2023 1st semester - Partial Differential Equation

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Contents

1 First Order Partial Differential Equation : Characteristic Curve

1.1 2nd Lecture

1.1.1 Picard Iteration

$$\begin{cases} \dfrac{df}{dt}(t) = F(t, f(t)) \\ f(t=0) = f_0 \end{cases}$$

We take initial function

$$f^{(0)}(t) = f_0$$

, and

$$f^{(n+1)}(t) = f_0 + \int_0^t F(s, f^{(n)}(s))ds$$

We want to prove the convergence by defining distance $D_n := \max_{t \in [0,T]} \left| f^{(n)}(t) - f^{(n-1)}(t) \right|$

Theorem 1. *If F is Lipshitz continuous in f and continuous in t then there exists some $T > 0$ such that there is a unique solution to Ordinary Differential Equation on some time interval $[0, T]$.*

First, $\{f^{(n)}(t)\}$ is Cauchy sequence of functions.

$$\begin{aligned} \left| f^{(n+1)}(t) - f^{(n)}(t) \right| &= \left| \int_0^t F(s, f^{(n)}(s)) - F(s, f^{(n-1)}(s))ds \right| \\ &\leq \left| \int_0^t C(f^{(n)}(s) - f^{(n-1)}(s))ds \right| \end{aligned}$$

So for sufficiently small-time interval, distance between two functions decreases. $T = \frac{1}{2C}$ is one value.

$$f^{(n)} \to g$$

Then g is a solution in some conditions since

$$\begin{aligned} g(t) = \lim_{n \to \infty} f^{(n)}(t) &= \lim_{n \to \infty} \left[f_0(t) + \int_0^t F(s, f^{(n)}(s))ds \right] \\ &= f_0(t) + \int_0^t F(s, g(s))ds \end{aligned}$$

The last equation holds because F is Lipshitz continuous on f.

Uniqueness could be proved by this way, if there are f_1, f_2 that are solution on $[0, T]$

$$\frac{d}{dt} \left| f_1(t) - f_2(t) \right| \leq C \left| f_1(t) - f_2(t) \right|$$

$$\frac{d}{dt} \left(e^{-ct} |f_1 - f_2| \right) \leq 0$$

Theorem 2 (Peano existence theorem)**.** *If F is continuous in t, f then there exists a solution to Ordinary Differential Equation.*

$$\begin{cases} \dfrac{df}{dt}(t) = F(t, f(t)) \\ f(t=0) = f_0 \end{cases}$$

Idea for proof is mollifying functions. This leads more friendly equation.

$$\begin{cases} \dfrac{df_\epsilon}{dt}(t) = F^\epsilon(t, f(t)) \\ f_\epsilon(t=0) = f_0 \end{cases}$$

Here $\{F^\epsilon\}$ is any sequence of Lipschitz functions converging to F in C^0. (It exists with help of mollifiers) Then f_ϵ 's satisfies following properties.

f_ϵ exists in a time interval $[0, T]$ and T is independent of ϵ
f_ϵ is uniformly bounded, i.e. $\sup_\epsilon \sup_t \left|f_\epsilon(t)\right| \le M_1$
f_ϵ is uniformly differentiable, i.e. $\sup_\epsilon \sup_t \left|\frac{d}{dt} f_\epsilon(t)\right| \le M_2$

Let we start proving these facts. Let $M = |F(0, f_0)|$. Then F is continuous so there exists some $\delta > 0$ such that $|t| \le \delta, |f - f_0| < \delta, \epsilon < \epsilon_0$ then $|F^\epsilon(t, f)| \le 2M$. Define $T = \frac{1}{2}\min\{\delta, \frac{\delta}{2M}\}$

If $|f_\epsilon(t) - f_0| \ge \delta$, $t^*_\epsilon = \inf_t(|f_\epsilon(t) - f_0| \ge \delta)$, then by continuity $|f_\epsilon(t^*_\epsilon) - f_0| = \delta$.

$$\delta \le \left| \int_0^{t^*_\epsilon} F^\epsilon(s, f_\epsilon(s)) ds \right| \le \int_0^{t^*_\epsilon} \left|F^\epsilon(s, f_\epsilon(s))\right| ds \le 2Mt^*_\epsilon < \delta$$

Contradiction.

Therefore, for this T, if $\epsilon < \epsilon_0$, $|f_\epsilon(t) - f_0| < \delta$. Since f_ϵ is integral of F^ϵ via t, it is differentiable and

$$\left| \frac{df_\epsilon}{dt} \right| = \left|F^\epsilon(t, f_\epsilon(t))\right| \le 2M$$

so uniformly differentiable.

Bounded ball in C^1 of C^0 norm is compact so there are subsequences $\{f_{\epsilon_k}\}$ that $f_{\epsilon_k} \to g$. Then g is solution because

$$f_{\epsilon_k}(t) = f_0 + \int_0^t F^{\epsilon_k}(s, f_{\epsilon_k}(s)) ds$$

inside the integral is continuous.

1.2 4th Lecture

1.2.1 Checking C^0 is Banach Space

First, $C^0[0, 1]$ is Banach space. We could think of counterexample $\{x^n\}(n \in \mathbb{N})$. But actually it is not Cauchy sequence.

For fixed m,

$$\frac{d}{dx}(x^n - x^m) = 0 \Leftrightarrow x^{n-m} = \frac{m}{n}$$

$$x^m - x^n = \left(1 - \frac{m}{n}\right)\left(\frac{m}{n}\right)^{m/(n-m)} \to 1$$

So it is not Cauchy sequence.

Now $C^0[0,1]$ is Banach. Consider cauchy sequence $\{f_n\}$. Then $f_n(x)$ is cauchy sequence for every x, so it converges i.e. $\forall x \in [0,1] \quad f_n(x) \to f(x)$. Define as this $f(x)$. Since $\{f_n\}$ is cauchy sequence, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, $||f_n(x) - f_m(x)|| < \epsilon/2$. Also for every $x \in [0,1]$ there exists $N_x \in \mathbb{N}$ that $|f_m(x) - f(x)| < \epsilon/2$ if $m \geq N_x$. So if $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for every $x \in [0,1]$. Thus f is a limit via norm.

f is continuous. To prove that, the idea is

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

Each three term could be small as possible.

1.2.2 Checking C^1 is Banach Space

If x_n is Cauchy sequence in C^1 norm, x_n' is Cauchy in C^0. By upper statement, $x_n' \to z$. Also, x_n is Cauchy in $C[0,1]$, $x_n \to y$

$$y(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \{x_n(0) + \int_0^t z_n(s)ds\} = y(0) + \int_0^t z(s)ds$$

because z_n is uniformly converging to z. Therefore, $y' = z$ and limit is differentiable and limit exists i.e. complete.

1.2.3 Checking $C^0([-T,T] \times \mathbb{R})$ is Banach Space

For fixed $x \in \mathbb{R}$ and by completeness of C^0, there exist $f_x(t;x) \in C^0([-T,T])$ such that

$$f_n(t,x) \to f_x(t,x)$$

There exist $N \in \mathbb{N}$ such that $n, m \geq N$ then $||f_n - f_m||_{C^0_{t,x}} < \epsilon/2$. However, for every t, x, $\exists N_{t,x} \in \mathbb{N}$ such that $|f_m(t,x) - f(t,x)| < \epsilon/2$ if $m \geq N_{t,x}$
Therefore, $||f - f_n||_{C^0_{t,x}} < \epsilon$ if $n \geq N$.

Continuity of f can be shown by

$$\begin{aligned}
|f(t,x) - f(s,y) &\leq |f_n(t,x) - f(t,x)| + |f_n(t,x) - f_n(s,y)| + |f_n(s,y) - f(s,y)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + |f_n(t,x) - f_n(s,y)| \\
&< \epsilon
\end{aligned}$$

If n is chosen above to satisfy inequality and $(t,x), (s,y)$ is close enough in f_n

1.2.4 Checking $C^1([-T,T]\times\mathbb{R})$ is a Banach Space

If $\{f_n\}$ is a Cauchy sequence via C^1 norm, $\{f_n\},\{\partial_t f_n\},\{\partial_x f_n\}$ is Cauchy sequence via C^0 norm. Because $C^0([-T,T]\times\mathbb{R})$ is Banach space,

$$f_n \to f,\ \partial_t f_n \to g_1,\ \partial_x f_n \to g_2$$

For arbitrary $x\in\mathbb{R}$,

$$\begin{aligned} f(t,x) &= \lim_{n\to\infty} f_n(x,t) = \lim_{n\to\infty}\left\{ f_n(x,0) + \int_0^t \partial_t f_n(x,s)ds \right\} \\ &= f(0,x) + \int_0^t g_1(s)ds \end{aligned}$$

Limit and integral could be exchanged because $\partial_t f_n$ converges uniformly. Therefore,

$$\frac{\partial}{\partial t} f(t,x) = g_1(t,x)$$

As similar way,

$$\frac{\partial}{\partial x} f(t,x) = g_2(t,x)$$

1.3 5th to 7th Lecture

1.3.1 Well-posedness of semi-linear PDE

Theorem 3. *Assume following functions have properties ; $g\in C^1(\mathbb{R}^n)$, $b\in C^1(\mathbb{R}\times\mathbb{R}^n)$, $f\in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})$ Domain $(t,x)\in\mathbb{R}^{n+1}$*
There exists some $T>0$ such that unique solution $u(t,x)$ of following Partial Differential Equation exists in the region $\{(t,x): |t|<T, x\in\mathbb{R}^n\}$

$$(*)\begin{cases} \partial_t u + b(t,x)\nabla_x u = f(t,x,u) \\ u(t=0,x) = g(x) \end{cases}$$

Lemma 4 (Characteristic curves)**.** *Assume that $b\in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$. For every t, below Ordinary Differential Equation has solution. Furthermore, for fixed t, solution for below Ordinary Differential Equation $X(t,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$ (i.e. X is C^1, X is invertible, X^{-1} is C^1)*

$$\begin{cases} \dfrac{d}{dt}X(t,z) = b(t,z,X(t,z)) \\ X(0,z) = z \end{cases}$$

Proof. **Step 1 : Proof of existence for each** z
Let we use Picard Iteration. $X^{(0)}(t;z) = z$,

$$X^{(n+1)}(t;z) = z + \int_0^t b(s,z,X^{(n)}(s;z))ds$$

Then,

$$||X^{(n+1)}(t;z) - X^{(n)}(t;z)|| = \int_0^t b(s,z,X^{(n)}(s;z)) - b(s,z,X^{(n-1)}(s;z))ds$$

$$\begin{aligned}
||X^{(n+1)}(t;z) - X^{(n)}(t;z)|| &\leq \int_0^t ||b(s,z,X^{(n)}(s;z)) - b(s,z,X^{(n-1)}(s;z))||ds \\
&\leq \int_0^t C||X^{(n)}(s;z) - X^{(n-1)}(s;z)||ds \\
&\leq \frac{1}{2}D_{n-1}(z)
\end{aligned}$$

where $D_n(z) = \max_{t\in[0,\frac{1}{2C}]} ||X^{(n+1)}(t;z) - X^n(t;z)||$.

So $0 \leq D_n(z) \leq \frac{1}{2}D_{n-1}(z)$ so $D_n(z)$ tends to zero. Since $C^0([-T,T]\times\mathbb{R}^n)$ is Banach Space, it converges to some $X(t;z)$ for every z.
Also, $X(t;z)$ satisfies Ordinary Differential Equation thus, differentiable on t.

Step 2 : Grownwall Inequality
Grownwall Inequality states that if $\frac{d}{dt}f \leq A(t)f + B(t)$,

$$f(t) \leq f(0)e^{\int_0^t A(s)ds} + \int_0^t B(s)e^{\int_s^t A(\tau)d\tau}ds$$

Proof is simple because

$$\begin{aligned}
\frac{d}{dt}\left[e^{-\int_0^t A(s)ds}f(t)\right] &= \frac{d}{dt}f\cdot e^{-\int_0^t A(s)} - f(t)\cdot A(t)e^{-\int_0^t A(s)ds} \\
&\leq B(t)e^{-\int_0^t A(s)ds}
\end{aligned}$$

Integrating both side attains Grownwall Inequality.

Step 3 : Bi-Lipshitz estimate
For the solution of Ordinary Differential Equation, let $z \neq z' \in \mathbb{R}^n$. $\frac{d}{dt}(X(t,z) - X(t,z')) = b(t,z,X(t,z)) - b(t,z',X(t,z'))$

$$\begin{aligned}
\left|\frac{d}{dt}||X-X'||^2\right| &= \left|2(b(t,z,X) - b(t,z',X'))\cdot(X-X')\right| \\
&\leq 2||X-X'||\big(||b(t,z,X) - b(t,z,X')|| + ||b(t,z,X') - b(t,z',X')||\big) \\
&\leq 2||X-X'||(C||X-X'|| + C||z-z'||) \\
&\leq 4C(||X-X'|| + ||z-z'||)
\end{aligned}$$

By Grownwall Inequality,

$$||X-X'||^2 \leq ||z-z'||^2e^{4Ct} + 4C||z-z'||^2te^{4Ct} = (1+4Cte^{4Ct})||z-z'||^2$$

$$-||X-X'||^2 \leq -||z-z'||^2 e^{-4Ct} + 4C||z-z'||^2 \int_0^t e^{-4C(t-s)} ds = -||z-z'||^2 e^{-4Ct}(2-e^{4Ct})$$

Therefore, if $T < \frac{\ln 2}{4C}$,

$$m(t) \leq \frac{||X-X'||}{||z-z'||} \leq M(t)$$

So X **is injective and continuous**

Step 4 : Surjectiveness
First, range of X is closed (in fixed t).
If $\{x_n\} \in Range(X)$ and Cauchy, by Lipshitzness proved above $\{z_n\}$ is cauchy on $\mathbb{R}^n$. Therefore, $z_n \to z$ for some z and satisfies

$$\lim_{n\to\infty} x_n = X(t,z) \in Range(X)$$

Second, range of X is open because of Invariance of domain lemma (This comes from Topological Sense).
Lemma : Invariance of domain If U is open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is injective, continuous map. Then $V = f(U)$ is open in $\mathbb{R}^n$ and f is homeomorphism between U and V.

Thus, Range of X is both open and closed on $\mathbb{R}^n$ so it is just full domain.

Step 5 : C^1 ness of X
By following lemma (will be proved after), X is C^1 in z.

Lemma 5. *Assume $F \in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n)$. Then for each t, solution of below equation : $V(t;\cdot)$ is C^1 in z*

$$\begin{cases} \frac{d}{dt}V(t;z) = F(t,z,V(t;z)) \\ V(0,z) = g(z) \end{cases}$$

This will be proven later.

Step 6 : X is homeomorphism
Since X is bijective, it is invertible. X is C^1 so X^{-1} is also C^1. □

Proof for Lemma 5. Consider following new system of Ordinary Differential Equations.

$$\begin{cases} \dfrac{d}{dt} w_{ij}(t;z) = \partial_{z_j} F_i(t,z,V) + \displaystyle\sum_{k=1}^{n} \partial_{V_k} F_i(t,z,V)\cdot w_{kj}(t;z) \\ w_{ij}(0;z) = \partial_{z_j} g_i(z) \end{cases}$$

And define measure how far w_{ij} is from our goal $\partial_{z_j} V_i$. (We do not know V is differentiable so we will use following measure with new variable h)

$$D_h(t;z)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{h}(V_i(t;z_{-j},z_j+h) - V_i(t;z)) - w_{ij}(t;z)\right)^2$$

Then we could find

$$
\begin{aligned}
&\left|\frac{1}{2}\frac{d}{dt}D_h(t;z)^2\right| \\
&= \Bigg|\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{h}(V_i(t;z_{-j},z_j+h)-V_i(t;z))-w_{ij}(t;z)\right)\cdot \\
&\bigg(\frac{1}{h}\big(F_i(t,z_{-j},z_j+h,V(t;z_{-j},z_j+h))-F_i(t,z,V(t;z))\big) \\
&\qquad\qquad -\partial_{z_j}F_i(t,z,V(t;z))-\sum_{k=1}^{n}\partial_{v_k}F_i(t,z,V(t;z))\cdot w_{kj}(t;z)\bigg)\Bigg|
\end{aligned}
$$

If we notate $F_{ij}(t;z,h)=F_i(t,z_{-j},z_j+h,V(t,z_{-j},z_j+h))$

$$
\begin{aligned}
&\frac{1}{h}(F_{ij}(t;z,h)-F_i(t,z,V(t;z)))-\partial_{z_j}F_i(t,z,V(t;z))-\sum_{k=1}^{n}\partial_{V_k}F_i(t,z,V(t;z))w_{kj}(t;z) \\
&=\sum_{k=1}^{n}\left(\frac{1}{h}(V_k(t,z_{-j},z_j+h)-V_k(t;z))-w_{kj}(t;z)\right)\partial_{V_k}F_i(t,z,V(t;z)) \\
&+\frac{1}{h}\bigg(F_i(t,z_{-j},z_j+h,V(t;z_{-j},z_j+h))-F_i(t,z,V(t;z))-h\partial_{z_j}F_i(t,z,V(t;z)) \\
&\qquad\qquad -\sum_{k=1}^{n}(V_k(t;z_{-j},z_j+h)-V_k(t;z))\partial_{V_k}F_i(t,z,V(t;z))\bigg)
\end{aligned}
$$

And second addition term is independent to w so denote it by $C_{ij}(t,h;z)$.

$$
\begin{aligned}
&\left|\frac{1}{2}\frac{d}{dt}D_h(t;z)^2\right| \\
&\leq \Bigg|\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{h}(V_i(t;z_{-j},z_j+h)-V_i(t;z))-w_{ij}(t;z)\right)\cdot \\
&\qquad\qquad \left(\frac{1}{h}(V_k(t;z_{-j},z_j+h)-V_k(t,z))-w_{kj}(t;z)\right) \\
&\qquad\qquad +\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}(t,h;z)\left(\frac{1}{h}(V_i(t;z_{-j},z_j+h)-V_i(t;z))-w_{ij}(t;z)\right)\Bigg| \\
&\leq \left|\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\big(\frac{1}{h}(V_i(t;z_{-j},z_j+h)-V_i(t;z))-w_{ij}(t;z)\big)\right)^2\right|+ \\
&\qquad\qquad \left|\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}(t,h;z)\left(\frac{1}{h}(V_i(t;z_{-j},z_j+h)-V_i(t;z))-w_{ij}(t;z)\right)\right| \\
&\leq nD_n(t;z)^2+\left|\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}(t,h;z)\left(\frac{1}{h}(V_i(t;z_{-j},z_j+h)-V_i(t;z))-w_{ij}(t;z)\right)\right|
\end{aligned}
$$

So, by Cauchy-Schwartz Inequality,

$$\frac{1}{2}\frac{d}{dt}D_h(t;z)^2 \le (n+1)D_h(t;z)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}(t,h;z)^2$$

By Grownwall Inequality,

$$D_h(t;z)^2 \le D_h(0;z)^2 \cdot e^{2(n+1)t} + \int_0^t 2C(s,h;z)e^{-2(n+1)s}ds \cdot e^{2(n+1)t}$$

If we look at $C_{ij}(t,h;z)$,

$$\begin{aligned} &C_{ij}(t,h;z) \\ &= \frac{1}{h}\bigg(F_i(t, z_{-j}, z_j+h, V(t; z_{-j}, z_j+h)) - F_i(t,z,V(t;z)) - h\partial_{z_j}F_i(t,z,V(t;z)) \\ &\qquad\qquad - \sum_{k=1}^{n}(V_k(t;z_{-j}, z_j+h) - V_k(t;z))\partial_{V_k}F_i(t,z,V(t;z))\bigg) \end{aligned}$$

It tends to zero if $h \to \infty$ so

$$\lim_{h\to 0} D_h(t;z)^2 = 0$$

This means V_i's j derivative is equal to w_{ij}. So X is C^1. □

Now we can prove Theorem, well-posedness of semi-linear Partial Differential Equation

Proof of Theoerem 3. We have proved X is diffeomorphism. Let $\bar{u}(t,z) = u(t, X(t,z))$. Then $\bar{u}$ satisfies

$$\frac{d}{dt}\bar{u}(t,z) = f(t, X(t,z), \bar{u}(t,z))$$

This $\bar{u}(t,z)$ exists on some $[-T,T]\times\mathbb{R}^n$ uniquely, and by Lemma 5 it is C^1 function.
If we define $u(t,x) = \bar{u}(t, X^{-1}(t,x))$, then $\partial_{x_i}\bar{u}(t, X^{-1}(t,x)) = \nabla_x \bar{u}(t, X^{-1}(t,x))\cdot\frac{dX^{-1}}{d_{x_i}}(t,x)$
So u is also C^1 function.

To see uniqueness of u, X, Z is independent of u so transform u and $\bar{u}$ is unique. Since $\bar{u}$ is unique, u is unique. □

1.3.2 Well-posedness of quasi-linear PDE

Theorem 6. *Assume following functions have properties ; $g \in C^1(\mathbb{R}^n)$, $b \in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})$, $f \in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})$ Domain $(t,x)\in\mathbb{R}^{n+1}$*
There exists some $T>0$ such that unique solution $u(t,x)$ of following Partial Differential Equation exists in the region $\{(t,x) : |t| < T, x \in \mathbb{R}^n\}$

$$(*)\begin{cases} \partial_t u + b(t,x,u)\nabla_x u = f(t,x,u) \\ u(t=0,x) = g(x) \end{cases}$$

Proof is similar to case of semi-linear case. So just introduce some key points different from semi-linear case.

First, let $X(t,z), \bar{u}(t,z)$ satisfies following system of Ordinary Differential Equation.

$$\begin{cases} \dfrac{dX}{dt}(t,z) = b(t, X(t,z), \bar{u}(t,z)) \\ \dfrac{d\bar{u}}{dt}(t,z) = f(t, X(t,z), \bar{u}(t,z)) \end{cases}$$

with initial data

$$\begin{cases} X(t=0,z) = z \\ \bar{u}(t=0,z) = g(z) \end{cases}$$

So we need to prove Solution exists, X is invertible, $\bar{u}, X$ is C^1

Existence

$X^{(0)}(t,z) = z, \bar{u}^{(0)}(t,z) = g(z)$ and let Picard Iteration with Following.

$$X^{(n+1)}(t,z) = z + \int_0^t b(s, X^{(n)}(s,z), \bar{u}^{(n)}(s,z))ds$$
$$\bar{u}^{(n+1)}(t,z) = g(z) + \int_0^t f(s, X^{(n)}(s,z), \bar{u}^{(n)}(s,z))ds$$

Then

$$\begin{aligned} &||X^{(n+1)}(t,z) - X^{(n)}(t,z)||^2 + ||\bar{u}^{(n+1)}(t,z) - \bar{u}^{(n)}(t,z)||^2 \\ &\leq 2C \int_0^t ||X^{(n)}(t,z) - X^{(n-1)}(t,z)||^2 + ||\bar{u}^{(n)}(t,z) - \bar{u}^{(n-1)}(t,z)||^2 \\ &\leq 2CTD_{n-1}(z) \end{aligned}$$

So if $T < 1/(4C)$, $D_n(z) \to 0$. So $X^{(n)}, \bar{u}^{(n)}$ is Cauchy, it lies on Banach space so there exists solution $X, \bar{u}$.

Bi-Lipshitz estimate

$$\begin{aligned} \left| \frac{d}{dt} ||X(t,z) - X(t,z')||^2 \right| &\leq 2||X - X'|| \cdot ||b(t,X,\bar{u}) - b(t,X',\bar{u}')|| \\ &\leq 2||X - X'|| \cdot (C||X - X'|| + C||\bar{u} - \bar{u}'||) \\ &\leq 2C(2||X - X'||^2 + ||\bar{u} - \bar{u}'||^2) \end{aligned}$$

Doing same for $||\bar{u} - \bar{u}'||^2$, by adding two terms,

$$\begin{aligned} \left| \frac{d}{dt} \Big(||X - X'||^2 + ||\bar{u} - \bar{u}'||^2 \Big) \right| &\leq 6C(||X - X'||^2 + ||\bar{u} - \bar{u}'||^2) \\ &< 6C(||X - X'||^2 + ||\bar{u} - \bar{u}'||^2) + ||z - z'||^2 \end{aligned}$$

Attaining Grownwall Inequality, we could find $c_1(t), c_2(t)$ such that

$$c_1(t) \leq \frac{||X - X'||^2 + ||\bar{u} - \bar{u}'||^2}{||z - z'||^2} \leq c_2(t)$$

If $X = X'$, $\bar{u} = \bar{u}'$ so it need to be $z = z'$. Therefore, X is injective. Also $X, \bar{u}$ is continuous.

Surjectiveness
Same as semi-linear case. Therefore, X is bijective, invertible.

C^1 ness of $X, \bar{u}$
Consider following system of Ordinary Differential Equation.

$$\begin{cases} \dfrac{dV_{ij}}{dt}(t,z) = \displaystyle\sum_{k=1}^{n} \partial_{x_k} b_i(t, X(t,z), \bar{u}(t,z))V_{kj}(t,z) + \partial_{\bar{u}} b_i(t, X(t,z), \bar{u}(t,z))W_j(t,z) \\ \dfrac{dW_j}{dt}(t,z) = \displaystyle\sum_{k=1}^{n} \partial_{x_k} f(t, X(t,z), \bar{u}(t,z))V_{kj}(t,z) + \partial_{\bar{u}} f(t, X(t,z), \bar{u}(t,z))W_j(t,z) \end{cases}$$

with initial data

$$\begin{cases} V_{ij}(0,z) = \delta_{ij} \\ W_j(0,z) = \partial_{z_j} g(z) \end{cases}$$

Using this measure, we could prove X and $\bar{u}$ is C^1.

$$D_h(t,z)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{1}{h}(X_i(t, z_{-j}, z_j + h) - X_i(t,z)) - V_{ij}(t,z) \right)^2 + \sum_{j=1}^{n} \left(\frac{1}{h}(\bar{u}(t, z_{-j}, z_j + h) - \bar{u}(t,z)) - W_j(t,z) \right)^2$$

End of proof Since $u(t,x) = \bar{u}(t, X^{-1}(t,x))$, it is C^1 satisfying original Equation. Uniqueness follows similar, showing $X, \bar{u}$ is unique and u is combination of those, thus unique.

1.4 8th Lecture

1.4.1 Fully nonlinear Partial Differential Equation

Most generalized theorem for first order PDE.

Theorem 7 (Fully nonlinear Partial Differential Equation)**.** *If Γ is hyperplane of $\mathbb{R}^n$ and Γ is C^3. Given initial value on Γ is C^2. Assume F satisfies algebraic root condition (so that $F(x, u(x), \nabla u(x)) = 0$ exists given $x, u(x)$ and derivatives of codimension 1) and C^2. Then solution exists on some neighborhood of Γ.*
Also, if initial data is C^∞ and F is C^∞ and Γ is C^∞ then solution is also C^∞.

$$F(x, u(x), Du(x)) = 0$$

For this case, characteristic curve $X(s,z)$ satisfies

$$q(s,z) := u(X(s,z))$$

$$r_i(s,z) := p_i(s,z) = (\partial_{x_i} u)(s,z)$$

$$\begin{cases} \frac{d}{ds}X = \nabla_p F(X, q, r) \\ \frac{d}{ds}r = -(\partial_u F(X, q, r))r - \nabla_x F(x, q, r) \\ \frac{d}{ds}q = r \cdot \nabla_p F(X, q, r) \end{cases}$$

To explain this, one need to think simple Idea. Fully nonlinear first order PDE is hard to solve, so instead we need to make quasi linear equation. This is possible when we derivate the equation $F(x, u, Du) = 0$. Let we first derivate only with x_1.

$$\partial_{x_1} F + \partial_u F \partial_{x_1} u + \partial_{p_1} F \partial_{x_1} \partial_{x_1} u + \partial_{p_2} F \partial_{x_1} \partial_{x_2} u + \cdots \partial_{p_n} F \partial_{x_1} \partial_{x_n} u = 0$$

Changing differential,

$$\partial_{x_1} F + \partial_u F(\partial_{x_1} u) + \partial_{p_1} F \partial_{x_1}(\partial_{x_1} u) + \partial_{p_2} F \partial_{x_2}(\partial_{x_1} u) + \cdots \partial_{p_n} F \partial_{x_n}(\partial_{x_1} u) = 0$$

This form is Quasci linear in $\partial_{x_1} u$.

$$\partial_{p_1} F \cdot \partial_{x_1} p_1 + \partial_{p_2} F \cdot \partial_{x_2} p_1 + \cdots + \partial_{p_n} F \cdot \partial_{x_n} p_1 = -\partial_{x_1} F - \partial_u F \cdot p_1$$

or

$$\nabla_p F \cdot \nabla_x p = -\nabla_x F - (\partial_u F)p$$

If we achieve characteristic curve that has gradient $\nabla_p F$, more easy to write. Thus we setted $X(s, z)$ which $X(0, z) = z$ satisfies

$$\frac{d}{ds}X(s, z) = \nabla_p F(X(s, z), q(s, z), r(s, z))$$

q, r is introduced since x value has changed into $X(s, z)$. $q(s, z) = u(X(s, z))$ and $r(s, z) = \nabla u(X(s, z)) = p(X(s, z))$.

Then above computation gives

$$\begin{aligned} \frac{d}{ds}r(s, z) &= \nabla_p F(X(s, z), q(s, z), r(s, z)) \cdot \nabla_x p(X(s, z)) \\ &= -\nabla_x F(X(s, z), q(s, z), r(s, z)) - (\partial_u F(X(s, z), q(s, z), r(s, z)))r \end{aligned}$$

Now remaining part is $q(s, z) = u(X(s, z))$. This is by simple derivation.

$$\frac{d}{ds}q(s, z) = \nabla_x u(s, z) \cdot \frac{dX}{ds}(s, z) = r(s, z) \cdot \nabla_p F(X(s, z), q(s, z), r(s, z))$$

1.4.2 Propagation of Regularity

I will prove only for semi-linear case. Quasci linear case will be similar.

Theorem 8 (Propagation of Regularity)**.** *For semi-linear PDE if $b, g, f \in C^\infty$, solution is also C^∞.*

$$\begin{cases} \partial_t u + b(t, x)\partial_x u = f(t, x, u) \\ u(0, x) = g(x) \end{cases}$$

Proof. Our goal is to make partial differential equation that will be the nth derivative of solution. Start with the second derivative.
While solving semi-linear PDE, we could find characteristic curve $X(t,z)$ satisfying

$$\frac{\partial}{\partial t}u(t,X(t,z)) = f(t,X(t,z),u(t,X(t,z)))$$

$$\frac{\partial}{\partial t}X(t,z) = b(t,x)$$

First equation enables to know that second derivatives, $\partial_{tt}, \partial_{tx}$ exists and continuous. Now only need to show is ∂_{xx} exists and continuous.
Motivation is derivate our differential equation. We do not know ∂_{xx} exists but if it exists, it will satisfy this differential equation.

$$\partial_t(\partial_x u) + b(t,x)\partial_x(\partial_x u) = \partial_x f(t,x,u) + \partial_u f(t,x,u)\partial_x u - \partial_x b(t,x)\partial_x u$$

So we could think of following differential equation.

$$\begin{cases} \partial_t v + b(t,x)\partial_x v = \partial_x f(t,x,u) + \partial_u f(t,x,u)\partial_x u - (\partial_x b)(\partial_x u) \\ v(0,x) = \partial_x g(x) \end{cases}$$

We know this is semi-linear Partial Differential Equation and within the characteristic curve, solution exists and C^1. **Important thing is, characteristic curve X is same as original PDE so solution exists on same region $[-T,T]\times\mathbb{R}$.**

$$\begin{aligned}\partial_t\big[v(t,X(t,z))\big] =& \partial_x f(t,X(t,z),u(t,X(t,z))) + \partial_u f(t,X(t,z),u(t,X(t,z)))\partial_x u(t,X(t,z)) \\ &- \partial_x b(t,X(t,z))\partial_x u(t,X(t,z))\end{aligned}$$

Now compare this result to derivative of original semi-linear PDE with characteristic curves.

$$\begin{aligned}\partial_t\big[\partial_z u(t,X(t,z))\big] = {}& \partial_x f(t,X(t,z),u(t,X(t,z)))\partial_z X(t,z) \\ &+ \partial_u f(t,X(t,z),u(t,X(t,z)))\partial_x u(t,X(t,z))\partial_z X(t,z)\end{aligned}$$

If we show $\partial_t v(t,X(t,z)) = \partial_t\partial_x u(t,X(t,z))$, then we are done because $v(0,z) = \partial_x u(0,z)$.

Now there only exists some computation.

$$\begin{aligned}\partial_t\big[\partial_z u(t,X(t,z))\big] &= \partial_t\big[\partial_x u(t,X(t,z))\partial_z X(t,z)\big] \\ &= \partial_t(\partial_x u(t,X(t,z)))\partial_z X(t,z) + \partial_x u(t,X(t,z))\partial_t\partial_z X(t,z)\end{aligned}$$

On the other hand,

$$\begin{aligned}\partial_t v(t,X(t,z)) =& \partial_x f(t,X(t,z),u(t,X(t,z))) + \partial_u f(t,X(t,z),u(t,X(t,z)))\partial_x u(t,X(t,z)) \\ &- \partial_x b(t,X(t,z))\partial_x u(t,X(t,z))\end{aligned}$$

Since $\partial_x b(t,X(t,z))\partial_z X(t,z) = \partial_z b(t,X(t,z)) = \partial_t\partial_z X(t,z)$,

$$\partial_t v(t,X(t,z))\partial_z X(t,z) = \partial_t\big[\partial_x u(t,X(t,z))\big]$$

$\partial_z X(t,z)$ is nonzero so We now know $v(t,x)$ is x-derivative of $u(t,x)$. So u is C^2. If we do same on differential equation of $v(t,x)$, we could know v is C^2, so u is C^3. Continuing this, we could find u is C^∞. $\square$

2 Heat Equation, Laplace Equation

2.1 Heat Equation

In this section, Heat Equation is main equation. We will look about existence and uniqueness of solution. Also important properties including maximum principal is useful.

2.1.1 Full Domain Homogeneuous Heat Equation

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u(t=0,x) = u_0(x) \end{cases}$$

$u(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ is full domain homogenuous heat equation. It is well known this equation have solution. To introduce it, we wll define **Heat Kernel** $\Phi(t,x)$

$$\Phi(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

Theorem 9 (Solution for Full Domain Homogenuous Heat Equation)**.** *If $u_0 \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ then $u(t,x) = \Phi(t,\cdot) * u_0$ is a solution to Heat Equation.*

Proof. Proof is about four steps.

STEP 1. u is bounded and uniformly continuous on x.

Boundness could be proved

$$|u(t,x)| \leq \int u_0(y)\Phi(t,x-y)dy \leq C\int \Phi(t,x-y)dy = C$$

Uniform continuous could be proved if we set $|x-x'| < \delta$ so that $|u_0(z_1) - u_0(z_2)| < \epsilon$ if $|z_1 - z_2| < \delta$

$$|u(t,x) - u(t,x')| = \left| \int_{\mathbb{R}^n} \Phi(t,x)(u_0(t,x-y) - u_0(t,x'-y))dy \right| \leq \int_{\mathbb{R}^n} \Phi(t,y)\epsilon dy = \epsilon$$

STEP 2. u is C^∞ in t,x

We know that $u = \Phi * u_0$ is C^∞ on x since $\Phi \in C^\infty$. To show u is C^∞ in t,

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \lim_{h\to 0}\int_{\mathbb{R}^n} \frac{1}{h}(\Phi(t+h,x-y) - \Phi(t,x-y))u_0(y)dy \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial t}\Phi(t,x-y)u_0(y)dy \end{aligned}$$

therefore, C^∞ in t also.

STEP 3. u satisfies $\partial_t u = \Delta u$

We show that

$$\partial_t = \partial_t \Phi(t,\cdot) * g$$

$$\Delta u = (\Delta\Phi(t,\cdot)) * g$$

With simple computation, $\Delta\Phi = \partial_t \Phi$.

STEP 4. $\lim_{t\downarrow 0} u(t,x) = u_0(x)$
$\forall\epsilon > 0$, let δ satisfies if $|y| < \delta$ then $|u_0(x-y) - u_0(x)| < \epsilon/2$

$$|u(t,x) - u_0(x)| = \left| \int_{\mathbb{R}^n} \Phi(t,y) \cdot (u_0(x-y) - u_0(x))dy \right| \leq \int_{\mathbb{R}^n} \Phi(t,y)|u_0(x-y) - u_0(x)|dy$$

We can divide into two terms.

$$\int_{B(0,\delta)} \Phi(t,y)|u_0(x-y) - u_0(x)|dy < \frac{\epsilon}{2}$$

$$\int_{\mathbb{R}^n - B(0,\delta)} \Phi(t,y)|u_0(x-y) - u_0(x)|dy \leq 2C \int_{\mathbb{R}^n - B(0,\delta)} \Phi(t,y)dy < \frac{\epsilon}{2}$$

for sufficiently small t.
Therefore, for $t < t_0$ $|u(t,x) - u_0(x)| < \epsilon$. □

This solution formula gives 'Infinite propagation speed'. This means, in any time $t > 0$, u is positive everywhere.

Lemma 10 (Infinite propagation speed property)**.** *If $u_0(x)$ compactly supported in $\mathbb{R}^n$, u is supported everywhere if $t > 0$*

Proof. Solution formula says

$$u(t,x) = \int_{y\in\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \times u_0(y)dy$$

so positive if $t > 0$ □

2.1.2 Full Domain Nonhomogenuous Heat Equation

$$\begin{cases} \partial_t u - \Delta u = f \\ u(t=0,x) = 0 \end{cases}$$

$u(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ is full domain homogenuous heat equation. It is also well known that solution exists under certain conditions.

Theorem 11 (Solution for Full Domain Nonhomogenuous Heat Equation)**.** *If $f \in C^1C^2(\mathbb{R}^+ \times \mathbb{R}^n)$ is compactly supported then solution exists for Full Domain Nonhomogenuous Heat Equation problem.*

Proof. We will assume certain solution formula. Consider following PDE.

$$\begin{cases} \partial_t u(t,x;s) - \Delta u(t,x;s) = 0 \\ u(t=s,x;s) = f(x,s) \end{cases}$$

This PDE has solution that we proved in homogenuous heat equation. Then we assume following $u(t,x)$ is a solution to nonhomogenuous heat equation.

$$u(t,x) = \int_0^t u(t,x;s)ds$$

There are two steps to prove this formula is solution.

STEP 1. u is $C^1C^2(\mathbb{R}^+ \times \mathbb{R}^n)$

$$\begin{aligned}
u(t,x) &= \int_0^t u(t,x;s)ds \\
&= \int_0^t \int_{\mathbb{R}^n} \Phi(t-s,y) f(x-y,s)dyds \\
&= \int_0^t \int_{\mathbb{R}^n} \Phi(s,y) f(x-y,t-s)dyds
\end{aligned}$$

So,

$$\partial_t u(t,x) = \int_{\mathbb{R}^n} \Phi(t,y) f(x-y,0)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(s,y) \partial_t f(x-y,t-s)dyds$$

$$\partial_{x_i x_j} u(t,x) = \int_0^t \int_{\mathbb{R}^n} \Phi(s,y) \partial_{x_i x_j} f(x-y,t-s)dyds$$

STEP 2. $\partial_t u - \Delta u = f$

$$\begin{aligned}
\partial_t u - \Delta u &= \int_0^t \int_{\mathbb{R}^n} \Phi(s,y)(\partial_t f(x-y,t-s) - \Delta_x f(x-y,t-s))dyds + \int_{\mathbb{R}^n} \Phi(t,y) f(x-y,0)dy \\
&= \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(s,y)\Big(-\frac{\partial}{\partial s} - \Delta_y\Big) f(x-y,t-s)dyds \\
&\qquad + \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(s,y)\Big(-\frac{\partial}{\partial s} - \Delta_y\Big) f(x-y,t-s)dyds + \int_{\mathbb{R}^n} \Phi(t,y) f(x-y,0)dy
\end{aligned}$$

We denote each terms I_ϵ, J_ϵ, K.

First, estimate I_ϵ.

$$\begin{aligned}
&\int_0^\epsilon \int_{\mathbb{R}^n} \Phi(s,y)\Big(-\frac{\partial}{\partial s} - \Delta_y\Big) f(x-y,t-s)dyds \\
&\qquad\qquad \le \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(s,y)(||\partial_t f||_{L^\infty} + ||D_x^2 f||_{L^\infty})dyds \le \epsilon C
\end{aligned}$$

Second, estimate J_ϵ

$$
\begin{aligned}
&\int_\epsilon^t \int_{\mathbb{R}^n} \Phi(s,y)\Big(-\frac{\partial}{\partial s} - \Delta_y\Big) f(x-y,t-s)dyds \\
&\qquad = \int_\epsilon^t \int_{\mathbb{R}^n} \partial_s\Phi(s,y)f(x-y,t-s) - \partial_s\big(\Phi(s,y)f(x-y,t-s)\big)dyds \\
&\qquad\qquad\qquad + \sum_{i=1}^n \int_\epsilon^t \int_{\mathbb{R}^n} \partial_{y_i}\Phi(s,y)\partial_{y_i}f(x-y,t-s) - \partial_{y_i}\big(\Phi(s,y)\cdot\partial_{y_i}f(x-y,t-s)\big)dyds \\
&\qquad = \int_\epsilon^t \int_{\mathbb{R}^n} \partial_s\Phi(s,y)f(x-y,t-s)dyds - \int_{\mathbb{R}^n} \Phi(t,y)f(x-y,0) - \Phi(\epsilon,y)f(x-y,t-\epsilon)dy \\
&\qquad\qquad\qquad\qquad - \sum_{i=1}^n \int_\epsilon^t \int_{\mathbb{R}^n} \partial_{y_iy_i}\Phi(s,y)f(x-y,t-s)dyds \\
&\qquad = \int_\epsilon^t \int_{\mathbb{R}^n} (\partial_s - \Delta_y)\Phi(s,y)f(x-y,t-s)dyds - \int_{\mathbb{R}^n} \Phi(t,y)f(x-y,0)dy \\
&\qquad\qquad\qquad\qquad + \int_{\mathbb{R}^n} \Phi(\epsilon,y)f(x-y,t-\epsilon)dy \\
&\qquad = -\int_{\mathbb{R}^n} \Phi(t,y)f(x-y,0)dy + \int_{\mathbb{R}^n} \Phi(\epsilon,y)f(x-y,t-\epsilon)dy
\end{aligned}
$$

So

$$J_\epsilon + K = \int_{\mathbb{R}^n} \Phi(\epsilon,y)f(x-y,t-\epsilon)dy \to f(x,t)$$

$\square$

2.1.3 Energy Method

Energy method is method for estimate many properties. Uniqueness is one application. There are various form of energy. In heat equation, we use energy as

$$E(t) = \int_{\mathbb{R}^n} \frac{1}{2}|u(t,x)|^2 dx$$

We do not know this quantity is not infinity. But if we assume u decays fast enough, we can prove uniqueness.

Theorem 12 (Uniqueness of Full Domain Homogenuous Heat Equation)**.** *For full domain homogenuous heat equation*

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u(t=0,x) = u_0(x) \end{cases}$$

There is upto one solution u that decays fast enough so $E(t) < \infty \quad \forall t \geq 0$

Proof. We can calculate derivative of energy.

$$
\begin{aligned}
\frac{d}{dt}E(t) &= \int_{\mathbb{R}^n} \partial_t u(t,x)u(t,x)dx \\
&= \int_{\mathbb{R}^n} u(t,x)\Delta u(t,x)dx \\
&= -\int_{\mathbb{R}^n} |\nabla u(t,x)|^2 dx \leq 0
\end{aligned}
$$

Thus, $E(t) \geq E(0)$. If we estimate energy method for $v-w$ which are two solutions v, w, $v-w$ is always zero. Therefore $v = w$ a.e. $\square$

However Energy method requires assumption. If we want to remove assumptions, we can use 'weighted' Energy method. Formula has changed with positve function $w(x) : \mathbb{R}^n \to \mathbb{R}^+$ that

$$E_w(t) = \frac{1}{2}\int_{\mathbb{R}^n} w(x)|u(t,x)|^2 dx$$

This w requires some conditions. $w(x)\cdot u(t,x)$ decays at infinity, $w > 0$, $|\nabla w(x)|^2 \leq C \cdot |w(x)|^2$. Then we could estimate derivative of weighted energy:

$$\begin{aligned}
\frac{d}{dt}E_w(t) &= -\int_{\mathbb{R}^n} \nabla(w(x)\cdot u(t,x))\nabla u(t,x)dx \\
&= -\int_{\mathbb{R}^n} w(x)|\nabla u(t,x)|^2 dx - \int_{\mathbb{R}^n} \nabla(w(x)\cdot \nabla u(t,x))u(t,x)dx
\end{aligned}$$

Second term could be estimated by Cauchy Schwartz Inequality,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n}(\nabla w(x)\cdot\nabla u(t,x))u(t,x)dx\right| &\leq \int_{\mathbb{R}^n} |u(t,x)|\cdot\frac{|\nabla w(x)|}{\sqrt{w(x)}}\sqrt{w(x)}|\nabla u(t,x)|dx \\
&\leq \int_{\mathbb{R}^n}\frac{4|u(t,x)|^2|\nabla w(x)|^2}{w(x)}dx + \int_{\mathbb{R}^n} w(x)|\nabla u(t,x)|^2 dx
\end{aligned}$$

Therefore,

$$\frac{d}{dt}E_w(t) \leq \int_{\mathbb{R}^n}\frac{4|u(t,x)|^2}{w(x)}|\nabla w(x)|^2 dx \leq 8CE_w(t)$$

$$E_w(t) \leq E_w(0)e^{8Ct}$$

Some examples for $w(x)$ is

$$w(x) = \frac{1}{(1+|x|^2)^N}$$

$$w(x) = e^{-|x|}$$

2.1.4 Mean Value Property, Maximum Principal

Mean Value Property is very important feature for Heat Equation and Laplace Equation. Briefly speaking, Mean Value Property is property that maximum is always gained in boundary.

Theorem 13 (Mean Value Theorem)**.** *If u satisfies $\partial_t u = \Delta u$ on some neighborhood of $D(t,x;r)$ which is*

$$D(t,x;r) = \left\{(s,y)\ \middle|\ s \leq t,\ \Phi(t-s, x-y) \geq \frac{1}{r^n}\right\}$$

Then following equation holds.

$$u(t,x) = \frac{1}{4r^n}\int_{D(t,x;r)} u(s,y)\cdot\frac{|x-y|^2}{(t-s)^2}dyds$$

Remark As theorem stating about Mean Value Property, only needed is heat equation holding on **some neighborhood**. So Mean Value Property is **Local Property**.

Proof. Actually we could shift coordinate so we only need to prove for $(t, x) = (0, 0)$. Define $\phi(r)$ as

$$\phi(r) = \frac{1}{4r^n} \int_{D(0,0;r)} u(s, y) \cdot \frac{|y|^2}{s^2} dyds$$

Change of variables could derive

$$\phi(r) = \frac{1}{4} \int_{D(0,0;1)} u(r^2 s, ry) \cdot \frac{|y|^2}{s^2} dyds$$

If we differentiate $\phi(r)$,

$$\begin{aligned}
\phi'(r) &= \frac{1}{4} \int_{D(0,0;1)} \left(2rs \cdot \partial_t u(r^2 s, ry) + y \cdot \nabla_x u(r^2 s, ry) \right) \frac{|y|^2}{s^2} dyds \\
&= \frac{1}{4r^{n+1}} \int_{D(0,0;r)} \left(\sum_{i=1}^{n} \partial_{x_i} u(s, y) y_i \frac{|y|^2}{s^2} + 2u_t(s, y) \cdot \frac{|y|^2}{s} \right) dyds
\end{aligned}$$

We separate two terms, each A, B.

Define

$$\psi(t, x) = \frac{n}{2} \log(-4\pi t) - \frac{|x|^2}{4t} - n \log r$$

Then $\psi \equiv 0$ in Boundary of $D(0, 0; r)$.

$$\begin{aligned}
B &= \int_{D(0,0;r)} 2u_t \cdot \frac{|y|^2}{s} dyds = \int_{D(0,0;r)} 4u_t \cdot \sum_{i=1}^{n} y_i \partial_{x_i} \psi(s, y) dyds \\
&= -\int_{D(0,0;r)} 4n\partial_t u(s, y)\psi(s, y) - 4 \sum_{i=1}^{n} \partial_t \partial_{x_i} u(s, y) y_i \psi(s, y) dyds \\
&= -\int_{D(0,0;r)} 4n\partial_t u(s, y)\psi(s, y) - 4 \sum_{i=1}^{n} \partial_{x_i} u(s, y) y_i \partial_t \psi(s, y) dyds \\
&= -\int_{D(0,0;r)} 4n\partial_t u(s, y)\psi(s, y) - 4 \sum_{i=1}^{n} \partial_{x_i} u(s, y) y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) dyds \\
&= -\int_{D(0,0;r)} \sum_{i=1}^{n} \partial_{x_i} u(s, y) \cdot y_i \cdot \frac{|y|^2}{s^2} dyds \\
&\qquad\qquad - \int_{D(0,0;r)} \left(4n\partial_t u(s, y)\psi(s, y) - \frac{2n}{s} \sum_{i=1}^{n} \partial_{x_i} u(s, y) \cdot y_i \right) dyds \\
&= -A - \int_{D(0,0;r)} \left(4n\Delta u(s, y)\psi(s, y) - \frac{2n}{s} \sum_{i=1}^{n} \partial_{x_i} u(s, y) \cdot y_i \right) dyds \\
&= -A - \int_{D(0,0;r)} \left(4n \sum_{i=1}^{n} \partial_{x_i} u(s, y) \partial_{x_i} \psi(s, y) - \frac{2n}{s} \sum_{i=1}^{n} \partial_{x_i} u(s, y) \cdot y_i \right) dyds \\
&= -A
\end{aligned}$$

We have $\phi'(r) = 0$ and as $r \downarrow 0$, value tends to $u(0,0)$ similar reason by Lebesgue point. $\square$

By upper proof, we gain Maximum Property.

Theorem 14 (Maximum Principal)**.** *If u satisfies $\partial_t u = \Delta u$ on domain $U_T = (0,T] \times U$*

$$\max_{\bar{U}_T} u = \max_{\bar{U}_T - U_T} u$$

Also if U is connected and there exists $(t_0, x_0) \in U_T$ achieves maximum then u is constant in $\bar{U}_{t_0}$

Proof. This is consequence of mean value property because if we achieve maximum at interior, we can draw $D(t,x;r)$ and mean value property says u is constant in this region. This can be extended until it intersects boundary. $\square$

2.1.5 Quantitive Bounds of Full Domain Heat Equation

We can estimate derivatives of solution.

Theorem 15. *For the solution of Full Domain Heat Equation, following holds.*

$$\sup_{x \in \mathbb{R}^n} |\partial_t^{(i)} \partial_x^{(k)} u(t,x)| \le \frac{C_{j,|k|}}{t^{j+|k|/2}} \sup_{x \in \mathbb{R}^n} |u_0(x)|$$

Proof. We have solution for Full Domain Heat Equation.

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) u_0(y) dy$$

So if we derivate this,

$$\begin{aligned}
\partial_t^{(j)} \partial_x^{(k)} u(t,x) &= \int_{\mathbb{R}^n} \partial_t^{(j)} \partial_x^{(k)} \Phi(t, x-y) u_0(y) dy \\
&\le \sup_{y \in \mathbb{R}^n} |u_0(y)| \cdot \int_{\mathbb{R}^n} \partial_t^{(j)} \partial_x^{(k)} \Phi(t, x-y) dy \\
&= \sup_{y \in \mathbb{R}^n} |u_0(y)| \cdot \int_{\mathbb{R}^n} \partial_t^{(j)} \partial_{x'}^{(k)} \Phi(t, \sqrt{t}(x'-z)) \cdot \frac{1}{(\sqrt{t})^{|k|}} \cdot \frac{1}{(\sqrt{t})^n} dz \\
&= \sup_{y \in \mathbb{R}^n} |u_0(y)| \cdot \int_{\mathbb{R}^n} \left(\partial_t^{(j)} \partial_{x'}^{(k)} \frac{1}{(4\pi t)^{n/2}} e^{-|x'-z|^2/4} \right) \cdot \frac{1}{(\sqrt{t})^{|k|}} \cdot \frac{1}{(\sqrt{t})^n} dz \\
&\le \frac{C_{j,|k|}}{t^{j+|k|/2}} \sup_{x \in \mathbb{R}^n} |u_0(x)|
\end{aligned}$$

$\square$

2.2 Laplace Equation

Laplace Equation is $\Delta u = 0$. We could find relationship between Laplace Equation and Heat Equation, we could interpret time limit of Heat Equation is Laplace Equation. We will again, begin with full domain problem.

2.2.1 Full Domain Laplace Equation

Theorem 16 (Full Domain Laplace Equation)**.**

$$\begin{cases} \Delta u = 0 \\ |u| \to 0 \quad as \quad |x| \to \infty \end{cases}$$

$x \in \mathbb{R}^n$. One solution for this problem is $\mathcal{N}(x)$

$$\mathcal{N}(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases}$$

This comes from simple computation.

2.2.2 Poisson's Equation

Poisson's Equation can solved by above $\mathcal{N}$ function.

Theorem 17 (Poisson's Equation)**.** *Poisson's Equation*

$$\begin{cases} \Delta u = -f \\ |u| \to 0 \quad as \quad |x| \to \infty \end{cases}$$

has solution $u(x) = (\mathcal{N} * f)(x)$ if $f \in C_c^2(\mathbb{R}^n)$

Proof. It needs careful calculation. First, by theorem of convolution,

$$\Delta u(x) = \int_{\mathbb{R}^n} \mathcal{N}(y) \Delta_x f(x-y) dy$$

We compute with separate terms

$$I_\epsilon = \int_{B(0,\epsilon)} \mathcal{N}(y) \Delta_x f(x-y) dy$$

$$J_\epsilon = \int_{\mathbb{R}^n - B(0,\epsilon)} \mathcal{N}(y) \Delta_x f(x-y) dy$$

First I_ϵ can be estimated:
For $n = 2$,

$$|I_\epsilon| \leq C ||D^2 f||_{L^\infty} \cdot \int_{B(0,\epsilon)} |\mathcal{N}(y)| dy \leq C\epsilon^2 |\log \epsilon|$$

For $n \geq 3$,

$$|I_\epsilon| \leq C ||D^2 f||_{L^\infty} \cdot \int_{B(0,\epsilon)} |\mathcal{N}(y)| dy \leq C\epsilon^2$$

Second J_ϵ can be estimated:

$$\begin{aligned}
J_\epsilon &= \int_{\mathbb{R}^n - B(0,\epsilon)} \mathcal{N}(y)\Delta_y f(x-y)dy \\
&= \int_{\mathbb{R}^n - B(0,\epsilon)} -\nabla_y \mathcal{N}(y) \cdot \nabla_y f(x-y)dy + \int_{\partial B(0,\epsilon)} \mathcal{N}(y)\frac{\partial f}{\partial \nu}(x-y)dS(y) \\
&= K_\epsilon + L_\epsilon
\end{aligned}$$

Each term could be estimated:

$$|L_\epsilon| \leq ||Df||_{L^\infty} \int_{\partial B(0,\epsilon)} \mathcal{N}(y)dS(y)$$

so for $n = 2$, $|L_\epsilon| \leq C\epsilon|\log \epsilon|$, $n \geq 3$ then $|L_\epsilon| \leq C\epsilon$

$$\begin{aligned}
K_\epsilon &= \int_{\mathbb{R}^n - B(0,\epsilon)} -\nabla_y \mathcal{N}(y) \cdot \nabla_y f(x-y)dy \\
&= \int_{\mathbb{R}^n - B(0,\epsilon)} \Delta \mathcal{N}(y) \cdot f(x-y)dy - \int_{\partial B(0,\epsilon)} \frac{\partial}{\partial \nu}\mathcal{N}(y)f(x-y)dS(y) \\
&= -\int_{\partial B(0,\epsilon)} \frac{\partial}{\partial \nu}\mathcal{N}(y)f(x-y)dS(y) \\
&= -\int_{\partial B(0,\epsilon)} \frac{1}{n\alpha(n)\epsilon^{n-1}} f(x-y)dS(y) \\
&= -\frac{1}{m^{n-1}(\partial B(0,\epsilon))} \int_{\partial B(0,\epsilon)} f(y)dS(y) \to -f(x)
\end{aligned}$$

We used Lebesgue point theorem and Gauss-Green theorem, Green's formula. □

2.2.3 Mean Value Property, Maximum Principal

Again, Mean Value Property is **Local Property** it can be applied into restricted domain or very small domain.

Theorem 18 (Mean Value Property). *$u \in C^2(U)$ and $\Delta u = 0$ then*

$$u(x) = \frac{1}{m^{n-1}(\partial B(x,r))} \int_{\partial B(x,r)} udS = \frac{1}{m^n(B(x,r))} \int_{B(x,r)} udy$$

for each $B(x,r) \subset U$

Proof. Proof is similar to Heat Equation's Mean Value Property. Let $\phi(r) = \frac{1}{m^{n-1}(\partial B(x,r))} u(y)dS(y)$. By easy computation,

$$\begin{aligned}
\phi'(r) &= \frac{1}{m^{n-1}(\partial B(0,1))} \int_{\partial B(0,1)} Du(x+rz) \cdot zdS(z) \\
&= \frac{1}{m^{n-1}(\partial B(x,r))} \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(y) \\
&= \frac{1}{m^{n-1}(\partial B(x,r))} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS(y) \\
&= \frac{r}{n} \frac{1}{m^n(B(x,r))} \int_{B(x,r)} \Delta u(y)dy = 0
\end{aligned}$$

So $\phi(r) = \lim_{t \to 0} \phi(t) = u(x)$

For second equation, we can integrate by polar and could derive it. $\square$

On the other hand, it is interesting that converse of Mean Value Property holds.

Theorem 19 (Converse of Mean Value Property). *If $u \in C^2(U)$ satisfies*

$$u(x) = \frac{1}{m^{n-1}(\partial B(x,r))} \int_{\partial B(x,r)} udS$$

for all $B(x,r) \subset U$, $\Delta u = 0$ on U.

Proof. If $\Delta u > 0$, since $u \in C^2(U)$, there is a ball $B(x,r)$ that $\Delta u > 0$ on this ball. By Mean Value Property proof's computation,

$$0 = \phi'(r) = \frac{r}{n} \frac{1}{m^n(B(x,r))} \int_{B(x,r)} \Delta u(y) dy > 0$$

contradiction. Same for $\Delta u < 0$. $\square$

Maximum Principal comes straight foward.

Theorem 20 (Maximum Principal). *$u \in C^2(U) \cap C(\bar{U})$ is harmonic in U then*
(1) $\max_{\bar{U}} u = \max_{\partial U} u$
(2) If U is connected and $x_0 \in U$ achieves maximum then u is constant in U

2.2.4 Regularity

Actually, for Laplace Equation, regularity imporves. To be more precise, if $u \in C(U)$ satisfies mean value property for all ball $B(x,r) \in U$ then u is actually $C^\infty(U)$

Theorem 21 (Improved Regularity). *If $u \in C(U)$ satisfies mean value property for all ball $B(x,r) \in U$ then $u \in C^\infty(U)$*

Proof. Key idea is mollification. Let U_ϵ be a set $U_\epsilon = \{x \in U \mid dist(x, \partial U) > \epsilon\}$. Mollification in this area is possible, so we gain $u^\epsilon \in C^\infty(U_\epsilon)$
Now our goal is proving $u \equiv u_\epsilon$ in U_ϵ. This could be computed as below

$$\begin{aligned}
u^\epsilon(x) &= \int_U \eta_\epsilon(x-y)u(y)dy \\
&= \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)dy \\
&= \frac{1}{\epsilon^n} \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) \int_{\partial B(x,r)} udSdr \\
&= \frac{1}{\epsilon^n} \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) n\alpha(n) r^{n-1} \cdot u(x) dr \\
&= u(x)
\end{aligned}$$

Therefore, $\epsilon \downarrow 0$ achieves our goal. $\square$

2.2.5 Quantitive Bounds of Laplace Equation

We can estimate derivatives of solution.

Theorem 22. *Assume u is Harmonic in U. Then following holds.*

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}}||u||_{L^1(B(x_0,r))}$$

for each $B(x_0,r) \subset U$, $|\alpha| = k$

Proof. For $k = 0$ it is just mean value property.
For $k = 1$

$$\begin{aligned}|\partial_{x_i} u(x_0)| &\leq \left|\frac{1}{|B(x_0,r/2)|}\int_{B(x_0,r/2)} \partial_{x_i} u dx\right| \\ &= \left|\frac{2^n}{\alpha(n)r^n}\int_{\partial B(x_0,r/2)} u\nu_i dS\right| \\ &\leq \frac{2n}{r}||u||_{L^\infty(\partial B(x_0,r/2))}\end{aligned}$$

We need estimate on u with $L^\infty(\partial B(x_0,r/2))$. If $x \in \partial B(x_0,r/2)$, $B(x,r/2) \subset B(x_0,r) \subset U$.

$$|u(x)| \leq \frac{1}{\alpha(n)}\left(\frac{2}{r}\right)^n ||u||_{L^1(B(x_0,r))}$$

so

$$|\partial_{x_i} u(x_0)| \leq \frac{2^{n+1}n}{\alpha(n)r^{n+1}}||u||_{L^1(B(x_0,r))}$$

For $k \geq 2$, we could repeat process of $k = 1$ since derivative $D^\alpha u$ is also harmonic. So for α, define β as $D^\alpha = \partial_{x_i} D^\beta$ and

$$\begin{aligned}|D^\alpha u(x_0)| &\leq \frac{nk}{r}||D^\beta u||_{L^\infty(\partial B(x_0,r/k))} \\ &\leq \frac{nk}{r}\cdot\frac{(2^{n+1}n(k-1))^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}}||u||_{L^1(B(x_0,r))} \\ &= \frac{(2^{n+1}nk)^k}{\alpha(n)r^{n+k}}||u||_{L^1(B(x_0,r))}\end{aligned}$$

□

2.2.6 Liouville's Theorem

Now using Quantitive Bounds of Laplace Equation, we can gain that in full domain laplace equation, only bounded solution is constant solution.

Theorem 23 (Liouville's Theorem)**.** *$u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded then u is constant*

Proof. Since quantitive bound of Laplace Equation,

$$|Du(x_0)| \leq \frac{\sqrt{n}}{r^{n+1}}C||u||_{L^1(B(x_0,r))} \leq \frac{\sqrt{n}C\alpha(n)}{r}||u||_{L^\infty(\mathbb{R}^n)}$$

as $r \to \infty$, this bound is zero. So derivative of u is zero, which means it is constant. □

2.2.7 Analyticity of Laplace Equation

From Regularity section, we proved Laplace Equation at boundary U, actually is C^∞. Moreover, we can prove solution u in U is analytic.

Theorem 24 (Analyticity)**.** *If $\Delta u = 0$ in U, then u is analytic in U.*

Proof. $x_0 \in int(U)$, define r as $\frac{1}{4}dist(x_0, \partial U)$. We know

$$M = \frac{1}{\alpha(n)r^n}||u||_{L^1(B(x_0,2r))} < \infty$$

For $x \in B(x_0, r)$, $B(x, r) \subset B(x_0, 2r) \subset U$. We can estimate $D^\alpha u$.

$$\begin{aligned}
||D^\alpha u||_{L^\infty(B(x_0,r))} &\leq M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|} \cdot |\alpha|^{|\alpha|} \\
&\leq CM\left(\frac{2^{n+1}n^2e}{r}\right)^{|\alpha|} \cdot |\alpha|!
\end{aligned}$$

So in Taylor Series

$$\sum_\alpha \frac{D^\alpha u(x_0)}{|\alpha|!}(x - x_0)^\alpha$$

We can estimate remainder term in

$$|x - x_0| < \frac{r}{2^{n+2}n^3e}$$

That

$$\begin{aligned}
R_N(x) = u(x) &- \sum_{k=0}^{N-1}\sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x - x_0)^\alpha}{|\alpha|!} \\
&= \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0)) \cdot (x - x_0)^\alpha}{|\alpha|!}
\end{aligned}$$

$$|R_N(x)| \leq CM\sum_{|\alpha|=N}\left(\frac{2^{n+1}n^2e}{r}\right)^N\left(\frac{r}{2^{n+2}n^3e}\right)^N \leq \frac{CM}{2^N} \to 0$$

□

3 Banach space Embedding

3.1 Banach Space Embedding(= Inequality) in Homogenuous norm

3.1.1 Function space

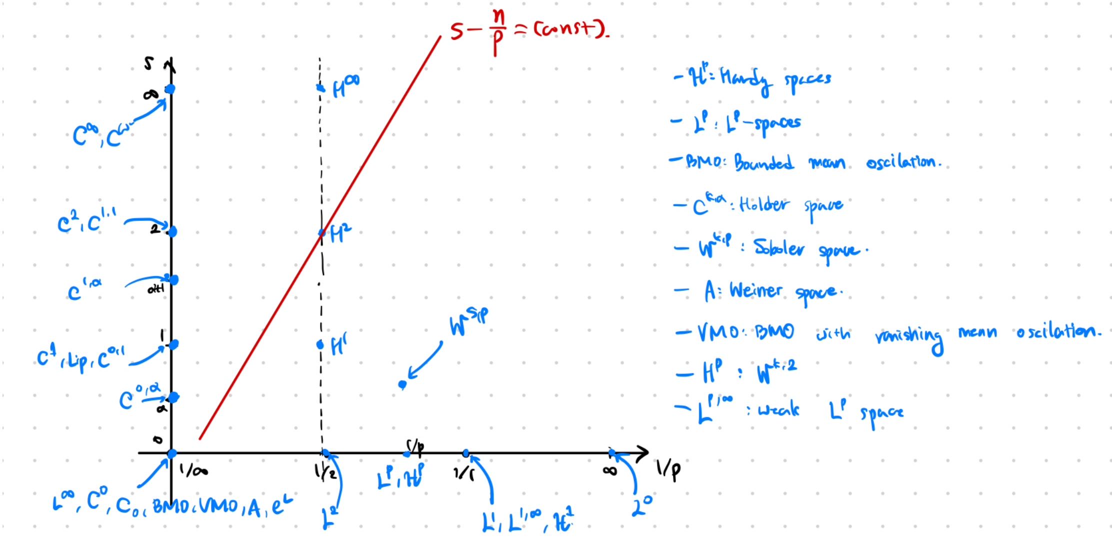


Figure 1: Integrability index via Differentiability index

We only consider $f \in L^1_{loc}$.

Hardy space :

$$||f||_{\mathcal{H}^p(U)} = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

L^p space :

$$||f||_{L^p} = \left(\int |f|^p \right)^{1/p}$$

BMO space :

$$||f||_{BMO} = \sup_{Q \subset U, Q \ a \ cube} \frac{1}{|Q|} \int_Q |f - f_Q|$$

Holder space : For $0 < \gamma \le 1$

$$||f||_{C^{k,\gamma}} = \sum_{|\alpha| \le k} ||D^\alpha f||_{C(\bar{U})} + \sum_{|\alpha| = k} \left\{ \sup_{x,y \in U} \left(\frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\alpha} \right) \right\}$$

Sobolev space :

$$||f||_{W^{k,p}} = \sum_{|\alpha| < k} ||D^\alpha f||_{L^p}$$

Weiner space :

$$||f||_A = \int_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi$$

VMO space : Vanishes at infinity and

$$||f||_{VMO} = \sup_{Q \subset U, Q \ a \ cube} \frac{1}{|Q|} \int_Q |f - f_Q|$$

H^p space : $W^{k,2}$
$L^{p,\infty}$: With $\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$

$$||f||_{L^{p,\infty}} = \sup_{\alpha > 0} (\alpha^p \lambda_f(\alpha))^{1/p}$$

C^k space :

$$||f||_{C^k} = \sum_{|\alpha| \le k} ||D^\alpha f||_{C(\bar{U})}$$

$B^s_{p,q}$ space :

$$||f||_{B^s_{p,q}} = \left(||f||^q_{W^{\lfloor s \rfloor, p}} + \int_0^\infty \left| \frac{\sup_{|h| \le t} ||f(\cdot + 2h) - 2f(\cdot + h) + f(\cdot)||_{L^p}}{t^{\{s\}}} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

3.1.2 Principals of homogenuous Embedding

We want an inequality $||f||_{\dot{Y}} \le C||f||_{\dot{X}}$ to obtain homogenuous Embedding $X \hookrightarrow Y$

1. Have same scale parameter $*_Y = *_X$

$$*_X = s_X - \frac{n}{p_X}$$

If Inequality established, $||f||_{\dot{Y}} \le C||f||_{\dot{X}}$. If we change f as $f(\lambda x)$

$$||f(\lambda x)||_{\dot{Y}} \le C||f(\lambda x)||_{\dot{X}}$$
$$\lambda^{*_X} ||f||_{\dot{Y}} \le C\lambda^{*_Y} ||f||_{\dot{X}}$$

2. Higher Differentiability could be embedded to lower differentiablility
For arbitrary function, multiply oscillating term :

$$f \to f e^{i\xi \cdot x}$$

Then for small ξ, inequality need to hold, $s_X \ge s_Y$ is needed.

3. Homogenuous Embedding might fail
It does not hold everytime.

3.2 Inhomogenuous Embedding

3.2.1 Banach Space Operation

Addition

$$X + Y = \{f : f = g + h, \quad g \in X,\ h \in Y\}$$
$$||f||_{X+Y} = \inf_{g,h}(||g||_X + ||h||_Y)$$

Intersection

$$X \cap Y = \{f : f \in X, Y\}$$
$$||f||_{X\cap Y} = ||f||_X + ||f||_Y$$

3.2.2 On the generous Inhomogenuous Embedding Situation

We now on interest $X_1 \cap X_2 \hookrightarrow Y$. I.e. $||f||_Y \le C||f||_{X_1}^{\gamma}||f||_{X_2}^{1-\gamma}$

1. **Have same scale parameter** $*_Y = *_X$

$$*_Y = *_1 \cdot \gamma + *_2 \cdot (1-\gamma)$$

2. **Differentiability condition**

$$\min(s_1, s_2) \le s_Y \le \max(s_1, s_2)$$

3. **Differentiability condition strengthen**

$$\min(s_1, s_2) < s_Y < \max(s_1, s_2)$$

Then, it is known that Inhomogenuous Embedding holds. **So Inhomogenuous Embedding is more robust.**

3.2.3 Young's inequality

Theorem 25 (Young's inequality)**.** *For* $1 \le p, q, r \le \infty$ *and satisfies* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $f \in L^p(\mathbb{R}^n),\ g \in L^q(\mathbb{R}^n)$

$$||f * g||_{L^r(\mathbb{R}^n)} \le C||f||_{L^p(\mathbb{R}^n)}||g||_{L^q(\mathbb{R}^n)}$$

3.2.4 HLS Inequality (Hardy Littlewood Sobolev Inequality)

Theorem 26 (HLS inequality)**.** *For* $f \in L^p(\mathbb{R}^n),\ g \in L^q(\mathbb{R}^n)$ *and* $1 < p, q$ *with* $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{n} = 2$ *then following inequality holds.*

$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)|x-y|^{-\lambda}g(y)dxdy \le C_{p,\lambda,n}||f||_{L^p}||g||_{L^q}$$

Or, in Banach Space it is useful to state other theorem.

Theorem 27 (HLS inequality)**.** *If* $g \in L^q(\mathbb{R}^n)$*,* $0 < \gamma < n$ *and* $1 + \frac{1}{p} = \frac{1}{q} + \frac{\gamma}{n}$ *then following inequality holds.*

$$\left|\left|g * \frac{1}{||\cdot||^\gamma}\right|\right|_{L^p} \leq C||g||_{L^q}$$

Proof uses **Hardy Littlewood maximal function**. First, we will look at Hardy Littlewood maximal function

3.2.5 Hardy Littlewood maximal function

Definition - Hardy Littlewood maximal funciton : For locally integrable function, $f \in L^1_{loc}(U)$

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

Useful theorem is well known.

Theorem 28 (Properties of Hardy Littlewood maximal funciton)**.** *If* $f \in L^p(\mathbb{R}^n)$*, for locally integrable function* $\omega : \mathbb{R}^n \to (0,\infty)$ *and within measure* $\omega(E) = \int_E \omega(y)dy$*,* M *is actually, mapping of followings :*
1. $M : L^1(M(\omega)dx) \to L^{1,\infty}(\omega dx)$ *i.e.*

$$\omega(\{x : |M(f)(x)| > \lambda\}) < \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)||M(\omega)(x)|dx$$

2. $M : L^p(M(\omega)dx) \to L^p(\omega dx)$ *for* $1 < p \leq \infty$ *i.e.*

$$\int_{\mathbb{R}^n} |M(f)(x)|^p \omega(x)dx < C \int_{\mathbb{R}^n} |f(x)|^p M(\omega)(x)dx$$

Proof for Theorem 12. Prove by three steps.

1. Proving $M : L^\infty(M(\omega)dx) \to L^\infty(\omega dx)$
If $f \in L^\infty(M(\omega)dx)$,

$$\begin{aligned}||Mf||_{L^\infty(\omega dx)} = \inf_{\omega(E)=0} \sup_{x \in E^c} |M(f)(x)| &\leq \inf_{|E|=0} \sup_{x\in E^c} |M(f)(x)| \\ &\leq \inf_{|E|=0} \sup_{x\in E^c} |f(x)| = \inf_{M(\omega)(E)=0} \sup_{x\in E^c} |f(x)| \\ &= ||f||_{L^\infty(M(\omega)dx)}\end{aligned}$$

2. Proving Theorem of case $p = 1$
Our goal is to prove for $f \in L^1(M(\omega)dx)$,

$$\omega(\{x : |M(f)(x)| > \lambda\}) < \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)||M(\omega)(x)|dx$$

Let $K \subseteq \{x : |M(f)(x)| > \lambda\}$ compact. If $x \in K$, there are some $r_x > 0$ that

$$\frac{1}{|B(x,r_x)|} \int_{B(x,r_x)} |f(y)|dy > \lambda$$

$K \subset \bigcup_{x \in K} B(x, r_x)$ so Finite cover exists and by Vitali's covering lemma, there are subcollection S that $K \subseteq \bigcup_{B \in S} 3B$ and $B \in S$ does not overlap.

Now, following inequality holds.

$$
\begin{aligned}
\omega(K) &\leq \sum_{B \in S} \omega(3B) = \sum_{B_j \in S} \int_{3B_j} \omega(x) dx \\
&\leq \sum_{B_j \in S} \int_{B_{4r_j}(y_j)} \omega(x) dx \qquad\qquad (y \in B_j) \\
&\leq \sum_{B_j \in S} M(\omega)(y) |B_{4r_j}(y)| = \sum_{B_j \in S} M(\omega)(y) \cdot 4^n \cdot |B_j| \\
&\leq C \sum_{B_j \in S} \frac{1}{\lambda} \frac{1}{|B_j|} \int_{B_j} |f(y)| M(\omega)(y) dy \cdot 4^n |B_j| \\
&\leq C \frac{1}{\lambda} \sum_{B_j \in S} \int_{B_j} |f(y)| M(\omega)(y) dy \\
&\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M(\omega)(y) dy = \frac{C}{\lambda} ||f||_{L^1(M(\omega)dx)}
\end{aligned}
$$

3. Proving for arbitrary p

This is by **Marcinkiewicz interpolation theorem**, Step 1, 2 implies for arbitrary $1 < p < \infty$, inequality holds. □

Within Hardy Littlewood maximal function, we can prove HLS inequality. For special case $\omega \equiv 1$ we can obtain for any $f \in L^p$, $M(f(x))$ finite almost everywhere.

Proof of HLS Inequality (Theorem 11). First, partitioning integral by this:

$$
\left(f * \frac{1}{|| \cdot ||^\gamma} \right)(x) = \int_{|x-y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy + \int_{|x-y| > R} \frac{f(y)}{|x-y|^\alpha} dy
$$

1. First integral part

$$
\begin{aligned}
\left| \int_{|x-y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \right| &\leq \sum_{r \in 2^{\mathbb{Z}}, r \leq R} \int_{r \leq |x-y| < 2r} \frac{|f(y)|}{|x-y|^\alpha} dy \\
&\leq \sum_{r \in 2^{\mathbb{Z}}} \frac{1}{r^\alpha} \int_{|x-y| < 2r} |f(y)| dy \\
&\leq C \sum_{r \in 2^{\mathbb{Z}}} \frac{1}{r^\alpha} r^n \frac{1}{|B(x, 2r)|} \int_{|x-y| < 2r} |f(y)| dy \leq C M(f)(x) \sum_{r \in 2^{\mathbb{Z}}} r^{n-\alpha} \\
&\leq R^{n-\alpha} M(f(x))
\end{aligned}
$$

This estimate holds almost everywhere because $M(f(x))$ exists almost everywhere.

2. Second integral part

$$\int_{|x-y|>R} \frac{f(y)}{|x-y|^\alpha} dy = \left(f * \left(\frac{1}{|x|^\alpha}\chi_{\{|x|>R\}}\right)\right)(x)$$

By Young's convolution inequality or Holder inequality,

$$\begin{aligned}
\left|\left|f * \left(\frac{1}{|x|^\alpha}\chi_{\{|x|>R\}}\right)\right|\right|_{L^\infty} &\leq ||f||_{L^p}\left|\left|\frac{1}{|x|^\alpha}\chi_{\{|x|>R\}}\right|\right|_{L^{p'}} \\
&= ||f||_{L^p}\Big(\int_{|x|>R} \frac{1}{|x|^{p'\alpha}} dx)^{\frac{1}{p'}} \\
&\leq C||f||_{L^p} R^{\frac{n}{p'}-\alpha}
\end{aligned}$$

3. Unite two result

We obtained

$$\int_{|x-y|\leq R} \frac{f(y)}{|x-y|^\alpha} dy \leq CM(f)(x)R^{n-\alpha}$$

$$\int_{|x-y|>R} \frac{f(y)}{|x-y|^\alpha} dy \leq C||f||_{L^p} R^{\frac{n}{p'}-\alpha}$$

Choosing $R \sim \left(\frac{||f||_{L^p}}{M(f)(x)}\right)^{\frac{p}{n}}$, we obtain by **Theorem 12**.

$$\left|\left|f * \frac{1}{|x|^\alpha}\right|\right|_{L^q} \leq C||f||_{L^p}^{1-\frac{p}{q}}||M(f)(x)||_{L^p}^{\frac{p}{q}} \leq C||f||_{L^p}$$

□

3.2.6 Gagliardo - Nirenberg Inequality

Statement is as following

Theorem 29 (Gagliardo-Nirenberg Inequality)**.** *Let $1 \leq q \leq \infty$ be a positive extended real quantity. Let j and m be non-negative integers such that $j < m$. Furthermore, let $1 \leq r \leq \infty$ be a positive extended real quantity, $p \geq 1$ be real, $\theta \in [0,1]$ such that relations*

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q}, \qquad \frac{j}{m} \leq \theta \leq 1$$

Then,

$$||D^j u||_{L^p(\mathbb{R}^n)} \leq C||D^m u||_{L^r(\mathbb{R}^n)}^\theta ||u||_{L^q(\mathbb{R}^n)}^{1-\theta}$$

for any $u \in L^q(\mathbb{R}^n)$ such that $D^m u \in L^r(\mathbb{R}^n)$.

Except two cases:

1. If $j = 0, q = \infty, rm < n$ then to establish inequality, additional assumption: u tends to 0 at infinity or $u \in L^s(\mathbb{R}^n)$ for some finite s.
2. If $r > 1$ and $m - j - \frac{n}{j}$ is a nonnegative integer, additional assumption : $\theta < 1$ is needed.

We could generalize this result into non integer s value.

Theorem 30 (Gagliardo-Nirenberg Inequality in Sobolev spaces)**.** *First generalize Sobolev spaces*
Define for $1 \leq p < \infty$

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{s+\frac{n}{p}}} \in L^p(\Omega \times \Omega) \right\}$$

in $s \in (0,1)$ *and for general* s,

$$W^{s,p}(\Omega) = \left\{ u \in W^{\lfloor s \rfloor, p}(\Omega) : D^{\lfloor s \rfloor} u \in W^{\{s\},p}(\Omega) \right\}$$

For $p = \infty$,

$$W^{s,\infty}(\Omega) = \left\{ u \in L^\infty(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{s}} \in L^\infty(\Omega \times \Omega) \right\}$$

in $s \in (0,1)$ *and for general* s,

$$W^{s,\infty}(\Omega) = \left\{ u \in W^{\lfloor s \rfloor, \infty}(\Omega) : D^{\lfloor s \rfloor} u \in W^{\{s\},\infty}(\Omega) \right\}$$

It's norm is defined as

$$||u||_{W^{s,p}(\Omega)} = \left(||u||^p_{L^p(\Omega)} + ||D^{\lfloor s \rfloor}||^p_{W^{\{s\},p}(\Omega)} \right)^{\frac{1}{p}}$$

$$||u||_{W^{s,\infty}(\Omega)} = ||u||_{L^\infty(\Omega)} + ||D^{\lfloor s \rfloor} u||_{W^{\{s\},\infty}(\Omega)}$$

Then for Ω *a whole space or a half-space or bounded Lipshitz domain and following holds :*

$$1 \leq p, p_1, p_2 \leq \infty, \quad s, s_1, s_2 \in \mathbb{R}^+ \cup \{0\}$$

If

$$\theta \in (0,1), \quad s_1 \leq s_2, \quad s = \theta s_1 + (1-\theta) s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

then following is equivalent.

1. *For* $u \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$

$$||u||_{W^{s,p}(\Omega)} \leq C ||u||^\theta_{W^{s_1,p_1}(\Omega)} ||u||^{1-\theta}_{W^{s_2,p_2}(\Omega)}$$

2. *At least one of them is false*

$$\begin{cases} s_2 \in \mathbb{N}, s_2 \geq 1 \\ p_2 = 1 \\ 0 < s_2 - s_1 \leq 1 - \dfrac{1}{p_1} \end{cases}$$

Furthermore, C *does not depend on* u.

3.3 Role of C_c^∞ technique in Embedding

First, this is some useful Theorem on Embedding

Theorem 31. *If C_c^∞ is dense in X, Y then $X \hookrightarrow Y$ is equivalent to $||f||_Y \leq C||f||_X$ for all $f \in C_c^\infty$ and some minor condition. (That is if $||f||_X$ small, $||f||_Y$ is small also)*

That is, we only need to check for compactly supported C^∞ function if this space is dense in X, Y. Additional condition does not require constant C for the ratio. It is type of epsilon - delta so this might be proved much easier.

Proof. Let $g_1, g_2, \cdots g_n, \cdots$ a C_c^∞ functions converging to $f \in X$. Then g_n is cauchy sequence on X, $||g_n - g_m||_Y \leq C||g_n - g_m||_X$ so g_n is cauchy sequence on Y also. X, Y are Banach space so Y has unique limit : h, $h = f$. Then we can find $N \in \mathbb{N}$ such that if $n \geq N$,

$$C||f||_X \geq C||g_n||_X - C||f - g_n||_X \geq ||g_n||_Y - \epsilon \geq ||f||_Y - ||f - g_n||_Y - \epsilon \geq ||f||_Y - 2\epsilon$$

So $||f||_Y \leq C||f||_X$. □

However, we need to be careful if C_c^∞ is not dense in given function space.

3.3.1 L^∞ space

$f(x) = sgn(x)$ is function that cannot expressed as limit of C_c^∞ function. If $\tilde{f} \in C_c^\infty$, $||f - \tilde{f}||_{L^\infty} \geq 1$ because at point 0 we cannot approximate by C^∞. So

$$\bar{C_c^\infty}^{L^\infty} \neq L^\infty$$

In fact, Completion via L^∞ norm is C^0 which contains continuous function that vanishes at infinity.

3.3.2 C^α space

Completion of C_c^∞ to C^α norm is not C^α space. In fact, it is function space c^α which contatins function that $\frac{|f(x)-f(y)|}{|x-y|^\alpha}$ is bounded and tends to 0 if $|x - y| \to 0$.

3.3.3 Non completion of C_c^∞ Embedding

We need to be careful if C_c^∞ is not dense on given function space. This is occured with two problems.

First, C^∞ has **Modulus of Integrability**. If we complete C^∞ via some norm, they could often strength conditions.

For example, for C^α, condition is strengthened like below.

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \quad bdd \quad \Rightarrow \quad \frac{|f(x) - f(y)|}{|x - y|^\alpha} \to 0$$

for L^∞,

$$esssup \quad exist \Rightarrow continuous$$

Second, there are functions that are 'just in' that function space.
For example, $sgn(x)$ only belongs to L^∞ but not in C^0, $g_\alpha(x) = \frac{|x|^\alpha}{1+|x|^2}$ only belongs to C^α but not in c^α.

In summary, C^∞ has **Modulus of Integrability**. This can cause phenomena that if two function spaces share its norm then, this **Modulus of Integrability** gives additional properties. This can cause problem for function that 'only' belongs to function space.

4 Sobolev Spaces

4.1 Naive Approach on Holder Spaces and Sobolev Spaces

4.1.1 Holder Spaces

$U \subset \mathbb{R}^n$ open and $0 < \gamma \leq 1$. if $u : U \to \mathbb{R}^n$ is bdd continuous,

$$||u||_{C(\bar{U})} := \sup_{x \in U} |u(x)|$$
$$||u||_{C^{\dot{0},\gamma}(\bar{U})} := \sup_{x,y \in U, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^r} \right\}$$
$$||u||_{C^{0,\gamma}(\bar{U})} := ||u||_{C(\bar{U})} + ||u||_{C^{\dot{0},\gamma}(\bar{U})}$$

Second norm is called Holder seminorm, and Third term is called Holder norm. Holder Space is defined by Holder norm,

$$C^{k,r}(\bar{U})$$

Is the Holder space, consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$||u||_{C^{k,r}(\bar{U})} := \sum_{|\alpha| \leq k} ||D^\alpha u||_{C(\bar{U})} + \sum_{|\alpha| = k} ||D^\alpha u||_{C^{\dot{0},\gamma}(\bar{U})}$$

This is Banach Space.

4.1.2 Sobolev Spaces

Weak Derivatives is crucial to define Sobolev Spaces. It remains Question **Why do we define 'Weak Derivatives'**

Weak derivatives are defined to satisfy integration by parts for multiplied function that is compact supported on U and infinitely differential functions.

If $u, v \in L^1_{loc}(U)$ and α multiindex, v is α th Weak partial derivative of u if

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

for all $\phi \in C_c^\infty(U)$

If Weak derivative exists, than it is known that it is uniquely defined up to a set of measure zero.

So the Sobolev space is defined in following norm for $1 \leq p < \infty$

$$||u||_{W^{k,p}(U)} := \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}}$$

and

$$||u||_{W^{k,\infty}(U)} := \sum_{|\alpha| \leq k} ess \sup_U |D^\alpha u|$$

If the norm is finite, the function belongs to $W^{k,p}(U)$

4.1.3 Understanding Holder spaces and Lp spaces by functions

Holder space

$$g_\alpha(x) = \frac{|x|^\alpha}{1+|x|^2}$$

$$g_\alpha \in C^\alpha - \left(\bigcup_{\beta>\alpha} C^\beta\right)$$

L^p space

First, consider function that having local integrability issue.

$$f_p(x) = \frac{|x|^{-n/p}}{(1+|x|^2)^N}$$

With sufficiently large N, it solves far-field decay problem.

$$f_p \notin L^p, \quad f_p \in \bigcap_{q<p} L^q$$

$$g_p(x) = f_p(x) \cdot \left[ln(\frac{1}{|x|})\right]^M$$

$$g_p \in L^p, \quad g_p \notin \bigcup_{p<r} L^r$$

Second, we need to consider function that only having far-field decay issue.

$$h_p(x) = (1+|x|^2)^{-n/(2p)}$$

$$h_p \notin L^p, \quad h_p \in \bigcap_{q>p} L^q$$

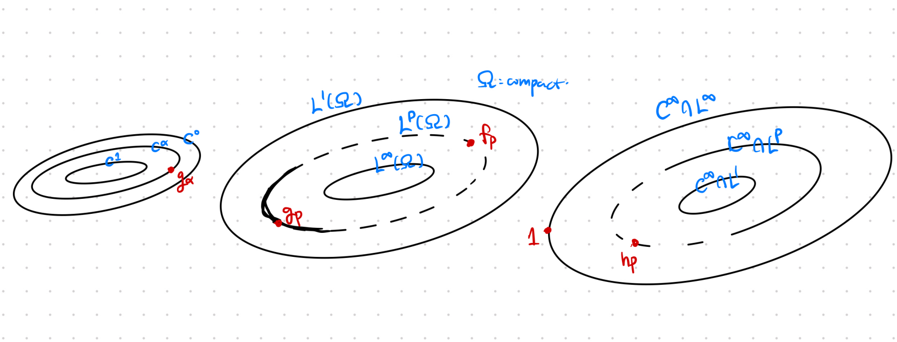


Figure 2: Holder, Lp functions

4.2 Distribution and Weak Derivatives

4.2.1 Distribution

Distribution is a continuous linear mapping $C_c^\infty(U) \to \mathbb{C}$. 'Continuous' requires topology, we define topology by convergence, convergence of $C_c^\infty(U)$ is defined convergence of support, convergence of each derivative.

We could define L^1_{loc} as distribution by following. Therefore, $L^1_{loc}(U) \hookrightarrow \mathcal{D}(U)$

$$< f, \phi >:= \int_U f\phi \qquad (f \in L^1_{loc}(U))$$

Now, **Function** is defined to satisfy L^1_{loc} condition. Two function are same if it is same in L^1_{loc} sense. Furthermore inclusion is injective since

$$\int h\phi \equiv 0 \Leftrightarrow h \equiv 0$$

4.2.2 Function $\neq$ Distribution

As we defined, Function is Distribution. But Distribution may not be a function.

Example is Dirac Delta δ_{x_0}. It is defined: $< \delta_{x_0}, \phi >= \phi(x_0)$. This is not a function.

Also, some functions that we used in ordinary sense is not considered "function" or "distribution" anymore. $f(x) = \frac{1}{x}$ defined in $\mathbb{R}$ is not L^1_{loc} so it is not a function nor distribution.

4.2.3 Weak Derivative $\equiv$ Derivate Distribution

For $u \in \mathcal{D}(U)$, we define $\partial^\alpha u \in \mathcal{D}(U)$,

$$< \partial^\alpha u, \phi >:= (-1)^{|\alpha|} < u, \partial^\alpha \phi >$$

There is useful theorem:

Theorem 32. *Any distribution is weak derivatives of some function*

For example, Dirac Delta is laplacian of some function. (laplacian is in this case, weak derivation)

$$\delta_0 = c_n \Delta\left(\frac{1}{|x|^{n-2}}\right) \qquad (n \geq 3)$$

4.2.4 Weak Derivative and Classical Derivative

Two concept is **NOT** same. There is a counterexample.

$\ln|x| \in L^1_{loc}(\mathbb{R}) \subset \mathcal{D}(\mathbb{R})$. Ordinary derivative of this function is $\frac{1}{x}$. We found $\frac{1}{x}$ is NOT a function nor distribution. Why this happened? It's because **Ordinary derivative does not coincide to Weak derivative**

Weak derivative of $\ln|x|$ is NOT $\frac{1}{x}$.

$$
\begin{aligned}
< \partial_x \ln|x|, \phi > &= - < \ln|x|, \partial_x \phi > \\
&= -\int_{\mathbb{R}} \ln|x| \partial_x \phi dx = -\int_0^\infty \ln x \partial_x \phi dx - \int_{-\infty}^0 \ln(-x) \partial_x \phi dx \\
&= -\lim_{\epsilon \downarrow 0} \left[\int_\epsilon^\infty \ln x \partial_x \phi dx + \int_{-\infty}^{-\epsilon} \ln(-x) \partial_x \phi dx \right] \\
&= \lim_{\epsilon \downarrow 0} \left[\ln \epsilon (\phi(\epsilon) - \phi(-\epsilon)) + \int_{\mathbb{R}-[-\epsilon,\epsilon]} \frac{\phi(x)}{x} dx \right] \\
&= \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}-[-\epsilon,\epsilon]} \frac{\phi(x)}{x} dx
\end{aligned}
$$

This is called **Principal Value**. This is distribution that

$$
< PV_{x_0}(f), \phi > := \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^n - B(x_0,\epsilon)} \phi(x) f(x) dx
$$

Furthermore, thinking about weak derivatives on $\ln|x|$, n-th derivative is

$$
\begin{aligned}
< \partial_x^{(n)} \ln|x|, \phi > &= (-1)^n < \ln|x|, \partial_x^{(n)} \phi > \\
&= \lim_{\epsilon \downarrow 0} (-1)^n \left[\int_\epsilon^\infty \ln x \partial_x^{(n)} \phi(x) dx + \int_{-\infty}^{\epsilon} \ln(-x) \partial_x^{(n)} \phi(x) dx \right] \\
&= \lim_{\epsilon \downarrow 0} (-1)^{n+1} \left[\int_\epsilon^\infty \frac{\partial_x^{(n-1)} \phi(x)}{x} dx + \int_{-\infty}^{-\epsilon} \frac{\partial_x^{(n-1)} \phi(x)}{x} dx \right] \\
&= \lim_{\epsilon \downarrow 0} (-1)^{n+1} \left[\int_\epsilon^\infty \frac{\partial_x^{(n-2)} \phi(x)}{x^2} dx + \int_{-\infty}^{-\epsilon} \frac{\partial_x^{(n-2)} \phi(x)}{x^2} dx \right] \\
&= \cdots \\
&= \\
&= \lim_{\epsilon \downarrow 0} (-1)^{n+1} (n-1)! \left[\int_\epsilon^\infty \frac{\phi(x)}{x^n} dx + \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x^n} dx \right] \\
&= < (-1)^{n+1} (n-1)! PV_0 \left(\frac{1}{x^n} \right), \phi >
\end{aligned}
$$

4.3 Sobolev Space

4.3.1 Definition

Sobolev space is **FUNCTION** space ($\subset L^1_{loc}(U)$) and $\partial^\alpha u$ coincides with some L^1_{loc} for all $|\alpha| \leq k$ and $\partial^\alpha u \in L^p(U)$. This L^p norm is defined:

$$
||f||_{L^p(U)} := \left[\lim_{V_n \to U, V_n \subset\subset U} \int_{V_n} |f|^p \right]^{\frac{1}{p}}, \qquad ||f||_{L^\infty(U)} = esssup_U f
$$

Now we will look at some properties of $W^{k,p}(U)$

4.3.2 Boundness

Interesting function: $|x|^{-\alpha}$

This function is in $W^{1,p}(B(0,1))$ iff $\alpha < \frac{n}{p} - 1$. So Sobolev space could contain unbounded function. Furthermore, it may contain function that is unbounded at any open set.

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{|x - r_k|^{\alpha}}$$

Unboundness could turn on to crazy functions. One may construct function $f_\gamma \in W^{1,2}(\mathbb{R}^3)$ that is unbounded in $\gamma \subset \mathbb{R}^3$ a C^1 curve. If $\gamma : z = h(x,y)$

$$f_\gamma = \frac{1}{(z - h(x,y))^N} \cdot \zeta(z - h(x,y))$$

which ζ is bump function supported on $[-1,1]$

4.3.3 Continuity (C^0 criteria)

$$W^{k,p} \hookrightarrow C^0$$

It's necessary condition is $k - \frac{n}{p} \geq 0$. Sufficient condition is $k - \frac{n}{p} > 0$. We will prove at **Sobolev inequality**.

4.3.4 C^∞ module

$u \in W^{k,p}(U)$ and $\phi \in C^\infty(\bar{U})$ then $\phi u \in W^{k,p}(U)$. (Here, $C^\infty(\bar{U})$ is function that all derivatives could continuously extended to $\bar{U}$)

4.3.5 Composition

If $\phi : \bar{U} \to \mathbb{R}^n$ and C^k,

$$||u \circ \phi||_{W^{k,p}(U)} \leq C||u||_{W^{k,p}(\phi(U))}$$

4.3.6 Transition continuity

$w \in \mathbb{R}^n$ be a vector, $\mathbb{T}_{\epsilon w}$ is an operator such that $(\mathbb{T}_{\epsilon w}(u))(x) := u(x + \epsilon w)$. Then

$$||\mathbb{T}_{\epsilon w} u||_{W^{k,p}(\mathbb{R}^n)} = ||u||_{W^{k,p}(\mathbb{R}^n)}$$

Also

$$\lim_{\epsilon \downarrow 0} ||\mathbb{T}_{\epsilon w} u - u||_{W^{k,p}(\mathbb{R}^n)}$$

4.4 Approximation

4.4.1 Smoothness of Boundary

We are interested in open set satisfying good property. We define 'smoothness' of boundary.

∂U is C^1 if for every $x_0 \in \partial U$, $\exists \delta(x_0) > 0$ such that after rotation in $\mathbb{R}^n$ that fixing x_0, following holds: $B(x_0, \delta(x_0)) \cap U = B(x_0, \delta(x_0)) \cap \{x_n > f(x_1, \cdots, x_{n-1})\}$ for some C^1 function f defined on $C^1(\bar{B}_{n-1}(x_0, \delta(x_0)))$
We can assume that $\nabla_{\mathbb{R}^{n-1}} f(x_0) = 0$

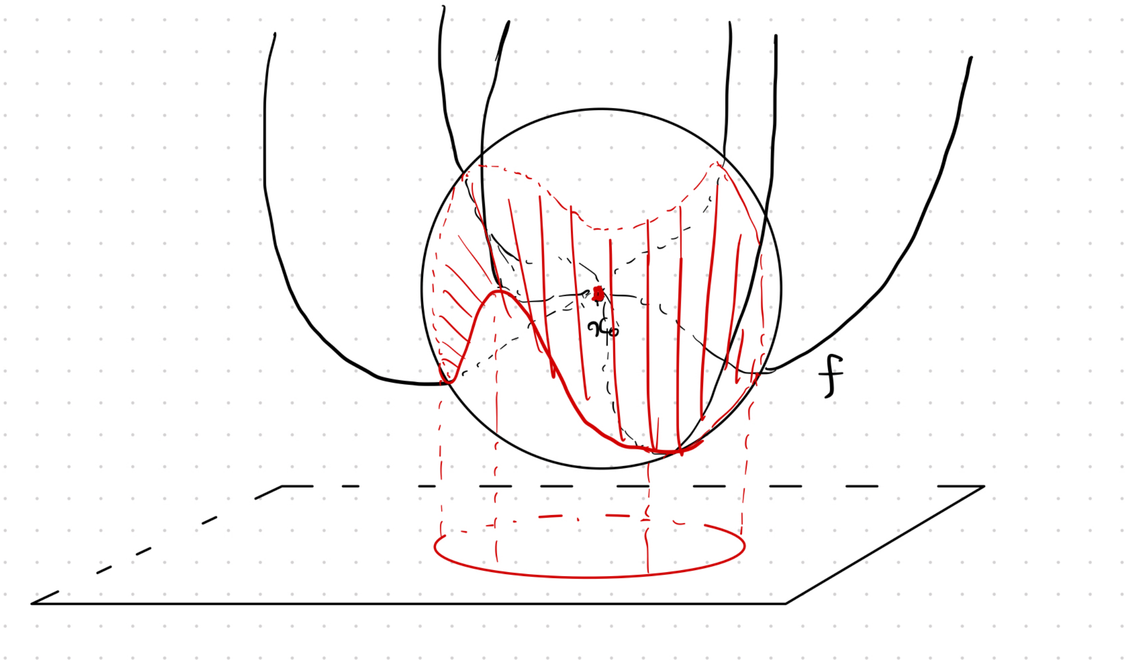


Figure 3: Smooth boundary

We also define Norm on boundary.

$$||\partial U||_{C^1} := \left(\min_{x_0 \in \partial U} \left(\max_{||\nabla f||_{C^0(\bar{B}_{n-1}(x_0, \delta(x_0)))} \leq 1} \delta(x_0) \right) \right)^{-1}$$

This indicates how complex is surface. Norm is big if boundary is Osciliating and Close each other.

4.4.2 Approximation Theorem

Theorem 33 (Approximation Theorem)**.** *There are three kinds of theorem.*
1. $u \in W^{k,p}(U)$ for $1 \leq p < \infty$ then $u^\epsilon = \eta_\epsilon * u$ is C^∞ approximation in sense of $W^{k,p}_{loc}(U)$.

$$u^\epsilon \in C^\infty(U_\epsilon), \quad ||u^\epsilon - u||_{W^{k,p}_{loc}} \to 0$$

2. Assume U is bounded. $u \in W^{k,p}(U)$ for $1 \le p < \infty$,

$$u_m \in C^\infty(U) \cap W^{k,p}(U), \quad ||u_m - u||_{W^{k,p}(U)} \to 0$$

3. Assume U is bounded and ∂U is C^1. $u \in W^{k,p}(U)$ for $1 \le p < \infty$.

$$u_m \in C^\infty(\bar{U}), \quad ||u_m - u||_{W^{k,p}(U)} \to 0$$

Especially third theorem is important. This means in U bounded and ∂U is C^1, then $W^{k,p}$ is completion of $C^\infty(\bar{(U)})$. It's because, first $W^{k,p}$ is Banach space and $C^\infty(\bar{U})$ is dense in $W^{k,p}$.

So, $W^{k,p}$ could be defined as two different way. First one is using 'Weak Derivative', define a norm and find functions that has finite norm. Second one is Completion of $C^\infty(\bar{U})$ function via Sobolev norm. This norm's derivative coincides with ordinary derivative in $C^\infty(\bar{U})$

Proof technique is mollifying and using partition of unity. Main Idea for proof is as following Figure. It is essential to prove rigorously the Approximation Theorem. Especially, third one.

Proof. First, consider some flatable small area. That is, $U \cap B(x^0, r)$ could written as $x_n > f(x_1, \cdots x_{n-1})$ and $B(x_0, r) \cap \{x_n > f(x_1, \cdots x_{n-1})\} \subset U$
In this area, for all $\epsilon > 0$, there exists r satisfies upper condition. Let $x^\epsilon = x + \lambda\epsilon e_n$. If λ satisfies $1 < \lambda < 2$ and $\lambda\epsilon < \frac{1}{2}r$. Then,

$$B(x^\epsilon, \epsilon) \subset U \cap B(x, r) \qquad \forall x \in V, \ \epsilon > 0$$

(V is defined by $U \cap B(x^0, r/2)$)

With defining $u_\epsilon(x) = u(x^\epsilon)$, we could mollify u_ϵ with ball ϵ. Mollifying u^ϵ, $v^\epsilon(x) = \eta_\epsilon * u_\epsilon \in C^\infty(\bar{V})$. We assume $v^\epsilon \to u$ with $W^{k,p}(V)$ norm.

$$||D^\alpha v^\epsilon - D^\alpha u||_{L^p(V)} \le ||D^\alpha v^\epsilon - D^\alpha u_\epsilon||_{L^p(V)} + ||D^\alpha u_\epsilon - D^\alpha u||_{L^p(V)}$$

We need to prove each term tend to zero as $\epsilon \downarrow 0$

-Lemma. $D^\alpha(\eta_\epsilon * u_\epsilon) = \eta_\epsilon * D^\alpha u_\epsilon$

$$\begin{aligned} D^\alpha(\eta_\epsilon * u_\epsilon) &= D^\alpha \int_U \eta_\epsilon(x-y) u_\epsilon(y) dy = \int_U D_x^\alpha \eta_\epsilon(x-y) u_\epsilon(y) dy \\ &= (-1)^{|\alpha|} \int_U [D_y^\alpha \eta_\epsilon(x-y)] u_\epsilon(y) dy = \int_U \eta_\epsilon(x-y) D_y^\alpha u_\epsilon(y) dy \\ &= \eta_\epsilon * D^\alpha u_\epsilon \end{aligned}$$

Now, by Lemma of mollifiers

$$||D^\alpha v^\epsilon - D^\alpha u_\epsilon||_{L^p(V)} = ||\eta_\epsilon * D^\alpha u_\epsilon - D_\epsilon^\alpha||_{L^p(V)} \to 0$$

Second term tends to zero means transition is continuous. By definition, $u_\epsilon(x) = u(x + \lambda\epsilon e_n)$. $D^\alpha u_\epsilon = (D^\alpha u)(x + \lambda\epsilon e_n)$

$$|(D^\alpha u)(x + \lambda\epsilon e_n) - (D^\alpha u)(x)| \to 0$$

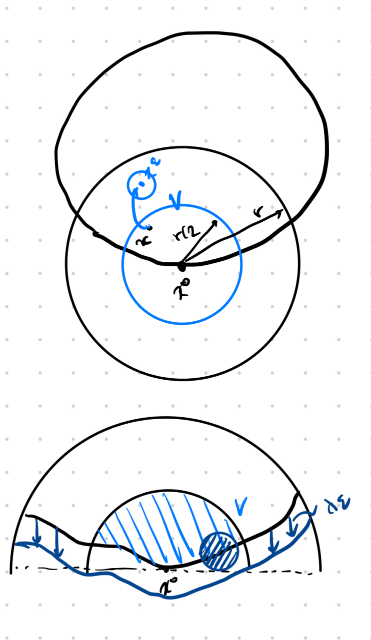


Figure 4: Approximation proof

pointwise and

$$|(D^\alpha u)(x+\lambda\epsilon e_n)-(D^\alpha u)(x)| \leq |(D^\alpha u)(x+\lambda\epsilon e_n)|+|D^\alpha u(x)|$$

That is, right hand side is finite in L^p norm. So by Lebesgue's Dominated Theorem as $\epsilon \downarrow 0$,

$$||D^\alpha u_\epsilon - D^\alpha u||_{L^p(V)} \to 0$$

We now found every small area, u could be Approximated. It is obvious that $\partial U \subset \bigcup_{x\in\partial U} B(x, r_x/2)$. ∂U is compact so there exists finite open covering :

$$B(x_1, r_1/2) \cup B(x_2, r_2/2) \cup \cdots \cup B(x_N, r_N/2)$$

For all $\delta > 0$, $V_i = U \cap B(x_i, \frac{r_i}{2})$ then we could find $v_i \in C^\infty(\bar{V}_i)$ that $||v_i - u||_{W^{k,p}(V_i)} \leq \delta$. Then there exists $V_0 \subset\subset U$ that $U \subset \bigcup_{i=0}^N V_i$. Within mollification $v_0 = \eta_\epsilon * u$, $v_0 \in C^\infty(\bar{V}_0)$ and $||v_0 - u||_{W^{k,p}(V_0)} \leq \delta$

There are partition of unity in $\bar{U}$ subordinate to $V_0, B(x_1, \frac{r_1}{2}) \cdots B(x_N, \frac{r_N}{2})$: $\{\zeta_i\}_{i=0}^N$

$$v = \sum_{i=0}^N \zeta_i v_i$$

For all multiindex α, $|\alpha| \leq k$

$$||D^\alpha v - D^\alpha u||_{L^p(U)} \leq C(N+1)\delta$$

$$||v-u||_{W^{k,p}(U)} \leq C\delta$$

$\square$

Summarising, this is main idea of Approximation Theorem.

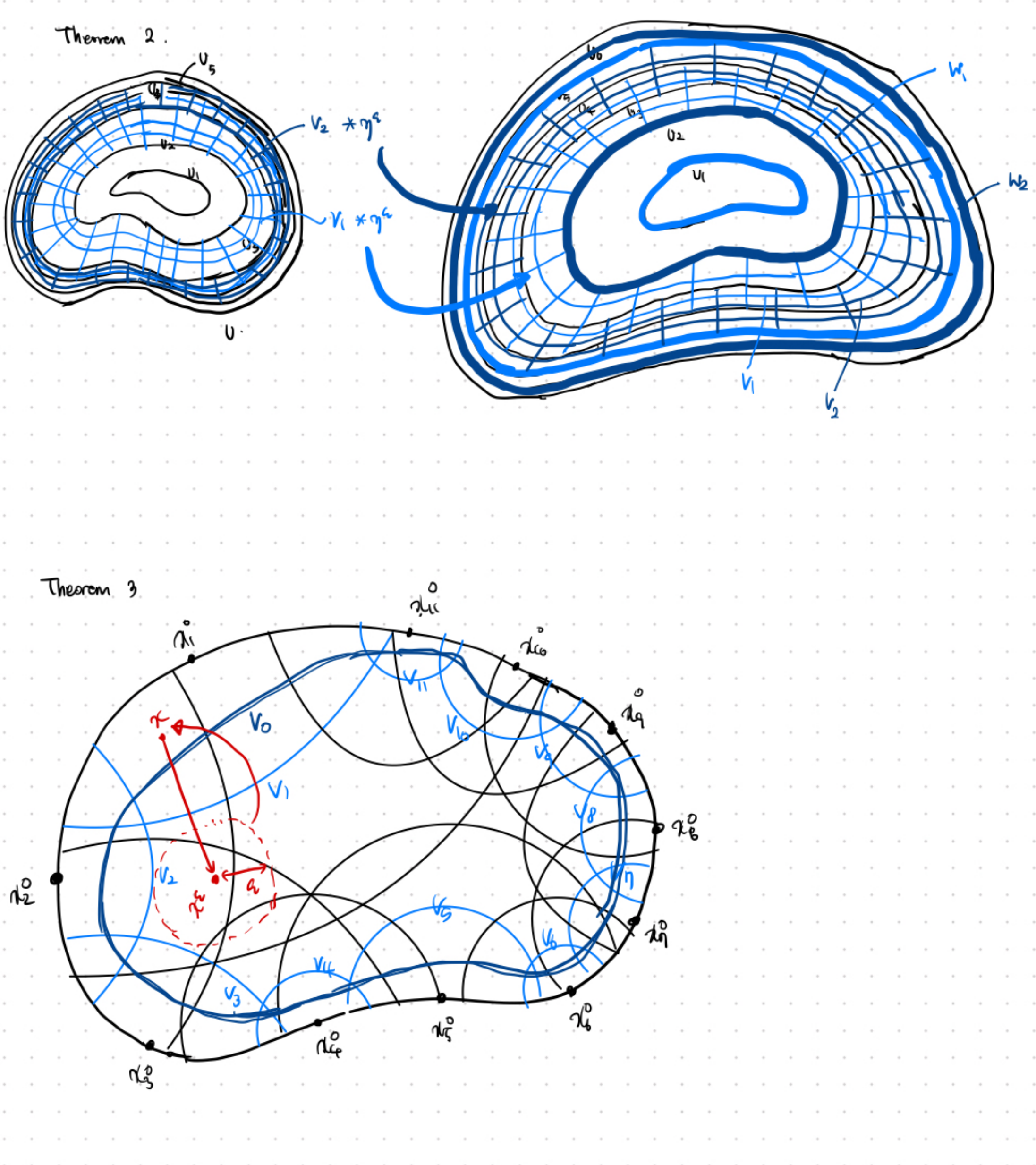


Figure 5: Main Idea for proof of Approximation Theorem

4.5 Extension

Theorem 34 (Extension Theorem)**.** *$1 \le p < \infty$, U is bounded ∂U is C^∞. If $U \subset\subset V$ then there exists Extension operator $E_V : W^{k,p}(U) \to W_0^{k,p}(V)$ such that $E_V(u)|_U = u$ and*

$$||E_V u||_{W^{k,p}(V)} \le C_{k,p,U,V}||u||_{W^{k,p}(U)}$$

Here, $W_0^{k,p}(U)$ is defined as completion of $C_c^\infty(U)$ in $W^{k,p}(U)$ norm.

If $k = 0$, $E_V u = \chi(U)u$ is objective extension since $W^{0,p} = L^p$. However, if $k \neq 0$ then $D(E_V u)$ is dirac delta in boundary thus not in a Sobolev space and also inequality does not hold. Therefore, we need to do more carefully for $k \ge 1$

Proof. First assume u is $C^\infty(\bar{U})$. After proving, we will use approximation theory and generalize to $W^{k,p}(U)$ functions.

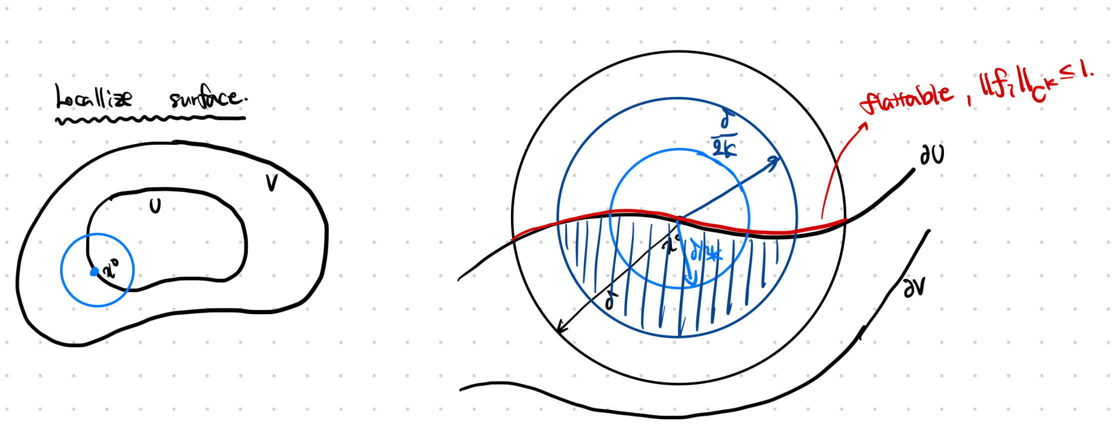


Figure 6: Extension proof

If $\{B(x_i, \delta)\}_{i=1}^N$ satisfies property $x_i \in \partial U$, $U \subset \bigcup_{i=1}^N B(x_i, \frac{\delta}{3k})$ and $B(x_i, \delta) \cap \partial V$ empty. Furthermore, if $\delta = \epsilon ||\partial U||^{-1}$ with small ϵ, in $B(x_i, \delta) \cap \partial U$ is examined $x_n = f^i(x_1, \cdots x_{n-1})$ that $||f^i||_{C^k} \le 1$.

In this situation, if extension

$$B(x^i, \frac{\delta}{2k}) \cap U \to B(x^i, \frac{\delta}{2k}) \cap V$$

from $C^\infty(\bar{U})$ to $C^\infty(\bar{V})$ possible, then partition of unity in $\bar{U}$ subordinate to $U_0(\subset\subset U), B(x_1, \frac{\delta}{2k}), \cdots B(x_N, \frac{\delta}{2k})$ exists: $\{\zeta_i\}_{i=0}^N$. Define

$$E_V(u) = \zeta_0 u + \sum_{i=1}^N \zeta_i (Eu)_{B(x_i, \frac{\delta}{2k})}$$

is $C^\infty(\bar{V})$, $E_V u = u$ a.e. U, $suppE_V \subset V$. Also

$$||E_V u||_{W^{k,p}(V)} \le ||u||_{W^{k,p}(U)} + N \times (\sup_n \sup_k ||D^\alpha \zeta_i||_{L^\infty}) \times k \times C||u||_{W^{k,p}(U)} \equiv C||u||_{W^{k,p}(U)}$$

For general $u \in W^{k,p}(U)$, by Approximation Theorem there exists $v_i \in C^\infty(\bar{U})$ that $v_1,\ v_2, \cdots \to u$ in $W^{k,p}(U)$ norm. $E_V v_1,\ E_V v_2, \cdots$ is now cauchy sequence on $W^{k,p}(V)$ by inequality, it converges

$$E_V u := \lim_{n \to \infty} E_V v_i$$

If $\{v_i^1\}, \{v_i^2\}$ both converging series to u defining $w_{2i-1} = v_i^1,\ w_{2i} = v_i^2$ is also cauchy so $E_V u$ is well-defined.

Now only task is to prove Extension is possible from $C^\infty(\bar{U})$ to $C^\infty(\bar{V})$

$$B(x^i, \frac{\delta}{2k}) \cap U \to B(x^i, \frac{\delta}{2k}) \cap V$$

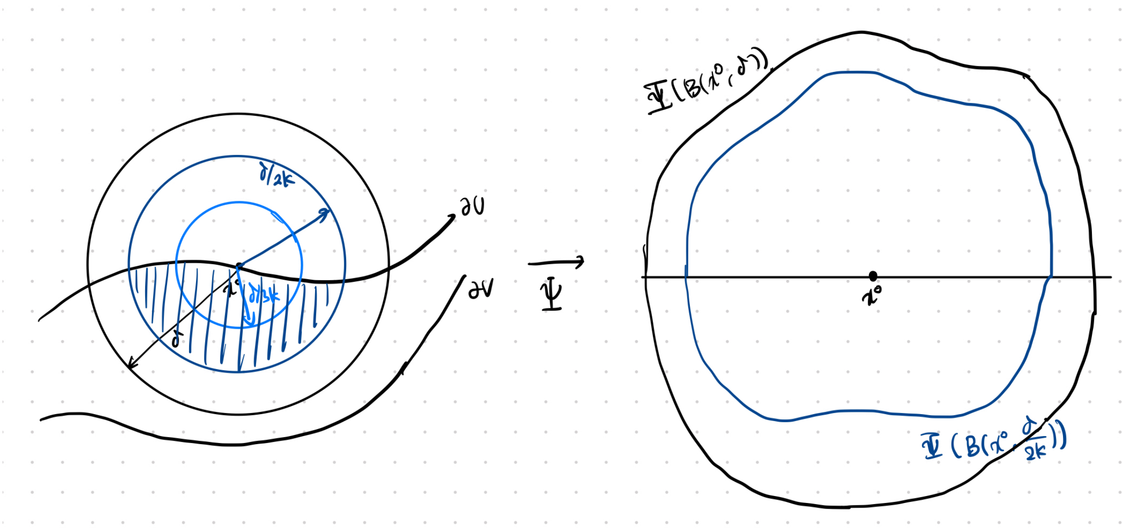


Figure 7: Extension proof

Define $\Psi(x) = \Psi(x',\ x_n) = (x',\ x_n - f(x'))$ $(\Psi : B(x^0, \delta) \to \mathbb{R}^n)$ and $\tilde{u}(\Phi(x)) = u(x)$. This is called **flatten technique**. The reason $\frac{\delta}{2k}$ is to ensure defining $E\tilde{u}$ makes sense. Applying Uryhson's lemma on $\bar{B}(x^0, \delta)$ to V, ζ exists : $\zeta \equiv 1$ in $\bar{B}(x^0, \delta)$ zero at outside.

$$(E\tilde{u})(x', -x_n) = \zeta\Big(a_1\tilde{u}(x', x_n) + a_2\tilde{u}(x', 2x_n) + \cdots + a_k\tilde{u}(x', kx_n)\Big)$$

in $(x', x_n) \in \Psi(B(x^0, \frac{\delta}{2k})) \cap \{x_n > 0\}$ This is possible because $(x', x_n) \in B(x^0, \delta/2k)$ then $(x', kx_n) \in \Psi(B(x^0, \delta) \cap U)$. a_j's satisfies

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & k \\ 1 & 4 & 9 & \cdots & k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^k & 3^k & \cdots & k^k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Then $D^\alpha(E\tilde{u}) = D^\alpha\tilde{u}$ on $\{x_n = 0\}$. So, this is obviously an extension.

$Eu(x) := (E\tilde{u})(\Psi(x))$ then since $||f||_{C^k} \leq 1$ by assumption, $||\Psi||_{C^k} < \infty$.

$$
\begin{aligned}
\partial_{\alpha_1} Eu(x) &= \sum_{i=1}^{n} \partial_i (E\tilde{u})(\Psi(x)) \cdot \partial_{\alpha_1} \Psi_i(x) \\
\partial_{\alpha_1,\alpha_2} Eu(x) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{ij} (E\tilde{u})(\Psi(x)) \cdot \partial_{\alpha_1} \Psi_i(x) \cdot \partial_{\alpha_2} \Psi_j(x) \\
&\qquad\qquad + \sum_{i=1}^{n} \partial_i (E\tilde{u})(\Psi(x)) \cdot \partial_{\alpha_1\alpha_2} \Psi_i(x) \\
&\vdots
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||Eu||_{W^{k,p}(B(x^0,\frac{\delta}{2k}))} &\leq C||\Psi||_{C^k} ||E\tilde{u}||_{W^{k,p}(\Psi(B(x^0,\frac{\delta}{2k})))} \\
&\leq C||\Psi||_{C^k} ||\tilde{u}||_{W^{k,p}(\Psi(B(x^0,\delta)))} \\
&\leq C||\Psi||_{C^k} ||\Psi^{-1}||_{C^k} ||u|||_{W^{k,p}(B(x^0,\delta))} \equiv C||u||_{W^{k,p}(B(x^0,\delta))}
\end{aligned}
$$

□

4.6 Trace

Theorem 35 (Trace Theorem). *U is bounded ∂U is C^∞. If $k > \frac{1}{2}$ then there exists Trace operator*

$$T : H^k(U) \to H^{k-\frac{1}{2}}(\partial U)$$

such that $Tu = u|_{\partial U}$ if $u \in C^\infty(\bar{U})$ and

$$||Tu||_{H^{k-\frac{1}{2}}(\partial U)} \leq C_{k,U}||u||_{H^k(U)}$$

Before doing so on trace operator, we need to define Sobolev Space for non-integer k

4.6.1 Non-integer k, Sobolev Spaces

$W^{k,p}$ could be defined in two different ways.

Using Fourier Series

$$W^{k,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{k}{2}}\mathcal{F}f\right] \in L^p(\mathbb{R}^n) \right\}$$

with norm defined

$$||f||_{W^{k,p}(\mathbb{R}^n)} = \left|\left|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{k}{2}}\mathcal{F}f\right]\right|\right|_{L^p(\mathbb{R}^n)}$$

Using Catersian product

$$W^{k,p}(\Omega) = \left\{ f \in W^{\lfloor k \rfloor,p}(\Omega) : \sup_{|\alpha|=\lfloor k \rfloor} \left(\int_{\Omega\times\Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{p\{k\}+n}} dxdy \right) < \infty \right\}$$

with norm defined

$$||f||_{W^{k,p}(\Omega)} = ||f||_{W^{\lfloor k \rfloor,p}(\Omega)} + \sup_{|\alpha|=\lfloor k \rfloor} \left(\int_{\Omega\times\Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{p\{k\}+n}} dxdy \right)$$

4.6.2 Trace Theorem

Assume Approximation $C^\infty(\bar{U})$ possible for Non-integer k Sobolev Spaces.

Proof for Trace Theorem. We flatten the boundary by $\Psi(x', x_n) = (x', x_n - f(x'))$. Then first, we can do on for $\varphi \in C^\infty(\bar{U})$. Define in this case, $(T\varphi)(x') = \varphi(x', 0)$

Here is a useful Lemma.
Lemma.

$$\widehat{T\varphi}(\xi') = \int_{\mathbb{R}} \hat{\varphi}(\xi', \xi_n) d\xi_n$$

Proof follows from definition of Fourier Transform.

$$
\begin{aligned}
\int_{\mathbb{R}} \hat{\varphi}(\xi', \xi_n) d\xi_n &= \int_{\mathbb{R}} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x', x_n) e^{-ix'\cdot\xi' - ix_n\xi_n} dx' dx_n d\xi_n \\
&= \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} \times \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x', x_n) e^{-ix_n\xi_n} dx_n d\xi_n dx' \\
&= \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} \times \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} \hat{\varphi_n}(\xi_n; x') d\xi_n dx' \\
&= \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} \times \frac{1}{(2\pi)^{(n-1)/2}} \varphi(x', 0) dx' \\
&= \widehat{T\varphi}(\xi')
\end{aligned}
$$

since

$$\int_{\mathbb{R}^n} \hat{\omega}(y) dy = (2\pi)^{n/2} \omega(0)$$

Now, by Cauchy Schwartz inequality

$$
\begin{aligned}
|\widehat{T\varphi}(\xi')| &= \int_{\mathbb{R}} \hat{\varphi}(\xi', \xi_n) d\xi_n \\
&\leq \left(\int_{\mathbb{R}} (1 + ||\xi'||^2 + |\xi_n|^2)^{-k} d\xi_n \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1 + ||\xi'||^2 + |\xi_n|^2)^{k} |\hat{\varphi}|^2 d\xi_n \right)^{\frac{1}{2}} \\
&\leq \left\{ (1 + ||\xi'||^2)^{\frac{1}{2}-k} \times \int_{\mathbb{R}} \frac{1}{(1+x^2)^k} dx \right\}^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1 + ||\xi'||^2 + |\xi_n|^2)^{k} |\hat{\varphi}|^2 d\xi_n \right)^{\frac{1}{2}}
\end{aligned}
$$

$\int_{\mathbb{R}} \frac{1}{(1+x^2)^k} dx$ converges if $k > \frac{1}{2}$
Integrating both side with ξ',

$$
\begin{aligned}
||T\varphi||_{H^{k-\frac{1}{2}}} &= \int_{\mathbb{R}^{n-1}} (1 + ||\xi'||^2)^{k-\frac{1}{2}} |\widehat{T\varphi}(\xi')|^2 d\xi' \\
&\leq C \left(\int_{\mathbb{R}} |\hat{\varphi}(\xi', \xi_n)|^2 (1 + ||\xi'||^2 + |\xi_n|^2)^k d\xi_n \right) d\xi' \\
&= C ||\varphi||_{H^k}
\end{aligned}
$$

Since $p = 2$. Flattening back will be bounded to $||\Psi||_{C^k} ||\Psi^{-1}||_{C^k}||$

Now, if $\{\zeta_i\}_{i=1}^N$ partition of unity on ∂U subordinate to $\{B(x_i, \delta)\}_{i=1}^N$, define

$$Tu = \sum_{i=1}^N \zeta_i T_i u$$

Then $Tu = u|_{\partial U}$ a.e. and inequality holds.

$$||Tu||_{H^{k-\frac{1}{2}}(\partial U)} \leq ||u||_{H^k(U)}$$

We need to prove for general H^k functions. By Approximation, $u \in H^k(U)$ then there exists $C^\infty(\bar{U})$ functions that converges to u in H^k norm.

By trace inequality, Tv_i is Cauchy sequence in $H^{k-\frac{1}{2}}(\partial U)$ thus converges to some function. Let we define this as Tu

$$Tu := \lim_{n\to\infty} Tv_n$$

Limit is defined in $H^{k-\frac{1}{2}}(\partial U)$ sense. Then also Tu satisfies following inequality.

$$||Tu||_{H^{k-\frac{1}{2}}(\partial U)} \leq ||u||_{H^k(U)}$$

□

4.6.3 Another Trace Theorem

Theorem 36 (Weak Trace Theorem)**.** *$1 \leq p < \infty$, U is bounded ∂U is C^∞.*

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ if $u \in C(\bar{U})$ and

$$||Tu||_{L^p(\partial U)} \leq C_{p,U}||u||_{W^{1,p}(U)}$$

Note that Tu coincides with u in boundary if $u \in C(\bar{U})$.

Proof. Using approximation and flatten technique.
Consider $u \in C(\bar{U})$. B is a ball that is flatable. $B = B(x^0, r)$ and $\hat{B} = B(x^0, r/2)$. Applying Uryhson's lemma, there exists $\zeta \in C_c^\infty(B)$ that $\zeta \equiv 1$ on $\hat{B}$ and nonnegative. $\Gamma = \partial U \cap \hat{B}$ then

$$\begin{aligned}\int_\Gamma |u|^p dx' \leq \int_{\{x_n=0\}} \zeta|u|^p dx' &= -\int_{B^+} (\zeta|u|^p)_{x_n} dX \\ &= -\int_{B^+} |u|^p\zeta_{x_n} + p|u|^{p-1}(sgn\ u)u_{x_n}\zeta dx \\ &\leq C\int_{B^+} |u|^p + |Du|^p dx\end{aligned}$$

By Young inequality.
Flattening gives factor $||\Psi||_{C^1}||\Psi^{-1}||_{C^1}$ but also a constant and within partition of unity as proof before, desired inequality outcomes for $C(\bar{U}) \cap W^{1,p}(U)$ functions.

For general $u \in W^{1,p}(U)$ functions, by approximation theorem there exists $v_1, v_2, \cdots$ which is $C^\infty(\bar{U})$ functions converging to u in $W^{1,p}(U)$ sense. Define Trace as a limit of Tv_i's because they form Cauchy Sequence by inequality proved before.

$$Tu := \lim_{n\to\infty} Tv_n$$

Limit in $L^p(\partial U)$. It satisfies inequality. □

4.6.4 Zero Trace in $W^{1,p}$

Theorem 37 (Zero Trace Theorem)**.** *U is bounded and ∂U is C^1. $u \in W^{1,p}(U)$.*

$$u \in W_0^{1,p}(U) \quad if\ and\ only\ if \quad Tu \equiv 0$$

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) Using partitions of unity and flattening ∂U, we can assume$u \in W^{1,p}(\mathbb{R}^n_+)$ and $Tu = 0$ on $\{x_n = 0\}$.

Step 1 Prove for $u \in W^{1,p}(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dt$$

Since $Tu = 0$ and $C^1(\bar{\mathbb{R}^n_+})$ is complete there exists $u_m \in C^1(\mathbb{R}^n_+)$ such that $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$ sense. By Weak Trace Theorem,

$$||T(u_m - u)||_{L^p(\mathbb{R}^{n-1})} \leq C||u_m - u||_{W^{1,p}(\mathbb{R}^\kappa_+)}$$

so $Tu_m \to 0$ in $L^p(\mathbb{R}^{n-1})$ sense.
If $x' \in \mathbb{R}^{n-1}$,

$$|u_m(x', x_n)| \leq |u_m(x', 0)| + \int_0^{x_n} |u_{m,x_n}(x', t)| dt$$

By Holder Inequality,

$$\left(\int_0^{x_n} |u_{m,x_n}(x', t)| dt \right)^p \leq \int_0^{x_n} |u_{m,x_n}(x', t)|^p dt \cdot x_n^{p-1}$$

so

$$\int_{\mathbb{R}^{n-1}} |u_m(x', x_n)|^p dx' \leq C \Bigg(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx' + x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du_m(x', t)|^p dx' dt \Bigg)$$

As $m \to \infty$,

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dt$$

Step 2 Construct $C_c^\infty(\mathbb{R}^n)$ function converging to u

ζ is defined to be $C^\infty(\mathbb{R}_+)$ which $\zeta \equiv 1$ on $[0, 1]$, $\zeta \equiv 0$ on $\mathbb{R}_+ - [0, 2]$, $0 \leq \zeta \leq 1$.
$\zeta_m(x) := \zeta(mx_n)$ and $w_m = u(x)(1 - \zeta_m)$. This procedure makes space from boundary by $\frac{1}{m}$. w_m is almost u but zero in $\{0 \leq x_n \leq \frac{1}{m}\}$. Clearly, $w_m \to u$ in $L^p(\mathbb{R}^n_+)$

Differential of w_m is $w_{m,x_n} = u_{x_n}(1 - \zeta_m) - mu\zeta'$, $D_{x'} w_m = (D_{x'} u)(1 - \zeta_m)$.

$$\begin{aligned}\int_{\mathbb{R}^n_+} |Dw_m - Du|^p dx &\leq C \int_{\mathbb{R}^n_+} |\zeta_m|^p |Du|^p dx + C \int_{\mathbb{R}^n_+} |mu\zeta'|^p dx \\ &= C \int_{\mathbb{R}^n_+} |\zeta_m|^p |Du|^p dx + Cm^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^p dx\end{aligned}$$

First part clearly tends to zero since $|\zeta_m|^p$ tends to zero.
Second part is by Step 1,

$$
\begin{aligned}
Cm^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^p dx &\le Cm^p \Big(\int_0^{2/m} t^{p-1} dt \Big) \Big(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dx_n \Big) \\
&\le C \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dx_n \to 0
\end{aligned}
$$

Thus $Dw_m \to Du$ in $L^p(\mathbb{R}^n_+)$ sense. Thus,

$$
w_m \to u
$$

in $W^{1,p}(\mathbb{R}^n_+)$ sense. Here w_m is not C^∞ but after mollifying w_m with radius $\frac{1}{2m}$, it turns to u_m which is $C_c^\infty(\mathbb{R}^n_+)$. □

4.7 Sobolev, Banach space Embeddings

4.7.1 Examples of Banach space Inhomogenuous Embeddings

- $L^1 \cap L^\infty \hookrightarrow L^2$

$$||f||_{L^2}^2 = \int |f|^2 \leq ||f||_{L^\infty} \int |f| = ||f||_{L^\infty} ||f||_{L^1}$$

- $\dot{C}^1 \cap \dot{C}^0 \hookrightarrow \dot{C}^{\frac{1}{2}}$

$$\left(\frac{|f(x) - f(y)|}{\sqrt{x-y}} \right)^2 \leq (|f(x) - f(y)|) \left(\frac{|f(x) - f(y)|}{|x-y|} \right)$$

-$\dot{H}^2 \cap L^2 \hookrightarrow \dot{H}^1$
-$\dot{C}^2 \cap \dot{C}^0 \hookrightarrow \dot{C}^1$

These Embeddings are useful and now, we are interested in homogenuous embeddings of Sobolev Spaces.

4.7.2 Examples of Sobolev space Embeddings within Fourier Approach

- $\dot{W}^{2,p} \hookrightarrow L^q$

q is determined with same scale parameter condition.

$$2 - \frac{n}{p} = *_{W^{2,p}} = *_{L^q} = -\frac{n}{q}$$

Proof Idea is, using Fourier transform properties. Briefly assume these properties. Afterall, Fourier transform is if $u \in L^1(\mathbb{R}^n)$, $\mathcal{F}u = \hat{u}$ defined to

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot y} u(x) dx$$

and inverse Fourier transform $\mathcal{F}^{-1}u = \check{u}$ defined as

$$\check{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot y} u(x) dx$$

Theorem 38 (Fourier transform properties)**.** *Assume* $u,\ v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *Then*
(1) $\check{u}, \hat{u} \in L^2(\mathbb{R}^n)$ *and*

$$||\check{u}||_{L^2(\mathbb{R}^n)} = ||\hat{u}||_{L^2(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)}$$

(2) For $D^\alpha u \in L^2(\mathbb{R}^n)$

$$\widehat{(D^\alpha u)} = (iy)^\alpha \hat{u}$$

(3)

$$\widehat{(u * v)} = (2\pi)^{n/2} \hat{u} \hat{v}$$

(4)

$$u = (\hat{u})\check{}$$

Now using (2) of above properties, we might think of proof idea that

$$\begin{aligned}
f &\longmapsto \hat{f}(\xi) = |\xi|^{-2}(|\xi|^2\hat{f}(\xi)) \\
&\longmapsto f = (|\xi|^{-2}|\xi|^2\hat{f}(\xi))\check{} = (|\xi|^{-2})\check{} * (\Delta f) = |\cdot|^{n-2} * (\Delta f)
\end{aligned}$$

This idea could be written differently as

$$f(x) = \Delta^{-1}\Delta f = c_n \int \frac{1}{|x-y|^{n-2}}(\Delta f)(y)dy$$

By H.L.S inequality

$$||f||_{L^q} \le C||\Delta f||_{L^p} \le C||f||_{\dot{W}^{2,p}}$$

- $\dot{W}^{1,p} \hookrightarrow L^q$

q is determined with same scale parameter condition.

$$1 - \frac{n}{p} = *_{W^{2,p}} = *_{L^q} = -\frac{n}{q}$$

Idea sketch is almost same

$$\begin{aligned}
f &\longmapsto \hat{f}(\xi) = |\xi|^{-1}(|\xi|\hat{f}(\xi)) \\
&\longmapsto f = (|\xi|^{-1}|\xi|\hat{f}(\xi))\check{} = (|\xi|^{-1})\check{} * (|\xi|\hat{f}(\xi))\check{} = |\cdot|^{n-1} * (|\xi|\hat{f}(\xi))\check{}
\end{aligned}$$

Now, $|\xi|\hat{f}(\xi) = \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}\hat{f}(\xi)$ is different with $|\xi_1|\hat{f}(\xi) + \cdots |\xi_n|\hat{f}(\xi)$ is problem. We could not directly convert $|\xi|\hat{f}(\xi)$ back.

To solve this is using partition of unity on **Surface of n-dimensional ξ ball**. Let $\chi_1,\ \chi_2 \cdots \chi_n$ partition of unity in S^{n-1}.

$$\chi_1 + \chi_2 + \cdots + \chi_n \equiv 1$$

and χ_i vanishes on some neighborhood of $\{\xi_i = 0\}$. Extend these χ to $\mathbb{R}^n \backslash \{0\}$ that $\chi_i(\xi) := \chi_i(\xi/||\xi||)$

Then,

$$\begin{aligned}
\hat{f}(\xi) &= \chi_1(\xi)\hat{f}(\xi) + \cdots + \chi_n(\xi)\hat{f}(\xi) \\
&= \frac{\chi_1(\xi)}{i\xi_1} i\xi_1\hat{f}(\xi) + \cdots + \frac{\chi_n(\xi)}{i\xi_n} i\xi_n\hat{f}(\xi) \\
&= \zeta_1(\xi)i\xi_1\hat{f}(\xi) + \cdots \zeta_n(\xi)i\xi_n\hat{f}(\xi)
\end{aligned}$$

Now inverse Fourier transform gives

$$\begin{aligned}
f &= \Big(\zeta_1(\xi)i\xi_1\hat{f}(\xi)\Big)\check{} + \cdots + \Big(\zeta_n(\xi)i\xi_n\hat{f}(\xi)\Big)\check{} \\
&= (\zeta_1(\xi))\check{} * \partial_1 f + \cdots + (\zeta_n(\xi))\check{} * \partial_n f
\end{aligned}$$

Since ζ_i are -1 homogenuous, $\check{\zeta}_i$ are -(n-1) homogenuous.

$$(\check{\zeta}_i(\xi))(tx) = \int \zeta_1(\xi)e^{itx\cdot\xi}d\xi = \int t\zeta_i(\xi')e^{ix\cdot\xi'}t^{-n}d\xi' = t^{1-n}(\check{\zeta}_i(\xi))(x)$$

So $||\cdot||^{n-1}\zeta_i(\xi)$ is bounded in $\mathbb{R}^n\backslash\{0\}$

$$||(\check{\zeta}_i(\xi)) * \partial_i f||_{L^q} = \left|\left|\frac{1}{||\cdot||^{n-1}}\big(||\cdot||^{n-1}(\check{\zeta}_i(\xi))\big) * \partial_i f\right|\right|_{L^q} \le C\left|\left|\frac{1}{||\cdot||^{n-1}} * \partial_i f\right|\right|_{L^q} \le C||\partial_1 f||_{L^p}$$

By HLS inequality.

4.7.3 Delicate Embedding on Sobolev spaces

Afterall, we will prove general sobolev embeddings but there are some strengthen Embeddings. For example, $n = 2$ then

$$W^{2,1} \hookrightarrow W^{1,2} \hookrightarrow BMO$$

by general sobolev embeddings but actually,

$$W^{2,1} \hookrightarrow C^0$$

Prove by following. By Approximation Theorem for all $\epsilon > 0$ $\exists v \in C_c^\infty(\mathbb{R}^2)$. v is obviously C^0

$$\begin{aligned}|u(x,y)| \le |u(x,y) - v(x,y)| + |v(x,y)| &\le ||v||_{W^{2,1}(\mathbb{R}^2)} + |u(x,y) - v(x,y)| \\ &\le ||u||_{W^{2,1}(\mathbb{R}^2)} + \epsilon + |u(x,y) - v(x,y)|\end{aligned}$$

Our objective is to prove $\forall \epsilon > 0$, $\exists \delta > 0$ and if $||w||_{W^{2,1}(\mathbb{R}^2)} < \delta$ then $|w(x,y)| < \epsilon$ a.e. (x,y). If this is true, applyihng $w = u - v$ completes proof.

Consider bump function on $[-1,1]$. $\psi : [-1,1] \to \mathbb{R}_+$ Assume $\int_{-\infty}^{\infty} \phi = 1$ and $\Psi(x) = \int_{-\infty}^{x} \phi(t)dt$ Define $\phi_n(x,y) = n^2\psi(nx)\psi(ny)$. Then

$$\int_{-\infty}^{x}\int_{-\infty}^{y} \phi_n(t,s)dsdt = \int_{-\infty}^{nx} \psi(t)dt \int_{-\infty}^{ny} \psi(t)dt = \Psi(nx)\Psi(ny) \le 1$$

$$\begin{aligned}\int\int w(x,y)\phi_n(x,y)dxdy &= \int\int \partial_{xy}w(x,y) \times \left(\int_{-\infty}^{x}\int_{-\infty}^{y} \phi_n(t,s)dsdt\right)dxdy \\ &\le ||\partial_{xy}w(x,y)||_{L^1} \\ &\le |||w||_{W^{2,1}(\mathbb{R}^2)} < \delta\end{aligned}$$

$\phi_n \to \delta_{(x,y)}$ so, a.e.

$$\int\int w(x,y)\phi_n(x,y)dxdy \to w(x,y)$$

4.8 General Sobolev Space Embeddings

There are three main Sobolev Embedding Theorems.

Theorem 39 ($W^{1,p}$)**.** *$W^{1,p}$ is Embedded as*
1. $p < n$ then

$$W^{1,p} \hookrightarrow L^q$$

2. $p = n$ then

$$W^{1,p} \hookrightarrow BMO$$

3. $p > n$ then

$$W^{1,p} \hookrightarrow C^\alpha$$

Theorem 40 ($W^{k,p}$)**.** *$W^{k,p}$ is Embedded as*
1. $p < n$ then

$$W^{k,p} \hookrightarrow W^{k-1,q}$$

2. $p = n$ then

$$W^{k,p} \hookrightarrow \bigcap_{r<\infty} W^{k-1,r}$$

3. $p > n$ then

$$W^{k,p} \hookrightarrow C^{k-1,\alpha}$$

Theorem 41 (Case $n = 1$)**.** *$W^{1,p}$ is Embedded as*
1. $p = 1$ then

$$W^{1,1} \hookrightarrow L^\infty$$

2. $p > 1$ then

$$W^{1,p} \hookrightarrow C^\alpha$$

4.8.1 Sobolev Embedding for $n = 1$

First, we prove $W^{1,1} \hookrightarrow L^\infty$

Proof. Consider $f \in C_c^\infty(\mathbb{R})$. $f(x) = \int_{-\infty}^{x} f'(y)dy$ so

$$|f(x)| \leq \int_{-\infty}^{\infty} |f'(y)|dy \leq ||f||_{W^{1,1}}$$

We now need to prove for general $W^{1,1}$ functions. By Approximation Theorem $\forall\epsilon > 0$, $\exists g \in C_c^\infty(\mathbb{R})$ that $||f - g||_{W^{1,1}} < \epsilon$ and $\sup_{\mathbb{R}-supp(g)} |f| < \epsilon$. Now

$$\begin{aligned} |f(x)| &\leq |f(x) - g(x)| + |g(x)| \leq ||g||_{W^{1,1}} + |f(x) - g(x)| \\ &\leq ||f||_{W^{1,1}} + \epsilon + |f(x) - g(x)| \end{aligned}$$

Thinking of test functions $\phi_n \in C_c(\mathbb{R})$ which derivatives are pointwise converging to Dirac delta on x

$$\frac{d}{dx}\phi_n \to \delta_x$$

and $supp(\phi_n) = [\phi^1, \phi^2]$, $\phi_n(\phi^2) = 1$, $\phi_n(\phi^1) = 0$ Now

$$\left| \int (f(x) - g(x)) \frac{d}{dx}\phi_n dx \right| = \left| \int \phi_n(f'(x) - g'(x))dx \right| \leq ||\phi_n||_{L^\infty} \cdot ||f'(x) - g'(x)||_{L^1} < 1 \cdot \epsilon$$

Left side of inequality converges to $|f(x) - g(x)|$ a.e. as $n \to \infty$ (It is Lebesgue Point)
$|f(x) - g(x)| < \epsilon$, $|f(x)| \leq ||f||_{W^{1,1}} + 2\epsilon$. $\square$

Remaining is another one.
Second, we prove $W^{1,p} \hookrightarrow C^\alpha$, $\alpha = 1 - \frac{1}{p}$.

Proof. $f \in W^{1,p}$ then thinking of $\phi_n \in C_c^\infty$ supported in $[x - \frac{1}{n}, y + \frac{1}{n}]$, $\phi_n|_{[x,y]} = 1$

$$\begin{aligned}
|f(x) - f(y)| &= \left| \lim_{n\to\infty} \int_U \phi_n'(x) f(x) dx \right| = \left| \lim_{n\to\infty} \int_U \phi_n(x) f'(x) dx \right| \\
&= \left| \int_x^y f'(t) dt \right| \\
&\leq \int_{\mathbb{R}} 1_{[x,y]} |f'(z)| dz \\
&\leq ||1_[x,y]||_{L^{p'}} ||f'||_{L^p} \leq C|x-y|^\alpha ||f||_{W^{1,p}}
\end{aligned}$$

$\square$

4.8.2 Comments on $n \neq 1$ cases

1. $W^{1,2} \hookrightarrow L^\infty$ **Does not satisfied if** $n = 2$
   Counter example is

$$f(r) = \left(\log \frac{1}{r} \right)^\alpha \chi(r)$$

For $\alpha < \frac{1}{2}$ and $\chi(r) = I(r \leq \frac{1}{2})$
- $f \notin L^\infty$: Around $r = 0$ it grows infinity.
- $f \in W^{1,2}$: Since two estimates established.

$$\int_0^{\frac{1}{2}} 2\pi r \left(\log \frac{1}{r} \right)^{2\alpha} dr = \int_{\log 2}^\infty 2\pi e^{-\theta} \theta^{2\alpha} e^{-\theta} d\theta < \infty$$

$$\int_0^{\frac{1}{2}} 2\pi r \left\{ \alpha \left(\log \frac{1}{r} \right) \frac{1}{r} \right\}^2 dr = \int_{\log 2}^\infty 2\pi e^{-2\theta} \frac{\alpha^2}{e^{-2\theta}} \theta^{2(\alpha-1)} d\theta < \infty$$

2. Above approach :

$$f(y) - f(x) = \int \nabla f$$

not essential. Since ∇f cannot be defined in line. We use, average of this quantity in a ball.

4.8.3 Sobolev Embedding for $W^{1,p}$

First we prove $W^{1,p} \hookrightarrow L^q$

Theorem 42 (Gagliardo-Nirenberg-Sobolev inequality)**.** *$1 \leq p < n$ then for $u \in C_c^1(\mathbb{R}^n)$ and $p^* = \frac{np}{n-p}$*

$$||u||_{L^{p^*}(\mathbb{R}^n)} \leq C_{p,n} ||Du||_{L^p(\mathbb{R}^n)}$$

Proof. If $p = 1$ then

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, x_2, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots x_n) dy_i$$

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, x_2, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots x_n)| dy_i$$

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, x_2, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots x_n)| dy_i \right)^{\frac{1}{n-1}}$$

Integrating above inequality and Holder inequality implies

$$\begin{aligned}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 &\leq \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &\leq \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \right)^{\frac{1}{n-1}}\end{aligned}$$

Integrating again,

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2 \leq &\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_1 dx_2 \right)^{\frac{1}{n-1}} \\ &\cdot \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}}\end{aligned}$$

Continuing,

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \leq \left(\int_{\mathbb{R}^n} |Du| dx \right)^{\frac{n}{n-1}}$$

as desired. For $p \neq 1$ apply above to $|u|^\gamma$ with $\gamma = \frac{p(n-1)}{n-p}$ □

Now, we prove $W^{1,p} \hookrightarrow L^q$

Proof. If $u \in W^{1,p}(U)$, there is an extension $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ compactly supported and inequality : $||\bar{u}||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)}$ holds. Now, we have $C_c^\infty(\mathbb{R}^n)$ sequence $\{u_m\}$ that converges to $\bar{u}$ in $W^{1,p}(\mathbb{R}^n)$ manner. By above Lemma,

$$||u_m||_{L^{p^*}(\mathbb{R}^n)} \leq C||Du_m||_{L^p(\mathbb{R}^n)}$$

Also,

$$||u_m - u_n||_{L^{p^*}(\mathbb{R}^n)} \leq C||Du_m - Du_n||_{L^p(\mathbb{R}^n)}$$

u_m is cauchy in L^{p^*}. $u_m \to u^*$ □

Second, we prove $W^{1,p} \hookrightarrow C^\alpha$. In this case we use **Moorey's Inequality**. Assume u is full domain $\mathbb{R}^n$

Theorem 43 (Moorey's Inequality)**.** *Assume $n < p \leq \infty$. Then there exists $C_{p,n}$ such that*

$$||u||_{C^{0,\gamma}(\mathbb{R}^n)} \leq C_{p,n}||u||_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$ and $\gamma = 1 - n/p$

Proof. We prove by integrating $u(y)-u(x)$ on some bounded ball. Remember estimate on $u(y)-u(x)$ by applying fundamental theorem of calculus is not possible in multidimensional cases since it cannot be defined. Instead, we integrate and estimate it.

$$\begin{aligned}
\int_{B(x,r)} |u(y)-u(x)|dy &= \int_0^r \int_{\partial B(x,s)} |u(z)-u(x)|dS(z)ds \\
&\leq \int_0^r s^{n-1} \int_{\partial B(0,1)} |u(x+sw)-u(x)|dS(w)ds \\
&\leq \int_0^r s^{n-1} \int_{\partial B(0,1)} \int_0^s |Du(x+tw)|dtdS(w)ds \\
&\leq \int_0^r s^{n-1} \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{t^{n-1}} dS(y)dtds \\
&= \int_0^r s^{n-1} \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}} dyds \leq \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy
\end{aligned}$$

As conclusion of computation,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \leq C \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy$$

□

Now using this, there are three steps.

Proof. STEP 1 : $\sup |u| < \infty$

Proof uses just above's estimate.

$$\begin{aligned}
|u(x)| &\leq \frac{1}{|B(x,1)|} \int_{B(x,1)} |u(y)-u(x)|dy + \frac{1}{|B(x,1)|} \int_{B(x,1)} |u(y)|dy \\
&\leq C \int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}} dy + C||u||_{L^p(\mathbb{R}^n)} \\
&\leq C \Big(\int_{\mathbb{R}^n} |Du|^p \Big)^{\frac{1}{p}} \Big(\int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)\frac{p}{p-1}}} \Big)^{\frac{(p-1)}{p}} + C||u||_{L^p(\mathbb{R}^n)} \\
&\leq C||u||_{W^{1,p}(\mathbb{R}^n)}
\end{aligned}$$

STEP 2 : $||u||_{\dot{C}^{0,\gamma}} < \infty$

Also using similar method. Denote $W = B(x,r) \cap B(y,r)$ where $r = ||x-y||$

$$\begin{aligned}
|u(x)-u(y)| &\leq \frac{1}{|W|} \Big(\int_W |u(x)-u(z)|dz + \int_W |u(y)-u(z)|dz \Big) \\
&\leq \frac{C}{|B(x,r)|} \Big(\int_{B(x,r)} |u(x)-u(z)|dz + \int_{B(y,r)} |u(y)-u(z)|dz \Big) \\
&\leq \frac{C}{|B(x,r)|} \Big(\int_{B(x,r)} |Du|^p dx \Big)^{\frac{1}{p}} \Big(\int_{B(x,r)} \frac{dz}{|x-z|^{(n-1)\frac{p}{p-1}}} \Big)^{\frac{p-1}{p}} \\
&\quad + \frac{C}{|B(y,r)|} \Big(\int_{B(y,r)} |Du|^p dx \Big)^{\frac{1}{p}} \Big(\int_{B(y,r)} \frac{dz}{|y-z|^{(n-1)\frac{p}{p-1}}} \Big)^{\frac{p-1}{p}} \\
&\leq Cr^\gamma ||Du||_{L^p(\mathbb{R}^n)}
\end{aligned}$$

So, following holds.

$$\frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \leq C||Du||_{L^p(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(\mathbb{R}^n)}$$

STEP 3 (Additional) :

Now for bounded domain, we extend into full domain $\bar{u} \in W^{1,p}(\mathbb{R}^n)$. Also approximate it by $C_c^\infty(\mathbb{R}^n)$ functions u_m. Apply STEP 1, 2 for u_m.

$$||u_m - u_l||_{C^{0,\gamma}(\mathbb{R}^n)} \leq C||u_m - u_l||_{W^{1,p}(\mathbb{R}^n)}$$

u_m is cauchy sequence on $C^{0,\gamma}(\mathbb{R}^n)$. Denote limit by u^*.

$$||u^*||_{C^{0,\gamma}(\mathbb{R}^n)} \leq ||u_m||_{C^{0,\gamma}(\mathbb{R}^n)}+\epsilon \leq C||u_m||_{W^{1,p}(\mathbb{R}^n)} \leq C||\bar{u}||_{W^{1,p}(\mathbb{R}^n)}+2\epsilon \leq C||u||_{W^{1,p}(U)}+2\epsilon$$

u^* is limit of u_m on $C^{0,\gamma}(\mathbb{R}^n)$ and $\bar{u}$ is limit of u_m on $W^{1,p}(\mathbb{R}^n)$. In U, those two limit will coincide. Therefore $u^* = \bar{u} = u$ a.e. on U.

$$||u||_{C^{0,\gamma}(U)} \leq C||u||_{W^{1,p}(U)}$$

□

Finally, we will prove $W^{1,n} \hookrightarrow BMO$.

Theorem 44 (Poincare's inequality)**.** *U is bounded, connected open subset of $\mathbb{R}^n$. ∂U is smooth. Assume $1 \leq p \leq \infty$. There exists $C_{n,p,U}$ such that*

$$||u - (u)_U||_{L^p(U)} \leq C_{n,p,U}||Du||_{L^p(U)}$$

for $u \in W^{1,p}(U)$

Proof. Proof by contradiction. Every integer k, $u_k \in W^{1,p}(U)$ satisfies

$$||u_k - (u_k)_U||_{L^p(U)} > k||Du_k||_{L^p(U)}$$

Define

$$v_k := \frac{u_k - (u_k)_U}{||u_k - (u_k)_U||_{L^p(U)}}$$

Then $(v_k)_U = 0$ and $||v_k||_{L^p(U)} = 1$. $||Dv_k||_{L^p(U)} < \frac{1}{k}$

Since Sobolev Embedding $W^{1,p}(U) \subset\subset L^p(U)$ is compact, there exists subsequence $\{v_{k_j}\}$ that converges to v in $L^p(U)$ sense. Then $(v)_U = 0$, $||v||_{L^p(U)} = 1$ holds.
However, test function $\phi \in C_c^\infty(U)$ and

$$\int_U v\phi_{x_i}dx = \lim_{k_j\to\infty}\int_U v_{k_j}\phi_{x_i}dx = -\lim_{k_j\to\infty}\int_U v_{k_j,x_i}\phi dx = 0$$

Therefore, v is weakly derivative, $v \in W^{1,p}(U)$ with $Dv = 0$ a.e. v is constant and $(v)_U = 0$ so $v \equiv 0$ but $||v||_{L^p(U)} = 0$ contradiction. □

We can prove Sobolev Embedding by Poincare's inequality. First, extend $u \in W^{1,n}(U)$ to $\bar{u} = Eu \in W^{1,n}(\mathbb{R}^n)$ with $||\bar{u}||_{W^{1,n}(\mathbb{R}^n)} \le C||u||_{W^{1,n}(U)}$

$$\begin{aligned}
\frac{1}{|B(x,r)|}\int_{B(x,r)} |\bar{u} - (\bar{u})_{B(x,r)}| &\le \frac{C}{r^n}||\bar{u} - (\bar{u})_{B(x,r)}||_{L^n(B(x,r))}\left(\int_{B(x,r)} 1\right)^{\frac{n-1}{n}} \\
&= \frac{C}{r}||\bar{u} - (\bar{u})_{B(x,r)}||_{L^n(B(x,r))} \\
&\le C||D\bar{u}||_{L^n(B(x,r))} \\
&\le C||\bar{u}||_{W^{1,n}(B(x,r))} \le C||\bar{u}||_{W^{1,n}(\mathbb{R}^n)} \le C||u||_{W^{1,n}(U)}
\end{aligned}$$

This proves Embedding to BMO.

4.8.4 Sobolev Embedding for $W^{k,p}$

Now in final, we need to prove $W^{k,p}$ embeddings. It is simple because if we apply sobolev embedding to derivatives from 0 to $k-1$, we can prove embeddings.
To be more precise $|\beta| \le k-1$, $D^\beta u \hookrightarrow L^{p^*}$ or $D^\beta u \hookrightarrow C^\alpha$. So to W^{k-1,p^*} and $C^{k-1,\alpha}$ embedding is true.

4.9 Compactness

We start from definition of Compactness in embedding and fundamental theorem : Arzela - Ascoli Theorem.

First, compactness is defined : $X \subset\subset Y$ if bounded sequence $\{f_n\}$ in X, then there exists a subsequence $\{f_{n_k}\}$ which is cauchy in Y. Compactness could be thought : In smaller scale (X), bounded sequence seems to be separate. However in large scale (Y), bounded sequence is quite dense and having converging subsequence. Fundamental theorem for compactness is Arzela - Ascoli Theorem.

4.9.1 Arzela - Ascoli Theorem

I state the theorem.

Theorem 45 (Arzela - Ascoli Theorem)**.** *$U \subset \mathbb{R}^n$ is bounded then $C^1(\bar{U}) \subset\subset C^0(\bar{U})$*

Proof. Let $\{f_n\}$ uniformly bounded sequence on $C^1(\bar{U})$. Let countable dense subset $\bar{\{x_m\}} = U$
Then we can apply **Cantor Diagonal Statement**

$\{f_n(x_1)\}$ is uniformly bounded so $f_{n_1^1}(x_1),\ f_{n_2^1}(x_1),\ \cdots,\ f_{n_k^1}(x_1)\cdots$ cauchy.
$\{f_{n^1}(x_2)\}$ is uniformly bounded so $f_{n_1^2}(x_2),\ f_{n_2^2}(x_2),\ \cdots,\ f_{n_k^2}(x_2)\cdots$ cauchy.
$\{f_{n^2}(x_3)\}$ is uniformly bounded so $f_{n_1^3}(x_3),\ f_{n_2^3}(x_3),\ \cdots,\ f_{n_k^3}(x_3)\cdots$ cauchy.
$\vdots$

Continuing, $\{f_{n_1^1}, f_{n_2^2}, f_{n_3^3}\cdots\} = \{f_{n_k}\}_{k\in\mathbb{N}}$ is cauchy in every x_m. We now define

$$g(x_m) = \lim_{k\to\infty} f_{n_k}(x_m)$$

There are some things to do more. First we need to extend g to U, this will by proving cauchy feature. Also, we need to prove $\{f_{n_k}\}$ converges to g in $C^0(\bar{U})$ sense.

Observe : $\forall n_k, m, m',\ |f_{n_k}(x_{m'}) - f_{n_k}(x_m)| = |f'_{n_k}(t)(x_m - x_{m'})| \leq C|x_m - x_{m'}|$ (since f'_{n_k} is bounded)
Now, $\forall\epsilon > 0, m, m'$ there exists K if $k \geq K$, $|f_{n_k}(x_{m'}) - g(x_{m'})| < \epsilon$, $|f_{n_k}(x_m) - g(x_m)| < \epsilon$.

$$\begin{aligned}|g(x_m) - g(x_{m'})| &\leq |g(x_m) - f_{n_k}(x_m)| + |f_{n_k}(x_m) - f_{n_k}(x_{m'})| + |f_{n_k}(x_{m'}) - g(x_{m'})| \\ &< 2\epsilon + C|x_m - x_{m'}|\end{aligned}$$

So, $\forall m, m'$ $|g(x_m) - g(x_{m'})| \leq C|x_m - x_{m'}|$. g is cauchy in $\{x_m\}$. Define

$$g(x) = \lim_{x_m\to x} g(x_m)$$

This was first step. Now second step is C^0 convergence.
As intermediate step, we prove g is cauchy in U. For every $\epsilon > 0$, $\forall x, y \in U$, there exists $z_x, z_y \in \{x_i\}$ such that $|x - z_x| < \frac{\epsilon}{C},\ \ |y - z_y| < \frac{\epsilon}{C}$ and $|g(x) - g(z_x)| < \epsilon,\ \ |g(y) - g(z_y)| < \epsilon$

$$|g(x) - g(y)| \leq |g(x) - g(z_x)| + |g(z_x) - g(z_y)| + |g(z_y) - g(y)| < 4\epsilon + C|x - y|$$

Now, for all $x \in U$ and $\epsilon > 0$, there exists $m = M$ such that $|x - x_m| < \frac{\epsilon}{C}$

$$\begin{aligned} |f_{n_k}(x) - g(x)| &\leq |f_{n_k}(x) - f_{n_k}(x_m)| + |f_{n_k}(x_m) - g(x_m)| + |g(x_m) - g(x)| \\ &\leq C|x - x_m| + C|x - x_m| + |f_{n_k}(x_m) - g(x_m)| \\ &< 2\epsilon + |f_{n_k}(m) - g(m)| < 3\epsilon \end{aligned}$$

For large $k \geq K_M$. Therefore, $f_{n_k} \to g$ in $C^0(\bar{U})$ □

4.9.2 Compact Embedding Theorem

Our goal is proving statement

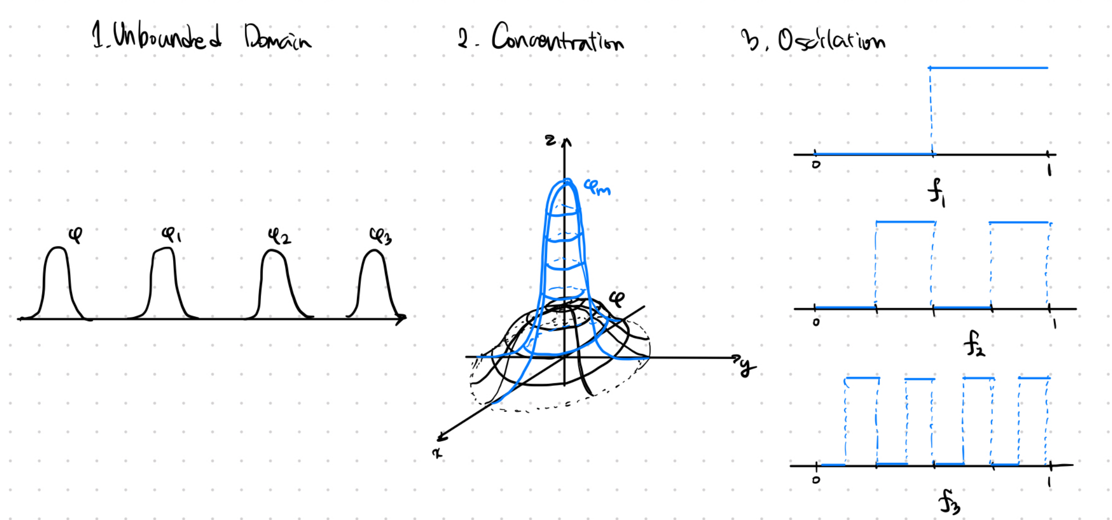
Theorem 46. *Assume completion of $X \cap C^\infty$ is X. f^δ is standard mollification of f length δ. If embedding $X \hookrightarrow Y$ satisties*

$$\lim_{\delta \to 0} \left[\sup_{0 \neq f \in X} \frac{||f - f^\delta||_Y}{||f||_X} \right] = 0$$

This Embedding is compact. $X \subset\subset Y$

To understand this theorem, we need to know about **Bottlenecks for Compactness** and **Standard Mollification**.

Bottlenecks for Compactness



There are three Bottlenecks : Unbounded Domain (Escaping to infinity), Concentration, Oscillation

1. Unbounded Domain : Issue comes if $\varphi \in C_c^\infty$, $\varphi_m(x) = \varphi(x - me_1)$ then it will never gain convergent subsequence.

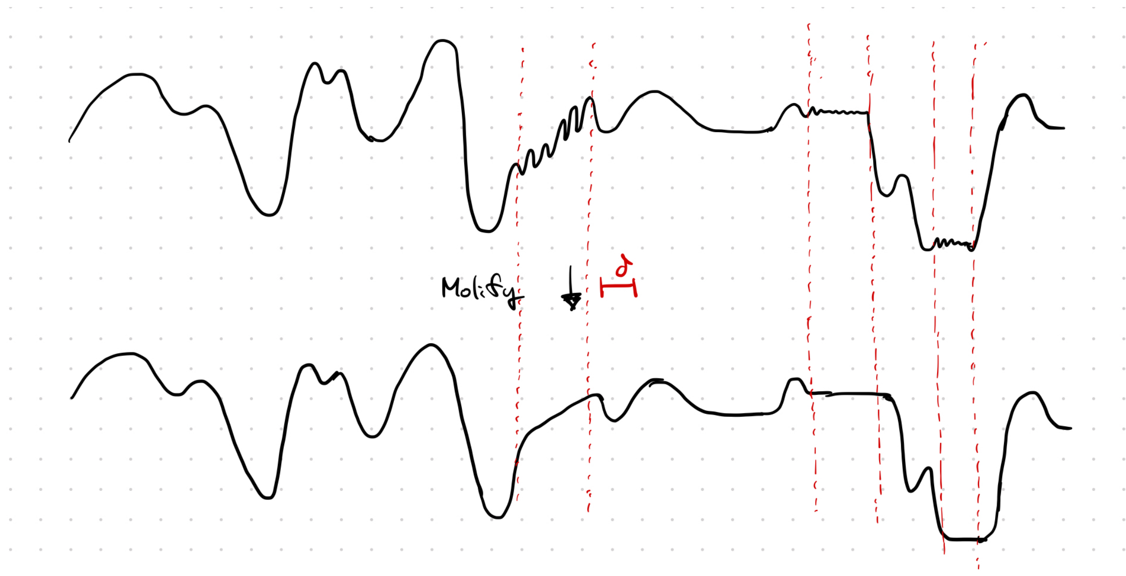
2. Concentration : If embedding $X \hookrightarrow Y$ is homogenuous embedding, $\varphi \in C_c^\infty$ and $\varphi_m(x) = m^\lambda \varphi(mx)$. Then $||\varphi_m||_X \leq C$ but $||\varphi_m||_Y \to 0$ is not true.
For example, $W^{1,1}(B) \hookrightarrow L^2(B)$ and $B = \{x^2 + y^2 < 1\}$. For $||\varphi_m||_{W^{1,1}(B)} \leq C$

$$||\varphi_m||^2_{L^2(B)} = \int \varphi^2(mx) m^2 dx = \int \varphi^2(x) dx \neq 0$$

3. Oscillation : For example, $L^\infty(B) \hookrightarrow L^1(B)$ might fail in compactness. There is sequence of $L^\infty(B)$ that is never cauchy on $L^1(B)$.

As a conclusion, <u>1. Domain should be bounded</u>, <u>2. Homogenuous Embedding might fail</u> and <u>3. Same differential index is impossible</u>

4.9.3 Standard Mollification



There are a lot of ways to mollify. However, important thing is **all mollification vanishes oscillation scale of** δ

- f compactly supported in $U \subset \mathbb{R}^n$

$$f^\delta(x) = (f * \varphi^\delta)(x) = \int f(y) \cdot \frac{1}{\delta^n} \varphi(\frac{x-y}{\delta}) dy$$

In this case, it is well known that following holds.

Theorem 47 (Properties of mollifiers). *If $f \in L^1_{loc}(U)$,*
(1) $f^\epsilon \in C^\infty(U_\epsilon)$
(2) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$
(3) If $f \in C(U)$ then $f^\epsilon \to f$ uniformly on compact subsets of U
(4) If $1 \leq p < \infty$ and $f \in L^p_{loc}(U)$ then $f^\epsilon \to f$ in $L^p_{loc}(U)$.

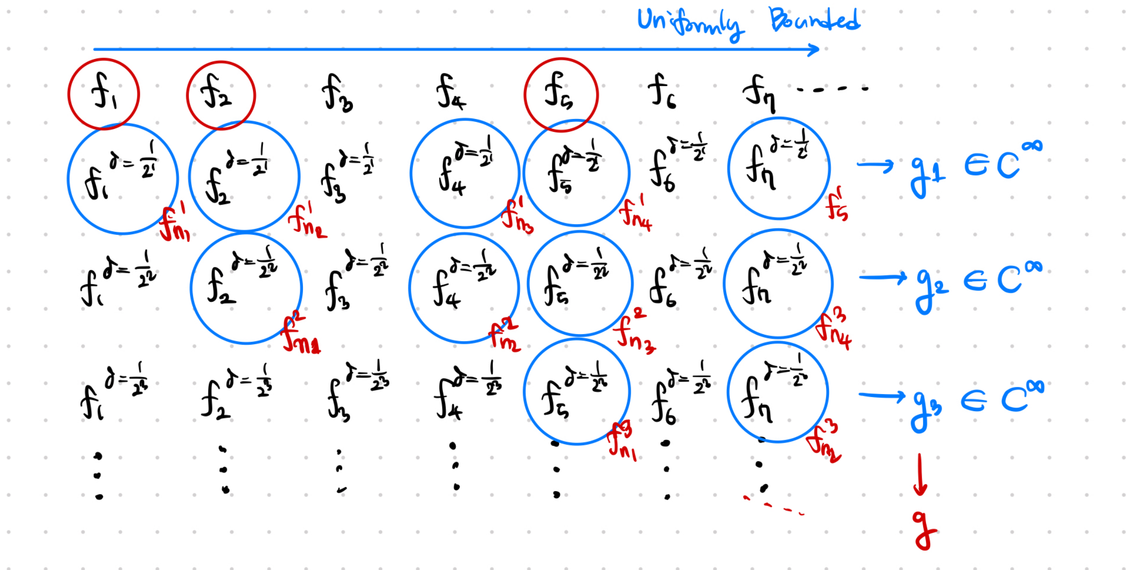
- f periodic in 1 and domain $\mathbb{T}^n$

$$f^\delta(x) = \sum_{k\in\mathbb{Z}^n,\ ||k||<\frac{1}{\delta}} \hat{f}_k e^{2\pi i k\cdot x}$$

We are ready to think of compact embedding theorem.

4.9.4 Proof for Compact Embedding Theorem

Proof for Compact Embedding Theoerem. Assume $f_1, f_2, \cdots$ uniformly bounded sequence on X.



We mollify f_i with $\delta = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}\cdots$.

$$f_1^{\delta=\frac{1}{2}},\ f_2^{\delta=\frac{1}{2}},\ f_3^{\delta=\frac{1}{2}},\cdots$$

are C^∞ and bounded.

By Arzela - Ascoli,
$f_{n_1^1}, f_{n_2^1}, f_{n_3^1}\cdots$ converges on C^0 sense. $\{n_i^1\}\subset\mathbb{Z}$
$f_{n_1^2}, f_{n_2^2}, f_{n_3^2}\cdots$ converges on C^1 sense. $\{n_i^2\}\subset\{n_i^1\}$
$f_{n_1^3}, f_{n_2^3}, f_{n_3^3}\cdots$ converges on C^2 sense. $\{n_i^3\}\subset\{n_i^2\}$
$\vdots$
Now, applying Cantor Diagonal Argument, $\{f_{n_1^1}, f_{n_2^2},\cdots\} = \{f_{n_k}\}$ is a subsequence such that $f_{n_k}\to g$ in C^∞ sense.

this could be headed to all mollified sequence with same δ. Now, notate if $\delta = \frac{1}{2^m}$, C^∞ converging subsequence by $\{f^m_{n_k}\}_{k\in\mathbb{N}}$ and limit g_m.

We will prove g_m **is Cauchy in** Y.

For $\epsilon > 0$, $a(m)$ defined below goes to zero if $m \to \infty$ by assumption.

$$a(m) = \sup_k \frac{||f_{n_k}^m - f_{n_k}||_Y}{||f_{n_k}||_X}$$

There exists M that $m \geq M$ then

$$a(m) < \frac{\epsilon}{4 \sup_k ||f_k||_X}$$

Now for this $m, m' \geq M$

$$\begin{aligned}||g_m - g_{m'}||_Y &\leq ||g_m - f_{n_k}^m||_Y + ||f_{n_k}^m - f_{n_k}||_Y + ||f_{n_k} - f_{n_k}^{m'}||_Y + ||f_{n_k}^{m'} - g_{m'}||_Y \\ &\leq \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon\end{aligned}$$

For large k.

g_m is Cauchy in Y. Therefore limit g exists. Remaining proof is showing $f_{n_k} \to g$ in Y.

For $\epsilon > 0$, M satisfies $m \geq M$ then $a(m) < \frac{\epsilon}{2 \sup_k ||f_k||_X}$ and M' satisfies if $m \geq M'$, $||g_m - g||_Y < \epsilon/2$

$$\begin{aligned}||f_{n_k} - g||_Y &\leq ||f_{n_k}^m - f_{n_k}||_Y + ||f_{n_k}^m - g_m||_Y + ||g_m - g||_Y \\ &\leq \epsilon/2 + ||f_{n_k}^m - g_m||_Y + \epsilon/2 \\ &\leq \epsilon + \epsilon = 2\epsilon\end{aligned}$$

Only last inequality uses $k \to \infty$. Therefore, $\{f_k\}$ is compact in Y. □

4.9.5 Torus Example

Let $a : \mathbb{R}^+ \to \mathbb{R}^+$, define norm with

$$||f||_a^2 = \sum_{k \in \mathbb{Z}^n} a(|k|) |\hat{f}_k|^2$$

X_a is function space with finite a norm. $X_a = \{f | dom f = \mathbb{T}^n,\ |\hat{f}_k| < \infty\ \forall k,\ ||f||_a < \infty\}$

Then a_1, a_2 with

$$\frac{a_2(|k|)}{a_1(|k|)} \longrightarrow 0$$

for $|k| \to \infty$ then $X_{a_1} \subset\subset X_{a_2}$

- Remark If $a(|k|) = (1 + |k|^2)^{s/2}$, $X_a = H^s$ and above theorem states if $s_2 < s_1$

$$H^{s_1}(\mathbb{T}^n) \subset\subset H^{s_2}(\mathbb{T}^n)$$

- Proof is simple. Applying Theorem above,

$$\lim_{k \to \infty} \frac{||f^{\delta_k} - f||_{a_2}^2}{||f||_{a_1}^2} = \lim_{k \to \infty} \frac{\sum_{|k| > K,\ k \in \mathbb{Z}^n} a_2(|k|) |\hat{f}_k|^2}{\sum_{k \in \mathbb{Z}^n} a_1(|k|) |\hat{f}_k|^2}$$

For all $\epsilon > 0$, some K^* exists that $K \geq K^*$ then $a_2(|k|) < \epsilon a_1(|k|)$. For appropriate δ,

$$\frac{||f^{\delta_k} - f||_{a_2}}{||f||_{a_1}} < \sqrt{\epsilon}$$

5 Weak Topology

We will now use terminology of 'weak topology' to solve Partial Differential Equations. Understanding about relation between normed topology and weak topology is important since we will use technique based on these relationships. For instance, bounded sequence in normed space have weakly converging sequence. It is same with 'Weak Topology is compact on Normed Topology'. Anyhow, for further discussion, weak topology is essential to deal with.

5.1 Definition of weak topology $\sigma\big(X, (f_i)_{i\in I}\big)$

We define weak topology. $U \subset X$ is open in weak topology $\sigma\big(X, (f_i)_{i\in I}\big)$ if U is union of elements in the collection $\mathcal{J}$.

$$\mathcal{J} = \left\{ \bigcap_{i\in J\subset I,\ |J|<\infty} f_i^{-1}(O_i) \middle| O_i \subset Y_i \quad \forall i \in J \right\}$$

Above, O_i is open set of Y_i. Remarkable point is, intersection is finite intersection and f_i do not need any arguments. Definition of weak topology makes each f_i continuous.

Theorem 48 (Weak Topology Convergence)**.** *$(x_n)_{n\in\mathbb{N}}$ is sequence in X. x_n converges to x in weak topology $\sigma\big(X, (f_i)_{i\in I}\big)$ if and only if $\forall i \in I$, $\lim_{n\to\infty} f_i(x_n) = f_i(x)$*

This is very important feature in weak topology. To see convergence on weak sense, we need to consider each function values. Notice that speed of convergence in each functions are not important.

Proof. We divide into two proofs.

First, suppose sequence converges in weak topology to some $x \in X$. Then since every f_i is continuous for $\sigma\big(X, (f_i)_{i\in I}\big)$,

$$\lim_{n\to\infty} f_i(x_n) = f_i(x_n)$$

Second, suppose there exists $x \in X$ such that $\forall i \in I$, $\lim_{n\to\infty} f_i(x_n)$ exists and equal to $f_i(x)$. Then for any open set O containing x, there exists a finite subset J of I and $(O_j)_{j\in J}$ such that $O_j \subset Y_j$ for all $j \in J$ satisfying

$$x \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(O_j)$$

Therefore, $\forall j \in J$, $f_j(x) \in O_j$. For each j, $f_i(x_n)$ converges to $f_i(x)$ so $N_j \in \mathbb{N}$ exists such that if $n \geq N_j$, $f_j(x_n) \in O_j$.

Let $N = \max_{j\in J} N_j$, then for $n \geq N$,

$$x_n \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(O_j) \subset O$$

This proves $(x_n)_{n\in\mathbb{N}}$ converges to x for weak Topology. □

We could think of continuous functions on weak topology. Here gives condition for function is continuous.

Theorem 49 (Weak topology Continuous funciton)**.** *Let $(Z, \mathcal{J})$ a topological space, $\varphi : Z \longrightarrow X$ map. φ is continuous for the topologies $\mathcal{J}$ and $\sigma\big(X, (f_i)_{i\in I}\big)$ if and only if for every $i \in I$, $f_i \circ \varphi$ is continuous.*

5.2 Weak topology for normed space

We define Weak topology for normed space by following.

$$\sigma(X, X^*) \equiv \sigma\big(X, (f)_{f\in X^*}\big)$$

We target f to be a bounded linear functional on X. Remember that X^* is space of bounded linear functional on X. This definition does not depend wheter X is Banach or not.

We are Interested in basic properties of weak topology $\sigma(X, X^*)$. First, it is Hausdorff.

Theorem 50 (Weak topology is Hausdorff)**.** *$\sigma(X, X^*)$ is Hausdorff*

Proof. Proof uses Hahn-Banach theorem. Let $x \neq y \in X$ then first, there exists $\epsilon > 0$ that $y \notin B(x, \epsilon)$. $B(x, \epsilon)$ is convex and open so by Hahn-Banach theorem, there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ that $\forall u \in B(x, \epsilon)$, $(f, u) < \alpha < (f, y)$. Then $f^{-1}((-\infty, \alpha))$ and $f^{-1}((\alpha, \infty))$ is weakly open set that separates x, y. $\square$

Now, we will meet most powerful tool that weak topology holds.

Proposition 1. *Following Holds.*
1. *Weak Topology is weaker than the normed topology.*
2. *$(x_n)_{n\in\mathbb{N}}$ weakly converges to x (we will denote by $(x_n)_{n\in\mathbb{N}} \rightharpoonup x$) if and only if $\forall f \in X^*$,*

$$\lim_{n\to\infty} (f, x_n) = \lim_{n\to\infty} (f, x)$$

3. *Strong converging sequence converges weakly.*
4. *$(x_n)_{n\in\mathbb{N}} \rightharpoonup x$ then $(x_n)_{n\in\mathbb{N}}$ is bounded and*

$$||x|| \leq \liminf_{n\to\infty} ||x_n||$$

5. *$(x_n)_{n\in\mathbb{N}} \rightharpoonup x$ and $(f_n)_{n\in\mathbb{N}} \to f$ in X^* then*

$$\lim_{n\to\infty} (f_n, x_n) = (f, x)$$

In PDE, fourth property is used often. Since weak convergence implies limit cannot grow at infinity, it sometimes give well defined functions. Proof of 1, 2 is obvious. 3 comes from inequality $|(f, x) - (f, x_n)| = |(f, x - x_n)| \leq ||f||||x - x_n|| \longrightarrow 0$ if x_n converges to x strongly.

4 can be proven by Banach-Steinhuas theorem. If x_n unbounded then we can apply Banach-Steinhaus theorem to completion of X^* to normed space X and bounded linear transformations $\{x_n\}_{n\in\mathbb{N}}$ which assigns (x_n, f) an value (f, x_n). Since

$$\sup_{x_n \in \{x_n\}_{n\in\mathbb{N}}} ||(x_n, f)|| < \infty$$

for every $f \in X^*$, it is only possible that $||x_n|| \leq M$ for some $M < \infty$. This proves boundness.

On the other hand, assume f is bounded linear functional on X then $|(f, x_n)| \leq ||f||||x_n||$ so by taking liminfs,

$$|(f, x)| \leq ||f|| \liminf_{n\to\infty} ||x_n||$$

Since

$$||x|| = \sup_{0\neq f\in X^*} \frac{|(f,x)|}{||f||}$$

$$||x|| \leq \liminf_{n\to\infty} ||x_n||$$

5 can be proved immediately since $|(f, x) - (f_n, x_n)| \leq |(f, x - x_n)| + ||f_n - f||||x_n||$. Righthand side tends to zero since x_n is bounded.

What will be the basis for weak topology $\sigma(X, X^*)$? following theorem tells basis of neighborhoods of certain point $x_0 \in X$

Theorem 51 (Basis for $\sigma(X, X^*)$). *$x_0 \in X$. Basis of neighborhoods of x_0 for weak topology is*

$$W_{\epsilon,f_1,\cdots f_n} = \left\{ x \in X \;\middle|\; \forall i \in \{1, 2, \cdots n\} \;\; |(f_i, x) - (f_i, x_n)| < \epsilon \right\}$$

$\forall n \in \mathbb{N},\ \epsilon > 0,\ f_1, \cdots f_n \in X^*$

Proof. $W_{\epsilon,f_1,\cdot f_n}$ is trivially, weakly open and contains x_0. If O is open in weak topology containing x,

$$\bigcap_{j\in\{1,2,\cdots n\}} f_j^{-1}(O_j) \subset O$$

for some f_j's. $(f_j, x_0) \in O_j$ so there exists $\epsilon_j > 0$ that $B((f_j, x_0), \epsilon_j) \subset O_j$. For $\epsilon = \min_{j\in\{1,\cdots n\}} \epsilon_j$, $W_{\epsilon,f_1,\cdots f_n} \subset O$ □

Then does weak topology and strong topology different? It depends on dimension. For infinite dimensional space, two topologies does not coincide. We can easily show if finite dimension, projection which is bounded linear functional allows weak topology equivalent to $||\cdot||_\infty$ norm and thus strong topology and weak topology coincides. Remaining part is, if X is infinite dimension, strong topology does not coincide with weak topology.

Proposition 2. *Weak topology and Strong topology does not coincide if dimension of X is infinite.*

Proof. Our claim is, $S = \{x \in X \mid ||x|| = 1\}$ then $0 \in \bar{S}$ in weak topology.

We will consider neighborhood of 0 in weak topology. By above theorem about basis for weak topology, $0 \in O$ open in weak topoology need to contain $W_{\epsilon,f_1,\cdots f_n}$. Then our definition of $W_{\epsilon,f_1,\cdots f_n}$ gives

$$W_{\epsilon,f_1,\cdots f_n} = \{x \in X \mid |(f_i, x)| < \epsilon\}$$

Now, mapping $\Phi : X \to \mathbb{R}^n$, $\Phi(x) = \big((f_1, x), (f_2, x), \cdots (f_n, x)\big)$ is linear. By rank nullity theorem,

$$Dim(Ker\Phi) + Dim(Im\Phi) = Dim(X) = \infty$$

Image is finite dimensional so $Dim(Ker\Phi) = \infty$. Some $0 \neq x \in X$ exists and $\Phi(x) = 0$. Then $\lambda x \in W \subset O$ for any lambda, choosing $1/||x||$ gives $\phi \neq W \cap S \subset O \cap S$.

As a consequence, closure of S via $\sigma(X, X^*)$ is $\bar{B}$. □

However we can build some conditions so that they play almost same role.

Theorem 52. *C is nonempty, convex set in X. Then C is strongly closed if and only if it is weakly closed*

Proof. We only need to prove if C is strongly closed than it is weakly closed. Assume set C is strongly closed.

This immitates proof of Hausdorffness. For $x \in C^c$, there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $(f, y) < \alpha < (f, x)$. Then $f^{-1}((\alpha, \infty))$ and $f^{-1}((-\infty, \alpha))$ seperates x and C so x does not belongs to closure of C regard to weak topology. □

Finally, we end with following theorem on Linear mapping between two Banach Spaces.

Theorem 53. *X, Y are Banach Spaces. $T : X \to Y$ is linear. Then T is continuous in both strong topology if and only if T is continuous in both weak topology.*

5.3 Weak$\star$ topology $\sigma(X^*, X)$

Weak$\star$ topology $\sigma(X^*, X)$ is quite different. Since $X^{**} = X$ does not hold generally, definition of $\sigma(X^*, X)$ is $\sigma(X^*, (x)_{x\in X})$. Note that we define (x, f) for $x \in X, f \in X^*$ by (f, x).

We can derive similar properties by mimicing many of proofs from weak topology $\sigma(X, X^*)$. First, Hausdorffness.

Theorem 54 (Hausdorffness of $\sigma(X^*, X)$)**.** *The topology $\sigma(X^*, X)$ is Hausdorff*

Proof. Proof became more simple. If $f \neq g$ in X^*, there exists $x \in X$ that $(f, x) \neq (g, x)$. We can find $\alpha \in \mathbb{R}$ such that $(f, x) < \alpha < (g, x)$ (without loss of generallity) and it gives $f \in x^{-1}((-\infty, \alpha))$ and $g \in x^{-1}((\alpha, \infty))$. □

We will go on some basic proposition and theorem as same as $\sigma(X, X^*)$

Proposition 3. *Following Holds.*

1. *Weak$\star$ Topology is weaker than the normed topology.*
2. *$(f_n)_{n\in\mathbb{N}}$ weak$\star$ convergent to f if and only if $\forall x \in X$,*

$$\lim_{n\to\infty} (f_n, x) = \lim_{n\to\infty} (f, x)$$

3. *Strong converging sequence in X^* converges weak$\star$ convergent.*
4. *$(f_n)_{n\in\mathbb{N}} \rightharpoonup f$ in weak$\star$ then $(f_n)_{n\in\mathbb{N}}$ is bounded and*

$$||f|| \leq \liminf_{n\to\infty} ||f_n||$$

5. *$(f_n)_{n\in\mathbb{N}} \rightharpoonup x$ in weak$\star$ and $(x_n)_{n\in\mathbb{N}} \to x$ in X then*

$$\lim_{n\to\infty} (f_n, x_n) = (f, x)$$

Theorem 55 (Basis for $\sigma(X^*, X)$)**.** *$f_0 \in X^*$. Basis of neighborhoods of f_0 for weak$\star$ topology is*

$$W_{\epsilon, x_1, \cdots x_n} = \left\{ f \in X^* \;\middle|\; \forall i \in \{1, 2, \cdots n\} \quad |(f, x_i) - (f_n, x_i)| < \epsilon \right\}$$

$\forall n \in \mathbb{N},\ \epsilon > 0,\ x_1, \cdots x_n \in X$

Difference comes now. Since $\sigma(X^*, X)$ define with bounded linear functionals $(x)_X$ only, we need to be more careful. Following two propositions highlights this.

Proposition 4. *Let $\varphi \in X^{**}$ and φ is weak$\star$ continuous. Then there exists $x \in X$ that $\forall f \in X^*$*

$$(\varphi, f) = (f, x)$$

It is important that **weak$\star$ continuous** cannot be removed.

Proof. Since φ is weak$\star$ continuous,

$$V = \{f \in X^* \mid |(\varphi, f)| < 1\}$$

is weak$\star$ open and contains 0. So there exists $x_1, \cdots x_n \in X, \epsilon > 0$ such that

$$W = \{f \in X^* \mid |(f, x_i)| < \epsilon \quad \forall 1 \leq i \leq n\} \subset V$$

If $f \in \bigcap_{i=1}^n Kerx_i$ then for all $\lambda \in \mathbb{R}$, $(\lambda f, x_i) = 0$ for all i. Therefore, $\lambda f \in W \subset V$ so $|\lambda||(\varphi, f)| < 1$ for all lambda. Therefore, $f \in Ker\varphi$.

There is a Lemma that if $f_1, \cdots f_n, f$ a linear functionals on vector space X then f is a linear combination of $f_1, \cdots f_n$ if and only if

$$\bigcap_{i=1}^n Kerf_i \subset Kerf$$

Therefore, φ is equal to linear combination if x_i's. □

Proposition 5. *Let H be a hyperplane in X^* and H is closed for weak$\star$ topology. Then there exists $x \in X$ and $\alpha \in \mathbb{R}$ such that*

$$H = \{f \in X^* \mid (f, x) = \alpha\}$$

Again, it is important that H **is closed for weak$\star$ topology** cannot be removed.

Proof. $H = \{f \in X^* \mid (\xi, f) = \alpha\}$ since H is hyperplane. Here, ξ is member of X^{**}.

Since H is weak$\star$ closed, for $f_0 \notin H$ there exists $x_1, \cdots, x_n \in X$ and $\epsilon > 0$ such that $W \cap H = \phi$ where

$$W = \{f \in X^* \mid |(f, x_i) - (f_0, x_i)| < \epsilon \quad \forall i \in \{1, \cdots, n\}\}$$

Assume $(\xi, f_0) < \alpha$.

We will show that no $f \in W$ satisfies $(\xi, f) > \alpha$. If not, for corresponding f, $\varphi : t \longmapsto (\xi, tf + (1-t)f_0)$ is continuous mapping on $[0, 1]$. (This is from weak topology's continuous mapping theorem). $\varphi(0) < \alpha$ and $\varphi(1) > \alpha$ so there exists $f' \in H$ so that $f' = tf + (1-t)f_0 \in W$. $f' \in W \cap H$ so contradiction.

Now we proved $\forall f \in W$, $(\xi, f) < \alpha$. Then $W - f_0 = \{f \in X^* \mid |(f, x_i)| < \epsilon \ \ \forall i \in \{1, \cdots, n\}\}$ is weak$\star$ neighborhood of 0. Then for all $f \in W - f_0$, $(\xi, f) = (\xi, f + f_0) - (\xi, f_0) < \alpha - (\xi, f_0)$. Since $-f$ also belongs to it, $|(\xi, f)| < \alpha - (\xi, f_0)$.

For other side, which is $(\xi, f_0) > \alpha$ we can do same with it and now we obtain for any $f_0 \notin H$, $\forall f \in W - f_0$

$$|(\xi, f)| < |\alpha - (\xi, f_0)|$$

It means ξ is weak$\star$ continuous on 0 and since ξ is bounded linear functional, ξ is weak$\star$ continuous. This proves that $(\xi, f) = (f, x)$ for all $f \in X^*$ for some $x \in X$ □

5.4 Banach-Alaoglu Theorem

Lastly, the major property of weak topology is compactness.

Theorem 56 (Banach-Alaoglu Theorem)**.** *The unit closed ball in X^* is weak$\star$ compact.*

Remark Unit ball in normed topology is never compact in infinite dimensional space.

Proof. First, let us check $\bar{B_{X^*}}$ is closed in weak$\star$ topology.

Let f_0 in closure of $\bar{B_{X^*}}$ via $\sigma(X^*, X)$. Then for all $\epsilon > 0$, there exists $x \in X$ that $||x|| = 1$ and $(f_0, x) > ||f_0|| - \epsilon$ by definition of $||f||$. If we set $W = \{f \in X^* \mid |(f, x) - (f_0, x) > \epsilon\}$ which is weak$\star$ neighborhood, since f_0 lies on closure of $\bar{B_{X^*}}$ on weak$\star$ topology, some $f \in W \cap \bar{B_{X^*}}$.

For this f, $||f|| \leq 1$ and $|(f, x) - (f_0, x)| < \epsilon$ thus $|(f, x)| > ||f_0|| - 2\epsilon$. We now gain $||f_0|| < 1 + 2\epsilon$ for all ϵ so $||f_0|| \leq 1$. This means $\bar{B_{X^*}}$ is closed on weak$\star$ topology.

Next, construct $Y = \mathbb{R}^X$ with product topology. Then $e_x : Y \to \mathbb{R}$ which assigns $w \in Y$ to $e_x(w) = w(x)$ is continuous. (Actually, Y is the smallest topology that makes these functions continuous)

Then $J : X^* \to Y$ that $f \longmapsto J(f)$ which $J(f)$ assignes $J(f)(x) = (f, x)$ is injection.

J is continuous if X^* hhas weak topology. This is because $(e_x \circ J)(f) = J(f)(x) = (f, x)$ for all $x \in X$, $f \in \bar{B_{X^*}}$. Also J^{-1} is continuous on $J(X^*)$ since $(J^{-1}(f), x) = (f, x) = e_x(f)$ for all $x \in X$, $f \in J(X^*)$.

Therefore, J is homeomorphism $X^* \to J(X^*)$ with weak topology and product topology. Now, $\forall x \in X$, $\forall f \in \bar{B_{X^*}}$

$$|J(f)(x)| = |(f, x)| \leq ||f|| ||x|| \leq ||x||$$

So

$$J(\bar{B}_{X^*}) \subset \prod_{x \in X} [-||x||, ||x||]$$

Righthand side is compact by Tychonoff's theorem and left hand side is closed since J^{-1} is continous and $\bar{B_{X^*}}$ is weak$\star$ closed. So $J(\bar{B_{X^*}})$ is compact and J^{-1} is again continuous so $\bar{B_{X^*}}$ is compact for the weak$\star$ topology. □

6 Korteweg-De Vries Equation

Solving well-posedness of Partial Differential Equation needs two seperate parts. Proving Existence and Uniqueness requires different proof technique.

KdV Equation is following

$$\begin{cases} \partial_t u + u\partial_x u + \partial_{xxx} u = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{T} \\ u(t=0,x) = u_0(x) \end{cases}$$

Our goal is proving well-posedness of KdV theorem.

Theorem 57 (Well-posedness of Korteweg-De Vries Equation)**.** *Given $u_0 \in H^s(\mathbb{T})$, $s > \frac{3}{2}$ we have*

1. *Existence : $\exists T$ depending on $||u_0||_{H^{\frac{3}{2}}}$, there is a solution in $C_{[0,T]}H^s$*
2. *Uniqueness : Solution is unique in $C_{[0,T]}H^s$*
3. *Continuity : Solution operator is continuous from bounded ball in H^s to $C_{[0,T]}H^s$*
4. *Propagation : If $u_0 \in C^\infty(\mathbb{T})$ solution is $C^\infty([0,T] \times \mathbb{T})$*

6.1 A Priori Estimates

Priori Estimates gives information for PDE and drives some property that we could think for target.
In this case, we first estimate with energy method.

$$E_n(t) := \int_{\mathbb{T}} |\partial_x^n u(t,x)|^2 dx$$

If solution is C^∞,

$$\begin{aligned} \frac{d}{dt}E_1(t) &= \int_{\mathbb{T}} 2\partial_x u(-(\partial_x u)^2 - u\partial_{xx}u - \partial_{xxxx}u)dx \\ &= \int_{\mathbb{T}} -2(\partial_x u)^3 dx + \int_{\mathbb{T}} (\partial_x u)^3 dx = -\int_{\mathbb{T}} (\partial_x u)^3 dx \end{aligned}$$

Therefore,

$$\left| \frac{d}{dt}E_1(t) \right| \leq CE_1(t)||\partial_x u(t)||_{L^\infty}$$

$$\begin{aligned} \frac{d}{dt}E_2(t) &= \int_{\mathbb{T}} 2(\partial_{xx}u)(-\partial_{xxxxx}u - u\partial_{xxx}u - 3\partial_x u\partial_{xx}u)dx \\ &= \int_{\mathbb{T}} -2(\partial_{xx}u)(u\partial_{xxx}u + 3\partial_x u\partial_{xx}u)dx \\ &= \int_{\mathbb{T}} \partial_x u(\partial_{xx}u)^2 dx - 6\int_{\mathbb{T}} \partial_x u\partial_{xx}u dx \\ &= -5\int_{\mathbb{T}} \partial_x u(\partial_{xx}u)^2 dx \end{aligned}$$

Therefore,

$$\left| \frac{d}{dt}E_2(t) \right| \leq CE_2(t)||\partial_x u(t)||_{L^\infty}$$

We could think that $||\partial_x u||_{L^\infty([0,T]\times\mathbb{T})}$ is important quantity. To ensure this quantity is finite, we need $H^s \to W^{1,\infty}$ for initial data. Comparing scale parameter $s - \frac{1}{2} \geq 1$. This gives $s \geq \frac{3}{2}$.

6.2 Existence

Key Idea is **Mollifying PDE**. We first do with **Galerkin Approximation**. **Galerkin Approximation** is kind of standard mollification.

$$\mathbb{P}_\delta f = \sum_{|k|<\frac{1}{\delta}} \hat{f}_k e^{2\pi ikx}$$

We mollify PDE.

$$\begin{cases} \partial_t u_\epsilon + \mathbb{P}_\epsilon(u_\epsilon \partial_x u_\epsilon) + \partial_{xxx} u_\epsilon = 0 \\ u_\epsilon(t=0) = \mathbb{P}_\epsilon u_0 \end{cases}$$

This is solvable since above equation is actually, system of ODE.

$$u_\epsilon = \sum_{|k|<\frac{1}{\epsilon}} \hat{u}_{\epsilon,k} e^{2\pi ikx}$$

Each Fourier coefficient is time dependent. This coefficient satisfies

$$\frac{d}{dt}\hat{u}_{\epsilon,k} = -(2\pi ik)^3 \hat{u}_{\epsilon,k} - \sum_{|\rho|<\frac{1}{\epsilon},\ |k-\rho|<\frac{1}{\epsilon}} \hat{u}_{\epsilon,\rho}\hat{u}_{\epsilon,k-\rho}(2\pi i(k-\rho))$$

This is system of ODE and solution exists since all coefficient is C^∞. Local solution exists and this solution is guaranteed $C^\infty(\mathbb{T})$ in every time t. That is, $||u_\epsilon(t)||_{C^\infty} < \infty \quad \forall\epsilon > 0$. Also,

$$\begin{aligned} \frac{d}{dt}\int |u_\epsilon(t,x)|^2 dx &= 2\int u_\epsilon(t,x)\partial_t u_\epsilon(t,x)dx \\ &= 2Re\int u_\epsilon[-\partial_{xxx}u_\epsilon - \mathbb{P}_\epsilon(u_\epsilon\partial_x u_\epsilon)]dx \\ &= 2Re\int u_\epsilon \mathbb{P}_\epsilon(u_\epsilon \partial_x u_\epsilon)dx \\ &= 2Re\int u_\epsilon^2(\partial_x u_\epsilon)dx = 0 \end{aligned}$$

So $||u_\epsilon(t,\cdot)||_{L^2(\mathbb{T})} = ||u_0||_{L^2(\mathbb{T})}$ and u_ϵ does not blow up at any time. Which makes that this solution is global solution.

We got $L^\infty_{\mathbb{R}^+}L^2_{\mathbb{T}}$ estimates. But, this does not imply (weak) solution for KdV equation. Even if $\sup_t ||u_{\epsilon_k} - u||_{L^2(\mathbb{T})} \to 0$,

$$\left|\int\int \partial_t\varphi(u_{\epsilon_k} - u)\right| \leq ||u_{\epsilon_k} - u||_{L^\infty_t L^2_x}||\partial_t\varphi||_{L^\infty} < \epsilon$$

$$\left|\int\int \partial_{xxx}\varphi(u_{\epsilon_k} - u)\right| \leq ||u_{\epsilon_k} - u||_{L^\infty_t L^2_x}||\partial_{xxx}\varphi||_{L^\infty} < \epsilon$$

BUT

$$\left|\int\int \partial_x\varphi(u_{\epsilon_k}^2 - u^2)\right| \longrightarrow 0$$

does not hold. We need to **upgrade** estimate. We call this Uniform Estimate.

6.2.1 Uniform Estimate

We need to upgrade estimate. In this process we use Gagliardo Nirenberg Inequality mentioned at 2.2.6. However, we will actually not use all statement but prove necessary condition for solving uniform estimates.

We are focused on following System of ODE.

$$\begin{cases} \partial_t u_\epsilon + \partial_{xxx}u_\epsilon + \mathbb{P}_\epsilon(u_\epsilon \partial_x u_\epsilon) = 0 \\ u_\epsilon(t=0) = \mathbb{P}_\epsilon u_0 \end{cases}$$

We want estimate :

$$\frac{1}{2}\frac{d}{dt}||u_\epsilon||_{\dot{H}^m} \leq C||u_\epsilon(t)||_{\dot{H}^m(\mathbb{T})} \cdot ||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})}$$

If this estimate is true and $u_0 \in H^{\frac{3}{2}}(\mathbb{T})$ then $u_0 \in W^{1,\infty}(\mathbb{T})$,

$$\frac{1}{2}\frac{d}{dt}||u_\epsilon||_{H^{\frac{3}{2}}(\mathbb{T})} \leq C||u_\epsilon(t)||_{H^{\frac{3}{2}}(\mathbb{T})} \cdot ||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})} \leq C||u_\epsilon(t)||^{\frac{3}{2}}_{H^{\frac{3}{2}}(\mathbb{T})}$$

(Notice $||u_\epsilon(t)||_{C^\infty}$ is continuously used)

So,

$$\left|\frac{1}{||u_\epsilon(t)||_{H^{\frac{3}{2}}(\mathbb{T})}} - \frac{1}{||\mathbb{P}_\epsilon u_0||_{H^{\frac{3}{2}}(\mathbb{T})}}\right| \leq Ct$$

If

$$t < \frac{C}{||\mathbb{P}_\epsilon u_0||_{H^{\frac{3}{2}}(\mathbb{T})}} = T \lesssim ||u_0||^{-\frac{1}{2}}_{H^{\frac{3}{2}}(\mathbb{T})}$$

Then in $t \in [0,T]$

$$||u_\epsilon(t)||_{H^{\frac{3}{2}}(\mathbb{T})} < \infty$$

$$||u_\epsilon(t)||_{H^m(\mathbb{T})} \leq ||\mathbb{P}_\epsilon u_0||_{H^m(\mathbb{T})} \times exp\left(\int_0^T ||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})}dt\right)$$

Therefore, we gain better estimate $u_\epsilon \in L_t^\infty H_x^m([0,T]\times\mathbb{T})$ if $u_0 \in H^m(\mathbb{T})$

To make it clear, we first have $u_\epsilon(t) \in C^\infty(\mathbb{T})$ every time (to infinity) and we sacrificed into finite time $[0,T]$, upgraded into $L_t^\infty H_x^m([0,T]\times\mathbb{T})$ if $u_0 \in H^m(\mathbb{T})$
Therefore, following estimate is needed.

$$\frac{1}{2}\frac{d}{dt}||u_\epsilon||_{\dot{H}^m} \leq C||u_\epsilon(t)||_{\dot{H}^m(\mathbb{T})} \cdot ||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})}$$

6.2.2 Gagliardo Nirenberg Inequality

$$\frac{1}{2}\frac{d}{dt}||u_\epsilon||_{\dot{H}^k}^2 = \int_{\mathbb{T}} \partial_x^k u_\epsilon \cdot \partial_t \partial_x^k u_\epsilon dx = \int_{\mathbb{T}} \partial_x^k u_\epsilon \cdot \partial_x^k(-\partial_x^3 u_\epsilon - u\partial_x u_\epsilon)dx$$

Since $\mathbb{T}$ is periodic, $\partial_x^3 u_\epsilon$ disappears. We need to estimate

$$\int_{\mathbb{T}} \partial_x^k u_\epsilon \cdot \partial(u_\epsilon \partial_x u_\epsilon)dx$$

For ensurance word 'disappears' mean

$$\int \partial^k u_\epsilon \partial^{k+3} u_\epsilon = -\int \partial^{k+1} u_\epsilon \partial^{k+2} u_\epsilon = \int \partial^{k+2} u_\epsilon \partial^{k+1} u_\epsilon = -\int \partial^{k+3} u_\epsilon \partial^k u_\epsilon = 0$$

We could apply weak derivative's Lipnitz Rule.

$$\partial_x^k(u_\epsilon \partial_x u_\epsilon) = \sum_{i=0}^{k} \binom{k}{i} \partial_x^i u_\epsilon \cdot \partial_x^{k-i+1} u_\epsilon$$

For $i = 0$

$$\left| \int_{\mathbb{T}} u_\epsilon \cdot \partial_x^{k+1} \cdot u_\epsilon \partial_x^k u_\epsilon dx \right| = \left| \frac{1}{2} \int_{\mathbb{T}} \partial_x u_\epsilon (\partial_x^k u_\epsilon)^2 dx \right| \le ||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})} \cdot ||u_\epsilon(t)||_{\dot{H}^m(\mathbb{T})}^2$$

For $i = 1, k$

$$\left| \int_{\mathbb{T}} \partial_x u_\epsilon (\partial_x^k u_\epsilon)^2 \right| \le ||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})} \cdot ||u_\epsilon(t)||_{\dot{H}^m(\mathbb{T})}^2$$

For $2 \le i \le k-1$ we cannot apply same logic. Here, Gagliardo Nirenberg Inequality plays a role. We first apply Holder inequality.

$$\int_{\mathbb{T}} \partial_x^k u_\epsilon \cdot \partial_x^i u_\epsilon \cdot \partial_x^{k-i+1} u_\epsilon dx \le ||\partial_x^k u_\epsilon(t)||_{L^2(\mathbb{T})} \cdot ||\partial_x^i u_\epsilon(t)||_{L^{p_i}(\mathbb{T})} \cdot ||\partial_x^{k-i+1} u_\epsilon(t)||_{L^{p_{k-i+1}}(\mathbb{T})}$$

We need to align $\partial_x^i u_\epsilon(t)$ a L^{p_i} norm. We will use $p_i = \frac{2(k-1)}{i-1}$. This is because of further calculation. This alignment makes p_i harmonic series from 2 to ∞.

Then following holds by Generalized Holder Inequality.

$$\begin{aligned} ||\partial_x^i u_\epsilon(t)||_{L^{p_i}}^{p_i} &= \int_{\mathbb{T}} |\partial_x^i u_\epsilon(t)|^{p_i} dx = \int_{\mathbb{T}} \partial_x(\partial_x^{i-1} u_\epsilon(t)) \cdot |\partial_x^i u_\epsilon(t)|^{p_i - 1} sgn(\partial_x^i u_\epsilon(t)) \\ &\le \int_{\mathbb{T}} \left| \partial_x^{i-1}(t) u_\epsilon \cdot (p_i - 1)(\partial_x^i u_\epsilon(t))^{p_i - 2} \cdot (\partial_x^{i+1} u_\epsilon(t)) \right| \\ &\le C_i ||\partial_x^{i-1} u_\epsilon(t)||_{L^{p_{i-1}}(\mathbb{T})} \cdot ||\partial_x^i u_\epsilon(t)||_{L^{p_i}(\mathbb{T})}^{p_i - 2} \cdot ||\partial_x^{i+1} u_\epsilon(t)||_{L^{p_{i+1}}(\mathbb{T})} \end{aligned}$$

For $C = \max_{2 \le i \le k-1} C_i$, we gained

$$||\partial_x^i u_\epsilon(t)||_{L^{p_i}(\mathbb{T})}^2 \le C ||\partial_x^{i-1} u_\epsilon(t)||_{L^{p_{i-1}}(\mathbb{T})} \cdot ||\partial_x^{i+1} u_\epsilon(t)||_{L^{p_{i+1}}(\mathbb{T})}$$

If we think of

$$a_i = \log\left(\frac{||\partial_x^i u_\epsilon(t)||_{L^{p_i}(\mathbb{T})}}{C} \right)$$

$2a_i \leq a_{i-1} + a_{i+1}$ so $a_i \leq \frac{k-i}{k-1}a_1 + \frac{i-1}{k-1}a_k$. We have a bound

$$||\partial_x^i u_\epsilon(t)||_{L^{p_i}(\mathbb{T})} \leq C||\partial_x u||_{L^\infty(\mathbb{T})}^{\frac{k-i}{k-1}} \cdot ||\partial_x^k u_\epsilon(t)||_{L^2(\mathbb{T})}^{\frac{i-1}{k-1}}$$

Therefore,

$$\int_{\mathbb{T}} \partial_x^k u_\epsilon \cdot \partial_x^i u_\epsilon \cdot \partial_x^{k-i+1} u_\epsilon dx \leq ||\partial_x^k u_\epsilon(t)||_{L^2(\mathbb{T})} \cdot ||\partial_x^i u_\epsilon(t)||_{L^{p_i}(\mathbb{T})} \cdot ||\partial_x^{k-i+1} u_\epsilon(t)||_{L^{p_{k-i+1}}(\mathbb{T})}$$
$$\leq C||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})} \cdot ||\partial_x^k u_\epsilon(t)||_{L^2(\mathbb{T})}^2 = C||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})} \cdot ||u_\epsilon(t)||_{\dot{H}^k(\mathbb{T})}^2$$

Then it is clear that

$$\frac{1}{2}\frac{d}{dt}||u_\epsilon||_{\dot{H}^k(\mathbb{T})}^2 \leq C||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})} \cdot ||u_\epsilon(t)||_{\dot{H}^k(\mathbb{T})}^2$$

So

$$\frac{1}{2}\frac{d}{dt}||u_\epsilon||_{\dot{H}^k(\mathbb{T})} \leq C||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})} \cdot ||u_\epsilon(t)||_{\dot{H}^k(\mathbb{T})}$$

6.2.3 Aubin Lion's Lemma

Aubin Lion's Lemma enables us to make solution from series of solutions of mollified equations. We do not take all of Aubin Lion's Lemma but following:

Lemma 58 (Aubin Lion's Lemma). *$\{u^k\}$ is sequence of C^∞ functions on each time in $[0,T] \times \mathbb{T}$. Assume*

$$\sup_{t\in[0,T]} ||u^k(t,\cdot)||_{H^2(\mathbb{T})} \leq C_1$$

$$\sup_{t\in[0,T]} ||\partial_t u^k(t,\cdot)||_{H^{-2}(\mathbb{T})} \leq C_2$$

Then there is subsequence $\{n_k\}$ such that $\{u_{n_k}\}$ is cauchy sequence in $C_{[0,T]}H^1$ and the limit u satisfies

$$||u||_{L^\infty_{[0,T]}H^2} \leq C_1$$

Proof. We divide into four steps to prove.

STEP 1. There is countable dense subset $\{t_l\} \subset [0,T]$. By Banach-Alaoglu Theorem, for all $l \in \mathbb{N}$ there exists subsequence $\{u^{n_k^l}(t_l)\}_{k\in\mathbb{N}}$ that converges weakly to some limit: $u(t_l)$. This is because for each t_l,

$$\sup ||u^k(t_l,\cdot)||_{H^2(\mathbb{T})} \leq C_1$$

Applying Cantor Diagonal Argument on subsequence $\{n_k^l\}$ (more precisely, we need to iterate and gain subsequence as l increases one at a time) there is subsequence $\{n_k'\}$ that

$$\{u^{n_k'}(t_l)\} \rightharpoonup u(t_l) \quad \forall t_l$$

Now, since $H_2 \subset\subset H_1$, subsequence of $\{n_k'\}$, $\{n_k\}$ exists so that

$$\{u^{n_k}(t_l)\} \to u(t_l)$$

its because weak convergence implies boundness in normed topology.

STEP 2. We will prove if $\{t_{l_m}\}_{m\in\mathbb{N}}$ then $\{u(t_{l_m})\}_{m\in\mathbb{N}}$ is Cauchy sequence in H^1

$$\begin{aligned}||u(t_{l_m}) - u(t_{l_{m'}})||_{H^1} \le\ & ||u(t_{l_m}) - u^{n_k}(t_{l_m})||_{H^1} \\ & + ||u^{n_k}(t_{l_m}) - u^{n_k}(t_{l_{m'}})||_{H^1} + ||u(t_{l_{m'}}) - u^{n_k}(t_{l_{m'}})||_{H^1}\end{aligned}$$

For all $\epsilon > 0$, there exists $K_1, K_2 \in \mathbb{N}$ that if $k \ge \max(K_1, K_2)$, first term and third term could be smaller than ϵ. Main concern is second term. We interpolate this term. (This will be discussed in 5.2.4)

Since $H^2 \subset H^1 \subset H^{-2}$ we can interpolate this term as

$$\begin{aligned}||u^{n_k}(t_{l_m}) - u^{n_k}(t_{l_{m'}})||_{H^1} &\le C||u^{n_k}(t_{l_m}) - u^{n_k}(t_{l_{m'}})||_{H^2}^{\frac{3}{4}}||u^{n_k}(t_{l_m}) - u^{n_k}(t_{l_{m'}})||_{H^{-2}}^{\frac{1}{4}} \\ &\le C(2C_1)^{\frac{3}{4}}\left|\left|\int_{t_{l_{m'}}}^{t_{l_m}} \partial_t u^{n_k}(s)ds\right|\right|_{H^{-2}}^{\frac{1}{4}} \\ &\le C(2C_1)^{\frac{3}{4}}C_2^{\frac{1}{4}}|t_{l_m} - t_{l_{m'}}|^{\frac{1}{4}} \\ &< \epsilon\end{aligned}$$

If in last term, $l_m, l_{m'}$ is sufficiently large. Therefore, for $l_m, l_{m'}$ is sufficiently large

$$||u(t_{l_m}) - u(t_{l_{m'}})||_{H^1} < 3\epsilon$$

Now, we are able to define u in any time. Which is by defining limit of cauchy sequence in H^1 norm.

STEP 3. We will prove u^{n_k} converges strongly to u in $L^\infty_{[0,T]}H^1$ norm.

This is quite simple because we can once more divide $||u(t) - u^{n_k}(t)||_{H^1}$ into three parts.

$$||u(t) - u^{n_k}(t)||_{H^1} \le ||u(t) - u(t_{l_m})||_{H^1} + ||u(t_{l_m}) - u^{n_k}(t_{l_m})||_{H^1} + ||u^{n_k}(t) - u^{n_k}(t_{l_m})||_{H^1}$$

for sufficiently large m, we can reduce first term and third term by ϵ due to STEP 2. Now, fix m and growing up k gives second term smaller than ϵ. We achieved for $k \ge K$, $||u(t) - u^{n_k}(t)||_{H^1} < 3\epsilon$ and done.

STEP 4. We will prove remaining two inequalities.

Since $\sup_{t\in[0,T]} ||u^{n_k}(t)||_{H^2} < \infty$, $u^{n_k}(t) \rightharpoonup \tilde{u}(t)$ in weak H^2 norm. But we gain $u^{n_k}(t) \to u(t)$ in H^1. Therefore, $\tilde{u} = u$ and $u^{n_k}(t) \rightharpoonup u(t)$ in weak H^2 norm. By proposition of weak topology,

$$||u(t)||_{H^2} \le \liminf_{k\to\infty} ||u^{n_k}(t)||_{H^2} \le C_1$$

□

6.2.4 Interpolation

Escaping for the proof of Existence of solution, one briefly summarise 'interpolation' technique.

In previous chapter, we discussed that inhomogenuous embedding is more robust. For example, we want to examine $||v||_{L^\infty}$ by H^0 and H^2 on $\mathbb{T}$.

We can use Fourier series. $v = \sum \hat{v}_k e^{2\pi ikx}$. Now, it is important : we interpolate with $v = v_1 + v_2$. H^2 can bound oscillating terms and H^0 will fill other parts. We may cut off k by M.

$$||v_{\leq M}||_{L^\infty} \leq \sum_{|k|\leq M} |\hat{v}_k| \leq \left(\sum_{|k|\leq M} |\hat{v}_k|^2 \right)^{1/2} (2M+1)^{1/2} \leq C_1 ||v||_{L^2} M^{1/2}$$

$$||v_{>M}||_{L^\infty} \leq \sum_{|k|>M} |\hat{v}_k| \leq \left(\sum_{|k|\leq M} |k|^4 |\hat{v}_k|^2 \right)^{1/2} \left(\sum_{|k|>M} \frac{1}{|k|^4} \right)^{1/2} \leq C_2 ||v||_{H^2} M^{-3/2}$$

Choosing

$$M \sim \left(\frac{||v||_{H^2}}{||v||_{H^0}} \right)^{1/2}$$

We gain

$$||v||_{L^\infty} \leq C ||v||_{H^0}^{3/4} ||v||_{H^2}^{1/4}$$

These Interpolation have advantages since we can apply Young's inequality and can delete higher estimate with other terms... esc. This cannot be done in homogenuous embeddings.

6.2.5 Proof of Existence

By Uniform Estimates, we gain $u_\epsilon \in L^\infty_{[0,T]} H^m_{\mathbb{T}}$ if $u_0 \in H^m(\mathbb{T})$. We proved Aubin Lion's Lemma for $m = 2$ but it could be easily generalized into $m > \frac{3}{2}$. Applying Aubin Lion's Lemma on u_ϵ enables to define $u \in L^\infty_{[0,T]} H^m_{\mathbb{T}}$ with $u_{\epsilon_j} \to u$ in $L^\infty_{[0,T]} H^1_{\mathbb{T}}$ norm. Now, upgraded regularity concludes proof.

$$\begin{aligned}
\left| \int\int (u_{\epsilon_j} - u) \partial_t \varphi \right| &\leq \left| \int_0^T \left[\int_{\mathbb{T}} (u_{\epsilon_j} - u)^2 \right]^{\frac{1}{2}} \left[\int_{\mathbb{T}} (\partial_t \varphi)^2 \right]^{\frac{1}{2}} \right| \\
&\leq ||u_{\epsilon_j} - u||_{L^\infty_{[0,T]} H^1_{\mathbb{T}}} \times T \times ||\partial_t \varphi||^{\frac{1}{2}}_{L^\infty_{[0,T]} L^\infty_{\mathbb{T}}} \to 0
\end{aligned}$$

$$\begin{aligned}
\left| \int\int (u_{\epsilon_j} - u) \partial_{xxx} \varphi \right| &\leq \left| \int_0^T \left[\int_{\mathbb{T}} (u_{\epsilon_j} - u)^2 \right]^{\frac{1}{2}} \left[\int_{\mathbb{T}} (\partial_{xxx} \varphi)^2 \right]^{\frac{1}{2}} \right| \\
&\leq ||u_{\epsilon_j} - u||_{L^\infty_{[0,T]} H^1_{\mathbb{T}}} \times T \times ||\partial_{xxx} \varphi||^{\frac{1}{2}}_{L^\infty_{[0,T]} L^\infty_{\mathbb{T}}} \to 0
\end{aligned}$$

$$\begin{aligned}
\left| \int\int u_{\epsilon_j}^2 \mathbb{P}_{\epsilon_j} \partial_x \varphi - u^2 \partial_x \varphi \right| &\leq \left| \int\int \sum_{|k| > \frac{1}{\epsilon_j}} \hat{\varphi}_k \cdot (2\pi ik) \cdot e^{2\pi ikx} u^2 \right| \\
&\quad + \left| \int\int \sum_{|k| < \frac{1}{\epsilon_j}} \hat{\varphi}_k \cdot (2\pi ik) \cdot e^{2\pi ikx} (u_{\epsilon_j}^2 - u^2) \right|
\end{aligned}$$

First term is,

$$\left| \int \int \sum_{|k|>\frac{1}{\epsilon_j}} \hat{\varphi}_k \cdot (2\pi i k) \cdot e^{2\pi i k x} u^2 \right| = \left| \int \int (\partial_x \varphi - \mathbb{P}_{\epsilon_j} \partial_x \varphi) u^2 \right|$$

$$\begin{aligned}
&\leq \left| \int ||(\partial_x \varphi - \mathbb{P}_{\epsilon_j} \partial_x \varphi)||_{L^\infty_{\mathbb{T}}} ||u||_{H^0_{\mathbb{T}}} \right| \\
&\leq T||(\partial_x \varphi - \mathbb{P}_{\epsilon_j} \partial_x \varphi)||_{L^\infty_{[0,T]} L^\infty_{\mathbb{T}}} ||u||_{L^\infty_{[0,T]} H^0_{\mathbb{T}}} \\
&< \epsilon \cdot T \cdot ||u||_{L^\infty_{[0,T]} H^0_{\mathbb{T}}}
\end{aligned}$$

for $\epsilon_j \to 0$. Second term is

$$\left| \int \int \sum_{|k|<\frac{1}{\epsilon_j}} \hat{\varphi}_k \cdot (2\pi i k) \cdot e^{2\pi i k x} (u_{\epsilon_j}^2 - u^2) \right| = \left| \int \int \mathbb{P}_{\epsilon_j} \partial_x \varphi (u_{\epsilon_j}^2 - u^2) \right|$$

$$\begin{aligned}
&\leq \left| \int ||\mathbb{P}_{\epsilon_j} \varphi||_{L^\infty_{\mathbb{T}}} \int_{\mathbb{T}} 2(u_{\epsilon_j} \partial_x u_{\epsilon_j} - u \partial_x u) \right| \\
&\leq \left| \int ||\mathbb{P}_{\epsilon_j} \varphi||_{L^\infty_{\mathbb{T}}} \int_{\mathbb{T}} 2\big(u_{\epsilon_j} (\partial_x u_{\epsilon_j} - \partial_x u) + \partial_x u (u_{\epsilon_j} - u)\big) \right| \\
&\leq 2||\mathbb{P}_{\epsilon_j} \varphi||_{L^\infty_{[0,T]} L^\infty_{\mathbb{T}}} \Bigg(||u_{\epsilon_j}||^2_{L^\infty_{[0,T]} H^0_{\mathbb{T}}} ||\partial_x (u_{\epsilon_j} - u)||^2_{L^\infty_{[0,T]} H^0_{\mathbb{T}}} \\
&\qquad + ||\partial_x u||^2_{L^\infty_{[0,T]} H^0_{\mathbb{T}}} ||u_{\epsilon_j} - u||^2_{L^\infty_{[0,T]} H^0_{\mathbb{T}}} \Bigg) \\
&\leq C||u_{\epsilon_j} - u||_{L^\infty_{[0,T]} H^1_{\mathbb{T}}} \to 0
\end{aligned}$$

Since

$$\int \int u_{\epsilon_j} \partial_t \varphi + u_{\epsilon_j} \partial_{xxx} \varphi + \frac{1}{2} u_{\epsilon_j}^2 (\mathbb{P}_{\epsilon_j} \partial_x \varphi) = 0$$

We can conclude

$$\int \int u \partial_t \varphi + u \partial_{xxx} \varphi + \frac{1}{2} u^2 \partial_x \varphi = 0$$

$u \in C_{[0,T]} H^1_{\mathbb{T}}$ is weak solution for KdV Equation.

6.3 Uniqueness

Proving on uniqueness of C^∞ solution is easy. This could motivate proof idea for uniqueness on other conditions.

Theorem 59 (Uniqueness of C^∞)**.** $u_0 \in C^\infty(\mathbb{T})$. *Then there exists upto one solution belonging to* $C^\infty([0,T] \times \mathbb{T})$

Proof. Let $u, \tilde{u}$ are solutions. Set $v = u - \tilde{u}$. v satisfies PDE :

$$\partial_t v + \partial_{xxx} v + u \partial_x v - v \partial_x \tilde{u} = 0$$

Then we can multiply v and integrate

$$\begin{aligned}
0 &= \int_{\mathbb{T}} v \Big(\partial_t v + \partial_{xxx} v + u \partial_x v - v \partial_x \tilde{u} \Big) \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} v^2 - \frac{1}{2} \int_{\mathbb{T}} \partial_x u \cdot v^2 + \int_{\mathbb{T}} \partial_x \tilde{u} \cdot v^2
\end{aligned}$$

So
$$\left|\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} v^2\right| = \left|\int_{\mathbb{T}}\left(\frac{1}{2}\partial_x u - \partial_x \tilde{u}\right)v^2\right| \le \left|\left|\frac{1}{2}\partial_x u - \partial_x \tilde{u}\right|\right|_{L^\infty_{\mathbb{T}}} \int_{\mathbb{T}} v^2$$
If $t=0$ then $\int v^2 = 0$, we can conclude
$$\int_{\mathbb{T}} v^2 = 0$$
for all $t \in [0,T]$ □

In this proof, we just need $C^1_t C^3_x$. But one is interested in reducing conditions. Here, we look at $C_{[0,T]}H^2_{\mathbb{T}}$. Important thing is, multiplication with v and integral process is not possible. Instead, we use idea to use mollified function v^ϵ.

Theorem 60 (Uniqueness of $C_{[0,T]}H^2_{\mathbb{T}}$)**.** $u_0 \in H^2_{\mathbb{T}}$. *Then there exists upto one solution belonging to* $C_{[0,T]}H^2_{\mathbb{T}}$

Proof. Let we mollify and cut off $v = u - \tilde{u}$
$$v^\epsilon_{t,x} = \left(\sum_{|k|<\frac{1}{\epsilon_1}} \hat{v}^{\epsilon_2}_k(t)e^{2\pi i k x}\right) \times \chi_{[t_0-\epsilon_3, t_0+\epsilon_3]}(t)$$
$\hat{v}^{\epsilon_2}_k$ is standard mollification on fourier coefficient. $\chi_{[t_0-\epsilon_3,t_0+\epsilon_3]}(t)$ is cut off function with bump function. This function is value 1 on $[t_0-\epsilon_3+\epsilon_3^2, t_0+\epsilon_3-\epsilon_3^2]$

Then we know
$$\int\int \partial_t v^\epsilon_{t,x} v + \partial_{xxx} v^\epsilon_{t,x} v + \partial_x(v^\epsilon_{t,x} u)v - (\partial_x \tilde{u})v^\epsilon_{t,x} v dxdt = 0$$
Thinking of difference between above left hand side and
$$\int\int \partial_t v^\epsilon_{t,x} v^\epsilon_{t,x} + \partial_{xxx} v^\epsilon_{t,x} v^\epsilon_{t,x} + \partial_x(v^\epsilon_{t,x} u)v^\epsilon_{t,x} - (\partial_x \tilde{u})v^\epsilon_{t,x} v^\epsilon_{t,x} dxdt$$
we need to estimate following terms. For easy notation, $v^{\epsilon_1}_x := \sum_{|k|<\frac{1}{\epsilon_1}} \hat{v}_k(t)e^{2\pi i k x}$
$$\begin{aligned}
\int_{t_0-\epsilon_3}^{t_0+\epsilon_3}\int_{\mathbb{T}} \partial_t v^\epsilon_{t,x}(v - v^\epsilon_{t,x}) &= \int_{t_0-\epsilon_3}^{t_0+\epsilon_3}\int_{\mathbb{T}} \partial_t v^\epsilon_{t,x}(v^{\epsilon_1}_x - v^\epsilon_{t,x}) \\
&= \int_{t_0-\epsilon_3+\epsilon_3^2}^{t_0+\epsilon_3-\epsilon_3^2}\left(\sum_{|k|<\frac{1}{\epsilon_1}} \partial_t \hat{v_k}^{\epsilon_2}(t)\cdot(\hat{v}_k - \hat{v}^{\epsilon_2}_k)\right) + \\
&+ \int_{[t_0-\epsilon_3,t_0+\epsilon_3]-[t_0-\epsilon_3+\epsilon_3^2,t_0+\epsilon_3-\epsilon_3^2]}\left(\sum_{|k|<\frac{1}{\epsilon_1}}(\partial_t \hat{v}^{\epsilon_2}_k)\cdot(\hat{v}_k - \hat{v}^{\epsilon_2}_k\chi)\right) \\
&\le \sum_{|k|<\frac{1}{\epsilon_1}} ||\partial_t \hat{v}^{\epsilon_2}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])} \times ||\hat{v}_k - \hat{v}^{\epsilon_2}_k||_{L^1([t_0-\epsilon_3,t_0+\epsilon_3])} \\
&+ \sum_{|k|<\frac{1}{\epsilon_1}} ||\partial_t \hat{v}^{\epsilon_2}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])} \times 4\epsilon_3^2 ||\hat{v}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])} + o(\epsilon_3) \\
&\le o(\epsilon_3) + 2\epsilon_3 \sum_{|k|<\frac{1}{\epsilon_1}}\Big\{||\partial_t \hat{v}^{\epsilon_2}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])} \times \\
&\quad \left(||\hat{v}_k - \hat{v}^{\epsilon_2}_k||_{L^2([t_0-\epsilon_3,t_0+\epsilon_3])} + 2\epsilon_3||\hat{v}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])}\right)\Big\}
\end{aligned}$$

Same computation is able to other terms.

$$\int_{t_0-\epsilon_3}^{t_0+\epsilon_3}\int_{\mathbb{T}}\partial_{xxx}v^{\epsilon}_{t,x}(v-v^{\epsilon}_{t,x}) \leq o(\epsilon_3) + 2\epsilon_3 \times \sum_{|k|<\frac{1}{\epsilon_1}}\Bigg\{||\hat{v}^{\epsilon_2}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])}\times(2\pi|k|)^3\times$$
$$\Bigg(||\hat{v}_k-\hat{v}^{\epsilon_2}_k||_{L^2([t_0-\epsilon_3,t_0+\epsilon_3])}+2\epsilon_3||\hat{v}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])}\Bigg)\Bigg\}$$

$$\int_{t_0-\epsilon_3}^{t_0+\epsilon_3}\int_{\mathbb{T}}\partial_x\tilde{u}\cdot v^{\epsilon}_{t,x}(v-v^{\epsilon}_{t,x}) \leq o(\epsilon_3) + ||\tilde{u}||_{L^\infty\dot{W}^{1,\infty}}\times 2\epsilon_3\times\sum_{|k|<\frac{1}{\epsilon_1}}\Bigg\{||\hat{v}^{\epsilon_2}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])}\times$$
$$\Bigg(||\hat{v}_k-\hat{v}^{\epsilon_2}_k||_{L^2([t_0-\epsilon_3,t_0+\epsilon_3])}+2\epsilon_3||\hat{v}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])}\Bigg)\Bigg\}$$

$$\int_{t_0-\epsilon_3}^{t_0+\epsilon_3}\int_{\mathbb{T}}\partial_x(v^{\epsilon}_{t,x}\cdot u)\cdot(v-v^{\epsilon}_{t,x}) = -\int_{t_0-\epsilon_3}^{t_0+\epsilon_3}\int_{\mathbb{T}}v^{\epsilon}_{t,x}\cdot u\cdot\partial_x(v-v^{\epsilon}_{t,x})$$
$$\leq o(\epsilon_3)+||u||_{L^\infty_{[0,T]}L^\infty_{\mathbb{T}}}\times 2\epsilon_3\times\sum_{|k|<\frac{1}{\epsilon_1}}\Bigg\{||\hat{v}^{\epsilon_2}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])}\times(2\pi|k|)\times$$
$$\Bigg(||\hat{v}_k-\hat{v}^{\epsilon_2}_k||_{L^2([t_0-\epsilon_3,t_0+\epsilon_3])}+2\epsilon_3||\hat{v}_k||_{L^\infty([t_0-\epsilon_3,t_0+\epsilon_3])}\Bigg)\Bigg\}$$

Therefore, divide by $2\epsilon_3$ and taking limit as $\epsilon_3 \downarrow 0$, all additional term disappears or remain with error $e_{\epsilon_1,\epsilon_2}(t)$. This error term tends to zero if $\epsilon_2 \downarrow 0$ whether the value ϵ_1 is.

For all $t \in [0,T]$, notate $v^{\epsilon_1,\epsilon_2}_{t,x} = \sum_{|k|<\infty}\hat{v}^{\epsilon_2}_k e^{2\pi ikx}$

$$\int_{\mathbb{T}}\partial_t v^{\epsilon_1,\epsilon_2}_{t,x}\cdot v^{\epsilon_1,\epsilon_2}_{t,x}+\partial_{xxx}v^{\epsilon_1,\epsilon_2}_{t,x}\cdot v^{\epsilon_1,\epsilon_2}_{t,x}+\partial_x(v^{\epsilon_1,\epsilon_2}_{t,x}u)v^{\epsilon_1,\epsilon_2}_{t,x}-\partial_x\tilde{u}\cdot v^{\epsilon_1,\epsilon_2}_{t,x}\cdot v^{\epsilon_1,\epsilon_2}_{t,x}dx+e_{\epsilon_1,\epsilon_2}(t)=0$$

Then we can do same as proof of C^∞!

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|v^{\epsilon_1,\epsilon_2}_{t,x}|^2 \leq \int_{\mathbb{T}}|v^{\epsilon_1,\epsilon_2}_{t,x}|^2\cdot(||\partial_x\tilde{u}||_{L^\infty}+\frac{1}{2}||\partial_x u||_{L^\infty})+|e_{\epsilon_1,\epsilon_2}(t)|$$

By Grownwall Inequality, $X^{\epsilon_1,\epsilon_2} = \int_{\mathbb{T}}|v^{\epsilon_1,\epsilon_2}_{t,x}|^2$ and $A(t) = ||\partial_x\tilde{u}||_{L^\infty}+\frac{1}{2}||\partial_x u||_{L^\infty}$, $E^{\epsilon_1,\epsilon_2} = |e_{\epsilon_1,\epsilon_2}|$

$$X^{\epsilon_1,\epsilon_2}(t) \leq X^{\epsilon_1,\epsilon_2}(0)\cdot exp\Bigg(\int_0^t A(s)ds\Bigg)+\int_0^t E^{\epsilon_1,\epsilon_2}(t)\cdot exp\Bigg(\int_s^t A(\tau)d\tau\Bigg)ds$$
$$\leq T\cdot||E^{\epsilon_1,\epsilon_2}||_{L^\infty([0,T])}\cdot exp\Bigg(\int_0^T|A(s)|ds\Bigg)$$

For any $\epsilon_1>0$, as $\epsilon_2\downarrow 0$ $E^{\epsilon_1,\epsilon_2}\downarrow 0$. That means for every $\epsilon_1>0$, $\int_{\mathbb{T}}|v^{\epsilon_1}_x|=0$. Therefore $v^{\epsilon_1}_x=0$ for all $\epsilon_1>0$, this means $v\equiv 0$ □

6.4 Propagation of Regularity

This states if $u_0\in C^\infty(\mathbb{T})$ solution is $C^\infty([0,T]\times\mathbb{T})$.

We have proven $u_0\in H^m(\mathbb{T})$ then solution exists on $[0,T]\times\mathbb{T}$ which belongs to $L^\infty_{[0,T]}H^m_{\mathbb{T}}$. Also, we have proven $T\lesssim||\partial_x u||^{-\frac{1}{2}}_{L^\infty(\mathbb{T})}$. We also have

$$\frac{1}{2}\frac{d}{dt}||u_\epsilon||_{\dot{H}^m}\leq C_m||u_\epsilon(t)||_{\dot{H}^m(\mathbb{T})}\cdot||\partial_x u_\epsilon(t)||_{L^\infty(\mathbb{T})}$$

by Gagliardo Nirenberg Inequality. Therefore,

$$||u||^2_{\dot{H}^m} \leq ||u_0||^2_{\dot{H}_m} \cdot exp\left(2C_m \int ||\partial_x u||_{L^\infty_{\mathbb{T}}} dt\right) < \infty$$

so H^m bound does not tend to infinity. We proved uniqueness so each solution coincides. Hence, at each time $u(t,\cdot)$ is C^∞. By PDE, it is also C^∞ in time. Combining this, $u \in C^\infty([0,T] \times \mathbb{T})$

7 Elliptic Regularity

Looking at Poisson Equation in bounded open domain U.

$$\begin{cases} -\Delta u = f & U \\ u = 0 & \partial U \end{cases}$$

If this is in Torus, we could do easily by Fourier transformation. We could easily find out solution is

$$u = \sum_{0 \neq k \in \mathbb{Z}^n} \frac{\hat{f}_k}{|k|^2} e^{ik \cdot x}$$

From this, we gain following inequality

$$||u||_{H^2(\mathbb{T}^n)} \leq C||f||_{L^2(\mathbb{T}^n)}$$

Which is type of **Elliptic Regularity**. Amazing part is, in the equation, only ∂_{ii} forms are considered but in the inequality above, estimate for all ∂_{ij} is considered.

However, if the domain is not Torus, we cannot apply Fourier Analysis. Also we cannot gain more regularity since boundary condition will break down. Boundary condition only assumes value of u is zero. Differentiation of u have no limit on boundary and can be nonzero.

We divide this Elliptic Regularity into three parts. First is proving existence and uniqueness of weak solution, second is proving regularity and go higher, finally we discuss more features.

7.1 Existence and Uniqueness

Terminology 'weak solution' is flexible. In this problem, we define it by:

7.1.1 Weak Solution

For $f \in L^2(U)$, if $\forall \phi \in C_c^\infty(U)$ satisfies

$$\int_U \nabla u \cdot \nabla \varphi - f\varphi dx = 0$$

then we call u a weak solution to PDE.

7.1.2 Infimum Estimate

Think of quantity

$$\inf_{v \in H_0^1(U)} \int_U \left[\frac{1}{2}|\nabla v|^2 - fv\right]$$

We could prove this quantity have lower bound.

$$\left|\int_U fv\right| \leq ||f||_{L^2(U)}||u||_{L^2(U)} \leq C||f||_{L^2(U)}||\nabla u||_{L^2(U)} \leq \frac{1}{2}||\nabla v||_{L^2(U)} + C||f||_{L^2(U)}$$

So it holds that

$$\int_U \frac{1}{2}|\nabla v|^2 - fv \geq -C||f||_{L^2(U)}$$

In the second inequality, we used $||u||_{L^2(U)} \leq C||\nabla u||_{L^2(U)}$ for $u \in H_0^1(U)$. This estimate is from Poincare's Inequality. In Section 4. Sobolev spaces, we proved that

$$||u - (u)_U||_{L^p(U)} \leq C||Du||_{L^p(U)}$$

for $u \in W^{1,p}(U)$. Plugging $p = 2$ and estimate

$$(u_k)_U \leq \int_U |Du_k| \leq ||Du_k||_{L^2(U)}$$

holds for $u_k \in C_c^\infty(U)$, then since $u \in H_0^1(U)$ we can find $u_k \to u$ in $H_0^1(U)$ sense. So $(u)_U \leq ||Du||_{L^2(U)}$ because $(u_k)_U \to (u)_U$ and $||Du_k||_{L^2(U)} \to ||Du||_{L^2(U)}$.

7.1.3 Argmin Estimate

We can find sequence achieving infimum. Let the sequence be $v_1, v_2, \cdots \in H_0^1(U)$ and infimum value α.

For some N, $n \geq N$ then

$$\alpha + 1 > \frac{1}{2}||\nabla v_n||_{L^2(U)}^2 - \int_U f v_n \geq \frac{1}{4}||\nabla v_n||_{L^2(U)}^2 - C||f||_{L^2(U)}^2$$

So $||\nabla v_n||_{L^2(U)} \leq C$ and by Poincare inequality, $||v_n||_{L^2(U)} \leq C$.

We gain uniformly bounded sequence $\{v_n\}$ in $H_0^1(U)$ norm. Therefore, in weak topology a subsequence converge to some limit; v

$$v_{n_k} \rightharpoonup v$$

Then v_{n_k} converges to v pointiwse a.e. Applying Lebesgue Dominated Theoerem,

$$\lim_{k\to\infty} \int_U f v_{n_k} = \int_U f v$$

Finally,

$$\int_U \frac{1}{2}|\nabla v|^2 - fv \leq \liminf_{n\to\infty} \int_U \frac{1}{2}|\nabla v_n|^2 - f v_n = \inf_{v \in H_0^1(U)} \int_U \frac{1}{2}|\nabla v|^2 - fv$$

7.1.4 Existence

Now we find v achieving minimum of the quantity. For arbitrary ϵ and $\varphi \in C_c^\infty(U)$,

$$\int_U \frac{1}{2}|\nabla v|^2 - fv \leq \int_U \frac{1}{2}|\nabla(v + \epsilon\varphi)|^2 - f(v + \epsilon\varphi)$$

After calculation,

$$\int_U \epsilon^2 \cdot \frac{1}{2}|\nabla\varphi|^2 + \epsilon(\nabla v \cdot \nabla\varphi - f\varphi) \geq 0$$

To hold this in arbitrary ϵ,

$$\int_U \nabla v \cdot \nabla\varphi - f\varphi = 0$$

We now have weak solution v to PDE.

7.1.5 Uniqueness

If we observe weak solution formula, actually we could extend $\varphi \in C_c^\infty(U)$ into $\varphi \in H_0^1(U)$ since for $H_0^1(U)$ function, we can generate $C_c^\infty(U)$ sequence converging to v. Applying Lebesgue Dominated Theorem solves this.

Therefore,

$$\int_U |\nabla v|^2 - fv = 0$$

Modifying this equality,

$$\int_U |\nabla v|^2 = \int_U fv \leq ||f||_{L^2(U)}||v||_{L^2(U)} \leq C||f||_{L^2(U)}||\nabla v||_{L^2(U)} \leq \frac{1}{2}||\nabla v||^2_{L^2(U)} + C||f||^2_{L^2(U)}$$

$$||\nabla v||_{L^2(U)} \leq C||f||_{L^2(U)}$$

.

Now suppose there are two solutions u_1, u_2. Then $w = u_1 - u_2$ satisfies

$$\begin{cases} -\Delta w = 0 & U \\ w = 0 & \partial U \end{cases}$$

Applying above estimate, $||\nabla w||_{L^2(U)} = 0$ so $w = 0$

7.2 Upgrading Regularity - Interior Regularity

Interesting part is, we can upgrade regularity. We have estimate $||\cdot||_{H_0^1(U)}$ but we can actually gain higher regularity; $||\cdot||_{H^2(U)}$. Furthermore, if $f \in C^\infty(U)$ solution is $C^\infty(U)$ also, and we will prove these facts in further sections.

First, we need to divide regularity on interior and boundary. The reason is, for interior we can estimate much easier. However in the boundary, some techniques are needed to resolve problem.

Interior case have motivation: For $V \subset\subset U$, can we upgrade regularity from H_0^1 to H^2?

Let we set K a compact set that satisfies $V \subset K \subset U$. Then there is bump function on V, ζ supported in K. In other words, $\zeta \equiv 1$ in V, $supp(\zeta) = K$. Define $w = \zeta u \in H_0^1(U)$. Then

$$\begin{aligned}
&\int_U \nabla w \cdot \nabla\varphi - \varphi(\zeta f - 2\nabla\zeta \cdot \nabla u - \Delta\zeta u) \\
&\qquad = \int_U (\zeta\nabla u + u\nabla\zeta)\cdot\nabla\varphi - \varphi\zeta f + 2\varphi\nabla\zeta\cdot\nabla u + \varphi u\Delta\zeta \\
&\qquad = \int_U \zeta(\nabla u\cdot\nabla\varphi - \varphi f) + u\sum_{i=1}^n(\partial_{x_i}\zeta\partial_{x_i}\varphi + \varphi\partial_{x_ix_i}\zeta) + 2\varphi\nabla\zeta\cdot\nabla u \\
&\qquad = \int_U \zeta(\nabla u\cdot\nabla\varphi - \varphi f) + \varphi\nabla f\cdot\nabla u + \sum_{i=1}^n \partial_{x_i}(\varphi\partial_{x_i}\zeta u) \\
&\qquad = \int_U \nabla u\cdot\nabla(\zeta\varphi) - f(\zeta\varphi) = 0
\end{aligned}$$

Therefore, w is weak solution to $-\Delta w = g$ where $g = \zeta f - 2\nabla\zeta \cdot \nabla u - \zeta u$. By proof of existence and uniqueness,

$$||w||_{H_0^1(U)} \leq C||g||_{L^2(U)} \leq C(||f||_{L^2(U)} + ||\nabla u||_{L^2(U)} + ||u||_{L^2(U)}) \leq C||f||_{L^2(U)}$$

Also, w can be mollified!! (Since $dist(K, \partial U) > 0$)

$$\begin{aligned}\int_U \nabla(\varphi^\epsilon * w) \cdot \nabla\varphi - \varphi(\varphi^\epsilon * g) &= \int_U (\varphi^\epsilon * \nabla w) \cdot \nabla\varphi - \varphi(\varphi^\epsilon * g) \\ &= \int_U \nabla w \cdot \nabla(\varphi^\epsilon * \varphi) - g(\varphi^\epsilon * \varphi) = 0\end{aligned}$$

In the last equation,

$$\int_U \int_{B(x,r)} \varphi(x)\varphi^\epsilon(x-y)g(y)dydx = \int_U \int_{B(y,\epsilon)} \varphi^\epsilon(y-x)\varphi(x)g(y)dydx$$

is used.

This means $w^\epsilon = w * \varphi^\epsilon$ is weak solution to $\Delta w^\epsilon = -g * \varphi^\epsilon$ and $w^\epsilon \in C_c^\infty$. So w is actually, solution. Surprisingly,

$$\begin{aligned}\int_U (g * \varphi^\epsilon)^2 &= \int_U (\Delta w^\epsilon)^2 \\ &= \int_U \left(\sum_{i=1}^n \partial_{x_i x_i} w^\epsilon\right)\left(\sum_{i=1}^n \partial_{x_i x_i} w^\epsilon\right) \\ &= \sum_{i,j=1}^n \int_U (\partial_{x_i x_j} w^\epsilon)^2 = ||w^\epsilon||_{\dot{H}^2(U)}\end{aligned}$$

So,

$$||w^\epsilon||_{\dot{H}^2(U)} = ||g * \varphi^\epsilon||_{L^2(U)} \leq C||g||_{L^2(U)} \leq C||f||_{L^2(U)}$$

Also

$$\begin{gathered}||w^\epsilon||_{H_0^1(U)} \leq C||w||_{H_0^1(U)} \leq C||g||_{L^2(U)} \leq C||f||_{L^2(U)} \\ ||w^\epsilon||_{H^2(U)} \leq C||f||_{L^2(U)}\end{gathered}$$

w^ϵ is bounded in $H^2(U)$ so there exists w^0 a weak limit of subsequence.

$$w^{\epsilon_j} \rightharpoonup w^0$$

in $H^2(U)$. Also $w^\epsilon \to w$ in $L^2(U)$ sense so $w \equiv w^0$ and

$$||w||_{H^2(U)} \leq \liminf_{j\to\infty} ||w^{\epsilon_j}||_{H^2(U)} \leq C||f||_{L^2(U)}$$

w coincides with u in K so,

$$||u||_{H^2(V)} \leq C|f||_{L^2(U)}$$

7.3 Upgrading Regularity - Boundary Regularity

7.3.1 Straighten Mapping

First, assume after rotating and shifting, $0 \in \partial U$ and $\exists r > 0$ s.t

$$U \cap B(0,r) = \{x \mid |x| < r,\ x_n > h(x')\}$$

As we seen at Sobolev space section, there is straightening mapping

$$\Psi : \mathbb{R}^n_x \to \mathbb{R}^n_y$$

$$\Psi(x', x_n) = (x', x_n - h(x'))$$

and inverse map

$$\Phi : \mathbb{R}^n_y \to \mathbb{R}^n_x$$

$$\Phi(y', y_n) = (y', y_n + h(y;))$$

Afterall, we will separate notation on original space and straighten space by notating each x, y. Then define

$$U_{x,r} = U \cap B(0,r), \qquad U_{y,r} = \Psi(U \cap B(0,r)), \qquad \tilde{u}(y', y_n) = u(y', y_n + h(y'))$$

We want to prove

$$||u||_{H^2(U \cap B(0,r))} \leq C||f||_{L^2(U)}$$

7.3.2 Weak Solution

We define Weak solution on straighten area. We define $\tilde{u}$ is weak solution to equation

$$-\Delta_y \tilde{u} - |\nabla_{y'} h|^2 \partial_{y_n y_n} \tilde{u} + \nabla_{y'}(\partial_{y_n} \tilde{u}) \cdot \nabla_{y'} h + (\Delta_{y'} h)\partial_{y_n} \tilde{u} = \tilde{f}$$

If for $\tilde{u} \in H^1_0(U_{y,r})$ and $\forall \varphi \in C^\infty_c(U_{y,r})$,

$$\int_{U_{y,r}} \nabla_y \tilde{u} \cdot \nabla_y \varphi - \tilde{f}\varphi + |\nabla_{y'} h|^2 (\partial_{y_n} \tilde{u})(\partial_{y_n} \varphi) - (\nabla_{y'} \tilde{u} \cdot \nabla_{y'} h)\partial_{y_n} \varphi + (\Delta_{y'} h)(\partial_{y_n} \tilde{u})\varphi = 0$$

Then if u is weak solution to $-\Delta_x u = f$ on $U_{x,r}$, $\tilde{u}(y', y_n) = u(y', y_n + h(y'))$ is weak solution on $U_{y,r}$. ($\tilde{f} = f(y', y_n + h(y'))$ is coordinate transformed f value)

Before proving this fact, notice $\tilde{u}$ is zero on $\{y_n = 0\} \cap U_{y,r}$ and $H^1(U_{y,r})$.

Now, to prove weak solution argument, we first derivate $\tilde{u}$ on each component.

$$\partial_{y_i} \tilde{u}(y', y_n) = \partial_{x_i} u(y', y_n + h(y')) + \partial_{x_n} u(y', y_n + h(y'))\partial_{y_i} h(y')$$

$$\partial_{y_n} \tilde{u}(y', y_n) = \partial_{x_n} u(y', y_n + h(y'))$$

Define for $\varphi \in C^\infty_c(U_{x,r})$, $\tilde{\varphi}(y', y_n) = \varphi(y', y_n + h(y'))$. Then $\tilde{\varphi} \in C^\infty_c(U_{y,r})$.

$$
\begin{aligned}
&\int_{U_{x,r}} \nabla_x u \cdot \nabla_x \varphi - f\varphi \\
&\quad = \int_{U_{y,r}} \partial_{y_n}\tilde{\varphi}(y',y_n)\partial_{y_n}\tilde{u}(y',y_n) - f(y',y_n+h(y'))\tilde{\varphi}(y',y_n) \\
&\quad + \int_{U_{y,r}} \sum_{i=1}^{n-1} \Big(\partial_{y_i}\tilde{\varphi}(y',y_n) - \partial_{y_n}\tilde{\varphi}(y',y_n)\partial_{y_i}h(y')\Big)\Big(\partial_{y_i}\tilde{u}(y',y_n) - \partial_{y_n}\tilde{u}(y',y_n)\partial_{y_i}h(y')\Big) \\
&\quad = \int_{U_{y,r}} \nabla_y\tilde{u}\cdot\nabla_y\tilde{\varphi} - \partial_{y_n}\tilde{\varphi}(\nabla_{y'}h\cdot\nabla_{y'}\tilde{u}) - \partial_{y_n}\tilde{u}(\nabla_{y'}\tilde{\varphi}\cdot\nabla_{y'}h) \\
&\qquad\qquad + (\partial_{y_n}\tilde{u})(\partial_{y_n}\varphi)|\nabla_{y'}h|^2 - f(y',y_n+h(y'))\tilde{\varphi}(y',y_n) \\
&\quad = 0
\end{aligned}
$$

Since $\varphi \leftrightarrow \tilde{\varphi}$ is one-to-one transform from $C_c^\infty(U_{x,r}) \leftrightarrow C_c^\infty(U_{y,r})$, $\tilde{u}$ is weak solution in equation

$$
-\Delta_y\tilde{u} - |\nabla_{y'}h|^2\partial_{y_ny_n}\tilde{u} + \nabla_{y'}(\partial_{y_n}\tilde{u})\cdot\nabla_{y'}h + (\Delta_{y'}h)\partial_{y_n}\tilde{u} = \tilde{f}
$$

7.3.3 Half Space Problem

Coming to the straighten region, we need to take care of Half space problem. Problem is

$$
\begin{cases}
-\Delta v = f & \mathbb{R}^n_+ \\
v = 0 & \partial\mathbb{R}^n_+
\end{cases}
$$

For compactly supported $f \in L^2(\mathbb{R}^n_+)$, solution v exists with $v \in H^2(\mathbb{R}^n_+)$

Key point is upgrading regularity. By previous estimate, $v \in H^1_0(\mathbb{R}^n_+)$ exists, since f is compactly supported.

Define $\varphi^{\epsilon_1} * \varphi^{\epsilon_1} * v$ be a mollification on scale ϵ_1 with first variable. Then

$$
\begin{aligned}
\int_{\mathbb{R}^n_+} \nabla_y v\cdot\nabla_y\partial_{y_1y_1}(\varphi^{\epsilon_1}*\varphi^{\epsilon_1}*v) &= \int_{\mathbb{R}^n_+} \nabla_y v\cdot(\varphi^{\epsilon_1}*(\nabla_y\partial_{y_1y_1}(\varphi^{\epsilon_1}*v))) \\
&= \int_{\mathbb{R}^n_+} (\varphi^{\epsilon_1}*\nabla_y v)\cdot(\nabla_y\partial_{y_1y_1}(\varphi^{\epsilon_1}*v)) \\
&= \int_{\mathbb{R}^n_+} \nabla_y(\varphi^{\epsilon_1}*v)\cdot(\nabla_y\partial_{y_1y_1}(\varphi^{\epsilon_1}*v)) \\
&= -\int_{\mathbb{R}^n_+} |\nabla_y\partial_{y_1}(\varphi^{\epsilon_1}*v)|^2
\end{aligned}
$$

Since $\varphi^{\epsilon_1}*\varphi^{\epsilon_1}*v \in H^1_0(\mathbb{R}^n_+)$,

$$
\int_{\mathbb{R}^n_+} \nabla_y v\cdot\nabla_y\partial_{y_1y_1}(\varphi^{\epsilon_1}*\varphi^{\epsilon_1}*v) = \int_{\mathbb{R}^n_+} f\cdot\partial_{y_1y_1}(\varphi^{\epsilon_1}*\varphi^{\epsilon_1}*v)
$$

Second term could be estimated

$$
\int_{\mathbb{R}^n_+} f\cdot\partial_{y_1y_1}(\varphi^{\epsilon_1}*\varphi^{\epsilon_1}*v) = \int_{\mathbb{R}^n_+} (\varphi^{\epsilon_1}*f)\cdot\partial_{y_1y_1}(\varphi^{\epsilon_1}*v)
$$

Therefore,

$$||\partial_{y_1}\nabla_y(\varphi^{\epsilon_1}*v)||^2_{L^2(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}||\partial_{y_1y_1}(\varphi^{\epsilon_1}*v)||_{L^2(\mathbb{R}^n_+)} \leq \frac{1}{2}||\partial_{y_1y_1}(\varphi^{\epsilon_1}*v)||^2_{L^2(\mathbb{R}^n_+)}+C||f||^2_{L^2(\mathbb{R}^n_+)}$$

and

$$||\partial_{y_1}\nabla_y(\varphi^{\epsilon_1}*v)||_{L^2(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}$$

$\partial_{y_1}\nabla_y(\varphi^{\epsilon_1}*v)$ is bounded in $L^2(\mathbb{R}^n_+)$. There is weak limit(Converging subsequence exists in weak topology) $w \in L^2(\mathbb{R}^n_+)$. Then we could define ∇v's y_1 weak derivative as w.

$$||\partial_{y_1}\nabla_y v||_{L^2(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}$$

In the same way, we can define $\nabla_y v$'s y_i derivative and there will be a bound

$$||\partial_{y_i}\nabla_y v||_{L^2(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}$$

Now, we can define weak y_ny_n derivative, by

$$\partial_{y_ny_n}v = f - \sum_{i=1}^{n-1}\partial_{y_iy_i}v$$

Also by previous calculation,

$$||\partial_{y_ny_n}v||_{L^2(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}$$

As summation,

$$||v||_{\dot{H}^2(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}$$

so with $||v||_{H^1_0(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}$,

$$||v||_{H^2(\mathbb{R}^n_+)} \leq C||f||_{L^2(\mathbb{R}^n_+)}$$

7.3.4 Boundary Regularity

In section 7.3.2 we proved $\tilde{u}$ is weak solution to PDE

$$-\Delta_y\tilde{u} - |\nabla_{y'}h|^2\partial_{y_ny_n}\tilde{u} + \nabla_{y'}(\partial_{y_n}\tilde{u})\cdot\nabla_{y'}h + (\Delta_{y'}h)\partial_{y_n}\tilde{u} = \tilde{f}$$

To apply 7.3.3 Half space problem solution, we define ζ supported in

$$U_{y,r} - \bigcup_{y\in(\partial U_{y,r}\setminus\{y_n=0\})} B(y,\epsilon)$$

We will do procedure on $\zeta\tilde{u}$.

$$\begin{aligned}
\Delta_y(\zeta\tilde{u}) &+ |\nabla_{y'}h|^2\partial_{y_ny_n}(\zeta\tilde{u}) + \nabla_{y'}(\partial_{y_n}(\zeta\tilde{u}))\cdot\nabla_{y'}h \\
&= \zeta\Delta_y\tilde{u} + \zeta|\nabla_{y'}h|^2\partial_{y_ny_n}\tilde{u} - \zeta\nabla_{y'}(\partial_{y_n}\tilde{u})\cdot\nabla_{y'}h \\
&+ \Bigg(\nabla_y\zeta\cdot\nabla_y\tilde{u} + \Delta_y\zeta\tilde{u} + |\nabla_{y'}h|^2(\partial_{y_n}\zeta)(\partial_{y_n}\tilde{u}) + |\nabla_{y'}h|^2(\partial_{y_ny_n}\zeta)\tilde{u} \\
&\qquad - \nabla_{y'}\zeta\cdot\nabla_{y'}h(\partial_{y_n}\tilde{u}) - \nabla_{y'}\tilde{u}\cdot\nabla_{y'}h(\partial_{y_n}\zeta) - \nabla_{y'}(\partial_{y_n}\zeta)\cdot\nabla_{y'}h(\tilde{u})\Bigg) \\
&= \zeta\Bigg(\Delta_y\tilde{u} + |\nabla_{y'}h|^2\partial_{y_ny_n}\tilde{u} - \nabla_{y'}(\partial_{y_n}\tilde{u})\cdot\nabla_{y'}h\Bigg) + E_{cutoff} \\
&= \zeta((\Delta_{y'}h)\partial_{y_n}\tilde{u} - \tilde{f}) + E_{cutoff} \\
&\equiv g
\end{aligned}$$

Defining g as above, cutoff term and ζ, h is just constant,

$$||g||_{L^2(U_{y,r})} \leq C||f||_{L^2(U_{y,r})}$$

We define $\zeta\tilde{u} = \tilde{u}^\star$ and want to estimate $\varphi^{\epsilon_1} * \varphi^{\epsilon_1} * \tilde{u}^\star$. Note that $\tilde{u}^\star$ is weak solution for

$$\Delta_y \tilde{u}^\star + |\nabla_{y'} h|^2 \partial_{y_n y_n} \tilde{u}^\star - \nabla_{y'}(\partial_{y_n} \tilde{u}^\star) \cdot \nabla_{y'} h = g$$

$$\begin{aligned}
\int_{U_{y,r}} & \nabla_y \tilde{u}^\star \cdot \nabla_y(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \varphi^{\epsilon_1} * \tilde{u}^\star)) + |\nabla_{y'} h|^2 (\partial_{y_n}(\varphi^{\epsilon_1} * \tilde{u}^\star))(\partial_{y_n}(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \varphi^{\epsilon_1} * \tilde{u}^\star))) \\
&- \nabla_{y'} \tilde{u}^\star \cdot \nabla_{y'} h(\partial_{y_n}(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \varphi^{\epsilon_1} * \tilde{u}^\star))) + g \cdot \partial_{y_1 y_1}(\varphi^{\epsilon_1} * \varphi^{\epsilon_1} * \tilde{u}^\star) \\
&= -\int_{U_{y,r}} |\partial_{y_1} \nabla_y (\varphi^{\epsilon_1} * \tilde{u}^\star)|^2 + \int_{U_{y,r}} |\nabla_{y'} h|^2 (\partial_{y_n}(\varphi^{\epsilon_1} * \tilde{u}^\star))(\partial_{y_n}(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \tilde{u}^\star))) \\
&- \int_{U_{y,r}} \nabla_{y'}(\varphi^{\epsilon_1} * \tilde{u}^\star) \cdot \nabla_{y'} h(\partial_{y_n}(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \tilde{u}^\star))) + \int_{U_{y,r}} g \cdot \partial_{y_1 y_1}(\varphi^{\epsilon_1} * \varphi^{\epsilon_1} * \tilde{u}^\star) \\
&= 0
\end{aligned}$$

Now we assumed $\nabla_{y'} h(0) = 0$, so $|\nabla_{y'} h(y')| \leq ||h||_{C^2(\mathbb{R}^{n-1})} r$ for $|y'| < r$. Setting $r = \epsilon / ||h||_{C^2(\mathbb{R}^{n-1})}$ for each $\epsilon > 0$

$$|\nabla_{y'} h(y')| \leq \epsilon$$

Then we could estimate intermediate term.

$$\begin{aligned}
\int_{U_{y,r}} & |\nabla_{y'} h|^2 (\partial_{y_n}(\varphi^{\epsilon_1} * \tilde{u}^\star))(\partial_{y_n}(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \tilde{u}^\star))) \\
&\leq \epsilon^2 \int_{U_{y,r}} \left| \partial_{y_n}(\varphi^{\epsilon_1} * \tilde{u}^\star) \partial_{y_n}(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \tilde{u}^\star)) \right| \\
&\leq \epsilon^2 \int_{U_{y,r}} \left| \partial_{y_1} \partial_{y_n}(\varphi^{\epsilon_1} * \tilde{u}^\star) \right|^2
\end{aligned}$$

$$\begin{aligned}
\int_{U_{y,r}} & \nabla_{y'}(\varphi^{\epsilon_1} * \tilde{u}^\star) \cdot \nabla_{y'} h(\partial_{y_n}(\partial_{y_1 y_1}(\varphi^{\epsilon_1} * \tilde{u}^\star))) \\
&\leq \epsilon \sum_{i=1}^{n-1} \int_{U_{y,r}} |\partial_{y_1 y_i}(\varphi^{\epsilon_1} * \tilde{u}^\star)|^2 + (n-1)\epsilon \int_{U_{y,r}} |\partial_{y_1 y_n}(\varphi^{\epsilon_1} * \tilde{u}^\star)|^2
\end{aligned}$$

Therefore, for small $\epsilon > 0$, we could gain

$$||\partial_{y_1} \nabla_y (\varphi^{\epsilon_1} * \tilde{u}^\star)||_{L^2(U_{y,r})} \leq C||f||_{L^2(U_{y,r})}$$

Similarly, we gain for all ∂_{y_i} for $1 \leq i \leq n-1$, $\partial_{y_n y_n} \tilde{u}^\star$ is defined from

$$\Delta_y \tilde{u}^\star + |\nabla_{y'} h|^2 \partial_{y_n y_n} \tilde{u}^\star - \nabla_{y'}(\partial_{y_n} \tilde{u}^\star) \cdot \nabla_{y'} h = g$$

Now, these estimate gives

$$||\zeta \tilde{u}||_{\dot{H}^2(U_{y,r})} \leq C||f||_{L^2(U_{y,r})}$$

So

$$||\tilde{u}||_{H^2(U_{y,r})} \leq C||f||_{L^2(U_{u,r})}$$

Therefore,

$$||u||_{H^2(U_{x,r})} \leq C||f||_{L^2(U_{y,r})}$$

7.3.5 Regularity

For all $\epsilon > 0$, there exists r that

$$r \precsim \frac{\epsilon}{||h||_{C^2(\mathbb{R}^{n-1})}}$$

and

$$U \subset \bigcup_{i=\{1,2,\cdots,n\}, x_i \in \partial U} B(x_i, r) \cup U_0$$

Interior Regularity and Boundary Regularity implies

$$||u||_{H^2(U)} \leq C||f||_{L^2(U)}$$

7.4 Higher Regularity

Higher Regularity is same. Dividing Interior, Boundary Regularity gives this.

7.5 Eigenvalue

We gain solution map $f \longmapsto u$ which is solution for

$$\begin{cases} -\Delta u = f & U \\ u = 0 & \partial U \end{cases}$$

$$(-\Delta)^{-1} : L^2(U) \to H_0^1(U) \cap H^2(U)$$

We say $v \in L^2(U)$ is an eigenfunction for $(-\Delta)^{-1}$ if $v \neq 0$ and there exists λ which is eigenvalue such that

$$(-\Delta)^{-1} v = \lambda v$$

By elliptic regularity, v is actually C^∞.

For example, $U = (-1, 1)$ then $(-\partial_{xx})^{-1} v = \lambda v$ if $\lambda = \frac{4}{\pi^2}$. This λ is eigenvalue.

One thing important is that eigenvalues are strictly positive.

$$\int_U v \cdot v = \lambda \int_U (-\Delta) v \cdot v$$

so

$$\int_U v^2 = \lambda \int_U |\nabla u|^2 \geq C\lambda \int_U v^2$$

Therefore, eigenvalue is strictly positive but also bounded. We say $(-\Delta)^{-1}$ a positive operator.

Also, $(-\Delta)^{-1}$ is self-adjoint. For any $f, g \in L^2(U)$

$$\int_U (-\Delta)^{-1} f \cdot g = \int_U f \cdot (-\Delta)^{-1} g$$

so eigenfunctions with different eigenvalues are orthogonal with L^2 inner product.

We now state maximum eigenvalue theorem.

Theorem 61 (Maximum eigenvalue theorem)**.** *For operator $(-\Delta)^{-1}$ following holds.*

$$\sup_{||v||_{L^2}=1} \int_U (-\Delta)^{-1} v \cdot v = \lambda_{max}$$

Proof. If u corresponding solution for $\Delta u = -v$, by previous discussion about elliptic regularity, it is bounded which only depends on domain. So suprenum exists. Now think of maximizing sequence. $\{v_k\}$. This sequecne is bounded in L^2 so weak limit $v_k \rightharpoonup v$ exists.

$$\int_U (-\Delta)^{-1} v_k \cdot v_k \to \int_U (-\Delta)^{-1} v \cdot v$$

We will show v is an eigenfunction.

$$\frac{\int_U (-\Delta)^{-1} v \cdot v}{\int_U v^2} \geq \frac{\int_U (-\Delta)^{-1}(v+\epsilon\varphi)\cdot(v+\epsilon\varphi)}{\int_U (v+\epsilon\varphi)^2}$$

Calculating derivative on $\epsilon = 0$

$$\left(\int_U (-\Delta)^{-1} v \cdot \varphi\right) ||v||_{L^2}^2 - \left(\int_U (-\Delta)^{-1} v \cdot v\right) \int_U v \cdot \varphi = 0$$

This means $(-\Delta)^{-1}v$ and v is parallel. v is eigenfunction. So

$$\sup_{||v||_{L^2}=1} \int_U (-\Delta)^{-1} v \cdot v = \lambda_{max}$$

□

Also another property holds. That is largest eigenfunction is simple and taken to be strictly positive in U.

Theorem 62. *For operator $(-\Delta)^{-1}$, largest eigenfunction is simple and can be taken to be strictly positive in U.*

Proof. First, prove $v > 0$.

If v has both positive and negative part, $v = v^+ + v^-$ then each v^+, v^- is eigenfunctions with same eigenvalue. Then

$$(-\Delta)v^+ = \frac{1}{\lambda_{max}} v +^{\geq} 0$$

By strong maximum principal, $v^+ > 0$ in U and so $v = v^+$.

Second, if two functions v_1, v_2 are eigenfunction with eigenvalue λ_{max}, we can choose α_1, α_2 that

$$\int_U \alpha_1 v_1 + \alpha_2 v_2 = 0$$

This $\alpha_1 v_1 + \alpha_2 v_2$ is also eigenfunction so $\alpha_1 v_1 + \alpha_2 v_2 = 0$. □

Now, we can decompose $L^2(U)$ by eigenfunctions.

$$\lambda_1 = \lambda_{max} > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_k \geq \cdots > 0$$

Also these eigenfunctions $\{v_{k,i}\}$ forms orthogonal basis since following holds.

$$\lambda_2 = \max_{v \in E_1^{\perp},\ ||v||_{L^2}=1} \int_U (-\Delta)^{-1} v \cdot v$$

We can keep decompose with eigenfunctions.