Convex Optimization

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Contents

Part I
Convex Optimization Problem

1 Introduction to Convex Optimization Problem

In this section, we define several things to define what Convex Optimization Problem is.

1.1 Convex Optimization Problem

$$\begin{aligned} &\textit{minimize} \;\; f_0(x) \\ &\textit{subject to} \;\; f_i(x) \leq b_i, \quad i = 1, \cdots, m \end{aligned}$$

where objective function, constraint functions are convex : $f_0, f_1, \cdots, f_m$. There are some definitions that make discussing Convex Optimization Problem concretely.

1.2 Convex Sets

1.2.1 Affine Set, Convex set

Affine set is set containing all line through any two distinct points in the set. Solution space of $Ax = b$ is Affine set. Hyperplane is also Affine set.
Convex set is set containing all segments through any two distinct points in the set. The word segment means for two points x_1, x_2 then all points in form $\theta x_1 + (1 - \theta)x_2$ where $0 \leq \theta \leq 1$

Not every set is convex set. But we can define convex set containing set S. This is called **convex hull** which contains all convex combinations (Convex combination of $x_1, \cdots x_k$ is $x = \theta_1 x_1 + \cdots \theta_k x_k$)

1.2.2 Convex Cone

Convex cone is set containing all conic combination in the set. The word conic combination means for $x_1, x_2, \cdots x_k$ is $x = \theta_1 x_1 + \theta_2 x_2 + \cdots \theta_k x_k$ which $\theta_1 \geq 0,\ \theta_2 \geq 0 \cdots \theta_k \geq 0$

1.2.3 Hyperplane and Halfspace

Hyperplane is set of form $\{x|a^T x = b\}$
Halfspace is set of the form $\{x|a^T x \leq b\}$
Hyperplanes are affine set, convex set. Halfspaces are convex.

1.2.4 Euclidean balls and Elipsoids

Euclidean Ball is set of form $B(x, r) = \left\{y \middle| ||y - x||_2 \leq r\right\}$
Ellipsoid is a set of form $\left\{y \middle| (y - x)^T P^{-1}(y - x) \leq 1\right\}$ for P symmetric positive definite.

1.2.5 Norm balls and Norm cones

This is generalization of balls and cones. We define ball and cone for arbitrary norm.
Norm ball is set of form $\{y \big| ||y-x|| \leq r\}$
Nrom cone is set of form $\{(x,t) \big| ||x|| \leq t\}$
These two norms are arbitrary.

1.3 Operations that preserve convexity

1.3.1 Operations that preserve convexity

1. Intersection
   Intersection of convex sets is convex.

2. Affine function
   Image and preimage of affine function is convex. Affine function is $f(x) = Ax + b$

3. Perspective function, Linear fractional function.
   Perspective function $P : \mathbb{R}^n \times \mathbb{R}^+ \longrightarrow \mathbb{R}^n$, $P(x,t) = x/t$ then images and preimages of convex sets are convex.
   Linear fractional function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s images and preimages of convex set is convex.

$$f(x) = \frac{Ax+b}{c^T x + d}$$

1.3.2 Example

1. Solution of set Matrix inequality.

$$\{x \big| x_1 A_1 + \cdots x_m A_m \prec B\}$$

It's because we can change $x_1, \cdots x_m$ to a vector.

2. Hyperbolic cone.

$$\{x \big| x^T P x \leq (c^T x)^2,\ c^T x \geq 0\}$$

with P symmetric positive definite matrix is convex. Since this is set of form

$$f(x) = \begin{pmatrix} P^{\frac{1}{2}} \\ c^T \end{pmatrix} x$$

Then for $K = \{(x,t) \big| ||x||_2 \leq t\}$, $f^{-1}(K)$ is Hyperbolic cone so it is convex.

1.4 Proper Cone

A convex cone $K \subseteq \mathbb{R}^n$ is proper cone if K is closed, solid (nonempty interior), pointed (contains no line).

Then we can define Generalized Inequality.

$$x \preceq_K y$$

means $y - x \in K$.

$$x \prec_K y$$

means $y - x \in int(K)$
Then we can define minimum element and minimal element.
$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

$$y \in S \Longrightarrow x \preceq_K y$$

$x \in S$ is **the minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \Longrightarrow y = x$$

1.5 Dual Cone

Dual cone of cone K is

$$K^* = \{y | y^T x \geq 0 \quad x \in K\}$$

Dual of proper cones are proper. We can define dual inequality.

$$y \succeq_{K^*} 0 \Longleftrightarrow y^T x \geq 0 \ \ \forall x \succeq_K 0$$

1.6 Separating hyperplane theorem and Supporting hyperplane theorem

1.6.1 Separating hyperplane theorem

If C, D are nonempty disjoint convex sets, there exist $a \neq 0, b$ that $\{x | a^T x = b\}$ separates two sets :

$$a^T x \leq b \quad (x \in C), \qquad a^T x \geq b \quad (x \in D)$$

1.6.2 Supporting hyperplane theorem

Supporting hyperplane is defined at boundary point. Supporting hyperplane to set C at on boundary point x_0 is

$$\{x | a^T x = a^T x_0\}$$

where $a^T x \leq a^T x_0$ for all $x \in C$.

Supporting Hyperplane Theorem is for C convex set, there exists a supporting hyperplane at every boundary point of C.

1.7 Convex functions

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if domain is convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in dom f$, $0 \leq \theta \leq 1$
Examples are

1. Affine function : $f(x) = Ax + b$
2. Exponential : $f(x) = e^{ax}$
3. Powers : x^α for $\alpha \geq 1$ or $\alpha < 0$
4. Negative entropy : $x \log x$
5. Logarithm : $-\log x$

1.8 Convex function testing technique

1. Restriction to the line.

   $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if every $g : \mathbb{R} \to \mathbb{R}$ with $g(t) = f(x + tv)$ is convex for any x in domain, $v \in \mathbb{R}^n$

   Example is $\log detX$ for symmetric positive definite matrix. We can find $\log det(X+tV)$ is convex for t

2. First order condition.

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \cdots \frac{\partial f(x)}{\partial x_n} \right)$$

exists on each domain, then differentiable f with convex domain is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

3. Second order condition.

   If Hessian $\nabla^2 f(x)$ exists, f with convex domain is convex if and only if

$$\nabla^2 f(x) \succeq 0$$

4. Epigraph.

   Epigraph of $f : \mathbb{R}^n \to \mathbb{R}$ is

$$epif = \left\{ (x, t) \in \mathbb{R}^{n+1} \middle| x \in domf, \ f(x) \leq t \right\}$$

f is convex if and only if $epif$ is convex set.

1.9 Operations that preserve convexity

1. Nonnegative multiple, Sum, Composition with Affine function.

2. Pointwise maximum.

   $f_1, f_2 \cdots f_m$ are convex then $f(x) = \max\{f_1(x), f_2(x), \cdots f_m(x)\}$ is convex.

3. Pointwise wupremum.

   $f(x, y)$ is convex in x for each $y \in \mathfrak{A}$ then

$$g(x) = \sup_{y \in \mathfrak{A}} f(x, y)$$

is convex.

4. Composition with scalar functions.

   Composition of $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ then

$$f(x) = h(g(x))$$

is convex if g convex, h convex, $\tilde{h}$ nondecreases or g concave, h convex, $\tilde{h}$ nonincreasing.

5. Vector composition.
   Composition of $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$ then

$$f(x) = h(g(x))$$

is convex if g_i convex, h convex, $\tilde{h}$ nondecreasing in each argument or g_i concave, h convex, $\tilde{h}$ nonincreasing in each argument.

6. Minimization.
   If $f(x, y)$ is convex in (x, y) and C is convex set then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex.

7. Perspective.
   Perspective of function $f : \mathbb{R}^n \to \mathbb{R}$ is function $g : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$

$$g(x, t) = tf(x/t)$$

g is convex if f is convex.

1.10 Conjugate function

Conjugate of a function f is

$$f^*(y) = \sup_{x \in dom f} (y^T x - f(x))$$

2 Convex Optimization Problem

2.1 Standard form of Convex Optimization Problem

Standard form of Optimization Problem is as following :

$$\textit{minimize} \;\; f_0(x)$$

$$\textit{subject to} \;\; f_i(x) \leq 0 \;\; (i = 1, 2, \cdots m), \;\; h_i(x) = 0 \;\; (i = 1, 2, \cdots p)$$

In general, optimal value is written as p^*,

$$p^* = \inf\{f_0(x) | f_i(x) \leq 0 \;\; (i = 1, 2, \cdots m), h_i(x) = 0 \;\; (i = 1, 2, \cdots p)\}$$

Feasibility is there exists $x \in \mathbb{R}^n$ satisfying constraints.

Now, we can say about convex optimization problem. Standard form of convex optimization problem is

$$\textit{minimize} \;\; f_0(x)$$

$$\textit{subject to} \;\; f_i(x) \leq 0 \;\; (i = 1, 2, \cdots m), \;\; a_i^T x = b_i \;\; (i = 1, 2, \cdots p)$$

Where $f_0, f_1, \cdots f_m$ are convex. Convex optimization problem is interestingly, local optimality implies global optimality. Therefore, our goal is finding local optimal value.

2.2 Basic Convex Optimization Problem

There are some gauranteed problems that is solved easily. Some reliable algorithm exists and solve these problems. In other words if we derive some convex optimization problem into this form, we are able to solve it. Although, more complex problem requires more time to solve it, it is important to reduce problem. Hierarchy of optimization problem is as following :

$$(LP) \subset (QP) \subset (SOCP) \subset (SDP)$$

2.2.1 Linear program (LP)

$$\textit{minimize} \;\; c^T x + d$$

$$\textit{subject to} \;\; Gx \preceq h, \;\; Ax = b$$

2.2.2 Quadratic program (QP)

$$\textit{minimize} \;\; \frac{1}{2} x^T P x + q^T x + r$$

$$\textit{subject to} \;\; Gx \preceq h, \;\; Ax = b$$

2.2.3 Quadratically constrained quadratic program (QCQP)

$$\textit{minimize} \;\; \frac{1}{2} x^T P_0 x + q_0^T x + r_0$$

$$\textit{subject to} \;\; \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0 \; (i = 1, 2, \cdots m), \;\; Ax = b$$

2.2.4 Second order cone programming (SOCP)

$$\textit{minimize} \;\; f^T x$$
$$\textit{subject to} \;\; ||A_i x + b_i||_2 \leq c_i^T + d_i \; (i = 1, 2, \cdots m), \; Fx = g$$

2.2.5 Semidefinite program (SDP)

$$\textit{minimize} \;\; c^T x$$
$$\textit{subject to} \;\; x_1 F_1 + x_2 F_2 + \cdots x_n F_n + G \preceq 0, \; Ax = b$$

3 Duality

Convex optimization has 'duality'. Term 'duality' has many different meanings but, in this case we say dual as dual problem. With concerning dual problem, we can gain some ensurence of solving equation. We will look foward on it but, dual problem gives lower bound of optimal value. So if algorithm reached unusual value via dual solution, there might be some error. Also, in some times, dual solution exactly become same with solution. We will look on these properties.

3.1 Lagrange Dual

On the standard form of optimization problem, denote domain $\mathcal{D}$ the intersection of domain of constraints and f_0. $\mathcal{D} = \bigcap_{i=0}^{m} dom f_i \cap \bigcap_{i=1}^{p} dom h_i$

$$minimize \;\; f_0(x)$$

$$subject\ to \;\; f_i(x) \leq 0 \;\; (i = 1, 2, \cdots m), \;\;\; h_i(x) = 0 \;\; (i = 1, 2, \cdots p)$$

Lagrangian of this problem is $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ with domain $\mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

We then define Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ which is

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \end{aligned}$$

if we look at second equality, optimization goal is affine on λ and ν. Therefore, $g(\lambda, \nu)$ is concave on λ, ν. In Lagrange dual function, lower bound property holds.

Lemma 1 (Lower bound property)**.** *If* $\lambda \succeq 0$

$$g(\lambda, \nu) \leq p^*$$

Proof is simple since if $\tilde{x}$ is feasible,

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

3.2 Dual Problem

On the Lagrange dual function, we can define Lagrange dual problem. Lagrange dual problem is

$$minimize \;\; g(\lambda, \nu)$$

$$subject\ to \;\; \lambda \succeq 0$$

We usually denote optimal value as d^*. Clearly by Lower bound property, $d^* \leq p^*$ and gives lower bound.

If we have property $d^* = p^*$, it is very good. We denote if this holds 'strong duality' holds. It does not hold in general but can hold for convex problems sometimes. Conditions that guarantee strong duality is interest. And we call the conditions as **constraint qualifications**

3.3 Slater's constraint qualification

Theorem 2 (Slater's constraint qualification). *Strong duality holds for a convex problem*

$$\textit{minimize} \quad f_0(x)$$

$$\textit{subject to} \quad f_i(x) \leq 0 \quad (i = 1, 2, \cdots m), \quad Ax = b$$

if it is strictly feasible. That is $\exists x \in int(\mathcal{D})$ *so that* $f_i(x) < 0$ *for all* $i = 1, 2, \cdots m$ *and* $Ax = b$*. Also it could be sharpened that if constraint affine, (kind of* $A_1x + b_1 \preceq 0$*) strict inequality does not have to hold.*

Slater's constraint can be applied into Linear Program and Quadratic Program. They satisfies Slater's constraint qualification so, dual solution equals to original solution. We discuss more on 'Examples'.

Next question is are there some conditions of solution for problem that strong duality holding. That is KKT conditions.

3.4 Karush Kuhn Tucker (KKT) conditions

Assume strong duality holds. Denote x^* primal optimal and (λ^*, ν^*) dual optimal. Then following holds.

$$\begin{aligned} f_0(x^*) = g(\lambda^*, \nu^*) &= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\ &\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \\ &\leq f_0(x^*) \end{aligned}$$

Therefore, we could obtain equality if strong duality hold that x^* minimizes $L(x, \lambda^*, \nu^*)$ and $\lambda_i^* f_i(x^*) = 0$ for $i = 1, 2, \cdots m$. So two of one holds.

$$\lambda_i^* = 0 \quad \textit{or} \quad f_i(x^*) = 0$$

This is called **Complementary Slackness**

KKT conditions are kind of converse. KKT conditions says that if $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ satisfies KKT condition, then they are optimal.

Theorem 3 (KKT condition on Convex Optimization problem). *If following condition holds (which is KKT condition) and problem is convex optimization problem then the* $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ *are optimal to original problem*

$$\textit{minimize} \quad f_0(x)$$

$$\textit{subject to} \quad f_i(x) \leq 0 \quad (i = 1, 2, \cdots m), \quad Ax = b$$

and dual problem.

$$\textit{minimize} \quad g(\lambda, \nu)$$

$$\textit{subject to} \quad \lambda \succeq 0$$

1. Primal Constraints : $f_i(\tilde{x}) \leq 0$, $i = 1, 2, \cdots m$ *and* $h_i(x) = 0$, $i = 1, 2, \cdots p$
2. Dual Constraints : $\tilde{\lambda} \succeq 0$
3. Complementary Slackness : $\tilde{\lambda}_i f_i(\tilde{x}) = 0$, $i = 1, 2, \cdots m$
4. Gradient of Lagrangian with respect to x vanishes.

$$\nabla f_0(\tilde{x}) + \sum_{i=1}^{m} \tilde{\lambda}_i \nabla f_i(\tilde{x}) + \sum_{i=1}^{p} \tilde{\nu}_i \nabla h_i(\tilde{x}) = 0$$

3.5 Proof of Slater's constraint qualification

Let us define set $\mathcal{A} \subseteq \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}$ that

$$\mathcal{A} = \{(u, v, t) | \exists x \in \mathcal{D},\ f_i(x) \leq u_i\ (i = 1, \cdots, m),\ h_i(x) = v_i\ (i = 1, \cdots, p),\ f_0(x) \leq t\}$$

this set is just kind of epigraph. Optimal value is

$$p^* = \inf_{(0,0,t) \in \mathcal{A}} t$$

Also, $g(\lambda, \nu)$ is

$$g(\lambda, \nu) = \inf_{(u,v,t) \in \mathcal{A}} (\lambda^T u + \nu^T v + t)$$

This set $\mathcal{A}$ is convex. If given problem is convex. Now define convex set $\mathcal{B}$ that

$$\mathcal{B} = \{(0, 0, s) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} | s < p^*\}$$

Then $\mathcal{A} \cap \mathcal{B} = \phi$ trivially. By Separating Hyperplane Theorem, there exists $(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$ and

$$(u, v, t) \in \mathcal{A} \Longrightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \geq \alpha$$
$$(u, v, t) \in \mathcal{B} \Longrightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \leq \alpha$$

For $\mathcal{A}$, u, t can increase, so $\tilde{\lambda} \succeq 0$ and $\mu \geq 0$. For $\mathcal{B}$, $\mu t \leq \alpha$ for all $t < p^*$ therefore, $\mu p^* \leq \alpha$.
From these facts if $x \in \mathcal{D}$, $(f_1(x), \cdots f_m(x), h_1(x), \cdots h_p(x), f_0(x)) \in \mathcal{A}$ so

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) + \mu f_0(x) \geq \alpha \geq \mu p^*$$

so if $\mu > 0$

$$L(x, \tilde{\lambda}/\mu, \tilde{\nu}/\mu) \geq p^*$$

and minimizing on $x \in \mathcal{D}$ implies $g(\tilde{\lambda}/\mu, \tilde{\nu}/\mu) \geq p^*$ which means $d^* = p^*$
We now consider cases $\mu = 0$. Then

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) \geq 0$$

for all $x \in \mathcal{D}$. Since there is a point $\tilde{x}$ satisfying Slater condition

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) = \sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) < 0$$

if any $\tilde{\lambda}_i > 0$. So $\tilde{\lambda} = 0$. $(\tilde{\lambda}, \tilde{\nu}, \tilde{\mu}) \neq 0$ so $\tilde{\nu} \neq 0$. However this is contradiction because $\tilde{\nu}^T (Ax - b) \geq 0$ but $\tilde{x}$ satisfies $\tilde{\nu}^T (A\tilde{x} - b) = 0$ and $\tilde{x} \in int\mathcal{D}$ so there is point with $\tilde{\nu}^T (Ax - b) < 0$ or $A^T \tilde{\nu} = 0$. Assuming A full rank, it is contradiction.

3.6 Proof of KKT condition

This is quite simple since $\tilde{x}$ is feasible because $f_i(\tilde{x}) \leq 0$ and $h_i(\tilde{x}) = 0$. $\tilde{\lambda} \succeq 0$ says $L(x, \tilde{\lambda}, \tilde{\nu})$ convex in x. Finally,

$$\nabla f_0(x) + \sum_{i=1}^{m} \tilde{\lambda}_i \nabla f_i(\tilde{x}) + \sum_{i=1}^{p} \tilde{\nu}_i \nabla h_i(\tilde{x}) = 0$$

so $L(x, \tilde{\lambda}, \tilde{\nu})$ is minimized at $\tilde{x}$.

We could conclude that

$$\begin{aligned} g(\tilde{\lambda}, \tilde{\nu}) &= L(\tilde{x}, \tilde{\lambda}, \tilde{\nu}) \\ &= f_0(\tilde{x}) + \sum_{i=1}^{m} \tilde{\lambda}_i f_i(\tilde{x}) + \sum_{i=1}^{p} \tilde{\nu}_i h_i(\tilde{x}) \\ &= f_0(\tilde{x}) \end{aligned}$$

We know $g(\lambda, \nu) \leq f_0(x)$ and equality holds if and only if strong duality holds and x, λ, ν optimal value.

3.7 Examples

3.7.1 Least norm solution of linear equations

$$\textit{minimize} \;\; x^T x$$

$$\textit{subject to} \;\; Ax = b$$

Lagrangian is $L(x, \nu) = x^T x + \nu^T (Ax - b)$. $\nabla_x L(x, \nu) = 2x + A^T \nu$ so $x = -\frac{1}{2} A^T \nu$ minimizes L.

$$g(\nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu$$

and this problem satisfies Slaters constraint qualification so strong duality holds. Also we gain Lower bound property

$$p^* \geq -\frac{1}{4} \nu^T A A^T \nu - b^T \nu \quad \forall \nu$$

3.7.2 Standard LP

$$\textit{minimize} \;\; c^T x$$

$$\textit{subject to} \;\; Ax = b, \; x \succeq 0$$

Lagrangian is $L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) - \lambda^T x$. This is affine on x so if $c + A^T \nu - \lambda \neq 0$, infinum of L is $-\infty$.

$$g(\lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & o.w. \end{cases}$$

and this problem satisfies Slaters constraint qualification so strong duality holds. Also we gain Lower bound property

$$p^* \geq \sup_{A^T \nu + c \succeq 0} -b^T \nu$$

3.7.3 Quadratic Program

$$\begin{aligned} &\textit{minimize} \;\; x^T P x \\ &\textit{subject to} \;\; Ax \preceq b \end{aligned}$$

Dual function is

$$g(\lambda) = \inf_x (x^T P x + \lambda^T (Ax - b)) = -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda - b^T \lambda$$

this problem satisfies Slaters constraint qualification so dual problem is equal to Quadratic program.

$$\begin{aligned} &\textit{maximize} \;\; -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda - b^T \lambda \\ &\textit{subject to} \;\; \lambda \succeq 0 \end{aligned}$$

Part II
Large Scale Convex Optimization Algorithm using Monotone Operators

In this section, we discuss algorithms that solve Convex Optimization Problem in some form.
For example, algorithm ADMM solves convex optimization problem of form

$$minimize\ f(x) + g(y)$$

$$subject\ to\ \ Ax + By = c$$

These methods are based on fixed point iteration. We will first look on basic definitions of monotone operators and prove fixed point iteration converges. By many kinds of tricks, we can modify fixed point iterations and gain some algorithms for some situation. We introduce algorithms but also interested in rigourous proof of algorithm convergence.
We lastly look on some computer architecture based view : a parallelism.