

Subsampling Confidence Bound for Persistent Diagram via Time-delay Embedding

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Abstract

Time-delay embedding is a fundamental technique in Topological Data Analysis (TDA) for reconstructing the phase space dynamics of time-series data. While persistent homology effectively identifies topological features, such as cycles associated with periodicity, a rigorous statistical framework for quantifying the uncertainty of these features has been lacking in this context. In this paper, we propose a subsampling-based method to construct confidence sets for persistence diagrams derived from time-delay embeddings. We establish finite-sample guarantees for the validity of these confidence bounds under regularity conditions—specifically for $C^{1,1}$ functions with positive reach—and prove their asymptotic convergence as the embedding dimension tends to infinity. This framework provides a principled statistical test for periodicity, enabling the distinction between true periodic signals and non-periodic approximations. Simulation studies demonstrate that our method achieves detection performance comparable to the Generalized Lomb-Scargle periodogram on periodic data while exhibiting superior robustness in distinguishing non-periodic signals with time-varying frequencies, such as chirp signals.

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1 Introduction

Time series data consist of observations ordered in time, and an important goal is to understand their dynamics and detect periodic behavior, since many scientific signals are driven by approximately repeating cycles [21]. Classical methods such as spectral analysis and autocorrelation are explicitly designed to identify such periodic patterns [15].

In many applications, time series are better viewed as realizations of random functions, and functional data analysis (FDA) provides basis-expansion and functional principal component analysis(FPCA) tools for dimension reduction and inference [17, 9, 23, 18, 10]. However, these approaches may miss geometric or topological regularities that are closely related to periodicity.

Topological Data Analysis (TDA) offers a complementary viewpoint by focusing on the geometric shape of the data. Persistent homology tracks the evolution of topological feature such as connected component, loop, and void across filtration parameter and summarizes



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them in a persistence diagram [22, 16]. Persistence diagrams are robust to noise and have been successfully used in time-series and signal analysis, where one-dimensional features often correspond to cyclic behavior [11].

For time-series data, a standard TDA pipeline applies a time-delay embedding to map one-dimensional signal into trajectory of higher-dimensional space, then computes the persistent homology of the resulting point cloud. The loops in this embedded trajectory correspond to periodic or quasi-periodic motion, so one-dimensional homology in the persistence diagram captures the presence of cycles [14]. This approach can detect periodic structure in smooth signals and has proved convergence of the associated persistence diagrams as the embedding dimension grows, but cannot provide a statistical framework for uncertainty in the diagram.

Our contribution We develop a subsampling-based method to construct confidence bounds for persistence diagrams arising from time-delay embeddings of time-series data that are expected to be periodic. Using Hausdorff distances between subsamples and embedded support, we obtain data-dependent radii with finite-sample guarantees under regularity and positive-reach conditions. These confidence sets measure the statistical significance of cycle associated with periodicity, leading to a principled periodicity test and detect periodicity of the data. Compared with previous sliding-window persistence methods, our framework provides the missing inferential layer by providing confidence-calibrated decisions directly on the persistence diagram.

2 Preliminaries

In this section, we introduce the mathematical and statistical framework for our analysis. We cover the essentials of time-delay embedding for transforming periodic time series into point clouds, the fundamentals of persistent homology for extracting topological features, and the subsampling methodology used for statistical inference.

2.1 Time-delay Embedding

Time-delay embedding is a method for reconstructing the phase space of a time-series, which is particularly effective for periodic data [13, 14, 20]. Given a continuous function $f \in C(\mathbb{T}, \mathbb{R})$, for an integer M and real number τ the sliding window embedding $SW_{M,\tau} : C(\mathbb{T}, \mathbb{R}) \rightarrow C(\mathbb{T}, \mathbb{R}^{M+1})$ of f is defined as:

$$SW_{M,\tau}f(t) = \begin{bmatrix} f(t) \\ f(t + \tau) \\ \vdots \\ f(t + M\tau) \end{bmatrix} \in \mathbb{R}^{M+1}$$

This process maps the time series to a point cloud X in \mathbb{R}^{M+1} . The embedding relies on two main parameters: the embedding dimension M and the time delay τ . Choosing appropriate values for M and τ is crucial to capture properties of time-series such as periodicity.

For a purely periodic signal, the reconstructed point cloud X generically traces a limit cycle that is topologically equivalent to a one-dimensional circle S^1 [19, 20, 14]. In practice, noise and finite sampling can thicken or distort this manifold, but its essential topology remains a robust descriptor of the system's periodic nature.

The geometric structure of time-delayed embeddings for periodic functions was rigorously analyzed by [14]. They considered periodic functions $f : \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}$. Using the notion of an L -periodic function, which is period of $2\pi/L$, they characterized the structure of the

sliding window embedding. Specifically, for a window size $\tau_N = \frac{2\pi}{L(2N+1)}$ and embedding dimension $M = 2N$, they proved that the sliding window embedding of trigonometric polynomials of degree at most N forms a composition of circle orbits that are mutually orthogonal.

Moreover, [14] introduced the pointwise centralize, normalizing operation. These maps are defined by $C(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x}^T \mathbf{1}}{\|\mathbf{1}\|^2} \mathbf{1}$ and $N(\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{x}}}$. Performing the sliding window embedding of finite samples $T \subset \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ to \mathbb{R}^{2N+1} with time delay $\tau_N = \frac{2\pi}{L(2N+1)}$ and doing the pointwise centralize normalize process, [14] proved the convergence results of those points \bar{X}_N .

► **Theorem 1** ([14] Theorem 6.6). *$f \in C^1(\mathbb{T})$ be an periodic function. $N \in \mathbb{N}$, $\tau_N = \frac{2\pi}{L(2N+1)}$ $T \subset \mathbb{T}$ finite and \bar{X}_N defined as above. Then the sequence of persistent diagrams $dgm(\bar{X}_N)$ is Cauchy with respect to d_B and*

$$\lim_{N \rightarrow \infty} dgm(\bar{X}_N) = dgm_\infty(f, T, \frac{2\pi}{L})$$

► **Theorem 2** ([14] Theorem 6.7). *Let $T, T' \subset \mathbb{T}$ be finite, and let $f \in C^1(\mathbb{T})$ be L -periodic with modulus of continuity ω . Then*

$$d_B(dgm_\infty(f, T, \frac{2\pi}{L}), dgm_\infty(f, T', \frac{2\pi}{L})) \leq 2\|f - \hat{f}(0)\|_2 \omega(d_H(T, T'))$$

2.2 Persistent Homology and Tameness

Given a topological space X and an integer k , we denote the k -th singular homology group by $H_k(X)$ and the k -th Betti number by $\beta_k(X) = \dim H_k(X)$ [12].

An $a \in \mathbb{R}$ is a homological critical value of a function $f : X \rightarrow \mathbb{R}$ if there exists an integer k such that for all sufficiently small $\varepsilon > 0$, the map $H_k(f^{-1}(-\infty, a-\varepsilon]) \rightarrow H_k(f^{-1}(-\infty, a+\varepsilon])$ induced by inclusion is not an isomorphism. In other words, homological critical values are levels where the homology of sub-level sets changes.

► **Definition 3.** *A function $f : X \rightarrow \mathbb{R}$ is tame if it has finitely many homological critical values and $H_k(f^{-1}(-\infty, a])$ is finite-dimensional for all $k \in \mathbb{Z}$ and $a \in \mathbb{R}$ [3].*

Distance functions on finite point clouds are tame [5].

2.3 Persistence Diagrams

For a tame function $f : X \rightarrow \mathbb{R}$, we write $F_x = H_k(f^{-1}(-\infty, x])$ and let $f_x^y : F_x \rightarrow F_y$ denote the map induced by inclusion for $x < y$. The image $F_x^y = \text{im } f_x^y$ is called a persistent homology group [6].

Let $(a_i)_{i=1..n}$ be the homological critical values of f . For $0 \leq i < j \leq n+1$, we define the multiplicity μ_i^j using persistent Betti numbers $\beta_x^y = \dim F_x^y$. Setting $a_0 = -\infty$ and $a_{n+1} = +\infty$, the multiplicity is defined as

$$\mu_i^j = (\beta_{a_i}^{a_j} - \beta_{a_{i-1}}^{a_j}) - (\beta_{a_i}^{a_{j+1}} - \beta_{a_{i-1}}^{a_{j+1}}).$$

The persistence diagram $\mathcal{P}(f) \subset \bar{\mathbb{R}}^2$ consists of points (a_i, a_j) with multiplicity μ_i^j , together with all diagonal points counted with infinite multiplicity [6].

To analyze embedded point clouds, we compute persistence diagrams using a distance filtration. Each topological feature (connected component, loop, void) is represented as a birth-death point (b, d) in the diagram, where b is the scale at which the feature appears and d is the scale at which it disappears [5].

2.4 Stability of Persistence Diagrams

A fundamental result establishes that persistence diagrams are stable under perturbations [4]. For two continuous tame functions f and g on a triangulable space X , the bottleneck distance between their persistence diagrams satisfies

$$d_B(\mathcal{P}(f), \mathcal{P}(g)) \leq \|f - g\|_\infty.$$

This stability ensures that small changes to the input function, such as noise or measurement error, produce only small changes in the persistence diagram.

In this work, we focus on 1-dimensional persistence diagrams, which capture the lifespans of loops. These features are particularly relevant for periodic signals, as they reveal the primary cyclic structure in time-delay embeddings [14].

2.5 Confidence bound on Persistent Diagram

There has been extensive research on formulating confidence bounds for persistence diagrams. [7] introduced a subsampling-based confidence bound construction. Assuming we observe a sample $S_n = \{X_1, \dots, X_n\}$ from a distribution P concentrated on a set \mathbb{M} , [7] proved that under regularity assumptions, subsampling-based confidence bounds are valid. When the subsample size b increases to infinity at a rate $b = o(n/\log n)$, they defined the function:

$$L_b(t) = \frac{1}{\binom{n}{b}} \sum_{j=1}^{\binom{n}{b}} \mathbb{I}(d_H(S_{b,n}^{(j)}, S_n) > t)$$

where $S_{b,n}^{(j)}$ denotes the j -th subsample of size b from S_n .

Defining $c_b = 2L_b^{-1}(\alpha)$, let $\hat{\mathcal{P}}$ be the persistence diagram of $\{X_1, \dots, X_n\}$ and \mathcal{P} be the persistence diagram of \mathbb{M} . The following theorem holds:

► **Theorem 4** ([7] Theorem 3). *Under the regularity condition of \mathbb{M} , for all large n ,*

$$\mathbb{P}(d_B(\hat{\mathcal{P}}, \mathcal{P}) > c_b) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

3 Assumptions and Statistical Models

We assume there exists a true sampling function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is periodic with period 1 and sampled points are independently sampled from uniform distribution on $[0, 2\pi]$. Also assume that we can get all the delayed data to construct sliding window map.

We assume there exists a true sampling function $f : \mathbb{R} \rightarrow \mathbb{R}$. We sample points t_1, \dots, t_n i.i.d. from uniform distribution on $\mathbb{T} = [T_{\min}, T_{\max}]$. And, for any m and τ , we assume that we can observe

$$\{SW_{m,\tau} f(t_j)\}_{1 \leq j \leq n} = \{(f(t_j), f(t_j + \tau), \dots, f(t_j + m\tau))\}_{1 \leq j \leq n}.$$

In the above setting, we can argue that image of sliding window for each sampled points are in fact, randomly sampled from the support, which is the closure of image of interval $[0, 2\pi]$ by the sliding window map. The underlying measure might be different from the Hausdorff measure, which is actually a pushforward of the uniform distribution on $[0, 2\pi]$.

We assume that either f is periodic or non-periodic, but not something in the middle. To ensure this, we define periodic and non-periodic as follows.

► **Definition 5.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function, and fix any $\Xi > 0$. We say f is Ξ -periodic if for any $t \in \mathbb{R}$, $f(t + \Xi) = f(t)$. We say f is periodic if f is Ξ -periodic for some $\Xi > 0$. We say f is Ξ -non-periodic if for any $t \in \mathbb{R}$, $f(t + \Xi) \neq f(t)$ or $f'(t + \Xi) \neq f'(t)$. We say f is non-periodic if f is Ξ -non-periodic for every $\Xi > 0$.

► **Assumption 1.** The true sampling function $f : \mathbb{R} \rightarrow \mathbb{R}$ is $C^{1,1}$, and for each $\Xi > 0$, f is either Ξ -periodic or Ξ -non-periodic.

However, even if we have valid confidence interval for the sampling function, sometimes non-periodic function can behave very similarly to a periodic function or vice versa, and they may not be distinguishable. To confront with this, for periodic testing, we need periodic or non-periodic to be ϵ -distinguishable, as follows:

► **Definition 6.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function, and fix any $\Xi > 0$. We say f is (Ξ, ϵ) -periodic if for any $t_1, t_2 \in \mathbb{R}$ with $\min_{n \in \mathbb{Z}} |t_1 - t_2 + n\Xi| \geq \frac{\pi\epsilon}{2\sup_t \|f'(t)\|_2}$, $|f(t_1) - f(t_2)| \geq \epsilon$ or $|f'(t_1) - f'(t_2)| \geq \epsilon$. We say f is ϵ -periodic if f is (Ξ, ϵ) -periodic for some $\Xi > 0$. We say f is (Ξ, ϵ) -non-periodic if for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| \geq \frac{\pi\epsilon}{2\sup_{t_1 \leq t \leq t_2} \|f'(t)\|_2}$, $|f(t_1) - f(t_2)| \geq \epsilon$ or $|f'(t_1) - f'(t_2)| \geq \epsilon$. We say f is ϵ -non-periodic if f is (Ξ, ϵ) -non-periodic for any $\Xi > 0$.

We also have the regularity condition tailored for periodicity detection.

► **Assumption 2.** The true sampling function $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 , and there exists some $\delta > 0$ such that for all $t \in \mathbb{R}$, either $|f'(t)| \geq \delta$ or $|f''(t)| \geq \delta$, and for all $t \in \mathbb{R}$, $|f''(t)| \leq L_2$.

4 Subsampling Confidence Bound for Sliding Window

4.1 Sliding Window Confidence Interval

Given a function $f : \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}$ and finitely many sampled points $T \subset \mathbb{T}$, we denote the sliding window embedded points of the sample by $X_N = SW_{2N, \tau_N} f(T)$ and the whole support by $M_N = SW_{2N, \tau_N} f([0, 2\pi])$. Here, the window size is $\tau_N = \frac{2\pi}{2N+1}$.

As in [14], we apply the centralization and normalization process pointwise to obtain \bar{X}_N and \bar{M}_N . The centralizing and normalizing maps $C, N : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are defined by $C(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x}^T \mathbf{1}}{\mathbf{1}^T \mathbf{1}} \mathbf{1}$ and $N(\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{x}}}$. Thus, $\bar{X}_N = N \circ C \circ X_N$ and $\bar{M}_N = N \circ C \circ M_N$. We first prove that the whole support is indeed an embedded manifold.

► **Theorem 7.** Suppose $f : \mathbb{T} \rightarrow \mathbb{R}$ is $C^{1,1}$ and nonconstant. There exists sufficiently large N_0 so that every $N \geq N_0$, M_N and \bar{M}_N are $C^{1,1}$ embedded manifold, homeomorphic to S^1

We formulate a method that generates confidence set of persistent diagram via sliding window method. Our goal is estimating the true persistent diagram for support M_N and \bar{M}_N . Accordance to [7], we use the subsampling method. We first introduce an algorithm and then state theoretical guarantee.

Algorithm 1 Implementation of Confidence bound

Input: $C^{1,1}$ periodic function $f : \mathbb{T} \rightarrow \mathbb{R}$, $T \subset \mathbb{T}$, n, b, N, α

1. Compute X_N , the sliding window embedding for each sample T to \mathbb{R}^{2N+1}

2. Use Monte Carlo method to compute the α quartile of all the Hausdorff distance of size b subsamples over $X_N : c_b/2$

Output: c_b

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Our next theorem states that we can construct a confidence set of homological features of M_N by using points X_N . Define the function as follows:

$$L_N(t) = \frac{1}{\binom{n}{b}} \sum_{j=1}^{\binom{n}{b}} I\left(d_H(X_N, X_{N,b}^{(j)}) > \sqrt{2N+1} \cdot t\right)$$

Here, $X_{N,b}^{(j)}$, $j = 1, 2, \dots, \binom{n}{b}$ are size b subsamples of points in X_N .

► **Theorem 8.** $f : \mathbb{T} \rightarrow \mathbb{R}$ be a $C^{1,1}$ function and assume that $\int_{\mathbb{T}} f = 0$ and $\int_{\mathbb{T}} f^2 = 1$. If $b = o(n/\log n)$ tending to infinity as $n \rightarrow \infty$. Define $c_{\alpha}^N = 2(L_N)^{-1}(\alpha)$. Then

$$P\left(d_B\left(\frac{1}{\sqrt{2N+1}}\mathcal{P}(X_N), \frac{1}{\sqrt{2N+1}}\mathcal{P}(M_N)\right) > c_{\alpha}^N\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

► **Remark 9.** In general, if $\int_{\mathbb{T}} f \neq 0$ and $\int_{\mathbb{T}} f^2 \neq 1$, then we can normalize f by $g(x) = \frac{f(x) - \int_{\mathbb{T}} f}{\int_{\mathbb{T}} (f - \int_{\mathbb{T}} f)^2}$. The sliding window embedded points of f and g are related by

$$SW_{M,\tau}g(t) = \frac{1}{\int_{\mathbb{T}} (f - \int_{\mathbb{T}} f)^2} \left(SW_{M,\tau}f(t) - \left(\int_{\mathbb{T}} f \right) \mathbf{1} \right)$$

We can state a corresponding theorem for the centralized, normalized points. Define the function similarly as

$$\bar{L}_N(t) = \frac{1}{\binom{n}{b}} \sum_{j=1}^{\binom{n}{b}} I\left(d_H(\bar{X}_N, \bar{X}_{N,b}^{(j)}) > t\right).$$

Here, $\bar{X}_{N,b}^{(j)}$, $j = 1, 2, \dots, \binom{n}{b}$ are size b subsamples of points in \bar{X}_N .

► **Theorem 10.** $f : \mathbb{T} \rightarrow \mathbb{R}$ be a $C^{1,1}$ function and assume $b = o(n/\log n)$ tending to infinity as $n \rightarrow \infty$. Define $\bar{c}_{\alpha}^N = 2(\bar{L}_N)^{-1}(\alpha)$. Then

$$P\left(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}(\bar{M}_N)) > \bar{c}_{\alpha}^N\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

4.2 Convergence of Confidence Bound

In this section, we prove that the confidence bound constructed in Section 4.1 converges as dimension tends to infinity and this limit gives a correct confidence bound for the limit of persistent diagrams for varying embedding dimensions.

► **Theorem 11.** Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is $C^{1,1}$ function and satisfies $\int_{\mathbb{T}} f = 0$ and $\int_{\mathbb{T}} f^2 = 1$. Assume $b = o(n/\log n)$ tending to infinity as $n \rightarrow \infty$. The confidence bound c_{α}^N and \bar{c}_{α}^N converges to the same limit as $N \rightarrow \infty$.

The next theorem shows that the limit $c_{\alpha} = \lim_{N \rightarrow \infty} c_{\alpha}^N = \lim_{N \rightarrow \infty} \bar{c}_{\alpha}^N$ also provides a confidence set for the limit of persistent diagrams. To do this, we first prove the existence of the limit of persistent diagrams.

► **Lemma 12.** Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is $C^{1,1}$ function and satisfies $\int_{\mathbb{T}} f = 0$ and $\int_{\mathbb{T}} f^2 = 1$. The persistent diagrams $\mathcal{P}(\frac{1}{\sqrt{2N+1}}X_N)$, $\mathcal{P}(\bar{X}_N)$, $\mathcal{P}(\frac{1}{\sqrt{2N+1}}M_N)$ and $\mathcal{P}(\bar{M}_N)$ forms Cauchy sequence under the bottleneck distance. Therefore, they converges to the limit

$$\lim_{N \rightarrow \infty} \mathcal{P}\left(\frac{1}{\sqrt{2N+1}}X_N\right) = \lim_{N \rightarrow \infty} \mathcal{P}(\bar{X}_N) = \mathcal{P}_{\infty}(\bar{X})$$

$$\lim_{N \rightarrow \infty} \mathcal{P}\left(\frac{1}{\sqrt{2N+1}} M_N\right) = \lim_{N \rightarrow \infty} \mathcal{P}(\bar{M}_N) = \mathcal{P}_{\infty}(\bar{M})$$

Now, we state that c_{α} provides a confidence band for limits of the persistent diagrams. For theoretical reasons, we impose additional assumption.

► **Theorem 13.** *Under the assumption below, the limit $c_{\alpha} = \lim_{N \rightarrow \infty} c_{\alpha}^N = \lim_{N \rightarrow \infty} \bar{c}_{\alpha}^N$ satisfies*

$$\lim_{N \rightarrow \infty} P\left(\frac{1}{\sqrt{2N+1}} d_B(\mathcal{P}(X_N), \mathcal{P}(M_N)) > c_{\alpha}\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

and

$$P\left(d_B(\mathcal{P}_{\infty}(\bar{X}), \mathcal{P}_{\infty}(\bar{M})) > c_{\alpha}\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

Assumption : The function $g(t) = \sum_{m=1}^{\infty} (\hat{f}(m)^2 + \hat{f}(-m)^2) \cos(mt)$ satisfies the following: for every $x \in \mathbb{R}$ the Lebesgue measure of $g^{-1}(x)$ is zero.

► **Remark 14.** Assumption on g is nontrivial but it will hold for almost all $C^{1,1}$ function f . Except for some degenerate cases, trigonometric series will not behave like a constant function in positive measure.

This theorem can be interpreted as the following idea. The limit of the persistence diagram is an “idealistic” situation. As the embedding dimension increases, it captures more precise information about the data points. From this viewpoint, limit of our subsampling bound also provides a correct probabilistic bound for the idealistic situation.

5 Periodicity Detection

In this section, we will show that under regular conditions we introduced before, non-periodic function is topologically trivial and periodic function is topologically a circle. And we will use this to test whether the sampling function is periodic or not.

First, we begin with that non-periodic function has its time-delay embedding having a nice manifold structure and topologically contractible. Having a manifold structure is important for the confidence interval to be valid for non-periodic function as well.

► **Theorem 15.** *Let f be an $C^{1,1}$ function defined on $f : \mathbb{R} \rightarrow \mathbb{R}$. If f satisfies for $x \neq y$ that $f(x) = f(y)$ then $f'(x) \neq f'(y)$. Then every compact subset $K \subset \mathbb{R}$ and $\tau_0 > 0$, there exists $N_0 \in \mathbb{N}$ such that $N \geq N_0$, $SW_{N, \tau_0/N} f(K)$ is a compact manifold homeomorphic to interval.*

To be used for testing periodicity, we need not only the trajectory of time-delayed embedding to be either contractible or circle, but we need its tubular neighborhood to be contractible or circle as well. For a set A and $t > 0$, let $A^t := \{x : d(x, A) < t\}$ be t -tubular neighborhood of A , where $d(x, A) = \max\{\|x - y\|_2 : y \in A\}$. A set A being homotopic equivalent to its tubular neighborhoods A^t , $t < t_0$, ensures that persistent homology of Čech complex, or distance function filtration, is empty when birth is strictly between 0 and t_0 .

We first delve into the non-periodic case. Under suitable regular condition, and when τ is small enough, then we have t -tubular neighborhood of the trajectory homotopic to interval, and in particular contractible.

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► **Theorem 16.** Suppose $f \in C^2$ satisfies Assumption 2 and ϵ -non-periodic. Then as $\tau \rightarrow 0$, $SW_{m,\tau}f$ satisfies that for any

$$0 < t < \frac{\sqrt{m+1}}{2} \min \left\{ \epsilon - o(\tau), \frac{(\delta - o(\tau))^2}{L_2} \right\},$$

we have that $(SW_{m,\tau}f(\mathbb{T}))^t$ is homotopic to interval, and in particular contractible.

This immediately implies that the 1-dimensional persistent homology of the trajectory is empty on the vertical strip $[0, t_0] \times \mathbb{R}$, for some $t_0 > 0$.

► **Corollary 17.** Suppose $f \in C^2$ satisfies Assumption 2 and ϵ -non-periodic. Let $\mathcal{P}_1(SW_{m,\tau}f)$ be 1-dimensional persistent homology of $SW_{m,\tau}f$ of Čech complex filtration, understood as a subset of \mathbb{R}^2 . Then

$$\mathcal{P}_1(SW_{m,\tau}f) \cap \left[0, \frac{\sqrt{m+1}}{2} \min \left\{ (\epsilon - C_\tau \tau), \frac{(\delta - C_\tau \tau)^2}{L_2} \right\} \right] \times \mathbb{R} = \emptyset.$$

Second, we look into periodic case. Again under suitable regular condition, and when τ is small enough, then we have t -tubular neighborhood of the trajectory homotopic to circle.

► **Theorem 18.** Let $\Xi > 0$, and Suppose $f \in C^2$ satisfies Assumption 2 and (Ξ, ϵ) -periodic. Then $SW_{m,\tau}f$ satisfies that for any

$$0 < t < \frac{\sqrt{m+1}}{2} \min \left\{ \epsilon - o(\tau), \frac{(\delta - o(\tau))^2}{L_2} \right\},$$

we have that $(SW_{m,\tau}f(\mathbb{T}))^t$ is homotopic to a circle S^1 .

This immediately implies that the 1-dimensional persistent homology of the trajectory has one point on the vertical strip $[0, t_0] \times \mathbb{R}$, for some $t_0 > 0$.

► **Corollary 19.** Let $\Xi > 0$, and suppose $f \in C^2$ satisfies Assumption 2 and (Ξ, ϵ) -periodic. Let $\mathcal{P}_1(SW_{m,\tau}f)$ be 1-dimensional persistent homology of $SW_{m,\tau}f$ of Čech complex filtration, understood as a subset of \mathbb{R}^2 . Then

$$\mathcal{P}_1(SW_{m,\tau}f) \cap \left[0, \frac{\sqrt{m+1}}{2} \min \left\{ (\epsilon - C_\tau \tau), \frac{(\delta - C_\tau \tau)^2}{L_2} \right\} \right] \times \mathbb{R} = \{(0, d)\},$$

where

$$d \geq \frac{\sqrt{m+1}}{2} \min \left\{ (\epsilon - C_\tau \tau), \frac{(\delta - C_\tau \tau)^2}{L_2} \right\}.$$

6 Simulation Study

6.1 Synthetic Data

In this section, we conduct simulations for both of our methodology, with and without normalization and centralization. Specifically, on the interval $[0, 4\pi]$ we sample $n = 500$ samples from the underlying function. Four types of error are added with the scale parameter increasing from 0.1 to 0.3: (1) Gaussian Additive (2) Gaussian Multiplicative (3) Laplacian

Additive (4) Laplacian Multiplicative. Denoising is done via moving average with the standard rule for parameter selection. (The odd number larger than 3 which is closest to \sqrt{n} is selected.) We suppose that L is known, i.e. the ideal delay τ is given. Linear interpolation is done for the time-delay embedding as we suppose that we do not know the true distribution. We compare our method with the result derived from Generalized Lomb-Scargle periodogram (GLS, [24]). N , which determines the embedding dimension, is chosen as $N = 10$. Subsampling is done for $b = 200$ samples, as Monte Carlo simulation is done 1000 times to estimate the confidence bound. Tables 1 show the result of time-delay embedding with subsampling and GLS. The result shows that our method detects the results are satisfactory when the error is not large. Next, we show that our method, when normalization and centralization is added, can effectively detect periodicity even when oscillation exists. Specifically, we test for periodic functions with damped oscillation, i.e. the product of a decreasing function and a periodic function. In this setting, we take $n = 200$ samples with $b = 50$ subsamples. The result is shown in Table 2. The result proves that our method detects significantly damped functions effectively. Finally, we show that our method rarely detects non-periodic functions as periodic. We test our function does not accept chirp functions to be periodic. In this setting, we take $n = 300$ samples with $b = 100$ subsamples. The result is shown in Table 3. We can find out that compared to the GLS method, which always detects function with period changing significantly to be periodic, our method successfully discriminates such non-periodic functions from periodic functions.

► Remark 20. The functions chosen have infinitely many non-zero Fourier coefficients, making our methods disadvantageous. Nevertheless, our method successfully detects periodicity.

■ **Table 1** Comparison for periodic function

$$(1) f(x) = \frac{3}{2-\cos x} \quad (2) f(x) = \log(5 + \sin(3x)) + e^{\cos(5x)}$$

Each value represents how much the method detected periodicity out of 100 simulations.

	no noise	GA			GM			LA			LM		
		0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
(1)	Tds	100	100	100	100	100	100	100	100	100	100	100	98
(1)	GLS	100	100	100	100	100	100	100	100	100	100	100	100
(2)	Tds	94	94	94	96	83	52	22	93	93	93	87	53
(2)	GLS	100	100	100	100	100	100	100	100	100	100	100	95

■ **Table 2** Time-delay with subsampling for oscillating periodic function

$$(1) f(x) = e^{-x/5} \frac{1}{2-\cos x} \quad (2) f(x) = e^{-x/5} (\log(5 + \sin(3x)) + e^{\cos(5x)})$$

Each value represents how much the method detected periodicity out of 50 simulations.

	no noise	GA			GM			LA			LM		
		0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
(1)	50	47	37	29	49	46	42	49	40	31	50	45	42
(2)	12	11	10	11	15	17	20	10	10	13	17	17	16

6.2 Real Data

In this section, we use the BIDMC dataset[2], which is a dataset with periodic signals, to check if our method is applicable to real data. Since each BIDMC set consists of 60000 samples, which is large than enough to apply our method, we only take the first 500 samples

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Table 3 Time-delay with subsampling for non periodic function
 $f(x) = \sin(10\sqrt{x})$

Each value represents how much the method detected periodicity out of 100 simulations.

		no noise			GA			GM			LA			LM		
		0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
Tds		26	7	9	11	10	13	12	10	12	16	8	11	13		
GLS		100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

and apply the test. If we want to test for specific periodicity, we may select the delay based on our domain knowledge. In our case, however, when the delay time τ was given sufficiently good, that is the window size is selected base on the period we want to test, it successfully detected periodicity from every dataset. Thus, we also did simulation when the time-delay is selected randomly, 5 times for each dataset. Then, if at least one of the cases is calculated to be periodic, then we consider the datasets to be periodic. Among the 53 datasets, 27 datasets were found to be periodic even when we only tested with 5 randomly chosen period.

7 Concluding Remarks

In this paper, we proposed a novel periodicity test for time series. We used time-delay embedding with subsampling to provide a confidence set for the significant features in the persistence diagram. We asymptotically proved that periodic functions must have a significant one-dimensional persistent homology lying in the confidence set, while non-periodic functions should not have a significant feature in the corresponding region for periodic function. In simulation settings, we used linear interpolation to calculate the unknown values. As a result, our method exhibited results comparable to those of the GLS method when the error level was not large. Our method also successfully captured periodic functions with significant damping, indicating that the function can detect oscillation. Meanwhile, our method successfully distinguished non-periodic functions, specifically chirp functions, in which the GLS method categorized as periodic. In real data, where we do not know the true period, our method successfully tested when we have a target period that we are curious about.

Future Work: There are several directions to extend the work. For theoretical approach, we may further extend the analysis for non-periodic functions to weaken the assumptions. We may extend the theory for periodic functions to quasiperiodic or almost periodic functions. For the practical implementation, we may find a better interpolation method, and a better way to test periodicity for any period, for we may not have a target period.

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A Statement on Section 2.5

In Section 2.5 we addressed the results from [7]. We describe the formal statement here. Assume the observed sample $\{X_1, \dots, X_n\}$ from a distribution P concentrated on the set \mathbb{M} . [7] assumed following assumptions.

Assumption A1. \mathbb{M} is a d -dimensional compact manifold without boundary, embedded in \mathbb{R}^D and $\text{reach}(\mathbb{M}) > 0$

Assumption A2. For each $x \in \mathbb{M}$, $\rho(x, t) = \frac{P(B(x, t/2))}{t^d}$ is bounded continuous function of t , differentiable for $t \in (0, t_0)$ and right differentiable at zero. $\rho(x, t)$ is bounded above from zero and infinity and there exists $t_0 > 0$ and some C that

$$\sup_x \sup_{0 \leq t \leq t_0} \left| \frac{\partial \rho(x, t)}{\partial t} \right| \leq C < \infty$$

Then, it is possible to construct the subsample bound from the Hausdorff distance between the subsample and the original sample. Define $L_b(t) = \frac{1}{\binom{n}{b}} \sum_{j=1}^{\binom{n}{b}} I(d_H(S_{b,n}^j, S_n) > t)$, which is the summation of the indicator function over all possible $\binom{n}{b}$ subsamples of size b . As remarked in [7], in practice we shall use the Monte Carlo method to approximate this function. Denote the persistence diagram of $\{X_1, \dots, X_n\}$ by $\hat{\mathcal{P}}$ and the persistence diagram of \mathbb{M} by \mathcal{P} . For b increasing to infinity as $n \rightarrow \infty$ and satisfying $b = o(\frac{n}{\log n})$,

► **Theorem 21** ([7] Theorem 3). *For all large n ,*

$$\mathbb{P}(d_B(\hat{\mathcal{P}}, \mathcal{P}) > c_b) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

B Proofs for Section 4.1

B.1 Proof of Theorem 7

We denote the centralizing and normalizing maps by C and N , respectively. To be precise, $C, N : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are the mappings defined by

$$C(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x}^T \mathbf{1}}{\mathbf{1}^T \mathbf{1}} \mathbf{1} \quad , \quad N(\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{x}}}$$

The normalizing and centralizing maps depend on the domain's dimension, but since the mappings are unambiguous, we will use N and C without specifying the dimensions.

Recall that \bar{M}_N is the image of S^1 under the mapping $N \circ C \circ SW_{2N, \tau_N}$. It follows that \bar{M}_N is an immersed submanifold. It remains to verify that this map is an injection to prove that \bar{M}_N is an embedded manifold.

If f is periodic with a period less than $1/L$, then the mapping is not an injection, but \bar{M}_N is still the image of $N \circ C \circ SW_{2N, \tau_N}([0, 1/L])$. Therefore, we assume f is periodic with minimum period 1. Under this condition, we prove injectivity.

Argue by contradiction. Suppose for each N , there exists $t_1^{(N)} \neq t_2^{(N)} \in [0, 2\pi]$ so that $N \circ C \circ SW_{2N, \tau_N}(t_1^{(N)}) = N \circ C \circ SW_{2N, \tau_N}(t_2^{(N)})$. Then

$$\begin{aligned} f(t_1^{(N)}) &= a_N f(t_2^{(N)}) + b_N \\ f(t_1^{(N)} + \tau_N) &= a_N f(t_2^{(N)} + \tau_N) + b_N \\ &\vdots \\ f(t_1^{(N)} + 2N\tau_N) &= a_N f(t_2^{(N)} + 2N\tau_N) + b_N \end{aligned}$$

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holds for some a_N, b_N . As there exists a subsequence of $\{t_1^{(N)}\}$ that converges to t_1 , we replace $\{t_1^{(N)}\}$ and $\{t_2^{(N)}\}$ with the subsequence. Similarly, we again replace the sequences with so that $t_2^{(N)}$ converges to t_2 .

We first claim that (a_N) and (b_N) are bounded. By the above equations,

$$\sum_{i=0}^{2N} (f(t_1^{(N)}) + i\tau_N) - m_1)^2 = a_N^2 \sum_{i=0}^{2N} (f(t_2^{(N)}) + i\tau_N) - m_2)^2$$

holds. Here, m_1 and m_2 are the means of $f(t_1^{(N)}), \dots, f(t_1^{(N)} + 2N\tau_N)$ and $f(t_2^{(N)}), \dots, f(t_2^{(N)} + 2N\tau_N)$, respectively. First, since

$$\frac{1}{2N+1} \sum_{i=0}^{2N} f(t + i\tau_N)^2 \rightarrow \int_{\mathbb{T}} f^2 = 1 \quad , \quad \frac{1}{2N+1} \sum_{i=0}^{2N} f(t + i\tau_N) \rightarrow \int_{\mathbb{T}} f = 0$$

uniformly in t , a_N converges to 1. Similarly, since

$$\frac{1}{2N+1} (f(t_1^{(N)}) + \dots + f(t_1^{(N)} + 2N\tau_N)) = \frac{a_N}{2N+1} (f(t_2^{(N)}) + \dots + f(t_2^{(N)} + 2N\tau_N)) + b_N$$

holds, b_N converges to 0. Now, for any $x \in [0, 2\pi]$, there exists $i^{(N)} \in \mathbb{Z}$ such that the sequence $t_1^{(N)} + i^{(N)}\tau_N$ converges to x . Since f is a $C^{1,1}$ function,

$$f(x) = \lim_{N \rightarrow \infty} f(t_1^{(N)} + i^{(N)}\tau_N) = \lim_{N \rightarrow \infty} a_N f(t_2^{(N)} + i^{(N)}\tau_N) + b_N = f(x + t_2 - t_1).$$

Since f has a minimal period 1, we conclude $t_1 = t_2$. Thus, $t_1^{(N)}, t_2^{(N)} \rightarrow t_1$, and

$$N \circ C \circ SW_{2N, \tau_N} f(t_1^{(N)}) = N \circ C \circ SW_{2N, \tau_N} f(t_2^{(N)})$$

holds. Finally, we show that the derivative of $N \circ C \circ SW_{2N, \tau_N} f$ at t_1 is nonzero to finish the proof by contradiction.

We claim that for sufficiently large N and for all $t \in [0, 2\pi]$, the derivative of $N \circ C \circ SW_{2N, \tau_N} f$ is nonzero. Denoting the Jacobians of the maps N and C by D_N and D_C respectively, this is equivalent to

$$D_N \cdot D_C \cdot SW_{2N, \tau_N} f' \neq 0.$$

Since C is a linear mapping, $D_C = C$. For $\mathbf{x}, \mathbf{v} \in \mathbb{R}^{2N+1}$, the Jacobian of the map N at \mathbf{x} satisfies $D_N \mathbf{v} = 0$ if and only if $\mathbf{v} = c\mathbf{x}$ for some $c \in \mathbb{R}$. Consequently, $C \circ SW_{2N, \tau_N} f'(t) = c \cdot C \circ SW_{2N, \tau_N} f(t)$ if $D_N \cdot D_C \cdot SW_{2N, \tau_N} f'(t) = 0$. Since $\ker C = \{a \cdot \mathbf{1} \mid a \in \mathbb{R}\}$, it follows that $SW_{2N, \tau_N} f'(t) = cSW_{2N, \tau_N} f(t) + a \cdot \mathbf{1}$.

If there exists a sequence $t^{(N)}$ such that $SW_{2N, \tau_N} f'(t^{(N)}) = c_N SW_{2N, \tau_N} f(t^{(N)}) + a_N \cdot \mathbf{1}$ holds, arguing similarly as above, $c_N \rightarrow \|f'\|_2$ and $a_N \rightarrow 0$. For any $x \in [0, 2\pi]$, there exists a sequence $\{t^{(N)} + i^{(N)}\tau_N\}$ that converges to x , and so

$$f'(x) = \lim_{N \rightarrow \infty} f'(t^{(N)} + i^{(N)}\tau_N) = \lim_{N \rightarrow \infty} c_N f(t^{(N)} + i^{(N)}\tau_N) + a_N = \|f'\|_2 f(x)$$

holds. Thus, f satisfies the functional equation $f' = b \cdot f$, and any $C^{1,1}$ periodic function satisfying this equation is constant, which is a contradiction.

Therefore, there exists N_0 such that $N \circ C \circ SW_{2N, \tau_N} f$ is locally injective for every $x \in \mathbb{T}$. Then for $N \geq N_0$, $N \circ C \circ SW_{2N, \tau_N} f$ must be globally injective, which implies that \bar{M}_N is a $C^{1,1}$ manifold.

Applying the same logic to M_N gives the result that if there exist sequences $t_1^{(N)} \neq t_2^{(N)}$ such that $SW_{2N, \tau_N} f(t_1^{(N)}) = SW_{2N, \tau_N} f(t_2^{(N)})$ for all N , then their accumulation points t_1 and t_2 must coincide. Now, the derivative of $SW_{2N, \tau_N} f$ is nonzero, thus, $SW_{2N, \tau_N} f$ is locally injective. Therefore, M_N is also a $C^{1,1}$ manifold.

B.2 Proof of Theorem 8 and Theorem 10

To prove Theorem 8 and 10, we first check the assumptions of Theorem 4

► **Lemma 22.** *For sufficiently large $N \in \mathbb{N}$, M_N, \bar{M}_N satisfies:*

- (1) $\text{reach}(M_N) > 0$, $\text{reach}(\bar{M}_N) > 0$ and M_N, \bar{M}_N is compact.
- (2) $\rho(x, t) = \frac{P(B(x, \frac{t}{2}))}{t}$ for $x \in M_N$, $\bar{\rho}(x, t) = \frac{P(B(x, \frac{t}{2}))}{t}$ for $x \in \bar{M}_N$ is differentiable for $t \in (0, t_0)$ and right differentiable at zero. Both $\rho, \bar{\rho}$ are bounded from zero and infinity. Moreover, there exists C_1, C_2 such that

$$\sup_x \sup_{0 \leq t \leq t_0} \left| \frac{\partial \rho(x, t)}{\partial t} \right| \leq C_1 < \infty \quad \sup_x \sup_{0 \leq t \leq t_0} \left| \frac{\partial \bar{\rho}(x, t)}{\partial t} \right| \leq C_2 < \infty$$

Proof. We provide the proof for \bar{M}_N . The proof for M_N is a slight modification.

- (1) Let us denote $N \circ C \circ SW_{2N, \tau_N} f = \gamma$ to emphasize that it is a closed curve. By Theorem 7, \bar{M}_N is an embedding of S^1 under the map γ . By [8], \bar{M}_N has positive reach if it is $C^{1,1}$.

Since the Jacobian of normalizing map N at $(x_1, \dots, x_d)^t$ is

$$D_N = \frac{1}{\sqrt{x_1^2 + \dots + x_d^2}} \left(I - \frac{1}{x_1^2 + \dots + x_d^2} \begin{pmatrix} x_1^2 & x_1 x_2 & \dots & x_1 x_d \\ \vdots & \vdots & \ddots & \vdots \\ x_d x_1 & x_d x_2 & \dots & x_d^2 \end{pmatrix} \right)$$

whose eigenvalues cannot exceed $\frac{1}{\sqrt{x_1^2 + \dots + x_d^2}}$. For the centralized sliding window points, a sufficiently large N implies $\|C \circ SW_{2N, \tau_N} f(t)\|_2 \geq \sqrt{2N+1}(\|f\|_2 - \epsilon) \geq \frac{1}{2}\sqrt{2N+1}$. On the other hand, the map C has eigenvalues 0 and 1, so

$$\begin{aligned} \|\gamma'(t_1) - \gamma'(t_2)\| &\leq \frac{2}{\sqrt{2N+1}} \sqrt{\sum_{n=0}^{2N} (f'(t_1 + n\tau_N) - f'(t_2 + n\tau_N))^2} \\ &\leq C|t_1 - t_2| \end{aligned}$$

Moreover, we proved in Theorem 7 that the derivative of $N \circ C \circ SW_{2N, \tau_N} f$ is nonzero. This concludes the proof.

- (2) Since the derivative of γ is nonzero (Theorem 7) and the domain is compact, the norm of the derivative is bounded from below and above. That is, $c_1 \leq \|\gamma'(t)\| \leq c_2$.

Now, let $t_0 > 0$ satisfy the following conditions:

1. $t_0 < \text{reach}(\bar{M}_N)$
2. For every $s \in [0, 2\pi]$, the distance $d_{\mathbb{R}^{2N+1}}(\gamma(s), \gamma(s + \delta))$ increases for $0 < \delta < t_0$ and decreases for $-t_0 < \delta < 0$.

For $x = \gamma(s_0)$, $t\bar{\rho}(x, t)$ is the inverse function of $d_{s_0}(t)$ that measures the geodesic length of γ on the time interval $[s_0 - t/2, s_0 + t/2]$.

$$d_{s_0}(t) = \int_{s_0 - t/2}^{s_0 + t/2} \|\gamma'\|_2 \quad d_{s_0}(t)\bar{\rho}(x, d_{s_0}(t)) = t$$

Since γ is $C^{1,1}$ curve,

$$d'_{s_0}(t) \left(\bar{\rho}(x, d_{s_0}(t)) + d_{s_0}(t) \frac{\partial \bar{\rho}(x, d_{s_0}(t))}{\partial t} \right) = 1$$

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$$\left| \frac{\partial \bar{\rho}(x, d_{s_0}(t))}{\partial t} \right| = \left| \frac{d_{s_0}(t) - td'_{s_0}(t)}{(d_{s_0}(t))^2} \right| = \left| \frac{\frac{1}{t^2} \int_0^t d'_{s_0}(s) - d'_{s_0}(t) ds}{(d_{s_0}(t))^2 / t^2} \right| \leq \frac{C}{2c_1^2}$$

where C is Lipschitz constant of γ'

Thus, the proof of Theorem 8 and Theorem 10 is direct from [7] Theorem 4.

Proof of Theorem 8, 10. Since the time-delayed embedded points are sampled from the support M_N or \bar{M}_N , by Theorem 3 in [7], our definitions of c_α^N and \bar{c}_α^N satisfy the confidence bound condition. Therefore,

$$P\left(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}(\bar{M}_N)) > \bar{c}_\alpha^N\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

$$P\left(d_B\left(\frac{1}{\sqrt{2N+1}}\mathcal{P}(X_N), \frac{1}{\sqrt{2N+1}}\mathcal{P}(M_N)\right) > c_\alpha^N\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

C Proofs for Section 4.2

C.1 Proof of Theorem 11

Proof of Theorem 11. We first recall results from [14]. Let $X_N = SW_{2N, \tau_N} f(T)$ and $Y_N = SW_{2N, \tau_N} S_N f(T)$, and denote the centralized, normalized versions by \bar{X}_N and \bar{Y}_N respectively (i.e., $\bar{X}_N = N \circ C \circ X_N$ and $\bar{Y}_N = N \circ C \circ Y_N$).

When $f \in C^{1,1}(\mathbb{T}, \mathbb{R})$ with $\int_{\mathbb{T}} f = 0$ and $\int_{\mathbb{T}} f^2 = 1$, the Hausdorff distance between these point clouds is bounded by (Proposition 4.2 and Theorem 5.6 in [14]):

$$d_H(X_N, Y_N) \leq 2\|R_N f'\|^2,$$

$$\bar{Y}_N = \frac{Y_N}{\sqrt{2N+1}\|S_N f\|_2},$$

$$\lim_{n \rightarrow \infty} d_H\left(\bar{X}_N, \frac{X_N}{\sqrt{2N+1}}\right) = 0.$$

In particular, we strengthen the third result to obtain a quantitative bound.

► **Lemma 23.** *If X_N and \bar{X}_N defined as above,*

$$d_H\left(\bar{X}_N, \frac{X_N}{\sqrt{2N+1}}\right) \leq \frac{C}{N}\|f'\|_\infty(1 + \|f\|_\infty)$$

which C is independent of N, f .

Combining three bounds, we get

$$d_H(\bar{X}_N, \bar{Y}_N) \leq C\left(\frac{1}{\sqrt{N}}\|R_N f'\|_2 + \frac{1}{N}(\|f\|_\infty + 1)\|f'\|_\infty\right)$$

Moreover, [14] showed that points in different dimensions can be related as follows:

$$|\|\bar{y}_1 - \bar{y}_2\| - \|\bar{y}'_1 - \bar{y}'_2\|| \leq 2\left(\frac{1}{\|S_N f\|_2} + \frac{1}{\|S_{N'} f\|_2}\right)\|S_{N'} f - S_N f\|_2$$

Define $\tilde{L}_N(t) = \frac{1}{\binom{n}{b}} \sum_{j=1}^{\binom{n}{b}} I(d_H(\bar{Y}_N, \bar{Y}_{N,b}^{(j)}) > t)$ and $\tilde{c}_\alpha^N = 2(\tilde{L}_N)^{-1}(\alpha)$. Then by the preceding inequalities, if the j -th subsample satisfies $d_H(\bar{X}_N, \bar{X}_{N,b}^{(j)}) > t$, then $d_H(\bar{Y}_N, \bar{Y}_{N,b}^{(j)}) > t - C(\frac{1}{\sqrt{N}} \|R_N f'\|_2 + \frac{1}{N} (\|f\|_\infty + 1) \|f'\|_\infty)$, and the same holds for the other direction. Therefore,

$$\left| d_H(\bar{X}_N, \bar{X}_{N,b}^{(j)}) - d_H(\bar{Y}_N, \bar{Y}_{N,b}^{(j)}) \right| < C \left(\frac{1}{\sqrt{N}} \|R_N f'\|_2 + \frac{1}{N} (\|f\|_\infty + 1) \|f'\|_\infty \right)$$

and

$$|\tilde{c}_\alpha^N - \tilde{c}_\alpha^N| < C \left(\frac{1}{\sqrt{N}} \|R_N f'\|_2 + \frac{1}{N} (\|f\|_\infty + 1) \|f'\|_\infty \right)$$

Applying the same method to \bar{Y}_N and $\bar{Y}_{N'}$, we obtain

$$|\tilde{c}_\alpha^N - \tilde{c}_\alpha^{N'}| < 2 \left(\frac{1}{\|S_N f\|_2} + \frac{1}{\|S_{N'} f\|_2} \right) \|S_{N'} f - S_N f\|_2$$

Combining these two inequalities,

$$\begin{aligned} |\tilde{c}_\alpha^N - \tilde{c}_\alpha^{N'}| &< C \left(\frac{1}{\sqrt{N}} \|R_N f'\|_2 + \frac{1}{N} (\|f\|_\infty + 1) \|f'\|_\infty + \frac{1}{\sqrt{N'}} \|R_{N'} f'\|_2 + \frac{1}{N'} (\|f\|_\infty + 1) \|f'\|_\infty \right) \\ &\quad + 2 \left(\frac{1}{\|S_N f\|_2} + \frac{1}{\|S_{N'} f\|_2} \right) \|S_{N'} f - S_N f\|_2 \end{aligned}$$

so $\{\tilde{c}_\alpha^N\}$ forms Cauchy sequence. Moreover, the same applied to $\frac{1}{\sqrt{2N+1}} X_N$ gives the convergence of $\{c_\alpha^N\}$. By Lemma 23, $|c_\alpha^N - \tilde{c}_\alpha^N| \leq \frac{C}{N} \|f'\|_\infty (1 + \|f\|_\infty)$ so they converge to the same limit. \blacktriangleleft

Proof of Lemma 23. We compute for the singleton point $x \in \mathbb{R}^{2N+1}$, which is image of the sliding window embedding. We calculate the norm

$$\left\| \frac{C(x)}{\|C(x)\|} - \frac{x}{\sqrt{2N+1}} \right\| \leq \left| 1 - \frac{\|C(x)\|}{\sqrt{2N+1}} \right| + \frac{1}{\sqrt{2N+1}} \|C(x) - x\|$$

Estimating each terms,

$$\begin{aligned} \frac{1}{2N+1} \|C(SW_{2N,\tau_N} f(t))\|^2 &= \frac{1}{2\pi} \sum_{n=0}^{2N} \tau_N \left(f(t + n\tau_N) - \frac{1}{2N+1} \sum_{m=0}^{2N} f(t + m\tau_N) \right)^2 \\ &= \frac{1}{2\pi} \sum_{n=0}^{2N} \tau_N f(t + n\tau_N)^2 - \frac{1}{2\pi} \tau_N \left(\frac{1}{2N+1} \sum_{n=0}^{2N} f(t + n\tau_N) \right)^2 \\ &= 1 - \frac{1}{2\pi} \left(\int_0^{2\pi} f(t)^2 dt - \sum_{n=0}^{2N} \tau_N f(t + n\tau_N)^2 \right) \\ &\quad - \frac{1}{2\pi} \tau_N \left(\frac{1}{2N+1} \sum_{n=0}^{2N} f(t + n\tau_N) \right)^2 \end{aligned}$$

Because for arbitrary C^1 function g the inequality

$$\left| \frac{1}{2N+1} \sum_{i=0}^{2N} g(t + \frac{i}{2N+1}) - \int_0^1 g(t) dt \right| \leq \frac{1}{2(2N+1)} \|g'\|_\infty$$



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holds, applying the inequality to f^2 and f gives

$$\left| 1 - \frac{\|C(SW_{2N,\tau_N} f(t))\|}{2N+1} \right| \leq \frac{C}{N} \|f'\|_\infty (1 + \|f\|_\infty)$$

On the other hand,

$$\|C(x) - x\| = \frac{1}{\sqrt{2N+1}} |\langle x, 1 \rangle|$$

so

$$\|C(SW_{2N,\tau_N} f(t)) - SW_{2N,\tau_N} f(t)\| = \frac{1}{\sqrt{2N+1}} \left| \sum_{i=0}^{2N} f(t + i\tau_N) \right| \leq \frac{C}{N^{1/2}} \|f'\|_\infty$$

Combining two results, for $x = SW_{2N,\tau_N} f(t)$,

$$\left\| \frac{C(x)}{\|C(x)\|} - \frac{x}{\sqrt{2N+1}} \right\| \leq \frac{C}{N} \|f'\|_\infty (1 + \|f\|_\infty)$$

as desired. \blacktriangleleft

C.2 Proof of Lemma 12 and Theorem 13

Let us define

$$E_N = C \left(\frac{1}{\sqrt{N}} \|R_N f'\|_2 + \frac{1}{N} (\|f\|_\infty + 1) \|f'\|_\infty \right) + 2 \left(\frac{1}{\|S_N f\|_2} + 1 \right) \|R_N f\|_2.$$

Then by Theorem 11, the inequalities $|c_\alpha^N - c_\alpha| < E_N$, $|\bar{c}_\alpha^N - c_\alpha| < E_N$, and $d_H(\bar{X}_N, \bar{Y}_N) < E_N$ hold. We first prove Lemma 12.

Proof of Lemma 12. As we proved in Appendix C, the inequalities $d_H(\bar{X}_N, \bar{Y}_N) \leq E_N$ and $d_H(\bar{X}_N, \frac{X_N}{\sqrt{2N+1}}) \leq E_N$ hold. Moreover, [14] showed that for N, N' there exists a projection map $P : \mathbb{R}^{N'} \rightarrow \mathbb{R}^{N'}$ and an isometry $Q : P(\mathbb{R}^{N'}) \rightarrow \mathbb{R}^N$ such that $Q \circ P(\bar{Y}_{N'}) = \bar{Y}_N$. Thus, [14] concluded that the persistence diagrams of \bar{X}_N and \bar{Y}_N are Cauchy with respect to the bottleneck distance, converging to the same limit.

From $d_H(\bar{X}_N, \frac{X_N}{\sqrt{2N+1}}) \leq E_N$ and $E_N \rightarrow 0$ as $N \rightarrow \infty$, we can conclude that $\mathcal{P}(\frac{1}{\sqrt{2N+1}} X_N)$ converges to the same limit.

Finally, the same logic can be applied to \bar{M}_N . The only subtle point is that the object is infinite, but the inequalities $d_H(\bar{X}_N, \bar{Y}_N) \leq E_N$ and $d_H(\bar{X}_N, \frac{X_N}{\sqrt{2N+1}}) \leq E_N$ hold pointwise, so $d_H(\bar{M}_N, \frac{1}{\sqrt{2N+1}} M_N) \leq E_N$ and $d_H(\bar{M}_N, \bar{M}_N^Y) \leq E_N$ hold. Here we denote the support of the truncated version by M^Y . Finally, the isometry and projection yield the same property, thus concluding Lemma 12. \blacktriangleleft

The next two lemmas play an important role in proving Theorem 13.

► **Lemma 24.** *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a $C^{1,1}$ function. For a set of n sampled points $T \subset \mathbb{T}$, define*

$$\epsilon_N(T) := \min_{\{i,j\} \neq \{k,l\}} \left| \|\bar{x}_{N,i} - \bar{x}_{N,j}\| - \|\bar{x}_{N,k} - \bar{x}_{N,l}\| \right|,$$

where $\bar{x}_{N,i}, \bar{x}_{N,j}, \bar{x}_{N,k}, \bar{x}_{N,l} \in \bar{X}_N$. Then for all $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ depending only on n and ϵ such that if $N \geq N_0$,

$$P(\epsilon_N(T) < 3E_N) < \epsilon.$$

Proof. By the maximal argument,

$$P(\epsilon_N(T) < a) \leq n^4 P(|\|\bar{x}_{N,i} - \bar{x}_{N,j}\| - \|\bar{x}_{N,k} - \bar{x}_{N,l}\|| < a).$$

So we can analyze this probability by randomly sampling four points from \bar{M}_N . Therefore,

$$P(\epsilon_N(T) < a) \leq n^4 P(|\|\bar{x}_1 - \bar{x}_2\| - \|\bar{x}_3 - \bar{x}_4\|| < a).$$

To handle this probability, we transform the points into the centralized, normalized versions of the sliding window for the **truncated** function. Let $\{y_i\}_{i=1,2,3,4}$ be the points corresponding to $\{x_i\}_{i=1,2,3,4}$, where $x_i = SW_{2N,\tau_N} f(t_i)$ and $y_i = SW_{2N,\tau_N} S_N f(t_i)$. Also, let $\{\bar{x}_i\}_{i=1,2,3,4}$ and $\{\bar{y}_i\}_{i=1,2,3,4}$ denote the centralized, normalized points of $\{x_i\}$ and $\{y_i\}$ respectively.

Based on our previous results, $\|\bar{x}_i^N - \bar{x}_j^N\| - \|\bar{y}_i^N - \bar{y}_j^N\| < E_N$ holds. Thus,

$$P(\epsilon_N(T) < a) \leq n^4 P(|\|\bar{x}_1 - \bar{x}_2\| - \|\bar{x}_3 - \bar{x}_4\|| < a) \leq n^4 P\left(|\|\bar{y}_1 - \bar{y}_2\| - \|\bar{y}_3 - \bar{y}_4\|| < a + 2E_N\right)$$

Since we assumed $\hat{f}(0) = 0$, $\|f\|_2 = 1$, $f = \sum_{m=0}^{\infty} a_m \cos(mt) + b_m \sin(mt)$ will satisfy $a_0 = b_0 = 0$. $\bar{y}_N = N \circ C \circ SW_{2N,\tau_N} S_N f(t)$ can be expressed as

$$\bar{y}_N = \sum_{m=1}^N (a_m \cos(mt) + b_m \sin(mt)) \frac{\mathbf{u}_m}{\|\mathbf{u}_m\|} + (b_m \cos(mt) - a_m \sin(mt)) \frac{\mathbf{v}_m}{\|\mathbf{v}_m\|}$$

Thus,

$$\|\bar{y}_1^N - \bar{y}_2^N\| = \sum_{m=1}^N (a_m^2 + b_m^2)(2 - 2 \cos(m(t_1 - t_2)))$$

Let $r_m^2 = a_m^2 + b_m^2$ and $g_N(t) = \sum_{m=1}^N r_m^2 - r_1^2 \cos(t) - r_2^2 \cos(2t) - \cdots - r_N^2 \cos(Nt)$, $g(t) = 1 - \sum_{m=1}^{\infty} r_m^2 \cos(mt)$. Since f is a $C^{1,1}$ function, $r_n = o(\frac{1}{n})$ holds so $g_N(t)$ uniformly converges to $g(t)$. Then $\|\bar{y}_1^N - \bar{y}_2^N\| = 2g_N(t_1 - t_2)$ so,

$$P(\epsilon_N(T) < a) \leq n^4 P\left(|g_N(t_1 - t_2) - g_N(t_3 - t_4)| < \frac{a}{2} + E_N\right)$$

Substituting $a = 3E_N$,

$$P(\epsilon_N(T) < 3E_N) \leq n^4 P\left(|g_N(t_1 - t_2) - g_N(t_3 - t_4)| < \frac{5}{2}E_N\right) \leq 2\pi n^4 P\left(|g_N(t) - g_N(s)| < \frac{5}{2}E_N\right)$$

because $P_{m \times m}(|t_1 - t_2| \in A) \leq 2\pi P_m(|t| \in A)$ holds for Lebesgue measure m on $[0, 2\pi]$. Using $|g_N(t) - g(t)| \leq \|R_N f\|_2^2 < E_N$,

$$P\left(\epsilon_N(T) < 3E_N\right) \leq 2n^4 P\left(|g(t) - g(s)| < \frac{9}{2}E_N\right)$$

By our assumption on g , $P(|g(t) - g(s)| = 0) = 0$. Therefore, for any $\epsilon > 0$ there exists $\delta > 0$ that $P(|g(t) - g(s)| < \delta) < \frac{\epsilon}{2n^4}$. Thus for $N \geq N_0$, so that $\frac{9}{2}E_N < \delta$,

$$P\left(\epsilon_N(T) < 3E_N\right) < \epsilon$$



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► **Lemma 25.** Let $f : T \rightarrow \mathbb{R}$ a $C^{1,1}$ function. For n -sampled points $T \subset \mathbb{T}$, define

$$\epsilon(T) := \min_{\{i,j\} \neq \{k,l\}} \left| \lim_{N \rightarrow \infty} \|\bar{x}_{N,i} - \bar{x}_{N,j}\| - \lim_{N \rightarrow \infty} \|\bar{x}_{N,k} - \bar{x}_{N,l}\| \right|$$

then $\epsilon(T) > 0$ with probability 1.

Proof. Note that since $\|\bar{x}_{N,1} - \bar{x}_{N,2}\|$ for $t_1, t_2 \in \mathbb{T}$ forms Cauchy sequence, the definition of $\epsilon(T)$ is well defined.

Since

$$\left| \left(\lim_{N' \rightarrow \infty} \|\bar{x}_{N',1} - \bar{x}_{N',2}\| \right) - \|\bar{x}_{N,1} - \bar{x}_{N,2}\| \right| < E_N$$

$|\epsilon_N(T) - \epsilon(T)| < 2E_N$. Thus, for $\epsilon > 0$, finding N_0 as Lemma 24, $\epsilon(T) > E_{N_0}$ with probability $1 - \epsilon$. Since ϵ is arbitrary, $\epsilon(T) > 0$ with probability 1. ◀

Proof of Theorem 13. For $\epsilon > 0$, let N_0 as Lemma 24. Then, we can decompose

$$P\left(d_B(\mathcal{P}_\infty(\bar{X}), \mathcal{P}_\infty(\bar{M}_N)) > c_\alpha\right) \leq P\left(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}_\infty(\bar{X})) > \frac{1}{3}\epsilon(T)\right) \quad (1)$$

$$+ P\left(d_B(\mathcal{P}_\infty(\bar{M}), \mathcal{P}(\bar{M}_N)) > \frac{1}{3}\epsilon(T)\right) \quad (2)$$

$$+ P\left(|c_\alpha^N - c_\alpha| > \frac{1}{3}\epsilon(T)\right) \quad (3)$$

$$+ P\left(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}(\bar{M}_N)) > c_\alpha^N - \epsilon(T)\right) \quad (4)$$

By Lemma 25, $\epsilon(T) > 0$ with probability 1. As $N \rightarrow \infty$, first three entries becomes zero. So there exists $N_1 > 0$ such that if $N > N_1$,

$$P\left(d_B(\mathcal{P}_\infty(\bar{X}), \mathcal{P}_\infty(\bar{M}_N)) > c_\alpha\right) \leq P\left(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}(\bar{M}_N)) > c_\alpha^N - \epsilon(T)\right)$$

We claim that if $\epsilon_N(T) \geq 3E_N$ then $c_\alpha^N - \epsilon(T) \geq c_{\alpha+3b/n}^N$ with at least probability $1 - \epsilon$. Above inequality is equivalent to

$$L_N\left(\frac{1}{2}(c_\alpha^N - \epsilon(T))\right) \leq \alpha + \frac{3b}{n}$$

To prove this, we count possible cases of subsamples that satisfies

$$\frac{1}{2}(c_\alpha^N - \epsilon(T)) < d_H(\bar{X}_{N,b}^{(j)}, \bar{X}_N) < \frac{1}{2}c_\alpha^N$$

Assume

$$\frac{1}{2}(c_\alpha^N - \epsilon(T)) < d_H(\bar{X}_{N,b}^{(j_1)}, \bar{X}_N) < d_H(\bar{X}_{N,b}^{(j_2)}, \bar{X}_N) < \dots < d_H(\bar{X}_{N,b}^{(j_M)}, \bar{X}_N) < \frac{1}{2}c_\alpha^N$$

then $(M-1)\epsilon_N(T) < \frac{1}{2}\epsilon(T)$ holds so

$$M < 1 + \frac{\epsilon(T)}{2\epsilon_N(T)} < 1 + \frac{\epsilon_N(T) + 2E_N}{2\epsilon_N(T)} = \frac{3}{2} + \frac{E_N}{\epsilon_N(T)} < 2$$

Thus if such $\frac{1}{2}(c_\alpha^N - \epsilon(T)) < d_H(\bar{X}_{N,b}^{(j)}, \bar{X}_N) < \frac{1}{2}c_\alpha^N$ subsample exists, its Hausdorff distance value can exist at most 1.

Now, such Hausdorff distance emerges as a distance of two points in \bar{X}_N . Every distance of two point will be different with probability 1, so the maximum number of subsamples that can have the distance is at most

$$\binom{n}{b} - \binom{n-2}{b} - \binom{n-2}{b-2} < \frac{3b}{n} \binom{n}{b}$$

since that two point need to locate at different group.

Thus the function L_N can vary by at most $\frac{3b}{n}$ so the claim holds. Therefore, for $N > \max\{N_0, N_1\}$

$$P\left(d_B(\mathcal{P}_\infty(\bar{X}), \mathcal{P}_\infty(\bar{M})) > c_\alpha\right) \leq P\left(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}(\bar{M}_N)) > c_\alpha^N - \epsilon(T)\right) \quad (5)$$

$$\leq P(\epsilon_N(T) < 3E_N) \quad (6)$$

$$+ P\left(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}(\bar{M}_N)) > c_\alpha^N - \epsilon(T), \epsilon_N(T) > 3E_N\right) \quad (7)$$

$$\leq \epsilon + P(d_B(\mathcal{P}(\bar{X}_N), \mathcal{P}(\bar{M}_N)) > c_{\alpha+3b/n}^N) \quad (8)$$

$$\leq \alpha + \epsilon + \frac{3b}{n} + O\left(\frac{b}{n}\right)^{1/4} \quad (9)$$

since ϵ is arbitrary,

$$P\left(d_B(\mathcal{P}_\infty(\bar{X}), \mathcal{P}_\infty(\bar{M})) > c_\alpha\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

As a consequence,

$$\lim_{N \rightarrow \infty} P\left(\frac{1}{\sqrt{2N+1}} d_B(\mathcal{P}(X_N), \mathcal{P}(M_N)) > c_\alpha\right) \leq \alpha + O\left(\frac{b}{n}\right)^{1/4}$$

◀

D Proofs for Section 5

Proof of Theorem 15. We imitate the proof of Theorem 7. Let $\tau_N = \frac{\tau_0}{N}$ and assume there exists $x_1^N \neq x_2^N \in K$ such that $SW_{N,\tau_N} f(x_1^N) = SW_{N,\tau_N} f(x_2^N)$. After taking a subsequence, $x_1^N \rightarrow x_1$ and $x_2^N \rightarrow x_2$. Then for any $x \in (x_1, x_1 + \tau_0)$ there exists sequence $\{x_1^N + i_N \tau_N\}$ converging to x . Thus, $SW_{N,\tau_N} f(x) = SW_{N,\tau_N} f(x + t_2 - t_1)$. Since $x \in (x_1, x_1 + \tau)$ is an interval, we can derivate both term to get $SW_{N,\tau_N} f'(x) = SW_{N,\tau_N} f'(x + t_2 - t_1)$ which is only possible when $t_1 = t_2$.

If there exists an sequence $t_1^N \neq t_2^N \rightarrow t_1$ but $SW_{N,\tau_N} f(t_1^N) = SW_{N,\tau_N} f(t_2^N)$. This is impossible since for large N , $SW_{N,\tau_N} f'(t_1) \neq 0$. Thus $SW_{N,\tau_N} f(K)$ is embedded manifold homeomorphic to interval. ◀

▷ **Claim 26.** Suppose $f \in C^1$, and f satisfies that for all t , either $|f(t)| \geq \epsilon$ or $|f'(t)| \geq \epsilon$ holds. Then then there exists some constant $C_\tau > 0$ such that

$$\|SW_{m,\tau} f(t)\|_2 \geq \sqrt{m+1} (\epsilon - C_\tau \tau),$$

and $C_\tau \rightarrow 0$ as $\tau \rightarrow 0$.



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Proof. Note that

$$SW_{m,\tau}f(t) = \begin{bmatrix} f(t) \\ f(t + \tau) \\ \vdots \\ f(t + m\tau) \end{bmatrix},$$

and hence

$$\|SW_{m,\tau}f(t)\|_2^2 = \sum_{j=0}^m (f(t + j\tau))^2.$$

For t, t_0 , let $R_1(f)(t)$ be the Taylor remainder term of f , i.e.,

$$R_1(f)(t) = f(t) - f(t_0) - (t - t_0)f'(t_0),$$

then by Taylor Remainder theorem, there exists some function $h : (0, \infty) \rightarrow (0, \infty)$ with $\lim_{x \rightarrow 0} h(x) = 0$ such that

$$|R_1(f)(t)| \leq h(|t - t_0|) |t - t_0|.$$

Now suppose that $t + j\tau$ and $t_{0,j}$ satisfy that $|t + j\tau - t_{0,j}| < \delta$. Then

$$\begin{aligned} |f(t + j\tau)| &= |f(t_{0,j}) + (t + j\tau - t_{0,j})f'(t_{0,j}) + R_1(f)(t + j\tau)| \\ &\geq |f(t_{0,j}) + (t + j\tau - t_{0,j})f'(t_{0,j})| - |R_1(f)(t + j\tau)|. \end{aligned}$$

Then

$$|R_1(f)(t + j\tau)| \leq \frac{L_2}{2} |t + j\tau - t_0|^2 \leq h(\delta) \delta,$$

And hence

$$|f(t + j\tau)| \geq |f(t_{0,j}) + (t + j\tau - t_0)f'(t_{0,j})| - h(\delta) \delta.$$

Therefore,

$$\begin{aligned} \|SW_{m,\tau}f(t)\|_2 &= \sqrt{\sum_{j=0}^m (f(t + j\tau))^2} \\ &\geq \sqrt{\sum_{j=0}^m (f(t_{0,j}) + (t + j\tau - t_0)f'(t_{0,j}))^2} - \sqrt{\sum_{j=0}^m (h(\delta) \delta)^2}. \end{aligned}$$

Now, we choose $\{t_{0,j}\}$ as follows: let $t'_0 < \dots < t'_{m'}$ be set as $t'_1 = t + \delta, t'_2 = t + 3\delta, \dots, t'_{m'} = t + (2m' - 1)\delta$, where $2m'\delta = m\tau$. Assume that $\frac{m\tau}{2\delta}$ is an integer for convenience. Then, the intervals $\{B_i\}$, where $B_i = [t'_i - \delta, t'_i + \delta]$ with exception $B_{m'} = [t'_{m'} - \delta, t'_{m'} + \delta]$ (i.e., only the last one is closed) are nonoverlapping and contains at least $\lfloor \frac{m+1}{m'} \rfloor \geq \frac{2\delta}{\tau}$ of $t + j\tau$'s. Then

$$\sum_{j=0}^m (f(t_{0,j}) + (t + j\tau - t_0)f'(t_{0,j}))^2 = \sum_{i=1}^{m'} \sum_{j:t+j\tau \in B_i} (f(t'_i) + (t + j\tau - t_0)f'(t'_i))^2.$$

Then for each i , either $|f(t'_i)| \geq \epsilon$ or $|f'(t'_i)| \geq \epsilon'$:

Suppose $|f(t'_i)| \geq \epsilon$ holds. Then,

$$\sum_{j:t+j\tau \in B_i} (f(t'_i) + (t + j\tau - t_0)f'(t'_i))^2 \geq |B_i| \epsilon^2.$$

And suppose $|f'(t'_i)| \geq \epsilon'$ holds. Then,

$$\sum_{j:t+j\tau \in B_i} (f(t'_i) + (t + j\tau - t_0)f'(t'_i))^2 \geq \frac{|B_i|^3}{12} \epsilon^2 \geq \frac{\epsilon^2 \delta^2}{3\tau^2}.$$

Hence for either cases,

$$\begin{aligned} & \sqrt{\sum_{j=0}^m \left(f(t_1 + \frac{m}{2}\tau) - f(t_2 + \frac{m}{2}\tau) + (j - \frac{m}{2})\tau \left(f'(t_1 + \frac{m}{2}\tau) - f'(t_2 + \frac{m}{2}\tau) \right) \right)^2} \\ & \geq \sqrt{\sum_{i=1}^{m'} |B_i| \min \left\{ \epsilon^2, \frac{\epsilon^2 \delta^2}{3\tau^2} \right\}} \\ & = \sqrt{m+1} \min \left\{ \epsilon, \frac{\epsilon \delta}{\sqrt{3}\tau} \right\}. \end{aligned}$$

And hence

$$\|SW_{m,\tau}f(t)\|_2 \geq \sqrt{m+1} \left(\min \left\{ \epsilon, \frac{\epsilon \delta}{\sqrt{3}\tau} \right\} - h(\delta)\delta \right).$$

This holds for any δ with $\frac{2\delta}{\tau} \geq 1$. Hence by conveniently choosing $\delta = \sqrt{3}\tau$ gives that

$$\|SW_{m,\tau}f(t)\|_2 \geq \sqrt{m+1} (\epsilon - C_\tau \tau),$$

and $C_\tau \rightarrow 0$ as $\tau \rightarrow 0$.

◀

▷ **Claim 27.** If f is ϵ -non-periodic, then for t_1 and t_2 with $|t_1 - t_2| \geq \frac{\pi\epsilon}{2 \sup_{t_1 \leq t \leq t_2} \|f'(t)\|_2}$,

$$\|SW_{m,\tau}f(t_1) - SW_{m,\tau}f(t_2)\|_2 \geq \sqrt{m+1} (\min\{\epsilon, \epsilon'\} - C_\tau \tau),$$

and $C_\tau \rightarrow 0$ as $\tau \rightarrow 0$.

Proof. Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(t) = f(t_1 + t) - f(t_2 - t)$. Then $g \in C^1$, and g satisfies that for all t , either $|g(t)| \geq \epsilon$ or $|g'(t)| \geq \epsilon'$ holds. Also,

$$\|SW_{m,\tau}f(t_1) - SW_{m,\tau}f(t_2)\|_2 = \|SW_{m,\tau}g(0)\|_2.$$

Then from above claim,

$$\|SW_{m,\tau}g(0)\|_2 \geq \sqrt{m+1} (\min\{\epsilon, \epsilon'\} - C_\tau \tau),$$

and $C_\tau \rightarrow 0$ as $\tau \rightarrow 0$.

◀

▷ **Claim 28.** Suppose f satisfies that for all t , either $|f'(t)| \geq \delta$ or $|f''(t)| \geq \delta$ holds. Then

$$\|(SW_{m,\tau}f)'\|_2 \geq \sqrt{m+1} (\delta - C_\tau \tau),$$

where C_τ is from above Claim.



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Proof. Observe that

$$(SW_{m,\tau}f)' = (SW_{m,\tau}f'),$$

and then it is trivial from the above Claim. \blacktriangleleft

\triangleright Claim 29. Let γ be arc-length parametrization of $SW_{m,\tau}f$. Then

$$\|\gamma''\|_2 \leq \frac{2 \|(SW_{m,\tau}f)''\|_2}{\|(SW_{m,\tau}f)'\|_2^2}.$$

In particular, suppose f satisfies that for all t , either $|f'(t)| \geq \delta$ or $|f''(t)| \geq \delta$ holds, and $\|f''(t)\|_2 \leq L_2$. Then

$$\|\gamma''\|_2 \leq \frac{2L_2}{\sqrt{m+1}(\delta - C_\tau\tau)^2},$$

where C_τ is from above Claim.

Proof. Let $\gamma(s) = SW_{m,\tau}f(t(s))$, then

$$\frac{d\gamma}{ds} = \frac{dSW_{m,\tau}f}{dt} \frac{dt}{ds} = (SW_{m,\tau}f)' \frac{dt}{ds}$$

Then $\left\| \frac{d\gamma}{ds} \right\|_2 = 1$, so

$$\frac{dt}{ds} = \|(SW_{m,\tau}f)'\|_2^{-1},$$

and

$$\frac{d\gamma}{ds} = \frac{(SW_{m,\tau}f)'}{\|(SW_{m,\tau}f)'\|_2}.$$

And then,

$$\frac{d^2\gamma}{ds^2} = \frac{\frac{d}{ds}(SW_{m,\tau}f)' \|(SW_{m,\tau}f)'\|_2 - (SW_{m,\tau}f)' \frac{d}{ds} \|(SW_{m,\tau}f)'\|_2}{\|(SW_{m,\tau}f)'\|_2^2}.$$

Then

$$\frac{d}{ds}(SW_{m,\tau}f)' = \frac{d}{dt}(SW_{m,\tau}f)' \frac{dt}{ds} = \frac{(SW_{m,\tau}f)''}{\|(SW_{m,\tau}f)'\|_2},$$

and

$$\frac{d}{ds} \|(SW_{m,\tau}f)'\|_2 = \frac{d}{dt} \|(SW_{m,\tau}f)'\|_2 \frac{dt}{ds} = \frac{\frac{d}{dt} \|(SW_{m,\tau}f)'\|_2^2}{2 \|(SW_{m,\tau}f)'\|_2^2}.$$

And hence

$$\frac{d^2\gamma}{ds^2} = \frac{(SW_{m,\tau}f)''}{\|(SW_{m,\tau}f)'\|_2^2} - \frac{(SW_{m,\tau}f)' \frac{d}{dt} \|(SW_{m,\tau}f)'\|_2^2}{2 \|(SW_{m,\tau}f)'\|_2^4}.$$

Now,

$$\frac{d}{dt} SW_{m,\tau} f(t) = \begin{bmatrix} f'(t) \\ f'(t+\tau) \\ \vdots \\ f'(t+m\tau) \end{bmatrix}, \quad \frac{d^2}{dt^2} SW_{m,\tau} f(t) = \begin{bmatrix} f''(t) \\ f''(t+\tau) \\ \vdots \\ f''(t+m\tau) \end{bmatrix},$$

and hence

$$\begin{aligned} \|(SW_{m,\tau} f)'\|_2^2 &= \sum_{j=0}^m f'(t+j\tau)^2, \\ \frac{d}{dt} \|(SW_{m,\tau} f)'\|_2^2 &= \sum_{j=0}^m 2f'(t+j\tau)f''(t+j\tau) = 2 \langle (SW_{m,\tau} f)', (SW_{m,\tau} f)'' \rangle. \end{aligned}$$

and therefore,

$$\frac{d^2\gamma}{ds^2} = \frac{(SW_{m,\tau} f)''}{\|(SW_{m,\tau} f)'\|_2^2} - \frac{(SW_{m,\tau} f)' \langle (SW_{m,\tau} f)', (SW_{m,\tau} f)'' \rangle}{\|(SW_{m,\tau} f)'\|_2^4}.$$

And then by Cauchy-Schwarz,

$$\|\gamma''\|_2 \leq \frac{\|(SW_{m,\tau} f)''\|_2}{\|(SW_{m,\tau} f)'\|_2^2} + \frac{\|(SW_{m,\tau} f)'\|_2^2 \|(SW_{m,\tau} f)''\|_2}{\|(SW_{m,\tau} f)'\|_2^4} = \frac{2 \|(SW_{m,\tau} f)''\|_2}{\|(SW_{m,\tau} f)'\|_2^2}.$$

Then

$$\|(SW_{m,\tau} f)''\|_2 = \sqrt{\sum_{j=0}^m f''(t+j\tau)^2} \leq L_2 \sqrt{m+1}.$$

Then from above Claim,

$$\|(SW_{m,\tau} f)'(t)\|_2 \geq (m+1)(\delta - C_\tau \tau)^2.$$

And hence we have that

$$\begin{aligned} \|\gamma''\|_2 &\leq \frac{2 \|(SW_{m,\tau} f)''\|_2}{\|(SW_{m,\tau} f)'\|_2^2} \\ &\leq \frac{2L_2}{\sqrt{m+1}(\delta - C_\tau \tau)^2}. \end{aligned}$$

◀

► **Lemma 30.** Suppose $f \in C^2$ satisfies Assumption 2 and ϵ -non-periodic. Then $SW_{m,\tau} f$ is injective and

$$\text{reach}(SW_{m,\tau} f(\mathbb{T})) \geq \frac{\sqrt{m+1}}{2} \min \left\{ (\epsilon - C_\tau \tau), \frac{(\delta - C_\tau \tau)^2}{L_2} \right\},$$

where C_τ is from above Claim.



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Proof. Let $\gamma : [0, S_{\max}]$ be arc-length parametrization of $SW_{m,\tau}f$ on \mathbb{T} . The reach of γ is lower bounded by R , when following condition holds [1]:

$$\text{for all } s \in \mathbb{R}, \quad \|\gamma''(s)\|_2 \leq \frac{1}{R},$$

$$\text{for all } s_1, s_2 \in \mathbb{R}, \quad |s_1 - s_2| \geq \pi R \text{ implies } \|\gamma(s_1) - \gamma(s_2)\|_2 \geq 2R.$$

Above Claim implies that

$$\|\gamma''(s)\|_2 \leq \frac{2L_2}{\sqrt{m+1}(\delta - C_\tau\tau)^2}.$$

Also, $\|(SW_{m,\tau}f)'\|_2$ can be bounded as

$$\|(SW_{m,\tau}f)'\|_2 \leq L_1\sqrt{m+1},$$

hence for $s_1, s_2 \in \mathbb{R}$, $|s_1 - s_2| \geq \pi \frac{\sqrt{m+1}}{2} (\min\{\epsilon, \epsilon'\} - C_\tau\tau)$ implies that

$$|t(s_1) - t(s_2)| \geq \frac{|s_1 - s_2|}{L_1\sqrt{m+1}} = \frac{\pi}{2L_1} (\epsilon - C_\tau\tau).$$

Hence from (ϵ, ϵ') -non-periodic condition, we have that

$$\|\gamma(s_1) - \gamma(s_2)\|_2 \geq \sqrt{m+1} (\epsilon - C_\tau\tau).$$

Hence we have that

$$\text{reach}(\gamma) \geq \frac{\sqrt{m+1}}{2} \min \left\{ \epsilon - C_\tau\tau, \frac{(\delta - C_\tau\tau)^2}{L_2} \right\}.$$

Also, γ satisfies that $|s_1 - s_2| \geq \pi R$ implying $\|\gamma(s_1) - \gamma(s_2)\|_2 \geq 2R$ implies that γ is injective, hence $SW_{m,\tau}f$ is injective as well. \blacktriangleleft

Proof of Theorem 2. Above Lemma implies that $SW_{m,\tau}f$ is injective on \mathbb{T} and

$$\text{reach}(SW_{m,\tau}f(\mathbb{T})) \geq \frac{\sqrt{m+1}}{2} \min \left\{ (\min\{\epsilon, \epsilon'\} - C_\tau\tau), \frac{(\min\{\delta', \delta''\} - C_\tau\tau)^2}{L_2} \right\}.$$

Hence for any $t < \frac{\sqrt{m+1}}{2} \min \left\{ (\min\{\epsilon, \epsilon'\} - C_\tau\tau), \frac{(\min\{\delta', \delta''\} - C_\tau\tau)^2}{L_2} \right\}$, $(SW_{m,\tau}f(\mathbb{T}))^t$ deformation retracts to $SW_{m,\tau}f(\mathbb{T})$, which is homeomorphic to the interval \mathbb{T} . Hence $(SW_{m,\tau}f(\mathbb{T}))^t$ is contractible. \blacktriangleleft

► **Lemma 31.** Let $\Xi > 0$, and Suppose $f \in C^2$ satisfies Assumption 2 and (Ξ, ϵ) -periodic. Then $SW_{m,\tau}f$ is injective on $[0, \Xi]$, and

$$\text{reach}(SW_{m,\tau}f(\mathbb{T})) \geq \frac{\sqrt{m+1}}{2} \min \left\{ (\epsilon - C_\tau\tau), \frac{(\delta - C_\tau\tau)^2}{L_2} \right\},$$

where C_τ is from above Claim.

Proof. Let $\gamma : [0, S_{\max}]$ be arc-length parametrization of $SW_{m,\tau}f$ on $[0, \Xi]$. The reach of γ is lower bounded by R , when following condition holds [1]:

$$\text{for all } s \in \mathbb{R}, \quad \|\gamma''(s)\|_2 \leq \frac{1}{R},$$

for all $s_1, s_2 \in \mathbb{R}$, $|s_1 - s_2| \geq \pi R$ and $S_{\max} - |s_1 - s_2| \geq \pi R$ implies $\|\gamma(s_1) - \gamma(s_2)\|_2 \geq 2R$.

Above Claim implies that

$$\|\gamma''(s)\|_2 \leq \frac{2L_2}{\sqrt{m+1}(\delta - C_\tau\tau)^2}.$$

Also, $\|(SW_{m,\tau}f)'\|_2$ can be bounded as

$$\|(SW_{m,\tau}f)'\|_2 \leq L_1\sqrt{m+1},$$

hence for $s_1, s_2 \in \mathbb{R}$, $|s_1 - s_2| \geq \pi \frac{\sqrt{m+1}}{2} (\min\{\epsilon, \epsilon'\} - C_\tau\tau)$ and $|s_1 - s_2| \leq S_{\max} - \pi R$ implies that

$$\min_{n \in \mathbb{N}} |t(s_1) - t(s_2) - n\Xi| \geq \frac{|s_1 - s_2|}{L_1\sqrt{m+1}} = \frac{\pi}{2L_1} (\epsilon - C_\tau\tau).$$

Hence from (Ξ, ϵ) -periodic condition, we have that

$$\|\gamma(s_1) - \gamma(s_2)\|_2 \geq \sqrt{m+1}(\epsilon - C_\tau\tau).$$

Hence we have that

$$\text{reach}(\gamma) \geq \frac{\sqrt{m+1}}{2} \min \left\{ \epsilon - C_\tau\tau, \frac{(\delta - C_\tau\tau)^2}{L_2} \right\}.$$

Also, γ satisfies that $|s_1 - s_2| \geq \pi R$ and $S_{\max} - |s_1 - s_2| \geq \pi R$ implying $\|\gamma(s_1) - \gamma(s_2)\|_2 \geq 2R$ implies that γ is injective on $[0, S_{\max}]$, hence $SW_{m,\tau}f$ is injective on $[0, \Xi]$ as well. ◀

Proof of Theorem 18. Above Lemma implies that $SW_{m,\tau}f$ is injective on $[0, \Xi]$ and

$$\text{reach}(SW_{m,\tau}f([0, \Xi])) \geq \frac{\sqrt{m+1}}{2} \min \left\{ (\min\{\epsilon, \epsilon'\} - C_\tau\tau), \frac{(\min\{\delta', \delta''\} - C_\tau\tau)^2}{L_2} \right\}.$$

Since $f(0) = f(\Xi)$, $SW_{m,\tau}f(0) = SW_{m,\tau}f(\Xi)$ as well. Hence $SW_{m,\tau}f$ can be understood as a homeomorphism from S^1 to $SW_{m,\tau}f([0, \Xi])$. Hence for any

$$t < \frac{\sqrt{m+1}}{2} \min \left\{ (\epsilon - C_\tau\tau), \frac{(\delta - C_\tau\tau)^2}{L_2} \right\},$$

$(SW_{m,\tau}f(\mathbb{T}))^t$ deformation retracts to $SW_{m,\tau}f(\mathbb{T})$, which is homeomorphic to the circle S^1 . ◀