Howe-Moore Theorem and Moore's Ergodicity Theorem

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1 Introduction

1.1 Dynamics on the Hyperbolic Plane

The motivation for homogeneous dynamics starts from the $SL_2\mathbb{R}$ action on the hyperbolic plane $\mathbb{H}^2$ ([EW11], Chapter 9). The hyperbolic plane is equipped with a Riemannian metric. The $SL_2\mathbb{R}$ action on $\mathbb{H}^2$ is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

The $SL_2\mathbb{R}$ action is an isometry, and the stabilizer of the point $i \in \mathbb{H}^2$ is the orthogonal group $SO(2)$. Also, we can regard $PSL_2\mathbb{R}$ as acting on the unit tangent bundle of the hyperbolic plane $T^1\mathbb{H}^2$; in this case, the action is simply transitive.

$$Dg(z,v) = \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}\right)$$

Fixing the reference vector $(z_0, v_0) = (i, i)$, we can identify $PSL_2\mathbb{R}$ with $T^1\mathbb{H}^2$.
Under this identification, the geodesic flow on $T^1\mathbb{H}^2$ is right multiplication by a diagonal element. Define $R_{a_t}(g) = ga_t^{-1}$ where $a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$. Similarly, the horocycle flow can be identified with right multiplication by $u^-(-s)$ where $u^-(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

1.2 Geodesic Flow on $\Gamma\backslash G$

We can give a measure on $T^1\mathbb{H}^2$ and, hence, on $PSL_2\mathbb{R}$: $dm = \frac{1}{y^2}dxdyd\theta$, where θ is the angle of the unit tangent vector. Since the $PSL_2\mathbb{R}$ action is an isometry and is conformal, the $PSL_2\mathbb{R}$ action preserves the measure.
For a lattice $\Gamma \leqslant PSL_2\mathbb{R}$, dm induces the R_g invariant finite measure m_X on $X = \Gamma\backslash G$. This is done by measuring the inverse image in the fundamental domain.

$$m_X(B) = m_G(\pi^{-1}(B) \cap F)$$

Defining the measure on the homogeneous space $X = \Gamma\backslash G$ and knowing the invariant action R_g, we can argue the ergodicity or mixing properties of some kinds of actions. One application is the relation to the Gauss Map. The Gauss map $T : [0,1] \to [0,1]$, $T(y) = \{\frac{1}{y}\}$ is intimately connected to the geodesic flow on $\Gamma\backslash G$; for example, an arbitrary invariant probability measure for the Gauss map lifts up to an invariant measure on $\Gamma\backslash G$. This demonstrates the difference between the geodesic flow and the unipotent flow. Unlike Ratner's theorem, which classifies ergodic invariant measures of unipotent flow, there exist many 'strange' ergodic invariant measures of the geodesic flow. ([EW11], Chapter 9)

2 Howe-Moore Theorem for $SL_2\mathbb{R}$

2.1 Associated Unitary Operators

For a measure-preserving transformation T of (X, μ), the associated operator $U_T : L^2_\mu(X) \to L^2_\mu(X)$ is $U_T(f) = f \circ T$ ([EW11], Chapter 2). The measure-preserving transformation (or more generally, the transformation with a quasi-invariant measure) is ergodic if $T^{-1}(A) = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$. The equivalent definition is that for $f : X \to \mathbb{C}$ measurable, $f \circ T = f$ almost everywhere if and only if f is constant almost everywhere. Expressed in terms of the associated unitary operator, T is ergodic if and only if there does not exist a nonconstant f satisfying $T(f) = f$, or 1 is not an eigenvalue of U_T.

The measure-preserving transformation is (strongly) mixing if $\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$ as $n \to \infty$ for any Borel sets A, B. In unitary language, this is equivalent to showing $\lim_{n\to\infty}\langle U_T^n f, g\rangle = \langle f, 1\rangle \cdot \langle g, 1\rangle$ for some dense subset of $L^2_\mu(X)$.
Generally, the unitary representation of a locally compact group G is $\pi_{\mathcal{H}} : G \to U(\mathcal{H}_\pi)$, where each image is a unitary transformation in the Hilbert space $\mathcal{H}_\pi$. We suppose that this representation is strongly continuous to guarantee that the map $(g, v) \mapsto \pi(g)v$ is continuous. The Associated Unitary Operator is one kind of unitary representation; in this case, the Hilbert space is $\mathcal{H} = L^2_\mu(X)$. Or, we can argue with the Hilbert space $\mathcal{H} = L^2_{\mu,0}(X)$ which are L^2 functions satisfying $\int_X f d\mu = 0$

2.2 Mautner's Phenomenon

Mautner's Phenomenon plays an important role in proving the Howe-Moore Theorem. It states that there exists some subgroup that fixes the vectors in a unitary representation under proper assumptions. The statement is ([BM00], Chapter 3):

Theorem 1 (Mautner's Phenomenon). *Let G be a locally compact group and $(\pi, \mathcal{H}_\pi)$ a strongly continuous unitary representation of G. Let $\xi, \xi_0 \in \mathcal{H}_\pi$ be such that*

$$\lim_{n\to\infty} \pi(a_n)\xi = \xi_0$$

Then

$$\pi(g)\xi_0 = \xi_0$$

for g in the closure of the subgroup generated by $S(\alpha)$, which we call N^+_α.

Here, $S(\alpha)$ for $\alpha = \{a_n\}$ is defined by the points g such that a subsequence of $a_n^{-1} g a_n$ accumulates to e. $S(\alpha) = \{g \in G \mid e \in \overline{\{a_n^{-1} g a_n \mid n \in \mathbb{N}\}}\}$. Also, the limit $\lim_{n\to\infty} \pi(a_n)\xi = \xi_0$ is defined in the weak* sense.

Proof. It is sufficient to prove this for $g \in S(\alpha)$; then by continuity, it will hold for $g \in N^+_\alpha$. Let a_{n_k} be a subsequence such that $\lim_{k\to\infty} a_{n_k}^{-1} g a_{n_k} = e$. Then

$$\begin{aligned} |\ \langle\pi(g)\xi_0, \eta\rangle - \langle\xi_0, \eta\rangle\ | &= \lim_{k\to\infty}\ |\ \langle\pi(g a_{n_k})\xi, \eta\rangle - \langle\pi(a_{n_k})\xi, \eta\rangle\ | \\ &= \lim_{k\to\infty}\ |\ \langle\pi(a_{n_k}^{-1} g a_{n_k})\xi, \pi(a_{n_k}^{-1})\eta\rangle - \langle\xi, \pi(a_{n_k}^{-1})\eta\rangle\ | \\ &\le \lim_{k\to\infty} \|(\pi(a_{n_k}^{-1} g a_{n_k}) - \mathrm{id}_{\mathcal{H}_\pi})\,\xi\|\|\eta\| = 0 \end{aligned}$$

□

2.2.1 Ergodicity of Geodesic Flow on $\Gamma\backslash SL_2\mathbb{R}$

Using Mautner's Phenomenon, we can prove the ergodicity of the geodesic flow on $\Gamma\backslash SL_2\mathbb{R}$. Recall that the geodesic flow corresponds to right multiplication by $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ ([EW11], Chapter 11). Applying Mautner's Phenomenon to $\alpha = \{(a_t)^n\}$,

$$\begin{pmatrix} e^{-nt/2} & 0 \\ 0 & e^{nt/2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{nt/2} & 0 \\ 0 & e^{-nt/2} \end{pmatrix} = \begin{pmatrix} a & e^{-nt}b \\ e^{nt}c & d \end{pmatrix}$$

This accumulates to e if and only if $c = 0, a = d = 1$; thus, this holds for the stable horocycle group U^-. So for $f \in L^2_\mu(X)$ that is a_t invariant, applying Mautner's Phenomenon to a_t, $t > 0$ and $t < 0$ respectively, f is fixed by U^- and U^+. But both U^- and U^+ generate $SL_2\mathbb{R}$ since $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
f is thus constant, finishing the proof.

2.2.2 Ergodicity of Horocycle Flow on $\Gamma\backslash SL_2\mathbb{R}$

In this case, Mautner's Phenomenon does not give further facts directly. $u_s^- = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, directly computing the element $S(\{u_{ns}^-\})$

$$\begin{pmatrix} 1 & -ns \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & ns \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - nsc & (a-nsc)ns + b - nsd \\ c & nsc + d \end{pmatrix}$$

The only matrices that accumulate to e are those where $c = 0, b = 0, a = d$, so $\{\pm I\}$. However, we can argue similarly as in Mautner's Phenomenon to deduce that the horocycle flow is also ergodic on $\Gamma \backslash G$.
Consider the function $\varphi(g) = \langle \pi(g)\xi_0, \xi_0 \rangle$. If $\xi_0 \in \mathcal{H}_\pi$ is invariant under the horocycle flow, then $\varphi(g) = \varphi(u_s^- g u_t^-)$ since

$$\langle \pi(u_s^- g u_t^-)\xi_0, \xi_0 \rangle = \langle \pi(g)\pi(u_t^-)\xi_0, (\pi(u_s^-))^{-1}\xi_0 \rangle = \langle \pi(g)\xi_0, \xi_0 \rangle$$

using unitarity and invariance. For arbitrary $\alpha \neq 0$,

$$\begin{pmatrix} 1 & \alpha\lambda_n^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\lambda_n^{-1} \\ \lambda_n & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\lambda_n^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \lambda_n & \alpha^{-1} \end{pmatrix}$$

so $\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right) = \lim_{n\to\infty} \varphi\left(\begin{pmatrix} 0 & -\lambda_n^{-1} \\ \lambda_n & 0 \end{pmatrix} \right)$.
This means that $\varphi(a)$ is constant where $a \in A$. Since for $a = I$, $\varphi(I) = \|\xi_0\|^2$, we have $\varphi(a) = \|\xi_0\|^2$, which means ξ_0 is also fixed by diagonal elements. The ergodicity of the geodesic flow completes the proof.

2.3 Howe-Moore's Theorem

We have shown the ergodicity of the geodesic flow on $\Gamma \backslash SL_2\mathbb{R}$. Indeed, it is more than ergodic; it is mixing. To show this, we investigate the set $S(\alpha)$ when α converges to ∞ (that is, for any compact subset $K \subseteq SL_2\mathbb{R}$, there exist only finitely many $a_n \in K$).

Lemma 1. *Let $\alpha = \{(g_n)\}$ be a sequence in $SL_2\mathbb{R}$ converging to ∞. Then there exists a non-trivial unipotent element in $S(\alpha)$.*

Proof. Consider the adjoint representation $Ad : SL_2\mathbb{R} \to GL(\mathfrak{sl}_2\mathbb{R})$. Since the center of $SL_2\mathbb{R}$ is $\{\pm I\}$, $Ad(G) \cong PSL_2\mathbb{R}$. As $g_n \to \infty$, $Ad(g_n)$ will also escape to infinity (with respect to the topology given at $Aut(\mathfrak{sl}_2\mathbb{R})$ by the operator norm). Thus we can choose vectors $v_n \in \mathfrak{sl}_2\mathbb{R}$ such that $\|v_n\| \to 0$ but $\|Ad(g_n)v_n\| = c > 0$. $h_n = \exp v_n \in SL_2\mathbb{R}$ forms a sequence that converges to I, but some subsequence $g_n h_n g_n^{-1}$ converges to u, which is not equal to I. Since the eigenvalues of $g_n h_n g_n^{-1}$ and h_n coincide, u must have only eigenvalue 1, so u is unipotent. □

Now, all preparation to prove the mixing property of the $SL_2\mathbb{R}$ action and the Howe-Moore Theorem is ready.

Theorem 2 (Howe-Moore Vanishing Theorem for $SL_2\mathbb{R}$)**.** *Let $\mathcal{H}_\pi$ be a Hilbert space carrying a unitary representation of $SL_2\mathbb{R}$ without any invariant vectors. Then for any $v, w \in \mathcal{H}_\pi$, the matrix coefficients vanish at ∞:*

$$\langle g_n v, w \rangle \to 0$$

Theorem 3 (Mixing property of $SL_2\mathbb{R}$ action)**.** *Let Γ be a lattice in $G = SL_2\mathbb{R}$. Then the action of $SL_2\mathbb{R}$ on $X = \Gamma \backslash G$ is mixing.*

Proof. Let $\alpha = \{g_n\}$ be a sequence of elements in $SL_2\mathbb{R}$ converging to ∞. Let $f \in L^2_\mu(X)$; then, since each element in $SL_2\mathbb{R}$ acts unitarily, there exists a subsequence $g_{n_k}(f)$ that converges in the weak* sense to some element f_0. Then by Lemma 1, $S(\alpha)$ contains a non-trivial unipotent element u, and Howe-Moore's Theorem implies $u(f_0) = f_0$. Using a similar argument to the ergodicity of the unipotent flow, f_0 must be constant and

$$\langle f_0, 1 \rangle = \lim_{k\to\infty} \langle g_{n_k} f, 1 \rangle = \int_X f dm_X$$

so $f_0 = \int_X f dm_X$.

$$\int_X f(xg_n^{-1})\bar{h}(x) dm_X = \langle g_n(f), h \rangle \to \langle \int_X f dm_X, h \rangle = \int_X f dm_X \int_X \bar{h} dm_X$$

proving the mixing property.
Doing the same on the general Hilbert space $\mathcal{H}_\pi$, all accumulation elements (in a weak* sense) of the sequence $g_{n_k} v$ must be fixed by all of $SL_2\mathbb{R}$. In the assumption of Howe-Moore's Theorem, we assume there are no nontrivial invariant vectors, so it must be zero. $g_n v \to 0$ in weak*, i.e.,

$$\langle g_n v, w \rangle \to 0$$

□

3 Howe-Moore Theorem for Semisimple Lie Groups

A semisimple Lie group is a Lie group whose Lie algebra is semisimple. A semisimple Lie algebra can be decomposed uniquely as the direct sum of simple Lie algebras, each of which is an ideal in $\mathfrak{g}$ (Theorem 1.54, [Kna02]).

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$$

The corresponding analytic subgroups of the Lie group G, denoted $S_1, \cdots, S_m$, become simple normal subgroups. Since the Lie algebras $\mathfrak{g}_1, \cdots, \mathfrak{g}_m$ are a direct sum, the Lie bracket between them is zero; hence, S_i and S_j commute. Moreover, since the only ideals of $\mathfrak{g}$ are direct sums of the $\mathfrak{g}_i$'s, each S_i is closed. (The closure of S_i is normal, so its Lie algebra $Lie(\overline{S_i})$ is an ideal containing $\mathfrak{g}_i$ and must be $\mathfrak{g}_i$). Finally, since G is connected,

$$G = S_1 \cdots S_m$$

We call these S_i simple factors. The Howe-Moore Vanishing Theorem for a general semisimple Lie group G is as follows ([BM00], Chapter 3):

Theorem 4 (Howe-Moore Vanishing Theorem)**.** *Let G be a connected semisimple Lie group with finite center. Let $(\pi, \mathcal{H}_\pi)$ be a strongly continuous unitary representation of G. Assume that the restriction of π to any non-compact simple factor S_i of G has no non-trivial invariant vectors. Then all the matrix coefficients of π vanish at infinity.*

For the proof of the Howe-Moore Vanishing Theorem, we need the structure theory of real semisimple Lie groups. The KAK decomposition ([Kna02], Theorem 7.39) states that every element in G has a decomposition $k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in A$. Here K is the compact Lie subgroup of G whose Lie algebra is $\mathfrak{k}$, and A is a simply connected Lie subgroup of G whose Lie algebra is $\mathfrak{a}$, the maximal abelian subspace of $\mathfrak{p}$ appearing in the Cartan decomposition. In this decomposition, a is unique up to conjugation by a member of the Weyl group $W(G, A)$. Thus,

$$G = K\overline{A^+}K$$

where $A^+ = \{\exp H \mid H \in \mathfrak{a}^+\}$ for a given ordering of restricted roots Σ. Then the elements tending to infinity actually matter only for the $\overline{A^+}$ part, since K is compact. More precisely:

Lemma 2 ([BM00], Lemma 1.3 of Chapter 3)**.** *Let $(\pi, \mathcal{H}_\pi)$ be a strongly continuous unitary representation. Suppose that for all matrix coefficients $\varphi_{\xi\eta}$ of π and for all sequences $\{a_n\}_n$ in $\overline{A^+}$ with $\lim_{n\to\infty} a_n = \infty$, we have $\lim_{n\to\infty} \varphi_{\xi\eta}(a_n) = 0$. Then all matrix coefficients of π vanish at infinity on G.*

The proof argues by contradiction. Decompose $g_n = k_n a_n h_n$. Using compactness, one can generate a subsequence such that $\pi(h_n)\xi$ and $\pi(k_n^{-1})\eta$ converge in norm to $\overline{\xi}, \overline{\eta}$ in $\mathcal{H}_\pi$. If $\lim_{n\to\infty} \varphi_{\xi\eta}(g_n) \neq 0$, then $\lim_{n\to\infty} \varphi_{\overline{\xi}\overline{\eta}}(a_n) \neq 0$, which is a contradiction.
Next, we argue that a sequence converging to infinity on $\overline{A^+}$ has an effect similar to powers of an appropriate $b \in \overline{A^+} \backslash \{e\}$.

Lemma 3 ([BM00] Lemma 1.8 of Chapter 3)**.**
(1) For $b \in A\backslash\{e\}$, let G_b be the closed subgroup generated by N_b^+ and $N_{b^{-1}}^+$. Then G_b is a non-discrete normal subgroup of G.
(2) For every sequence α in $\overline{A^+}$ converging to infinity, there exists $b \in \overline{A^+}\backslash\{e\}$ such that $N_\alpha^+ = N_b^+$.

Proof. (1) Let $b = \exp_G H$ (since A is simply connected, such an $H \in \mathfrak{a}$ exists) and classify the restricted roots by their value on H.

$$\mathfrak{g}_b^- = \sum_{\lambda(H)<0} \mathfrak{g}^\lambda,\ \mathfrak{g}_b^+ = \sum_{\lambda(H)>0} \mathfrak{g}^\lambda,\ \mathfrak{g}_b^0 = \sum_{\lambda(H)=0} \mathfrak{g}^\lambda$$

Let $\mathfrak{g}_b$ be the subalgebra generated by $\mathfrak{g}_b^+$ and $\mathfrak{g}_b^-$; it is clearly an ideal. Also, $b^n \exp_G X b^{-n} = \exp_G(Ad(b)^n X) = \exp_G((\exp(adH))^n X) = \exp_G(\exp(n\, adH)X) = \exp_G(e^{n\lambda(H)}X)$. So $\exp_G \mathfrak{g}_b^+ \subset N_b^+$ and $\exp_G \mathfrak{g}_b^- \subset N_{b^{-1}}^+$. Thus, $\exp_G(\mathfrak{g}_b) \subseteq G_b$.
On the other hand, if $x \in S((b^n))$, then $b^{-n}xb^n \in U$ for some neighborhood U of e. The exponential mapping is a local diffeomorphism, so $U = U^- U^0 U^+$, where each component is a neighborhood of e in $\exp_G \mathfrak{g}_b^-$, $\exp_G \mathfrak{g}_b^0$, and $\exp_G \mathfrak{g}_b^+$, respectively. Thus $x = b^n \exp_G y b^{-n}$ with $y = y^- + y^0 + y^+$. As we have seen above, $b^{-n}xb^n$ can accumulate to e only if $y^- = 0$ and $y^0 = 0$. Thus $x \in \exp_G(\mathfrak{g}_b^+)$, and

$$S((b^n)_n) \cup S((b^{-n})_n) \subseteq \exp_G(\mathfrak{g}_b) \subseteq N_b^+ \cup N_{b^{-1}}^+$$

G_b is thus generated by $\exp_G(\mathfrak{g}_b)$. It is normal (since $\mathfrak{g}_b$ is an ideal in $\mathfrak{g}$) and non-discrete since $\mathfrak{g}_b \neq 0$.

(2) For the simple roots of the restricted root system Σ, denoted R, there exists an element $\lambda \in R$ such that

$$\limsup_{n\to\infty} \lambda(a_n) = \infty$$

Here, we suppose $\exp_G(a_n)$ converges to infinity, with $a_n \in \overline{\mathfrak{a}^+}$. Otherwise, $Ad(\exp_G(a_n)) = \exp(ad\, a_n)$ would be bounded, but $Ad : G \to Ad(G)$ is a finite covering map, so this contradicts the assumption. For such simple roots R_α, define b by the exponential of the element $H \in \overline{\mathfrak{a}^+}$ such that

$$\lambda(H) = \begin{cases} 1 & \text{if } \lambda \in R_\alpha \\ 0 & \text{if } \lambda \notin R_\alpha \end{cases}$$

We claim that $\exp_G(\mathfrak{g}_b^+) = S(\alpha)$, where $\mathfrak{g}_b^+ = \sum_{\lambda(H)>0} \mathfrak{g}^\lambda$.
First, $\exp_G \mathfrak{g}^\lambda \subseteq S(\alpha)$ for $\lambda \in R_\alpha$ since $(\exp_G a_n)^{-1} \exp_G X \exp_G a_n = \exp_G(-\exp(ad\, a_n)X) = \exp_G(-\lambda(a_n)X)$ for $X \in \mathfrak{g}^\lambda$.
Conversely, if $x \in S(\alpha)$, then $(\exp_G(a_n))^{-1} x \exp_G(a_n) = \exp_G X_n^+ \exp_G X_n^0 \exp_G X_n^-$ using the decomposition $U = U^- U^0 U^+$.

$$x = \exp_G(\exp(ad\, a_n)X_n^+) \exp_G(\exp(ad\, a_n)X_n^0) \exp_G(\exp(ad\, a_n)X_n^-)$$

Taking the limit as $n \to \infty$, $x = \lim_{n\to\infty} \exp_G(Ad(\exp_G a_n)X_n^+) \in \exp_G(\mathfrak{g}_b^+)$.

$\square$

I would like to add an explicit example for **Lemma 3** above. Let us suppose $G = SL_3\mathbb{R}$. In this case, the root system and positive Weyl Chamber are shown in Figure 1.

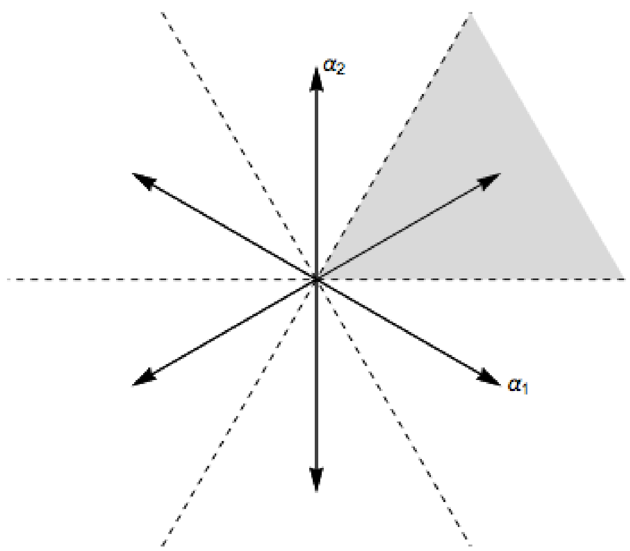

Figure 1: Root system and positive Weyl Chamber of $\mathfrak{sl}_3\mathbb{R}$ [Mat24]

$$\bar{\mathfrak{a}}^+ = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \middle| a_1 + a_2 + a_3 = 0, a_1 \geq a_2 \geq a_3 \right\}$$

and its exponential is

$$\bar{A}^+ = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \middle| a_1 a_2 a_3 = 1, a_1 \geq a_2 \geq a_3 > 0 \right\}$$

For the sequence $\alpha = \{\mathrm{diag}(e^{a_{n1}}, e^{a_{n2}}, e^{a_{n3}})\}_n$ in $\bar{A}^+$, given the standard lexicographic ordering, the simple positive roots are $R = \{e_1 - e_2, e_2 - e_3\}$.
So our element b is chosen as follows:

- If $\limsup_{n\to\infty}(a_{n1}-a_{n2})=\infty$ and $\limsup_{n\to\infty}(a_{n2}-a_{n3})=\infty$, then

$$b=\exp_{SL_3\mathbb{R}}(\mathrm{diag}(1,0,-1))=\mathrm{diag}(e,1,e^{-1})$$

- If $\limsup_{n\to\infty}(a_{n1}-a_{n2})=\infty$ and $\limsup_{n\to\infty}(a_{n2}-a_{n3})<\infty$, then

$$b=\exp_{SL_3\mathbb{R}}(\mathrm{diag}(\frac{2}{3},-\frac{1}{3},-\frac{1}{3}))=\mathrm{diag}(e^{2/3},e^{-1/3},e^{-1/3})$$

- If $\limsup_{n\to\infty}(a_{n1}-a_{n2})<\infty$ and $\limsup_{n\to\infty}(a_{n2}-a_{n3})=\infty$, then

$$b=\exp_{SL_3\mathbb{R}}(\mathrm{diag}(\frac{1}{3},\frac{1}{3},-\frac{2}{3}))=\mathrm{diag}(e^{1/3},e^{1/3},e^{-2/3})$$

The Howe-Moore Vanishing Theorem is now approachable. Let $(\pi,\mathcal{H}_\pi)$ be a strongly continuous unitary representation. Since compact simple factors do not affect the vanishing of the matrix coefficient, we can assume the S_i are all non-compact. Let $\alpha=\{a_n\}_n$ be a sequence in $\overline{A^+}$ tending to infinity and $\xi_0\in\mathcal{H}_\pi$ be such that $\lim_{n\to\infty}\pi(a_n)\xi=\xi_0$ in the weak* sense. If we prove $\xi_0=0$, then by Lemma 2, all matrix coefficients will vanish at infinity.
Mautner's Phenomenon implies that ξ_0 is invariant under N_α^+. By Lemma 3 (2), there exists $b\in\overline{A^+}\backslash\{e\}$ such that $N_\alpha^+=N_b^+$.
For each $\lambda\in\Sigma^+$ such that $\mathfrak{g}^\lambda\subseteq\mathfrak{g}_b^+$, and $X\in\mathfrak{g}^\lambda\backslash\{0\}$, the set $\{X,\theta(X),H_\lambda\}$ generates a Lie algebra $\mathfrak{g}_X$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. ($H_\lambda$ is an element satisfying $\lambda(H)=B(H,H_\lambda)$ for $H\in\mathfrak{a}$). Let G_X be the analytic subgroup of G with Lie algebra $\mathfrak{g}_X$. Since G has a finite center, G_X has a finite center (considering the Adjoint map $Ad:G\to GL(\mathfrak{g})$) and is closed (as the inverse image of the closed subgroup $Ad(G_X)$ in $GL(\mathfrak{g})$).
ξ_0 is invariant under N_b^+; hence, by $\exp_G\mathbb{R}X$, we need a statement from Chapter 2.2.2 (Ergodicity of Horocycle flow on $\Gamma\backslash SL_2\mathbb{R}$) for a general G locally isomorphic to $SL(2,\mathbb{R})$. If this holds, ξ_0 is invariant under G_X, and hence under all of G_b. Since Lemma 3 (1) says G_b is a non-discrete normal subgroup of G, it must contain some simple factor of G, which contradicts the hypothesis that every non-compact simple factor does not have an invariant vector.

Lemma 4 ([BM00] Theorem 1.6 of Chapter 3)**.** *Let G be a group locally isomorphic to $SL(2,\mathbb{R})$ and with a finite center, and let $(\pi,\mathcal{H}_\pi)$ be a unitary representation of G. Let ξ_0 be a vector in $\mathcal{H}_\pi$ which is invariant under $N=\exp_G\mathbb{R}X$. Then ξ_0 is invariant under G.*

Proof. We proved this for $SL(2,\mathbb{R})$ already. For $G=PSL(2,\mathbb{R})$, π can be lifted to a representation of $SL(2,\mathbb{R})$, so it holds. Generally, G has a map $p:G\to PSL(2,\mathbb{R})$. $\mathcal{H}_\pi$ splits into Z-isotypic components, determined by the character of Z.

$$\mathcal{H}_\pi=\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_r$$

Z is the center, so each component is invariant under G. Therefore, if $\xi_0=\xi_0^{(1)}+\cdots+\xi_0^{(r)}$, then each $\xi_0^{(i)}$ is fixed by N. Defining $\psi_i(g)=|\langle\pi(g)\xi_0^{(i)},\xi_0^{(i)}\rangle|^2$, we have $\pi(z)\xi_0^{(i)}=\chi_i(z)\xi_0^{(i)}$, so ψ_i can be considered as a function on $PSL(2,\mathbb{R})$. Thus, applying the result for $PSL(2,\mathbb{R})$, ψ_i is constant on G. $\pi(g)\xi_0^{(i)}=\chi(g)\xi_0^{(i)}$ for some character χ. The kernel $ker(\chi)$ will be a normal subgroup of G. Since $PSL(2,\mathbb{R})$ is simple, $ker(\chi)\subseteq Z$ or $ker(\chi)=G$. If $ker(\chi)\subseteq Z$, then $S^1=im(\chi)\cong(G/Z)/(Z/H)$ is abelian, which is a contradiction. Thus $\chi\equiv 1$, and ξ_0 is fixed under G. □

4 Moore's Ergodicity Theorem

When G acts on the probability space (X,μ), applying the Howe-Moore Vanishing Theorem to $(\pi,L^2_{\mu,0}(X))$ gives Moore's Ergodicity Theorem.

Theorem 5 (Moore's Ergodicity Theorem ([BM00] Theorem 2.1 of Chapter 3))**.** *Let G be a semisimple Lie group with finite center. Let G act on a probability space (X,μ) and assume that the restriction of this action to any simple non-compact factor of G is ergodic. Let H be a subgroup of G with a non-compact closure. Then the action of H on X is strongly mixing.*

One application of Moore's Ergodicity Theorem is the action of a closed subgroup H on $\Gamma\backslash G$. For a unimodular locally compact group G and a closed subgroup H, the homogeneous space $X=G/H$ has a unique G-invariant Borel measure $d\nu_X$ that satisfies the Weil Formula

$$\int_G f(x)d\nu_G(x) = \int_X \int_H f(xh)d\nu_H(h)d\nu_X(\dot{x})$$

We define a lattice in G as a discrete subgroup Γ such that G/Γ has a finite G-invariant Borel measure.
By Moore's Ergodicity Theorem, if G is a simple Lie group with finite center and H is a non-compact closed subgroup, then H acts ergodically on $\Gamma\backslash G$. Also, by Moore's Duality Theorem, which states that if H_1, H_2 are closed subgroups of a locally compact group G, then H_1 is ergodic on G/H_2 if and only if H_2 is ergodic on G/H_1, Γ acts ergodically on G/H (here, equipped with the quasi-invariant Borel measure of G/H that satisfies the Weil Formula).
One example is $G = SL(n,\mathbb{R})$ and $H = Stab((1,0,\cdots,0)^t)$ for the action $\pi : G \to GL(\mathbb{R}^n\backslash\{0\})$, with $\Gamma = SL(n,\mathbb{Z})$. Then H is non-compact and Γ is a lattice, so the $SL(n,\mathbb{Z})$ action on $G/H \cong \mathbb{R}^n\backslash\{0\}$ is ergodic.

4.1 Applications

We apply Moore's Ergodicity Theorem to a compact Riemannian surface of genus $g \geq 2$. It is known that such a surface Σ is covered by the hyperbolic plane. Thus, Σ can be identified with $\Gamma\backslash\mathbb{H}^2$. Under this identification, the unit tangent bundle of Σ corresponds to $\Gamma\backslash PSL(2,\mathbb{R})$, so the geodesic flow on $T^1(\Sigma)$ can be analyzed by analysis on $\Gamma\backslash PSL(2,\mathbb{R})$. By Moore's Ergodicity Theorem, the geodesic flow is ergodic.
More generally, we can apply Moore's Ergodicity Theorem to Riemannian symmetric spaces. First, a globally symmetric Riemannian space is a Riemannian manifold (M,Q) equipped with an involutive isometry s_p at each point $p \in M$ (s_p makes p an isolated fixed point). Riemannian symmetric spaces have a one-to-one correspondence with Riemannian symmetric pairs (G,K), where G is a connected Lie group and K is a closed subgroup such that $\mathrm{Ad}(K)$ is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$, and there exists an involutive automorphism σ satisfying $(G_\sigma)_0 \subseteq K \subseteq G_\sigma$. In the Riemannian symmetric space G/K, the geodesic flow is given as

$$t \mapsto \varphi_t((g,v)) = (g\exp(tv), v)$$

for $v \in \{X \in \mathfrak{g} \mid d\sigma(X) = -X\} \cap \{Q_K(X,X) = 1\}$ and $g \in G$.
Let G/K be a Riemannian symmetric space of rank 1 and Γ be a lattice in G. By Moore's Ergodicity Theorem, the geodesic flow of $\Gamma\backslash G/K$ is ergodic and mixing. The rank 1 assumption is essential; if the rank is ≥ 2, then the geodesic flow is not ergodic.

4.2 Extensions

4.2.1 Strongly Mixing Property

The dynamical properties of the geodesic flow extend beyond mixing to mixing of all orders. Let G be a connected semisimple Lie group with a finite center and no compact factors, acting on (X,μ) in a measure-preserving manner. This action is mixing of all orders if for all $k \geq 2$ and measurable subsets $B_1, \cdots, B_k$ of X, the sequences $\{(g_{1n}, \cdots, g_{kn})\}$ satisfying $\lim_{n\to\infty} g_{in}^{-1} g_{jn} = \infty$ for all $1 \leq i < j \leq k$ satisfy

$$\lim_{n\to\infty} \mu\left(\bigcap_{i=1}^k g_{in} \cdot B_i\right) = \prod_{i=1}^k \mu(B_i)$$

Mozes proved that ergodic actions of semisimple Lie groups are mixing of all orders.

Theorem 6 (Mozes, 1992 [Moz95])**.** *Let G be a connected semisimple Lie group with a finite center and no compact factors. If G acts on a Lebesgue probability measure space (X,μ) in a continuous, measure-preserving way and is ergodic in each non-central normal subgroup, then the Lie group action is mixing of all orders.*

4.2.2 Algebraic Group Extensions

The Howe-Moore Theorem holds in more general settings. In Howe and Moore's original paper, they proved the vanishing result for connected reductive algebraic groups over a local field of any characteristic ([HM79]). I would like to introduce a much friendlier example: the p-adic field case.
For closed linear p-adic groups $G \subset SL_d(\mathbb{Q}_p)$, we can perform an analysis similar to that of closed linear matrix groups. For example, defining the norm on $v \in \mathfrak{gl}_d(\mathbb{Q}_p)$ as $\|v\| = \max_{i,j} |v_{i,j}|_p$, the exponential

$$\exp(v) = \sum_{n=0}^{\infty} \frac{1}{n!} v^n$$

converges for a small norm. The inverse is given by

$$\log(g) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(g - I)$$

for $g \in GL_d(\mathbb{Q}_p)$ that is sufficiently close to I [EW25].
For $G = SL_d(\mathbb{Q}_p)$, the KAK decomposition is given by $K = SL_d(\mathbb{Z}_p)$ and

$$A = \left\{ \begin{pmatrix} p^{\alpha_1} & & \\ & \ddots & \\ & & p^{\alpha_d} \end{pmatrix} \middle| \alpha_1, \cdots, \alpha_d \in \mathbb{Z},\ \alpha_1 + \cdots + \alpha_d = 0 \right\}$$

The Howe-Moore theorem holds for $G = SL_d(\mathbb{Q}_p)$. If G acts unitarily on a Hilbert space $\mathcal{H}$, assuming the action has no non-trivial fixed vector, then the matrix coefficients vanish at infinity.

5 Conclusion

In this report, we discussed Howe-Moore's theorem on $SL_2\mathbb{R}$ and more general semisimple Lie groups. Howe-Moore's theorem helps us to analyze the dynamical properties of Lie group actions; in fact, Moore's Ergodicity Theorem shows that under suitable conditions, the action is ergodic and strongly mixing.
In the proof, we examined how the structure theory of Lie groups and Lie algebras can be used. Specifically, the KAK decomposition and root space decomposition allow us to prove the Howe-Moore theorem. Finally, we concluded with the applications and extensions of this topic. The ergodicity and mixing properties hold for the geodesic flow on locally symmetric Riemannian spaces. The mixing property can be strengthened to mixing of all orders in certain situations, and the Howe-Moore theorem holds in the general setting of connected reductive algebraic groups.

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