Algebraic topology

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This post illustrates fundamental results on Hatcher's Algebraic topology book [Hat02]

1 Homology : definition and technique

1.1 Relative Homology

Relative homology defined as $H_n(X, A)$ is interpreted as following:

It is represented by relative cycle $\varphi \in C_n(X)$ that $\partial\varphi \in C_n(A)$. So draw an image cycle whose boundary belongs to A.

This relative cycle is trivial if $\varphi = \partial\psi + \varphi^A$ where $\psi \in C_{n+1}(X)$, $\varphi^A \in C_n(A)$.

Homology is hard to imagine what it states for. It's because homology is the form $ker\partial/im\partial$. We need to imagine homology by specific cycle which is much easier to believe. Only aware of the fact that there are several other cycles that match with the cycle by equivalent class.

1.2 Techniques for Calculating Homology

Calculation of homology depends on some techniques. Long exact sequence is the main one. Mayer Vietoris argument is another.

1.2.1 Mayer Vietoris

Mayer Vietoris argument is useful because **We know all maps in exact sequence**

$$\cdots H_n(A \cap B) \xrightarrow{(i_{1*}, i_{2*})} H_n(A) \oplus H_n(B) \xrightarrow{j_{1*} - j_{2*}} H_n(X) \xrightarrow{\partial_*} H_n(A \cap B) \cdots$$

Note that third map is quite clear. Cycle of X can be decomposed into sum of chain in A and chain in B by barycentric subdivision. These two chain cancels, $\varphi = \varphi_A + \varphi_B$ then $0 = \partial\varphi_A + \partial\varphi_B$. So third map sends φ into $\partial\varphi_A = -\partial\varphi_B$.

1.2.2 Excision

Excision is an isomorphism but its fundamental is from **inclusion map**. $i : (X - Z, A - Z) \hookrightarrow (X, A)$ induces $H_n(X - Z, A - Z) \xrightarrow{i_*} H_n(X, A)$.

This fact may be important if excision relates to **naturality**.

Another thing to note is the proof technique. Constructing chain homotopy yields the same induced map is crucial component in chain argument.
$\partial P + P\partial = f_* - g_*$ then f, g generates the same induced map.

1.3 Naturality

Naturality states if $f : (X, A) \to (Y, B)$ then

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Third commutative diagram is important. Induced map commutes with boundary map is key of naturality.

1.4 Cellular Homology

1.4.1 Mapping Telescope Argument

Hatcher Lemma 2.34 uses mapping telescope argument for infinite dimensional CW complex. I haven't seen this type of argument applied to other situation but following proof was interesting.

1.4.2 Cellular Boundary Map

Strength of cellular homology is that it is able to compute explicitly the boundary map. Understanding following diagram is important.

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial"] \arrow[d, "\Phi_{\alpha *}"] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"] \arrow[d, "\approx"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\approx"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

Cellular boundary map sends $H_n(X^n, X^{n-1})$ to $H_{n-1}(X^{n-1}, X^{n-2})$. We need to see how actually each class send to others.

$$\begin{tikzcd}
[D^n_\alpha] \arrow[r, "\partial"] \arrow[d, "\Phi_{\alpha *}"] & [\partial D^n_\alpha] \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & [d_{\alpha\beta} e^{n-1}_\beta] \\
[e^n_\alpha] \arrow[r, "\partial_n"] \arrow[dr] & [*] \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & [\sum_\beta d_{\alpha\beta} e^{n-1}_\beta] \arrow[u, "q_{\beta *}"] \arrow[d, "\approx"] \\
 & [\sum_\beta d_{\alpha\beta} e^{n-1}_\beta] \arrow[r, "\approx"] & [\sum_\beta d_{\alpha\beta} e^{n-1}_\beta]
\end{tikzcd}$$

Therefore, the formula holds.

1.4.3 Application

Applying cellular homology to various CW complexes are remarkable. Len's space, Real and Complex Projective space... and exercises of chapter 2.2

1.5 Homology is an abelianization of fundamental group

The statement that homology is an abelianization of fundamental group is itself useful but I want to point out that this statement helps us to realize what homology represents.

Like example 2A.2. on Algebraic hatcher, we can express first homology group as 'loops'. Also, by calculation on fundamental group, we can identify whether two loops determine the same homology group.

1.6 Simplicial Approximation theorem of CW complex

Proof of simplicial approximation theorem is quite hard. I want to describe details of the proof.

The main bottleneck is that ordinary mapping cone does not apply to simplicial objects. Building mapping cylinder for simplicial map is as following:

- First remark that $f : K \to L$ simplicial map can be extended to $f : K' \to L$ where K' is barycentric subdivision of K. This is by choosing 'lowest index' for each barycentric division. In the proof, we want to make $M(f)$ that has retraction onto L with $r_1|_{K'} = f$. Right hand side f mean f applyed to K'.
- First construct $M(f|_{K^0})$. K^0 maps into L^0 by f so connect those points by Δ^1.

$$M(f|_{K^0}) = L \cup (\cup_{K^0} \Delta^1)$$

- Existing $M(f|_{K^{n-1}})$, n-simplex σ of K and $\tau = f(\sigma)$. Then within the barycentric subdivision of σ, $M(f : \sigma \to \tau)$ can be think of cone whose vertex is barycenter and underlying space $M(f : \partial\sigma \to \tau)$. Note that f extended to barycenter of σ sends barycenter to one of the 0 simplex of L. Thus, cone is perfectly defined.
- Attach $M(f : \sigma \to \tau)$ to $M(f|_{K^{n-1}})$. Then we obtain $M(f|_{K^n})$ that deformation retracts to $M(f|_{K^{n-1}})$. Here, deformation retract is defined for the 'realization' of mapping cylinders.

- Thus there can occur the problem that deformation retract does not equal f. However, this can be corrected by linear homotopy.
- To prove CW complex X, there is a simplicial complex Y that is homotopy equivalence, we'll construct CW complexes. Z_n contains X^n as a deformation retract and contains Y_n as 'subcomplex' and 'simplicial complex' which is also deformation retract. (Note that X^n means n-skeleton of X while Y_n and Z_n are just indices)
- Given attaching map $\varphi_\alpha : S^n \to X^n$, it is homotopic to $f_\alpha : S^n \to Y_n$ which is simplicial map. Define $W_n = Z_n \cup_\alpha M(f_\alpha)$. Then

$$S^n_\alpha \hookrightarrow M(f_\alpha) \to Y_n \hookrightarrow Z_n \to X^n$$

Map exists, $S^n_\alpha \to Y_n$ is f_α and $Z_n \to X^n$ is deformation retract. This map is homotopic to φ_α
- Z_{n+1} defined as $D^{n+1}_\alpha \times I$ attached to X^n and W_n. Attaching $D^{n+1}_\alpha \times 0$ to X^n is by φ_α and attaching $D^{n+1}_\alpha \times 1$ to W_n is by inclusion to mapping cylinder $M(f_\alpha)$ in W_n. $D^{n+1}_\alpha \times (0,1)$ is by homotopy of φ_α and f_α.
- One side of Z_{n+1} which consists of $D^{n+1}_\alpha \times 0$ becomes X^{n+1}
- The other side, for each mapping cylinder $M(f_\alpha)$ the one added is a disk attached to S^n_α. So mapping cone became mapping cone, thus become $Y_n \cup_\alpha C(f_\alpha)$ which is a simplicial complex. This will be Y_{n+1}
- Projecting $D^{n+1}_\alpha \times I$ from 0×0 (or saying $D^{n+1} \times I$ deform retract onto $\partial D^{n+1} \times I \cup D^{n+1} \times 1$) gives Z_{n+1} deformation retract to $Z_n \cup Y_{n+1}$ so Y_{n+1}. The other side gives deformation retract to $W_n \cup X^{n+1}$ so X^n.
- Y, Z defined as union of Y_n and Z_n 's satisfy the condition.

2 Cohomology and Homology

2.1 Intuitive understanding of Cohomology

Cohomology is Quotient group which lies on $C^n(X)$. Cohomology can be realized as a **function** assigning value to each map $\sigma : \Delta^n \to X$. Cocyle is one of the function that satisfies $\delta\varphi = 0$.
Relative cohomology $H^n(X, A)$ could be realized as some function assigning value to each map that is cocycle and assign 0 to map σ whose $Image(\sigma) \subset A$.

2.2 Universal Coefficient Theorem

Universal Coefficient Theorem is useful when we know homology. This sequence splits and also **natural**.

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C; G) \xrightarrow{h} Hom(H_n(C), G) \to 0$$

Not all cohomology group can be computed as universal coefficient theorem. Trivial isomorphism

$$H_n(X; \mathbb{Z}_2) = H^n(X; \mathbb{Z}_2)$$

holds. It's because assigning value 1 to σ can be interpreted as just adding σ. Thus one to one correspondence on chain and cochain exists in $\mathbb{Z}_2$ coefficient.

2.3 Similar techniques hold for cohomology

Mayer Vietoris, Excision... also holds for cohomology. Two things are remarkable.

2.3.1 Naturality of h

This is nontrivial that following diagram is commutative.

$$\begin{tikzcd}
H^n(A;G) \arrow[r, "\delta"] \arrow[d, "h"] & H^{n+1}(X,A;G) \arrow[d, "h"] \\
Hom(H_n(A),G) \arrow[r, "\partial^*"] & Hom(H_{n+1}(X,A),G)
\end{tikzcd}$$

Proof is just a diagram chasing but, we can learn what cohomology really is. Realizing as a function assigning values to chain is key ingredient of this proof.

2.3.2 Cellular cochain is exactly dual to the cellular chain complex

Hatcher Theorem 3.5. Thus we can easily use cellular cochain complex for CW complexes if their cellular chain complex is known.

2.4 Cup Product

2.4.1 Cup product commutes with induced map

For $f : X \to Y$,

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

This might be viewed not that powerful, but actually by this property, we can check following diagram is commutative. (This is also a solution for Hatcher exercise 2). Suppose $f : X \to Y$ that $f(A) \subset C$, $f(B) \subset D$.

$$\begin{array}{ccc} H^k(X,A) \times H^l(X,B) & \xrightarrow{\smile} & H^{k+l}(X, A \cup B) \\ \uparrow{\scriptstyle f^* \times f^*} & & \uparrow{\scriptstyle f^*} \\ H^k(Y,C) \times H^l(Y,D) & \xrightarrow{\smile} & H^{k+l}(Y, C \cup D) \end{array}$$

This is by diagram chasing

$$\begin{array}{ccc} [f^*\phi] \times [f^*\psi] & \xrightarrow{\smile} & [f^*\phi] \smile [f^*\psi] = [f^*(\phi \smile \psi)] \\ \uparrow{\scriptstyle f^* \times f^*} & & \uparrow{\scriptstyle f^*} \\ [\phi] \times [\psi] & \xrightarrow{\smile} & [\phi \smile \psi] \end{array}$$

2.4.2 Cup product is skew commutative

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha$$

2.5 Cohomology Ring

Cohomology ring is homotopy invariant, containing a lot of informations.

2.5.1 Proof detail on cohomology ring of Real projective space

At first glance, I haven't 'really' understand the proof. I'll describe more details on the proof of Hatcher's Theorem 3.12.

2.6 Orientation

2.6.1 Orientation as homology or as section

Orientation could be understand as a generator of homology $H_n(X|x)$.It may satisfy 'continuity' that for some ball $x \in B$, $H_n(X|A) \to H_n(X|x)$ maps generator to generator.
We can think in this way, why Mobius band is not orientable. Following to the circle (deform retraction of Mobius band), generator points opposite way but the same point.

More elegant way to understand this is thinking about covering space M_R, topologized by above 'continuity'

$$M_R = \{\mu_x | x \in M, \mu_x \in H_n(X|x; R)\}$$

If M is R orientable then each section will correspond to element of R. Identity of R corresponds to R-orientation. However, if M is not R-orientable, then element in $H_n(X|x; R)$ need to equal to its minus. Thus $2r = 0$ condition is needed. This fact is illustrated in Theorem 3.26 and Lemma 3.27 of Hatcher's book. Lemma 3.27 is itself useful in many situations.

2.6.2 Orientation of manifold with boundary

If M is compact manifold with boundary, orientation is defined by orientation of $M - \partial M$.
Existence of collar neighborhood of ∂M gives isomorphism

$$H_n(M, \partial M; R) \approx H_n(M - \partial M, \partial M \times (0, \epsilon); R)$$

Since $H_n(M - \partial M, \partial M \times (0, \epsilon); R) = H_n(M - \partial M | K; R)$, it can be viewed as orientation of M. It means $H_n(M, \partial M; R)$ represents orientation of M.
Crucial fact is that connecting homomorphism maps fundamental class of M into fundamental class of ∂M

$$\partial : H_n(M, \partial M) \xrightarrow{\approx} H_{n-1}(\partial M)$$

We can prove this by following method.
First we prove that $H_n(M) = 0$. There are two maps

$$i : M - \partial M \times [0, \epsilon] \hookrightarrow M$$

$$h : M \to M - \partial M \times [0, \epsilon]$$

Where h follows by proof of existence of collar neighborhood. M is homeomorphic to $(M - \partial M) \times [0, 3\epsilon)$. h sends $(M - \partial M) \times [0, 2\epsilon]$ into itself and $\partial M \times [0, 2\epsilon]$ into projection $\partial M \times 2\epsilon$.
Then $h \circ i$ and $i \circ h$ is homotopic to identity map. Thus M and $M - \partial M \times [0, \epsilon]$ are homotopy equivalent and later one is noncompact manifold without boundary. Which implies $H_n(M) = 0$.
Now, for the long exact sequence

$$0 \to H_n(M, \partial M) \xrightarrow{\partial} H_{n-1}(\partial M) \to H_{n-1}(M)$$

We know M is homotopic equivalent to orientable noncompact manifold without boundary so $H_{n-1}(M)$ is free. Thus ∂ map must be surjective. That means ∂ map is an isomorphism.
As a corollary we can prove Hatcher Ex. 33. If compact manifold retracts onto its boundary, then $r : M \to \partial M$ and $i : \partial M \to M$ satisfies $i \circ r = id$. However for the long exact sequence

$$H_n(M, \partial M) \xrightarrow{\approx} H_{n-1}(\partial M) \xrightarrow{0} H_{n-1}(M)$$

So inclusion map induces zero map in homology. There cannot exist a retraction.

2.7 Diagram chasing within cohomology with compact support

I want to describe diagram chasing that contains cohomology with compact support.

2.7.1 Lemma 3.36 from Hatcher

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^k(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U) \oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H_c^k(M) \arrow[r] \arrow[d, "D_M"] & H_c^{k+1}(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

First Square: for $K \subset U$ and $L \subset V$ compact,

$$\begin{tikzcd}
H^k(M|K \cap L) \arrow[r] \arrow[d] & H^k(M|K) \oplus H^k(M|L) \arrow[d] \\
H^k(U \cap V|K \cap L) \arrow[d] & H^k(U|K) \oplus H^k(V|L) \arrow[d] \\
H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V)
\end{tikzcd}$$

for representative cochain φ

$$\begin{tikzcd}
[\varphi] \arrow[r] \arrow[d] & [(\varphi, -\varphi)] \arrow[d] \\
[\varphi|_{U\cap V}] \arrow[d] & [(\varphi|_U, -\varphi|_V)] \arrow[d] \\
[\mu_{K\cap L} \frown \varphi|_{U\cap V}] \arrow[r] & *
\end{tikzcd}$$

$$* = (\mu_K \frown \varphi|_U, -\mu_L \frown \varphi|_V) = (\mu_{K\cap L} \frown \varphi|_{U\cap V}, -\mu_{K\cap L} \frown \varphi|_{U\cap V})$$

is what we have to show and it holds because of cap product commutes with induced map.

2.8 Appendix A : Universal Coefficient Theorem

We have two universal coefficient theorems : Universal coefficient theorem for cohomology and Universal coefficient theorem for homology.
Universal coefficient theorem for cohomology states

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C; G) \to Hom(H_n(C), G) \to 0$$

and is split exact, natural.
Universal coefficient theorem for homology states

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to Tor(H_{n-1}(C), G) \to 0$$

and is split exact, natural.

2.9 Appendix B : General Kunneth formula

2.9.1 Homology as cell of CW complex

In the Appendix B, it introduces how to 'interpret' homology in CW structure. Then, we will be able to define 'product' of cycles. For cohomology, we defined cup product which $\varphi \smile \psi$ applied separately on Δ - complexes. The cellular cross product is defined to satisfy $d(e^i \times e^j) = de^i \times e^j + (-1)^i e^i \times de^j$. Which e^i, e^j corresponds to i cell and j cell of X and Y respectively. More precisely, these e^i are the generator of $H_i(X^i, X^{i-1})$. However, we need an attention to determine orientation of e^i. This orientation is equivalent to linear algebraic orientation. $H_i(I^i, \partial I^i)$ can be determined by orientation of basis elements, which is -1 or 1 according to determinant.
Then, product map of homology is well-defined and cellular maps also satisfies 'good' property. See Lemma 3B.2.
Now we can think of homology for CW complexes that corresponds to one i cells. This gives an easy explanation why $H_n(X \times S^k; \mathbb{Z}) \approx H_n(X; \mathbb{Z}) \oplus H_{n-k}(X; \mathbb{Z})$

2.9.2 Algebraic Kunneth formula

Algebraic Kunneth formula is formula that determines homology of tensor product of two chains. This applied to homology of CW complexes induce topological Kunneth formula.

2.9.3 Cohomology as cell of CW complex

Cohomology could be also defined within 'cell' in CW complex. It assigns each i cell a R value. Then as homology does, we can define product of cohomology.

$$(\varphi \times_2 \psi)(e^k_\alpha \times e^l_\beta) = \varphi(e^k_\alpha)\psi(e^l_\beta)$$

and assigns 0 to other $k + l$ cells which are not the product of k-cell and l-cell of X and Y.
Important fact of this definition is, this definition of product coincides with product defined by cup product of cohomology.

$$\varphi \times_1 \psi = p_1^*(\varphi) \smile p_2^*(\psi)$$

2.10 Appendix C : Hopf Algebras

We have looked at interesting algebraic structure on cohomology ring. Within the 'cup product' cohomology ring became graded algebra. If given topological space satisfies additional structure : H-space, then more structure could be induced on cohomology and homology.

2.10.1 Cohomology : Hopf algebra structure

Hopf algebra is a graded algebra over a commutative ring R with special operation : coproduct.
(1) There is an identity element $1 \in A^0$ (2) Coproduct map $\Delta : A \to A \otimes A$ is homomorphism of graded algebras and $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_i \alpha'_i \otimes \alpha''_i$
Since H-space have continuous multiplication map μ, this induces cohomology graded ring $H^*(X; R)$ a Hopf algebra structure.

2.10.2 Homology : Pontryagin product (Sometimes, Pontryagin ring)

With the multiplication map, it induces

$$H_*(X;R) \otimes H_*(X;R) \xrightarrow{\times} H_*(X \times X;R) \xrightarrow{\mu_*} H_*(X;R)$$

This Pontryagin product is not necessarily associative but it holds for most of the situations.

2.10.3 Hopf algebra and dual Hopf algebra

Proposition 3C.10 states the dual of hopf algebra is hopf algebra.
A be a Hopf algebra over R and is f.g. free R-module. Then the product $\pi : A \otimes A \to A$ and coproduct $\Delta : A \to A \otimes A$ which is in cohomology ring, the cup product map and coproduct map, have duals π and Δ over A and this induces hopf algebra.

References

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