Rudin - Real and Complex Analysis

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Contents

1 Chapter 1. Measure

1.1 Exercise 12 : Power of simple functions and Lebegue's Monotone Convergence Theorem

Problem. Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$

I solved this challenging problem by this way. First, by Theorem 1.17 there exist simple measurable functions s_n on X such that

$$0 \leq s_1 \leq s_2 \leq \cdots \leq |f|$$
$$s_n(x) \to f(x) \text{ as } n \to \infty, \text{ for every } x \in X$$

And by Lebesgue's Monotone Convergence Theorem,

$$\int_X s_n d\mu \to \int_X |f| d\mu$$

For every ϵ, there exists some $N \in \mathbb{N}$ such that

$$0 \leq \int_E (|f| - s_N) d\mu \leq \int_X (|f| - s_N) d\mu < \epsilon/2$$

for every $E \in \mathfrak{M}$. Now s_N has only finite values, denote by $n_1, n_2, \cdots n_M$, if we define

$$\delta = \frac{\epsilon}{2 * max(|n_1|, |n_2|, \cdots, |n_M|) + 1}$$

then for any $E \subseteq X$ with $\mu(E) < \delta$,

$$\int_E s_N d\mu \leq \mu(E) * max(|n_1|, |n_2|, \cdots, |n_M|) < \epsilon/2$$

Therefore, $\int_X |f| d\mu < \epsilon$ for every $\mu(E) < \delta$.

Lebesgue's Monotone Convergence Theorem only requires increasing property and point-wise Convergence property. Increasing property is for guarantee f is measurable. The point-wise convergence property is delicate but could be interpreted as this way. Integral is defined with the sequence of simple functions. Seeing the proof of Lebesgue's Monotone Convergence Theorem, it uses the fact that since point-wise convergence property, simple function will cover all part of f's.

By this proof, we could see the great property, powerful tool from measure and integrals.
(1) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, and $A_1 \subset A_2 \subset A_3 \subset \cdots$
(2) $f : X \to [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that

(a) $0 \leq s_1 \leq s_2 \leq \cdots \leq f$
(b) $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$

Remark This could be compared to Absolute continuity for complex measures. Actually $f d\mu = d\lambda$ then $\lambda \ll \mu$ so following holds in special case of Lemma on complex measure.

2 Chapter 2. Riesz Representation Theorem

2.1 Section 2.7 : Converse of $K \subset V \subset \bar{V} \subset U$

I have a question if the converse holds for this Theorem.

Theorem (2.7)**.** *Suppose U is open in locally compact Hausdorff space X, $K \subset U$ and K compact. Then there is an open set V with compact closure that*

$$K \subset V \subset \bar{V} \subset U$$

Converse is, if any K compact, U open, there exists open set V with compact closure such that $K \subset V \subset \bar{V} \subset U$ Then X is locally compact, Hausdorff space.

Answer is yes for locally compactness. Because any singleton set is compact $\{x\}$. Applying hypothesis for $U = X$, there exists compact set $\bar{V}$ and so X is locally compact.

However, it is not yes for Hausdorff space because there is a counterexample, $X = \{a, b\}$ with topology ϕ, X

2.2 Section 2.12 : Urysohn Lemma

Lemma (2.12 Urysohn Lemma)**.** *Suppose X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K is compact. Then there exists an $f \in C_c(X)$ such that*

$$K \prec f \prec V$$

On the proof of Urysohn's lemma, we constructed two functions; $f = \sup_r f_r, g = \inf_s g_s$. (What f_r and g_s is,)

I was curious why they constructed two functions. The reason was to make sure $f = g$ and conclude a function is continuous. f is lower semicontinuous, g is upper semicontinuous, so a function implies continuity.

2.3 Section 2.13 : Partition of unity

I found out this step of proof is quite non-obvious. First state the Partition of unity Lemma.

Lemma (2.13 Partition of Unity Lemma)**.** *Suppose $V_1, V_2 \cdots V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and*

$$K \subset V_1 \cup \cdots \cup V_n$$

Then there exist functions $h_i \prec V_i$ $(i = 1, 2, \cdots n)$ such that

$$h_1(x) + h_2(x) + \cdots + h_n(x) = 1 \quad (x \in K)$$

$\{h_1, h_2, \cdots h_n\}$ is called a partition of unity on K subordinate to the cover $\{V_1, V_2, \cdots V_n\}$

$$W_{x_1} \cup W_{x_2} \cup \cdots \cup W_{x_n} \supset K$$

Why did they assert this? Simply answering, it's because $K \nsubseteq V_i$. So with compactness, we make finite sets contained by some V's and union contains K.

Then is K could be replaced into general subset of X? Is there a weaker condition?

I found counterexample for this,

$S = (-1, 1), V_1 = (-1/2, 1), V_2 = (-1, 1/2)$.

For $(1/2, 1)$, $h_1(x) = 1$ However h_1 needs to vanish out of V_1, so it is contradiction that h_1 is continuous.

2.4 Section 2.14 : Riesz Representation Theorem

Theorem (2.14 Riesz Representation Theorem)**.** *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that*

$$\Lambda f = \int_X f d\mu$$

for every $f \in C_c(X)$

The big view of this prove is first, defining measure of an open set. Second, define measure on any set from measure on open sets. Third, showing measure on compact set only with itself, without using open set. Fourth, every lemma within measure is lower estimated by compact set, upper estimated by open set.

$$\begin{aligned}(\mu(V) = \sup\{\Lambda f : f \prec V\}) &\implies (\mu(E) = \inf\{\mu(V) : E \subset V\}) \\ &\implies (\mu(K) = \inf\{\Lambda f : K \prec f\}) \\ &\implies (Following \ \ lemmas)\end{aligned}$$

More detailed steps are as following.

2.4.1 Definition

$$\mu(V) = \sup\{\Lambda f : f \prec V\}$$

$$\mu(E) = \inf\{\mu(V) : E \subset V\}$$

$$\mathfrak{M}_F : E \ s.t. \ \mu(E) < \infty, \ \mu(E) = \sup\{\mu(K) : K \subset E\}$$

$$\mathfrak{M} : E \ s.t. \ \forall K, \ E \cap K \in \mathfrak{M}$$

2.4.2 Steps

Step 1. If $E_1, E_2, \cdots$ arbitrary subsets of X

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

Main idea is lower estimating each subset's measure by proper open sets.

$$\downarrow$$

Step 2. If K is compact, then $K \in \mathfrak{M}_F$ and

$$\mu(K) = \inf\{\Lambda f : K \prec f\}$$

This is really important fact.
Think outside of box to the view of squeezing measure value with compact set and open set

$$\downarrow$$

Step 3. V open set always satisfy

$$\mu(V) = \sup\{\mu(K) : K \subset V\}$$

$$\downarrow$$

Step 4. If $E_1, E_2, \cdots$ are pairwise disjoint members of $\mathfrak{M}_F$. Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Main Idea is showing other inequality (contrast to Step 1) by compact sets.

$$\downarrow$$

Step 5. If $E \in \mathfrak{M}_F$ and $\epsilon > 0$, there is a compact set K and an open set V such that $K \subset E \subset V$ and $\mu(V - K) < \epsilon$

$$\downarrow$$

Step 6. If $A, B \in \mathfrak{M}_F$, then is $A - B, A \cup B, A \cap B$

$$\downarrow$$

Step 7. $\mathfrak{M}$ is an σ - algebra in X contains all Borel sets.

$$\downarrow$$

Step 8. $\mathfrak{M}_F$ consists of sets $E \in \mathfrak{M}$ for which $\mu(E) < \infty$

$$\downarrow$$

Step 9. μ is a measure on $\mathfrak{M}$

$$\downarrow$$

Step 10. For every $f \in C_c(X), \Lambda f = \int_X f d\mu$

2.5 More on Riesz Representation Theorem's measure function

First, lemma from the proof of Riesz Representation Theorem is important. Following lemma and theorem states how to approximate measure of sets by specialized one : Open set and closed set, G_δ or F_σ set.

Lemma (2.14 Riesz Representation Theorem)**.** *If $E \in \mathfrak{M}_F$ and $\epsilon > 0$, there is a compact set K and an open set V such that $K \subset E \subset V$ and $\mu(V - K) < \epsilon$*

Theorem (2.17 Approximation of measure on set). *Suppose X is a locally compact, σ-compact Hausdorff space. Then $\mathfrak{M}$ and μ have following properties. (σ-compact set if set is countable union of compact spaces)*
(a) $E \in \mathfrak{M}$ and $\epsilon > 0$, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V - F) < \epsilon$
(b) μ is regular Borel measure on X.
(c) If $E \in \mathfrak{M}$, there are sets A and B such that A is an F_σ, B is a G_δ. $A \subset E \subset B$ and $\mu(B - A) = 0$

The point is, in Riesz Representation Theorem, $K \subset E \subset V$ and $\mu(V - K) < \epsilon$ is $E \in \mathfrak{M}$, i.e. $\mu(E) < \infty$. However, this Theorem assumes if X is σ-compact, for any measurable sets, we could find closed set and opens set that is close to measurable set.

Also, (c) is crucial. Because we could exactly measure set within G_δ or F_σ set.

Theorem (2.18 Theorem). *Suppose X is a locally compact, each open set is σ-compact, Hausdorff space. Then for any positive Borel measure λ on X, which $\lambda(K) < \infty$ for every compact set K is regular.*

2.5.1 Why compact cannot play role on 2.17 (a)

$E = \bigcup_{n \in \mathbb{Z}} [2n, 2n+1]$. Then for every compact set, $\mu(E - K) = \infty$. So, for infinite measured set, compact set cannot be close to them because in Riesz Representation Theorem, every compact set has finite measure.

2.6 Approximating functions by continuous functions

2.6.1 2.24 Lusin's Theorem

Approximating measurable function by continuous funciton.

Theorem (Lusin's Theorem). *Suppose f is a complex measurable function on X, $\mu(A) < \infty$, $f(x) = 0$ if $x \notin A$ and $\epsilon > 0$. There exists a $g \in C_c(X)$ such that*

$$\mu(\{x : f(x) = g(x)\}) < \epsilon$$

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$$

2.6.2 2.25 The Vitali-Caratheodory Theorem

Approximating L^1 function by lower semicontinuous and upper semicontinuous functions.

Theorem (The Vitali-Carathedory Theorem). *Suppose $f \in L^1(\mu)$, f is real-valued, and $\epsilon > 0$. Then there exists upper semicontinuous and bounded above function u, lower semicontinuous and bounded below function v, such that $u \leq f \leq v$ and*

$$\int_X (v - u) d\mu < \epsilon$$

2.7 Exercise 8 : Tricky construction of Borel Set

Problem. Consturct a Borel set $E \in \mathbb{R}^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment I. Is possible to have $m(E) < \infty$ for such a set?

To me, this is very hard problem. I find some facts that is elementary. Such E do not contain interval and E^c do not contain interval.

While trying out constructing such set, I found some ridiculous wrong proof that such E cannot exist.

Since $\mathbb{R}^1$ is σ-compact set, by theorem 2.17, $A \subset E \subset B$ such that B is G_δ So, $m(E \cap I) = m(B \cap I) - m((B - E) \cap I)$. Every term is finite and $m(B - E) \leq m(B - A) = 0$ so $m(E \cap I) = m(B \cap I)$.

We could think G_δ set instead of E; an arbitrary set. Every open set on $\mathbb{R}^1$ is countable union of open intervals. Therefore, assume

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (a_{m,n}, b_{m,n})$$

E^c should not contain open interval so, each $\bigcap_{m=1}^{\infty} (a_{m,n}, b_{m,n})^c$ should not contain open interval. Then, each set's complement is organized with countable singleton points. E is intersection of those sets so E^c is set that is organized by countable singletons. Thus $m(E^c) = 0$, for every interval I, $m(E \cap I) = m(I)$ contradiction.

This logic is almost true except the last sentence, "$\bigcap_{m=1}^{\infty} (a_{m,n}, b_{m,n})^c$ should not contain open interval. Then, each set's complement is organized with countable singleton points."

Counterexample is contour set. Countable intersection of intervals could be uncountable set. So that sentence is false.

But we could notice that we only need to work with G_δ set.

2.7.1 My try

Actually, I failed to solve this problem. But I want to record my solution and furthermore have an opportunity to reprove if this construction works.

My try was making borel set on $[0, 1]$ and copy and paste to all $[n, n+1]$. For $[0, 1]$ making E, borel set with $\mu(E) = \frac{1}{2}$ is possible with this way:

$$\frac{1}{2} = \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^{i+j}}$$

So, for G_δ set $E = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (a_{m,n}, b_{m,n})$, making each $\bigcup_{n=1}^{\infty} (a_{m,n}, b_{m,n})$ to be measure $1 - \frac{1}{2^{m+1}}$.
Then,

$$\mu(E^c) \geq \sum_{m=1}^{\infty} \left(1 - \sum_{n=1}^{\infty} (b_{m,n} - a_{m,n})\right) = \sum_{m=1}^{\infty} \frac{1}{2^{m+1}} = \frac{1}{2}$$

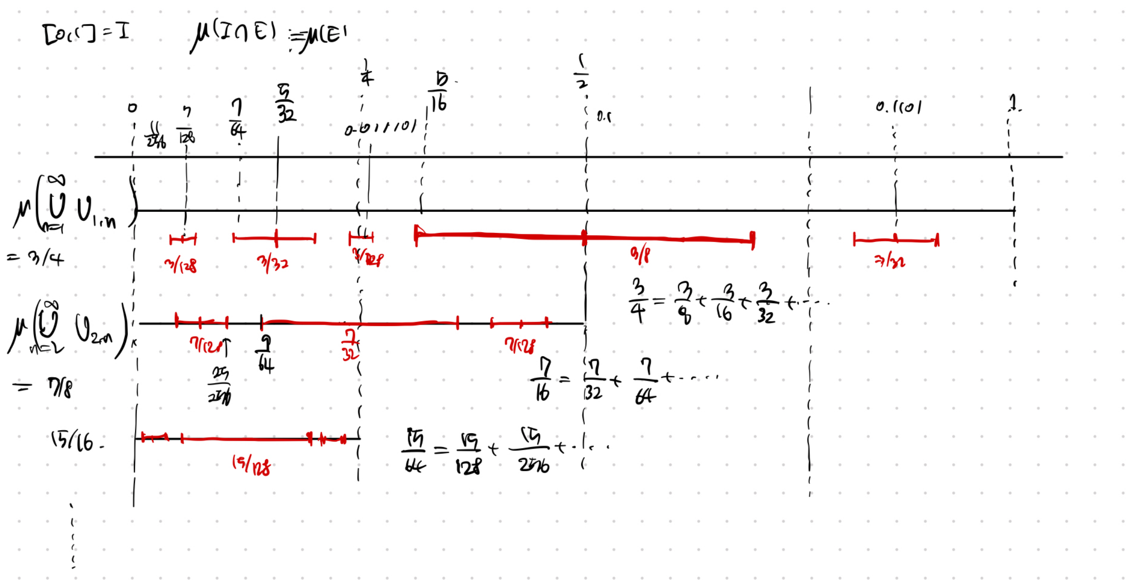

Figure 1: Constructing Borel set working on [0,1]

More descriptive construction is as following. For each n, partition $[0,1]$ into 2^{n-1} equal length interval. Now work with m, starting from the middle of interval, remove $\frac{1}{2}\bigg(1-\frac{1}{2^{n+1}}\bigg)$. After eliminating, remove from the middle of interval, each interval length is decreasing into $\frac{1}{4}$. The figure will be as Figure 1 above.
Because we construct in this way, it is easy to check $\mu\big(\big[\frac{k}{2^n},\frac{k+1}{2^n}\big]\big) > 0$ because in $n+1$-th step, we work in interval $\big[\frac{k}{2^n},\frac{k+1}{2^n}\big]$
But it was quite difficult to prove

$$\mu\big(\big[\frac{k}{2^n},\frac{k+1}{2^n}\big]\big) < \frac{1}{2^n}$$

.
If this proven, every interval contains some $\big[\frac{k}{2^n},\frac{k+1}{2^n}\big]$ so $0 < m(E\cap I) < m(I)$

2.7.2 Solution from StackExchange

Solution 1

CTDP : compact totally disconnected subset of $I=[0,1]$, having positive measure.
$< I_n >$: enumeration of all segments in I whose endpoints are rational.

Construct sequences $< A_n >, < B_n >$ of CTDP's. Start with disjoint CTDP's A_1, B_1 in I_1. If $A_1, B_1, A_2, B_2, \cdots, A_{n-1}, B_{n-1}$ are chosen, their union C_n is CTDP. (Because A_m^c, B_m^c are all open sets so finite intersection of open set is open. Compactness on $\mathbb{R}^1$ is same as boundness, closedness.)
However, I cannot prove that C_n is totally disconnected.

$$A = \bigcup_{n=1}^{\infty} A_n$$

Then if $V \subset I$ open, nonempty, for some n, $I_n \subset V$ so

$$0 < m(A_n) \leq m(A \cap V) < m(A \cap V) + m(B_n) \leq m(V)$$

Solution 2

This is more visualized proof, $\{r_n\}$ is an enumeration of the rationals,

$$V_n = (r_n - 3^{-n-1}, r_n + 3^{-n-1}), \quad W_n = V_n - \bigcup_{k=1}^{\infty} V_{n+k}$$

Then,

$$m(W_n) > m(V_n) - \sum_{k=1}^{\infty} m(V_{n+k}) = \frac{m(V_n)}{2}$$

(The strict inequality is very important to prove this statement)
For each n, K_n be a Borel set in V_n with measure $m(K_n) = m(V_n)/2$. (The equality is very important to prove this statement).

$$A_n = W_n \cap K_n, \quad A = \bigcup_{n=1}^{\infty} A_n$$

First, we need to prove for every V_n, $0 < m(A \cap V_n)$. We could prove $m(W_n \cap K_n) > 0$ because

$$m(W_n \cup K_n) \leq m(V_n) < m(W_n) + m(K_n) = m(W_n \cap K_n) + m(W_n \cup K_n)$$

Therefore, for every V_n, $m(A \cap V_n) \geq m(A_n \cap V_n) = m(A_n) = m(W_n \cap K_n) > 0$
Next, we need to prove for every V_n, $m(A \cap V_n) < m(V_n)$.

$$\begin{aligned} m(A \cap V_n) &= m\left(\bigcup_{k=0}^{\infty} A_{n+k} \cap V_n\right) \leq \sum_{k=0}^{\infty} m(K_{n+k} \cap V_n) \\ &< \sum_{k=0}^{\infty} m(K_{n+k}) = \sum_{k=0}^{\infty} \frac{m(V_n)}{2^{k+1}} = m(V_n) \end{aligned}$$

And this A satisfies finite measure.

2.8 Exercise 14, 24 : Approximating functions

Problem. f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$. Prove that there exist Borel functions g and h such tat $g(x) = h(x)$ a.e. $[m]$, and $g(x) \le f(x) \le h(x)$ for every x

Key idea : for every Lebesgue measurable set E, there exists Borel set A, B (especially G_δ and F_σ set) such that $A \subset E \subset B$ and $m(B - A) = 0$

More rigorous construction is mimic step functions on Theorem 1.17. First assume f is positive. (Geometric figure is as follows Figure 2)

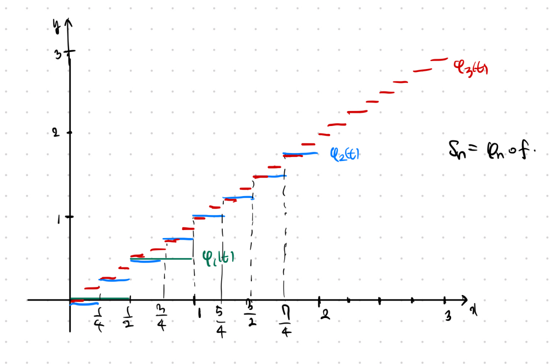

Figure 2: Constructing step functions converging pointwise to f

For the nth step, we consider range $[0, n]$. Then $f^{-1}\big([\frac{k}{2^n}, \frac{k+1}{2^n}]\big)$ is Lebesgue measurable for every $0 \le k < n \times 2^n$. So there are Borel sets $A_{k,n}, B_{k,n}$ that satisfy $A_k \subset f^{-1}\big([\frac{k}{2^n}, \frac{k+1}{2^n}]\big) \subset B_k$ and $m(B_k - A_k) = 0$. Therefore,

$$\sum_{k=0}^{2^n \cdot n - 1} \frac{k}{2^n} \chi_{A_{k,n}} \le f \le \sum_{k=0}^{2^n \cdot n - 1} \frac{k+1}{2^n} \chi_{B_{k,n}}, \quad x \in [0, n]$$

Each supremum and infimum of borel measurable functions are borel measurable.

$$g = \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n \cdot n - 1} \frac{k}{2^n} \chi_{A_{k,n}}$$

$$h = \inf_{n \in \mathbb{N}} \sum_{k=0}^{2^n \cdot n - 1} \frac{k+1}{2^n} \chi_{B_{k,n}}$$

Then $g(x) \le f(x) \le h(x)$ is obvious and

$$x \notin \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2^n \cdot n - 1} (B_{k,n} - A_{k,n}) \Rightarrow g(x) = h(x)$$

Therefore, $g(x) = h(x)$ a.e. $[m]$.

Problem. A step function is a finite linear combination of characteristic functions of bounded intervals in $\mathbb{R}^1$. Assume $f \in L^1(\mathbb{R}^1)$ and prove that there is a sequence $\{g_n\}$ of step functions so that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f(x)-g_n(x)|dx = 0$$

Key idea : $f \in L^1(\mathbb{R}^1)$ there are simple function sequence converging to f, also step function could be very close to simple function.

For every $\epsilon > 0$, $f \in L^1(\mathbb{R}^1)$ so there exists $R > 0$ such that $\int_{\mathbb{R}^1-(-R,R)}|f|dx < \epsilon/3$. By Lebesgue dominated convergence theorem, there are simple function sequence $\{s_n\}$ converging to f. Since $f \in L^1(\mathbb{R}^1)$,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f(x)-s_n(x)|dx = 0$$

Also there exists N such that $n \geq N$ then $\int_{-\infty}^{\infty}|f(x)-s_n(x)|dx < \epsilon/3$. s_N is simple function so $s_N = \sum_{i=1}^{i(N)} c_i\chi_{E_i}$. Let $C = \sup_{i\in\{1,2,\cdots i(N)\}}|c_i|$. Each set E_i is Lebesgue measurable so there exists open set V_i satisfying $m(V_i - E_i) < \frac{\epsilon}{3C\cdot 2^{i+2}}$. Then since every open interval is countable union of open intervals, $V_i \cap (-R,R) = \bigcup_{j=1}^{\infty}(a_{i,j},b_{i,j})$ there exists $j(i)$ that $m(V_i - \bigcup_{j=1}^{j(i)}(a_{i,j},b_{i,j})) < \frac{\epsilon}{3C\cdot 2^{i+2}}$
In summary,

$$m\big(\bigcup_{j=1}^{j(i)}(a_{i,j},b_{i,j}) - E_i\cap(-R,R)\big) + m\big(E_i\cap(-R,R) - \bigcup_{j=1}^{j(i)}(a_{i,j},b_{i,j})\big) < \frac{\epsilon}{3C\cdot 2^{i+1}}$$

Now if g_ϵ is step function that assigns value in two ways; $g_\epsilon(x) = \max_{i\in I_x}|c_i|$ where $I_x = \big\{i | x \in (a_{i,j},b_{i,j}) \ \ \exists 1 \leq j \leq j(i)\big\}$.
Then g_ϵ almost agrees value with s_N and to be precise,

$$\int_{(-R,R)}|s_N - g_\epsilon|dx \leq C\cdot\sum_{i=1}^{i(N)}\frac{2\epsilon}{3C\cdot 2^{i+1}} < \epsilon/3$$

Therefore,

$$\int_{-\infty}^{\infty}|f-g_\epsilon|dx \leq \int_{\mathbb{R}^1-(-R,R)}|f-g_\epsilon|dx + \int_{(-R,R)}|f-s_N|dx + \int_{(-R,R)}|s_N-g_\epsilon|dx < \epsilon$$

As we desired.

2.9 Exercise 17 : No inner regularity for infinite measure subset

This problem has meaning that

If $\mu(E) = \infty$, inner regularity may not hold

Problem. Define the distance between points (x_1, y_1) and (x_2, y_2) to be

$$|y_1 - y_2| \text{ if } x_1 = x_2, \quad 1 + |y_1 - y_2| \text{ if } x_1 \neq x_2$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, x_2, \cdots x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y) dy$$

Let μ be the measure associated with this Λ by Riesz Representation Theorem. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

Noticing if a set E is compact in this metric, x values are finite number. (Since if two points differ x, their distance is at least 1).
First, $\mu(K) = 0$ because $\mu(K) = \Lambda\big(\sum_{i=1}^{n} \chi_{(x_i,0)}\big) = 0$
Second, we will show $\mu(E) = \infty$. If V is an open set containing x-axis,

$$S_n = \left\{ x \middle| \left\{ (x, y) ||y| < \frac{1}{n} \right\} \subset V \right\}$$

then $\bigcup_{n=1}^{\infty} S_n = E$. Therefore, it is impossible that S_n are all finite. (Then union of them is countable) Then for a set S_N which has infinite elements, $x_1, x_2, \cdots x_n \in S_N$,

$$f(x, y) = (1 - Ny)\chi_{x \in \{x_1, \cdots x_n\}}$$

then $\Lambda f = \frac{n}{N}$. As n be arbitrary natural number, $\sup\{\Lambda f | f \prec V\} = \infty = \mu(V)$. Therefore, $\mu(E) = \inf\{\mu(V) | E \subset V\} = \infty$

2.10 Exercise 18 : No regularity for non σ-compact

This shows if not every open set is σ-compact, measure on 2.18 could not be regular

Sub problem 1. X is well-ordered uncountable set which has a last element ω_1, such that every predecessor of ω_1 has at most countable many predecessors. For $\alpha \in X$, let $P_\alpha[S_\alpha]$ be the set of all predecessors(successors) of α, and call a subset of X open if it is a P_α or an S_β or a $P_\alpha \cap S_\beta$ or a union of such sets. Prove that X is then a compact Hausdorff space.

Showing Hausdorff space is easy because $x < y$ then, if there is an element z that $x < z < y$, P_z, S_z is it. If it does not exist, P_y, S_x is it.

Compactness is more tricky. Thinking about open cover of X ($\{U_\alpha\}$), it covers ω_1 so $S_{\alpha_1} \in \{U_\alpha\}$. $\alpha_1 \in \bigcup_\alpha U_\alpha$ so if $\alpha_1 \in P_\gamma \in \{U_\alpha\}$, we have finite open cover $\{S_{\alpha_1}, P_\gamma\}$. However if α_1 is covered by set formed as S_{α_2} or $S_{\alpha_2} \cap P_\gamma$ then α_2 do same thing to α_2. This process generates infinite decreasing sequence of X. Since well-ordered set do not have infinite decreasing sequence, this process terminates. Therefore, there exists finite open cover of X. X is compact.

Sub problem 2. Prove that the complement of the point ω_1 is an open set which is not σ-compact.

$X-\{\omega_1\}=P_{\omega_1}$ so it is open set. If $X-\{\omega_1\}$ is σ-compact, since X is compact, closedness and compactness corresponds. So if $X-\{\omega_1\}$ is countable union of closed sets, ω_1 is countable intersection of open sets so

$$\{\omega_1\}=\bigcap_{n=1}^{\infty} S_{\alpha_n}$$

This is contradiction because

$$X=\{\omega_1\}\cup\left(\bigcup_{n=1}^{\infty} P_{\alpha_n}\right)$$

and it is countable while X is uncountable.

Also this proves following lemma.

[**Lemma**] If sequence $\alpha_1,\alpha_2,\cdots$ increases, least element of $\left\{x\middle|\alpha_n < x \quad \forall n\in\mathbb{N}\right\}$ is not ω_1

Sub problem 3. Prove that to every $f\in C(X)$ there corresponds an $\alpha\neq\omega_1$ such that f is constant on S_α.

For every $\epsilon>0$, $f^{-1}(f(\omega_1)-\epsilon,f(\omega_1)+\epsilon)$ is open set on X. So $S_{\alpha_\epsilon}\subset f^{-1}(f(\omega_1)-\epsilon,f(\omega_1)+\epsilon)$ for some α_ϵ. Considering $\beta_n=\max_{\epsilon\in\{1,1/2,\cdots 1/n\}}\alpha_\epsilon$, by Lemma above $\beta\neq\omega_1$ satisfies $\beta>\beta_n \quad \forall n\in\mathbb{N}$. Then $f(S_\beta)=\{f(\omega_1)\}$ so β is desired element.

[**Lemma**] Compact set is uncountable iff it contains ω_1 and ω_1 is clustering point.

proof.

If compact set is uncountable, ω_1 should be clustering point. Otherwise, least member of $\left\{x\middle|z<x \quad \forall z\in K\right\}$ is not equal to ω_1 and $K\subset P_{\omega_1}$ countable. And if K does not contain ω_1, then $\{P_x\}_{x\neq\omega_1}$ is open cover of K so there exists $K\subset P_\alpha$ thus countable.

If compact set contains ω_1 and it is clustering point, if set is countable,

$$X=\{\omega_1\}\cup\bigcup_{x\in K} P_x$$

X is also countable. Contradiction.

Sub problem 4. Prove that the intersection of every countable collection $\{K_n\}$ of uncountable compact subsets of X is uncountable.

By Lemma, each K_n contains ω_1 and ω_1 is clustering point for each set. We will show ω_1 is clustering point of intersection of collection.

First choose $\alpha_{1,1}\in K_1$. Since ω_1 is clustering point of K_2, $\alpha_{1,1}<\alpha_{2,1}\in K_2$. Since ω_1 is clustering point of K_1, $\alpha_{2,1}<\alpha_{1,2}\in K_1$. Repeating this process generates sequence $\alpha_{1,1}<\alpha_{2,1}<\alpha_{1,2}<\alpha_{3,1}<\alpha_{2,2}<\alpha_{1,3}<\cdots$. By lemma, there are α that $\alpha>\alpha_{m,n}$ for all m,n.

We could do this progress for starting bigger than α. Therefore ω_1 is clustering point of $\bigcap K_n$. $\bigcap K_n$ is uncountable compact set. (Compactness is trivial because closeness corresponds to compactness)

Main problem. $\mathfrak{M}$ be the collection of all $E \subset X$ such that either $E \cup \{\omega_1\}$ or $E^c \cup \{\omega_1\}$ contains an uncountable compact set; in the first case, define $\lambda(E) = 1$; in the second case, define $\lambda(E) = 0$. Prove that $\mathfrak{M}$ is a σ-algebra which contains all Borel sets in X, that λ is a measure on $\mathfrak{M}$ which is not regular and that

$$f(\omega_1) = \int_X f d\lambda$$

for every $f \in C(X)$. Describe the regular μ which associates with this linear functional.

Step I. $\mathfrak{M}$ is σ-algebra, containing Borel sets.

It is obvious $\phi, X \in \mathfrak{M}$ and $E \in \mathfrak{M} \Rightarrow E^c \in \mathfrak{M}$. If $A_1, A_2 \cdots \in \mathfrak{M}$, in the case $A_m \cup \{\omega_1\}$ contains uncountable compact subset for some m, so does $\bigcup A_n$. On the other case, $\bigcap \left(A_n^c \cup \{\omega_1\}\right)$ contains countable intersection of uncountable compact sets that is uncountable, compact. Therefore $\bigcup A_n \in \mathfrak{M}$.

Step II. λ is measure on $\mathfrak{M}$.

First, λ is well-defined. If $E \cup \{\omega_1\}$, $E^c \cup \{\omega_1\}$ both contains uncountable compact set, their intersection must exist (that is not equal to ω_1). Contradiction.
If pairwise disjoint $A_1, A_2 \cdots \in \mathfrak{M}$, and $\lambda(A_n) = 0$, by Step I, $\lambda\big(\bigcup A_n\big) = 0$, $\lambda(A_m) = 1$ for some m, then the number of measure 1 is at most 1 (since pairwise disjoint condition) and by Step I, $\lambda\big(\bigcup A_n\big) = 1$ Therefore, λ is measure on $\mathfrak{M}$.

Step III.

For continuous function f, f is constant in some S_α. Therefore for any step function converging to f, its integral via λ is equal to $f(\omega_1)$. Therefore,

$$f(\omega_1) = \int_X f d\lambda$$

Step IV. Furthermore features

λ is not regular. Because every open set containing ω_1 has measure 1, but $\lambda(\omega_1) = 0$.
Also for the linear functional, there are regular measure μ that is point measure of ω_1 i.e. measure of set is 1 iff it contains ω_1.

2.11 Compact set plays important on measure

Compact set is important object to understand. Good topological based understanding of Compact set is important.

3 Chapter 3. L^p spaces

3.1 $L^p(\mu)$ is complete

Main idea for proof is Lebesgue's Monotone Convergence Theorem. Especially Fatou's Lemma.

If $\{f_n\}$ is cauchy sequence, we find subsequence $\{f_{n_i}\}$ such that

$$||f_{n_{i+1}} - f_{n_i}||_p < 2^{-i}$$

Defining

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|, g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

By Minkowski inequality, $||g_k||_p \leq \sum_{i=1}^{k} ||f_{n_{i+1}} - f_{n_i}||_p \leq 1$. So Fatou's lemma shows $||g||_p \leq 1$. This means

$$f_{n_1}(x) + \sum_{i=1}^{\infty}(f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges for a.e X. That is function f that these cauchy sequence converging.

For every $\epsilon > 0$, there exists N if $m, n \geq N$, $||f_m - f_n||_p < \epsilon$ so

$$\int_X |f - f_m|^p d\mu \leq \liminf_{i \to \infty} \int_X |f_{n_i} - f_m|^p d\mu < \epsilon^p$$

Also by Fatau's Lemma. f is now L^p limit of $\{f_n\}$.

3.1.1 Fundamentally, why does Fatou's Lemma working?

Fatou's lemma states $f_n : X \to [0, \infty]$ is measurable, for each positive integer n then

$$\int_X \left(\liminf_{n \to \infty} f_n \right) d\mu \leq \liminf_{n \to \infty} \int_X f_n d\mu$$

Proof follows from Monotnone convergence theorem applied on $g_k(x) = \inf_{i \geq k} f_i(x)$. Then why does Lebesgue Monotnone Convergence Theorem holds fundamentally.

3.1.2 Fundamentally, why does Lebesgue Monotone Convergence Theorem holds?

I didn't find fundamental reason but, I have thought about two conditions.
(a) $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$
(b) $f_n(x) \to f(x)$

Lebesgue Monotnone Theorem says if this holds,

$$\int_X f_n d\mu \to \int_X f d\mu$$

It is quite obvious (b) needs to hold. But why (a) is needed?
If increasing statement omitted, there is counterexample, $f_n(x) = \frac{1}{x+n}$
If positiveness statement omitted, there is counterexample, $f_n(x) = -\frac{1}{x+n}$

I think Lebesgue's Monotnone Convergence Theorem is somehow elementary theorem. That is obviously comes from definition of integral and measure. I like to emphasize feature of integral : **Integral is organized with summation and multiplication**. Also I want to emphasize important component for measure and integral : **positivity and increasing feature**

Theorem that could be so called 'Theorem' is Lebesgue's Dominated Convergence Theorem.

3.1.3 Lebesgue's Dominated Convergence Theorem

Condition for increasing is omitted in this statement. Instead, L^1 condition is added. i.e.

$$f_n(x) \to f(x)$$

and

$$|f_n(x)| \leq g(x), \quad g \in L^1(\mu)$$

Interesting thing is integration limit only need pointwise convergence and some small assumptions

3.1.4 Pointwise covergence with L^p spaces

Furthermore, one could find following Theorem which states about L^p space with pointwise convergence.

Theorem. *If $1 \leq p \leq \infty$ and $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$, with limit j then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to $f(x)$*

This means if $f_n \to f$ in $L^p(\mu)$ then we have subsequence f_{n_k} which converges to f pointwise. Actually this is relation between **Normed Topology and Weak Topology**.

3.2 Approximation

3.2.1 L^p space with $1 \leq p < \infty$

There are two main approximation

Theorem. *For $1 \leq p < \infty$*
(1) $C_c(X)$ is dense in $L^p(\mu)$.
(2) S is class of all complex, measurable, simple functions on X such that

$$\mu(\{x : s(x) \neq 0\}) < \infty$$

Then S is dense in $L^p(\mu)$

That is, first we could **approximate L^p by continuous functions** that has compact support. Second, we could **approximate L^p by simple functions** that is nonzero on finite measure.

In other view, $L^p(\mathbb{R}^k)$ is the completion of the metric space which is obtained by endowing $C_c(\mathbb{R}^k)$ with L^p-metric.

3.2.2 L^∞ space

$C_c(X)$ is not dense in this space. $sgn(x)$ is important example. $C_c(X)$, completion via supremum norm is space $C_0(X)$ that is, vanishing at infinity.

Theorem. *Completion of $C_c(X)$ via supremum norm is, $C_0(X)$ that is continuous function space that vanishes at infinity.*

3.3 Containment between different L^r, L^s spaces

Assume $0 < r < s$.

3.3.1 For what condition does $L^s \subset L^r$

$$\mu(X) < \infty$$

($\Rightarrow$) If $\mu(X) < \infty$ and f is $L^s(\mu)$, by Holder Inequality

$$\begin{aligned}\int_X |f|^r d\mu &\leq \left(\int_X (|f|^r)^{\frac{s}{r}} d\mu\right)^{\frac{r}{s}} \left(\int_X 1^{\frac{s}{s-r}} d\mu\right)^{\frac{s-r}{s}} \\ &= ||f||_s^r \cdot \mu(X)^{\frac{s-r}{s}} < \infty\end{aligned}$$

($\Leftarrow$) If the statement holds, set $s = \infty$ then function 1 is in $L^1(\mu)$. Therefore $\mu(X) = \int_X 1 d\mu < \infty$.

3.3.2 For what condition does $L^r \subset L^s$

$$\exists c, \quad \mu(E) > 0 \Rightarrow \mu(E) > c$$

($\Rightarrow$) If above condition holds and c exists, let f is $L^r(\mu)$ function. E_n is the set $E_n = \{x \in X : f(x) \geq n\}$. Then as $n \to \infty$, $\mu(E_n) \to 0$. So there exists n_0 such that $\mu(E_{n_0}) = 0$. Therefore,

$$\int_X |f|^s d\mu \leq \int_X n_0^{s-r} |f|^r d\mu \leq ||f||_r^r \cdot n_0^{s-r}$$

f is $L^s(\mu)$

($\Leftarrow$) Think controversial statement : If every $\epsilon > 0$ we can find measurable set E with $\mu(E) < \epsilon$. Then let we construct following sets.
First, $\mu(E_1) > 0$. Then E_{n+1} is defined to $0 < \mu(E_{n+1}) < \frac{1}{3}\mu(E_n)$. $W_n = E_n - \bigcup_{k=1}^\infty E_{n+k}$. Then $0 < \mu(W_n) < \frac{1}{3^{n-1}}\mu(E_1)$.

Now defining f as

$$f = \sum_{n=1}^\infty \frac{1}{\mu(W_n)^{1/r}} \times 3^{-(n-1)(s-r)/s^2}$$

Then
$$\int_X |f|^r d\mu = \sum_{n=1}^{\infty} 3^{-(n-1)(s-r)r/s^2} < \infty$$
but
$$\int_X |f|^s d\mu = \sum_{n=1}^{\infty} \left(\frac{1}{\mu(W_n^{(s-r)/s})} \times 2^{-i} \right) \geq \sum_{n=1}^{\infty} \frac{3^{(n-1)(s-r)/s}}{\mu(E_1)^{(s-r)/s}} \times 3^{-(n-1)(s-r)/s} = \infty$$

3.3.3 Some Counterexamples

1. Counting measure on $\mathbb{N}$ and $f(n) = \frac{1}{n^{\frac{2}{r+s}}}$ then $f \in L^r$ but $f \notin L^s$.
2. Lebesgue measure on $[0,1]$ and $f(x) = \frac{1}{x^{\frac{2}{r+s}}}$ then $f \in L^r$ but $f \notin L^s$.

3.3.4 Remarks

Remarkable fact is, for $r < s$, $L^r \subset L^s$ iff measure cannot be extremely small, $L^s \subset L^r$ iff measure cannot be extremely big.
How can we interpret this?

3.4 Mapping against L^p spaces

3.4.1 Exercise 13. Integration: $f \to \int_0^x f$

For $1 < p < \infty$, $f \in L^p((0,\infty))$ relative to Lebesgue measure and
$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

Then Hardy's inequality says
$$||F||_p \leq \frac{p}{p-1}||f||_p$$

Proof as follows. Suppose f is positive and $f \in C_c((0,\infty))$ then by Holder Inequality,
$$\begin{aligned}\int_0^\infty F^p(x)dx &= -p\int_0^\infty F^{p-1}(x)xF'(x)dx \\ &= -p\int_0^\infty F^{p-1}(x)(f(x) - F(x))dx \\ &\geq -p||F||_p^{p-1}||f||_p + p\int_0^\infty F^p(x)dx\end{aligned}$$

So Desired result comes. We could extend into noncompact support, non sign-condition functions easily.

This means mapping $f \to F$ gives L^p to L^p mapping for $1 < p < \infty$. Unfortunately, considering $p = 1$ does not gives such mapping.

To say differently, if $f > 0$ and $f \in L^1$ then $F \notin L^1$. If f is norm zero on L^1, f needs to be zero a.e. So, we could notice f has positive norm and so $\int_0^N f(t)dt > \epsilon$ for some N and ϵ. Then,
$$\int_0^\infty \int_0^x f(t)dtdx \geq \int_N^\infty \epsilon dx = \infty$$
Therefore, F is not L^1

3.4.2 Exercise 24. $f \to f^p$ is continuous mapping

Exercise 24 says $f, g \in L^p(\mu)$ then $f \to f^p$ is continuous mapping from $L^p(\mu)$ to $L^1(\mu)$. There are two inequalities proving this.

For $0 < p < 1$,

$$\int ||f|^p - |g|^p| d\mu \leq \int |f - g|^p d\mu$$

For $1 \leq p < \infty$, $||f||_p \leq R$ and $||g||_p \leq R$ then

$$\int ||f|^p - |g|^p| d\mu \leq 2pR^{p-1}||f - g||_p$$

Proof follows from some calculation on complex numbers.

3.5 Egoroff's theorem

Theorem (Egoroff's theorem)**.** *If $\mu(X) < \infty$, if $\{f_n\}$ is a sequence of complex measurable functions which converges pointwise at every point of X, and if $\epsilon > 0$, there is a measurable set $E \subset X$, with $\mu(X - E) < \epsilon$, such that $\{f_n\}$ converges uniformly on E.*

This theorem means **for pointwise converging functions, we could make it into uniformly converging functions with deleting some small-measure space**.

3.5.1 Proof of Egoroff's theorem

$$S(n, k) = \bigcap_{i,j>n} \left\{ x : |f_i(x) - f_j(x)| < \frac{1}{k} \right\}$$

Then for each k, $S(1, k) \subset S(2, k) \subset S(3, k) \subset \cdots$. Since $\{f_n\}$ converges pointwise, $\bigcup_n S(n, k) = X$. By basic property of measure

$$\mu(S(n, k)) \to \mu(X)$$

Now choose n_k as following : $n_k > n_{k-1}$ and $\mu(S(n_k, k)) \geq \mu(X) - \frac{\epsilon}{2^k}$. Then

$$\mu\left(\bigcap_k S(n_k, k) \right) \geq \mu(X) - \epsilon$$

Also, denote E as $E = \bigcap_k S(n_k, k)$, E is measurable and for those x in E and $\epsilon > 0$, we can find $\frac{2}{N} < \epsilon$, for $n \geq n_N$, $|f_n - f| < \epsilon$. Therefore, E satisfies desired property.

However, this does not hold when $\mu(X) = \infty$ even if it is σ-finite spaces. Counterexample for this is

$$f_n(x) = \begin{cases} 0 & x < n + 1 \\ \frac{1}{x-n} & x \geq n + 1 \end{cases}$$

$\{f_n\}$ pointwisely converges to zero function but $||f_n - 0|| = \infty$ for all n so does not converge uniformly.

3.5.2 Lemma following from Egoroff's theorem

Lemma : μ is a positive measure on X, $0 < p < \infty$, $f \in L^p(\mu), f_n \in L^p(\mu), f_n(x) \to f(x)$ a.e., and $||f_n||_p \to ||f||_p$ as $n \to \infty$. Then

$$\lim ||f - f_n||_p = 0$$

First, to comment on this Lemma, most of the hypothesis are seemed to be necessary except the condition $||f_n||_p \to ||f||_p$. However, if this is omitted following counterexample exists.

$$f_n(x) = \begin{cases} \sqrt{\frac{n}{x}} & x < \frac{1}{n} \\ 0 & x \geq \frac{1}{n} \end{cases}$$

For all n, $||f_n||_1 = 2$ and f_n converges to zero on $(0, 1)$. But $||f - f_n||_1 = ||f_n||_1 = 2$, nonzero.

3.5.3 Proof of Lemma

1. First we prove $X = A \cup B$ with $\int_A |f|^p < \epsilon$, $\mu(B) < \infty$, and $f_n \to f$ uniformly on B.

$E_n = \left\{x : |f(x)| > \frac{1}{n}\right\}$ and $E_0 = \left\{x : f(x) = 0\right\}$ then $E_0 \bigcup \bigcup_n E_n = X$. For all $n \in \mathbb{N}$, $\mu(E_n) < \infty$. By definition of $||f||_p$,

$$\int_{E_N} |f|^p d\mu \geq ||f||_p^p - \epsilon/2$$

Then we have $N \in \mathbb{N}$ such that above inequality holds and $\mu(E_N) < \infty$.

Append Egoroff's theorem on E_N. Then we could find set F_m such that in F, $\{f_n\}$ converges uniformly to f and $\mu(E_N - F) < \frac{1}{m}$ (for all m).

$$A_m = X - \bigcup_{1 \leq m' \leq m} F_{m'}, \quad B_m = \bigcup_{1 \leq m' \leq m} F_{m'}$$

$\{f_n\}$ uniformly converges on B_m, $\mu(B_m) < \infty$ and $\mu(E_N - F_m) \to 0$. Therefore, we could find M such that

$$\int_{B_M} |f|^p d\mu \geq \int_{E_N} |f|^p d\mu - \epsilon/2$$

Now if

$$A = X - B_M, \quad B = B_M$$

desired property comes.

2. $\limsup \int_A |f_n|^p d\mu \leq \epsilon$

Fatou's lemma applied to $\int_B |f_n|^p$ leads

$$\int_B |f|^p d\mu = \int_B \liminf_n |f_n|^p d\mu$$
$$\leq \liminf_n \int_B |f_n|^p d\mu \leq ||f||_p^p - \limsup_n \int_A |f_n|^p d\mu$$

So $\limsup_n \int_A |f_n|^p d\mu \leq \epsilon$ obtained. (But we do not use fact that $\{f_n\}$ uniformly converges to f on B)

3. $\lim ||f - f_n||_p = 0$

Apply Fatou's lemma to $h_n = \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$ where $\gamma_p = \max(1, 2^{p-1})$. (This γ_p satisfies following inequality between arbitrary complex numbers α and β, $|\alpha - \beta|^p \leq \gamma_p(|\alpha|^p + |\beta|^p)$)
First, in B, f_n converges f uniformly and $\mu(B) < \infty$, so

$$\int_B |f_n - f|^p d\mu \leq \epsilon^p \mu(B)$$

for sufficiently large n.
Second, for A,

$$\begin{aligned}
2\gamma_p \int_A |f|^p d\mu &= \int_A \liminf_n h_n d\mu \\
&\leq \gamma_p \int_A |f|^p d\mu + \liminf_n \int_A \big(\gamma_p |f_n|^p - |f - f_n|^p\big) d\mu \\
&\leq \gamma_p \int_A |f|^p d\mu + \gamma_p \limsup_n \left(\int_A |f_n|^p d\mu \right) - \limsup_n \int_A |f - f_n|^p d\mu
\end{aligned}$$

So following inequality holds.

$$\limsup_n \int_A |f - f_n|^p d\mu < \limsup_n \left(\int_A |f_n|^p d\mu - \int_A |f|^p d\mu \right) \leq \epsilon$$

Therefore, $||f_n - f||_p \to 0$ because

$$\limsup_n \int_X |f - f_n|^p d\mu < \epsilon + \epsilon^p \mu(B)$$

4 Chapter 4. Hilbert space

4.1 Basic Hilbert space properties

H is complex vector space with inner product space. H is actually a metric space because Schwarz Inequality and Triangle Inequality holds.

There are some properties like this.

4.1.1 Continuous functions

$$x \to (x, y)$$

$$x \to (y, x)$$

$$x \to ||x||$$

Is continuous functions.

4.1.2 Smallest norm property

Every nonempty, closed, convex set E in a Hilbert space H contains a unique element of smallest norm.

4.1.3 Orthogonal Decomposition

M be a closed subspace of a Hilbert space H. Then every $x \in H$ has a unique decomposition $x = Px + Qx$, that is $Px \in M,\ Qx \in M^{\perp}$

4.1.4 Continuous linear functional Representation

L be a continuous linear functional on H, there is a unique $y \in H$ such that

$$Lx = (x, y)$$

4.2 Orthonormal sets

For orthonormal set $\{u_\alpha\}$, $\alpha \in A$ then, for **each** $x \in H$ we could make **complex function** $\hat{x} : A \to \mathbb{F}$ such that

$$\hat{x}(\alpha) = (x, u_\alpha)$$

This $\hat{x}$ is kind of function that assign α a index into coordinate value. Interesting part is, $x \to \hat{x}$ is **mapping from Hilbert space H onto Hilbert space $l^2(A)$ (So called Riesz Fischer theorem)** (with counting measure on set A)

Theorem. *Let $\{u_\alpha : \ \alpha \in A\}$ be an orthonormal set in H. Each of the following implies other three.*
(1) $\{u_\alpha\}$ is a maximal orthonormal set in H
(2) P of all finite linear combination of $\{u_\alpha\}$ is dense in H
*(3) **[Bessel inequality]** Equality folds for every $x \in H$*

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 = ||x||^2$$

*(4) **[Parseval's identity]** Equality folds for all $x, y \in H$*

$$\sum_{\alpha \in A} \hat{x}(\alpha)\bar{\hat{y}}(\alpha) = (x, y)$$

By Hausdorff Maximality Theorem, any orthonormal set could extended to orthonormal basis. Essentially, if $\{u_\alpha\}$ is orthonormal set, $x \to \hat{x}$ is onto mapping. Moreover, mapping $x \to \hat{x}$ is Hilbert space isomorphism if $\{u_\alpha\}$ is orthonormal basis.

Therefore, by defining orthonormal basis we could interpret Hilbert space as coordinate on orthonormal basis with l^2 space

4.3 Fourier series

T is unit circle on complex space. We will almost use equally T as $[0, 2\pi]$ with converting $f(t) = F(e^{it})$. That is equalize domain T function to 2π periodic functions. $L^p(T)$ is function space which for $1 \leq p < \infty$,

$$||f||_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p dt \right\}^{\frac{1}{p}}$$

or

$$||f||_\infty = \sup_t |f(t)|$$

Trigonometric polynomial is finite sum of form

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n cos(nt) + b_n sin(nt)) = \sum_{n=-N}^{N} c_n e^{int} = \sum_{n=-N}^{N} c_n u_n(t)$$

4.3.1 Completeness of Trigonometric System $\equiv L^2(T)$ has orthonormal basis $u_n(t)$

First, $C(T)$ dense in $L^2(T)$. It's because $C_c(X)$ is dense in $L^p(\mu)$ (3.2 Approximation in this note). $u_n(t)$ is definitely, orthonormal set. By proving $C_c(X)$ could be approximated by finite sum of $u_n(t)$, we could prove completeness of the Trigonometric system.

Goal : $||f - P||_\infty < \epsilon$ for P a Trigonometric polynomial.
Defining Trigonometric polynomials $Q_1, Q_2, \cdots$

$$Q_k(t) = c_k \left(\frac{1 + cos(t)}{2} \right)^k$$

$$P_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t - s) Q_k(s) ds$$

Then $\forall \epsilon > 0$, $\exists n$ then $\forall t \ \ |f(t) - P_n(t)| < \epsilon$

Therefore, we got **orthonormal basis $u_n(t)$ for $L^2(T)$.**

4.3.2 Fourier Series

For any $f \in L^1(T)$, define Fourier coefficients of f : $\hat{f}(n)$ (which is **Exactly same as Hilbert space Fourier coefficients**)

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

Fourier series of f is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$$

Partial sum of f Is

$$s_N(t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}$$

Properties gained by Hilbert space theorems, $f \in L^2(T)$ then
[Riesz-Fischer theorem] If $\{c_n\}$ a complex numbers such that $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$ then there exists an $f \in L^2(T)$ such that

$$\hat{f}(n) = c_n$$

[Parseval theorem] $f, g \in L^2(T)$ then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\bar{\hat{g}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\bar{g(t)}dt$$

Also $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$ so s_N **converges to** $f \in L^2(T)$ **in** L^2 **sense.**

$$\lim_{N\to\infty} ||f - s_N||_2 = 0$$

4.4 Exercise 6 : Hilbert Cube

$\{u_n\}$ is orthonormal set in H. Then Hilbert cube is set of form

$$x = \sum_{n=1}^{\infty} c_n u_n$$

where $|c_n| \leq \frac{1}{n}$. This set Q is compact although H is not locally compact.

H is not locally compact since open ball $B(0,1)$ has sequence $u_1,\ u_2,\ u_3 \cdots$ that any subsequence does not converge.

Theorem (Hilbert Cube). *If $\{\delta_n\}$ is sequence of positive numbers, and Q be a set of all $x \in H$ of the form*

$$x = \sum_{n=1}^{\infty} c_n u_n$$

where $|c_n| \leq \delta_n$. Then Q is compact if and only if $\sum_{n=1}^{\infty} \delta_n^2 < \infty$

Proof as follows. H is normed space so, sequential compactness is equivalent to compactness. If $x_1, x_2, \cdots$ a sequence on Q,

$$x_1 = \sum_{n=1}^{\infty} c_{1n} u_n$$

$$x_2 = \sum_{n=1}^{\infty} c_{2n} u_n$$

$$x_3 = \sum_{n=1}^{\infty} c_{3n} u_n$$

Then c_{1n} is complex sequence that is bounded by δ_1, it has converging subsequence. $c_{1n_1^1},\ c_{2n_2^1},\ \cdots$ converging to c_1.
$c_{2n_m^1}$ is complex sequence that is bounded by δ_2, it has converging subsequence. $c_{2n_1^2},\ c_{2n_2^2},\ \cdots$ converging to c_2.

Repeating these steps, we can generate $c_1,\ c_2,\ c_3 \cdots$. Let $x = \sum_{k=1}^{\infty} c_k u_k$ then $|c_k| \leq \delta_k$ so $X \in Q$.

Now, $x_1^1,\ x_2^2,\ \cdots$ converge to x. For all $\epsilon > 0$, let N be a number satisfying $\sum_{n=N+1}^{\infty} \delta_n^2 < \epsilon$

If $k \geq N$,

$$\begin{aligned} ||x_k^k - x||^2 &= \sum_{n=1}^{N} |c_{n,n_k^k} - c_n|^2 + \sum_{n=N+1}^{\infty} |c_{n,n_k^k} - c_n|^2 \\ &< N \times \left(\sqrt{\frac{\epsilon}{N}}\right)^2 + 4 \sum_{n=N+1}^{\infty} \delta_n^2 \\ &< 5\epsilon \end{aligned}$$

If $\sum_{n=1}^{\infty} \delta_n^2 = \infty$ then, define n_k a sequence that

$$\sum_{n=n_k+1}^{n_{k+1}} \delta_n^2 \geq 1$$

For

$$x^k = \sum_{n=1}^{n_k} \delta_n u_n$$

$$||x^k - x^l|| \geq 1$$

for all $k \neq l$, there are no converging sequence.

4.5 Exercise 3, 4 : Separability

Separability is defined : A space is separable if it contains countable dense subset.

As discussed on Chapter 3, $L^p \quad (1 \leq p < \infty)$ and L^∞ differs in same sense for separability.

4.5.1 L^p is Separable

Trigonometric functions are dense in L^p spaces. Think of sets

$$S_{N,M} = \left\{ P \middle| P = \sum_{n=-N}^{M} r_{\sigma_{N,M}(n)} u_n \right\}$$

$S_{N,M} = \mathbb{Q} \times \mathbb{Q} \times \cdots$ for $N + M$ times, so countable. Also,

$$P_{\mathbb{Q}} = \bigcup_{N,M \geq 1} S_{N,M}$$

so rational-coefficient Trigonometric polynomial are countable.

$P_{\mathbb{Q}}$ is dense on P, the space of Trigonometric polynomial. Therefore, L^p has countable dense subset thus Separable.

4.5.2 L^∞ is not Separable

Think of function $\delta(x) = \chi_{[0,\pi)}(x)$. Functions $\delta_t(x) := \delta(x+t)$ are all L^∞ functions for all $0 \leq t < 2\pi$.

$$||\delta_t - \delta_s||_{L^\infty} = 1$$

If there are countable dense subset S of L^∞, we could find functions $g_{t,n} \in S$ such that

$$g_{t,n} \to \delta_t$$

Especially,

$$||g_{t,n} - \delta_t|| < 1/2$$

Then it is contradiction because S is countable and $\{\delta_t\}_t$ is uncountable, some $g_{t,n} = g_{s,m}$ but

$$1 = ||\delta_t - \delta_s||_{L^\infty} \leq ||g_{t,n} - \delta_t|| + ||g_{s,m} - \delta_s|| < \frac{1}{2} + \frac{1}{2} = 1$$

By these observation, we could claim following theorem for separability.

Theorem (Separability). *Hilbert space H is separable if and only if H contains a maximal orthonormal system which is at most countable.*

Proof is by thinking of orthonormal basis with index set A then

$$H \equiv l^2(A)$$

If orthonormal basis are uncountable, similar to L^∞ argument, H is not separable.

Also, if orthonormal basis is countable, similar to L^p argument, we could construct by assigning rational numbers for coefficient.

4.6 Exercise 9, 10, 13 : On orthonormal basis u_n, sine, cosine

These exercise gives more understanding on sine, cosine, and u_n functions.

[Lemma 1] If $A \subset [0, 2\pi]$, A is measurable then

$$\lim_{n\to\infty} \int_A cos(nx)dx = \lim_{n\to\infty} \int_A sin(nx)dx = 0$$

Proof comes by thinking of $A \subset V \subset [-\pi, 3\pi]$ and V is open set. Then $V = U_1 \cup U_2 \cup \cdots$. There exists m such that

$$m\left(\bigcup_{i=m+1}^{\infty} U_i\right) < \epsilon/2$$

For $U_1 = (a_1, b_1), U_2 = (a_2, b_2), \cdots U_m = (a_m, b_m)$, for sufficiently large N, if $n \geq N$

$$\left|\int_{(a_j,b_j)} sin(nx)dx\right| < \frac{\epsilon}{2N}$$

for all $j \in \{1, 2, \cdots m\}$.

Therefore, for all $A \subset V \subset [-\pi, 3\pi]$ there exists N such that for $n \geq N$

$$\left|\int_V sin(nx)dx\right| < \epsilon$$

Using regularity of Lebesgue measure, there exists open set $A \subset W$ that $m(W - A) < \epsilon$. Then

$$\left|\int_A sin(nx)dx\right| < \left|\int_W sin(nx)dx\right| + m(W - A) < 2\epsilon$$

That is

$$\lim_{n\to\infty} \int_A cos(nx)dx = \lim_{n\to\infty} \int_A sin(nx)dx = 0$$

[Lemma 2] Let $n_1 < n_2 < n_3 < \cdots$ be positive integers, E be set of all $x \in [0, 2\pi]$ which $\{sin n_k x\}$ converges. Then $m(E) = 0$

Proof comes from this equation : $2sin^2\alpha = 1 - cos2\alpha$. So $cos(2n_kx)$ converges in E. For $F \subset E$ a measurable set,

$$\lim_{k\to\infty} \int_F cos(2n_kx)dx = 0$$

By Lebesgue's Dominated Convergence Theorem,

$$\int_F \lim_{k\to\infty} cos(2n_kx)dx = 0$$

That is,

$$\int_F \lim_{k\to\infty} sin^2(n_kx)dx = 1$$

So $sin(n_kx) \to \pm\frac{1}{\sqrt{2}}$ a.e. on E.

Also,

$$\int_F \lim_{k\to\infty} sin(n_k x)dx = 0$$

so $m(F) = 0$. That is, $m(E) = 0$

Remember, limit exchange on integral is able within Lebesgue Dominated Conversion Theorem or requires increasing sequence.

[**Lemma 3**] Suppose f is a continuous function on $\mathbb{R}$ with period 1, then for every irrational number α,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt$$

Proof comes from doing with

$$f(t) = e^{2\pi ikt} \qquad\qquad k = 0, \pm 1, \pm 2, \cdots$$

This is trivial because for $k = 0$, it naturally comes and for $k \neq 0$,

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ik(n\alpha)} = \frac{1}{N}\frac{e^{2\pi ik\alpha}(1 - e^{2\pi ik(N\alpha)})}{1 - e^{2\pi ik\alpha}} \to 0 = \int_0^1 e^{2\pi ikt}dt$$

Now for continuous function $f \in C([0,1])$, there exists Trigonometric polynomial P such that

$$P = \sum_{k=-K}^{K} c_k e^{2\pi ikt}, \qquad ||f - P||_\infty < \epsilon/3$$

Then we know for sufficiently large N,

$$\left|\int_0^1 f(t)dt - \int_0^1 P(t)dt\right| < \frac{\epsilon}{3}$$
$$\left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) - \frac{1}{N}\sum_{n=1}^{N} P(n\alpha)\right| < \frac{\epsilon}{3}$$
$$\left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) - \int_0^1 P(t)dt\right| < \frac{\epsilon}{3}$$

Therefore, the equation hold by triangle inequality.

We need to think more about this equation. What does this equation means? If α is rational, left hand side will be kind of mean for finite values. For irrational α, we now know $n\alpha$ can play a role as Riemann integral. So this equation means $\{n\alpha\}_{n\in\mathbb{N}}$ **could play role that is uniformly distributed on [0,1] for continuous functions if α irrational**

5 Chapter 5. Banach Space

Banach space is in one sentence, complex vector space that has norm and is complete. Three important theorems (actually four) are important. First define a norm of linear transformation $\Lambda : X \to Y$

$$||\Lambda|| = \sup\{||\Lambda x|| : x \in X, ||x|| \leq 1\}$$

Boundness of Λ and Continuity of Λ coincides.

Now introduce four major theorems for Banach space : Baire's Theorem, Banach-Steinhaus Theorem, Open Mapping Theorem, Hahn-Banach Theorem.

5.1 Baire's Theorem

Theorem (Baire's Theorem)**.** *If X is complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X*

Before proving statement, thinking of meaning is quite important. Completeness plays important role. Briefly thinking, every open subset is dense on X. So every finite intersection of open subset is still dense. Problem occurs on countable intersection. To resolve this, completeness is needed. If sequentially finite intersection is dense, one can extract points that belong to intersection of arbitrary open set and finite intersection of open subsets. These points will gather, they will form Cauchy sequence. Completeness implies limit exists and this point will make subset dense.

Proof as follows : $V_1, V_2, V_3 \cdots$ dense open set in X. If arbitrary open set $W \neq \phi$, $W \cap \bigcap_n V_n \neq \phi$ then $\bigcap_n V_n$ is dense.

Using notation

$$S(x, r) = \{y \in X : \rho(x, y) < r\}$$

V_1 dense so there exists

$$\bar{S}(x_1, r_1) \subset W \cap V_1, \qquad 0 < r_1 < 1$$

V_2 dense so $V_2 \cap S(x_1, r_1) \neq \phi$ there exists

$$\bar{S}(x_2, r_2) \subset S(x_1, r_1) \cap V_2, \qquad 0 < r_2 < \frac{1}{2}$$

Repeating this,

$$\bar{S}(x_n, r_m) \subset S(x_{n-1}, r_{n-1}) \cap V_n, \qquad 0 < r < \frac{1}{n}$$

Then $\{x_n\}$ is cauchy sequence, x is desired point : $x \in W$, $x \in V_n$ for all n

Following is consequence theorem.

Theorem. *In a complete metric space X which has no isolated points, no countable dense set is G_δ*

Proof : If E is dense, G_δ and $x_1, x_2, \cdots$ numbering possible, $E = \cap V_n$ then $V_n - \{x_1, x_2, \cdots x_n\}$ is dense open, but intersection is not dense in E.

Notice "no isolated" argument used to verify $V_n - \{x_1, x_2, \cdots x_n\}$ is dense.

Also, this means if X has no isolated points, G_δ set is uncountable.

[Corollary] $\mathbb{Q}$ is not G_δ

5.2 Banach-Steinhaus Theorem

Theorem (Banach-Steinhaus Theorem)**.** *X is Banach space, Y is normed linear space. $\{\Lambda_\alpha\}$ is collection of bounded linear transformations of X into Y. Then only one of two holds*
(1) $\exists M < \infty$ such that $||\Lambda_\alpha|| \leq M$
(2) $\sup_{\alpha\in A} ||\Lambda_\alpha x|| = \infty$ for some dense G_δ set in X

Proof Idea is

$$\varphi(x) = \sup_{\alpha\in A} ||\Lambda_\alpha x||, \qquad V_n = \{x : \varphi(x) > n\}$$

If some V_N fails to dense in X, (1) holds, if not (2) holds.

Main usage is to define Λ_α and prove (1) or (2) might fail. Then other holds.

5.3 Open Mapping Theorem

This theorem is more tricky. **Understanding this theorem is, in Banach space onto bounded linear transform, image for unit ball does not vanish anywhere**. It is not rigorous but understanding like : Image for boundary of unit ball do not tend to zero anywhere. **It states Zero is impossible but also tending to zero is impossible**

Theorem (Open Mapping Theorem)**.** *U, V are open unit balls of the Banach spaces X, Y. Every bounded linear transformation Λ of X **onto** Y there corresponds a $\delta > 0$ so that*

$$\Lambda(U) \supset \delta V$$

Proof is first, within $y \in Y$ onto implies $\exists x \in X$ that $\Lambda x = y$. If $||x|| < k$, $y \in \Lambda(kU)$. So

$$Y = \bigcup_{k\in\mathbb{N}} \Lambda(kU)$$

By Baire theorem, some kU contains open set W. We can prove for each $y \in W$, $\eta > 0$ exists such that $y + y_0 \in W$ if $||y_0|| < \eta$ and

$$||y|| < \delta \Longrightarrow \{x_i\} \ s.t. \ ||x_i|| < 2k \ \ and \ \ \Lambda x_i \to y$$

Therefore, we have a bound if $y \in Y$, $0 < \epsilon < 1$, there exists $x \in X$ such that

$$||x|| \leq \delta^{-1}||y|| \ \ and \ \ ||y - \Lambda x|| < \epsilon$$

Now, recursively construct x_i's for $y \in \delta V$.

$$||y - \Lambda x_1|| < \frac{1}{2}\delta\epsilon, \qquad ||x_1|| < 1$$

$$||y - \Lambda x_1 - \Lambda x_2|| < \frac{1}{2^2}\delta\epsilon, \qquad ||x_1|| < \frac{1}{2}\epsilon$$

$$||y - \Lambda x_1 - \Lambda x_2 - \Lambda x_3|| < \frac{1}{2^3}\delta\epsilon, \qquad ||x_1|| < \frac{1}{2^2}\epsilon$$

$$\vdots$$

For $x = x_1 + x_2 \cdots$, $||x|| < 1 + \epsilon$ and one could check $y = \Lambda x$. Thus

$$\Lambda((1+\epsilon)U) \supset \delta V$$

$$\Lambda(U) \supset (1+\epsilon)^{-1}\delta V$$

Theorem (Open Mapping Theorem applied to inverse transform). *X, Y is Banach space. Every bounded linear transformation Λ of X **one-to-one and onto** Y there is a $\delta > 0$ so that*

$$||\Lambda x|| \geq \delta ||x||$$

So Λ^{-1} is also bounded linear transformation of Y onto X.

Now, summarising open mapping theorem

In Banach Space onto bounded linear mapping, vanishing to 0 does not occur

5.4 Hahn-Banach Theorem

Theorem (Hahn-Banach Theoeremm). *If M is subspace (not necessarily closed) of normed linear space X. f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X such that*

$$||F|| = ||f||$$

Proof is first prove if one vector is added to M, extension is possible and using Hausdorff Maximality Theorem for extend until X.

This theorem gives us **concept of duality on X and bounded linear functional on X**. First encode property of some points to subspace of X by function, and Hahn-Banach space extends this function to X with norm preserving then we have encoded information on bounded linear funcitonals.

Consequences are given :

Lemma. *M a linear subspace of normed linear space X, and $x_0 \in X$. x_0 is in $\bar{M}$ if and only if there is no bounded linear functional f on X such that $f(x) = 0$ for $x \in M$ but $f(x_0) \neq 0$*

Lemma. *X is normed linear space and if $x_0 \in X, \quad x_0 \neq 0$ there is a bounded linear functional f on X of norm 1 with $f(x_0) = ||x_0||$*

5.5 Bounded linear functional space X^*

X^* is collection of all bounded linear functional on X.

Some properties of X^* is, it is non trivial vector space. For $x_1 \neq x_2$ then there is linear functional f with $f(x_1 - x_2) = x_1 - x_2$ so it separates points on X. Also

$$||x|| = \sup\{|f(x)| : f \in X^*, \quad ||f|| = 1\}$$

so, $f \longmapsto f(x)$ is bounded linear functional of norm $||x||$.

Actually, X^* is Banach space. Normed condition could checked easily, and for completeness if $\{f_n\}$ Cauchy sequence $\forall ||x|| \leq 1$, $||f_n - f_m|| \geq ||f_n(x) - f_m(x)|$ so $f_n(x)$ Cauchy. This ocnverges to $f(x)$ then, $|f_n(x) - f_m(x)| \leq ||f_n - f_m|| < \epsilon$ implies $|f_n(x) - f(x)| < 2\epsilon$ for each x when $m \to \infty$. Therefore, $||f_n - f|| \to 0$. f is limit of $\{f_n\}$.

There are more Consequences obtained by encoding X's property to X^*.

Lemma. *$\{||x_n||\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$*

Proof. If we define $\Lambda_i f : f \longmapsto f(x_i)$, $||\Lambda_i|| = ||x_i||$. So, $\{||x_n||\}$ is bounded if and only if Λ_i's are uniformly bounded.

Applying **Banach-Steinhaus Theoerem**, Λ_n are uniformly bounded or $\sup_{n\in\mathbb{N}} |\Lambda_n f| = \infty$ for some f. This means $\sup_n |f(x_n)| = \infty$ on some $f \in X^*$ which is contradiction to assumption. □

5.5.1 Examples of X^*

c_0, l^1, l^∞ is Banach spaces consisting of all complex sequences $x = \{\xi_i\}$ defined as

$$x \in l^1 \ if\ and\ only\ if\ ||x||_1 = \sum |\xi_i| < \infty$$

$$x \in l^\infty \ if\ and\ only\ if\ ||x||_\infty = \sup |\xi_i| < \infty$$

$$c_0 \subset l^\infty \ \ and\ satisfies\ \xi_i \to 0$$

In these spaces,

(1) $(c_0)^* = l^1$

Proof by considering $y = \{\eta_i\} \in l^1$ and $\Lambda x = \sum \xi_i \eta_i$ for $x \in c_0$. This is bounded linear functional because, first $\sum \xi_i \eta_i$ converges.

$$\sum_{|n| \geq N} |\xi_i \eta_i| < \epsilon ||x||_{l^1} \to 0$$

so tail parts tend to zero and

$$|\xi_i \eta_i| \leq ||x||_{l^\infty} \sum |\eta_i| < \infty$$

Now this Λ is bounded since $|\Lambda x| \leq ||y||_1 ||x||_\infty$. Actually norm is $||y||_1$ ($||\Lambda|| = ||y||_1$) because for

$$\zeta_n = \left(\frac{\bar{\eta_1}}{|\eta_1|}, \frac{\bar{\eta_2}}{|\eta_2|}, \cdots \frac{\bar{\eta_n}}{|\eta_n|} \right)$$

$||\Lambda \zeta_n|| \to ||y||_1$ (more precisely, we could make them close to $\forall \epsilon > 0$)

If $\Lambda \in (c_0)^*$, define $\eta_i := \Lambda e_i$ then $\sum_{n=-N}^{N} |\eta_n| \leq ||\Lambda||$ so $\eta \in l^1$.

(2) $(l^1)^* = l^\infty$

Also for $\Lambda x = \sum \xi_i \eta_i$ for $x \in l^1$ $y \in l^\infty$ then Λx is bounded linear functional. Then for bounded linear functional Λ, define $\eta_i := \Lambda e_i$ then $||\eta_i|| \leq ||\Lambda||$ so $\eta \in l^\infty$.

(3) $(l^\infty)^* \neq l^1$

In this case, it induces for $\Lambda \in (l^\infty)^*$ and $\Lambda e_i = \eta_i$

$$\sum_{n\in\mathbb{Z}} |\eta_n| < ||\Lambda||$$

This differs from

$$\forall n \in \mathbb{N}, \quad \sum_{|n|\leq\mathbb{N}} |\eta_n| < ||\Lambda||$$

Also, note that c_0 and l^1 are separable but l^∞ is not. c_0 and l^1 have countable dense subset: finite sums of $\{r_j e_i\}_{i\in\mathbb{N},\ j\in\mathbb{Q}}$ but l^∞ is Hilbert space and does not have countable dense subset.

5.6 Fourier Analysis

5.7 Exercise 13 : Application of Baire's Theorem

$\{f_n\}$ be a sequence of continuous complex functions on a complete metric space X, such that for every $x \in X$, there exists $f(x) = \lim_{n\to\infty} f_n(x)$
(1) There is an open set $V \neq \phi$ and number $M < \infty$ such that $|f_n(x)| < M$ for all $x \in V$.
(2) If $\epsilon > 0$, there is an open set $V \neq \phi$ and integer N such that $|f(x) - f_n(x)| \leq \epsilon$ if $x \in V$ and $n \geq N$.

This is, **in complete metric space pointwise convergence implies uniform boundness and uniform convergence in some open set.**

Also, this

5.8 Exercise 14 : Nowhere differentiable functions

5.9 Exercise 16 : Closed Graph Theorem

5.10 Exercise 20 : Application of Baire's Theorem 2