Riemannian Mainfold by Lee

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I will summarize important theorems in Riemannian Manifold by John Lee [Lee18]

Chapter 2. Review of Tensors, Manifolds, and Vector Bundles

I will cover small portion of this chapter, since a lot of things are precisely explained at Smooth Manifold book.

Lemma 1 (Lemma 2.1). *Let V be a finite-dimensional vector space. There is a **natural** isomorphism between $T^k_{l+1}(V)$ and $\mathrm{Multi}(V^* \times \cdots \times V^* \times V \times \cdots \times V, V)$. Here V^* appears l times and V appears k times.*

For $k = 1$ and $l = 0$, we can let $\Phi : End(V) \to T^1_1(V)$ by

$$(\Phi(A))(\omega, X) = \omega(AX)$$

This map sends endomorphism A to multilinear map. Also, it is vector space isomorphism thus to show injectiveness, if $\Phi(A) = 0$ then it is trivial that $A = 0$. To show surjectiveness, with any basis $\{e^i\}$ and its dual $\{E_i\}$, for $F \in T^1_1(V)$ we can let $A^j_i = F(E_i, e^j)$.
Similarly, we can set

$$\Phi : \mathrm{Multi}(V^* \times \cdots \times V^* \times V \times \cdots \times V, V) \to T^k_{l+1}(V)$$

$$\Phi(A)(\omega^1, \cdots, \omega^{l+1}, X_1, \cdots, X_k) = \omega^{l+1}(A(\omega^1, \cdots, \omega^l, X_1, \cdots, X_k))$$

We can define **Trace** operator.

$$tr : T^{k+1}_{l+1} \to T^k_l(V)$$

$$(tr\, F)(\omega^1, \cdots, \omega^l, V_1, \cdots, V_k) = tr(F(\omega^1, \cdots, \omega^l, \cdot, V_1, \cdots, V_k, \cdot))$$

where right hand side trace operator is trace defined on endomorphism.
Tensor defined on manifold is defined by vector bundle over a manifold. We call this a **Tensor Bundle**. Following **Tensor Characterization Lemma** states tensor field can be characterized by showing $C^\infty(M)$ linearity.

Lemma 2 (Lemma 2.4). *A map*

$$\tau : \mathscr{T}^1(M) \times \cdots \times \mathscr{T}^1(M) \times \mathscr{T}(M) \times \cdots \times \mathscr{T}(M) \to C^\infty(M)$$

is induced by a (k, l) tensor field if and only if it is multilinear over $C^\infty(M)$.

Chapter 3. Riemannian Metrics

g is Riemannian metric if it is element of $\mathscr{T}^2(M)$ and symmetric positive definite. Simple definitions are:
- Isometry if $\varphi : (M, g) \to (\tilde{M}, \tilde{g})$ and

$$\varphi^* \tilde{g} = g$$

- $\mathcal{J}(M)$ be isometry group of M.
- In local frame (not essentially coordinate chart)

$$g = g_{ij} \varphi^i \otimes \varphi^j$$

- Induced Metric. For $i : M \hookrightarrow (\tilde{M}, \tilde{g})$ then we can define riemannian metric on M by

$$g = i^* \tilde{g}$$

- Product Manifold. We can define natural Riemannian metric on $M_1 \times M_2$ where (M_1, g_1), (M_2, g_2) are Riemannian manifolds. Natural isomorphism $T_{(p_1,p_2)}(M_1 \times M_2) \approx T_{p_1}M_1 \oplus T_{p_2}M_2$ exists so we can define $g = g_1 \oplus g_2$
- Covering space. For $\pi : \tilde{M} \to (M, g)$ a covering map, we can define $\tilde{g} = \pi^* g$. Then for covering transform $\varphi : \tilde{M} \to \tilde{M}$, $\varphi^*\tilde{g} = \varphi^*\pi^* g = \pi^* g = \tilde{g}$ holds. Converse also holds that if $\pi : (\tilde{M}, \tilde{g}) \to M$ is smooth covering map that $\tilde{g}$ is invariant under all covering transformations then there exists g that is π related to $\tilde{g}$

Raising and Lowering Indices

We define Flat and Sharp operation.

$$\flat : TM \to T^*M$$

$$X^\flat(Y) = g(X, Y)$$

In local coordinate $X_j = g_{ij}X^i$ for $X^\flat = X_j dx^j$.

$$\sharp : T^*M \to TM$$

In local coordinate $\omega^i = g^{ij}\omega_j$ where g^{ij} is ij component of g^{-1}.

Inner product of Tensors

This is quite stunning result. **Riemann Metric naturally induces fiber metric on arbitrary dimensional tensor bundles**

Lemma 3 (Lemma 3.1). *Let g be a Riemannian metric on a manifold M. There is a unique fiber metric on each tensor bundle $T^k_l M$ that if $(E_1, \cdots, E_n)$ is an orthonormal basis for T_pM and $(\varphi^1, \cdots, \varphi^n)$ is dual basis then $E_{j_1} \otimes \cdots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}$ forms orthonormal basis for $T^k_l(T_pM)$*

We prove it by defining inner product by

$$\langle F, G\rangle = g^{i_1 r_1} \cdots g^{i_k r_k} g_{j_1 s_1} \cdots g_{j_l s_l} F^{j_1 \cdots j_l}_{i_1 \cdots i_k} G^{s_1 \cdots s_l}_{r_1 \cdots r_k}$$

And we will show this definition satisfies orthonormal property.

$$\begin{aligned}
&\langle E_{j_1} \otimes \cdots E_{j_l} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}, E_{\tilde{j}_1} \otimes \cdots \otimes E_{\tilde{j}_l} \otimes \varphi^{\tilde{i}_1} \otimes \cdots \otimes \varphi^{\tilde{i}_k}\rangle \\
&= E^{r_1}_{j_1} \cdots E^{r_l}_{j_l} (E^{-1})^{i_1}_{s_1} \cdots (E^{-1})^{i_k}_{s_k} E^{\tilde{r}_1}_{\tilde{j}_1} \cdots E^{\tilde{r}_l}_{\tilde{j}_l} (E^{-1})^{\tilde{i}_1}_{\tilde{s}_1} \cdots (E^{-1})^{\tilde{i}_k}_{\tilde{s}_k} g^{s_1 \tilde{s}_1} \cdots g^{s_k \tilde{s}_k} g_{r_1 \tilde{r}_1} \cdots g_{r_l \tilde{r}_l} \\
&= (E^{r_1}_{j_1} E^{\tilde{r}_1}_{\tilde{j}_1} g_{r_1 \tilde{r}_1}) \cdots (E^{r_l}_{j_l} E_{\tilde{j}_l \tilde{r}_l} g_{r_l \tilde{r}_l})((E^{-1})^{i_1}_{s_1} (E^{-1})^{\tilde{i}_1}_{\tilde{s}_1} g^{s_1 \tilde{s}_1}) \cdots ((E^{-1})^{i_k}_{s_k} (E^{-1})^{\tilde{i}_k}_{\tilde{s}_k} g^{s_k \tilde{s}_k}) \\
&= \langle E_{j_1}, E_{\tilde{j}_1}\rangle_g \cdots \langle E_{j_l}, E_{\tilde{j}_l}\rangle_g \langle (\varphi^{i_1})^\sharp, (\varphi^{\tilde{i}_1})^\sharp\rangle_g \langle (\varphi^{i_k})^\sharp, (\varphi^{\tilde{i}_k})^\sharp\rangle_g \cdots \\
&= \delta_{j_1 \tilde{j}_1} \cdots \delta_{j_l, \tilde{j}_l} \delta_{i_1 \tilde{i}_1} \cdots \delta_{i_k \tilde{i}_k}
\end{aligned}$$

Moreover, we can know that if ω, η are covariant 1 tensors, then

$$\langle \omega, \eta\rangle = \langle \omega^\sharp, \eta^\sharp\rangle$$

The Volume Form

There exists a unique n-form dV on oriented Riemannian Manifold such that for orthonormal basis(oriented) $(E_1, \cdots, E_n)$, $dV(E_1, \cdots, E_n) = 1$

Model Spaces of Riemannian Geometry

0.0.1 Euclidean space

V be an f.d.v.s with inner product then defining $g(X, Y) = \langle X, Y\rangle$ is Riemannian metric.

0.0.2 Spheres

S_R^n is sphere with radius R in R^{n+1}. Equipped with the induced Riemannian metric $\overset{\circ}{g}_R$. Here are the properties
- Homogeneous: Sphere admits Lie group to act smoothly and transitively by isometries
- Isotropic: Exists a Lie group acting smoothly such that the isotropy subgrouop acts transitively on T_pM
Actually, the Lie group is $O(n+1)$. Identifying $T_pS_R^n$ to subspace of T_pR^{n+1}, $O(n+1)$ is the group taking $(p/R, \{E_i\})$ to $(\tilde{p}/R, \{\partial_i\})$
- $S_R^n - \{p\}$ conformally equivalent to R^n.
Here, conformal equivalence is defined: for (M, g) and $(\tilde{M}, \tilde{g})$, diffeomorphism $\varphi : M \to \tilde{M}$ exists that $\varphi^*\tilde{g} = fg$, $f \in C^\infty(M)$. Or, it just preserves angles.
Calculation by stereographic projection. In differential geometry, **calculation** is important. We know the stereographic projection formula. $\sigma : S_R^n - \{N\} \to R^n$ is defined

$$\sigma(\xi, \tau) = \frac{R\xi}{R - \tau}$$

$$\sigma^{-1}(u) = \left(\frac{2R^2u}{|u|^2 + R^2}, \frac{|u|^2 - R^2}{|u|^2 + R^2}R\right)$$

0.0.3 Hyperbolic spaces

There exists three different models for hyperbolic space.
Hyperboloid Model

$$H_R^n = \{(\xi^1, \cdots, \xi^n, \tau) | \tau > 0, \tau^2 - |\xi|^2 = R^2\}$$

with the metric

$$h_R^1 = i^*m$$

. Where $i : H_R^n \to R^{n+1}$ is inclusion, $m = (dx^1)^2 + \cdots + (dx^n)^2 - (dy)^2$
Poincare Ball Model B_R^n is the ball of radius R in R^n with the metric

$$h_R^2 = 4R^4\frac{(du^1)^2 + \cdots + (du^n)^2}{(R^2 - |u|^2)^2}$$

Poincare Half-Space Model U_R^n is the upper half space in R^n with the metric

$$h_R^3 = R^2\frac{(dx^1)^2 + \cdots + (dx^{n-1})^2 + dy^2}{y^2}$$

It is quite a huge calculation to prove these models are isometric.
¡br /¿
Hyperbolid Model $\Leftarrow$ Poincare Ball Model

$$\pi(\xi, \tau) = \frac{R}{R + \tau}\xi$$

$$\pi^{-1}(u) = \left(\frac{2R^2u}{R^2 - |u|^2}, \frac{R^2 + |u|^2}{R^2 - |u|^2}R\right)$$

π is a diffeomorphism. Remaining is $(\pi^{-1})^*h_R^1 = h_R^2$.
For $V \in T_qB_R^n$, $(\pi^{-1})^*h_R^1(V, V) = m(\pi_*^{-1}V, \pi_*^{-1}V)$. Calculation gives

$$V\xi^j = \frac{2R^2V^j}{R^2 - |u|^2} + \frac{4R^2u^j(\sum V^iu^i)}{(R^2 - |u|^2)^2}$$

$$V\tau = \frac{4R^3(\sum V^iu^i)}{(R^2 - |u|^2)^2}$$

$$m(\pi_*^{-1}V, \pi_*^{-1}V) = \sum_{j=1}^{m}(V\xi^j)^2 - (V\tau)^2 = \frac{4R^4|V|^2}{(R^2 - |u|^2)^2} = h_R^2(V, V)$$

Poincare Ball Model $\Leftarrow$ Poincare Half Plane Model

$$\kappa(u, v) = \left(\frac{2R^2u}{|u|^2 + (v - R)^2}, \frac{R^2 - |u|^2 - v^2}{|u|^2 + (v - R)^2}R\right)$$

$$\kappa^{-1}(x,y) = \left(\frac{2R^2x}{|x|^2+(y+R)^2}, \frac{|x|^2+y^2-R^2}{|x|^2+(y+R)^2}R\right)$$

For $V \in T_pB_R^n$ we calculate κ_*V
Now, H_R^n have symmetry group $O_+(n,1)$ this group preserves Minkowski metric and taking $\tau > 0$ to itself. This group action gives H_R^n be homogeneous and isotropic.
Let $p \in H_R^n$ and $\{E_i\}$ orthonormal basis, $\{E_1, \cdots, E_n, E_{n+1} = p/R\}$ is a basis to R^{n+1}. We claim

$$m = (\varphi^1)^2 + \cdots + (\varphi^n)^2 - (\varphi^{n+1})^2$$

If we set $(x^1, \cdots, x^{n+1})$ a coordinate of R^{n+1},

$$(E_1, \cdots, E_{n+1})^T = A(\partial_1, \cdots, \partial_{n+1})^T$$

for the matrix A

$$A = \begin{pmatrix} a_1^1 & \cdots & a_1^{n+1} \\ \vdots & \ddots & \vdots \\ p^1/R & \cdots & p^{n+1}/R \end{pmatrix}$$

$$(\varphi^1, \cdots, \varphi^{n+1})^T = (A^{-1})^T(dx^1, \cdots, dx^{n+1})^T$$

Now,

$$(\varphi^1)^2 + \cdots + (\varphi^n)^2 - (\varphi^{n+1})^2 = (dx^1, \cdots, dx^{n+1})A^{-1}\begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}(A^{-1})^T(dx^1, \cdots, dx^{n+1})^T$$

and explicit calculation gives

$$A^{-1}\begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}(A^{-1})^T = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}$$

So $m = (\varphi^1)^2 + \cdots + (\varphi^n)^2 - (\varphi^{n+1})^2$

References

[Lee18] John M. Lee. *Introduction to Riemannian Manifolds*. 2nd. Springer, Cham, 2018. ISBN: 978-3-319-91754-2.