l(M)$ then

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X\eta)$$

Vector field along S is the map $N : S \to TM$. N is traverse if N_p and T_pS spans T_pM.

Proposition 14 (Proposition 13.12). *M is oriented smooth n-manifold and S is immersed hypersurface of M. N is traverse vector field along S then S has a unique orientation where $(N \lrcorner \Omega)|_S$ is the orientation form.*

Orientation of Boundary

The same logic can be applied to boundary. In the boundary, we find **traverse vector field to be smooth outward-pointing vector field**

Lemma 5 (Lemma 13.16). *M is smooth manifold with boundary. Then there exists smooth outward-pointing vector field along ∂M.*

As a corollary,

Proposition 15 (Proposition 13.17). *M be an oriented smooth manifold with boundary. Then ∂M is orientable.*

The Riemannian Volume Form

In **Orientable Riemannian Manifold** the Riemannian volume form exists. This is the smooth orientation form $\Omega \in \mathscr{A}^n(M)$ such that

$$\Omega(E_1, \cdots, E_n) = 1$$

for every oriented local orthonormal frame (E_i) for M. In local coordinate,

$$dV_g = \sqrt{det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$$

Hypersurfaces in Riemannian manifold, the Riemannian volume form for hypersurface is determined by (N is smooth unit normal vector field along S)

$$dV_{\tilde{g}} = (N \lrcorner dV_g)|_S$$

Boundary of Riemannian manifold still holds the same formula. For N the outward unit normal vector field along ∂M

$$dV_{\tilde{g}} = (N \lrcorner dV_g)|_{\partial M}$$

Chapter 14. Integration on Manifolds

I will briefly skip over the defining process of integration. By partition of unity making integration to occur on Euclidean spaces.
One of the most beautiful theorem is **Stoke's Theorem**
Theorem 14.9 (Stoke's Theorem) M be a smooth, oriented n-dimensional manifold with boundary, ω be a compactly supported smooth $(n-1)$-form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega$$

Integration on Riemannian Manifolds

Integration was possible only on n forms. We want to integrate $f : M \to \mathbb{R}$. But the problem is **there is no consistent transform of 0-form f to n-form**. In **Riemannian Manifold we can define consistent transform**. (M, g) be oriented Riemannian Manifold, then $f : M \to \mathbb{R}$ can be integrated by

$$\int_M f dV_g$$

The Divergence Theorem

$* : C^\infty(M) \to \mathscr{A}^n(M)$ by $*f = f dV_g$. Divergence operator is div : $\mathscr{T}(M) \to C^\infty(M)$ by

$$\text{div} X = *^{-1} d(X \lrcorner dV_g)$$

One application of Stoke's Theorem is
Theorem 14.23 (The Divergence Theorem) M be an oriented Riemannian manifold with boundary. For any compactly supported smooth vector field X on M,

$$\int_M (divX) dV_g = \int_{\partial M} \langle X, N \rangle_g dV_{\tilde{g}}$$

Integration on Lie Groups

Here is 'easy' version of Haar measure existence.

Proposition 16 (Proposition 14.25)**.** *G be a compact Lie group endowed with a left-invariant orientation. Then G has a unique left-invariant orientation form Ω that*

$$\int_G \Omega = 1$$

As an analogue of **consistent transform of 0-form to n-form** on Riemannaian manifold, since above **Haar volume is unique $f \to f dV$ is consistent transform**.

Densities

The density originally defined on vector space is a function

$$\mu : V \times \cdots \times V \to \mathbb{R}$$

satisfying $\mu(TX_1, \cdots, TX_n) = |\text{det}T|\mu(X_1, \cdots, X_n)$
Density is not a tensor because nonlinear. However the space of densities $\Omega(V)$ is 1-dimensional vector space.
Adaptation to manifold is routine process.

$$\Omega M = \coprod_{p \in M} \Omega(T_p M)$$

Then there exists a smooth positive density on M.
The purpose of defining density is to integrate by densities. Integration of density in vector space: $\mu = f|dx^1 \wedge \cdots \wedge dx^n|$

$$\int_D \mu = \int_D f dV$$

The Riemannian Density (M, g) be a Riemannian manifold (with/without boundary). There is a unique smooth positive density μ on M that for any local orthonormal frame (E_i)

$$\mu(E_1, \cdots, E_n) = 1$$

Version of divergence theorem holds for Riemannian densities.

Theorem 15 (Theorem 14.34)**.** *(M, g) a Riemannian manifold with boundary. For any compactly supported smooth vector field X on M, $dV_g, dV_{\tilde{g}}$ a Riemannian densities*

$$\int_M (\text{div}X) dV_g = \int_{\partial M} \langle X, N \rangle_g dV_{\tilde{g}}$$

Chapter 17 to 19 gives new insight to understand vector fields. Moreover, this insight provides new view of Lie bracket operation.

Chapter 17. Integral Curves and Flows

Integral curve of V smooth vector field on M is smooth curve $\gamma : J \to M$ such that

$$\gamma'(t) = V_{\gamma(t)}$$

for all $t \in J$.

The fundamental theorem of flows

Theorem 16 (Theorem 17.8)**.** *Let V be a smooth vector field on a smooth manifold M. There is a unique maximal smooth flow $\theta : \mathcal{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:*
(1) For each $p \in M$ the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.
(2) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{\theta(s,p)}$ is the interval $\mathcal{D}^{(p)} - s$
(3) For each $t \in \mathbb{R}$ the set

$$M_t = \{p \in M : (t, p) \in \mathcal{D}\}$$

is open in M and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}
(4) For each $(t, p) \in \mathcal{D}$,

$$(\theta_t)_* V_p = V_{\theta_t(p)}$$

The proof uses ODE theorem.
(1) **Existence** : $U \subset \mathbb{R}^n$ be an open set, $V : U \to \mathbb{R}^n$ is Lipshitz continuous. Let $(t_0, x_0) \in \mathbb{R} \times U$ given. There exist an open interval $J_0 \subset \mathbb{R}$ containing t_0, an open set $U_0 \subset U$ containing x_0, and for each $x \in U_0$ a C^1 curve $\gamma : J_0 \to U$ satisfies

$$(\gamma^i)'(t) = V^i(\gamma(t))$$

$$\gamma^i(t_0) = x^i$$

(2) **Uniqueness** : $U \subset \mathbb{R}^n$ an open set, and $V : U \to \mathbb{R}^n$ is Lipschitz continuous. For any $t_0 \in \mathbb{R}$, any two solutions to initial value problem are equal on their common domain.

(3) **Smoothness** $U \subset \mathbb{R}^n$ is an open set and $V : U \to \mathbb{R}^n$ is Lipschitz continuous. $U_0 \subset U$ is an open set, $J_0 \subset \mathbb{R}$ is an open interval containing t_0, and $\theta : J_0 \times U_0 \to U$ is a map that $\gamma(t) = \theta(t, x)$ solves the initial value problem for each $x \in U_0$. If V is C^k for some $k \geq 0$, then θ is C^k.

Proof of the ODE theorem is important enough. It can be compared to the proof of implicit function theorem and inverse function theorem.

Proof Ideas:
(1) **Existence** : Define iteration

$$I_\gamma(t) = x + \int_{t_0}^{t} V(\gamma(s))ds$$

by finding complete metric space and proving I is contraction, fixed point is the solution

(2) **Uniqueness** : Is direct application of the comparison lemma

(3) **Smoothness** : Triky part. First is showing θ continuous. This is by comparison lemma. Applying comparison lemma gives jointly continuity.
Next is C^1. Nicely, by initial value problem statement,

$$\frac{\partial \theta^i}{\partial t}(t, x) = V^i(\theta(t, x))$$

so $\frac{\partial \theta^i}{\partial t}$ exists and continuous. Remaining part is showing $\frac{\partial \theta^i}{\partial x^j}$ exists and continuous.
First define $1 \leq i, j \leq n$, $(\Delta_h)^i_j : \bar{J}_1 \times \bar{U}_1 \to \mathbb{R}$ be

$$(\Delta_h)^i_j(t, x) = \frac{\theta^i(t, x + he_j) - \theta^i(t, x)}{h}$$

By proving $(\Delta_h)^i_j$ converges uniformly on $\bar{J}_1 \times \bar{U}_1$ as $h \to 0$, C^1 version will be proved.
Taylor formula gives $t \in \bar{J}_1$, $y \in \bar{U}_1$, $v \in \bar{B}_r(0)$

$$V^i(y + v) = V^i(y) + v^k \frac{\partial V^i}{\partial y^k}(y) + v^k G^i_k(y, v)$$

then

$$\frac{\partial}{\partial t}(\Delta_h)^i_j(t, x) = \left(\frac{\partial V^i}{\partial y^k}(\theta(t, x)) + G^i_k(y, v)\right)(\Delta_h)^k_j(t, x)$$

$$\left|\frac{\partial}{\partial t}(\Delta_h(t, x) - \Delta_{\tilde{h}}(t, x))\right| \leq \sup|DV| \cdot |\Delta_h(t, x) - \Delta_{\tilde{h}}(t, x)| + 2\epsilon n e^{CT}$$

Now C^k version is induction. $\frac{\partial \theta^i}{\partial t}$ clearly holds. Using the derivative by t, smoothness can be easily interpreted as new system of ODE.

Complete Vector Field

When does the flow is global? That means, $\mathcal{D}^{(p)}$ is $\mathbb{R}$.

Lemma 6 (Lemma 17.10 (Escape Lemme))**.** *V be a smooth vector field on a smooth manifold M. If γ is an integral curve of V whose maximal domain is not all of $\mathbb{R}$, then the image of γ cannot lie in any compact subset of M.*

The flow need to **'Escape'** when it is not global.

Canonical coordinate of vector field

V be a vector field on M, $p \in M$ is a singular point if $V_p = 0$ and regular point otherwise. If we start from singular point, $\theta^{(p)}$ will be constant curve. If we start from regular point, then $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is an immersion.

Theorem 17 (Theorem 17.13)**.** *Let V be a smooth vector field on a smooth manifold M, and $p \in M$ be a regular point for V. There exist smooth coordinates (u^i) on some neighborhood of p which V has the coordinate representation $\frac{\partial}{\partial u^1}$*

Idea is, by choosing smooth coordinates $(x^1, \cdots, x^n)$ centered at p, $\theta : \mathcal{D} \to U$ the flow of V, the new coordinate is

$$\psi(t, u^2, \cdots, u^n) = \theta_t(0, u^2, \cdots, u^n)$$

This ψ will satisfy

$$\psi_* : \left(\frac{\partial}{\partial t}\bigg|_{(0,0)}, \frac{\partial}{\partial u^2}\bigg|_{(0,0)}, \cdots, \frac{\partial}{\partial u^n}\bigg|_{(0,0)} \right) \to \left(V_p, \frac{\partial}{\partial x^2}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right)$$

Time Dependent Vector Fields

Time Dependent Vector Field version still holds.
Theorem 17.15 (Flows of Time-Dependent Vector Fields) M be a smooth manifold, $V : J \times M \to TM$ be a smooth time-dependent vector field on M. There exist an open set $\mathcal{E} \subset J \times J \times M$ and smooth map $\theta : \mathcal{E} \to M$ such that $s \in J$ and $p \in M$, the set

$$\mathcal{E}^{(s,p)} = \{t \in J : (t, s, p) \in \mathcal{E}\}$$

containing s the smooth curve $\gamma : \mathcal{E}^{(s,p)} \to M$ by
$gamma(t) = \theta(t, s, p)$ is the unique maximal solution to

$$\gamma'(t) = V(t, \gamma(t))$$

$$\gamma(s) = p$$

The proof is minor modification of proof of time independent vector field.

Classification of 1-manifolds

Using the flow of vector fields, we can prove the classification of 1 manifolds... Look at exercise.

References

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