

2025 年循人中学高三期末考兼统考预试

高中组

高级数学 (III)

(SC007)

训练集

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前言

此训练集的雏形要追溯到 2019 年,依稀记得为当时的备赛爬了大量的 AMC 历届考题,创了“AMC 10 精选”的文档,其中我认为是属于统考范围的难题,便收录于另一个文档。

尘封已久,直到今年 5 月,我在整理电脑资料时偶然翻出,看着当年的内容,心想:读了一个数学系,难免觉得发展空间实属不少。在这几个月内,我随性地在网络上寻找、采集、整理更有趣(恶心)、更值得深思(烧脑)的题目,于是诞生了——高级数学 (III) 训练集。

起初,我并没想过要为每一题整理出解析,但发现到不少当年的题目只附答案没附解法。可惜的是,我的脑袋算是停止运作了一年,发现有一些题目对于现在的我实在是难以下咽。好吧,此乃下下策,但我决定用 LaTeX 开始动手为每一题补上解法、重写排版、增加新的内容。也好,既是备忘录,也是为了防止未来的我看到这些题目时会再次怀疑人生:「咦?是不是百尺竿头,更蠢一步?」

依据统考范围,我把训练集大致分为代数、组合数学、几何及微积分。若你想自行动手尝试,可以在 preamble 中将 `\printanswers` 这行注释掉。

绝大部分解法都由官方或我提供,只有某些是 ChatGpt 生成后由我再校对,之中一定会有纰漏,欢迎批评指正,也希望这份训练集能助你一路披荆斩棘、越战越勇!

——冬恒

目录



代数

一元二次方程、多项式	4
因式定理、余式定理	32
根式、绝对值	45
指数与对数	55
方程组	71
取整	103
函数	122
不等式	149
数列与级数	208
二项展开式	268
泰勒展开式	291
矩阵	306
行列式	324
复数	348
数学归纳法	387

数论

整除	425
同余	428
不定方程	433

组合数学

排列与组合	442
概率、期望值	468
统计	495

几何

解三角形	501
三角函数	612
反三角函数	676
平面向量	687
直角坐标	698
圆锥曲线	726
坐标变换	802
轨迹方程式、参数方程式	808
极坐标	839
立体几何、空间向量	855

微积分

极限	909
微分	934
积分	959
积分技巧	985
微分方程	1146

总题数: 1656

微积分

极限

考点: 极限的直观意义、性质、基本极限 ($\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e, \dots$)、对数微分法、洛必达法则、幂级数的展开、夹挤定理

1.

$$\lim_{x \rightarrow 0} [x]$$

由于

$$\lim_{x \rightarrow 0^+} [x] = 0, \quad \lim_{x \rightarrow 0^-} [x] = -1$$

极限 $\lim_{x \rightarrow 0} [x]$ 不存在。

2.

$$\lim_{x \rightarrow 3} ([x] - [x])$$

由于

$$\lim_{x \rightarrow 3^+} ([x] - [x]) = \lim_{x \rightarrow 3^+} [x] - \lim_{x \rightarrow 3^+} [x] = 4 - 3 = 1$$

且

$$\lim_{x \rightarrow 3^-} ([x] - [x]) = \lim_{x \rightarrow 3^-} [x] - \lim_{x \rightarrow 3^-} [x] = 3 - 2 = 1$$

故

$$\lim_{x \rightarrow 3} ([x] - [x]) = 1$$

3.

$$\lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right]$$

由于

$$\lim_{x \rightarrow 0^+} \left[\frac{\sin x}{x} \right] = \lim_{x \rightarrow 0^-} \left[\frac{\sin x}{x} \right] = 0$$

故

$$\lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right] = 0$$

4.

$$\lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x}$$

有

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x} &= \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

5.

$$\lim_{\theta \rightarrow 0} \frac{\theta - \theta \cos \theta}{\sin \theta \tan \theta}$$

有

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\theta - \theta \cos \theta}{\sin \theta \tan \theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\frac{\sin \theta}{\theta}} \cdot \frac{1}{\tan \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} \cdot \frac{1}{\frac{\sin \theta}{\theta}} \cdot \frac{\theta}{\tan \theta} \cdot \theta \\ &= \frac{1}{2} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot 0 \\ &= 0 \end{aligned}$$

6.

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \left(\frac{1 - \cos \theta}{2} \right)}{\theta^4}$$

有

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{1 - \cos\left(\frac{1 - \cos \theta}{2}\right)}{\theta^4} &= \lim_{\theta \rightarrow 0} \left[\frac{1 - \cos\left(\frac{1 - \cos \theta}{2}\right)}{\left(\frac{1 - \cos \theta}{2}\right)^2} \cdot \left(\frac{1 - \cos \theta}{\theta^2}\right)^2 \cdot \frac{1}{4} \right] \\ &= \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{4} = \frac{1}{32}\end{aligned}$$

其中

$$\lim_{x \rightarrow 0} \frac{1 + \cos x}{x^2} = \frac{1}{4} \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} = \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}$$

7.

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^{-1} x \tan x}$$

有

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^{-1} x \tan x} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^{-1} x \cdot \frac{\sin x}{\cos x}} \\ &= \lim_{x \rightarrow 0} \left(-\frac{1 - \cos x}{x^2} \right) \cdot \frac{x}{\sin^{-1} x} \cdot \frac{x}{\sin x} \cdot \cos x \\ &= -\frac{1}{2} \cdot 1 \cdot 1 \cdot 1 \\ &= -\frac{1}{2}\end{aligned}$$

8. 求极限

$$\lim_{x \rightarrow 1} \left[\cos\left(\frac{\pi}{2}x\right) + x \cos\left(\frac{\pi}{2}x\right) + x^2 \cos\left(\frac{\pi}{2}x\right) + \dots \right]$$

注意到

$$\begin{aligned}
 & \lim_{x \rightarrow 1} \left[\cos \left(\frac{\pi}{2} x \right) + x \cos \left(\frac{\pi}{2} x \right) + x^2 \cos \left(\frac{\pi}{2} x \right) + \dots \right] \\
 &= \lim_{x \rightarrow 1} \frac{\cos \left(\frac{\pi}{2} x \right)}{1 - x} \\
 &\stackrel{H}{=} \lim_{x \rightarrow 1} \frac{-\frac{\pi}{2} \sin \left(\frac{\pi}{2} x \right)}{-1} \\
 &= \frac{\pi}{2} \sin \frac{\pi}{2} \\
 &= \frac{\pi}{2}
 \end{aligned}$$

9. 计算极限

$$\lim_{x \rightarrow 0} \left(\sqrt{\frac{1}{x(x-1)} + \frac{1}{4x^2}} - \frac{1}{2x} \right)$$

有理化得

$$\begin{aligned}
 \lim_{x \rightarrow 0} \left(\sqrt{\frac{1}{x(x-1)} + \frac{1}{4x^2}} - \frac{1}{2x} \right) &= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{x(x-1)} + \frac{1}{4x^2} \right) - \frac{1}{4x^2}}{\sqrt{\frac{1}{x(x-1)} + \frac{1}{4x^2}} + \frac{1}{2x}} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{1}{x-1}}{\sqrt{\frac{x}{x-1} + \frac{1}{4}} + \frac{1}{2}} \\
 &= \frac{-1}{\frac{1}{2} + \frac{1}{2}} \\
 &= -1
 \end{aligned}$$

10. 试求

$$\lim_{x \rightarrow \infty} \left(\sqrt[5]{x^5 + 3x^4 + 4x^3 + 3x} - \sqrt[3]{x^3 + 3x^2 + 4x + 1} \right).$$

由洛必达法则,

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} \left(\sqrt[5]{x^5 + 3x^4 + 4x^3 + 3x} - \sqrt[3]{x^3 + 3x^2 + 4x + 1} \right) \\
 &= \lim_{x \rightarrow \infty} \frac{\sqrt[5]{1 + \frac{3}{x} + \frac{4}{x^2} + \frac{3}{x^4}} - \sqrt[3]{1 + \frac{3}{x} + \frac{4}{x^2} + \frac{1}{x^3}}}{\frac{1}{x}} \\
 &\stackrel{H}{=} - \lim_{x \rightarrow \infty} x^2 \left(\frac{1}{5} \left(1 + \frac{3}{x} + \frac{4}{x^2} + \frac{3}{x^4} \right)^{-\frac{4}{5}} \left(-\frac{3}{x^2} - \frac{8}{x^3} - \frac{12}{x^5} \right) \right. \\
 &\quad \left. - \frac{1}{3} \left(1 + \frac{3}{x} + \frac{4}{x^2} + \frac{1}{x^3} \right)^{-\frac{2}{3}} \left(-\frac{3}{x^2} - \frac{8}{x^3} - \frac{3}{x^4} \right) \right) \\
 &= \frac{3}{5} - 1 = -\frac{2}{5}
 \end{aligned}$$

11. 已知等差数列 $\{a_n\}, \{b_n\}$ 公差均不为 0, 且 $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 5$, 求

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_n}{n \cdot b_{2n}}.$$

设 $\{a_n\}, \{b_n\}$ 公差为 d_1, d_2 , 则

$$a_n = a_1 + (n-1)d_1, \quad b_n = b_1 + (n-1)d_2.$$

由已知极限得

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{a_1 + (n-1)d_1}{b_1 + (n-1)d_2} = \frac{d_1}{d_2} = 5.$$

所以 $d_1 = 5d_2$, 因此所求极限为

$$\lim_{n \rightarrow \infty} \frac{\frac{n}{2}(2a_1 + (n-1)d_1)}{n(b_1 + (2n-1)d_2)} = \lim_{n \rightarrow \infty} \frac{2a_1 + (n-1)d_1}{2(b_1 + (2n-1)d_2)} = \frac{d_1}{4d_2} = \frac{5}{4}$$

12. 设 $a_1 + a_2 + \cdots + a_n = n^3 - 2n$, 试求

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{a_3 + a_6 + \cdots + a_{3n}} - \sqrt[3]{a_2 + a_4 + \cdots + a_{2n}}}{n}$$

由

$$S(n) = \sum_{k=1}^n a_k = n^3 - 2n$$

得

$$a_n = S(n) - S(n-1) = n^3 - 2n - [(n-1)^3 - 2(n-1)] = 3n^2 - 3n - 1$$

于是

$$a_{3k} = 27k^2 - 9k - 1, \quad a_{2k} = 12k^2 - 6k - 1,$$

且

$$\sum_{k=1}^n a_{3k} = \frac{27}{6}n(n+1)(2n+1) - \frac{9}{2}n(n+1) - n = 9n^3 + 9n^2 - n,$$

$$\sum_{k=1}^n a_{2k} = 2n(n+1)(2n+1) - 3n(n+1) - n = 4n^3 + 3n^2 - 2n$$

故极限为

$$\lim_{n \rightarrow \infty} \left(\sqrt[3]{9 + \frac{9}{n} - \frac{1}{n^2}} - \sqrt[3]{4 + \frac{3}{n} - \frac{2}{n^2}} \right) = \sqrt[3]{9} - \sqrt[3]{4}$$

13.

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1+3^{2n}} + \sqrt[n]{3^{2n}+5^{2n}} + \sqrt[n]{5^{2n}+7^{2n}} + \cdots + \sqrt[n]{(2m-1)^{2n}+(2m+1)^{2n}}}{m^3}$$

首先有

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\sqrt[n]{1+3^{2n}} + \sqrt[n]{3^{2n}+5^{2n}} + \cdots + \sqrt[n]{(2m-1)^{2n}+(2m+1)^{2n}} \right) \\ &= \lim_{n \rightarrow \infty} \left(3^2 \sqrt[n]{\left(\frac{1}{3}\right)^{2n}} + 1 + 5^2 \sqrt[n]{\left(\frac{3}{5}\right)^{2n}} + 1 + \cdots + (2m+1)^2 \sqrt[n]{\left(\frac{2m-1}{2m+1}\right)^{2n}} + 1 \right) \\ &= \sum_{k=1}^m (2k+1)^2 \\ &= \sum_{k=1}^m (4k^2 + 4k + 1) \\ &= \frac{2}{3}m(m+1)(2m+1) + 2m(m+1) + m \end{aligned}$$

因此, 原式

$$\lim_{m \rightarrow \infty} \frac{\frac{2}{3}m(m+1)(2m+1) + 2m(m+1) + m}{m^3} = \frac{4}{3}$$

14.

$$\lim_{x \rightarrow 0} \frac{\int_{\cos x}^1 e^{-y^2} dy}{x \sin x}$$

由洛必达法则,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\int_{\cos x}^1 e^{-y^2} dy}{x \sin x} &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin x \cdot e^{-(\cos x)^2}}{\sin x + x \cos x} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x(1 + 2 \sin^2 x) \cdot e^{-(\cos x)^2}}{2 \cos x - x \sin x} \\ &= \frac{1}{2e} \end{aligned}$$

15.

$$\lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 - \sqrt[3]{x})}{1 + \cos \pi x}$$

消去根号, 令 $x = t^6$, 当 $x \rightarrow 1$ 时, $t \rightarrow 1$, 由洛必达法则,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 - \sqrt[3]{x})}{1 + \cos \pi x} &= \lim_{t \rightarrow 1} \frac{(1 - t^3)(1 - t^2)}{1 + \cos \pi t^6} \\ &= \lim_{t \rightarrow 1} \frac{1 - t^2 - t^3 + t^5}{1 + \cos \pi t^6} \\ &\stackrel{H}{=} \lim_{t \rightarrow 1} \frac{-2t - 3t^2 + 5t^4}{-6\pi t^5 \sin \pi t^6} \\ &\stackrel{H}{=} \lim_{t \rightarrow 1} \frac{-2 - 6t + 20t^3}{-30\pi t^4 \sin \pi t^6 - 6\pi t^5 \cdot \cos \pi t^6 \cdot 6\pi t^5} \\ &= \frac{-2 - 6(1) + 20(1)^3}{-6\pi \cdot (-1) \cdot 6\pi} \\ &= \frac{1}{3\pi^2} \end{aligned}$$

16.

$$\lim_{x \rightarrow 0} \frac{\left(\int_0^x t \cos t^2 dt \right)^2}{\int_0^x \sin t^2 dt}$$

首先发现到

$$\int_0^x t \cos t^2 dt = \left[\frac{1}{2} \sin t^2 \right]_0^x = \frac{1}{2} \sin x^2$$

将积分结果代入原极限式:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\left(\int_0^x t \cos t^2 dt \right)^2}{\int_0^x \sin t^2 dt} &= \lim_{x \rightarrow 0} \frac{\frac{1}{4} \sin^2 x^2}{\int_0^x \sin t^2 dt} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{4} \cdot 2 \sin x^2 \cdot \cos x^2 \cdot 2x}{\sin x^2} \\ &= \lim_{x \rightarrow 0} (x \cos x^2) \\ &= 0 \cdot 1 = 0 \end{aligned}$$

17. 求实数 a, b , 使得

$$\lim_{x \rightarrow 0} \frac{\sqrt{ax+b}-2}{x} = 1.$$

有理化得

$$\lim_{x \rightarrow 0} \frac{\sqrt{ax+b}-2}{x} \cdot \frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x \rightarrow 0} \frac{ax+b-4}{x(\sqrt{ax+b}+2)}$$

当 $x \rightarrow 0$ 时分母趋于 0, 欲使极限存在, 分子也必须趋于 0, 因此

$$a \cdot 0 + b - 4 = 0,$$

从而 $b = 4$, 于是

$$\lim_{x \rightarrow 0} \frac{a}{\sqrt{ax+4}+2} = \frac{a}{\sqrt{4}+2} = 1,$$

解得

$$a = 4$$

18. 已知

$$\lim_{x \rightarrow -1} \frac{ax^2 + 3x + b}{x + 1} = 2, \quad a, b \in \mathbb{R}$$

试求 (a, b) 。

当 $x \rightarrow -1, x+1 \rightarrow 0$; 欲使该极限存在且等于 2, 分子 $ax^2 + 3x + b$ 在 $x \rightarrow -1$ 也应趋于 0, 即

$$\lim_{x \rightarrow -1} (ax^2 + 3x + b) = 0 \Rightarrow a(-1)^2 + 3(-1) + b = 0 \Rightarrow a + b = 3 \quad (1)$$

又原极限中分子分母处处可导, 且极限

$$\lim_{x \rightarrow -1} \frac{(ax^2 + 3x + b)'}{(x+1)'} = \lim_{x \rightarrow -1} \frac{2ax + 3}{1} = -2a + 3$$

存在, 则由洛必达法则,

$$\lim_{x \rightarrow -1} \frac{ax^2 + 3x + b}{x+1} \stackrel{H}{=} -2a + 3 = 2 \Rightarrow (a, b) = \left(\frac{1}{5}, \frac{14}{5}\right)$$

19. 已知 a, b 为正整数, 设函数

$$f(x) = \lim_{n \rightarrow \infty} \frac{2x^{2n+1} + ax^2 + bx - 1}{2x^{2n} + 3},$$

若 $\forall x \in \mathbb{R}, f(x)$ 为连续函数, 求序对 (a, b) 。

首先注意到

$$f(x) = \lim_{n \rightarrow \infty} \frac{2x^{2n+1} + ax^2 + bx - 1}{2x^{2n} + 3} = \lim_{n \rightarrow \infty} \frac{x + \frac{ax^2+bx-1}{2x^{2n}}}{1 + \frac{3}{2x^{2n}}} = \begin{cases} x, & |x| > 1, \\ \frac{ax^2 + bx - 1}{3}, & |x| < 1. \end{cases}$$

由连续性,

$$\lim_{x \rightarrow 1^+} f(x) = 1, \quad \lim_{x \rightarrow 1^-} f(x) = \frac{a+b-1}{3}, \quad f(1) = \lim_{n \rightarrow \infty} \frac{2+a+b-1}{2+3} = \frac{a+b+1}{5}$$

可得

$$\frac{a+b+1}{5} = 1 = \frac{a+b-1}{3} \Rightarrow a+b=4 \quad (1)$$

同理,

$$\lim_{x \rightarrow -1^+} f(x) = \frac{a-b-1}{3}, \quad \lim_{x \rightarrow -1^-} f(x) = -1, \quad f(-1) = \frac{-2+a-b-1}{5} = \frac{a-b-3}{5}$$

可得

$$\frac{a-b-3}{5} = -1 = \frac{a-b-1}{3} \Rightarrow a-b=-2 \quad (2)$$

由 (1), (2) 得

$$(a, b) = (1, 3)$$

20. 若

$$\lim_{x \rightarrow 0} \frac{\sqrt{f(x) \sin x + 1} - 1}{e^{4x} - 1} = 2,$$

求

$$\lim_{x \rightarrow 0} f(x)$$

当 $x \rightarrow 0$ 时, 原式为不定型 $\frac{0}{0}$, 可用洛必达法则:

$$\lim_{x \rightarrow 0} \frac{\sqrt{f(x) \sin x + 1} - 1}{e^{4x} - 1} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{f(x) \sin x + 1}} \cdot [f'(x) \sin x + f(x) \cos x]}{4e^{4x}}.$$

当 $x \rightarrow 0$,

$$\sin x \rightarrow 0, \cos x \rightarrow 1, e^{4x} \rightarrow 1, \sqrt{f(x) \sin x + 1} \rightarrow 1$$

上式化简为

$$\lim_{x \rightarrow 0} \frac{f(x)}{8} = 2 \Rightarrow \lim_{x \rightarrow 0} f(x) = 16$$

21. 定义序列 $\{x_n\}_{n=2}^{\infty}$ 如下:

$$(n + x_n)[\sqrt{2} - 1] = \ln 2.$$

求 $\lim_{n \rightarrow \infty} x_n$.

由方程可解得

$$x_n = \frac{\ln 2}{2^{\frac{1}{n}} - 1} - n,$$

极限形式为 $\infty - \infty$ 。令 $u = \frac{1}{n}$, 并使用洛必达法则两次, 有

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \frac{\ln 2 - n \cdot 2^{\frac{1}{n}} + n}{2^{\frac{1}{n}} - 1} \\ &= \lim_{u \rightarrow 0} \frac{u \ln 2 - 2^u + 1}{u 2^u - u} \\ &\stackrel{H}{=} \lim_{u \rightarrow 0} \frac{\ln 2 - 2^u \ln 2}{2^u - 1 + u 2^u \ln 2} \\ &\stackrel{H}{=} \lim_{u \rightarrow 0} \frac{-2^u (\ln 2)^2}{2^u \ln 2 + 2^u \ln 2 + u 2^u (\ln 2)^2} \\ &= \frac{-(\ln 2)^2}{2 \ln 2} \\ &= -\frac{1}{2} \ln 2 \end{aligned}$$

22. 计算极限

$$\lim_{x \rightarrow 0} \left[\frac{1}{x^3} \int_0^x \frac{t \ln(t+1)}{t^4 + \frac{1}{6}} dt \right]$$

使用两次洛必达法则,

$$\begin{aligned} \lim_{x \rightarrow 0} \left[\frac{1}{x^3} \int_0^x \frac{t \ln(t+1)}{t^4 + \frac{1}{6}} dt \right] &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{x \ln(x+1)}{x^4 + \frac{1}{6}}}{3x^2} \\ &= \lim_{x \rightarrow 0} \frac{\ln(x+1)}{3x^5 + \frac{x}{2}} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{x+1}}{15x^4 + \frac{1}{2}} \\ &= \frac{1}{0 + \frac{1}{2}} \\ &= 2 \end{aligned}$$

23. 求极限

$$\lim_{x \rightarrow 0^+} (\sin x)^{\frac{1}{\ln x}}$$

此极限属于 0^0 型。利用对数恒等式将原式重写为以 e 为底的形式:

$$(\sin x)^{\frac{1}{\ln x}} = \exp \left(\frac{\ln(\sin x)}{\ln x} \right)$$

由于 e^x 是连续函数, 只需计算指数部分的极限

$$L = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\ln x}$$

属于 $\frac{\infty}{\infty}$ 型, 由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{x}{\sin x} \cdot \cos x = 1 \cdot 1 = 1$$

因此原极限为

$$e^1 = e$$

24. 求极限

$$\lim_{x \rightarrow \infty} (x + e^x)^{\frac{1}{x}}$$

此极限属于 ∞^0 型, 由

$$(x + e^x)^{\frac{1}{x}} = \exp \left(\frac{\ln(x + e^x)}{x} \right)$$

设

$$L = \lim_{x \rightarrow \infty} \frac{\ln(x + e^x)}{x}$$

此极限属于 $\frac{\infty}{\infty}$ 型, 由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1+e^x}{x+e^x}}{1} = \lim_{x \rightarrow \infty} \frac{1+e^x}{x+e^x}$$

再由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{1+e^x} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{e^x} + 1} = \frac{1}{0+1} = 1$$

因此, 原极限为

$$e^1 = e$$

25. 求

$$\lim_{x \rightarrow \infty} \left[\sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} \right]^x$$

首先注意到

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{(x^2 + 2x - 1) - (x^2 - 1)}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1}} \\ &= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 + \frac{2}{x} - \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}} \\ &= \frac{2}{1+1} = 1 \end{aligned}$$

故原极限为 1^∞ 型, 于是

$$\lim_{x \rightarrow \infty} \left[\sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} \right]^x = \exp \left[\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} - 1 \right) \right]$$

其中

$$\begin{aligned}\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} - 1 \right) &= \lim_{x \rightarrow \infty} x \cdot \frac{(x^2 + 2x - 1) - (\sqrt{x^2 - 1} + 1)^2}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1} + 1} \\&= \lim_{x \rightarrow \infty} x \cdot \frac{2x - 2\sqrt{x^2 - 1}}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1} + 1} \\&= \lim_{x \rightarrow \infty} x \cdot \frac{\frac{2}{x + \sqrt{x^2 - 1}} - 1}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1} + 1} \\&= \lim_{x \rightarrow \infty} \frac{\frac{2}{x + \sqrt{x^2 - 1}} - 1}{\sqrt{1 + \frac{2}{x} - \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}} + \frac{1}{x^2}} \\&= \frac{0 - 1}{1 + 1} \\&= -\frac{1}{2}\end{aligned}$$

故原极限为

$$\lim_{x \rightarrow \infty} \left[\sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} \right]^x = \frac{1}{\sqrt{e}}$$

26. 设

$$\lim_{x \rightarrow 0} \frac{\sin 6x + xf(x)}{x^3} = 0,$$

求

$$\lim_{x \rightarrow 0} \frac{6 + f(x)}{x^2}$$

原极限为

$$\lim_{x \rightarrow 0} \frac{\sin 6x + xf(x)}{x^3} = 0.$$

利用幂级数展开

$$\sin 6x = 6x - \frac{(6x)^3}{6} + o(x^3) = 6x - 36x^3 + o(x^3),$$

代入得

$$\frac{6x - 36x^3 + xf(x)}{x^3} = \frac{6x + xf(x)}{x^3} - 36 + o(1) = \frac{6 + f(x)}{x^2} - 36 + o(1).$$

极限存在且为 0, 则

$$\lim_{x \rightarrow 0} \left(\frac{6 + f(x)}{x^2} - 36 \right) = 0 \implies \lim_{x \rightarrow 0} \frac{6 + f(x)}{x^2} = 36.$$

27. 若 $f(x)$ 为满足

$$\lim_{x \rightarrow 1} \frac{f(x)}{x-1} = 36, \quad \lim_{x \rightarrow -1} \frac{f(x)}{x+1} = -36, \quad \lim_{x \rightarrow 2} \frac{f(x)}{x-2} = 0, \quad \lim_{x \rightarrow -2} \frac{f(x)}{x+2} = 0$$

的最低次多项式, 求 $f(3)$ 。

设

$$f(x) = (ax+b)(x-1)(x+1)(x-2)^2(x+2)^2$$

由

$$\lim_{x \rightarrow 1} \frac{f(x)}{x-1} = 36, \quad \lim_{x \rightarrow -1} \frac{f(x)}{x+1} = -36$$

得方程式

$$\begin{cases} (a+b) \cdot 2 \cdot 1^2 \cdot 3^2 = 36 \\ (-a+b) \cdot (-2) \cdot (-3)^2 \cdot 1^2 = -36 \end{cases} \Rightarrow a=0, b=2$$

因此

$$f(x) = 2(x^2-1)(x-2)^2(x+2)^2 \Rightarrow f(3) = 400$$

28. 若

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^4+3} - [A + B(x-1) + C(x-1)^2]}{(x-1)^2} = 0,$$

求常数 A, B, C 。

设

$$L = \lim_{x \rightarrow 1} \frac{\sqrt{x^4+3} - [A + B(x-1) + C(x-1)^2]}{(x-1)^2}$$

由于分母当 $x \rightarrow 1$ 时趋于 0, 而极限存在且为 0, 故分子必趋于 0:

$$\lim_{x \rightarrow 1} \left(\sqrt{x^4+3} - [A + B(x-1) + C(x-1)^2] \right) = \sqrt{1+3} - A = 2 - A = 0$$

解得

$$A = 2$$

此时原式变为 $\frac{0}{0}$ 型, 由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{\frac{2x^3}{\sqrt{x^4+3}} - B - 2C(x-1)}{2(x-1)} = 0$$

同样地, 由于分母趋于 0, 分子必趋于 0:

$$\lim_{x \rightarrow 1} \left(\frac{2x^3}{\sqrt{x^4+3}} - B - 2C(x-1) \right) = \frac{2}{2} - B = 1 - B = 0$$

解得

$$B = 1$$

再次应用洛必达法则处理该 $\frac{0}{0}$ 型极限:

$$L \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \left(\frac{2x^3}{\sqrt{x^4+3}} - 1 - 2C(x-1) \right)}{\frac{d}{dx} 2(x-1)} = 0$$

计算分子导数:

$$\frac{d}{dx} \left(\frac{2x^3}{\sqrt{x^4+3}} \right) = \frac{6x^2\sqrt{x^4+3} - 2x^3 \cdot \frac{4x^3}{2\sqrt{x^4+3}}}{x^4+3} = \frac{6x^2(x^4+3) - 4x^6}{(x^4+3)\sqrt{x^4+3}}$$

代入 $x = 1$ 得到

$$\frac{6(4) - 4}{4\sqrt{4}} = \frac{20}{8} = \frac{5}{2}$$

因此解得

$$\frac{\frac{5}{2} - 2C}{2} = 0 \Rightarrow C = \frac{5}{4}$$

综上, 常数分别为

$$A = 2, \quad B = 1, \quad C = \frac{5}{4}$$

由于极限式

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^4+3} - [A + B(x-1) + C(x-1)^2]}{(x-1)^2} = 0$$

符合函数 $f(x)$ 在 $x = 1$ 处的二阶泰勒展开定义, 其中多项式

$$P_2(x) = A + B(x-1) + C(x-1)^2$$

必须等于 $f(x)$ 的二阶泰勒多项式。设 $f(x) = \sqrt{x^4+3}$, 根据泰勒公式可知:

$$A = f(1), \quad B = f'(1), \quad C = \frac{f''(1)}{2!}$$

由

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x^4+3}} \cdot 4x^3 = \frac{2x^3}{\sqrt{x^4+3}} \\ f''(x) &= \frac{6x^2\sqrt{x^4+3} - 2x^3 \cdot \frac{2x^3}{\sqrt{x^4+3}}}{x^4+3} = \frac{6x^2(x^4+3) - 4x^6}{(x^4+3)\sqrt{x^4+3}} \end{aligned}$$

代入得

$$f(1) = 2, \quad f'(1) = 1, \quad f''(1) = \frac{5}{2}$$

因此

$$A = 2, \quad B = 1, \quad C = \frac{5}{4}$$

29.

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{3n^2 + 1}} + \frac{1}{\sqrt{3n^2 + 2}} + \cdots + \frac{1}{\sqrt{3n^2 + 2n}} \right)$$

令

$$L = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{3n^2 + 1}} + \frac{1}{\sqrt{3n^2 + 2}} + \cdots + \frac{1}{\sqrt{3n^2 + 2n}} \right)$$

由夹挤定理,

$$\frac{2}{\sqrt{3}} = \lim_{n \rightarrow \infty} \left(\frac{2n}{\sqrt{3n^2}} \right) < L < \lim_{n \rightarrow \infty} \left(\frac{2n}{\sqrt{3n^2 + 2n}} \right) = \frac{2}{\sqrt{3}} \Rightarrow L = \frac{2\sqrt{3}}{3}$$

30. 设 $a_n = \sqrt{1 \cdot 2} + \sqrt{2 \cdot 3} + \cdots + \sqrt{n(n+1)}$, 求 $\lim_{n \rightarrow \infty} \frac{a_n}{n^2}$ 的值。

发现到

$$\sum_{k=1}^n \sqrt{k^2} < a_n < \sum_{k=1}^n \sqrt{(k+1)^2} \Rightarrow \frac{n(n+1)}{2} < a_n < \frac{n(n+3)}{2}$$

于是有

$$\lim_{n \rightarrow \infty} \frac{n(n+1)}{2n^2} < \lim_{n \rightarrow \infty} \frac{a_n}{n^2} < \lim_{n \rightarrow \infty} \frac{n(n+3)}{2n^2} \Rightarrow \frac{1}{2} < \lim_{n \rightarrow \infty} \frac{a_n}{n^2} < \frac{1}{2}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{a_n}{n^2} = \frac{1}{2}$$

31. 设 $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$, 求 $\lim_{n \rightarrow \infty} a_n$ 之值。

由

$$(2n-1)(2n+1) < (2n)^2$$

令 $S = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, 则

$$(2n+1)S^2 = (1 \cdot 3)(3 \cdot 5)(5 \cdot 7) \cdots ((2n-3)(2n-1))((2n-1)(2n+1)) < 2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2$$

即

$$\sqrt{2n+1}S < 2 \cdot 4 \cdot 6 \cdots (2n)$$

于是

$$\Rightarrow 0 < a_n = \frac{S}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{2n+1}}$$

由夹挤定理,

$$0 < \lim_{n \rightarrow \infty} a_n < \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n+1}} = 0 \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

32. 求极限

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{kn}}$$

由放缩法,

$$\sqrt{k-1} + \sqrt{k} < 2\sqrt{k} < \sqrt{k} + \sqrt{k+1}$$

有理化得

$$2(\sqrt{k+1} - \sqrt{k}) < \frac{2}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1})$$

于是有

$$\frac{2}{\sqrt{n}} \sum_{k=1}^n (\sqrt{k+1} - \sqrt{k}) < \frac{2}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{k}} < \frac{2}{\sqrt{n}} \sum_{k=1}^n (\sqrt{k} - \sqrt{k-1})$$

累加得

$$\frac{2\sqrt{n+1} - 2}{\sqrt{n}} < \frac{2}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{k}} < \frac{2\sqrt{n}}{\sqrt{n}} = 2$$

因为

$$\lim_{n \rightarrow \infty} \frac{2\sqrt{n+1} - 2}{\sqrt{n}} = 2$$

由夹挤定理得,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{kn}} = 2$$

33. 求极限

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2 + n + 1} + \frac{2}{n^2 + n + 2} + \cdots + \frac{n}{n^2 + n + n} \right)$$

由放缩法,

$$\frac{1 + 2 + \cdots + n}{n^2 + n + n} < \sum_{k=1}^n \frac{k}{n^2 + n + k} < \frac{1 + 2 + \cdots + n}{n^2 + n + 1}$$

即

$$\frac{n(n+1)}{2(n^2 + 2n)} < \sum_{k=1}^n \frac{k}{n^2 + n + k} < \frac{n(n+1)}{2(n^2 + n + 1)}$$

由于

$$\lim_{n \rightarrow \infty} \frac{n(n+1)}{2(n^2 + 2n)} = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2(n^2 + n + 1)} = \frac{1}{2}$$

由夹挤定理得

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2 + n + k} = \frac{1}{2}$$

34. 求极限

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \sqrt{k^2 + n + k}$$

对于整数 $1 \leq k \leq n$, 有

$$\sqrt{k^2} < \sqrt{k^2 + n + k} < \sqrt{k^2 + 2n}$$

于是

$$\frac{1}{n^2} \sum_{k=1}^n k < \frac{1}{n^2} \sum_{k=1}^n \sqrt{k^2 + n + k} \leq \frac{1}{n^2} \sum_{k=1}^n (k + \sqrt{2n})$$

由于

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}$$

且

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n (k + \sqrt{2n}) &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k + \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \sqrt{2n} \\ &= \frac{1}{2} + \lim_{n \rightarrow \infty} \frac{n\sqrt{2n}}{n^2} \\ &= \frac{1}{2} \end{aligned}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \sqrt{k^2 + n + k} = \frac{1}{2}$$

35. 求极限

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+\sqrt{2}} + \frac{1}{n+\sqrt{3}} + \cdots + \frac{1}{n+\sqrt{n}} \right)$$

由放缩法得

$$\frac{n}{n+\sqrt{n}} < \sum_{k=1}^n \frac{1}{n+\sqrt{k}} < \frac{n}{n+1}$$

由于

$$\lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{1+1/n} = 1$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+\sqrt{2}} + \cdots + \frac{1}{n+\sqrt{n}} \right) = 1$$

36. 求极限

$$\lim_{n \rightarrow \infty} \frac{n^k \sin^2(n!)}{n+2}, \quad 0 < k < 1$$

由于

$$0 \leq \sin^2 x \leq 1, \quad \forall x \in \mathbb{R}$$

得到不等式

$$0 \leq \frac{n^k \sin^2(n!)}{n+2} \leq \frac{n^k}{n+2}$$

当 $0 < k < 1$ 时, 上界的极限为

$$\lim_{n \rightarrow \infty} \frac{n^k}{n+2} = 0$$

故由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{n^k \sin^2(n!)}{n+2} = 0$$

37. 求极限

$$\lim_{n \rightarrow \infty} \sqrt[n]{3^n + (2|\sin(n^n)|)^n}$$

由于

$$0 \leq |\sin(n^n)| \leq 1$$

得到上下界:

$$\sqrt[n]{3^n} \leq \sqrt[n]{3^n + (2|\sin(n^n)|)^n} \leq \sqrt[n]{3^n + 2^n} \leq \sqrt[n]{2 \cdot 3^n}$$

由于

$$\lim_{n \rightarrow \infty} \sqrt[n]{3^n} = \lim_{n \rightarrow \infty} \sqrt[n]{2 \cdot 3^n} = 3$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \sqrt[n]{3^n + (2|\sin(n^n)|)^n} = 3$$

38. 已知数列 $\{a_n\}$ 满足

$$\frac{2n^2 - 7}{4n + 5} < a_n < \frac{3n^2 + 8}{6n - 1}, \quad n \in \mathbb{Z}^+$$

求极限

$$\lim_{n \rightarrow \infty} \frac{3na_n}{(n+1)^2}$$

将不等式两边同时乘以 $\frac{3n}{(n+1)^2}$:

$$\frac{3n}{(n+1)^2} \cdot \frac{2n^2 - 7}{4n + 5} < \frac{3na_n}{(n+1)^2} < \frac{3n}{(n+1)^2} \cdot \frac{3n^2 + 8}{6n - 1}$$

取极限得

$$\frac{3}{2} = \lim_{n \rightarrow \infty} \frac{3n(2n^2 - 7)}{(4n + 5)(n + 1)^2} \leq \lim_{n \rightarrow \infty} \frac{3na_n}{(n + 1)^2} \leq \lim_{n \rightarrow \infty} \frac{3n(3n^2 + 8)}{(6n - 1)(n + 1)^2} = \frac{3}{2}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{3na_n}{(n+1)^2} = \frac{3}{2}$$

39. 已知 $x_1 = 1$, 且对每个正整数 $n \geq 2$,

$$n(x_n)^2 - x_{n-1} - n = 0, \quad x_n \geq 0.$$

求 $\lim_{n \rightarrow \infty} x_n$ 或证明 $\{x_n\}$ 发散。

由等式得对所有 $n \geq 2$, 有

$$x_n = \sqrt{1 + \frac{x_{n-1}}{n}}$$

首先证明 $x_n \leq 2$: 有 $x_1 = 1 \leq 2$ 。假设 $n \geq 2$ 有 $x_n \leq 2$ 成立, 则

$$x_n = \sqrt{1 + \frac{x_n}{n+1}} \leq \sqrt{1+2} < 2$$

故由数学归纳法得对所有 $n \geq 2$ 都皆有 $x_n \leq 2$, 故

$$1 \leq x_n = \sqrt{1 + \frac{x_{n-1}}{n}} \leq \sqrt{1 + \frac{2}{n}}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} x_n = 1$$

40. (a) 证明

$$\int_0^1 \left(1 + \sin \frac{\pi}{2} x\right)^n dx > \frac{2^{n+1} - 1}{n+1} \quad (n = 1, 2, \dots)$$

由

$$\sin x \geq \frac{2}{\pi} x, \quad 0 \leq x \leq \frac{\pi}{2}$$

得

$$\sin\left(\frac{\pi}{2}x\right) \geq x, \quad 0 \leq x \leq 1,$$

从而

$$\int_0^1 \left(1 + \sin \frac{\pi}{2} x\right)^n dx \geq \int_0^1 (1+x)^n dx = \frac{(1+x)^{n+1}}{n+1} \Big|_0^1 = \frac{2^{n+1} - 1}{n+1}.$$

因此原不等式成立。

(b) 求极限:

$$\lim_{n \rightarrow \infty} \left[\int_0^1 \left(1 + \sin \frac{\pi}{2} x\right)^n dx \right]^{\frac{1}{n}}$$

注意到:

$$\frac{2^n}{n+1} < \frac{2^{n+1}-1}{n+1} \leq \int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \leq \int_0^1 2^n dx = 2^n.$$

对上述不等式取 n 次方根, 得:

$$2(n+1)^{-\frac{1}{n}} < \left[\int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \right]^{\frac{1}{n}} \leq 2.$$

因为 $(n+1)^{-\frac{1}{n}} \rightarrow 1$, 由夹挤定理可得:

$$\lim_{n \rightarrow \infty} \left[\int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \right]^{\frac{1}{n}} = 2$$

41. (a) 设 n 是正整数, 计算

$$\int_0^{n\pi} x \sin^2 x dx$$

对非负整数 k ,

$$\int_{k\pi}^{(k+1)\pi} x \sin^2 x dx = \int_0^\pi (k\pi + x) \sin^2 x dx$$

利用恒等式 $\sin^2 x = \frac{1-\cos 2x}{2}$, 得:

$$= \frac{1}{2} \int_0^\pi (k\pi + x)(1 - \cos 2x) dx = \frac{1}{2} \left((k\pi + x) \Big|_0^\pi - \int_0^\pi (k\pi + x) \cos 2x dx \right)$$

由于 $\int_0^\pi \cos 2x dx = 0$ 且 $\int_0^\pi x \cos 2x dx = 0$, 所以最后只剩:

$$\int_{k\pi}^{(k+1)\pi} x \sin^2 x dx = \frac{1}{2} \int_0^\pi (k\pi + x) dx = \frac{1}{2} \left(k\pi^2 + \frac{\pi^2}{2} \right) = \frac{\pi^2(2k+1)}{4}$$

所以,

$$\int_0^{n\pi} x \sin^2 x dx = \sum_{k=0}^{n-1} \frac{\pi^2(2k+1)}{4} = \frac{\pi^2}{4} \sum_{k=0}^{n-1} (2k+1) = \frac{\pi^2}{4} \cdot n^2$$

因为 $\sum_{k=0}^{n-1} (2k+1) = n^2$, 所以原式为

$$\frac{\pi^2 n^2}{4}$$

(b) 证明对任何正实数 p , 函数极限

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} \int_0^x t |\sin t|^p dt$$

存在.

记

$$S_0 = \int_0^\pi |\sin t|^p dt, \quad S_1 = \int_0^\pi t |\sin t|^p dt.$$

对任意非负整数 k , 有

$$\int_{k\pi}^{(k+1)\pi} t |\sin t|^p dt = \int_0^\pi (k\pi + x) |\sin x|^p dx = k\pi S_0 + S_1.$$

若设 $x = (n + \alpha)\pi$ 且 $0 \leq \alpha < 1$, 则

$$\int_0^x t |\sin t|^p dt = \sum_{k=0}^{n-1} (k\pi S_0 + S_1) + \int_{n\pi}^x t |\sin t|^p dt.$$

利用估计 (注意 $\int_{n\pi}^x t |\sin t|^p dt \leq (n+1)\pi \cdot S_0$), 得上下界:

$$\frac{\frac{1}{2}n(n-1)\pi S_0 + nS_1}{(n+1)^2\pi^2} \leq \frac{1}{x^2} \int_0^x t |\sin t|^p dt \leq \frac{\frac{1}{2}n(n+1)\pi S_0 + (n+1)S_1}{n^2\pi^2}.$$

随着 $n \rightarrow \infty$, 上下界极限均趋于 $\frac{S_0}{2\pi}$, 所以由夹挤定理知极限存在, 且为

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} \int_0^x t |\sin t|^p dt = \frac{S_0}{2\pi}$$

42. (a) 求

$$\int_0^\pi \frac{1}{1 + \cos^2 x} dx$$

及

$$\int_0^\pi \frac{\sin^2 x}{1 + \cos^2 x} dx$$

设

$$I_1 = \int_0^\pi \frac{1}{1 + \cos^2 x} dx, \quad I_2 = \int_0^\pi \frac{\sin^2 x}{1 + \cos^2 x} dx.$$

发现

$$I_1 = \frac{1}{2} \int_0^\pi \frac{1 + \sin^2 x + \cos^2 x}{1 + \cos^2 x} dx = \frac{1}{2} (I_2 + \pi) \Rightarrow I_2 = 2I_1 - \pi$$

所以只需计算 I_1 即可得出两个积分。发现被积函数在 $x = \frac{\pi}{2}$ 处不连续, 将 I_1 写成

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos^2 x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{1 + \cos^2 x} dx$$

其中第一个积分变为

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{2 + \tan^2 x} dx = \int_0^{\infty} \frac{1}{2 + u^2} du = \left[\frac{1}{\sqrt{2}} \arctan \left(\frac{u}{\sqrt{2}} \right) \right]_0^{\infty} = \frac{1}{\sqrt{2}} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{2\sqrt{2}}.$$

对于第二个积分如下, 设 $x = \pi - y, dx = -dy$,

$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{1 + \cos^2 x} dx = \int_{\frac{\pi}{2}}^0 \frac{1}{1 + \cos^2(\pi - y)} (-dy) = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos^2 y} dy = \frac{\pi}{2\sqrt{2}}$$

所以

$$I_1 = \frac{\pi}{2\sqrt{2}} + \frac{\pi}{2\sqrt{2}} = \frac{\pi\sqrt{2}}{2}$$

$$I_2 = 2I_1 - \pi = 2 \cdot \frac{\pi\sqrt{2}}{2} - \pi = \pi(\sqrt{2} - 1)$$

(b) 证明

$$\lim_{x \rightarrow \infty} \frac{\int_0^x \frac{\sin^2 t}{1 + \cos^2 t} dt}{\int_0^x \frac{1}{1 + \cos^2 t} dt} = 2 - \sqrt{2}$$

取 $n = \left\lfloor \frac{x}{\pi} \right\rfloor$, 则

$$\int_0^x \frac{1}{1 + \cos^2 t} dt = \sum_{k=1}^n \int_{(k-1)\pi}^{k\pi} \frac{1}{1 + \cos^2 t} dt + \int_{k\pi}^x \frac{1}{1 + \cos^2 t} dt$$

于是

$$\frac{n\pi}{\sqrt{2}} \leq \int_0^x \frac{1}{1 + \cos^2 t} dt \leq \frac{(n+1)\pi}{\sqrt{2}}$$

同理

$$n\pi(\sqrt{2} - 1) \leq \int_0^x \frac{1}{1 + \cos^2 t} dt \leq (n+1)\pi(\sqrt{2} - 1)$$

于是

$$\frac{n}{n+1}(2 - \sqrt{2}) \leq \frac{\int_0^x \frac{\sin^2 t}{1 + \cos^2 t} dt}{\int_0^x \frac{1}{1 + \cos^2 t} dt} \leq \frac{n+1}{n}(2 - \sqrt{2}),$$

令 $x \rightarrow \infty$ 便得结论。

43. 对每个正整数 n , 设

$$R_n = \{(x, y) \mid 0 \leq x \leq n, 0 \leq y \leq \sqrt{x}\},$$

记 $N(n)$ 为 R_n 中坐标都是整数的点的个数。求

$$\lim_{n \rightarrow \infty} \frac{N(n)}{n^{\frac{3}{2}}}.$$

对每个正整数 n , 满足 $0 \leq y \leq \sqrt{k}$ 的整数 y 个数为 $[\sqrt{k}] + 1$, 所以

$$N(n) = \sum_{k=0}^n ([\sqrt{k}] + 1) = (n+1) + \sum_{k=1}^n [\sqrt{k}]$$

由于 $\sqrt{k} - 1 < [\sqrt{k}] \leq \sqrt{k}$, 并且

$$\int_0^n \sqrt{x} dx \leq \sum_{k=1}^n [\sqrt{k}] \leq \int_0^{n+1} \sqrt{x} dx$$

所以

$$1 + \int_0^n \sqrt{x} dx \leq N(n) \leq (n+1) + \int_0^{n+1} \sqrt{x} dx$$

即

$$1 + \frac{2}{3}n^{\frac{3}{2}} \leq N(n) \leq (n+1) + \frac{2}{3}(n+1)^{\frac{3}{2}}$$

由夹挤定理, 得到

$$\lim_{n \rightarrow \infty} \frac{N(n)}{n^{\frac{3}{2}}} = \frac{2}{3}$$

微分

考点: 导数、微分法则、链导法、切线与法线、函数的增减性、函数的极值、变率与相关变率、隐函数的微分法、曲线的凸向及拐点及曲线的渐近线、最值定理、介值定理、拉格朗日中值定理、罗尔定理、柯西中值定理、积分中值定理、达布定理

1. 已知 $f(x) = x^{2013} - x^{2012} + x^{2011} - x^{2010} + 1$, 求

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1-h)}{h}$$

计算导数:

$$f'(x) = 2013x^{2012} - 2012x^{2011} + 2011x^{2010} - 2010x^{2009}$$

于是

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(1+h) - f(1-h)}{h} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} + \lim_{h \rightarrow 0} \frac{f(1) - f(1-h)}{h} \\ &= f'(1) + f'(1) \\ &= 2 \cdot (2013 - 2012 + 2011 - 2010) \\ &= 4 \end{aligned}$$

2. 已知 $f'(1) = 12$, 求

$$\lim_{h \rightarrow 0} \frac{f(1+4h) - f(1-2h)}{3h}$$

改写

$$\begin{aligned} \frac{f(1+4h) - f(1-2h)}{3h} &= \frac{f(1+4h) - f(1)}{3h} + \frac{f(1) - f(1-2h)}{3h} \\ &= \frac{4}{3}f'(1) + \frac{2}{3}f'(1) \\ &= 24 \end{aligned}$$

3. 求

$$\lim_{h \rightarrow 0} \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \frac{2\sqrt{\sin x}}{\pi h} dx \right]$$

先提取常数

$$\lim_{h \rightarrow 0} \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \frac{2\sqrt{\sin x}}{\pi h} dx \right] = \frac{2}{\pi} \lim_{h \rightarrow 0} \left[\frac{1}{h} \int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \sqrt{\sin x} dx \right]$$

令

$$F(x) = \int \sqrt{\sin x} dx \Rightarrow F'(x) = \sqrt{\sin x}$$

则

$$\frac{1}{h} \int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \sqrt{\sin x} dx = \frac{F\left(\frac{\pi}{6}+h\right) - F\left(\frac{\pi}{6}\right)}{h}$$

当 $h \rightarrow 0$ 时, 上式为导数定义:

$$\lim_{h \rightarrow 0} \frac{F\left(\frac{\pi}{6}+h\right) - F\left(\frac{\pi}{6}\right)}{h} = F'\left(\frac{\pi}{6}\right) = \sqrt{\sin \frac{\pi}{6}}$$

因此原极限为

$$\frac{2}{\pi} \cdot \sqrt{\sin \frac{\pi}{6}} = \frac{2}{\pi} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\pi}$$

4. 由导数定义, 证明

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

由导数的定义,

$$\begin{aligned}
 \frac{d}{dx}(\sec x) &= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos x - \cos(x+h)}{h \cos x \cos(x+h)} \\
 &= \lim_{h \rightarrow 0} \frac{2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}}{h \cos x \cos(x+h)} \\
 &= \lim_{h \rightarrow 0} \left[\frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \frac{\sin\left(x + \frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\
 &= 1 \cdot \frac{\sin x}{\cos^2 x} \\
 &= \sec x \tan x
 \end{aligned}$$

5. 通过导数的定义, 求

$$f(x) = \frac{1}{\sqrt{x^2 - 1}}, \quad |x| > 1$$

的导数 $f'(x)$ 。

由导数的定义,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{(x+h)^2 - 1}} - \frac{1}{\sqrt{x^2 - 1}}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x^2 - 1} - \sqrt{(x+h)^2 - 1}}{h \sqrt{x^2 - 1} \sqrt{(x+h)^2 - 1}} \\
 &= \lim_{h \rightarrow 0} \frac{(x^2 - 1) - ((x+h)^2 - 1)}{h \sqrt{x^2 - 1} \sqrt{(x+h)^2 - 1} (\sqrt{x^2 - 1} + \sqrt{(x+h)^2 - 1})} \\
 &= \lim_{h \rightarrow 0} \frac{-2xh - h^2}{h \sqrt{x^2 - 1} \sqrt{(x+h)^2 - 1} (\sqrt{x^2 - 1} + \sqrt{(x+h)^2 - 1})} \\
 &= \lim_{h \rightarrow 0} \frac{-2x - h}{\sqrt{x^2 - 1} \sqrt{(x+h)^2 - 1} (\sqrt{x^2 - 1} + \sqrt{(x+h)^2 - 1})} \\
 &= \frac{-2x}{\sqrt{x^2 - 1} \sqrt{x^2 - 1} (2\sqrt{x^2 - 1})} \\
 &= -\frac{x}{(x^2 - 1)^{\frac{3}{2}}}
 \end{aligned}$$

6. 设

$$y = \frac{(x^4 + 2)(x^3 - 2)}{(x + 2)(x^2 - 2)},$$

求 $\frac{dy}{dx}$ 。

对等式两边取自然对数,

$$\ln y = \ln(x^4 + 2) + \ln(x^3 - 2) - \ln(x + 2) - \ln(x^2 - 2)$$

对 x 求导,

$$\frac{1}{y} \frac{dy}{dx} = \frac{4x^3}{x^4 + 2} + \frac{3x^2}{x^3 - 2} - \frac{1}{x + 2} - \frac{2x}{x^2 - 2}$$

整理得

$$\frac{dy}{dx} = \frac{4x^3(x^3 - 2)}{(x + 2)(x^2 - 2)} + \frac{3x^2(x^4 + 2)}{(x + 2)(x^2 - 2)} - \frac{(x^4 + 2)(x^3 - 2)}{(x + 2)^2(x^2 - 2)} - \frac{2x(x^4 + 2)(x^3 - 2)}{(x + 2)(x^2 - 2)^2}$$

7. 已知曲线方程

$$y = 2^{3e^{2x}}, \quad x \in \mathbb{R}.$$

以 y 表示 $\frac{dy}{dx}$ 。

由链导法,

$$\frac{dy}{dx} = 2^{3e^{2x}} \cdot \ln 2 \cdot 6e^{2x} = y \cdot \ln 2 \cdot 6e^{2x}$$

又由原式取自然对数得

$$\ln y = 3e^{2x} \ln 2 \Rightarrow 2 \ln y = 6e^{2x} \ln 2$$

因此得到

$$\frac{dy}{dx} = 2y \ln y$$

不如一开始就取 \ln ,

$$\ln y = 3e^{2x} \ln 2.$$

对 x 求导,

$$\frac{1}{y} \frac{dy}{dx} = \ln 2 \cdot 6e^{2x}$$

同样代入 $\ln y = 3e^{2x} \ln 2$ 得

$$\frac{dy}{dx} = y \cdot (\ln 2) \cdot 6e^{2x} = 2y \ln y$$

8. 已知曲线由方程

$$x^m y^n = (x+y)^{m+n}, \quad x \neq 0, y \neq 0, x+y \neq 0, my - nx \neq 0$$

隐式定义, 其中 m, n 为有理数。证明

$$\frac{dy}{dx} = \frac{y}{x}$$

两边取对数,

$$m \ln x + n \ln y = (m+n) \ln(x+y)$$

对 x 求导,

$$\frac{m}{x} + \frac{n}{y} \frac{dy}{dx} = \frac{m+n}{x+y} \left(1 + \frac{dy}{dx} \right)$$

整理得

$$\frac{my - nx}{x(x+y)} = \frac{my - nx}{y(x+y)} \frac{dy}{dx}$$

由于 $x+y \neq 0, my - nx \neq 0$, 故

$$\frac{dy}{dx} = \frac{y}{x}$$

9. 已知曲线 C 的隐函数方程为

$$y = xe^y, \quad x \neq 0, y \neq 1, y \neq 2.$$

证明

$$(1-y) \frac{d^2 y}{dx^2} = (2-y) \left(\frac{dy}{dx} \right)^2.$$

对 $y = xe^y$ 两边求导,

$$\frac{dy}{dx} = e^y + xe^y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} (1-y) = e^y$$

再次求导,

$$\frac{d^2y}{dx^2}(1-y) - \left(\frac{dy}{dx}\right)^2 = e^y \frac{dy}{dx}$$

代入 $e^y = \frac{dy}{dx}(1-y)$,

$$\frac{d^2y}{dx^2}(1-y) - \left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 (1-y)$$

即

$$\frac{d^2y}{dx^2}(1-y) = \left(\frac{dy}{dx}\right)^2 (2-y)$$

10. 求曲线 C

$$x^2 + 3xy - 2y^2 + 17 = 0$$

的极值。

设 $F(x, y) = x^2 + 3xy - 2y^2 + 17$ 。对方程 $F(x, y) = 0$ 两边求导,

$$2x + 3y + 3xy' - 4yy' = 0$$

整理得:

$$y' = -\frac{2x + 3y}{3x - 4y}$$

令 $y' = 0$, 得到

$$y = -\frac{2}{3}x$$

代入原方程解得

$$x^2 + 3x\left(-\frac{2}{3}x\right) - 2\left(-\frac{2}{3}x\right)^2 + 17 = 0 \Rightarrow x_1 = 3, x_2 = -3$$

得临界点 $P_1(3, -2), P_2(-3, 2)$ 。再求导得,

$$y''(3x - 4y) + y'(3 - 4y') = -(2 + 3y')$$

在临界点处 $y' = 0$, 化简得

$$y''(3x - 4y) = -2 \Rightarrow y'' = \frac{-2}{3x - 4y}$$

由于

$$y''(3, -2) = \frac{-2}{3(3) - 4(-2)} = -\frac{2}{17} < 0, \quad y''(-3, 2) = \frac{-2}{3(-3) - 4(2)} = \frac{2}{17} > 0$$

故 $y_1 = -2$ 是极大值, $y_2 = 2$ 是极小值。该曲线在 $x = 3$ 处取得极大值 -2 , 在 $x = -3$ 处取得极小值 2 。

11. 设函数 $f(x)$ 处处可导, 且 $f'(0) = 1$, 并对任意实数 x, h 恒有

$$f(x+h) = f(x) + f(h) + 2hx,$$

求 $f'(x)$ 。

已知 $\forall x, h \in \mathbb{R}$,

$$f(x+h) = f(x) + f(h) + 2hx$$

对 h 微分,

$$f'(x+h) = f'(h) + 2x$$

令 $h = 0$, 则

$$f'(x) = f'(0) + 2x = 1 + 2x$$

12. 设 $f(x) = x(x-1)(x-2)\cdots(x-2023)$, $g(x) = f(f(x))$, 求 $g'(1)$ 。

求导得

$$f'(x) = (x-1)P(x) + x(x-2)(x-3)\cdots(x-2023) = xQ(x) + (x-1)(x-2)\cdots(x-2023)$$

所以有

$$f(1) = 0, \quad f'(1) = 2022!, \quad f'(0) = -2023!$$

由于 $g(x) = f(f(x))$, 有

$$g'(x) = f'(f(x)) \cdot f'(x) \Rightarrow g'(1) = f'(f(1)) \cdot f'(1) = f'(0) \cdot 2022! = -2023! \cdot 2022!$$

13. 设 $f(x)$ 为正实函数且为可微分函数, 对任意实数 x, y , 满足 $f(x+y) = 2f(x)f(y)$, 若 $f'(0) = 2$, 试求 $\frac{f''(x)}{f(x)}$ 。

由 $f(x+y) = 2f(x)f(y)$ 代入 $x = y = 0$ 得

$$f(0) = 2f(0)f(0) \Rightarrow f(0) = \frac{1}{2}$$

考虑 $f'(x)$,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2f(x)f(h) - f(x)}{h} \\ &= 2f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = 2f(x)f'(0) = 2f(x) \cdot 2 = 4f(x) \end{aligned}$$

对两边求导得

$$f''(x) = 4f'(x) = 4 \cdot 4f(x) \Rightarrow \frac{f''(x)}{f(x)} = 16$$

14. 已知 $f(x)$ 在 $x=0$ 连续, 且

$$\lim_{x \rightarrow 0} \frac{\ln(f(x) + 2)}{x - \sin x} = 1,$$

求 $f'(0)$ 。

因为 $f(x)$ 在 $x=0$ 连续, 所以

$$\lim_{x \rightarrow 0} \ln(f(x) + 2) = \ln(f(0) + 2).$$

又因 $x - \sin x \rightarrow 0$, 分母趋于 0, 极限存在, 需有分子也趋于 0, 即

$$\ln(f(0) + 2) = 0 \Rightarrow f(0) = -1.$$

所以

$$\lim_{x \rightarrow 0} \frac{\ln(f(x) + 2)}{x - \sin x} = \lim_{x \rightarrow 0} \frac{f(x) + 1}{x - \sin x} \quad (\text{因 } \ln(1+y) \sim y).$$

又 $x - \sin x \sim \frac{x^3}{6}$, 于是

$$\lim_{x \rightarrow 0} \frac{f(x) + 1}{x^3} = \frac{1}{1/6} = 6.$$

因此

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(x) + 1}{x} = \lim_{x \rightarrow 0} \frac{f(x) + 1}{x^3} \cdot x^2 = 6 \cdot 0 = 0.$$

故 $f'(0) = \boxed{0}$ 。待验证

15. 已知 a, b 为实数, 若函数 $f(x) = 2ax^3 - 3ax^2 + b$ 在 $0 \leq x \leq 2$ 时有最大值 7, 最小值 -3, 求数对 (a, b) 。

导数为

$$f'(x) = 6ax^2 - 6ax = 6ax(x - 1)$$

临界点为 $x = 0, 1$, 二阶导为

$$f''(x) = 12ax - 6a = 6a(2x - 1)$$

当 $a > 0$, 则

$$f(x) \begin{cases} \text{递增} & x \geq 1 \\ \text{递减} & 0 \leq x \leq 1 \\ \text{递增} & x \leq 0 \end{cases}$$

极小值在 $x = 1$, 极大值在 $x = 2$ 或 $x = 0$, 解得

$$\begin{cases} f(2) = 4a + b = 7 \\ f(1) = -a + b = -3 \end{cases} \Rightarrow a = 2, b = -1$$

当 $a < 0$, 则

$$f(x) \begin{cases} \text{递减} & x \geq 1 \\ \text{递增} & 0 \leq x \leq 1 \\ \text{递减} & x \leq 0 \end{cases}$$

极大值在 $x = 1$, 极小值在 $x = 0$ 或 $x = 2$, 同理解得

$$\begin{cases} f(1) = -a + b = 7 \\ f(2) = 4a + b = -3 \end{cases} \Rightarrow a = -2, b = 5$$

$\therefore (a, b) = (2, -1)$ 或 $(-2, 5)$

16. 函数 f 定义为

$$f(n, y) = \sum_{x=1}^n \frac{x^2 y^x}{k}, \quad n \in \mathbb{N}, \quad y \in \mathbb{R}$$

其中

$$k = \sum_{r=1}^n r^2,$$

若已知数列求和公式

$$\sum_{r=1}^n r^4 = \frac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1),$$

证明

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n}{20(2n+1)^2}.$$

首先, 利用标准求和公式:

$$k = \sum_{x=1}^n x^2 = \frac{n(n+1)(2n+1)}{6}.$$

既然 k 与 x 无关, 可以将其作为常数因子提取到求和号外

$$f(n, y) = \frac{1}{k} \sum_{x=1}^n x^2 y^x.$$

对 y 求导,

$$\frac{df}{dy} = \frac{1}{k} \sum_{x=1}^n x^3 y^{x-1}, \quad \left. \frac{df}{dy} \right|_{y=1} = \frac{1}{k} \sum_{x=1}^n x^3.$$

再次对 y 求导,

$$\frac{d^2 f}{dy^2} = \frac{1}{k} \sum_{x=1}^n x^3 (x-1) y^{x-2} = \frac{1}{k} \sum_{x=1}^n x^4 y^{x-2} - \frac{1}{k} \sum_{x=1}^n x^3 y^{x-2}.$$

在 $y=1$ 时,

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{1}{k} \sum_{x=1}^n x^4 - \frac{1}{k^2} \left(\sum_{x=1}^n x^3 \right)^2.$$

代入求和公式得

$$\begin{aligned} & \frac{6}{n(n+1)(2n+1)} \cdot \frac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1) - \frac{36}{n^2(n+1)^2(2n+1)^2} \cdot \left[\frac{n^2(n+1)^2}{4} \right] \\ &= \frac{6n^3 + 9n^2 + n - 1}{5(2n+1)} - \frac{9}{4(2n+1)^2} \\ &= \frac{3n^4 + 6n^3 - n^2 - 4n}{20(2n+1)^2}. \end{aligned}$$

因此,

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n}{20(2n+1)^2},$$

得证。

17. (a) 设

$$f(x) = x^3 + x,$$

且 $g(x)$ 为 $f(x)$ 的反函数, 求 $g'(10)$ 。

由于 f 与 g 为反函数, $f(g(x)) = x$, 对两边求导得到

$$f'(g(x))g'(x) = 1$$

又因为 $f(2) = 10$, 所以 $g(10) = 2$ 。令 $x = 10$, 则

$$f'(2)g'(10) = 1$$

而 $f'(x) = 3x^2 + 1$, 所以 $f'(2) = 13$, 因此

$$g'(10) = \frac{1}{13}$$

(b) 对 $x > 0$, 定义

$$h(x) = \frac{1}{f(x)},$$

证明函数 $f(x) + h(x)$ 在 $x = g(1)$ 处取得最小值。

令

$$F(x) = f(x) + h(x) = f(x) + \frac{1}{f(x)}$$

则

$$F'(x) = f'(x) - \frac{f'(x)}{[f(x)]^2} = f'(x) \frac{[f(x)]^2 - 1}{[f(x)]^2}$$

因为 $f'(x) > 1$, 所以在临界点有 $f(x_0)^2 - 1 = 0$, 即 $f(x_0) = 1$, 因此 $x_0 = g(1)$, 再次求导,

$$F''(x) = f''(x) - \frac{[f(x)]^2 f''(x) - 2f(x)[f'(x)]^2}{[f(x)]^4}$$

在 $x = x_0$ 时, $f(x_0) = 1$, $f''(x_0) = 6g(1)$, $f'(x_0) = 3(g(1))^2 + 1$, 所以

$$F''(x_0) = 6g(1) - (6g(1) - 2[3(g(1))^2 + 1]^2) = 2[3(g(1))^2 + 1]^2 > 0$$

因此 $x = x_0$ 为极小值。由于

$$\lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow \infty} F(x) = \infty,$$

故 $x = g(1)$ 为 $F(x)$ 的最小值。

18. 已知对任意正整数 k , 方程

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$$

恰有一个正实根, 设该方程唯一正实根为 α_k , 试证该无穷数列 $\langle \alpha_k \rangle$ 收敛, 且

$$\lim_{k \rightarrow \infty} \alpha_k = 2$$

原方程可化为

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = x^k - (x^{k-1} + x^{k-2} + \cdots + 1) = x^k - \frac{x^k - 1}{x - 1}$$

令

$$f_k(x) = x^{k+1} - 2x^k + 1$$

则 α_k 为 $f_k(x) = 0$ 在 $(1, 2)$ 内的唯一正实根 (注意 $\alpha_k \neq 1$, 当 $k > 1$), 考虑

$$f_k(0) = 1, \quad f_k(1) = 0, \quad f_k(2) = 1$$

对 $f_k(x)$ 求导得

$$f'_k(x) = (k+1)x^k - 2kx^{k-1} = x^{k-1}[(k+1)x - 2k]$$

导函数零点为

$$x = 0 \quad \text{或} \quad x = \frac{2k}{k+1} = 2 - \frac{2}{k+1}$$

因此

$$f_k(x) \text{ 在 } \left(0, 2 - \frac{2}{k+1}\right) \text{ 上递减, } \left(2 - \frac{2}{k+1}, \infty\right) \text{ 上递增}$$

由于 $f_k(1) = 0$, $f_k(2) = 1$ 且 α_k 是唯一正实根, 结合单调性可得:

$$2 - \frac{2}{k+1} < \alpha_k < 2$$

故由夹挤定理,

$$\lim_{k \rightarrow \infty} \alpha_k = 2$$

因此 α_k 收敛, 且收敛至 2, 故得证。

19. 已知 $a < b < c$, $f'(x)$ 在 (a, c) 上严格递增, 且 $f(x)$ 在 $[a, c]$ 上连续, 证明

$$(b-a)f(c) + (c-b)f(a) > (c-a)f(b).$$

由拉格朗日中值定理, 因此存在 α 和 β 使得

$$\frac{f(c) - f(b)}{c - b} = f'(\beta), \quad b < \beta < c$$

以及

$$\frac{f(b) - f(a)}{b - a} = f'(\alpha), \quad a < \alpha < b$$

由于 f' 严格递增, 我们有 $f'(\beta) > f'(\alpha)$, 因此

$$\frac{f(c) - f(b)}{c - b} > \frac{f(b) - f(a)}{b - a}$$

即

$$(b - a)f(c) + (c - b)f(a) > (c - a)f(b)$$

20. 设 $f, g: \mathbb{R} \rightarrow \mathbb{R}$ 是连续函数, 且 g 可导, 若

$$(f(0) - g'(0))(g'(1) - f(1)) > 0,$$

证明存在实数 $c \in (0, 1)$, 使得 $f(c) = g'(c)$ 。

令

$$F(x) = \int_0^x f(t) dt, \quad h(x) = F(x) - g(x).$$

由 f 的连续性, F 可导且 $F' = f$, 因此

$$h'(x) = F'(x) - g'(x) = f(x) - g'(x).$$

故由 $(f(0) - g'(0))(g'(1) - f(1)) > 0$ 知

$$h'(0)(-h'(1)) > 0,$$

因此 $h'(0)$ 和 $h'(1)$ 异号。由达布定理, 存在 $c \in (0, 1)$ 使得

$$h'(c) = 0 \Rightarrow f(c) = g'(c).$$

21. 设 $f: (0, \infty) \rightarrow \mathbb{R}$ 为连续可微函数, $b > a > 0$ 且 $f(a) = f(b) = k$ 。证明存在 $\xi \in (a, b)$ 使得

$$f(\xi) - \xi f'(\xi) = k.$$

若考虑函数 $g(x) = \frac{f(x)}{x}$, 则

$$g'(x) = \frac{xf'(x) - f(x)}{x^2},$$

其中分子与已知正好对应。考虑在 $[a, b]$ 上的函数

$$g(x) = \frac{f(x)}{x}, \quad h(x) = \frac{1}{x}$$

由柯西中值定理, 存在 $\xi \in (a, b)$ 使得

$$\frac{g(a) - g(b)}{h(a) - h(b)} = \frac{g'(\xi)}{h'(\xi)}$$

注意到

$$\frac{g'(\xi)}{h'(\xi)} = \frac{\frac{\xi f'(\xi) - f(\xi)}{\xi^2}}{-\frac{1}{\xi^2}} = f(\xi) - \xi f'(\xi)$$

而

$$\frac{g(a) - g(b)}{h(a) - h(b)} = \frac{\frac{f(a)}{a} - \frac{f(b)}{b}}{\frac{1}{a} - \frac{1}{b}} = k$$

故原命题得证。

22. 设 $f: \mathbb{R} \rightarrow \mathbb{R}$ 为二阶可导函数, 且 $f(0) = 0$ 。证明: 存在 $\xi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ 使得

$$f''(\xi) = f(\xi)(1 + 2 \tan^2 \xi).$$

设 $g(x) = f(x) \cos x$, 由于

$$g\left(-\frac{\pi}{2}\right) = g(0) = g\left(\frac{\pi}{2}\right) = 0$$

由罗尔定理, 存在

$$\xi_1 \in \left(-\frac{\pi}{2}, 0\right), \quad \xi_2 \in \left(0, \frac{\pi}{2}\right)$$

使得

$$g'(\xi_1) = g'(\xi_2) = 0$$

考虑函数

$$h(x) = \frac{g'(x)}{\cos^2 x} = \frac{f'(x) \cos x - f(x) \sin x}{\cos^2 x}$$

显然有

$$h(\xi_1) = h(\xi_2) = 0$$

再由罗尔定理, 存在

$$\xi \in (\xi_1, \xi_2) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

使得 $h'(\xi) = 0$, h 导数为

$$\begin{aligned} 0 = h'(\xi) &= \frac{g''(\xi) \cos^2 \xi + 2 \cos \xi \sin \xi g'(\xi)}{\cos^4 \xi} \\ &= \frac{(f''(\xi) \cos \xi - 2f'(\xi) \sin \xi - f(\xi) \cos \xi) \cos \xi + 2 \sin \xi (f'(\xi) \cos \xi - f(\xi) \sin \xi)}{\cos^3 \xi} \\ &= \frac{f''(\xi) \cos^2 \xi - f(\xi)(\cos^2 \xi + 2 \sin^2 \xi)}{\cos^3 \xi} \end{aligned}$$

整理得

$$0 = \frac{1}{\cos \xi} (f''(\xi) - f(\xi)(1 + 2 \tan^2 \xi))$$

因此得证

$$f''(\xi) = f(\xi)(1 + 2 \tan^2 \xi)$$

23. 证明当 $0 < \theta < \frac{\pi}{4}$ 时, 有

$$\tan \theta < \frac{4\theta}{\pi}.$$

设

$$f(x) = \tan x, \quad x = \theta \in \left(0, \frac{\pi}{4}\right)$$

因为 f 在 $\left[0, \frac{\pi}{4}\right]$ 上连续, $\left(0, \frac{\pi}{4}\right)$ 上可导, 由拉格朗日中值定理, 存在

$$0 < \xi < \theta, \theta < \xi' < \frac{\pi}{4}$$

使得

$$\sec^2 \xi = \frac{\tan \theta - \tan 0}{\theta - 0}, \quad \sec^2 \xi' = \frac{\tan \frac{\pi}{4} - \tan \theta}{\frac{\pi}{4} - \theta}$$

由于 $\sec^2 x$ 在 $\left(0, \frac{\pi}{4}\right)$ 上单调递增, 因此 $\sec^2 \xi' > \sec^2 \xi$, 即

$$\frac{1 - \tan \theta}{\frac{\pi}{4} - \theta} > \frac{\tan \theta}{\theta} \Rightarrow \frac{4\theta}{\pi} > \tan \theta$$

令

$$f(\theta) = \frac{4\theta}{\pi} - \tan \theta.$$

则

$$f'(\theta) = \frac{4}{\pi} - \sec^2 \theta$$

因此可解得唯一的 $\xi \in (0, \frac{\pi}{4})$ 使得 $f'(\xi) = 0$, 又

$$f'(\theta) < 0, \forall \theta \in (0, \xi), \quad f'(\theta) > 0, \forall \theta \in (\xi, \frac{\pi}{4})$$

因此 f 在 $(0, \xi)$ 上严格递减, 在 $(\xi, \frac{\pi}{4})$ 上严格递增. 又 $f(0) = f(\frac{\pi}{4}) = 0$, 所以对所有 $(0, \frac{\pi}{4})$ 皆有 $f(\theta) < 0$, 即

$$\tan \theta < \frac{4\theta}{\pi}, \quad 0 < \theta < \frac{\pi}{4}$$

24. 证明对于所有非负整数 n , 有

$$2(3n-1)^n \geq (3n+1)^n$$

当 $n=0$ 时, 不等式显然成立, 因为

$$2(3 \cdot 0 - 1)^0 = 2 \cdot 1 = 2 \geq 1 = (3 \cdot 0 + 1)^0.$$

考虑 $n \geq 1$, 命题等价于

$$2(3n-1)^n \geq (3n+1)^n \iff \left(\frac{3n-1}{3n+1}\right)^n \geq \frac{1}{2} \iff n \ln \frac{3n-1}{3n+1} \geq -\ln 2$$

定义函数

$$f(x) = x \ln \frac{3x-1}{3x+1}, \quad x \geq 1.$$

对 $f(x)$ 求导,

$$f'(x) = \ln \frac{3x-1}{3x+1} + \frac{6x}{9x^2-1}, \quad f''(x) = \frac{6(3x^2-1)}{(9x^2-1)^2}$$

当 $x \geq 1$ 时, $f''(x) > 0$, 故 $f'(x)$ 在 $[1, +\infty)$ 上严格递增. 又因为

$$\lim_{x \rightarrow +\infty} f'(x) = \lim_{x \rightarrow +\infty} \left(\ln \frac{3-1/x}{3+1/x} + \frac{6/x}{9-1/x^2} \right) = 0$$

由 $f'(x)$ 严格递增且趋于 0 可知, 在 $[1, +\infty)$ 上 $f'(x) < 0$ 。因此 $f(x)$ 在 $[1, +\infty)$ 上严格递减。根据其极限:

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \ln \left(1 - \frac{2}{3x+1} \right)^x = -\frac{2}{3}$$

由于 $f(x)$ 严格递减且下界为 $-\frac{2}{3}$, 而 $-\frac{2}{3} > -0.693 \approx -\ln 2$ 。故对于所有 $n \geq 1$, 有 $f(n) > -\frac{2}{3} > -\ln 2$, 原不等式成立。(待验证)

25. 当 $x \in (0, \frac{\pi}{2})$, 比较 $\tan(\sin x)$ 及 $\sin(\tan x)$ 的大小。

设

$$f(x) = \tan(\sin x) - \sin(\tan x)$$

求导得,

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan x)}{\cos^2 x} = \frac{\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x)}{\cos^2 x \cdot \cos^2(\tan x)}$$

对 $0 < x < \arctan \frac{\pi}{2}$, 由与 \cos 在 $(0, \frac{\pi}{2})$ 上凸, 由琴生不等式及 AM-GM 不等式,

$$\sqrt[3]{\cos(\tan x) \cdot \cos^2(\sin x)} < \frac{1}{3} [\cos(\tan x) + 2 \cos(\sin x)] \leq \cos \left(\frac{\tan x + 2 \sin x}{3} \right) < \cos x$$

其中最后一个不等式由

$$\left(\frac{\tan x + 2 \sin x}{3} \right)' = \frac{1}{3} \left(\frac{1}{\cos^2 x} + 2 \cos x \right) \geq 1$$

得到, 据此有

$$\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x) > 0 \Rightarrow f'(x) > 0$$

因此 f 在 $\left[0, \arctan \frac{\pi}{2}\right]$ 上单调递增。又注意到

$$\tan \left(\sin \left(\arctan \frac{\pi}{2} \right) \right) = \tan \frac{\frac{\pi}{2}}{\sqrt{1 + \frac{\pi^2}{4}}} > \tan \frac{\pi}{4} = 1$$

因此对于 $x \in \left[\arctan \frac{\pi}{2}, \frac{\pi}{2}\right]$, 有 $\tan(\sin x) > 1$, 所以 $f(x) > 0$ 。综上, 对于所有 $x \in (0, \frac{\pi}{2})$,

$$\tan(\sin x) > \sin(\tan x).$$

26. 证明: 对半开区间 $\left(0, \frac{\pi}{2}\right]$ 内的任意 x , 有

$$\left(\frac{\sin x}{x} \right)^3 > \cos x.$$

在 $\left(0, \frac{\pi}{2}\right]$ 上, 有泰勒展开的不等式:

$$\sin x > x - \frac{x^3}{6}, \quad \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

因此

$$\left(\frac{\sin x}{x}\right)^3 > \left(1 - \frac{x^2}{6}\right)^3$$

展开右边并与 $\cos x$ 比较:

$$\left(1 - \frac{x^2}{6}\right)^3 = 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{216} > 1 - \frac{x^2}{2} + \frac{x^4}{24} > \cos x$$

其中

$$0 < x \leq \frac{\pi}{2} < 3 \Rightarrow x^2 < 9 \Rightarrow \frac{x^4}{12} - \frac{x^6}{216} > \frac{x^4}{24}$$

故原不等式得证。

27. 证明不等式: 当 $0 < x \leq 1$ 时, 有

$$\sin x + \arcsin x > 2x.$$

$\sin x$ 与 $\arcsin x$ 的麦克劳林展开式分别为

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \cdots$$

因此, 对任意 $0 < x \leq 1$, 存在 θ_1, θ_2 , 满足 $0 < \theta_1 < x, 0 < \theta_2 < x$, 使得

$$\sin x = x - \frac{x^3}{3!} + \frac{\theta_1^5}{5!}, \quad \arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\theta_2^5}{5}$$

于是

$$\sin x + \arcsin x = 2x + \frac{\theta_1^5}{5!} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\theta_2^5}{5} > 2x$$

故得证。

设

$$f(x) = \sin x + \arcsin x,$$

则 $f(0) = 0$, 且

$$f'(x) = \cos x + \frac{1}{\sqrt{1-x^2}},$$

从而

$$f'(0) = 2,$$

故直线 $y = 2x$ 是曲线 $y = f(x)$ 在点 $(0, 0)$ 处的切线。又

$$f''(x) = -\sin x + \frac{x}{(1-x^2)^{\frac{3}{2}}}.$$

当 $0 < x < 1$ 时, 有

$$\frac{x}{(1-x^2)^{\frac{3}{2}}} > \sin x,$$

因此 $f''(x) > 0$, 函数 $f(x)$ 在区间 $(0, 1)$ 上上凸, 即 $f(x)$ 在 $(0, 1]$ 上位于其在点 $(0, 0)$ 处的切线 $y = 2x$ 之上, 故得证

$$\sin x + \arcsin x > 2x$$

28. 设 $f: \mathbb{R} \rightarrow \mathbb{R}$ 为二次可微函数, 满足 $f(0) = 1$, $f'(0) = 0$, 且对所有 $x \in [0, \infty)$ 有

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

证明对所有 $x \in [0, \infty)$ 有

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

设 $x \in [0, \infty)$, 令 $g(x) = f'(x) - 2f(x)$. 则

$$g'(x) - 3g(x) \geq 0$$

即

$$(g(x)e^{-3x})' \geq 0$$

因此

$$g(x)e^{-3x} \geq g(0) = f'(0) - 2f(0) = -2$$

即

$$f'(x) - 2f(x) \geq -2e^{3x}$$

两边同乘 e^{-2x} 并整理得

$$(f(x)e^{-2x})' \geq -2e^x$$

即

$$(f(x)e^{-2x} + 2e^x)' \geq 0$$

于是

$$f(x)e^{-2x} + 2e^x \geq f(0) + 2 = 3$$

即

$$f(x) \geq 3e^{2x} - 2e^{3x}$$

29. 设函数

$$F(x) = e^{-x} - \left(1 - \frac{x}{n}\right)^n,$$

证明: 当 $n \geq 2$ 且 $x \in [0, n]$ 时,

$$0 \leq F(x) \leq \frac{e^{-1}}{n}.$$

函数 $F(x)$ 在区间 $[0, n]$ 上连续且可微, 因此在该区间内必有最大值与最小值, 且只能出现在端点或临界点处。有 $F(0) = 1 - 1 = 0$, $F(n) = e^{-n}$, 且

$$F'(x) = -e^{-x} + \left(1 - \frac{x}{n}\right)^{n-1}$$

先证明 $F(x) \geq 0$ 。只需证明

$$\left(1 - \frac{x}{n}\right)^n \leq e^{-x}, \quad x \in [0, n]$$

等价于

$$0 \leq 1 - \frac{x}{n} \leq e^{-x/n}, \quad x \in [0, n]$$

令 $t = \frac{x}{n}$, 则 $t \in [0, 1]$, 上述不等式化为

$$1 - t \leq e^{-t}, \quad t \in [0, 1]$$

这是显然成立的, 因为直线 $y = 1 - t$ 是曲线 $y = e^{-t}$ 在 $(0, 1)$ 处的切线, 而 $y = e^{-t}$ 在 $[0, 1]$ 上向下凸, 因此图像始终位于其切线之上, 故 $F(x) \geq 0$, 最小值在 $x = 0$ 处取得。接下来确定最大值, 由于

$$F'(n) = -e^{-n} < 0,$$

最大值不可能出现在 $x = n$, 因此必存在 $x_0 \in (0, n)$ 使得 $F'(x_0) = 0$, 这意味

$$e^{-x_0} = \left(1 - \frac{x_0}{n}\right)^{n-1}$$

于是

$$\begin{aligned} F(x_0) &= e^{-x_0} - \left(1 - \frac{x_0}{n}\right)^n \\ &= e^{-x_0} - \left(1 - \frac{x_0}{n}\right)^{n-1} \left(1 - \frac{x_0}{n}\right) \\ &= e^{-x_0} - e^{-x_0} \left(1 - \frac{x_0}{n}\right) \\ &= e^{-x_0} \frac{x_0}{n} \end{aligned}$$

设 $g(x) = xe^{-x}$, 可推导得知 $g(x)$ 在 $x = 1$ 处取得最大值 e^{-1} , 因此

$$F(x_0) \leq \frac{e^{-1}}{n}$$

综上所述 $n \geq 2$ 且 $x \in [0, n]$ 时,

$$0 \leq F(x) \leq \frac{e^{-1}}{n}$$

30. 已知在 $(-\infty, +\infty)$ 上具有二阶连续导数的函数 $f(x)$ 满足方程

$$x^2 f''(x) - 2x \sin x f'(x) = e^x + e^{-x} - 2,$$

若 $f(x)$ 在 $x = a$ 处取极值, 问 $f(a)$ 是函数 $f(x)$ 的极大值还是极小值? 请说明理由。

因为 $f(x)$ 在 $x = a$ 处取极值, 所以

$$f'(a) = 0.$$

代入原方程得

$$a^2 f''(a) = e^a + e^{-a} - 2.$$

由 AM-GM 不等式,

$$e^a + e^{-a} \geq 2,$$

等号成立当且仅当 $a = 0$, 所以

$$a^2 f''(a) = e^a + e^{-a} - 2 \geq 0,$$

若 $a \neq 0$, 则 $a^2 > 0$, 因此

$$f''(a) = \frac{e^a + e^{-a} - 2}{a^2} > 0,$$

所以 $f(x)$ 在 $x = a$ 处为极小值; 若 $a = 0$, 则

$$f''(0) = \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{x^2}$$

由洛必达法则,

$$f''(0) = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{2} = 1 > 0.$$

因此 $f''(0) > 0$ 仍为极小值, 故 $f(a)$ 是函数 $f(x)$ 的极小值。

31. 已知函数 $f(t)$ 在 $[a, x]$ 上可微, 且 $f'(t)$ 可微。对所有 $x > a$, 存在 c_x 满足 $a < c_x < x$ 且

$$\int_a^x f(t) dt = f(c_x)(x - a).$$

假设 $f'(a) \neq 0$, 证明

$$\lim_{x \rightarrow a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

设

$$F(x) = \int_a^x f(t) dt$$

利用 $F(x)$ 的泰勒展开, 有

$$F(x) = F(a) + (x - a)F'(a) + \frac{(x - a)^2}{2}F''(\theta_x)$$

其中 $a < \theta_x < x$, 且当 $x \rightarrow a$ 时, $\theta_x \rightarrow a$ 。又 $F(a) = 0$, $F'(x) = f(x)$, $F''(x) = f'(x)$, 所以

$$F(x) = 0 + (x - a)f(a) + \frac{(x - a)^2}{2}f'(\theta_x)$$

由定义有

$$f(c_x) = \frac{F(x)}{x - a} = f(a) + \frac{x - a}{2}f'(\theta_x)$$

因此

$$\frac{f(c_x) - f(a)}{x - a} = \frac{1}{2}f'(\theta_x)$$

另一方面可以写成

$$\frac{f(c_x) - f(a)}{x - a} = \frac{f(c_x) - f(a)}{c_x - a} \cdot \frac{c_x - a}{x - a}$$

取极限得到

$$\lim_{x \rightarrow a} \frac{1}{2} f'(\theta_x) = \lim_{x \rightarrow a} \frac{f(c_x) - f(a)}{c_x - a} \cdot \lim_{x \rightarrow a} \frac{c_x - a}{x - a}$$

由此可得

$$\frac{1}{2} f'(a) = f'(a) \cdot \lim_{x \rightarrow a} \frac{c_x - a}{x - a}$$

即

$$\lim_{x \rightarrow a} \frac{c_x - a}{x - a} = \frac{1}{2}$$

32. (a) 已知 $f(x)$ 在 $[0, +\infty)$ 上连续且单调增加, 且 $f(0) \geq 0$, 证明:

$$F(x) = \begin{cases} \frac{1}{x^n} \int_0^x t^{n-1} f(t) dt, & x > 0, \\ 0, & x = 0 \end{cases}$$

在 $[0, +\infty)$ 上连续且单调增加, 其中 $n > 0$ 。

先证连续性。由于 f 在 $[0, +\infty)$ 上连续, 且 $f(0) \geq 0$, 有:

$$\lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow 0^+} \frac{1}{x^n} \int_0^x t^{n-1} f(t) dt = 0 = F(0),$$

所以 $F(x)$ 在 $x = 0$ 处连续。

再证单调性。对于 $x > 0$, 由积分中值定理可得:

$$\int_0^x t^{n-1} f(t) dt = f(\xi) \int_0^x t^{n-1} dt = f(\xi) \cdot \frac{x^n}{n}, \quad \text{其中 } \xi \in (0, x).$$

因此,

$$F(x) = \frac{1}{x^n} \cdot \int_0^x t^{n-1} f(t) dt = \frac{1}{x^n} \cdot f(\xi) \cdot \frac{x^n}{n} = \frac{f(\xi)}{n}.$$

因为 f 单调递增, $\xi \in (0, x)$, 所以 $x \uparrow \Rightarrow \xi \uparrow \Rightarrow f(\xi) \uparrow$, 故 $F(x)$ 单调递增。

(b) 设 $f(x)$ 在 $[0, 1]$ 上二阶可导, 且满足 $|f''(x)| \leq 1$ 。已知 $f(x)$ 在 $(0, 1)$ 内取最大值为 $\frac{1}{4}$, 证明:

$$|f(0)| + |f(1)| \leq 1.$$

设 $x_0 \in (0, 1)$, 使得 $f(x_0) = \frac{1}{4}$, 且 $f'(x_0) = 0$ 。对 $f(0)$ 做 Taylor 展开 (Lagrange 余项形式) 得:

$$f(0) = f(x_0) + f'(x_0)(0 - x_0) + \frac{1}{2} f''(\xi)(0 - x_0)^2 = \frac{1}{4} + \frac{1}{2} f''(\xi) x_0^2,$$

其中 $\xi \in (0, x_0)$ 。同理,

$$f(1) = \frac{1}{4} + \frac{1}{2}f''(\eta)(1-x_0)^2, \quad \eta \in (x_0, 1).$$

因为 $|f''(x)| \leq 1$, 故

$$|f(0)| \leq \left| \frac{1}{4} + \frac{1}{2}f''(\xi)x_0^2 \right| \leq \frac{1}{4} + \frac{1}{2}x_0^2,$$

$$|f(1)| \leq \left| \frac{1}{4} + \frac{1}{2}f''(\eta)(1-x_0)^2 \right| \leq \frac{1}{4} + \frac{1}{2}(1-x_0)^2.$$

相加得:

$$|f(0)| + |f(1)| \leq \frac{1}{2} + \frac{1}{2}(x_0^2 + (1-x_0)^2).$$

注意 $x_0^2 + (1-x_0)^2 = 1 - 2x_0(1-x_0) \leq 1$, 因此:

$$|f(0)| + |f(1)| \leq \frac{1}{2} + \frac{1}{2} \cdot 1 = 1.$$

33. 设函数 $f(x)$ 在 $x=0$ 处可导, 且 $f(0)=0, f'(0)=1$, 设

$$F(x) = \int_0^x t^{n-1} f(x^n - t^n) dt,$$

求

$$\lim_{x \rightarrow 0} \frac{F(x)}{x^{2n}}.$$

由题设 $f(0)=0$, 且 f 在 0 可导, 考虑换元 $t=xu$, 则

$$F(x) = \int_0^x t^{n-1} f(x^n - t^n) dt = x^n \int_0^1 u^{n-1} f(x^n - x^n u^n) du.$$

注意 $x^n - x^n u^n = x^n(1-u^n)$, 故

$$F(x) = x^n \int_0^1 u^{n-1} f(x^n(1-u^n)) du.$$

因 $f(0)=0$, 且 f 在 0 处可导, 令 $h=x^n(1-u^n)$, 有

$$f(h) = f(0) + f'(0)h + o(h) = f'(0)x^n(1-u^n) + o(x^n).$$

所以

$$F(x) = x^n \int_0^1 u^{n-1} (f'(0)x^n(1-u^n) + o(x^n)) du = f'(0)x^{2n} \int_0^1 u^{n-1}(1-u^n) du + o(x^{2n}).$$

计算积分:

$$\int_0^1 u^{n-1}(1-u^n)du = \int_0^1 u^{n-1}du - \int_0^1 u^{2n-1}du = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

故

$$\lim_{x \rightarrow 0} \frac{F(x)}{x^{2n}} = \frac{1}{2n}$$

积分

考点: 不定积分、定积分、面积计算、旋转体的体积、直线运动问题

1. 已知 $p > 0$, 证明

$$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \frac{1}{p+1}.$$

设

$$S_n = \sum_{k=1}^n k^p$$

则

$$\frac{S_n}{n^{p+1}} = \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^p.$$

这是函数 $f(x) = x^p$ 在 $[0, 1]$ 的黎曼和, 故

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{p+1}} = \int_0^1 x^p dx = \frac{1}{p+1}.$$

2. 求

$$\lim_{k \rightarrow \infty} \left(\frac{2017^{\frac{1}{k}}}{k+1} + \frac{2017^{\frac{2}{k}}}{k+\frac{1}{2}} + \cdots + \frac{2017^{\frac{k}{k}}}{k+\frac{1}{k}} \right)$$

写成黎曼和:

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k 2017^{\frac{j}{k}} = \int_0^1 2017^x dx = \left[\frac{2017^x}{\ln 2017} \right]_0^1 = \frac{2016}{\ln 2017}.$$

(待验证, $2017^{\frac{j}{k}}/(1+0?)$)

3. 设

$$a_n = \frac{2}{n} \left[(2^2 + 1) + \left(2 + \frac{2}{n}\right)^2 + 1 + \cdots + \left(2 + \frac{2n-2}{n}\right)^2 + 1 \right]$$

求 $\lim_{n \rightarrow \infty} a_n$

发现

$$a_n = \sum_{k=1}^n \frac{2}{n} \left[\left(2 + \frac{2k-2}{n} \right)^2 + 1 \right].$$

是函数 $f(x) = (2+x)^2 + 1$ 在 $[0, 2]$ 上的黎曼和, 故

$$\lim_{n \rightarrow \infty} a_n = \int_0^2 [(2+x)^2 + 1] dx = \left[\frac{x^3}{3} + 2x^2 + 5x \right]_0^2 = \frac{62}{3}.$$

4. 试求

$$\lim_{n \rightarrow \infty} \frac{5}{n^2} \left[\sqrt{4n^2 - 2 \cdot 1^2} + \sqrt{4n^2 - 2 \cdot 2^2} + \cdots + \sqrt{4n^2 - 2 \cdot n^2} \right]$$

将其写成黎曼和, 得

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{5}{n} \sqrt{4 - 2 \left(\frac{k}{n} \right)^2} = \int_0^1 5\sqrt{4 - 2x^2} dx.$$

可得原极限为

$$\left[\frac{5}{\sqrt{2}} \left(x\sqrt{2 - x^2} + 2 \sin^{-1} \frac{x}{\sqrt{2}} \right) \right]_0^1 = \frac{5\sqrt{2}(\pi + 2)}{4}.$$

5. 设函数 $f: (0, \infty) \rightarrow \mathbb{R}$, 且当 $x > 0$ 时, 恒有

$$\lim_{n \rightarrow \infty} \frac{1}{3n-1} \left(f\left(\frac{x^2}{n}\right) + f\left(\frac{2x^2}{n}\right) + \cdots + f\left(\frac{nx^2}{n}\right) \right) = \frac{\sqrt[3]{7+x}}{x},$$

求 $f(1)$ 。

先写成黎曼和:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{3n-1} \left(f\left(\frac{x^2}{n}\right) + f\left(\frac{2x^2}{n}\right) + \cdots + f\left(\frac{nx^2}{n}\right) \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{3n-1} \sum_{k=1}^n f\left(\frac{kx^2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{n}{3n-1} \sum_{k=1}^n \frac{1}{n} f\left(\frac{kx^2}{n}\right) \\ &= \frac{1}{3} \int_0^1 f(x^2 t) dt = \frac{\sqrt[3]{7+x}}{x}. \end{aligned}$$

于是

$$\int_0^1 x^2 f(x^2 t) dt = 3x \sqrt[3]{7+x}.$$

取 $u = x^2 t \Rightarrow du = x^2 dt$, 则

$$\begin{aligned} \int_0^{x^2} f(u) du &= 3x \sqrt[3]{7+x}, \\ \frac{d}{dx} \left(\int_0^{x^2} f(u) du \right) &= \frac{d}{dx} \left(3x \sqrt[3]{7+x} \right), \\ 2x f(x^2) &= 3 \sqrt[3]{7+x} + x(7+x)^{-\frac{2}{3}}. \end{aligned}$$

令 $x = 1$, 则

$$2f(1) = 3 \cdot 2 + \frac{1}{4} \Rightarrow f(1) = \frac{25}{8}.$$

6.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sin \frac{\pi}{12n} + \sin \frac{3\pi}{12n} + \sin \frac{5\pi}{12n} + \cdots + \sin \frac{(2n-1)\pi}{12n} \right)$$

写成黎曼和,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sin \frac{(2k-1)\pi}{12n} &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{k=1}^{2n} \sin \frac{k\pi}{12n} - \sum_{k=1}^n \sin \frac{2k\pi}{12n} \right) \\ &= \int_0^2 \sin \frac{\pi}{12} x dx - \int_0^1 \sin \frac{\pi}{6} x dx \end{aligned}$$

可得

$$\left[-\frac{12}{\pi} \cos \frac{\pi}{12} x \right]_0^2 - \left[-\frac{6}{\pi} \cos \frac{\pi}{6} x \right]_0^1 = \frac{6-3\sqrt{3}}{\pi}$$

7. 求

$$f(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{x(1-x)}{k + (n-k)x}, \quad x \in [0, 1]$$

显然 $f(0) = f(1) = 0$, 对于 $x \in [0, 1]$,

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{x(1-x)}{k + (n-k)x} &= x(1-x) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\left(\frac{k}{n}\right)(1-x) + x} \cdot \frac{1}{n} \\ &= x(1-x) \int_0^1 \frac{1}{(1-x)y + x} dy \\ &= x \cdot [\ln |(1-x)y + x|]_0^1 \\ &= -x \ln(x)\end{aligned}$$

8. 求极限

$$\lim_{n \rightarrow \infty} \left(\sum_{k=10}^{n+9} \frac{2^{11(k-9)/n}}{\log_2 e^{n/11}} - \sum_{k=0}^{n-1} \frac{58}{\pi \sqrt{(n-k)(n+k)}} \right)$$

首先有

$$\lim_{n \rightarrow \infty} \sum_{k=10}^{n+9} \frac{2^{11(k-9)/n}}{\log_2 e^{n/11}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n 2^{11\left(\frac{k}{n}\right)} \ln \left(\frac{11}{n} \right) \ln 2 = \int_0^{11} 2^x \ln 2 \, dx$$

且

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{58}{\pi \sqrt{(n-k)(n+k)}} = \frac{58}{\pi} \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{1}{n \sqrt{1 - \left(\frac{k}{n}\right)^2}} = \frac{58}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx$$

故所求为

$$\int_0^{11} 2^x \ln 2 \, dx - \frac{58}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = [2^x]_0^{11} - \frac{58}{\pi} [\arcsin x]_0^1 = 2047 - 29 = 2018$$

9. 求极限

$$\lim_{t \rightarrow 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n},$$

注意到

$$\lim_{t \rightarrow 1^-} \frac{1-t}{-\ln t} = 1$$

因此

$$\lim_{t \rightarrow 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n} = \lim_{t \rightarrow 1^-} (-\ln t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n} = (-\ln t) \sum_{n=1}^{\infty} \frac{1}{1+e^{-n \ln t}}$$

设 $h = -\ln t$, 则当 $t \rightarrow 1^-$ 时 $h \rightarrow 0^+$, 于是上述极限化为

$$\lim_{h \rightarrow 0^+} h \sum_{n=1}^{\infty} \frac{1}{1 + e^{nh}}$$

这是一个黎曼和, 对应的积分为

$$\int_0^{\infty} \frac{dx}{1 + e^x} = \ln 2$$

10. 对于 $n = 1, 2, \dots$, 定义

$$S_n = \log \left(\sqrt[n^2]{1^1 \cdot 2^2 \cdot \dots \cdot n^n} \right) - \log(\sqrt{n}),$$

其中 \log 表示自然对数。求 $\lim_{n \rightarrow \infty} S_n$ 。

将 S_n 变形如下:

$$\begin{aligned} S_n &= \frac{1}{n^2} \sum_{k=1}^n k \log k - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n} \left(\log \frac{k}{n} + \log n \right) \right) - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} + \frac{\log n}{n^2} \sum_{k=1}^n k - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} + \frac{\log n}{n^2} \cdot \frac{n(n+1)}{2} - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} + \frac{\log n}{2n} \end{aligned}$$

当 $n \rightarrow \infty$ 时, 最后一项 $\frac{\log n}{2n}$ 趋于 0, 而第一项是函数 $f(x) = x \log x$ 在区间 $[0, 1]$ 上的黎曼和, 因此

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} = \int_0^1 x \log x \, dx$$

利用分部积分法计算得

$$\int_0^1 x \log x \, dx = \left[\frac{x^2}{2} \log x - \frac{x^2}{4} \right]_0^1 = -\frac{1}{4}$$

所以极限 $\lim_{n \rightarrow \infty} S_n$ 存在, 且

$$\lim_{n \rightarrow \infty} S_n = -\frac{1}{4}$$

11. 求极限

$$\lim_{n \rightarrow \infty} \frac{(1^2 + 2^2 + \cdots + n^2)(1^5 + 2^5 + \cdots + n^5)}{(1^3 + 2^3 + \cdots + n^3)(1^4 + 2^4 + \cdots + n^4)}$$

首先看到

$$a_n = \frac{\left(\sum_{k=1}^n \left(\frac{k}{n}\right)^2\right) \left(\sum_{k=1}^n \left(\frac{k}{n}\right)^5\right)}{\left(\sum_{k=1}^n \left(\frac{k}{n}\right)^3\right) \left(\sum_{k=1}^n \left(\frac{k}{n}\right)^4\right)} = \frac{\left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^2\right) \left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^5\right)}{\left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^3\right) \left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^4\right)}$$

故

$$\lim_{n \rightarrow \infty} a_n = \frac{\left(\int_0^1 x^2 dx\right) \left(\int_0^1 x^5 dx\right)}{\left(\int_0^1 x^3 dx\right) \left(\int_0^1 x^4 dx\right)} = \frac{\frac{1}{3} \cdot \frac{1}{6}}{\frac{1}{4} \cdot \frac{1}{5}} = \frac{10}{9}$$

12. 计算极限

$$\lim_{n \rightarrow \infty} \frac{(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n})^2 (1^3 + 2^3 + \cdots + n^3)}{(\sqrt[3]{1} + \sqrt[3]{2} + \cdots + \sqrt[3]{n})^3 (1^2 + 2^2 + \cdots + n^2)}$$

由

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

可得

$$f(n) = \frac{n^2 \left(\sum_{k=1}^n \frac{1}{n} \sqrt{\frac{k}{n}}\right)^2}{n^3 \left(\sum_{k=1}^n \frac{1}{n} \sqrt[3]{\frac{k}{n}}\right)^3} \cdot \frac{3n(n+1)}{2(2n+1)}$$

由积分定义,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sqrt{\frac{k}{n}} = \int_0^1 \sqrt{x} dx = \frac{2}{3}, \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sqrt[3]{\frac{k}{n}} = \int_0^1 x^{\frac{1}{3}} dx = \frac{3}{4}$$

因此

$$\lim_{n \rightarrow \infty} f(n) = \frac{\left(\frac{2}{3}\right)^2}{\left(\frac{3}{4}\right)^3} \cdot \frac{3}{4} = \frac{64}{81}$$

13. 求极限

$$L = \lim_{n \rightarrow \infty} \left(\frac{(2n)!}{n! n^n} \right)^{\frac{1}{n}}$$

首先有

$$\ln L = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\ln(2n)! - \ln n! - n \ln n \right)$$

其中

$$\ln(2n)! - \ln n! = \sum_{k=n+1}^{2n} \ln k = \sum_{k=1}^n \ln(n+k)$$

因此

$$\ln L = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{k=1}^n \ln(n+k) - n \ln n \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left(1 + \frac{k}{n} \right)$$

这是在 $[0, 1]$ 上的黎曼和, 得

$$\ln L = \int_0^1 \ln(1+x) dx = \left[(1+x) \ln(1+x) - x \right]_0^1 = 2 \ln 2 - 1$$

所以

$$L = e^{2 \ln 2 - 1} = \frac{4}{e}$$

14. 设

$$A_n = \{\sin(\ln 1), \sin(\ln 2), \dots, \sin(\ln n)\}, \quad B_n = \{\cos(\ln 1), \cos(\ln 2), \dots, \cos(\ln n)\}$$

且 \bar{a}_n, \bar{b}_n 分别为 A_n, B_n 的算术平均值. 求

$$\lim_{n \rightarrow \infty} (\bar{a}_n \cos(\ln n) - \bar{b}_n \sin(\ln n)).$$

由积分定义,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{i=1}^n (\sin(\ln i) \cos(\ln n) - \cos(\ln i) \sin(\ln n)) \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{i=1}^n \sin(\ln i - \ln n) \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \sin \left(\ln \left(\frac{i}{n} \right) \right) \\
 &= \int_0^1 \sin(\ln x) dx \\
 &= \Im \int_0^1 (e^{i \ln x}) dx \\
 &= \Im \int_0^1 x^i dx \\
 &= \Im \left(\frac{1}{i+1} \right) = -\frac{1}{2}
 \end{aligned}$$

15. 设函数 $f(x)$ 在闭区间 $[0, 1]$ 上具有连续导数, $f(0) = 0, f(1) = 1$ 。证明:

$$\lim_{n \rightarrow \infty} n \left(\int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right) = -\frac{1}{2}.$$

考察括号内的表达式:

$$\begin{aligned}
 & \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \\
 &= \sum_{k=1}^n \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x) dx - \frac{1}{n} f\left(\frac{k}{n}\right) \right) \\
 &= \sum_{k=1}^n \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(f\left(\frac{k}{n}\right) + \left(x - \frac{k}{n}\right) f'\left(\frac{k}{n}\right) + o\left(x - \frac{k}{n}\right) \right) dx - \frac{1}{n} f\left(\frac{k}{n}\right) \right) \\
 &= \sum_{k=1}^n \left(\left[f\left(\frac{k}{n}\right) x + \frac{f'\left(\frac{k}{n}\right)}{2} \left(x - \frac{k}{n}\right)^2 + o\left(\left(x - \frac{k}{n}\right)^2\right) \right]_{\frac{k-1}{n}}^{\frac{k}{n}} - \frac{1}{n} f\left(\frac{k}{n}\right) \right) \\
 &= \sum_{k=1}^n \left(-\frac{f'\left(\frac{k}{n}\right)}{2n^2} + o\left(\frac{1}{n^2}\right) \right)
 \end{aligned}$$

将上述结果乘以 n 并代入原极限中, 利用黎曼和定义可得原式为

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(-\frac{1}{2} \cdot \frac{1}{n} \sum_{k=1}^n f' \left(\frac{k}{n} \right) + o \left(\frac{1}{n} \right) \right) &= -\frac{1}{2} \int_0^1 f'(x) dx \\ &= -\frac{1}{2} [f(x)]_0^1 \\ &= -\frac{1}{2} (1 - 0) = -\frac{1}{2}\end{aligned}$$

证毕。

16. 证明

$$\frac{\pi}{4} < \int_0^1 \frac{1}{1+x^8} dx < 1$$

在 $x \in (0, 1)$ 上, 有 $0 < x^8 < 1$, 于是

$$\frac{1}{1+x^8} < 1 \implies \int_0^1 \frac{1}{1+x^8} dx < \int_0^1 1 dx = 1$$

又 $x^8 < x^2$, 所以

$$\frac{1}{1+x^8} > \frac{1}{1+x^2} \implies \int_0^1 \frac{dx}{1+x^8} > \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

17. 若 $f(x)$ 为一实系数多项式函数, 已知

$$\int_0^1 f(x)f'(x) dx = 1, \quad \int_0^1 (f(x))^3 f'(x) dx = 2,$$

求

$$\int_0^1 (f(x))^5 f'(x) dx$$

的值。

有

$$1 = \int_0^1 f(x)f'(x) dx = \left[\frac{1}{2}(f(x))^2 \right]_0^1 = \frac{1}{2}([f(1)]^2 - [f(0)]^2) \quad (1)$$

同理,

$$2 = \int_0^1 (f(x))^3 f'(x) dx = \left[\frac{1}{4}(f(x))^4 \right]_0^1 = \frac{1}{4}([f(1)]^4 - [f(0)]^4) \quad (2)$$

由 (1), (2) 解得

$$[f(0)]^2 = 1, \quad [f(1)]^2 = 3$$

于是

$$\int_0^1 (f(x))^5 f'(x) dx = \left[\frac{1}{6} (f(x))^6 \right]_0^1 = \frac{1}{6} (f(1)^6 - f(0)^6) = \frac{13}{3}$$

18. 设 f 为实数系上的连续函数, 若

$$\frac{d}{dx} \int_{-x}^x f(t) dt = x^2 + 1,$$

求

$$\int_{-1}^1 f(x) dx$$

两边积分得

$$g(x) = \int_{-x}^x f(t) dt = \int (x^2 + 1) dx = \frac{1}{3}x^3 + x + C$$

令 $x = 0$, 得

$$g(0) = \int_0^0 f(t) dt = 0 \Rightarrow C = 0 \Rightarrow g(x) = \frac{1}{3}x^3 + x$$

所以

$$\int_{-1}^1 f(x) dx = g(1) = \frac{4}{3}$$

19. 已知一连续实函数 $f(x)$ 满足 $f(2x) = 3f(x), \forall x \in \mathbb{R}$, 若 $\int_0^1 f(x) dx = 1$, 求

$$\int_0^2 f(x) dx$$

由 $f(2x) = 3f(x)$ 得

$$\int_0^1 f(x) dx = \frac{1}{3} \int_0^1 f(2x) dx = 1$$

令 $u = 2x$, 则 $du = 2dx$, 于是

$$\int_0^2 f(u) du = 6$$

故

$$\int_0^2 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx = 1 + \int_1^2 f(x) dx = 6 \Rightarrow \int_1^2 f(x) dx = 5$$

20. 设多项式函数 $y = f(x)$ 满足

$$f(x) = 4x^3 - 12x^2 + 8x + 20 - \int_1^x f(t) dt,$$

求 $f(x)$ 。

对两边求导得

$$f'(x) + f(x) = 12x^2 - 24x + 8$$

故 $f(x)$ 为二次多项式, 设

$$f(x) = ax^2 + bx + c, f'(x) = 2ax + b$$

代入得

$$f(x) + f'(x) = ax^2 + bx + c + 2ax + b = ax^2 + (b + 2a)x + (b + c)$$

比较系数, 解得

$$a = 12, b = -48, c = 56 \Rightarrow f(x) = 12x^2 - 48x + 56$$

21. 设

$$f(x) = \int_0^x g(t) dt + 1, \quad g(x) = 12x^2 - 6x + \int_0^1 [f(t) + g'(t)] dt,$$

求 $g(0)$ 。

由题意有

$$f(0) = 1, \quad g(0) = \int_0^1 [f(t) + g'(t)] dt,$$

又

$$f(x) = \int_0^x [12t^2 - 6t + g(0)] dt + 1 = 4x^3 - 3x^2 + xg(0) + 1$$

代入 $g(x)$ 得

$$g(x) = 12x^2 - 6x + \int_0^1 (4t^3 - 3t^2 + tg(0) + 1) dt + g(1) - g(0)$$

其中

$$\int_0^1 (4t^3 - 3t^2 + tg(0) + 1)dt = \left[t^4 - t^3 + \frac{1}{2}g(0)t^2 + t \right]_0^1 = \frac{1}{2}g(0) + 1$$

因此

$$g(x) = 12x^2 - 6x + 1 + g(1) - \frac{1}{2}g(0)$$

令 $x = 1$, 得

$$g(0) = 14$$

22. 设

$$f(x) = x + 3 + \int_0^x g(t) dt, \quad g(x) = 2x - 9 + \int_0^x f(t) dt,$$

试求 $f(3)$ 。

设

$$a = \int_0^1 g(x) dx, \quad b = \int_0^2 f(x) dx.$$

由题得

$$f(x) = x + 3 + a, \quad g(x) = 2x - 9 + b.$$

因此

$$b = \int_0^2 f(x) dx = \int_0^2 (x + 3 + a) dx = \left[\frac{1}{2}x^2 + (3 + a)x \right]_0^2 = 8 + 2a,$$

$$a = \int_0^1 g(x) dx = \int_0^1 (2x - 9 + b) dx = \left[x^2 + (b - 9)x \right]_0^1 = b - 8.$$

联立得

$$b = 8 + 2a, \quad a = b - 8.$$

代入消元得

$$b = 8 + 2(b - 8) \Rightarrow b = 8, \quad a = b - 8 = 0.$$

所以

$$f(x) = x + 3 + a = x + 3.$$

因此

$$f(3) = 3 + 3 = 6.$$

(待验证, 为什么设 a, b ?)

23. 设 $0 < a < b$, 证明

$$\int_a^b (x^2 + 1)e^{-x^2} dx > e^{-a^2} - e^{-b^2}.$$

令

$$f(x) = \int_0^x (t^2 + 1)e^{-t^2} dt, \quad g(x) = -e^{-x^2}$$

两函数在 $(0, \infty)$ 上均递增。由柯西中值定理, 存在 $x \in (a, b)$ 使得

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)} = \frac{(x^2 + 1)e^{-x^2}}{2xe^{-x^2}} = \frac{1}{2} \left(x + \frac{1}{x} \right) \geq \sqrt{x \cdot \frac{1}{x}} = 1$$

于是

$$\int_a^b (x^2 + 1)e^{-x^2} dx = f(b) - f(a) \geq g(b) - g(a) = e^{-a^2} - e^{-b^2}$$

由不等式 $(x - 1)^2 \geq 0 \Rightarrow x^2 + 1 \geq 2x$,

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geq \int_a^b 2xe^{-x^2} dx = [-e^{-x^2}]_a^b = e^{-a^2} - e^{-b^2}$$

24. 设 f 是定义在区间 $[0, 1]$ 上的连续实值函数。证明存在 $\xi \in [0, 1]$, 使得

$$\int_0^1 x^2 f(x) dx = \frac{1}{3} f(\xi).$$

由于 f 连续, 由最值定理, f 必在 $[0, 1]$ 上取得最小值和最大值, 分别设之为 $f(a), f(b)$, 其中 $a, b \in [0, 1]$, 于是

$$f(a) \int_0^1 x^2 dx \leq \int_0^1 x^2 f(x) dx \leq f(b) \int_0^1 x^2 dx,$$

即

$$f(a) \leq 3 \int_0^1 x^2 f(x) dx \leq f(b).$$

由拉格朗日中值定理, 存在 $\xi \in [0, 1]$, 使得

$$f(\xi) = 3 \int_0^1 x^2 f(x) dx.$$

即得证命题。

25. 设 $f: [0, 1] \rightarrow (0, \infty)$ 是可积函数, 且对所有 $x \in [0, 1]$ 有

$$f(x) \cdot f(1-x) = 1,$$

证明

$$\int_0^1 f(x) dx \geqslant 1.$$

对任意 $x \in [0, 1]$, 由 AM-GM 不等式,

$$f(x) + f(1-x) \geqslant 2\sqrt{f(x)f(1-x)} = 2$$

在区间 $[0, \frac{1}{2}]$ 上积分得

$$\int_0^1 f(x) dx = \int_0^{\frac{1}{2}} f(x) dx + \int_0^{\frac{1}{2}} f(1-x) dx = \int_0^{\frac{1}{2}} (f(x) + f(1-x)) dx \geqslant \int_0^{\frac{1}{2}} 2 dx = 1$$

由条件可得

$$\int_0^1 f(x) dx = \int_0^1 f(1-x) dx = \int_0^1 \frac{1}{f(x)} dx$$

于是由柯西不等式,

$$\left(\int_0^1 f(x) dx \right)^2 = \int_0^1 f(x) dx \cdot \int_0^1 \frac{1}{f(x)} dx \geqslant \left(\int_0^1 1 dx \right)^2 = 1$$

因此

$$\int_0^1 f(x) dx \geqslant 1$$

26. 设 $f: (-1, 1) \rightarrow \mathbb{R}$ 是二阶可导函数, 满足

$$2f'(x) + xf''(x) \geqslant 1, \quad x \in (-1, 1).$$

证明

$$\int_{-1}^1 xf(x) dx \geqslant \frac{1}{3}.$$

令

$$g(x) = xf(x) - \frac{x^2}{2}$$

则

$$g''(x) = 2f'(x) + xf''(x) - 1 \geq 0$$

因此 g 是凸函数, 以 g 在 $x=0$ 处的切线估计 g 。设 $g'(0) = a$, 则由凸性知

$$g(x) \geq g(0) + g'(0)x = ax$$

于是

$$\int_{-1}^1 xf(x) dx = \int_{-1}^1 \left(g(x) + \frac{x^2}{2} \right) dx \geq \int_{-1}^1 \left(ax + \frac{x^2}{2} \right) dx = \frac{1}{3}$$

这就证明了结论。

27. 设 f 为闭区间 $[0, 1]$ 上的实值连续函数。已知

$$\int_0^1 f(x) dx = \frac{\pi}{4},$$

证明存在 y 满足 $0 < y < 1$ 且

$$\frac{1}{1+y} < f(y) < \frac{1}{2y}.$$

注意到

$$\int_0^1 \frac{1}{1+x^2} dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

令

$$g(x) = f(x) - \frac{1}{1+x^2}$$

若 $g(x) \equiv 0 \quad \forall x \in (0, 1)$, 则 $f(y) = \frac{1}{1+y^2}$ 对所有 $y \in (0, 1)$ 都成立; 反之, 由于

$$\int_0^1 g(x) dx = 0$$

存在 $x_0, x_1 \in (0, 1)$ 使得 $g(x_0) < 0$ 且 $g(x_1) > 0$ 。由 g 在 $(0, 1)$ 上连续, 由介值定理, 存在 y 满足 $x_0 < y < x_1$ 使得 $g(y) = 0$, 于是 $f(y) = \frac{1}{1+y^2}$ 。

综上, 皆存在 $y \in (0, 1)$ 使得 $f(y) = \frac{1}{1+y^2}$ 。由于 $0 < y < 1$, 有

$$2y < 1 + y^2 < 1 + y$$

从而得到

$$\frac{1}{1+y} < f(y) < \frac{1}{2y}$$

证毕。

28. 设函数 f 在区间 $[a, b]$ 上具有连续导数, 且满足 $f(a) = f(b) = 0$ 。证明

$$\max_{a \leq x \leq b} |f'(x)| \geq \frac{4}{(b-a)^2} \left| \int_a^b f(x) dx \right|.$$

由拉格朗日中值定理, 对任意 $x \in (a, b)$, 存在 $\xi_1 \in (a, x), \xi_2 \in (x, b)$ 使得

$$f(x) - f(a) = f'(\xi_1)(x-a), \quad f(x) - f(b) = f'(\xi_2)(x-b)$$

设

$$M = \max_{a \leq x \leq b} |f'(x)|$$

则

$$|f(x)| \leq M(x-a), \quad |f(x)| \leq M(b-x).$$

因此

$$\begin{aligned} \frac{4}{(b-a)^2} \int_a^b |f(x)| dx &\leq \frac{4}{(b-a)^2} \left(\int_a^{(a+b)/2} M(x-a) dx + \int_{(a+b)/2}^b M(b-x) dx \right) \\ &= \frac{4}{(b-a)^2} \left(\frac{(b-a)^2}{8} M + \frac{(b-a)^2}{8} M \right) = M \end{aligned}$$

又因为

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx,$$

于是得证

$$\max_{a \leq x \leq b} |f'(x)| \geq \frac{4}{(b-a)^2} \left| \int_a^b f(x) dx \right|$$

29. 设 a 为实数, 已知函数

$$f(x) = 4x^2 - 3ax + 4 \int_0^1 (tf(t)) dt,$$

及

$$g(x) = x^2 + 4x + a - \int_0^x ((t+1)g'(t)) dt.$$

若方程 $f(x) - x \cdot g(x) = 0$ 有两相异实根 α, β , 且 $\alpha < \beta$, 求

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} (3x^2 - 2ax + a^2) dx$$

的最小值。

设常数

$$C = \int_0^1 t f(t) dt,$$

则

$$f(x) = 4x^2 - 3ax + 4C.$$

计算 C :

$$C = \int_0^1 t(4t^2 - 3at + 4C) dt = \int_0^1 (4t^3 - 3at^2 + 4Ct) dt = [t^4 - at^3 + 2Ct^2]_0^1 = 1 - a + 2C.$$

整理得

$$C = 1 - a + 2C \implies C = a - 1.$$

因此

$$f(x) = 4x^2 - 3ax + 4a - 4.$$

对 $g(x)$, 由条件有

$$g'(x) = 2x + 4 - (x + 1)g'(x),$$

整理得

$$g'(x) + (x + 1)g'(x) = 2x + 4 \implies (x + 2)g'(x) = 2x + 4,$$

即

$$g'(x) = \frac{2x + 4}{x + 2} = 2.$$

计算积分

$$\int_0^x (t + 1)g'(t) dt = \int_0^x 2(t + 1) dt = x^2 + 2x,$$

代入 $g(x)$:

$$g(x) = x^2 + 4x + a - (x^2 + 2x) = 2x + a.$$

因此

$$f(x) - xg(x) = (4x^2 - 3ax + 4a - 4) - x(2x + a) = 4x^2 - 3ax + 4a - 4 - 2x^2 - ax = 2x^2 - 4ax + 4a - 4.$$

设两根为 α, β , 满足

$$\alpha + \beta = 2a, \quad \alpha\beta = 2a - 2.$$

计算

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} (3x^2 - 2ax + a^2) dx = \frac{1}{\beta - \alpha} [x^3 - ax^2 + a^2x]_{\alpha}^{\beta} = \frac{1}{\beta - \alpha} (\beta^3 - \alpha^3 - a(\beta^2 - \alpha^2) + a^2(\beta - \alpha)).$$

利用因式分解:

$$\beta^3 - \alpha^3 = (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2), \quad \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha),$$

因此表达式为

$$(\beta^2 + \alpha\beta + \alpha^2) - a(\beta + \alpha) + a^2.$$

利用已知根的和与积:

$$\beta + \alpha = 2a, \quad \alpha\beta = 2a - 2,$$

以及

$$\beta^2 + \alpha^2 = (\beta + \alpha)^2 - 2\alpha\beta = (2a)^2 - 2(2a - 2) = 4a^2 - 4a + 4.$$

所以

$$\beta^2 + \alpha\beta + \alpha^2 = (\beta^2 + \alpha^2) + \alpha\beta = (4a^2 - 4a + 4) + (2a - 2) = 4a^2 - 2a + 2.$$

代入原式:

$$4a^2 - 2a + 2 - a \cdot 2a + a^2 = 4a^2 - 2a + 2 - 2a^2 + a^2 = 3a^2 - 2a + 2.$$

令

$$h(a) = 3a^2 - 2a + 2 = 3\left(a - \frac{1}{3}\right)^2 + \frac{5}{3}.$$

因此最小值为

$$\frac{5}{3}.$$

(待验证)

30. 已知

$$\int_0^2 f(x) dx = 1, \quad f(2) = \frac{1}{2}, \quad f'(2) = 0,$$

计算

$$\int_0^1 x^2 f''(2x) dx$$

进行变量代换, 令 $t = 2x$, 则 $dt = 2dx$, 于是

$$I = \int_0^1 x^2 f''(2x) dx = \frac{1}{2} \int_0^2 \left(\frac{t}{2}\right)^2 f''(t) dt = \frac{1}{8} \int_0^2 t^2 f''(t) dt$$

由分部积分法,

$$I = \frac{1}{8} \left([t^2 f'(t)]_0^2 - \int_0^2 2t f'(t) dt \right) = -\frac{1}{4} \int_0^2 t f'(t) dt$$

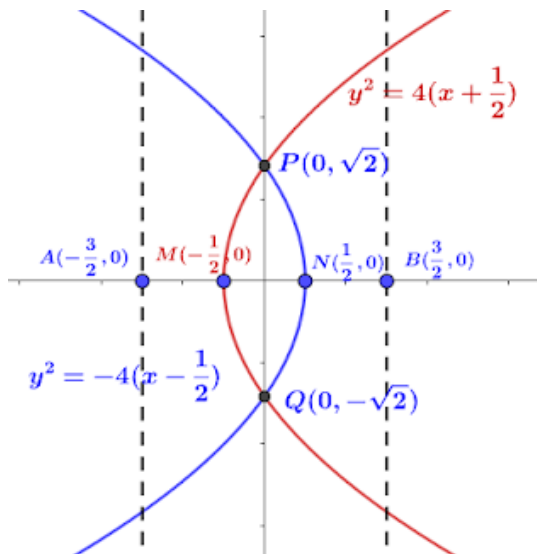
再由分部积分法,

$$I = -\frac{1}{4} ([t f(t)]_0^2 - \int_0^2 f(t) dt) = -\frac{1}{4} (2 \cdot \frac{1}{2} - 1) = 0$$

31. 坐标平面上, A, B 两点分别在直线

$$L_1: x = -\frac{3}{2}, \quad L_2: x = \frac{3}{2}$$

上。已知 $AB \perp L_1$, 且 M, N 为 AB 的三等分点, 并满足 $AM = MN = NB$ 。设 Γ_1 为以 L_1 为准线, N 为焦点的抛物线; Γ_2 为以 L_2 为准线, N 为顶点的抛物线, 求 Γ_1 与 Γ_2 所围区域的面积。



所围面积为

$$I = 4 \int_0^{\frac{1}{2}} \sqrt{-4 \left(x - \frac{1}{2} \right)} dx = 8 \int_0^{\frac{1}{2}} \sqrt{\frac{1}{2} - x} dx$$

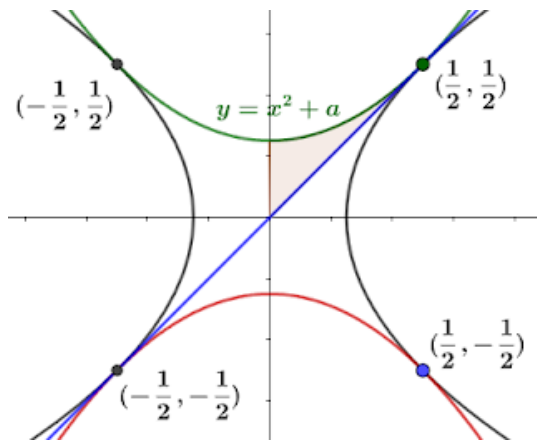
令 $u = \frac{1}{2} - x, du = -dx$, 则

$$I = 8 \int_0^{\frac{1}{2}} \sqrt{u} du = 8 \left[\frac{2}{3} u^{\frac{3}{2}} \right]_0^{\frac{1}{2}} = \frac{4\sqrt{2}}{3}$$

32. 已知四条抛物线

$$\Gamma_1: y = x^2 + a, \quad \Gamma_2: y = -x^2 - a, \quad \Gamma_3: y^2 = x - a, \quad \Gamma_4: y^2 = -x - a,$$

其中 a 为正实数, 若任相邻两条抛物线均相切, 试求这四条抛物线所围成之区域面积。



Γ_1, Γ_3 的共切点在 $x = y$ 上, 故知

$$x^2 - x + a = 0$$

恰有一根, 其中判别式为

$$1 - 4a = 0 \Rightarrow a = \frac{1}{4}$$

故四个切点为

$$\left(\frac{1}{2}, \frac{1}{2} \right), \left(\frac{1}{2}, -\frac{1}{2} \right), \left(-\frac{1}{2}, \frac{1}{2} \right), \left(-\frac{1}{2}, -\frac{1}{2} \right),$$

所围面积为

$$8 \int_0^{\frac{1}{2}} \left(x^2 - x + \frac{1}{4} \right) dx = 8 \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4} \right]_0^{\frac{1}{2}} = \frac{1}{3}$$

33. 已知曲线

$$y^2 + 2xy + 2x^2 = 50,$$

求被 x 轴和曲线上 $y \geq 0$ 部分围成的区域面积。

将曲线改写为

$$y^2 + 2xy + 2x^2 = 50 \Rightarrow y = -x \pm \sqrt{50 - x^2}$$

可知曲线与坐标轴交于 $(0, \pm 5\sqrt{2})$, $(\pm 5, 0)$, 设曲线与某垂直线切于点 P , 对曲线求导,

$$4x + 2y + 2x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x + y}{x + y}$$

此时 $\frac{dy}{dx}$ 不存在, 故有

$$x + y = 0 \Rightarrow y = -x$$

代入曲线方程解得

$$2x^2 + 2x(-x) + (-x)^2 = 50 \Rightarrow x = -5\sqrt{2}$$

所以垂直切线点为 $P = (-5\sqrt{2}, 5\sqrt{2})$, 故所围面积为

$$A = \int_{-5\sqrt{2}}^5 (-x + \sqrt{50 - x^2}) dx - \int_{-5\sqrt{2}}^{-5} (-x - \sqrt{50 - x^2}) dx$$

其中由换元 $x = \sqrt{50} \sin \theta$ 知不定积分

$$\begin{aligned} \int \sqrt{50 - x^2} dx &= \int \sqrt{50 - \sin^2 \theta} \cdot \sqrt{50} \cos \theta d\theta \\ &= 50 \int \cos^2 \theta d\theta \\ &= 25 \int (1 + \cos 2\theta) d\theta \\ &= 25 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \end{aligned}$$

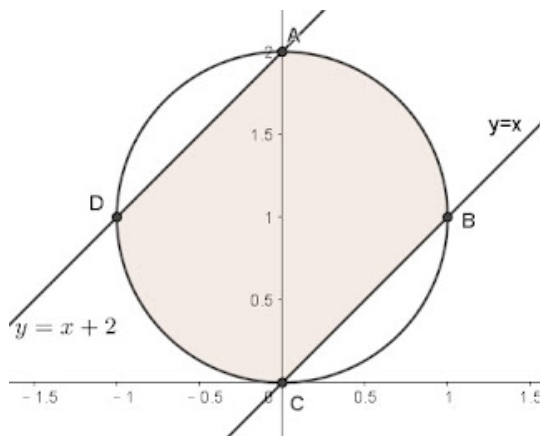
于是

$$\begin{aligned} A &= \left[-\frac{1}{2}x^2 \right]_{-5\sqrt{2}}^5 + \left[\frac{1}{2}x^2 \right]_{-5\sqrt{2}}^{-5} + \left[25\theta + \frac{25 \sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + \left[25\theta + \frac{25 \sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \\ &= 25\pi \end{aligned}$$

34. 坐标平面上, 满足联立不等式

$$\begin{cases} x^2 + (y-1)^2 \leq 1 \\ x-y \leq 0 \\ x-y \geq -2 \end{cases}$$

的解区域为 S , 求区域 S 绕 x 轴旋转一圈所得立体的体积。



联立 $x^2 + (y-1)^2 = 1$ 与 $y = x + 2$ 得

$$A(0, 2), D(-1, 1)$$

联立 $x^2 + (y-1)^2 = 1$ 与 $y = x$ 得

$$B(1, 1), C(0, 0)$$

如上图, 左半部绕 x 轴旋转体的体积为

$$\begin{aligned} V_1 &= \pi \int_{-1}^0 \left[(x+2)^2 - (1 - \sqrt{1-x^2})^2 \right] dx \\ &= \pi \int_{-1}^0 \left[2x^2 + 4x + 2 + 2\sqrt{1-x^2} \right] dx \\ &= \pi \left[\frac{2}{3}x^3 + 2x^2 + 2x + (\sqrt{1-x^2} + \arcsin x) \right]_{-1}^0 = \frac{1}{2}\pi^2 + \frac{5}{3}\pi \end{aligned}$$

上图右半部绕 x 轴旋转体的体积为

$$\begin{aligned} V_2 &= \pi \int_0^1 \left[(1 + \sqrt{1-x^2})^2 - x^2 \right] dx \\ &= \pi \int_0^1 \left[2 - 2x^2 + 2\sqrt{1-x^2} \right] dx \\ &= \pi \left[2x - \frac{2}{3}x^3 + \sqrt{1-x^2} + \arcsin x \right]_0^1 = \frac{1}{2}\pi^2 + \frac{1}{3}\pi \end{aligned}$$

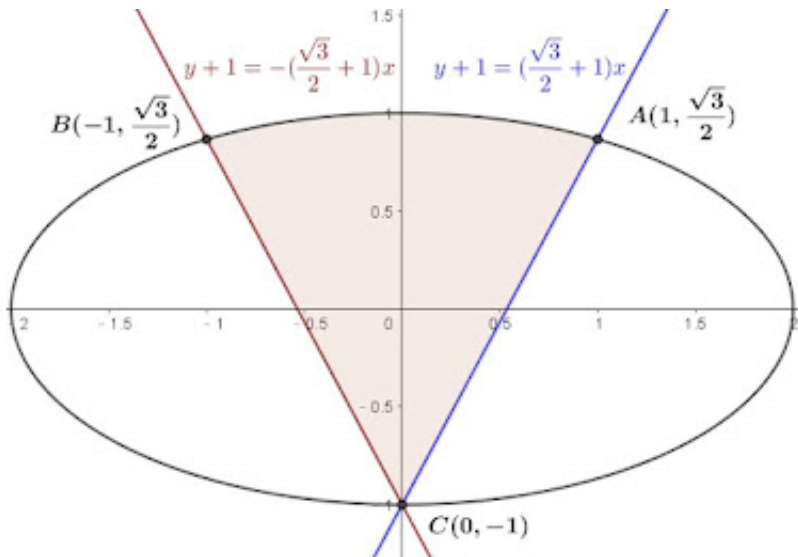
因此总体积为

$$V = V_1 + V_2 = \pi^2 + 2\pi$$

35. 在坐标平面上, 求由不等式

$$\frac{x^2}{4} + y^2 \leq 1, \quad y + 1 \geq \left(\frac{\sqrt{3}}{2} + 1\right)x, \quad y + 1 \geq -\left(\frac{\sqrt{3}}{2} + 1\right)x$$

所围成的图形面积。



三方程的交点为

$$A\left(1, \frac{\sqrt{3}}{2}\right), \quad B\left(-1, \frac{\sqrt{3}}{2}\right), \quad C(0, -1)$$

图形关于 y 轴对称, 于是所求面积为

$$S = 2 \left[\int_0^1 \sqrt{1 - \frac{x^2}{4}} dx - \int_0^1 \left(\frac{\sqrt{3} + 2}{2}x - 1 \right) dx \right]$$

令 $x = 2 \sin \theta, dx = 2 \cos \theta d\theta$, 则

$$\int_0^1 \sqrt{1 - \frac{x^2}{4}} dx = \int_0^{\frac{\pi}{6}} \sqrt{1 - \sin^2 \theta} \cdot 2 \cos \theta d\theta = \int_0^{\frac{\pi}{6}} (1 + \cos 2\theta) d\theta = \frac{\sqrt{3}}{4} + \frac{\pi}{6}$$

且

$$\int_0^1 \left(\frac{\sqrt{3} + 2}{2}x - 1 \right) dx = \left[\frac{\sqrt{3} + 2}{4}x^2 - x \right]_0^1 = \frac{\sqrt{3} - 2}{4}$$

所以

$$S = 2 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}-2}{4} \right) = 1 + \frac{\pi}{3}$$

36. 计算

$$\lim_{x \rightarrow 0^+} \int_x^{2x} \frac{\sin^m t}{t^n} dt, \quad m, n \in \mathbb{N}.$$

注意到当 $t \rightarrow 0^+$ 时 $\frac{\sin t}{t} \rightarrow 1$, 且函数 $\frac{\sin t}{t}$ 在 $(0, \pi)$ 上单调递减。因此, 对于 $x \in (0, \frac{\pi}{2})$ 且 $t \in [x, 2x]$, 有

$$\frac{\sin 2x}{2x} < \frac{\sin t}{t} < 1.$$

于是

$$\left(\frac{\sin 2x}{2x} \right)^m \int_x^{2x} t^{m-n} dt < \int_x^{2x} \frac{\sin^m t}{t^n} dt < \int_x^{2x} t^{m-n} dt.$$

对积分做换元 $t = xu$ 得

$$\int_x^{2x} t^{m-n} dt = x^{m-n+1} \int_1^2 u^{m-n} du.$$

注意到 $\left(\frac{\sin 2x}{2x} \right)^m \rightarrow 1$, 于是极限取决于 x^{m-n+1} 的幂:

$$\lim_{x \rightarrow 0^+} \int_x^{2x} \frac{\sin^m t}{t^n} dt = \begin{cases} 0, & m-n+1 > 0 \\ \ln 2, & m-n+1 = 0 \\ +\infty, & m-n+1 < 0 \end{cases}$$

37. (a) 已知 a, b 是相异实数, λ 为非零实参数。证明

$$\left[\int_a^b f(x)g(x) dx \right]^2 \leq \left[\int_a^b [f(x)]^2 dx \right] \left[\int_a^b [g(x)]^2 dx \right]$$

由关系 $[\lambda f(x) + g(x)]^2 \geq 0$, 展开得

$$\lambda^2 [f(x)]^2 + 2\lambda f(x)g(x) + [g(x)]^2 \geq 0$$

对 x 从 a 到 b 积分知

$$\lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x) dx + \int_a^b [g(x)]^2 dx \geq 0$$

即关于 λ 的二次不等式, 要求判别式为非正:

$$4\lambda^2 \left[\int_a^b f(x)g(x) dx \right]^2 - \left[2\lambda \int_a^b [f(x)]^2 dx \right] \left[2\lambda \int_a^b [g(x)]^2 dx \right] \leq 0$$

消去 λ 即得证。

(b) 试证明

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \leq \sqrt{\frac{\pi}{2}}$$

令 $f(x) = \sqrt{\sin x}$, $g(x) = 1$, 则

$$\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \right]^2 \leq \left[\int_0^{\frac{\pi}{2}} \sin x dx \right] \left[\int_0^{\frac{\pi}{2}} 1 dx \right] = [-\cos x]_0^{\frac{\pi}{2}} \cdot [x]_0^{\frac{\pi}{2}} = 1 \cdot \frac{\pi}{2}$$

故

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \leq \sqrt{\frac{\pi}{2}}$$

(c) 试证明

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \geq \frac{16}{25} \cdot \frac{4}{\pi} = \frac{64}{25\pi}$$

令 $f(x) = (\sin x)^{\frac{1}{4}}$, $g(x) = \cos x$, 则

$$\left[\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{4}} \cos x dx \right]^2 \leq \left[\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{2}} dx \right] \left[\int_0^{\frac{\pi}{2}} \cos^2 x dx \right]$$

左式积分可用代换 $u = \sin x$, $du = \cos x dx$:

$$\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{4}} \cos x dx = \frac{4}{5}$$

而右式第二个积分为

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx = \frac{\pi}{4}$$

代入不等式得

$$\left(\frac{4}{5}\right)^2 \leq \left[\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \right] \cdot \frac{\pi}{4}$$

即

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \geq \frac{64}{25\pi}$$

积分技巧

考点: 换元积分法、部分分式积分法、三角函数的积分法、三角代换法、分部积分法、降阶公式、广义积分、对称性、国王法、函数的奇偶性、积分符号内取微分、其他

1.

$$\int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx$$

改写次方, 注意到

$$\frac{1}{x^2(1+x^4)^{\frac{3}{4}}} = x^{-5}(1+x^{-4})^{-\frac{3}{4}}$$

设 $u = 1 + x^{-4}$, 则 $du = -4x^{-5} dx$, 因此

$$\begin{aligned} \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx &= \int_{81}^{16} -\frac{1}{4} u^{-\frac{3}{4}} du \\ &= \frac{1}{4} \left[4u^{\frac{1}{4}} \right]_{16}^{81} \\ &= 1 \end{aligned}$$

2.

$$\int_0^1 \frac{1}{(x^{\frac{7}{6}} + 4x^{\frac{2}{3}})^{\frac{3}{4}}} dx$$

首先注意到

$$\frac{1}{(x^{\frac{7}{6}} + 4x^{\frac{2}{3}})^{\frac{3}{4}}} = \frac{1}{[x^{\frac{2}{3}}(x^{\frac{1}{2}} + 4)]^{\frac{3}{4}}} = \frac{1}{x^{\frac{1}{2}}(x^{\frac{1}{2}} + 4)^{\frac{3}{4}}}$$

令 $u = x^{\frac{1}{2}}$, 则 $x = u^2, dx = 2u du$, 代入得

$$\begin{aligned} \int_0^1 \frac{1}{(x^{\frac{7}{6}} + 4x^{\frac{2}{3}})^{\frac{3}{4}}} dx &= \int_0^1 \frac{1}{u(u+4)^{\frac{3}{4}}} \cdot 2u du \\ &= \left[8(u+4)^{\frac{1}{4}} \right]_0^1 \\ &= 8(\sqrt[4]{5} - \sqrt{2}) \end{aligned}$$

3.

$$\int_0^4 x^3 \sqrt{9+x^2} dx$$

设 $u = 9 + x^2$, $du = 2x dx$, 则

$$\begin{aligned} \int_0^4 x^3 \sqrt{9+x^2} dx &= \frac{1}{2} \int_9^{25} (u-9) \sqrt{u} du \\ &= \left[\frac{1}{5} u^{\frac{5}{2}} - 3u^{\frac{3}{2}} \right]_9^{25} \\ &= 282.4 \end{aligned}$$

4.

$$\int \frac{1}{1+x^4} dx$$

无中生有, 凭空捏造, 将被积函数拆分成

$$\int \frac{1}{1+x^4} dx = \frac{1}{2} \int \frac{x^2+1}{x^4+1} dx - \frac{1}{2} \int \frac{x^2-1}{x^4+1} dx$$

对于第一部分, 分子分母同除以 x^2 :

$$\frac{1}{2} \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \frac{1}{2} \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} = \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x-\frac{1}{x}}{\sqrt{2}} \right)$$

对于第二部分, 同理可得

$$\frac{1}{2} \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \frac{1}{2} \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2} = \frac{1}{4\sqrt{2}} \ln \left| \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right|$$

合并结果得到

$$\int \frac{1}{1+x^4} dx = \frac{\sqrt{2}}{4} \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) - \frac{\sqrt{2}}{8} \ln \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| + C$$

5.

$$\int \frac{4}{e^{3x} \sqrt{e^{2x}+4}} dx$$

设 $t = \frac{1}{e^x}$, 则 $dx = -\frac{dt}{t}$, 代入积分:

$$I = \int \frac{4}{e^{3x} \sqrt{e^{2x} + 4}} dx = \int 4t^3 \cdot \frac{1}{\sqrt{t^{-2} + 4}} \cdot \left(-\frac{1}{t}\right) dt = -4 \int \frac{t^3}{\sqrt{1 + 4t^2}} dt$$

再设 $u = \sqrt{1 + 4t^2}$, 则 $u^2 = 1 + 4t^2, u du = 4t dt$,

$$I = -4 \int \frac{t^2}{u} \cdot \frac{u}{4} du = \frac{1}{4} \int (1 - u^2) du = \frac{1}{4} \left(u - \frac{1}{3} u^3 \right) + C$$

将 $u = \sqrt{1 + 4t^2}, t = e^{-x}$ 代回得

$$I = \frac{(1 - 2t^2)\sqrt{1 + 4t^2}}{6} + C = \frac{(e^{2x} - 2)\sqrt{e^{2x} + 4}}{6e^{3x}} + C$$

6. 求

$$\int_0^\pi \frac{1 + x \cos x}{x + e^{-\sin x}} dx$$

应用配凑法, 对被积函数上下乘以 $e^{\sin x}$,

$$I = \int_0^\pi \frac{1 + x \cos x}{x + e^{-\sin x}} dx = \int_0^\pi \frac{e^{\sin x} + x e^{\sin x} \cos x}{x e^{\sin x} + 1} dx$$

注意到分母的导数为

$$\frac{d}{dx}(x e^{\sin x} + 1) = e^{\sin x} + x(e^{\sin x} \cdot \cos x) = e^{\sin x}(1 + x \cos x)$$

因此

$$I = \int_0^\pi \frac{d(x e^{\sin x} + 1)}{x e^{\sin x} + 1} = \left[\ln |x e^{\sin x} + 1| \right]_0^\pi = \ln(1 + \pi)$$

7.

$$\int_0^{\ln 2} \sqrt{e^x - 1} dx$$

设 $u = \sqrt{e^x - 1} \Rightarrow u^2 = e^x - 1$, 则

$$2u \, du = e^x \, dx \Rightarrow dx = \frac{2u}{u^2 + 1} \, du$$

原积分变为:

$$\begin{aligned} \int_0^{\ln 2} \sqrt{e^x - 1} \, dx &= \int_0^1 \frac{2u^2}{u^2 + 1} \, du \\ &= \int_0^1 \left(2 - \frac{2}{u^2 + 1} \right) \, du \\ &= \left[2u - 2 \arctan u \right]_0^1 \\ &= 2 - \frac{\pi}{2} \end{aligned}$$

8.

$$\int_0^\infty \frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} \, dx$$

观察到被积函数可被拆成部分分式:

$$\frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} = \frac{1}{e^{2x} + 3} - \frac{1}{e^{8x} + 3}$$

于是各积分可配凑成

$$\begin{aligned} \int_0^\infty \frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} \, dx &= \int_0^\infty \left(\frac{1}{e^{2x} + 3} - \frac{1}{e^{8x} + 3} \right) \, dx \\ &= \int_0^\infty \left(\frac{e^{-2x}}{1 + 3e^{-2x}} - \frac{e^{-8x}}{1 + 3e^{-8x}} \right) \, dx \\ &= \left[-\frac{1}{6} \ln(1 + 3e^{-2x}) - \frac{1}{24} \ln(1 + 3e^{-8x}) \right]_0^\infty \\ &= \frac{1}{4} \ln 2 \end{aligned}$$

9. 计算

$$\int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} \, dx$$

尝试将分母调整为 $1 + (xe^{-x})^2$,

$$I = \int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} dx = \int_0^1 \frac{e^{-x}(1-x)}{(xe^{-x})^2 + 1} dx$$

此时设 $u = xe^{-x}$, 则 $du = (1-x)e^{-x} dx$, 因此积分变为

$$I = \int_0^{e^{-1}} \frac{1}{u^2 + 1} du = \left[\arctan u \right]_0^{e^{-1}} = \arctan \frac{1}{e}$$

10. 已知

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

求

$$\int_e^\infty e^{-x^2} \ln(x^2) x^{(\ln(x^{-x^2})+2x^2)} dx$$

发现

$$\begin{aligned} I &= \int_e^\infty e^{-x^2} \ln(x^2) x^{(\ln(x^{-x^2})+2x^2)} dx \\ &= \int_e^\infty e^{-x^2} \ln(x^2) e^{\ln x(-x^2 \ln x + 2x^2)} dx \\ &= 2 \int_e^\infty e^{-(x \ln x - x)^2} \ln(x) dx \end{aligned}$$

设 $u = x \ln x - x$, 则 $du = \left(\ln x + x \cdot \frac{1}{x} - 1 \right) dx = \ln x dx$, 于是

$$I = 2 \int_0^\infty e^{-u^2} du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

11.

$$\int \left(\frac{\ln x - 1}{1 + (\ln x)^2} \right)^2 dx$$

设 $u = \ln x$, 则 $x = e^u, dx = e^u du$, 积分变为

$$\begin{aligned} I &= \int \left(\frac{\ln x - 1}{1 + (\ln x)^2} \right)^2 dx \\ &= \int \left(\frac{u - 1}{1 + u^2} \right)^2 e^u du \\ &= \int \left[\frac{1}{1 + u^2} - \frac{2u}{(1 + u^2)^2} \right] e^u du \end{aligned}$$

由性质

$$\int [f(u) + f'(u)]e^u du = f(u)e^u + C$$

其中

$$f(u) = \frac{1}{1 + u^2}, \quad f'(u) = -\frac{2u}{(1 + u^2)^2}$$

因此

$$I = \frac{1}{1 + u^2} e^u + C = \frac{x}{1 + (\ln x)^2} + C$$

12.

$$\int x\sqrt{x+19} dx$$

配凑得

$$\begin{aligned} \int x\sqrt{x+19} dx &= \int (x+19-19)(x+19)^{\frac{1}{2}} dx \\ &= \int (x+19)^{\frac{3}{2}} dx - 19 \int (x+19)^{\frac{1}{2}} dx \\ &= \frac{2}{5}(x+19)^{\frac{5}{2}} - \frac{38}{3}(x+19)^{\frac{3}{2}} + C \end{aligned}$$

由分部积分,

$$\begin{aligned} \int x\sqrt{x+19} dx &= \frac{2}{3}x(x+19)^{\frac{3}{2}} - \frac{2}{3} \int (x+19)^{\frac{3}{2}} dx \\ &= \frac{2}{3}x(x+19)^{\frac{3}{2}} - \frac{4}{15}(x+19)^{\frac{5}{2}} + C \end{aligned}$$

注: 两者答案皆正确。

13.

$$\int_3^6 (\sqrt{x + \sqrt{12x - 36}} + \sqrt{x - \sqrt{12x - 36}}) dx$$

设 $x = 3 + t, dx = dt$, 则

$$\sqrt{12x - 36} = \sqrt{12(x - 3)} = 2\sqrt{3t}$$

$$\begin{aligned} \int_3^6 (\sqrt{x + \sqrt{12x - 36}} + \sqrt{x - \sqrt{12x - 36}}) dx &= \int_0^3 \left(\sqrt{3 + t + 2\sqrt{3t}} + \sqrt{3 + t - 2\sqrt{3t}} \right) dt \\ &= \int_0^3 (\sqrt{3} + \sqrt{t} + \sqrt{3} - \sqrt{t}) dt \\ &= \int_0^3 2\sqrt{3} dt \\ &= 6\sqrt{3} \end{aligned}$$

14.

$$\int \frac{1}{(3x + 7)\sqrt{x + 2}} dx$$

设 $u = \sqrt{x + 2}$, 则 $x = u^2 - 2, dx = 2u du$, 代入积分,

$$\begin{aligned} \int \frac{1}{(3x + 7)\sqrt{x + 2}} dx &= \int \frac{2u}{(3u^2 + 1)u} du \\ &= \int \frac{2}{3u^2 + 1} du \\ &= \frac{2}{\sqrt{3}} \tan^{-1}(\sqrt{3}u) + C \\ &= \frac{2\sqrt{3}}{3} \tan^{-1} \sqrt{3x + 6} + C \end{aligned}$$

15.

$$\int \frac{\sqrt{4 + x}}{x} dx$$

设 $u = \sqrt{4+x}$, 则 $x = u^2 - 4, dx = 2u du$:

$$\begin{aligned}\int \frac{\sqrt{4+x}}{x} dx &= \int \frac{u}{u^2-4} \cdot 2u du \\&= 2 \int \left(1 + \frac{4}{u^2-4}\right) du \\&= 2u + 2 \int \left(\frac{1}{u-2} - \frac{1}{u+2}\right) du \\&= 2u + 2 \ln \left| \frac{u-2}{u+2} \right| + C \\&= 2\sqrt{4+x} + 2 \ln \left| \frac{\sqrt{4+x}-2}{\sqrt{4+x}+2} \right| + C\end{aligned}$$

16.

$$\int_1^2 \frac{x^2-1}{x^3 \sqrt{2x^4-2x^2+1}} dx$$

作代换

$$x = \frac{1}{u}, \quad dx = -\frac{1}{u^2} du$$

于是

$$\begin{aligned}I &= \int_1^2 \frac{x^2-1}{x^3 \sqrt{2x^4-2x^2+1}} dx = \int_1^{\frac{1}{2}} \frac{\frac{1}{u^2}-1}{\frac{1}{u^3} \sqrt{\frac{2}{u^4}-\frac{2}{u^2}+1}} \left(-\frac{1}{u^2}\right) du \\&= \int_{\frac{1}{2}}^1 \frac{\frac{1-u^2}{u^2} \cdot \frac{1}{u^2}}{\frac{1}{u^3} \cdot \frac{\sqrt{u^4-2u^2+2}}{u^2}} du \\&= \int_{\frac{1}{2}}^1 \frac{u(1-u^2)}{\sqrt{u^4-2u^2+2}} du\end{aligned}$$

再设

$$w = u^4 - 2u^2 + 2 \implies dw = (4u^3 - 4u) du = -4u(1-u^2) du$$

于是

$$I = \int_{\frac{25}{16}}^1 \frac{-\frac{1}{4} dw}{\sqrt{w}} = \frac{1}{4} \int_1^{\frac{25}{16}} w^{-\frac{1}{2}} dw = \frac{1}{4} \left[2\sqrt{w} \right]_1^{\frac{25}{16}} = \frac{1}{8}$$

17.

$$\int \frac{9}{(9-x^2)^{\frac{3}{2}}} dx$$

设 $x = 3 \sin \theta$, 则 $dx = 3 \cos \theta d\theta$, 原积分变为

$$\begin{aligned} \int \frac{9}{(9-x^2)^{\frac{3}{2}}} dx &= \int \frac{9}{(9-9\sin^2 \theta)^{\frac{3}{2}}} \cdot 3 \cos \theta d\theta \\ &= \int \frac{27 \cos \theta}{(9 \cos^2 \theta)^{\frac{3}{2}}} d\theta \\ &= \int \sec^2 \theta d\theta \\ &= \tan \theta + C \\ &= \frac{x}{\sqrt{9-x^2}} + C \end{aligned}$$

18.

$$\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} dx$$

设 $x = \sin \theta$, $dx = \cos \theta d\theta$, 则

$$\begin{aligned} \int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} dx &= \int \frac{\sin^2 \theta}{(1-\sin^2 \theta)^{\frac{3}{2}}} \cos \theta d\theta \\ &= \int \frac{\sin^2 \theta}{\cos^3 \theta} \cos \theta d\theta = \int \tan^2 \theta d\theta \\ &= \int (\sec^2 \theta - 1) d\theta \\ &= \tan \theta - \theta + C \\ &= \frac{x}{\sqrt{1-x^2}} - \sin^{-1} x + C \end{aligned}$$

19.

$$\int \sqrt{\frac{x+1}{x+3}} dx$$

有理化处理得

$$I = \int \sqrt{\frac{x+1}{x+3}} dx = \int \frac{x+1}{\sqrt{(x+1)(x+3)}} dx$$

注意到分母 $(x+1)(x+3) = (x+2)^2 - 1$, 于是可凑得

$$I = \int \frac{x+2}{\sqrt{(x+2)^2 - 1}} dx - \int \frac{1}{\sqrt{(x+2)^2 - 1}} dx$$

分别计算两个积分。对于第一个积分,

$$\int \frac{x+2}{\sqrt{(x+2)^2 - 1}} dx = \frac{1}{2} \int ((x+2)^2 - 1)^{-\frac{1}{2}} d((x+2)^2 - 1) = \sqrt{(x+2)^2 - 1} + C_1$$

对于第二个积分, 设 $x+2 = \sec \theta$, 则 $dx = \sec \theta \tan \theta d\theta$,

$$\begin{aligned} \int \frac{1}{\sqrt{(x+2)^2 - 1}} dx &= \int \frac{\sec \theta \tan \theta}{\sqrt{\sec^2 \theta - 1}} d\theta \\ &= \int \frac{\sec \theta \tan \theta}{\tan \theta} d\theta \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C_2 \\ &= \ln |x+2 + \sqrt{(x+2)^2 - 1}| + C_2 \end{aligned}$$

最后合并得

$$I = \sqrt{x^2 + 4x + 3} - \ln |x+2 + \sqrt{x^2 + 4x + 3}| + C, \quad C = C_1 + C_2$$

20.

$$\int_0^{\sqrt{3}} \frac{x}{x^4 + 9} dx$$

令 $x^2 = 3 \tan \theta$, 则 $2x dx = 3 \sec^2 \theta d\theta$, 代入积分:

$$\int_0^{\sqrt{3}} \frac{x}{x^4 + 9} dx = \int_0^{\frac{\pi}{4}} \frac{3 \sec^2 \theta}{2((3 \tan \theta)^2 + 9)} d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{6} d\theta = \left[\frac{\theta}{6} \right]_0^{\frac{\pi}{4}} = \frac{\pi}{24}$$

21.

$$\int \frac{dx}{(x^2 + 9)^3}$$

设 $x = 3 \tan \theta, dx = 3 \sec^2 \theta d\theta$:

$$\begin{aligned}\int \frac{dx}{(x^2 + 9)^3} &= \int \frac{3 \sec^2 \theta}{(9 \tan^2 \theta + 9)^3} d\theta \\&= \frac{1}{243} \int \cos^4 \theta d\theta \\&= \frac{1}{243} \int \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta \\&= \frac{1}{972} \int (1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta \\&= \frac{1}{972} \int \left(\frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta \\&= \frac{1}{648} \theta + \frac{1}{972} \sin 2\theta + \frac{1}{7776} \sin 4\theta + C \\&= \frac{1}{648} \tan^{-1} \frac{x}{3} + \frac{1}{972} \cdot \frac{2 \cdot \frac{x}{3}}{1 + \frac{x^2}{9}} + \frac{1}{3888} \cdot \frac{2 \cdot \frac{x}{3}}{1 + \frac{x^2}{9}} \cdot \frac{1 - \frac{x^2}{9}}{1 + \frac{x^2}{9}} + C \\&= \frac{1}{648} \tan^{-1} \frac{x}{3} + \frac{x}{162(x^2 + 9)} + \frac{x(9 - x^2)}{648(x^2 + 9)^2} + C\end{aligned}$$

22.

$$\int \frac{x}{(x^2 - 4x + 13)^2} dx$$

凑得

$$\begin{aligned}I &= \int \frac{x}{(x^2 - 4x + 13)^2} dx = \frac{1}{2} \int \frac{2x - 4 + 4}{(x^2 - 4x + 13)^2} dx \\&= \frac{1}{2} \int \frac{2x - 4}{(x^2 - 4x + 13)^2} dx + 2 \int \frac{1}{((x - 2)^2 + 9)^2} dx\end{aligned}$$

对于第一个积分,

$$\frac{1}{2} \int \frac{d(x^2 - 4x + 13)}{(x^2 - 4x + 13)^2} = -\frac{1}{2(x^2 - 4x + 13)} + C_1$$

对于第二个积分, 设 $x - 2 = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, 则

$$\begin{aligned}
 2 \int \frac{1}{((x-2)^2 + 9)^2} dx &= 2 \int \frac{3 \sec^2 \theta}{(9 \sec^2 \theta)^2} d\theta \\
 &= \frac{2}{27} \int \cos^2 \theta d\theta \\
 &= \frac{1}{27} \int (1 + \cos 2\theta) d\theta \\
 &= \frac{1}{27} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C_2 \\
 &= \frac{1}{27} \left(\tan^{-1} \frac{x-2}{3} + \frac{1}{2} \cdot \frac{2 \cdot \frac{x-2}{3}}{1 + \left(\frac{x-2}{3}\right)^2} \right) + C_2 \\
 &= \frac{1}{27} \tan^{-1} \frac{x-2}{3} + \frac{x-2}{9(x^2 - 4x + 13)} + C_2
 \end{aligned}$$

于是

$$\begin{aligned}
 I &= \frac{1}{27} \tan^{-1} \frac{x-2}{3} + \frac{x-2}{9(x^2 - 4x + 13)} - \frac{1}{2(x^2 - 4x + 13)} + C \\
 &= \frac{1}{27} \tan^{-1} \frac{x-2}{3} + \frac{2x-13}{18(x^2 - 4x + 13)} + C, \quad C = C_1 + C_2
 \end{aligned}$$

23.

$$\int_1^{\sqrt[3]{2}} \frac{\sqrt{x^3 - 1}}{x} dx$$

设 $\tan \theta = \sqrt{x^3 - 1}$, 则

$$x^3 = 1 + \tan^2 \theta = \sec^2 \theta, \quad 3x^2 dx = 2 \sec^2 \theta \tan \theta d\theta$$

原积分化为

$$\begin{aligned}\int_1^{\sqrt[3]{2}} \frac{\sqrt{x^3-1}}{x} dx &= \int_0^{\frac{\pi}{4}} \frac{\tan \theta}{x} \cdot \frac{2 \sec^2 \theta \tan \theta}{3x^2} d\theta \\&= \frac{2}{3} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta \tan^2 \theta}{1 + \tan^2 \theta} d\theta \\&= \frac{2}{3} \int_0^{\frac{\pi}{4}} \tan^2 \theta d\theta \\&= \frac{2}{3} \int_0^{\frac{\pi}{4}} (\sec^2 \theta - 1) d\theta \\&= \frac{2}{3} \left[\tan \theta - \theta \right]_0^{\frac{\pi}{4}} \\&= \frac{2}{3} - \frac{\pi}{6}\end{aligned}$$

24.

$$\int \sqrt{(1+x)(5-x)} dx$$

由配方得

$$I = \int \sqrt{(1+x)(5-x)} dx = \int \sqrt{9 - (x-2)^2} dx$$

设 $x-2 = 3 \sin \theta$, $dx = 3 \cos \theta d\theta$, 则

$$\begin{aligned}I &= \int \sqrt{9 - 9 \sin^2 \theta} \cdot 3 \cos \theta d\theta \\&= 9 \int \cos^2 \theta d\theta \\&= \frac{9}{2} \int (1 + \cos 2\theta) d\theta \\&= \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta + C \\&= \frac{9}{2} \theta + \frac{9}{2} \sin \theta \cos \theta + C \\&= \frac{9}{2} \sin^{-1} \frac{x-2}{3} + \frac{9}{2} \cdot \frac{x-2}{3} \cdot \frac{\sqrt{9 - (x-2)^2}}{3} + C \\&= \frac{9}{2} \sin^{-1} \frac{x-2}{3} + \frac{1}{2} (x-2) \sqrt{(1+x)(5-x)} + C\end{aligned}$$

25.

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx$$

有理化得

$$I = \int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx = \int_0^1 \frac{\sqrt{1-x}(1+\sqrt{x})}{1-x} dx = \int_0^1 \frac{1+\sqrt{x}}{\sqrt{1-x}} dx$$

设 $u = \sqrt{1-x}$, 则 $x = 1-u^2$, $dx = -2u du$, 于是

$$I = \int_1^0 \frac{1+\sqrt{1-u^2}}{u} (-2u) du = 2 \int_0^1 (1+\sqrt{1-u^2}) du = 2 + 2 \int_0^1 \sqrt{1-u^2} du$$

设 $u = \sin \theta$, 则

$$\int_0^1 \sqrt{1-u^2} du = \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta = \frac{\pi}{4}$$

故

$$I = 2 + \frac{\pi}{2}$$

26.

$$\int \sqrt{\frac{x}{1-x}} dx$$

设 $\sqrt{x} = \sin \theta$, 则 $x = \sin^2 \theta$, $dx = 2 \sin \theta \cos \theta d\theta$, 于是

$$\begin{aligned} \int \sqrt{\frac{x}{1-x}} dx &= \int \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} \cdot 2 \sin \theta \cos \theta d\theta \\ &= \int \frac{\sin \theta}{\cos \theta} \cdot 2 \sin \theta \cos \theta d\theta \\ &= \int 2 \sin^2 \theta d\theta \\ &= \int (1 - \cos 2\theta) d\theta \\ &= \theta - \frac{1}{2} \sin 2\theta + C \\ &= \theta - \sin \theta \cos \theta + C \\ &= \arcsin \sqrt{x} - \sqrt{x-x^2} + C \end{aligned}$$

27.

$$\int_7^9 \sqrt{\frac{x-7}{11-x}} dx$$

设 $x = 7 \cos^2 \theta + 11 \sin^2 \theta = 7 + 4 \sin^2 \theta$, 则 $dx = 8 \sin \theta \cos \theta d\theta$ 。于是

$$\begin{aligned} \int_7^9 \sqrt{\frac{x-7}{11-x}} dx &= \int_0^{\frac{\pi}{4}} \sqrt{\frac{4 \sin^2 \theta}{4 \cos^2 \theta}} \cdot 8 \sin \theta \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} 8 \sin^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} 4(1 - \cos 2\theta) d\theta \\ &= \left[4\theta - 2 \sin 2\theta \right]_0^{\frac{\pi}{4}} \\ &= \pi - 2 \end{aligned}$$

令 $x = 9 - 2 \sin \theta$, 则 $dx = -2 \cos \theta d\theta$ 。于是

$$\begin{aligned} \int_7^9 \sqrt{\frac{x-7}{11-x}} dx &= \int_{\frac{\pi}{2}}^0 \sqrt{\frac{(9-2 \sin \theta)-7}{11-(9-2 \sin \theta)}} (-2 \cos \theta) d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{2-2 \sin \theta}{2+2 \sin \theta}} \cos \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-\sin \theta)^2}{1-\sin^2 \theta}} \cos \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{1-\sin \theta}{\cos \theta} \cos \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} (1-\sin \theta) d\theta \\ &= \left[2\theta + 2 \cos \theta \right]_0^{\frac{\pi}{2}} \\ &= \pi - 2 \end{aligned}$$

28.

$$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx$$

令 $x+1 = \frac{1}{u}$, 则 $dx = -\frac{1}{u^2} du$, 且

$$x = \frac{1}{u} - 1 \Rightarrow x^2 + 4x + 2 = (x+2)^2 - 2 = \left(\frac{1}{u} + 1\right)^2 - 2 = \frac{1}{u^2} + \frac{2}{u} - 1$$

于是

$$\begin{aligned} \int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx &= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1+2u-u^2}{u^2}}} \left(-\frac{1}{u^2}\right) du \\ &= -\int \frac{1}{\sqrt{1+2u-u^2}} du \\ &= -\int \frac{1}{\sqrt{2-(u-1)^2}} du \\ &= -\arcsin \frac{u-1}{\sqrt{2}} + C \\ &= -\arcsin \frac{\frac{1}{x+1}-1}{\sqrt{2}} + C \\ &= \arcsin \frac{x}{\sqrt{2}(x+1)} + C \end{aligned}$$

29.

$$\int_0^4 \frac{16}{3(3x^2+16)^{\frac{5}{2}}} dx$$

设 $\sqrt{3}x = 4\tan\theta$, 则 $dx = \frac{4}{\sqrt{3}}\sec^2\theta d\theta$,

$$\begin{aligned} \int_0^4 \frac{16}{3(3x^2+16)^{\frac{5}{2}}} dx &= \int_0^{\frac{\pi}{3}} \frac{16}{3(16\sec^2\theta)^{\frac{5}{2}}} \cdot \frac{4}{\sqrt{3}}\sec^2\theta d\theta \\ &= \frac{1}{48\sqrt{3}} \int_0^{\frac{\pi}{3}} \cos^3\theta d\theta \\ &= \frac{1}{48\sqrt{3}} \int_0^{\frac{\pi}{3}} (1-\sin^2\theta) d(\sin\theta) \\ &= \frac{1}{48\sqrt{3}} \left[\sin\theta - \frac{1}{3}\sin^3\theta \right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{128} \end{aligned}$$

30.

$$\int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx$$

设 $u = \sqrt{x}$, 则 $x = u^2, dx = 2u du$, 代入积分得

$$\begin{aligned} I &= \int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx = \int \frac{(3u^4 + 5u^2)u}{(u^2 + 1)^2} \cdot 2u du \\ &= \int \frac{6u^6 + 10u^4}{(u^2 + 1)^2} du \\ &= \int (6u^2 - 2) du + \int \frac{-2u^2 + 2}{(u^2 + 1)^2} du \end{aligned}$$

对于第二部分积分, 设 $u = \tan \theta$, 则 $du = \sec^2 \theta d\theta$,

$$\begin{aligned} \int \frac{-2u^2 + 2}{(u^2 + 1)^2} du &= \int \frac{2(1 - \tan^2 \theta)}{(\sec^2 \theta)^2} \cdot \sec^2 \theta d\theta \\ &= \int 2(\cos^2 \theta - \sin^2 \theta) d\theta \\ &= \int 2 \cos(2\theta) d\theta \\ &= \sin(2\theta) + C_1 \\ &= \frac{2u}{u^2 + 1} + C_1 \end{aligned}$$

故

$$\begin{aligned} I &= 2u^3 - 2u + \frac{2u}{u^2 + 1} + C \\ &= 2x\sqrt{x} - 2\sqrt{x} + \frac{2\sqrt{x}}{x+1} + C \\ &= \frac{2x^2\sqrt{x}}{x+1} + C \end{aligned}$$

31.

$$\int \frac{2-x}{\sqrt{x}(x+2)^2} dx$$

设 $x = 2 \tan^2 \theta$, 则 $dx = 4 \tan \theta \sec^2 \theta d\theta$, 于是

$$\begin{aligned}
 I &= \int \frac{2 - 2 \tan^2 \theta}{\sqrt{2} \tan \theta (2 \sec^2 \theta)^2} \cdot 4 \tan \theta \sec^2 \theta d\theta \\
 &= \sqrt{2} \int \frac{1 - \tan^2 \theta}{\sec^2 \theta} d\theta \\
 &= \sqrt{2} \int (\cos^2 \theta - \sin^2 \theta) d\theta \\
 &= \sqrt{2} \int \cos 2\theta d\theta \\
 &= \frac{\sqrt{2}}{2} \sin 2\theta + C \\
 &= \sqrt{2} \sin \theta \cos \theta + C \\
 &= \sqrt{2} \cdot \sqrt{\frac{x}{x+2}} \cdot \sqrt{\frac{2}{x+2}} + C \\
 &= \frac{2\sqrt{x}}{x+2} + C
 \end{aligned}$$

32.

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx$$

令 $x = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$, 于是有

$$\begin{aligned}
 I &= \int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = \int_2^{\frac{1}{2}} \frac{\frac{1}{t^4} - 1}{\frac{1}{t^2} \sqrt{\frac{1}{t^4} + 1}} \left(-\frac{1}{t^2}\right) dt \\
 &= \int_{\frac{1}{2}}^2 \frac{1 - t^4}{t^2 \sqrt{t^4 + 1}} dt = -I
 \end{aligned}$$

由 $I = -I$ 知 $2I = 0$, 故

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = 0$$

33.

$$\int \frac{1 + x^2}{(1 - x^2) \sqrt{1 + x^4}} dx$$

分子分母同时除以 x^2 , 变形为

$$I = \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}} dx = \int \frac{1+\frac{1}{x^2}}{(\frac{1}{x}-x)\sqrt{x^2+\frac{1}{x^2}}} dx$$

注意到分母根号内可配方为

$$x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2$$

令 $t = x - \frac{1}{x}$, 则 $dt = \left(1 + \frac{1}{x^2}\right) dx$, 积分变为

$$I = - \int \frac{1}{t\sqrt{t^2+2}} dt$$

对该式进行倒代换, 令 $t = \frac{\sqrt{2}}{z}$, 则 $dt = -\frac{\sqrt{2}}{z^2} dz$, 代入得

$$\begin{aligned} I &= - \int \frac{1}{\frac{\sqrt{2}}{z}\sqrt{\frac{2}{z^2}+2}} \cdot \left(-\frac{\sqrt{2}}{z^2}\right) dz \\ &= \int \frac{1}{\sqrt{2z^2+2}} dz = \frac{1}{\sqrt{2}} \ln |z + \sqrt{z^2+1}| + C \end{aligned}$$

代回变量 $z = \frac{\sqrt{2}}{x - \frac{1}{x}}$, 化简整理即得结果。

34.

$$\int \frac{1+x+\sqrt{x^2+x}}{\sqrt{x}+\sqrt{x+1}} dx$$

别急, 先注意到

$$1+x+\sqrt{x^2+x} = \sqrt{x+1}(\sqrt{x+1}+\sqrt{x})$$

代入原积分式, 分母被巧妙约去,

$$\begin{aligned} I &= \int \frac{\sqrt{x+1}(\sqrt{x+1}+\sqrt{x})}{\sqrt{x}+\sqrt{x+1}} dx \\ &= \int \sqrt{x+1} dx \\ &= \frac{2}{3}(x+1)^{\frac{3}{2}} + C \end{aligned}$$

35. (a) 利用复角公式 $\cos(A+B)$ 证明

$$\cos \frac{5\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

注意到

$$\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6},$$

应用复角公式:

$$\begin{aligned}\cos \frac{5\pi}{12} &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

(b) 利用适当的三角代换, 求

$$\int_{\sqrt{2}}^{\sqrt{\sqrt{6}+\sqrt{2}}} \frac{2}{x\sqrt{x^4-1}} dx$$

设 $x^2 = \sec \theta$, 则 $2x dx = \sec \theta \tan \theta d\theta$, 当 $x = \sqrt{2}$, 得 $\theta = \frac{\pi}{3}$, 当 $x = \sqrt{\sqrt{6}+\sqrt{2}}$,

$$\cos \theta = \frac{1}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4},$$

由 (a) 部分知 $\theta = \frac{5\pi}{12}$, 代入积分式,

$$\begin{aligned}I &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{2}{x\sqrt{\sec^2 \theta - 1}} \cdot \frac{\sec \theta \tan \theta}{2x} d\theta \\ &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{\sec \theta}{\sec \theta} \cdot \frac{1}{2} d\theta \\ &= \left[\frac{1}{2} \theta \right]_{\frac{\pi}{3}}^{\frac{5\pi}{12}} = \frac{\pi}{24}\end{aligned}$$

36.

$$\int_{-1}^1 \frac{1}{(e^x+1)(1+x^2)} dx$$

巧妙换元 $u = -x$,

$$\begin{aligned}\int_{-1}^1 \frac{dx}{(e^x + 1)(1 + x^2)} &= \int_{-1}^0 \frac{dx}{(e^x + 1)(1 + x^2)} + \int_0^1 \frac{dx}{(e^x + 1)(1 + x^2)} \\&= -\int_1^0 \frac{du}{(e^{-u} + 1)(1 + u^2)} + \int_0^1 \frac{dx}{(e^x + 1)(1 + x^2)} \\&= \int_0^1 \frac{e^u du}{(1 + e^u)(1 + u^2)} + \int_0^1 \frac{dx}{(e^x + 1)(1 + x^2)} \\&= \int_0^1 \frac{du}{1 + u^2} = \frac{\pi}{4}\end{aligned}$$

37. 使用代换 $u = \sqrt{\frac{1+x}{1-x}}$, 计算

$$\int_0^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2}-3(1-x^2)} dx$$

由代换 $u^2 = \frac{1+x}{1-x}$ 可得 $x = \frac{u^2-1}{u^2+1}$, 对 x 求导得

$$dx = \frac{4u}{(u^2+1)^2} du$$

将被积式中的各项用 u 表示:

$$\sqrt{1-x^2} = \sqrt{\frac{(u^2+1)^2 - (u^2-1)^2}{(u^2+1)^2}} = \frac{2u}{u^2+1}, \quad 1-x^2 = \frac{4u^2}{(u^2+1)^2}$$

$$4x+5 = 4\left(\frac{u^2-1}{u^2+1}\right) + 5 = \frac{9u^2+1}{u^2+1}$$

当 $x = 0$ 时 $u = 1$; 当 $x = \frac{1}{4}$ 时 $u = \sqrt{\frac{5}{3}}$, 代入原积分:

$$\begin{aligned} I &= \int_1^{\sqrt{\frac{5}{3}}} \frac{3}{\frac{9u^2+1}{u^2+1} \cdot \frac{2u}{u^2+1} - 3 \cdot \frac{4u^2}{(u^2+1)^2}} \cdot \frac{4u}{(u^2+1)^2} du \\ &= \int_1^{\sqrt{\frac{5}{3}}} \frac{12u}{2u(9u^2+1) - 12u^2} du = \int_1^{\sqrt{\frac{5}{3}}} \frac{6}{9u^2 - 6u + 1} du \\ &= \int_1^{\sqrt{\frac{5}{3}}} \frac{6}{(3u-1)^2} du = \left[-\frac{2}{3u-1} \right]_1^{\sqrt{\frac{5}{3}}} \\ &= 1 - \frac{2}{\sqrt{15}-1} = \frac{6-\sqrt{15}}{7} \end{aligned}$$

38. 已知实数 $a > b > 0$, 使用代换 $x = \frac{ab}{t}$, 求积分

$$\int_0^\infty \frac{\ln x}{(x+a)(x+b)} dx$$

设原积分为 I , 作代换 $x = \frac{ab}{t}$, 则 $dx = -\frac{ab}{t^2} dt$,

$$I = \int_\infty^0 \frac{\ln(ab/t)}{(\frac{ab}{t}+a)(\frac{ab}{t}+b)} \left(-\frac{ab}{t^2}\right) dt = \int_0^\infty \frac{\ln(ab) - \ln t}{(t+b)(t+a)} dt$$

整理得

$$I = \ln(ab) \int_0^\infty \frac{1}{(t+a)(t+b)} dt - I \implies 2I = \ln(ab) \int_0^\infty \frac{1}{(t+a)(t+b)} dt$$

对被积函数作部分分式分解:

$$2I = \frac{\ln(ab)}{a-b} \int_0^\infty \left(\frac{1}{t+b} - \frac{1}{t+a} \right) dt = \frac{\ln(ab)}{a-b} \left[\ln \left(\frac{t+b}{t+a} \right) \right]_0^\infty = \frac{\ln(ab) \ln \left(\frac{a}{b} \right)}{a-b}$$

故

$$I = \frac{\ln(ab) \ln \left(\frac{a}{b} \right)}{2(a-b)}$$

39. 使用代换 $u = \sec x + \sqrt{\tan x}$, 求

$$\int \frac{2 \sin x \sqrt{\tan x} + 1}{2 \cos x \sqrt{\tan x} (\cos x \sqrt{\tan x} + 1)} dx$$

设 $u = \sec x + \sqrt{\tan x}$, 则

$$du = \left(\sec x \tan x + \frac{\sec^2 x}{2\sqrt{\tan x}} \right) dx = \frac{2 \sin x \sqrt{\tan x} + 1}{2 \cos^2 x \sqrt{\tan x}} dx$$

整理原积分:

$$\begin{aligned} I &= \int \frac{2 \sin x \sqrt{\tan x} + 1}{2 \cos^2 x \sqrt{\tan x} (\sqrt{\tan x} + \sec x)} dx \\ &= \int \frac{1}{\sqrt{\tan x} + \sec x} \cdot \frac{2 \sin x \sqrt{\tan x} + 1}{2 \cos^2 x \sqrt{\tan x}} dx \\ &= \int \frac{1}{u} du = \ln |u| + C \end{aligned}$$

代回变量得

$$I = \ln |\sec x + \sqrt{\tan x}| + C$$

40.

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx$$

设 $u = \sqrt{x+2}$, 则 $x = u^2 - 2, dx = 2u du$ 。代入得

$$I = \int \frac{16}{(u^2+4)(u^2-4)u} \cdot 2u du = \int \frac{32}{(u^2+4)(u^2-4)} du$$

应用部分分式分解:

$$\frac{32}{(u^2+4)(u^2-4)} = \frac{4}{u^2-4} - \frac{4}{u^2+4}$$

积分得

$$\begin{aligned} I &= \int \left(\frac{1}{u-2} - \frac{1}{u+2} - \frac{4}{u^2+4} \right) du \\ &= \ln \left| \frac{u-2}{u+2} \right| - 2 \arctan \left(\frac{u}{2} \right) + C \end{aligned}$$

代回 $u = \sqrt{x+2}$ 得

$$I = \ln \left| \frac{\sqrt{x+2}-2}{\sqrt{x+2}+2} \right| - 2 \arctan \left(\frac{\sqrt{x+2}}{2} \right) + C$$

41.

$$\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx$$

应用配凑法

$$I = \int \frac{x^2 - \frac{1}{x^2}}{\sqrt{x^4 + 1}} dx = \int \frac{x - \frac{1}{x^3}}{\sqrt{x^2 + \frac{1}{x^2}}} dx$$

此时注意到

$$\frac{d}{dx} \left(x^2 + \frac{1}{x^2} \right) = 2x - \frac{2}{x^3} = 2 \left(x - \frac{1}{x^3} \right)$$

于是原积分可化为

$$I = \frac{1}{2} \int \frac{d(x^2 + \frac{1}{x^2})}{\sqrt{x^2 + \frac{1}{x^2}}} = \sqrt{x^2 + \frac{1}{x^2}} + C = \frac{\sqrt{x^4 + 1}}{x} + C$$

42. 已使用代换 $x^2 + x + 2 = (u - x)^2$, 求

$$\int \frac{1}{x \sqrt{x^2 + x + 2}} dx$$

由已知关系得

$$x^2 + x + 2 = u^2 - 2ux + x^2 \Rightarrow x = \frac{u^2 - 2}{2u + 1}$$

则

$$dx = \frac{2(u^2 + u + 2)}{(2u + 1)^2} du$$

注意到

$$\sqrt{x^2 + x + 2} = u - x = u - \frac{u^2 - 2}{2u + 1} = \frac{u^2 + u + 2}{2u + 1}$$

代入积分得

$$\begin{aligned} I &= \int \frac{1}{\frac{u^2-2}{2u+1} \cdot \frac{u^2+u+2}{2u+1}} \cdot \frac{2(u^2+u+2)}{(2u+1)^2} du \\ &= \int \frac{2}{u^2-2} du \\ &= \frac{1}{\sqrt{2}} \ln \left| \frac{u-\sqrt{2}}{u+\sqrt{2}} \right| + C \\ &= \frac{1}{\sqrt{2}} \ln \left| \frac{x+\sqrt{x^2+x+2}-\sqrt{2}}{x+\sqrt{x^2+x+2}+\sqrt{2}} \right| + C \end{aligned}$$

43.

$$\int_{0.2}^{0.5} \frac{\sqrt{x-x^2}}{x^4} dx$$

将被积函数变形:

$$\frac{\sqrt{x-x^2}}{x^4} = \frac{1}{x^3} \sqrt{\frac{1}{x} - 1}$$

设 $u = \sqrt{\frac{1}{x} - 1}$, 则

$$x = \frac{1}{u^2+1} \Rightarrow dx = -\frac{2u}{(u^2+1)^2} du$$

于是

$$\begin{aligned} I &= \int_2^1 (u^2+1)^3 \cdot u \cdot \frac{-2u}{(u^2+1)^2} du \\ &= \int_1^2 2u^2(u^2+1) du = \int_1^2 (2u^4 + 2u^2) du \\ &= \left[\frac{2}{5}u^5 + \frac{2}{3}u^3 \right]_1^2 = \frac{256}{15} \end{aligned}$$

44.

$$\int \frac{dx}{x^2(a+bx)^2}, \quad a, b \in \mathbb{R}$$

设 $u = \frac{a+bx}{x}$, 则

$$x = \frac{a}{u-b}, \quad dx = \frac{-a du}{(u-b)^2}, \quad a+bx = xu = \frac{au}{u-b}$$

代入原式:

$$\begin{aligned} \int \frac{dx}{x^2(a+bx)^2} &= \int \frac{-a du}{(u-b)^2} \cdot \frac{1}{x^2(a+bx)^2} \\ &= \int \frac{-a du}{(u-b)^2} \cdot \frac{(u-b)^4}{a^4 u^2} \\ &= -\frac{1}{a^3} \int \frac{(u-b)^2}{u^2} du \\ &= -\frac{1}{a^3} \int \left(1 - \frac{2b}{u} + \frac{b^2}{u^2}\right) du \\ &= -\frac{1}{a^3} \left(u - 2b \ln |u| - \frac{b^2}{u}\right) + C \end{aligned}$$

代回 $u = \frac{a+bx}{x}$ 得

$$\int \frac{dx}{x^2(a+bx)^2} = -\frac{1}{a^3} \left(\frac{a+bx}{x} - 2b \ln \left|\frac{a+bx}{x}\right| - \frac{b^2 x}{a+bx}\right) + C$$

45. 使用代换 $\sqrt{5-4x-x^2} = (1-x)u$, 其中 $x \neq 1$, 求

$$\int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx$$

代换给出

$$5-4x-x^2 = (1-x)(x+5) = (1-x)^2 u^2 \Rightarrow x = \frac{u^2-5}{u^2+1} \Rightarrow dx = \frac{12u}{(u^2+1)^2} du$$

由 $1 - x = \frac{6}{u^2 + 1}$ 得,

$$\begin{aligned} I &= \int \frac{x}{(5 - 4x - x^2)^{\frac{3}{2}}} dx = \int \frac{u^2 - 5}{u^2 + 1} \cdot \frac{1}{\left(\frac{6u}{u^2 + 1}\right)^3} \cdot \frac{12u}{(u^2 + 1)^2} du \\ &= \int \frac{u^2 - 5}{u^2 + 1} \cdot \frac{(u^2 + 1)^3}{216u^3} \cdot \frac{12u}{(u^2 + 1)^2} du \\ &= \frac{1}{18} \int \frac{u^2 - 5}{u^2} du \\ &= \frac{1}{18} \left(u + \frac{5}{u} \right) + C \end{aligned}$$

代回 $u = \frac{\sqrt{5 - 4x - x^2}}{1 - x}$, 整理得

$$I = \frac{5 - 2x}{9\sqrt{5 - 4x - x^2}} + C$$

46.

$$\int \frac{\sqrt{1 + \sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$$

注意到

$$d(\sqrt[3]{x}) = \frac{1}{3\sqrt[3]{x^2}} dx$$

于是

$$\int \frac{\sqrt{1 + \sqrt[3]{x}}}{\sqrt[3]{x^2}} dx = 3 \int (1 + \sqrt[3]{x})^{\frac{1}{2}} d(\sqrt[3]{x}) = 2(1 + \sqrt[3]{x})^{\frac{3}{2}} + C$$

47.

$$\int \frac{1}{x(1 + x^{119})} dx$$

分子分母同乘 x^{118} , 构造 $\frac{f'}{f}$ 形式:

$$I = \int \frac{1}{x(1 + x^{119})} dx = \int \frac{x^{118}}{x^{119}(1 + x^{119})} dx$$

设 $t = x^{119}$, 则 $dt = 119x^{118} dx$,

$$\begin{aligned} I &= \frac{1}{119} \int \frac{1}{t(1+t)} dt = \frac{1}{119} \int \left(\frac{1}{t} - \frac{1}{1+t} \right) dt \\ &= \frac{1}{119} \ln \left| \frac{t}{1+t} \right| + C = \frac{1}{119} \ln \left| \frac{x^{119}}{1+x^{119}} \right| + C \end{aligned}$$

48.

$$\int_{-1}^1 \frac{x^2}{1+2^x} dx$$

设 $u = -x$, 则

$$I = \int_{-1}^1 \frac{u^2}{1+2^{-u}} du = \int_{-1}^1 \frac{u^2 2^u}{1+2^u} du = \int_{-1}^1 \frac{x^2 2^x}{1+2^x} dx$$

于是有

$$2I = \int_{-1}^1 \frac{x^2(2^x+1)}{1+2^x} dx = \int_{-1}^1 x^2 dx = \frac{2}{3} \Rightarrow I = \frac{1}{3}$$

49.

$$I = \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx$$

应用倒数代换法, 设 $x = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$, 整理得

$$I = \int_2^{\frac{1}{2}} \frac{\ln \frac{1}{t}}{1+(\frac{1}{t})^2} \left(-\frac{1}{t^2} \right) dt = \int_2^{\frac{1}{2}} \frac{\ln t}{t^2+1} dt = -I$$

由于 $I = -I$, 故 $I = 0$ 。

50.

$$\int_0^{\sqrt{3}} \frac{1}{1+x^2} \sin^{-1} \frac{2x}{1+x^2} dx$$

设 $x = \tan \theta$, 则 $dx = \sec^2 \theta d\theta$ 。原积分变形为:

$$I = \int_0^{\frac{\pi}{3}} \sin^{-1}(\sin 2\theta) d\theta$$

注意到 $\sin^{-1}(\sin \alpha)$ 的值取决于 α 所在的区间: 当 $0 \leq 2\theta \leq \frac{\pi}{2}$, 即 $0 \leq \theta \leq \frac{\pi}{4}$ 时,

$$\sin^{-1}(\sin 2\theta) = 2\theta;$$

当 $\frac{\pi}{2} < 2\theta \leq \frac{2\pi}{3}$, 即 $\frac{\pi}{4} < \theta \leq \frac{\pi}{3}$ 时,

$$\sin^{-1}(\sin 2\theta) = \pi - 2\theta$$

因此, 积分需分段计算:

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} 2\theta d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\pi - 2\theta) d\theta \\ &= \left[\theta^2 \right]_0^{\frac{\pi}{4}} + \left[\pi\theta - \theta^2 \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \frac{\pi^2}{16} + \frac{2\pi^2}{9} - \frac{3\pi^2}{16} \\ &= \frac{7\pi^2}{72} \end{aligned}$$

51.

$$I = \int_{\frac{1}{2}}^2 \frac{x^{2012} - 1}{x^{2014} + 1} dx$$

设 $x = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$, 代入积分得:

$$\begin{aligned} I &= \int_2^{\frac{1}{2}} \frac{\left(\frac{1}{t}\right)^{2012} - 1}{\left(\frac{1}{t}\right)^{2014} + 1} \left(-\frac{1}{t^2}\right) dt \\ &= \int_{\frac{1}{2}}^2 \frac{1 - t^{2012}}{1 + t^{2014}} \cdot \frac{t^{2014}}{t^{2012} \cdot t^2} dt \\ &= - \int_{\frac{1}{2}}^2 \frac{t^{2012} - 1}{t^{2014} + 1} dt = -I \end{aligned}$$

由 $I = -I$ 可知 $I = 0$ 。

52. 若正整数 k 满足

$$\left(\int_0^1 x^{2018} (2019 + kx^{10}) \sqrt{1 + x^{10}} dx \right)^2 = 8,$$

求 k 的最大值。

由于被积函数随 k 严格递增, 则当

$$\int_0^1 x^{2018} (2019 + kx^{10}) \sqrt{1 + x^{10}} dx = \sqrt{8}$$

时 k 取最大。欲写成形如

$$\int_0^1 (2019x^{2018-n} + kx^{2028-n}) \sqrt{x^{2n} + x^{2n+10}} dx$$

的积分, 考虑强迫

$$2n - 1 = 2018 - n,$$

可得 $n = 673$, 此时设 $u = x^{1346} + x^{1356}$, 则 $du = 1346x^{1345} + 1356x^{1355}dx$, 于是

$$\begin{aligned} I &= \int_0^1 (2019x^{1345} + kx^{1355}) \sqrt{x^{1346} + x^{1356}} dx \\ &= \frac{3}{2} \int_0^1 (1346^{1345} + \frac{2}{3}kx^{1355}) \sqrt{x^{1346} + x^{1356}} dx \end{aligned}$$

当 $\frac{2}{3}k = 1356$ 即 $k = 2034$ 时, 有

$$I = \frac{3}{2} \int_0^2 \sqrt{u} du = \frac{3}{2} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_0^2 = \sqrt{8}$$

53.

$$\int \frac{dx}{x^3 + 1}$$

部分分式分解得

$$\frac{1}{x^3 + 1} = \frac{1}{3(x+1)} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2 - x + 1}$$

于是

$$\int \frac{dx}{x^3 + 1} = \frac{1}{3} \int \frac{dx}{x+1} - \frac{1}{3} \int \frac{x}{x^2 - x + 1} dx + \frac{2}{3} \int \frac{dx}{x^2 - x + 1}$$

其中

$$\int \frac{dx}{x+1} = \ln|x+1|, \quad \int \frac{x}{x^2-x+1} dx = \frac{1}{2} \ln(x^2-x+1)$$
$$\int \frac{dx}{x^2-x+1} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

故

$$\int \frac{dx}{x^3+1} = \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

54.

$$\int \frac{x^4+1}{x^6+1} dx$$

注意到 $x^6+1=(x^2+1)(x^4-x^2+1)$, 将被积函数分子进行配凑:

$$\frac{x^4+1}{x^6+1} = \frac{(x^4-x^2+1)+x^2}{x^6+1} = \frac{1}{x^2+1} + \frac{x^2}{x^6+1}$$

于是积分拆分为:

$$I = \int \frac{x^4+1}{x^6+1} dx = \int \frac{1}{x^2+1} dx + \int \frac{x^2}{(x^3)^2+1} dx$$

第一部分为 $\tan^{-1} x$; 第二部分通过 $d(x^3) = 3x^2 dx$ 得:

$$\frac{1}{3} \int \frac{d(x^3)}{(x^3)^2+1} = \frac{1}{3} \tan^{-1}(x^3)$$

故结果为:

$$I = \tan^{-1} x + \frac{1}{3} \tan^{-1}(x^3) + C$$

55.

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx$$

进行多项式除法,

$$\frac{x^4(1-x)^4}{x^2+1} = \frac{x^8-4x^7+6x^6-4x^5+x^4}{x^2+1} = x^6-4x^5+5x^4-4x^2+4-\frac{4}{x^2+1}$$

逐项积分得

$$\begin{aligned} I &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2 + 1} \right) dx \\ &= \left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4 \tan^{-1} x \right]_0^1 \\ &= \frac{22}{7} - \pi \end{aligned}$$

56.

$$\int_0^1 \frac{(x^2 + 1)(x^2 + 4)}{(x^2 + 3)(x^2 - 4)} dx$$

将被积函数化为真分式:

$$\frac{x^4 + 5x^2 + 4}{x^4 - x^2 - 12} = 1 + \frac{6x^2 + 16}{(x^2 + 3)(x - 2)(x + 2)}$$

对真分式部分设待定系数:

$$\frac{6x^2 + 16}{(x^2 + 3)(x - 2)(x + 2)} = \frac{Ax + B}{x^2 + 3} + \frac{C}{x - 2} + \frac{D}{x + 2}$$

两边同乘以分母得

$$6x^2 + 16 = (Ax + B)(x^2 - 4) + C(x^2 + 3)(x + 2) + D(x^2 + 3)(x - 2)$$

利用特殊值法求解:

- 令 $x = 2: 6(4) + 16 = C(7)(4) \Rightarrow C = \frac{10}{7}$ 。
- 令 $x = -2: 6(4) + 16 = D(7)(-4) \Rightarrow D = -\frac{10}{7}$ 。
- 令 $x = 0: 16 = B(-4) + 2(3)C - 2(3)D \Rightarrow B = \frac{2}{7}$ 。
- 比较 x^3 项系数: $0 = A + C + D$, 故 $A = 0$ 。

于是被积函数为

$$1 + \frac{2}{7(x^2 + 3)} + \frac{10}{7(x - 2)} - \frac{10}{7(x + 2)}$$

积分得

$$\begin{aligned} I &= \left[x + \frac{2}{7\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{10}{7} \ln \left| \frac{x-2}{x+2} \right| \right]_0^1 \\ &= \left(1 + \frac{2}{7\sqrt{3}} \cdot \frac{\pi}{6} + \frac{10}{7} \ln \frac{1}{3} \right) - 0 \\ &= 1 + \frac{\pi}{21\sqrt{3}} - \frac{10}{7} \ln 3 \end{aligned}$$

57.

$$\int_0^{\frac{1}{3}} \frac{32x^2}{(x^2-1)(x+1)^3} dx$$

化简分母得 $\frac{32x^2}{(x-1)(x+1)^4}$ 。设 $u = x+1$, 则 $dx = du$, 于是

$$I = \int_1^{\frac{4}{3}} \frac{32(u-1)^2}{(u-2)u^4} du = \int_1^{\frac{4}{3}} \frac{32u^2 - 64u + 32}{u^4(u-2)} du$$

进行部分分式分解:

$$\frac{32u^2 - 64u + 32}{u^4(u-2)} = -\frac{16}{u^4} + \frac{24}{u^3} - \frac{4}{u^2} - \frac{2}{u} + \frac{2}{u-2}$$

逐项积分得

$$\begin{aligned} I &= \left[\frac{16}{3u^3} - \frac{12}{u^2} + \frac{4}{u} - 2 \ln |u| + 2 \ln |u-2| \right]_1^{\frac{4}{3}} \\ &= \left(\frac{9}{4} - \frac{27}{4} + 3 - 2 \ln \frac{4}{3} + 2 \ln \frac{2}{3} \right) - \left(\frac{16}{3} - 12 + 4 \right) \\ &= \frac{7}{6} - 2 \ln 2 \end{aligned}$$

58.

$$\int \frac{1}{\sqrt[4]{1+x^4}} dx$$

设 $(tx)^4 = 1 + x^4$, 则 $x = (t^4 - 1)^{-\frac{1}{4}}$, 对 x 求导得 $dx = -t^3(t^4 - 1)^{-\frac{5}{4}} dt$, 代入得

$$I = \int \frac{1}{tx} \cdot dx = \int (t^4 - 1)^{\frac{1}{4}} \cdot \frac{1}{t} \cdot [-t^3(t^4 - 1)^{-\frac{5}{4}}] dt = - \int \frac{t^2}{t^4 - 1} dt$$

利用部分分式拆分,

$$I = -\frac{1}{2} \int \left(\frac{1}{t^2 - 1} + \frac{1}{t^2 + 1} \right) dt = -\frac{1}{4} \ln \left| \frac{t-1}{t+1} \right| - \frac{1}{2} \tan^{-1} t + C$$

其中

$$t = \frac{\sqrt[4]{1+x^4}}{x}$$

59.

$$\int \frac{2}{\sqrt{x} - 3\sqrt[4]{x} + 2} dx$$

设 $t = \sqrt[4]{x}$, 则 $x = t^4$, $dx = 4t^3 dt$ 。

$$I = \int \frac{2}{\sqrt{x} - 3\sqrt[4]{x} + 2} dx = \int \frac{8t^3}{t^2 - 3t + 2} dt$$

进行多项式除法与部分分式分解, 可得

$$\frac{8t^3}{(t-1)(t-2)} = 8t + 24 + \frac{64}{t-2} - \frac{8}{t-1}$$

逐项积分得:

$$\begin{aligned} I &= 4t^2 + 24t + 64 \ln |t-2| - 8 \ln |t-1| + C \\ &= 4\sqrt{x} + 24\sqrt[4]{x} + 64 \ln |\sqrt[4]{x} - 2| - 8 \ln |\sqrt[4]{x} - 1| + C \end{aligned}$$

60.

$$\int \frac{3x^3}{x^4 + x^3 + x + 1} dx$$

观察到

$$x^4 + x^3 + x + 1 = (x^3 + 1)(x + 1) = (x + 1)^2(x^2 - x + 1)$$

设待定系数:

$$\frac{3x^3}{(x+1)^2(x^2-x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1}$$

求得 $A = 2, B = -1, C = 1, D = -1$,

$$\begin{aligned} I &= \int \frac{2}{x+1} dx - \int \frac{1}{(x+1)^2} dx + \int \frac{x-1}{x^2-x+1} dx \\ &= 2 \ln|x+1| + \frac{1}{x+1} + \frac{1}{2} \ln(x^2-x+1) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + C \\ &= \ln \left| (x+1)^2 \sqrt{x^2-x+1} \right| - \frac{\sqrt{3}}{3} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + \frac{1}{x+1} + C \end{aligned}$$

61.

$$\int \frac{x^2(x^4+1)}{\sqrt{x^4+2}} dx$$

注意到

$$I = \int \frac{x^2(x^4+1)}{\sqrt{x^4+2}} dx = \int \frac{x^6+x^2}{\sqrt{x^4+2}} dx = \int \frac{x^7+x^3}{\sqrt{x^8+2x^4}} dx$$

此时有 $\frac{d}{dx}(x^8+2x^4) = 8x^7+8x^3 = 8(x^7+x^3)$, 于是

$$I = \frac{1}{8} \int \frac{d(x^8+2x^4)}{\sqrt{x^8+2x^4}} = \frac{1}{4} \sqrt{x^8+2x^4} + C = \frac{x^2}{4} \sqrt{x^4+2} + C$$

62.

$$\int \frac{x^2+1}{(x^2-2x+2)^3} dx$$

设 $x - 1 = \tan \theta$, 则 $dx = \sec^2 \theta d\theta$, $x^2 - 2x + 2 = \sec^2 \theta$, 代入得:

$$\begin{aligned} I &= \int \frac{\tan^2 \theta + 2 \tan \theta + 2}{\sec^6 \theta} \sec^2 \theta d\theta = \int (\sin^2 \theta \cos^2 \theta + 2 \sin \theta \cos^3 \theta + 2 \cos^4 \theta) d\theta \\ &= \int \left[\frac{1}{4} \sin^2 2\theta + 2 \sin \theta \cos^3 \theta + 2 \left(\frac{1 + \cos 2\theta}{2} \right)^2 \right] \\ &= \int \left[\frac{1}{8} - \frac{1}{8} \cos 4\theta + 2 \sin \theta \cos^3 \theta + \frac{1}{2} + \cos 2\theta + \frac{1}{4} (1 + \cos 4\theta) \right] d\theta \\ &= \int \left[\frac{7}{8} + \cos 2\theta + \frac{1}{8} \cos 4\theta + 2 \sin \theta \cos^3 \theta \right] d\theta \\ &= \frac{7}{8} \theta + \frac{1}{2} \sin 2\theta + \frac{1}{32} \sin 4\theta - \frac{1}{2} \cos^4 \theta + C \end{aligned}$$

还原变量: 由 $\tan \theta = x - 1$, 知

$$\cos^2 \theta = \frac{1}{1 + \tan^2 \theta} = \frac{1}{x^2 - 2x + 2}$$

利用倍角公式:

- $\sin 2\theta = 2 \sin \theta \cos \theta = 2 \tan \theta \cos^2 \theta = \frac{2(x-1)}{x^2 - 2x + 2}$
- $\sin 4\theta = 2 \sin 2\theta (2 \cos^2 \theta - 1) = \frac{4(x-1)(2x-x^2)}{(x^2 - 2x + 2)^2}$
- $\cos^4 \theta = \left(\frac{1}{x^2 - 2x + 2} \right)^2$

代入整理得

$$\begin{aligned} I &= \frac{7}{8} \tan^{-1}(x-1) + \frac{x-1}{x^2 - 2x + 2} + \frac{(x-1)(2x-x^2)}{8(x^2 - 2x + 2)^2} - \frac{4}{8(x^2 - 2x + 2)^2} + C \\ &= \frac{7}{8} \tan^{-1}(x-1) + \frac{7x^3 - 21x^2 + 30x - 20}{8(x^2 - 2x + 2)^2} + C \end{aligned}$$

63.

$$\int \frac{dx}{\sqrt{x+1} + \sqrt[4]{x+1}}$$

设 $x+1=u^4$, 则 $dx=4u^3 du$ 。原积分变为:

$$\begin{aligned}\int \frac{dx}{\sqrt{x+1}+\sqrt[4]{x+1}} &= \int \frac{4u^3}{u^2+u} du \\ &= 4 \int \frac{u^2}{u+1} du \\ &= 4 \int \left(u-1+\frac{1}{u+1}\right) du \\ &= 2u^2-4u+4\ln(u+1)+C \\ &= 2\sqrt{x+1}-4\sqrt[4]{x+1}+4\ln(\sqrt[4]{x+1}+1)+C\end{aligned}$$

64.

$$\int \frac{1+x^2}{x\sqrt{x^4-x^2+1}} dx$$

分子分母同时除以 x^2 :

$$I = \int \frac{1+x^2}{x\sqrt{x^4-x^2+1}} dx = \int \frac{1+\frac{1}{x^2}}{\sqrt{(x-\frac{1}{x})^2+1}} dx$$

设 $t=x-\frac{1}{x}$, 则 $dt=(1+\frac{1}{x^2})dx$,

$$I = \int \frac{dt}{\sqrt{t^2+1}} = \ln|t+\sqrt{t^2+1}|+C = \ln\left|\frac{x^2-1+\sqrt{x^4-x^2+1}}{x}\right|+C$$

65.

$$\int \frac{1}{(2x^2+3)\sqrt{5x^2+4}} dx$$

设 $x=\frac{1}{u}$, 则 $dx=-\frac{1}{u^2}du$,

$$I = \int \frac{-\frac{1}{u^2} du}{(\frac{2}{u^2}+3)\sqrt{\frac{5}{u^2}+4}} = \int \frac{-u du}{(2+3u^2)\sqrt{5+4u^2}}$$

设 $t = \sqrt{5 + 4u^2}$, 则 $u^2 = \frac{t^2 - 5}{4}$, $t dt = 4u du$ 。代入 $2 + 3u^2 = 2 + 3\left(\frac{t^2 - 5}{4}\right) = \frac{3t^2 - 7}{4}$:

$$I = \int \frac{-\frac{1}{4}t dt}{\frac{3t^2 - 7}{4} \cdot t} = - \int \frac{dt}{3t^2 - 7} = \frac{1}{3} \int \frac{dt}{\frac{7}{3} - t^2}$$

利用公式

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right|$$

有

$$I = \frac{1}{3} \cdot \frac{1}{2\sqrt{\frac{7}{3}}} \ln \left| \frac{\sqrt{\frac{7}{3}} + t}{\sqrt{\frac{7}{3}} - t} \right| + C = \frac{\sqrt{21}}{42} \ln \left| \frac{\sqrt{7} + \sqrt{3}t}{\sqrt{7} - \sqrt{3}t} \right| + C, \quad t = \sqrt{5 + \frac{4}{x^2}}$$

66.

$$\int \frac{x + x^{\frac{2}{3}} + x^{\frac{1}{6}}}{x(1 + x^{\frac{1}{3}})} dx$$

设 $x = u^6$, 则 $dx = 6u^5 du$, 代入原积分:

$$\begin{aligned} \int \frac{x + x^{\frac{2}{3}} + x^{\frac{1}{6}}}{x(1 + x^{\frac{1}{3}})} dx &= \int \frac{u^6 + u^4 + u}{u^6(1 + u^2)} \cdot 6u^5 du \\ &= 6 \int \frac{u^5 + u^3 + 1}{1 + u^2} du \\ &= 6 \int \left(u^3 + \frac{1}{1 + u^2} \right) du \\ &= 6 \left(\frac{1}{4} u^4 + \tan^{-1} u \right) + C \\ &= \frac{3}{2} x^{\frac{2}{3}} + 6 \tan^{-1}(x^{\frac{1}{6}}) + C \end{aligned}$$

67.

$$\int \frac{\sqrt[3]{1 + \sqrt[4]{x}}}{\sqrt{x}} dx$$

设 $1 + x^{\frac{1}{4}} = t^3$, 则 $x = (t^3 - 1)^4$, 且 $dx = 12t^2(t^3 - 1)^3 dt$:

$$I = \int \frac{t}{(t^3 - 1)^2} \cdot 12t^2(t^3 - 1)^3 dt = 12 \int (t^6 - t^3) dt$$

积分得:

$$I = \frac{12}{7}t^7 - 3t^4 + C = \frac{12}{7}(1 + \sqrt[4]{x})^{\frac{7}{3}} - 3(1 + \sqrt[4]{x})^{\frac{4}{3}} + C$$

68.

$$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx$$

分子分母同乘 $(x+1)$ 构造如下:

$$I = \int \frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}} dx = \int \frac{1-\frac{1}{x^2}}{(x+2+\frac{1}{x})\sqrt{x+1+\frac{1}{x}}} dx$$

此时可设 $u = \sqrt{x+1+\frac{1}{x}}$, 则

$$u^2 = x+1+\frac{1}{x} \Rightarrow 2u du = \left(1 - \frac{1}{x^2}\right) dx,$$

代入得

$$I = \int \frac{2u du}{(u^2+1) \cdot u} = 2 \int \frac{du}{u^2+1} = 2 \tan^{-1} u + C = 2 \tan^{-1} \sqrt{x+1+\frac{1}{x}} + C$$

69.

$$\int \frac{x}{(x-1)\sqrt{x^2-2x}} dx$$

设 $x-1=t$, 则 $dx=dt$, $x^2-2x=t^2-1$, 原积分变为

$$I = \int \frac{t+1}{t\sqrt{t^2-1}} dt = \int \frac{dt}{\sqrt{t^2-1}} + \int \frac{dt}{t\sqrt{t^2-1}}$$

第一项为 $\ln|t + \sqrt{t^2-1}|$; 第二项令 $t = \sec \theta$ 则 $dt = \sec \theta \tan \theta d\theta$, 且 $\sqrt{t^2-1} = \tan \theta$:

$$\int \frac{\sec \theta \tan \theta}{\sec \theta \tan \theta} d\theta = \int 1 d\theta = \theta + C_2$$

故

$$I = \ln |t + \sqrt{t^2 - 1}| + \sec^{-1} |t| + C = \ln |x - 1 + \sqrt{x^2 - 2x}| + \sec^{-1} |x - 1| + C$$

70.

$$\int \frac{3 \sin 2x + 4 \cos x e^{\sin x} + 8 \cos x}{3 \sin x + e^{\sin x} + 1} dx$$

利用二倍角公式 $\sin 2x = 2 \sin x \cos x$ 并提取分子中的 $\cos x$:

$$I = \int \frac{(6 \sin x + 4e^{\sin x} + 8) \cos x}{3 \sin x + e^{\sin x} + 1} dx$$

令 $u = \sin x$, 则 $du = \cos x dx$ 。代入原式:

$$I = \int \frac{6u + 4e^u + 8}{3u + e^u + 1} du = 2 \int \frac{3u + 2e^u + 4}{3u + e^u + 1} du$$

观察分母 $f(u) = 3u + e^u + 1$, 其导数为 $f'(u) = e^u + 3$, 于是

$$\begin{aligned} I &= 2 \left(\int \frac{3u + e^u + 1}{3u + e^u + 1} du + \int \frac{e^u + 3}{3u + e^u + 1} du \right) \\ &= 2u + 2 \ln |3u + e^u + 1| + C \\ &= 2 \sin x + 2 \ln |3 \sin x + e^{\sin x} + 1| + C \end{aligned}$$

71.

$$\int \frac{\sqrt{x}}{\sqrt[3]{x} - 1} dx$$

设 $x = u^6$, 则 $dx = 6u^5 du$ 。原积分变为:

$$I = \int \frac{u^3}{u^2 - 1} \cdot 6u^5 du = 6 \int \frac{u^8}{u^2 - 1} du$$

进行多项式除法:

$$\frac{u^8}{u^2 - 1} = u^6 + u^4 + u^2 + 1 + \frac{1}{u^2 - 1}$$

逐项积分得

$$\begin{aligned} I &= 6 \left(\frac{u^7}{7} + \frac{u^5}{5} + \frac{u^3}{3} + u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| \right) + C \\ &= \frac{6}{7} x^{\frac{7}{6}} + \frac{6}{5} x^{\frac{5}{6}} + 2x^{\frac{1}{2}} + 6x^{\frac{1}{6}} + 3 \ln \left| \frac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{6}} + 1} \right| + C \end{aligned}$$

72.

$$\int \frac{(x+1)^3(x-1)}{(x^2+1)^2\sqrt{x^4+x^2+1}} dx$$

分子写成

$$(x+1)^2(x+1)(x-1) = (x^2+2x+1)(x^2-1)$$

于是

$$I = \int \frac{(1 + \frac{2}{x} + \frac{1}{x^2})(1 - \frac{1}{x^2})}{(1 + \frac{1}{x^2})^2 \sqrt{1 + \frac{1}{x^2} + \frac{1}{x^4}}} dx = \int \frac{(x + \frac{1}{x} + 2)(1 - \frac{1}{x^2})}{(x + \frac{1}{x})^2 \sqrt{(x + \frac{1}{x})^2 - 1}}$$

设 $t = x + \frac{1}{x}$, 则 $dt = (1 - \frac{1}{x^2})dx$, 原式化为:

$$I = \int \frac{t+2}{t^2\sqrt{t^2-1}} dt = \int \frac{1}{t\sqrt{t^2-1}} dt + 2 \int \frac{1}{t^2\sqrt{t^2-1}} dt$$

第一部分设 $t = \sec \theta \Rightarrow I_1 = \sec^{-1} |t| + C$, 第二部分设 $t = \frac{1}{\cos \theta}$ 或 $u = \frac{1}{t}$ 还原得 $I_2 = \frac{2\sqrt{t^2-1}}{t} + C$ 。代回 $t = x + \frac{1}{x}$:

$$I = \sec^{-1} \left| x + \frac{1}{x} \right| + \frac{2\sqrt{x^4+x^2+1}}{x^2+1} + C$$

73.

$$\int_0^{\frac{\pi}{2}} \frac{e^x(\sin x + \cos x - 2)}{(\cos x - 2)^2} dx$$

有

$$\int_0^{\frac{\pi}{2}} \frac{e^x(\sin x + \cos x - 2)}{(\cos x - 2)^2} dx = \int_0^{\frac{\pi}{2}} e^x \left(\frac{\sin x}{(\cos x - 2)^2} + \frac{1}{\cos x - 2} \right) dx$$

发现被积函数形如 $e^x(f(x) + f'(x))$, 其中

$$f(x) = \frac{1}{\cos x - 2}, f'(x) = \frac{\sin x}{(\cos x - 2)^2}$$

所以

$$\int_0^{\frac{\pi}{2}} e^x \left(\frac{1}{\cos x - 2} + \frac{\sin x}{(\cos x - 2)^2} \right) dx = \left[\frac{e^x}{\cos x - 2} \right]_0^{\frac{\pi}{2}} = 1 - \frac{e^{\frac{\pi}{2}}}{2}$$

74.

$$\int e^x \left(\frac{1}{\sqrt{x^2 + 1}} + \frac{1 - 2x^2}{(x^2 + 1)^{\frac{5}{2}}} \right) dx$$

观察被积函数, 尝试配凑 $e^x(f(x) + f'(x))$ 的形式。令

$$f(x) = \frac{1}{\sqrt{x^2 + 1}} + \frac{x}{(x^2 + 1)^{\frac{3}{2}}}$$

计算其导数:

$$\begin{aligned} f'(x) &= -\frac{1}{2}(x^2 + 1)^{-\frac{3}{2}}(2x) + \left[1 \cdot (x^2 + 1)^{-\frac{3}{2}} + x \cdot \left(-\frac{3}{2} \right) (x^2 + 1)^{-\frac{5}{2}}(2x) \right] \\ &= -\frac{x}{(x^2 + 1)^{\frac{3}{2}}} + \frac{1}{(x^2 + 1)^{\frac{3}{2}}} - \frac{3x^2}{(x^2 + 1)^{\frac{5}{2}}} \\ &= -\frac{x}{(x^2 + 1)^{\frac{3}{2}}} + \frac{1 - 2x^2}{(x^2 + 1)^{\frac{5}{2}}} \end{aligned}$$

于是原被积函数可改写为:

$$e^x \left[\left(\frac{1}{\sqrt{x^2 + 1}} + \frac{x}{(x^2 + 1)^{\frac{3}{2}}} \right) + \left(-\frac{x}{(x^2 + 1)^{\frac{3}{2}}} + \frac{1 - 2x^2}{(x^2 + 1)^{\frac{5}{2}}} \right) \right] = e^x[f(x) + f'(x)]$$

根据积分性质

$$\int e^x[f(x) + f'(x)] dx = e^x f(x) + C,$$

得

$$I = e^x \left(\frac{1}{\sqrt{x^2 + 1}} + \frac{x}{(x^2 + 1)^{\frac{3}{2}}} \right) + C$$

75.

$$\int \sqrt{x + \sqrt{x^2 + 2}} dx$$

设 $x = \sqrt{2} \sinh t$, 则 $dx = \sqrt{2} \cosh t dt$, 由

$$\sqrt{x^2 + 2} = \sqrt{2} \cosh t, \quad \cosh t + \sinh t = e^t$$

有

$$I = \int \sqrt{\sqrt{2}e^t} \cdot \sqrt{2} \cosh t dt = 2^{\frac{3}{4}} \int e^{\frac{t}{2}} \frac{e^t + e^{-t}}{2} dt = 2^{-\frac{1}{4}} \int (e^{\frac{3t}{2}} + e^{-\frac{t}{2}}) dt$$

积分得

$$I = 2^{-\frac{1}{4}} \left(\frac{2}{3} e^{\frac{3t}{2}} - 2e^{-\frac{t}{2}} \right) + C$$

由 $e^t = \frac{x + \sqrt{x^2 + 2}}{\sqrt{2}}$, 得 $e^{\frac{t}{2}} = \frac{\sqrt{x + \sqrt{x^2 + 2}}}{2^{\frac{1}{4}}}$, 代入各项,

$$\begin{aligned} I &= 2^{-\frac{1}{4}} \cdot \frac{2}{3} \cdot \left(\frac{\sqrt{x + \sqrt{x^2 + 2}}}{2^{\frac{1}{4}}} \right)^3 + -2 \cdot 2^{-\frac{1}{4}} \cdot \left(\frac{\sqrt{x + \sqrt{x^2 + 2}}}{2^{\frac{1}{4}}} \right)^{-1} \\ &= \frac{1}{3} (x + \sqrt{x^2 + 2})^{\frac{3}{2}} - \frac{2}{\sqrt{x + \sqrt{x^2 + 2}}} + C \end{aligned}$$

76.

$$\int \frac{\sin \theta \cos \theta}{(3 + 5 \sin^2 \theta - 2 \cos^2 \theta)^{\frac{1}{3}}} d\theta$$

注意到分母可写成

$$3 + 5 \sin^2 \theta - 2(1 - \sin^2 \theta) = 1 + 7 \sin^2 \theta$$

设 $u = 1 + 7 \sin^2 \theta$, 则 $du = 14 \sin \theta \cos \theta d\theta$, 原积分变为

$$I = \frac{1}{14} \int u^{-\frac{1}{3}} du = \frac{3}{28} u^{\frac{2}{3}} + C = \frac{3}{28} (3 + 5 \sin^2 \theta - 2 \cos^2 \theta)^{\frac{2}{3}} + C$$

77. 使用代换 $u = 1 + \cos^2 x$ 或其他方法, 求

$$\int \frac{4 \cot x}{1 + \cos^2 x} dx$$

作代换 $u = 1 + \cos^2 x$, 则 $du = -2 \sin x \cos x dx$, 注意到

$$\frac{4 \cos x}{\sin x(1 + \cos^2 x)} = \frac{2 \cdot (2 \sin x \cos x)}{(1 - \cos^2 x)(1 + \cos^2 x)}$$

于是原积分为

$$I = \int \frac{-2}{(2-u)u} du = \int \left(\frac{1}{u-2} - \frac{1}{u} \right) du$$

积分得

$$I = [\ln |u-2| - \ln |u|] + C = \ln \left| \frac{u-2}{u} \right| + C$$

代回 $u = 1 + \cos^2 x$,

$$I = \ln \left| \frac{\cos^2 x - 1}{1 + \cos^2 x} \right| + C = \ln \left(\frac{\sin^2 x}{1 + \cos^2 x} \right) + C$$

78. 使用代换 $u = \sin x + x \tan x$ 或其他方法, 求

$$\int \frac{2x + \sin 2x + 2 \cos^3 x}{(x + \cos x) \sin 2x} dx$$

由代换 $u = \sin x + x \tan x$, 则有

$$\frac{du}{dx} = \cos x + \tan x + x \sec^2 x$$

原积分变为

$$\begin{aligned} I &= \int \frac{2x + 2 \sin x \cos x + 2 \cos^3 x}{(x + \cos x)(2 \sin x \cos x)} \cdot \frac{1}{\cos x + \tan x + x \sec^2 x} du \\ &= \int \frac{x + \sin x \cos x + \cos^3 x}{(x + \cos x) \sin x \cos x} \cdot \frac{\cos^2 x}{\cos^3 x + \sin x \cos x + x} du \\ &= \int \frac{\cos^2 x}{(x + \cos x) \sin x \cos x} du \\ &= \int \frac{1}{x \tan x + \sin x} du \\ &= \int \frac{1}{u} du \end{aligned}$$

积分得

$$I = \ln |u| + C = \ln |\sin x + x \tan x| + C$$

79. 使用代换 $u = 1 + x^2 \csc x$ 或其他方法, 求积分

$$\int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx$$

作代换 $u = 1 + x^2 \csc x$, 对 x 求导得

$$\frac{du}{dx} = 2x \csc x - x^2 \csc x \cot x = x \csc x (2 - x \cot x)$$

代入积分式:

$$\begin{aligned} \int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx &= \int \frac{x(2 - x \cot x)}{x^2 + \sin x} \cdot \frac{du}{x \csc x (2 - x \cot x)} \\ &= \int \frac{1}{x^2 \csc x + 1} du \\ &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |1 + x^2 \csc x| + C \end{aligned}$$

80. (a) 证明

$$\frac{d}{d\theta}(\tan^3 \theta) = 3 \tan^4 \theta + 3 \sec^2 \theta - 3,$$

据此, 求

$$\int_0^{\frac{\pi}{4}} \tan^4 \theta d\theta$$

有

$$\begin{aligned} \frac{d}{d\theta} \tan^3 \theta &= 3 \tan^2 \theta \sec^2 \theta \\ &= 3 \tan^2 \theta (\tan^2 \theta + 1) \\ &= 3 \tan^4 \theta + 3 \tan^2 \theta \\ &= 3 \tan^4 \theta + 3(\sec^2 \theta - 1) \\ &= 3 \tan^4 \theta + 3 \sec^2 \theta - 3 \end{aligned}$$

故得证, 于是

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \tan^4 \theta d\theta &= \frac{1}{3} \int_0^{\frac{\pi}{4}} (3 \tan^4 \theta) d\theta \\
 &= \frac{1}{3} \int_0^{\frac{\pi}{4}} \left(\frac{d}{d\theta} \tan^3 \theta - 3 \sec^2 \theta + 3 \right) d\theta \\
 &= \left[\frac{1}{3} (\tan^3 \theta - 3 \tan \theta + 3\theta) \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{3} \left[1 - 3 + \frac{3\pi}{4} \right] = \frac{\pi}{4} - \frac{2}{3}
 \end{aligned}$$

(b) 证明

$$\frac{d}{d\theta} \tan^3 \theta = 3 \sec^4 \theta - 3 \sec^2 \theta$$

由此计算

$$\int_0^{\frac{\pi}{4}} \sec^4 \theta d\theta$$

有

$$\frac{d}{d\theta} \tan^3 \theta = 3 \tan^2 \theta \sec^2 \theta = 3(\sec^2 \theta - 1)(\sec^2 \theta) = 3 \sec^4 \theta - 3 \sec^2 \theta$$

故得证, 于是

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \sec^4 \theta d\theta &= \frac{1}{3} \int_0^{\frac{\pi}{4}} (3 \sec^4 \theta) d\theta \\
 &= \frac{1}{3} \int_0^{\frac{\pi}{4}} \left(\frac{d}{d\theta} \tan^3 \theta + 3 \sec^2 \theta \right) d\theta \\
 &= \left[\frac{1}{3} (\tan^3 \theta + 3 \tan \theta) \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{3} (1 + 3) = \frac{4}{3}
 \end{aligned}$$

81.

$$\int \sin^3 2x \cos^2 3x dx$$

利用倍角公式

$$\sin^3 \theta = \frac{3 \sin \theta - \sin 3\theta}{4}, \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

展开积分得

$$\begin{aligned} & \int \sin^3 2x \cos^2 3x dx \\ &= \frac{1}{8} \int (3 \sin 2x - \sin 6x)(1 + \cos 6x) dx \\ &= \frac{1}{8} \int (3 \sin 2x + 3 \sin 2x \cos 6x - \sin 6x - \sin 6x \cos 6x) dx \\ &= \frac{1}{8} \int \left(3 \sin 2x + \frac{3}{2}(\sin 8x - \sin 4x) - \sin 6x - \frac{1}{2} \sin 12x \right) dx \\ &= -\frac{3}{16} \cos 2x + \frac{3}{64} \cos 4x + \frac{1}{48} \cos 6x - \frac{3}{128} \cos 8x + \frac{1}{192} \cos 12x + C \end{aligned}$$

82.

$$\int_0^{\frac{\pi}{4}} \frac{3 - 4 \cos 2x + \cos 4x}{3 + 4 \cos 2x + \cos 4x} dx$$

注意到

$$3 - 4 \cos 2x + \cos 4x = 3 - 4 \cos 2x + (2 \cos^2 2x - 1) = 2(1 - \cos 2x)^2 = 8 \sin^4 x$$

同理可得 $3 + 4 \cos 2x + \cos 4x = 8 \cos^4 x$, 于是原积分变为

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \frac{8 \sin^4 x}{8 \cos^4 x} dx \\ &= \int_0^{\frac{\pi}{4}} \tan^4 x dx \\ &= \int_0^{\frac{\pi}{4}} \tan^2 x (\sec^2 x - 1) dx \\ &= \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x dx - \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) dx \\ &= \left[\frac{1}{3} \tan^3 x - \tan x + x \right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4} - \frac{2}{3} \end{aligned}$$

83.

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2} dx$$

分母展开得

$$(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2 = 1 + 1 + 2 \cos 6x = 4 \cos^2 3x$$

故原积分变为

$$I = \frac{1}{4} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 3x}{\cos^2 3x} dx = \left[\frac{1}{12} \sec 3x \right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = \frac{1}{12} \left(\sec \frac{3\pi}{4} - \sec \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{6}$$

84.

$$\int_0^{\frac{\pi}{4}} \left(\frac{\sin^2 5\theta}{\sin^2 \theta} - \frac{\cos^2 5\theta}{\cos^2 \theta} \right) d\theta$$

先化简被积函数, 有

$$\begin{aligned} \frac{\sin^2 5\theta}{\sin^2 \theta} - \frac{\cos^2 5\theta}{\cos^2 \theta} &= \frac{\sin^2 5\theta \cos^2 \theta - \cos^2 5\theta \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{(\sin 5\theta \cos \theta - \cos 5\theta \sin \theta)(\sin 5\theta \cos \theta + \cos 5\theta \sin \theta)}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{\sin(5\theta - \theta) \sin(5\theta + \theta)}{\sin^2 \theta \cos^2 \theta} \\ &= 4 \cdot \frac{\sin 4\theta \sin 6\theta}{\sin^2 2\theta} \\ &= 4 \cdot \frac{2 \sin 2\theta \cos 2\theta \cdot (3 \sin \theta - 4 \sin^3 \theta)}{\sin^2 2\theta} \\ &= 8 \cos 2\theta (3 - 4 \sin^2 2\theta) \\ &= 8 \cos 2\theta [3 - 2(1 - \cos 4\theta)] \\ &= 8(\cos 2\theta + 2 \cos 4\theta \cos 2\theta) \\ &= 8(\cos 2\theta + \cos 6\theta + \cos 2\theta) \\ &= 8(2 \cos 2\theta + \cos 6\theta) \end{aligned}$$

积分得

$$\int_0^{\frac{\pi}{4}} \left(\frac{\sin^2 5\theta}{\sin^2 \theta} - \frac{\cos^2 5\theta}{\cos^2 \theta} \right) d\theta = \left[8 \sin 2\theta + \frac{4}{3} \sin 6\theta \right]_0^{\frac{\pi}{4}} = \frac{20}{3}$$

85.

$$\int \cos^5 x \sin 5x \, dx$$

利用恒等式 $\cos^5 x = \frac{1}{16}(\cos 5x + 5 \cos 3x + 10 \cos x)$, 积分变为

$$\begin{aligned} I &= \frac{1}{16} \int \sin 5x \cos 5x + 5 \sin 5x \cos 3x + 10 \sin 5x \cos x \, dx \\ &= \frac{1}{32} \int \sin 10x + 5(\sin 8x + \sin 2x) + 10(\sin 6x + \sin 4x) \, dx \\ &= -\frac{1}{32} \left(\frac{1}{10} \cos 10x + \frac{5}{8} \cos 8x + \frac{5}{3} \cos 6x + \frac{5}{2} \cos 4x + \frac{5}{2} \cos 2x \right) + C \end{aligned}$$

86.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cot^3 x}{\csc x} \, dx$$

注意到

$$\frac{\cot^3 x}{\csc x} = \frac{\cos^3 x}{\sin^2 x},$$

设 $u = \sin x$, 则 $du = \cos x \, dx$,

$$I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1-u^2}{u^2} \, du = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} (u^{-2} - 1) \, du$$

积分得:

$$I = \left[-\frac{1}{u} - u \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{15 - 7\sqrt{3}}{6}$$

87.

$$\int \frac{\sqrt{\cos 2x}}{\sin x} \, dx$$

将原积分写成

$$I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx = \int \frac{\sqrt{2\cos^2 x - 1}}{\sin^2 x} \sin x dx$$

设 $t = \cos x$, 则 $dt = -\sin x dx$,

$$I = \int \frac{\sqrt{2t^2 - 1}}{t^2 - 1} dt = \int \frac{2(t^2 - 1) + 1}{(t^2 - 1)\sqrt{2t^2 - 1}} dt$$

拆分为两个积分:

$$I = \int \frac{2}{\sqrt{2t^2 - 1}} dt + \int \frac{1}{(t^2 - 1)\sqrt{2t^2 - 1}} dt$$

对于第一部分积分 I_1 易知

$$I_1 = \sqrt{2} \int \frac{1}{\sqrt{t^2 - \frac{1}{2}}} dt = \sqrt{2} \ln \left| t + \sqrt{t^2 - \frac{1}{2}} \right| + C_1 = \sqrt{2} \ln(\sqrt{2}t + \sqrt{2t^2 - 1}) + C_1$$

第二部分积分 I_2 , 令 $t = \frac{1}{z}$, 则 $dt = -\frac{1}{z^2} dz$:

$$I_2 = \int \frac{1}{(\frac{1}{z^2} - 1)\sqrt{\frac{2}{z^2} - 1}} \left(-\frac{1}{z^2}\right) dz = \int \frac{z}{(z^2 - 1)\sqrt{2 - z^2}} dz$$

再令 $w = \sqrt{2 - z^2}$, 则 $w^2 = 2 - z^2, z dz = -w dw$:

$$I_2 = \int \frac{-w}{(2 - w^2 - 1)w} dw = \int \frac{1}{w^2 - 1} dw = \frac{1}{2} \int \left(\frac{1}{w - 1} - \frac{1}{w + 1} \right) dw$$

积分得

$$I_2 = \frac{1}{2} \ln \left| \frac{w - 1}{w + 1} \right| + C_2 = \frac{1}{2} \ln \left| \frac{\sqrt{2 - z^2} - 1}{\sqrt{2 - z^2} + 1} \right| + C_2 = \frac{1}{2} \ln \left| \frac{\sqrt{2t^2 - 1} - t}{\sqrt{2t^2 - 1} + t} \right| + C_2$$

合并 I_1, I_2 ,

$$I = \sqrt{2} \ln(\sqrt{2}t + \sqrt{2t^2 - 1}) + \frac{1}{2} \ln \left| \frac{\sqrt{2t^2 - 1} - t}{\sqrt{2t^2 - 1} + t} \right| + C$$

还原变量 $t = \cos x$ 且 $\sqrt{2\cos^2 x - 1} = \sqrt{\cos 2x}$:

$$I = \sqrt{2} \ln(\sqrt{2} \cos x + \sqrt{\cos 2x}) + \frac{1}{2} \ln \left| \frac{\sqrt{\cos 2x} - \cos x}{\sqrt{\cos 2x} + \cos x} \right| + C$$

88.

$$\int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx$$

设 $t = \sec x + \tan x$, 则

$$dt = (\sec x \tan x + \sec^2 x) dx = t \sec x dx$$

且由恒等式 $\sec^2 x - \tan^2 x = (\sec x + \tan x)(\sec x - \tan x) = 1$,

$$\sec x - \tan x = \frac{1}{t}$$

将 $\sec x + \tan x = t$ 与 $\sec x - \tan x = \frac{1}{t}$ 相加得

$$2 \sec x = t + \frac{1}{t} \Rightarrow \sec x = \frac{1}{2} \left(t + \frac{1}{t} \right)$$

代入原积分得

$$\begin{aligned} I &= \int \frac{\sec^2 x}{t^{\frac{9}{2}}} \cdot \frac{dt}{t \sec x} \\ &= \int \frac{1}{2} (t + t^{-1}) t^{-\frac{11}{2}} dt \\ &= \frac{1}{2} \int (t^{-\frac{9}{2}} + t^{-\frac{13}{2}}) dt \\ &= -\frac{1}{7} t^{-\frac{7}{2}} - \frac{1}{11} t^{-\frac{11}{2}} + C \\ &= -\frac{1}{7(\sec x + \tan x)^{\frac{7}{2}}} - \frac{1}{11(\sec x + \tan x)^{\frac{11}{2}}} + C \end{aligned}$$

89.

$$\int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx$$

作代换 $u = 1 + \sin^2 x$, 则 $\frac{du}{dx} = 2 \sin x \cos x$, 积分变为

$$I = \int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx = \int_1^{\frac{3}{2}} \frac{4 \tan x}{u} \cdot \frac{du}{2 \sin x \cos x} = \int_1^{\frac{3}{2}} \frac{2}{u(2-u)} du$$

部分分式分解, 积分得

$$I = \int_1^{\frac{3}{2}} \left(\frac{1}{u} + \frac{1}{2-u} \right) du = \left[\ln |u| - \ln |2-u| \right]_1^{\frac{3}{2}} = \ln 3$$

90. 使用代换 $u = 1 - \tan^2 x$, 求定积分

$$\int_0^{\frac{\pi}{6}} \tan x \sec 2x dx$$

由代换 $u = 1 - \tan^2 x$, 有 $\frac{du}{dx} = -2 \tan x \sec^2 x$, 且

$$\sec 2x = \frac{1 + \tan^2 x}{1 - \tan^2 x} = \frac{\sec^2 x}{u}$$

积分化为

$$\int_0^{\frac{\pi}{6}} \tan x \sec 2x dx = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{u} du = \left[\frac{1}{2} \ln |u| \right]_{\frac{2}{3}}^1 = \frac{1}{2} \ln \frac{3}{2}$$

91.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x} dx$$

由恒等式 $\cos^2 x + \sin^2 x = 1$, 原积分化为

$$\begin{aligned} I &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x} dx = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x + \sin^2 x}{2 \sin^2 x \cos x} dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\cos x}{\sin x} + \sec x \right) dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\cot x \csc x + \sec x) dx \end{aligned}$$

积分得

$$I = \frac{1}{2} \left[-\csc x + \ln |\sec x + \tan x| \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2} \ln \left(\frac{2 + \sqrt{3}}{\sqrt{3}} \right) + 1 - \frac{\sqrt{3}}{3}$$

92.

$$\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2} \sin^2 2x} dx$$

从化简被积函数开始,

$$\begin{aligned} \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2} \sin^2 2x} &= \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2 \sin^2 x \cos^2 x} \\ &= \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} \\ &= \frac{(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x)}{\sin^4 x + \cos^4 x} \\ &= \sin^2 x - \cos^2 x \\ &= -\cos 2x \end{aligned}$$

故所求积分为

$$-\frac{1}{2} \sin 2x + C$$

93.

$$\int \frac{3 \sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx$$

重写被积函数,

$$I = \int \frac{3 \sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx = \int 3 \tan^2 x \sec^2 x (1 - \tan^3 x)^{-2} dx$$

设 $u = \tan x \implies du = \sec^2 x dx$, 代入:

$$I = \int 3u^2(1 - u^3)^{-2} du = (1 - u^3)^{-1} + C = \frac{1}{1 - \tan^3 x} + C$$

94.

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

先拆分分数,

$$\begin{aligned} I &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos^2 x} + \frac{\cos^4 x}{\sin^2 x} dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{(1 - \cos^2 x)^2}{\cos^2 x} + \frac{(1 - \sin^2 x)^2}{\sin^2 x} dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} (\sec^2 x - 2 + \cos^2 x + \csc^2 x - 2 + \sin^2 x) dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} (\sec^2 x + \csc^2 x - 3) dx \end{aligned}$$

积分得

$$I = \left[\tan x - \cot x - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} = 2 - \frac{3\pi}{8}$$

95. 求积分

$$\int_0^{\pi/2} \frac{\sin 2x}{\sqrt{4 - \sin^4 x}} dx$$

令 $u = \sin^2 x$, 则 $du = 2 \sin x \cos x dx = \sin 2x dx$, 原积分变为

$$\int_0^{\pi/2} \frac{\sin 2x}{\sqrt{4 - \sin^4 x}} dx = \int_0^1 \frac{1}{\sqrt{4 - u^2}} du = \left[\arcsin \frac{u}{2} \right]_0^1 = \frac{\pi}{6}$$

96.

$$\int_0^{\frac{\pi}{2}} \frac{1 + \cos x + \sin x - \tan x}{1 + \tan x} dx$$

上下乘以 $\cos x$, 得

$$\begin{aligned}& \int_0^{\frac{\pi}{2}} \frac{1 + \cos x + \sin x - \tan x}{1 + \tan x} dx \\&= \int_0^{\frac{\pi}{2}} \frac{\cos x + \cos^2 x + \sin x \cos x - \sin x}{\cos x + \sin x} dx \\&= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x + \sin x} + \cos x dx \\&= \left[\ln |\cos x + \sin x| + \sin x \right]_0^{\frac{\pi}{2}} \\&= 1\end{aligned}$$

97.

$$\int \frac{\sin x \cos x}{\sin x + \cos x} dx$$

由 $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$,

$$\begin{aligned}\int \frac{\sin x \cos x}{\sin x + \cos x} dx &= \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x} dx \\&= \frac{1}{2} \int \left(\sin x + \cos x - \frac{1}{\sin x + \cos x} \right) dx\end{aligned}$$

现考虑积分

$$I = \int \frac{1}{\sin x + \cos x} dx$$

由于 $\sin x + \cos x = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$, 变为

$$\begin{aligned}I &= \int \frac{1}{\sqrt{2} \sin \left(x + \frac{\pi}{4} \right)} dx \\&= \frac{1}{\sqrt{2}} \int \csc \left(x + \frac{\pi}{4} \right) dx \\&= -\frac{1}{\sqrt{2}} \ln \left| \csc \left(x + \frac{\pi}{4} \right) + \cot \left(x + \frac{\pi}{4} \right) \right| + C_1\end{aligned}$$

代入原式可得

$$\int \frac{\sin x \cos x}{\sin x + \cos x} dx = \frac{1}{2}(\sin x - \cos x) + \frac{1}{2\sqrt{2}} \ln \left| \csc \left(x + \frac{\pi}{4} \right) + \cot \left(x + \frac{\pi}{4} \right) \right| + C$$

由 $\sin x + \cos x = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$,

$$\int \frac{\sin x \cos x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \int \frac{\sin x \cos x}{\sin \left(x + \frac{\pi}{4} \right)} dx$$

令 $u = x + \frac{\pi}{4}$, $dx = du$, 则

$$\sin x = \frac{\sin u - \cos u}{\sqrt{2}}, \quad \cos x = \frac{\sin u + \cos u}{\sqrt{2}}$$

$$\begin{aligned} \therefore \int \frac{\sin x \cos x}{\sin x + \cos x} dx &= \frac{1}{\sqrt{2}} \int \frac{\frac{1}{2}(\sin u - \cos u)(\sin u + \cos u)}{\sin u} du \\ &= \frac{1}{2\sqrt{2}} \int \frac{2\sin^2 u - 1}{\sin u} du \\ &= \frac{1}{\sqrt{2}} \int \sin u du - \frac{1}{2\sqrt{2}} \int \csc u du \\ &= -\frac{\cos(x + \frac{\pi}{4})}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln \left| \csc \left(x + \frac{\pi}{4} \right) + \cot \left(x + \frac{\pi}{4} \right) \right| + C \end{aligned}$$

98.

$$\int_0^{\pi/4} \frac{10}{2 - \tan x} dx$$

令 $u = \tan x$, 则 $du = \sec^2 x dx = (1 + u^2) dx$, 积分变为

$$I = \int_0^1 \frac{10}{(2 - u)(1 + u^2)} du$$

使用部分分式分解,

$$\frac{10}{(2 - u)(1 + u^2)} = \frac{4 + 2u}{u^2 + 1} + \frac{2}{2 - u}$$

积分可写为

$$\begin{aligned} I &= \int_0^1 \left(\frac{4 + 2u}{u^2 + 1} + \frac{2}{2 - u} \right) du \\ &= \int_0^1 \left(\frac{4}{u^2 + 1} + \frac{2u}{u^2 + 1} + \frac{2}{2 - u} \right) du \\ &= \left[4 \arctan u + \ln(u^2 + 1) - 2 \ln |2 - u| \right]_0^1 \\ &= \pi + 3 \ln 2 \end{aligned}$$

99.

$$\int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2 \cos x)(\sin x + 3 \cos x)} dx$$

将被积函数写成

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{\sin^2 x + 5 \sin x \cos x + 6 \cos^2 x} dx = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx$$

令 $u = \tan x$, 则 $du = \sec^2 x dx$ 。当 $x = \arcsin \frac{3}{5}$ 时, $u = \frac{3}{4}$, 当 $x = \arccos \frac{3}{5}$ 时, $u = \frac{4}{3}$, 积分化为:

$$I = \int_{\frac{3}{4}}^{\frac{4}{3}} \frac{1}{u^2 + 5u + 6} du = \int_{\frac{3}{4}}^{\frac{4}{3}} \frac{1}{(u+2)(u+3)} du$$

部分分式分解:

$$\frac{1}{(u+2)(u+3)} = \frac{1}{u+2} - \frac{1}{u+3}$$

积分得

$$I = \left[\ln(u+2) - \ln(u+3) \right]_{\frac{3}{4}}^{\frac{4}{3}} = \ln \frac{150}{143}$$

100. 若常数 $|k| \neq 1$, 求

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + k^2 \tan^2 x} dx,$$

令 $u = \tan x$, 则 $dx = \frac{du}{1+u^2}$, 积分变为:

$$I = \int_0^{\infty} \frac{1}{1 + k^2 u^2} \cdot \frac{1}{1 + u^2} du = \int_0^{\infty} \frac{1}{(u^2 + 1)(k^2 u^2 + 1)} du$$

使用部分分式分解:

$$\frac{1}{(u^2 + 1)(k^2 u^2 + 1)} = \frac{1}{k^2 - 1} \left(\frac{k^2}{k^2 u^2 + 1} - \frac{1}{u^2 + 1} \right)$$

由于

$$\int_0^{\infty} \frac{k^2}{k^2 u^2 + 1} du = k \arctan(ku) \Big|_0^{\infty} = \frac{k\pi}{2}, \quad \int_0^{\infty} \frac{1}{u^2 + 1} du = \arctan u \Big|_0^{\infty} = \frac{\pi}{2}$$

代回并化简得

$$I = \frac{1}{k^2 - 1} \left(\frac{k\pi}{2} - \frac{\pi}{2} \right) = \frac{(k-1)\pi}{2(k-1)(k+1)} = \frac{\pi}{2(k+1)}$$

101. 使用代换 $u = \sin x + \csc x$, 求

$$\int \frac{\cos^3 x}{(1 + \sin x) \sin x} dx$$

由代换 $u = \sin x + \csc x$, 得 $du = (\cos x - \cot x \csc x) dx$, 代入积分

$$\begin{aligned} \int \frac{\cos^3 x}{(1 + \sin x) \sin x} dx &= \int \frac{\cos^3 x}{(1 + \sin x) \sin x} \cdot \frac{1}{\cos x - \cot x \csc x} du \\ &= \int \frac{\cos^3 x}{(1 + \sin x) \sin x} \cdot \frac{\sin^2 x}{\cos x (\sin^2 x - 1)} du \\ &= \int \frac{\cos^2 x}{1 + \sin x} \cdot \frac{\sin x}{-\cos^2 x} du \\ &= \int -\frac{1}{\csc x + \sin x} du \\ &= \int -\frac{1}{u} du \\ &= \ln \left| \frac{1}{u} \right| + C \\ &= \ln \left| \frac{1}{\sin x + \csc x} \right| + C \end{aligned}$$

102. 求

$$\int \frac{\sin x + \sin 3x}{\cos x \sqrt{4 - \cos^4 x}} dx$$

和差化积得

$$I = \int \frac{\sin x + \sin 3x}{\cos x \sqrt{4 - \cos^4 x}} dx = \int \frac{2 \sin 2x \cos x}{\cos x \sqrt{4 - \cos^4 x}} dx = \int \frac{4 \sin x \cos x}{\sqrt{4 - \cos^4 x}} dx$$

设 $u = \cos^2 x$, 则 $du = -2 \sin x \cos x dx$ 。

$$I = \int \frac{-2}{\sqrt{4 - u^2}} du = -2 \arcsin \frac{u}{2} + C = -2 \arcsin \left(\frac{\cos^2 x}{2} \right) + C$$

103. 求积分

$$\int_0^{\pi/4} \frac{1}{\cos^2 x + 25 \sin^2 x} dx$$

注意到

$$\frac{1}{\cos^2 x + 25 \sin^2 x} = \frac{\sec^2 x}{1 + 25 \tan^2 x}$$

设 $u = \tan x$, 则 $du = \sec^2 x dx$, 原积分变为:

$$\int_0^{\pi/4} \frac{1}{\cos^2 x + 25 \sin^2 x} dx = \int_0^1 \frac{1}{1 + 25u^2} du = \frac{5}{25} [\arctan(5u)]_0^1 = \frac{1}{5} \arctan 5$$

104.

$$\int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + \sin 2x \tan \frac{x}{2}} dx$$

使用半角代换 (万能公式):

$$t = \tan \frac{x}{2} \implies dx = \frac{2}{1+t^2} dt$$

当 $x = 0$ 时 $t = 0$, 当 $x = \frac{\pi}{3}$ 时 $t = \frac{\sqrt{3}}{3}$, 且有

$$\sin 2x = 2 \sin x \cos x = 2 \left(\frac{2t}{1+t^2} \right) \left(\frac{1-t^2}{1+t^2} \right) = \frac{4t(1-t^2)}{(1+t^2)^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

积分得

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + \sin 2x \tan \frac{x}{2}} dx &= \int_0^{\frac{\sqrt{3}}{3}} \frac{6\sqrt{3} \cdot \frac{1-t^2}{1+t^2}}{4 + \frac{4t(1-t^2)}{(1+t^2)^2} \cdot t} \cdot \frac{2}{1+t^2} dt \\ &= \int_0^{\frac{\sqrt{3}}{3}} \frac{3\sqrt{3}(1-t^2)}{1+3t^2} dt \\ &= \sqrt{3} \int_0^{\frac{\sqrt{3}}{3}} \left(\frac{4}{1+3t^2} - 1 \right) dt \\ &= \sqrt{3} \left[\frac{4}{\sqrt{3}} \arctan(\sqrt{3}t) - t \right]_0^{\frac{\sqrt{3}}{3}} \\ &= \pi - 1 \end{aligned}$$

105.

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\sin x}} dx$$

使用万能代换 $t = \tan \frac{x}{2}$, 则

$$dx = \frac{2 dt}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$$

代入原积分,

$$I = \int_0^1 \sqrt{\frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2}}} \cdot \frac{2}{1+t^2} dt = \int_0^1 \sqrt{\frac{2t^2}{1+t^2+2t}} \cdot \frac{2}{1+t^2} dt = 2\sqrt{2} \int_0^1 \frac{t}{(t+1)(t^2+1)} dt$$

利用部分分式分解得

$$\frac{t}{(t+1)(t^2+1)} = \frac{1}{2} \left(\frac{t+1}{t^2+1} - \frac{1}{t+1} \right) = \frac{1}{2} \left(\frac{t}{t^2+1} + \frac{1}{t^2+1} - \frac{1}{t+1} \right)$$

于是

$$I = \sqrt{2} \left[\frac{1}{2} \ln(t^2+1) + \arctan t - \ln|t+1| \right]_0^1 = \sqrt{2} \left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right)$$

106.

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

令 $u = \sin x - \cos x$, 则 $du = (\cos x + \sin x) dx$, 利用恒等式 $u^2 = (\sin x - \cos x)^2 = 1 - \sin 2x$, 得到 $\sin 2x = 1 - u^2$, 分母变形为

$$9 + 16 \sin 2x = 9 + 16(1 - u^2) = 25 - 16u^2$$

代入原积分

$$I = \int_{-1}^0 \frac{1}{25 - 16u^2} du = \int_{-1}^0 \frac{1}{(5-4u)(5+4u)} du$$

分解得

$$\frac{1}{(5-4u)(5+4u)} = \frac{1}{10} \left(\frac{1}{5-4u} + \frac{1}{5+4u} \right)$$

故

$$I = \frac{1}{10} \left[-\frac{1}{4} \ln|5-4u| + \frac{1}{4} \ln|5+4u| \right]_{-1}^0 = \frac{1}{20} \ln 3$$

107.

$$\int_0^{\frac{\pi}{8}} \frac{\sqrt{3}}{2 + \sin 4x} dx$$

使用万能公式, 设 $u = \tan 2x$, 则 $du = 2 \sec^2 2x dx$, 原积分变为

$$\begin{aligned} I &= \int_0^1 \frac{\sqrt{3}}{2 + \frac{2u}{1+u^2}} \cdot \frac{du}{2(1+u^2)} \\ &= \frac{\sqrt{3}}{4} \int_0^1 \frac{1}{u^2 + u + 1} du \\ &= \frac{\sqrt{3}}{4} \int_0^1 \frac{1}{(u + \frac{1}{2})^2 + \frac{3}{4}} du \\ &= \frac{\sqrt{3}}{4} \cdot \frac{4}{3} \int_0^1 \frac{1}{\left[\frac{2}{\sqrt{3}}(u + \frac{1}{2})\right]^2 + \frac{3}{4}} du \\ &= \frac{\sqrt{3}}{4} \cdot \frac{4}{3} \cdot \frac{\sqrt{3}}{2} \int_0^1 \frac{\frac{2}{\sqrt{3}}}{\left[\frac{2}{\sqrt{3}}(u + \frac{1}{2})\right]^2 + \frac{3}{4}} du \\ &= \frac{1}{2} \left[\arctan \left(\frac{2}{\sqrt{3}} \left(u + \frac{1}{2} \right) \right) \right]_0^1 \\ &= \frac{\pi}{12} \end{aligned}$$

108.

$$\int \frac{1}{1 + \sin^2 x} dx$$

分子分母除以 $\cos^2 x$,

$$I = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx = \int \frac{\sec^2 x}{1 + 2 \tan^2 x} dx$$

设 $u = \sqrt{2} \tan x$, 则 $du = \sqrt{2} \sec^2 x dx$, 代入积分式:

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{u^2 + 1} du = \frac{1}{\sqrt{2}} \arctan u + C = \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + C$$

109.

$$\int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2 + \cos 3x} dx$$

使用万能代换 $t = \tan \frac{3x}{2}$, 则 $dx = \frac{2}{3(1+t^2)} dt$, 积分变为:

$$I = \int_1^{\sqrt{3}} \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{3(1+t^2)} dt = \frac{2}{3} \int_1^{\sqrt{3}} \frac{1}{t^2 + 3} dt$$

积分得

$$I = \frac{2}{3} \cdot \frac{1}{\sqrt{3}} \left[\arctan \frac{t}{\sqrt{3}} \right]_1^{\sqrt{3}} = \frac{\pi\sqrt{3}}{54}$$

110.

$$\int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx$$

设 $u = \sec x + \tan x$, 则

$$du = (\sec x \tan x + \sec^2 x) dx = \sec x (\tan x + \sec x) dx,$$

又由 $\sec^2 x - \tan^2 x = 1$ 可知 $\sec x - \tan x = \frac{1}{u}$ 。将 $\sec x + \tan x = u$ 与 $\sec x - \tan x = \frac{1}{u}$ 相加得:

$$2 \sec x = u + \frac{1}{u} \implies \sec x = \frac{1}{2} \left(u + \frac{1}{u} \right)$$

于是

$$\begin{aligned} I &= \int \frac{\sec x \cdot \sec x}{\sqrt{\sec x + \tan x}} dx = \int \frac{\frac{1}{2} \left(u + \frac{1}{u} \right) \cdot \frac{1}{u}}{\sqrt{u}} du \\ &= \frac{1}{2} \int \left(u^{-\frac{1}{2}} + u^{-\frac{5}{2}} \right) du \\ &= u^{\frac{1}{2}} - \frac{1}{3} u^{-\frac{3}{2}} + C \\ &= \sqrt{\sec x + \tan x} - \frac{1}{3(\sec x + \tan x)^{\frac{3}{2}}} + C \end{aligned}$$

111. 设

$$I = \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx$$

(a) 证明:

$$I + \pi = \int_0^{\frac{\pi}{2}} \frac{4}{1 + \cos^2 x} dx$$

首先对 I 的分子进行变形:

$$I = \int_0^{\pi} \frac{(1 + \cos^2 x) - (1 - \sin^2 x + \cos^2 x)}{1 + \cos^2 x} dx = \int_0^{\pi} \left(1 - \frac{2 \cos^2 x}{1 + \cos^2 x} \right) dx$$

于是

$$\begin{aligned} I + \pi &= \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{1 + \cos^2 x}{1 + \cos^2 x} dx \\ &= \int_0^{\pi} \frac{1 + \sin^2 x + \cos^2 x}{1 + \cos^2 x} dx \\ &= \int_0^{\pi} \frac{2}{1 + \cos^2 x} dx \end{aligned}$$

由于被积函数关于 $x = \frac{\pi}{2}$ 对称,

$$I + \pi = 2 \int_0^{\frac{\pi}{2}} \frac{2}{1 + \cos^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{4}{1 + \cos^2 x} dx$$

(b) 据此, 求 I 的值。

分子分母同乘 $\sec^2 x$:

$$I + \pi = \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{\sec^2 x + 1} dx = \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{2 + \tan^2 x} dx$$

设 $t = \tan x$, 则 $dt = \sec^2 x dx$:

$$I + \pi = \int_0^{\infty} \frac{4}{2 + t^2} dt = 4 \cdot \frac{1}{\sqrt{2}} \arctan \left[\frac{t}{\sqrt{2}} \right]_0^{\infty} = \pi \sqrt{2}$$

故

$$I = \pi(\sqrt{2} - 1)$$

(c) 用另一种方法验证 (b) 的结果, 先将被积函数写成 $\cot^2 x$ 的函数。

将分子分母同除以 $\sin^2 x$, 写成 $\cot^2 x$ 的函数:

$$I = \int_0^\pi \frac{1}{\csc^2 x + \cot^2 x} dx = \int_0^\pi \frac{1}{1 + 2 \cot^2 x} dx$$

设 $u = \cot x$, 则 $dx = -\frac{du}{1+u^2}$, 于是

$$I = \int_\infty^{-\infty} \frac{1}{1+2u^2} \left(-\frac{du}{1+u^2}\right) = \int_{-\infty}^{\infty} \frac{1}{(1+2u^2)(1+u^2)} du$$

利用部分分式分解:

$$\frac{1}{(1+2u^2)(1+u^2)} = \frac{2}{1+2u^2} - \frac{1}{1+u^2}$$

积分得

$$I = \left[\sqrt{2} \arctan(\sqrt{2}u) - \arctan u \right]_{-\infty}^{\infty} = \pi(\sqrt{2} - 1)$$

结论与 (b) 部分一致。

112. (a) 证明

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

从 $\tan 3\theta = \tan(2\theta + \theta)$ 开始:

$$\begin{aligned} \tan 3\theta &= \tan(2\theta + \theta) \\ &= \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} \\ &= \frac{\frac{2 \tan \theta}{1 - \tan^2 \theta} + \tan \theta}{1 - \frac{2 \tan \theta}{1 - \tan^2 \theta} \tan \theta} \\ &= \frac{2 \tan \theta + \tan \theta(1 - \tan^2 \theta)}{1 - \tan^2 \theta - 2 \tan^2 \theta} = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \end{aligned}$$

故得证。

(b) 计算

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx$$

考虑积分:

$$I = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(1-3x^2)(x^2+1)} dx$$

令 $x = \tan \theta \implies dx = \sec^2 \theta d\theta$,

$$I = \int_0^{\frac{\pi}{12}} \frac{6 \tan 3\theta}{\tan^2 \theta + 1} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{12}} 6 \tan 3\theta d\theta$$

积分得

$$I = \left[2 \ln |\sec 3\theta| \right]_0^{\frac{\pi}{12}} = \ln 2$$

113.

$$\int \frac{\sin^3 \frac{x}{2}}{\cos \frac{x}{2} \sqrt{\cos x + \cos^2 x + \cos^3 x}} dx$$

注意到

$$\begin{aligned} \frac{\sin^3 \frac{x}{2}}{\cos \frac{x}{2} \sqrt{\cos x + \cos^2 x + \cos^3 x}} &= \frac{\sin^2 \frac{x}{2} \cdot \sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2} \sqrt{\cos x + \cos^2 x + \cos^3 x}} \\ &= \frac{\sin x (1 - \cos x)}{2(1 + \cos x) \sqrt{\cos x + \cos^2 x + \cos^3 x}} \end{aligned}$$

考虑

$$\int \frac{\sin x (1 - \cos x)}{2(1 + \cos x) \sqrt{\cos x + \cos^2 x + \cos^3 x}} dx$$

设 $t = \cos x$, $dt = -\sin x dx$, 则

$$I = \frac{1}{2} \int \frac{t-1}{(t+1)\sqrt{t+t^2+t^3}} dt = \frac{1}{2} \int \frac{1-\frac{1}{t^2}}{(t+2+\frac{1}{t})\sqrt{t+1+\frac{1}{t}}} dt$$

设 $u = \sqrt{t+1+\frac{1}{t}}$, 则 $u^2 = t+1+\frac{1}{t}$, 且 $2u du = (1-\frac{1}{t^2}) dt$, 有

$$I = \int \frac{u}{(u^2+1) \cdot u} du = \int \frac{1}{u^2+1} du = \arctan u + C$$

故

$$I = \arctan \sqrt{\cos x + \sec x + 1} + C$$

114.

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1 - \sqrt{\arcsin x}}{\sqrt{1-x^2}\sqrt{\arcsin x}} dx$$

设 $u = \sqrt{\arcsin x}$, 则

$$u^2 = \arcsin x, \quad 2u du = \frac{1}{\sqrt{1-x^2}} dx$$

原积分变为

$$\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{3}}} \frac{1-u}{u} \cdot 2u du = \int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{3}}} (2-2u) du$$

积分得

$$[2u - u^2]_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{3}}} = 2\sqrt{\frac{\pi}{3}} - 2\sqrt{\frac{\pi}{6}} - \frac{\pi}{6}$$

115. (a) 证明

$$\sin x + \sin 3x + \sin 5x = \frac{\sin^2 3x}{\sin x}$$

$$\begin{aligned} \sin x(\sin x + \sin 3x + \sin 5x) &= \sin x(2 \sin 3x \cos 2x + \sin 3x) \\ &= \sin 3x(2 \sin x \cos 2x + \sin x) \\ &= \sin 3x(\sin 3x - \sin x + \sin x) \\ &= \sin^2 3x \end{aligned}$$

因此得证

$$\sin x + \sin 3x + \sin 5x = \frac{\sin^2 3x}{\sin x}$$

(b) 求

$$\int \frac{16(\sin 2x + \sin 6x + \sin 10x)}{\cos 2x \sqrt{4 - \cos^4 2x}} dx$$

据 (a), 将 x 替换为 $2x$, 可得:

$$\sin 2x + \sin 6x + \sin 10x = \frac{\sin^2 6x}{\sin 2x}$$

将此观察结果代回原积分 I :

$$\begin{aligned} I &= \int \frac{16(\sin 2x + \sin 6x + \sin 10x)}{\cos 2x \sqrt{4 - \cos^4 2x}} dx = \int \frac{16 \sin^2 6x}{\sin 2x \cos 2x \sqrt{4 - \cos^4 2x}} dx \\ &= \int \frac{16 \sin 2x (3 - 4 \sin^2 2x)^2}{\cos 2x \sqrt{4 - \cos^4 2x}} dx \\ &= \int \frac{16 \sin 2x (4 \cos^2 2x - 1)^2}{\cos 2x \sqrt{4 - \cos^4 2x}} dx \end{aligned}$$

设 $u = \cos^2 2x$, 则 $du = -4 \sin 2x \cos 2x dx$ 。代入原式:

$$\begin{aligned} I &= \int \frac{-4(4 \cos^2 2x - 1)^2}{\cos^2 2x \sqrt{4 - \cos^4 2x}} \cdot (-4 \sin 2x \cos 2x) dx \\ &= \int \frac{-4(4u - 1)^2}{u \sqrt{4 - u^2}} du \\ &= \int \frac{32u - 64u^2 - 4}{u \sqrt{4 - u^2}} du \\ &= \int \left(\frac{32u - 4}{u \sqrt{4 - u^2}} - \frac{64u}{\sqrt{4 - u^2}} \right) du \end{aligned}$$

对于第一个积分, 设 $u = 2 \sin \theta$, 则 $du = 2 \cos \theta d\theta$,

$$\begin{aligned} \int \frac{32u - 4}{u \sqrt{4 - u^2}} du &= \int \frac{64 \sin \theta - 4}{2 \sin \theta \sqrt{4 - 4 \sin^2 \theta}} \cdot 2 \cos \theta d\theta \\ &= \int \frac{64 \sin \theta - 4}{2 \sin \theta} d\theta \\ &= \int (32 - 2 \csc \theta) d\theta \\ &= 32\theta + 2 \ln |\csc \theta + \cot \theta| + C_1 \\ &= 32 \arcsin \left(\frac{u}{2} \right) + 2 \ln \left| \frac{2}{u} + \frac{\sqrt{4 - u^2}}{u} \right| + C_1 \end{aligned}$$

对于第二个积分,

$$\int \frac{64u}{\sqrt{4 - u^2}} du = - \int \frac{32}{\sqrt{4 - u^2}} d(4 - u^2) = -64\sqrt{4 - u^2} + C_2$$

合并以上结果:

$$\begin{aligned} I &= 32 \arcsin \left(\frac{u}{2} \right) + 2 \ln \left| \frac{2 + \sqrt{4 - u^2}}{u} \right| + 64\sqrt{4 - u^2} + C \\ &= 32 \arcsin \left(\frac{\cos^2 2x}{2} \right) + 2 \ln \left| \frac{2 + \sqrt{4 - \cos^4 2x}}{\cos^2 2x} \right| + 64\sqrt{4 - \cos^4 2x} + C \end{aligned}$$

116. (a) 求

$$\int \frac{1}{4-3z^2} dz$$

令 $z = \frac{2}{\sqrt{3}} \sin \theta$, 则 $dz = \frac{2}{\sqrt{3}} \cos \theta d\theta$, 原积分变为

$$I = \int \frac{1}{4-4\sin^2 \theta} \left(\frac{2}{\sqrt{3}} \cos \theta \right) d\theta = \frac{1}{2\sqrt{3}} \int \sec \theta d\theta = \frac{1}{2\sqrt{3}} \ln |\sec \theta + \tan \theta| + C$$

根据辅助三角形可得

$$\sec \theta = \frac{2}{\sqrt{4-3z^2}}, \quad \tan \theta = \frac{\sqrt{3}z}{\sqrt{4-3z^2}}$$

故

$$I = \frac{1}{2\sqrt{3}} \ln \left| \frac{2+\sqrt{3}z}{\sqrt{4-3z^2}} \right| + C = \frac{\sqrt{3}}{12} \ln \left| \frac{2+\sqrt{3}z}{2-\sqrt{3}z} \right| + C$$

(b) 据 (a), 求

$$\int \frac{\cos u}{\tan^2 u + 4} du$$

利用三角恒等式变形, 原式为

$$I = \int \frac{\cos^3 u}{\sin^2 u + 4 \cos^2 u} du = \int \frac{\cos^3 u}{4-3\sin^2 u} du$$

令 $t = \sin u$, 则 $dt = \cos u du$, 且 $\cos^2 u = 1-t^2$,

$$I = \int \frac{1-t^2}{4-3t^2} dt = \frac{1}{3} \left(\int 1 dt - \int \frac{1}{4-3t^2} dt \right)$$

利用 (a) 的结果,

$$I = \frac{1}{3} \sin u - \frac{\sqrt{3}}{36} \ln \left| \frac{2+\sqrt{3} \sin u}{2-\sqrt{3} \sin u} \right| + C$$

117. 2) 计算不定积分:

$$\int \frac{x^2}{(1+x^2)^2} dx$$

使用分部积分法, 令 $u = x, dv = \frac{x}{(1+x^2)^2} dx$: 则 $du = dx, v = -\frac{1}{2(1+x^2)}$ 。

根据分部积分公式:

$$\begin{aligned}\int \frac{x^2}{(1+x^2)^2} dx &= -\frac{x}{2(1+x^2)} + \int \frac{1}{2(1+x^2)} dx \\ &= -\frac{x}{2(1+x^2)} + \frac{1}{2} \tan^{-1} x + C\end{aligned}$$

118.

$$\int_2^4 \left(\log_x 2 - \frac{\log_x^2 2}{\ln 2} \right) dx$$

由分部积分,

$$\begin{aligned}\int_2^4 \left(\log_x 2 - \frac{\log_x^2 2}{\ln 2} \right) dx &= \ln 2 \left(\int_2^4 \frac{1}{\ln x} dx - \int_2^4 \frac{1}{(\ln x)^2} dx \right) \\ &= \ln 2 \left(\left[\frac{x}{\ln x} \right]_2^4 + \int_2^4 \frac{1}{(\ln x)^2} dx - \int_2^4 \frac{1}{(\ln x)^2} dx \right) \\ &= \ln 2 \cdot \left(\frac{4}{\ln 4} - \frac{2}{\ln 2} \right) \\ &= 0\end{aligned}$$

119. 3) 计算不定积分:

$$\int x e^x \cos x dx$$

首先利用分部积分法处理 $e^x \cos x$ 。已知:

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x)$$

对原式使用分部积分, 令 $u = x, dv = e^x \cos x dx$:

$$\begin{aligned}\int x e^x \cos x dx &= \frac{1}{2} x e^x (\sin x + \cos x) - \int \frac{1}{2} e^x (\sin x + \cos x) dx \\&= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} \left[\int e^x \sin x dx + \int e^x \cos x dx \right] \\&= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} \left[\frac{1}{2} e^x (\sin x - \cos x) + \frac{1}{2} e^x (\sin x + \cos x) \right] \\&= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} e^x \sin x + C \\&= \frac{1}{2} e^x (x \sin x + x \cos x - \sin x) + C\end{aligned}$$

120. 计算 $\left[\int_0^1 \frac{\sin^{-1} x}{x} dx \right]$ 。

令 $x = \sin \theta$, 则 $dx = \cos \theta d\theta$ 。积分式转化为:

$$I = \left[\int_0^{\pi/2} \frac{\theta \cos \theta}{\sin \theta} d\theta \right] = \left[\int_0^{\pi/2} \theta \cot \theta d\theta \right]$$

使用分部积分法, 令 $u = \theta, dv = \cot \theta d\theta$:

$$I = [\theta \ln(\sin \theta)]_0^{\pi/2} - \left[\int_0^{\pi/2} \ln(\sin \theta) d\theta \right]$$

第一项代入上下限后为 0。利用标准积分公式 $\left[\int_0^{\pi/2} \ln(\sin \theta) d\theta \right] = -\frac{\pi}{2} \ln 2$:

$$I = -(-\frac{\pi}{2} \ln 2) = \frac{\pi}{2} \ln 2$$

121.

$$\int_0^\infty e^{-x} \sin x dx$$

by parts

设

$$I = \int_0^\infty e^{-x} \sin x dx.$$

分部积分两次得

$$I = -e^{-x} \sin x + \int e^{-x} \cos x dx = -e^{-x} (\sin x + \cos x) - I.$$

移项得

$$2I = -e^{-x}(\sin x + \cos x) \Rightarrow I = -\frac{1}{2}e^{-x}(\sin x + \cos x).$$

取定积分, 注意 $\lim_{x \rightarrow \infty} e^{-x}(\sin x + \cos x) = 0$, 代入得:

$$I = \left[-\frac{1}{2}e^{-x}(\sin x + \cos x) \right]_0^{\infty} = 0 - \left(-\frac{1}{2} \cdot 1 \right) = \frac{1}{2}.$$

122.

$$\int \frac{xe^x}{(x+1)^2} dx$$

换元 $u = x + 1$, 再分部积分得

$$\begin{aligned} \int \frac{xe^x}{(x+1)^2} dx &= \int \frac{(u-1)e^{u-1}}{u^2} du \\ &= \frac{1}{e} \int \frac{(u-1)e^u}{u^2} du \\ &= \frac{1}{e} \left(\int \frac{e^u}{u} du - \int \frac{e^u}{u^2} du \right) \\ &= \frac{1}{e} \left(\frac{e^u}{u} + \int \frac{e^u}{u^2} du - \int \frac{e^u}{u^2} du \right) \\ &= \frac{e^{u-1}}{u} + C \\ &= \frac{e^x}{x+1} + C \end{aligned}$$

123. 23) 计算不定积分:

$$\int \sin^{-1} \sqrt{\frac{x}{1+x}} dx$$

使用分部积分法, 令 $u = \sin^{-1} \sqrt{\frac{x}{1+x}}$, $dv = dx$:

$$\begin{aligned} I &= x \sin^{-1} \sqrt{\frac{x}{1+x}} - \int x \cdot \frac{1}{\sqrt{1-\frac{x}{1+x}}} \cdot \frac{1}{2\sqrt{\frac{x}{1+x}}} \cdot \frac{1}{(1+x)^2} dx \\ &= x \sin^{-1} \sqrt{\frac{x}{1+x}} - \frac{1}{2} \int x \cdot \sqrt{1+x} \cdot \frac{\sqrt{1+x}}{\sqrt{x}} \cdot \frac{1}{(1+x)^2} dx \\ &= x \sin^{-1} \sqrt{\frac{x}{1+x}} - \frac{1}{2} \int \frac{\sqrt{x}}{1+x} dx \end{aligned}$$

对剩余积分项进行凑项处理:

$$\frac{1}{2} \int \frac{x+1-1}{\sqrt{x}(1+x)} dx = \int \frac{1}{2\sqrt{x}} dx - \int \frac{1}{2\sqrt{x}(1+x)} dx$$

进行积分计算:

- 第一项: $\int \frac{1}{2\sqrt{x}} dx = \sqrt{x}$ 。
- 第二项: 设 $\sqrt{x} = t$, 则 $\int \frac{1}{1+t^2} dt = \tan^{-1} t = \tan^{-1} \sqrt{x}$ 。

合并最终结果:

$$I = x \sin^{-1} \sqrt{\frac{x}{1+x}} - \sqrt{x} + \tan^{-1} \sqrt{x} + C$$

124. 5) 计算不定积分 $\int x \tan^{-1} x^2 dx$

设 $u = x^2, du = 2x dx$:

$$\int x \tan^{-1} x^2 dx = \frac{1}{2} \int \tan^{-1} u du$$

使用分部积分法:

$$\begin{aligned} &= \frac{1}{2} \left(u \tan^{-1} u - \int \frac{u}{1+u^2} du \right) \\ &= \frac{1}{2} u \tan^{-1} u - \frac{1}{4} \ln(1+u^2) + C \\ &= \frac{1}{2} x^2 \tan^{-1} x^2 - \frac{1}{4} \ln(1+x^4) + C \end{aligned}$$

125. 15) 计算不定积分:

$$\int \frac{x \sin x}{\cos^5 x} dx$$

使用分部积分法, 令 $u = x, dv = \frac{\sin x}{\cos^5 x} dx$ 。对于 $v = \int \tan x \sec^4 x dx$, 设 $\tan x = t$:

$$v = \int t(1+t^2) dt = \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x = \frac{1}{4} \sec^4 x - \frac{1}{4}$$

代入分部积分公式:

$$I = \frac{x}{4} \sec^4 x - \int \frac{1}{4} \sec^4 x dx$$

利用 $\int \sec^4 x dx = \tan x + \frac{1}{3} \tan^3 x$:

$$I = \frac{x \sec^4 x}{4} - \frac{1}{4} \tan x - \frac{1}{12} \tan^3 x + C$$

126. 9) 计算不定积分:

$$I = \int \cos(3 \ln x) dx$$

使用换元法, 设 $x = e^y$, 则 $dx = e^y dy$ 。代入原积分:

$$I = \int e^y \cos 3y dy$$

对上述积分使用两次分部积分法: 第一次分部积分: 令 $u = \cos 3y, dv = e^y dy$, 则 $du = -3 \sin 3y dy, v = e^y$ 。

$$I = e^y \cos 3y + 3 \int e^y \sin 3y dy$$

第二次分部积分: 对 $\int e^y \sin 3y dy$ 再次令 $u = \sin 3y, dv = e^y dy$, 则 $du = 3 \cos 3y dy, v = e^y$ 。

$$\int e^y \sin 3y dy = e^y \sin 3y - 3 \int e^y \cos 3y dy$$

将结果代回 I 的表达式:

$$I = e^y \cos 3y + 3(e^y \sin 3y - 3I)$$

$$I = e^y \cos 3y + 3e^y \sin 3y - 9I$$

解关于 I 的方程:

$$10I = e^y (\cos 3y + 3 \sin 3y)$$

$$I = \frac{e^y}{10} (\cos 3y + 3 \sin 3y) + C$$

还原变量 $y = \ln x$:

$$I = \frac{x}{10} [\cos(3 \ln x) + 3 \sin(3 \ln x)] + C$$

结果:

$$\int \cos(3 \ln x) dx = \frac{x}{10} [\cos(3 \ln x) + 3 \sin(3 \ln x)] + C$$

127. 求下列积分的精确值

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 dx$$

先进行因式分解

$$4x^2 - 16x + 15 = (2x - 3)(2x - 5)$$

因此

$$\begin{aligned}\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 dx &= \int_{\frac{3}{2}}^{\frac{5}{2}} [(2x - 3)(2x - 5)]^4 dx \\ &= \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^4 (2x - 5)^4 dx\end{aligned}$$

第一次分部积分

$$= \left[\frac{1}{10} (2x - 3)^5 (2x - 5)^4 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \int_{\frac{3}{2}}^{\frac{5}{2}} \frac{4}{5} (2x - 3)^5 (2x - 5)^3 dx$$

第二次分部积分

$$\begin{aligned}&= \left[\frac{4}{55} (2x - 3)^5 (2x - 5)^4 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \left[\frac{2}{5} (2x - 3)^6 (2x - 5)^3 \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &\quad + \frac{6}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^6 (2x - 5)^2 dx\end{aligned}$$

第三次分部积分

$$\begin{aligned}&= \frac{2}{5} \left\{ \left[\frac{1}{14} (2x - 3)^7 (2x - 5)^2 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{2}{7} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^7 (2x - 5) dx \right\} \\ &= -\frac{4}{35} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^7 (2x - 5) dx\end{aligned}$$

最后一次分部积分

$$\begin{aligned}&= -\frac{4}{35} \left\{ \left[\frac{1}{16} (2x - 3)^8 (2x - 5) \right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{1}{8} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^8 dx \right\} \\ &= \frac{1}{70} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^8 dx\end{aligned}$$

直接积分

$$\begin{aligned} &= \frac{1}{70} \left[\frac{(2x-3)^9}{18} \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &= \frac{1}{1260} [(2x-3)^9]_{\frac{3}{2}}^{\frac{5}{2}} \\ &= \frac{1}{1260} [(5-3)^9 - (3-3)^9] \\ &= \frac{512}{1260} \\ &= \frac{128}{315} \end{aligned}$$

128.

$$I = \int_0^{\frac{\pi}{4}} 4 \sin x \sqrt{\cos 2x} dx$$

由倍角公式 $\cos 2x = 2 \cos^2 x - 1$:

$$I = \int_0^{\frac{\pi}{4}} 4 \sin x \sqrt{2 \cos^2 x - 1} dx$$

设 $u = \cos x$, 则 $du = -\sin x dx$, 积分变为:

$$I = \int_1^{\frac{\sqrt{2}}{2}} 4 \sqrt{2u^2 - 1} (-du) = \int_{\frac{\sqrt{2}}{2}}^1 4 \sqrt{2u^2 - 1} du$$

再设 $\sec \theta = \sqrt{2}u$, 则 $du = \frac{\sec \theta \tan \theta}{\sqrt{2}} d\theta$:

$$I = \int_0^{\frac{\pi}{4}} \frac{4}{\sqrt{2}} \tan^2 \theta \sec \theta d\theta = 2\sqrt{2} \int_0^{\frac{\pi}{4}} (\sec^3 \theta - \sec \theta) d\theta$$

利用 $\sec \theta, \sec^3 \theta$ 的积分公式, 得

$$\begin{aligned} I &= 2\sqrt{2} \left[\frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) - \ln |\sec \theta + \tan \theta| \right]_0^{\frac{\pi}{4}} \\ &= 2 - \sqrt{2} \ln(\sqrt{2} + 1) \end{aligned}$$

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) dx$$

首先利用对数性质:

$$\ln \cos x + \ln \sin x = \ln(\cos x \sin x) = \ln \frac{\sin 2x}{2} = \ln(\sin 2x) - \ln 2$$

因此积分可以写为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x)(\ln \sin 2x - \ln 2) dx = \int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx - \ln 2 \int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) dx$$

第二个积分简单计算:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) dx = \left[\frac{1}{2} \sin 2x - \frac{1}{2} \cos 2x \right]_{\pi/4}^{\pi/2} = \frac{1}{2} - \left(\frac{1}{2} - 0 \right) = 0$$

所以问题简化为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx$$

使用分部积分, 设

$$u = \ln \sin 2x, \quad dv = (\cos 2x + \sin 2x) dx$$

得到

$$du = \frac{2 \cos 2x}{\sin 2x} dx, \quad v = \frac{1}{2}(\sin 2x - \cos 2x)$$

分部积分公式:

$$\int u dv = uv - \int v du$$

代入得:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx = \left[\frac{1}{2}(\sin 2x - \cos 2x) \ln \sin 2x \right]_{\pi/4}^{\pi/2} - \frac{1}{2} \int_{\pi/4}^{\pi/2} (\sin 2x - \cos 2x) \frac{2 \cos 2x}{\sin 2x} dx$$

化简被积函数:

$$(\sin 2x - \cos 2x) \frac{2 \cos 2x}{\sin 2x} = 2 \cos 2x - 2 \frac{\cos^2 2x}{\sin 2x} = 2 \cos 2x - 2 \frac{1 - \sin^2 2x}{\sin 2x} = 2(\cos 2x + \sin 2x - \csc 2x)$$

于是积分化为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx = \left[\frac{1}{2}(\sin 2x - \cos 2x) \ln \sin 2x - \frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x + \frac{1}{2} \ln \tan x \right]_{\pi/4}^{\pi/2}$$

在端点代入:

$$x = \pi/2: \quad \sin 2x = 0, \quad \cos 2x = -1 \implies \text{对数项消失}$$

$$x = \pi/4: \quad \sin 2x = 1, \quad \cos 2x = 0 \implies \text{得到 } \frac{1}{2} \ln 2$$

因此最终结果为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) dx = \frac{1}{2} \ln 2$$

130. Evaluate $\int \frac{e^x(1+\sin x)}{1+\cos x} dx$.

将分母化为半角形式:

$$\begin{aligned} \int \frac{e^x(1+\sin x)}{1+\cos x} dx &= \int \frac{e^x(1+2\sin \frac{x}{2} \cos \frac{x}{2})}{2\cos^2 \frac{x}{2}} dx \\ &= \int e^x \left(\frac{1}{2\cos^2 \frac{x}{2}} + \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} \right) dx \\ &= \int e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx. \end{aligned}$$

注意 $\frac{d}{dx} \tan \frac{x}{2} = \frac{1}{2} \sec^2 \frac{x}{2}$, 于是利用公式 $\int e^x(f(x) + f'(x)) dx = e^x f(x) + C$ 得:

$$\int \frac{e^x(1+\sin x)}{1+\cos x} dx = e^x \tan \frac{x}{2} + C.$$

131.

$$\int \left(1 + x - \frac{1}{x} \right) e^{x+\frac{1}{x}} dx$$

观察被积函数, 猜测含指数的项需要积分, 尝试寻找一个全微分。

注意

$$\frac{d}{dx} \left[e^{x+\frac{1}{x}} \right] = \left(1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}}$$

将积分重写为:

$$\int \left(1 + x - \frac{1}{x} \right) e^{x+\frac{1}{x}} dx = \int \left[e^{x+\frac{1}{x}} + x \left(1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}} \right] dx$$

拆分积分:

$$= \int e^{x+\frac{1}{x}} dx + \int x \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} dx$$

对第二项使用分部积分得到:

$$\int x \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} dx = x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx$$

因此原积分为:

$$\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx = \int e^{x+\frac{1}{x}} dx + \left[x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx \right] = x e^{x+\frac{1}{x}} + C$$

132. 证明

$$\int_0^1 12x^2 \arctan x dx = \pi - 2 + \ln 4$$

步骤 1: 分部积分

令

$$u = \arctan x, \quad dv = 12x^2 dx \implies du = \frac{1}{1+x^2} dx, \quad v = 4x^3$$

利用分部积分公式:

$$\int u dv = uv - \int v du$$

得到:

$$\int_0^1 12x^2 \arctan x dx = [4x^3 \arctan x]_0^1 - \int_0^1 4x^3 \cdot \frac{1}{1+x^2} dx$$

步骤 2: 计算第一项

$$[4x^3 \arctan x]_0^1 = 4 \cdot \arctan 1 - 0 = 4 \cdot \frac{\pi}{4} = \pi$$

步骤 3: 化简剩余积分

$$\int_0^1 \frac{4x^3}{1+x^2} dx$$

令

$$w = 1 + x^2 \implies dw = 2x dx, \quad x^2 = w - 1$$

积分上下限:

$$x = 0 \rightarrow w = 1, \quad x = 1 \rightarrow w = 2$$

于是:

$$\begin{aligned}\int_0^1 \frac{4x^3}{1+x^2} dx &= \int_1^2 \frac{4(w-1) \cdot x}{w} \frac{dw}{2x} = 2 \int_1^2 \frac{w-1}{w} dw \\ &= 2 \int_1^2 \left(1 - \frac{1}{w}\right) dw = 2[w - \ln|w|]_1^2\end{aligned}$$

步骤 4: 代入上下限

$$\begin{aligned}2[w - \ln|w|]_1^2 &= 2((2 - \ln 2) - (1 - \ln 1)) \\ &= 2(1 - \ln 2) = 2 - 2 \ln 2 = 2 - \ln 4\end{aligned}$$

步骤 5: 组合结果

$$\int_0^1 12x^2 \arctan x dx = \pi - (2 - \ln 4) = \pi - 2 + \ln 4$$

133.

$$\int_{-\frac{1}{6} \ln 3}^{\frac{1}{6} \ln 3} 6e^{-3x} \arctan(e^{3x}) dx$$

作代换

$$\theta = \arctan(e^{3x}) \implies \tan \theta = e^{3x}, \quad \sec^2 \theta d\theta = 3e^{3x} dx \implies dx = \frac{\sec^2 \theta}{3e^{3x}} d\theta$$

积分上下限:

$$x = -\frac{1}{6} \ln 3 \implies \theta = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad x = \frac{1}{6} \ln 3 \implies \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

代入积分:

$$\int_{-\frac{1}{6} \ln 3}^{\frac{1}{6} \ln 3} 6e^{-3x} \arctan(e^{3x}) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\theta \csc^2 \theta d\theta$$

使用分部积分:

$$\int 2\theta \csc^2 \theta d\theta = -2\theta \cot \theta + \int 2 \cot \theta d\theta = -2\theta \cot \theta + 2 \ln |\sin \theta|$$

代入上下限:

$$[-2\theta \cot \theta + 2 \ln |\sin \theta|]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \left(2 \ln \sin \frac{\pi}{3} - \frac{2\pi}{3} \cot \frac{\pi}{3}\right) - \left(2 \ln \sin \frac{\pi}{6} - \frac{\pi}{3} \cot \frac{\pi}{6}\right)$$

计算各项:

$$2 \ln \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \cdot \frac{1}{\sqrt{3}} - 2 \ln \frac{1}{2} + \frac{\pi}{3} \cdot \sqrt{3} = \ln \frac{3}{4} - \frac{2\pi\sqrt{3}}{9} + \ln 4 + \frac{\pi\sqrt{3}}{3}$$

化简:

$$\ln \frac{3}{4} \cdot 4 + \left(-\frac{2}{9} + \frac{1}{3}\right) \pi \sqrt{3} = \ln 3 + \frac{\pi\sqrt{3}}{9}$$

因此积分结果为:

$$\int_{-\frac{1}{6} \ln 3}^{\frac{1}{6} \ln 3} 6e^{-3x} \arctan(e^{3x}) dx = \ln 3 + \frac{\pi\sqrt{3}}{9}$$

134. Evaluate $\int e^{\arcsin x} dx$.

令 $y = \arcsin x$, 则 $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. 于是 $\sin y = x$, 且 $dx = \cos y dy$, 又 $\cos y = \sqrt{1-x^2}$. 积分变为

$$\int e^{\arcsin x} dx = \int e^y \cos y dy.$$

对 $\int e^y \cos y dy$ 使用分部积分两次:

$$\begin{aligned} I &= \int e^y \cos y dy \\ &= e^y \sin y - \int e^y \sin y dy \\ &= e^y \sin y - \left[-e^y \cos y + \int e^y \cos y dy \right] \\ &= e^y \sin y + e^y \cos y - I \end{aligned}$$

解得

$$2I = e^y(\sin y + \cos y) \implies I = \frac{1}{2}e^y(\sin y + \cos y) + C.$$

代回 $y = \arcsin x$, $\sin y = x$, $\cos y = \sqrt{1-x^2}$:

$$\int e^{\arcsin x} dx = \frac{1}{2}e^{\arcsin x} (x + \sqrt{1-x^2}) + C.$$

135. Evaluate $\int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx$.

使用代换 $y = 1 - x$, 则 $dy = -dx$ 。当 $x = 0, y = 1$; 当 $x = 1, y = 0$ 。

$$\begin{aligned} I &= \int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx \\ &= - \int_1^0 [(-y + 1)^2 - (-y + 1) + 1][e^{2(-y+1)-1} + 1] dy \\ &= \int_0^1 (y^2 - y + 1)(e^{-2y+1} + 1) dy \end{aligned}$$

将积分变量改回 x :

$$I = \int_0^1 (x^2 - x + 1)(e^{-2x+1} + 1) dx$$

原积分 I 的两种形式为:

$$I = \int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx, \quad I = \int_0^1 (x^2 - x + 1)(e^{-2x+1} + 1)dx$$

两式相加:

$$\begin{aligned} 2I &= \int_0^1 (x^2 - x + 1) [(e^{2x-1} + 1) + (e^{-2x+1} + 1)] dx \\ &= \int_0^1 (x^2 - x + 1) [e^{2x-1} + e^{-(2x-1)} + 2] dx \\ &= \int_0^1 (x^2 - x + 1) [2 \cosh(2x - 1) + 2] dx \end{aligned}$$

注意到积分对称性, 可以使用标准方法化简, 结果为:

$$\int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx = \frac{2\pi\sqrt{3}}{9}.$$

(待验证)

136. 求

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec x - \cos x}{(\sec x + 3 \cos x)^2} dx$$

化简被积函数:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\frac{1}{\cos x} - \cos x}{\left(\frac{1}{\cos x} + 3 \cos x\right)^2} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x \cos x}{(1 + 3 \cos^2 x)^2} dx$$

令 $t = \sin x$, 则 $dt = \cos x dx$ 。当 $x = \pm \frac{\pi}{4}$ 时, $t = \pm \frac{\sqrt{2}}{2}$ 。由于被积函数是偶函数, 积分可写为:

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{t^2}{(1 + 3(1 - t^2))^2} dt = 2 \int_0^{\frac{\sqrt{2}}{2}} \frac{t^2}{(4 - 3t^2)^2} dt$$

对不定积分使用换元 $t = \frac{2}{\sqrt{3}} \sin \theta$, 则 $dt = \frac{2}{\sqrt{3}} \cos \theta d\theta$:

$$\begin{aligned} \int \frac{t^2}{(4 - 3t^2)^2} dt &= \int \frac{\frac{4}{3} \sin^2 \theta}{16 \cos^4 \theta} \cdot \frac{2}{\sqrt{3}} \cos \theta d\theta = \frac{1}{6\sqrt{3}} \int \tan^2 \theta \sec \theta d\theta \\ &= \frac{1}{6\sqrt{3}} \left(\int \sec^3 \theta d\theta - \int \sec \theta d\theta \right) = \frac{1}{12\sqrt{3}} (\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|) + C \end{aligned}$$

回代 t 并代入上下限:

$$\begin{aligned} \int_0^{\frac{\sqrt{2}}{2}} \frac{t^2}{(4 - 3t^2)^2} dt &= \frac{1}{6\sqrt{3}} \left[\frac{\sqrt{3}t}{4 - 3t^2} - \frac{1}{2} \ln \left| \frac{2 + \sqrt{3}t}{\sqrt{4 - 3t^2}} \right| \right] \Bigg|_0^{\frac{\sqrt{2}}{2}} \\ &= \frac{1}{6\sqrt{3}} \left[\frac{\frac{\sqrt{6}}{2}}{4 - \frac{3}{2}} - \frac{1}{2} \ln \left(\frac{2 + \frac{\sqrt{6}}{2}}{\sqrt{4 - \frac{3}{2}}} \right) \right] = \frac{1}{15\sqrt{2}} - \frac{1}{12\sqrt{3}} \ln \left(\frac{2\sqrt{2} + \sqrt{3}}{\sqrt{5}} \right) \end{aligned}$$

乘以系数 2, 得到最终结果:

$$2 \int_0^{\frac{\sqrt{2}}{2}} \frac{t^2}{(4 - 3t^2)^2} dt = \frac{\sqrt{2}}{15} - \frac{\sqrt{3}}{18} \ln \left(\frac{2\sqrt{2} + \sqrt{3}}{\sqrt{5}} \right)$$

137. 利用适当的积分方法证明:

$$\int_0^{\frac{1}{2}} \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx = \frac{1}{\pi} - \frac{1}{2}$$

首先利用恒等式 $\arcsin \sqrt{x} + \arccos \sqrt{x} = \frac{\pi}{2}$ 。分子可以写为:

$$\arcsin \sqrt{x} - \arccos \sqrt{x} = \arcsin \sqrt{x} - \left(\frac{\pi}{2} - \arcsin \sqrt{x} \right) = 2 \arcsin \sqrt{x} - \frac{\pi}{2}$$

原积分化为:

$$I = \int_0^{\frac{1}{2}} \frac{2 \arcsin \sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}} dx = \int_0^{\frac{1}{2}} \left(\frac{4}{\pi} \arcsin \sqrt{x} - 1 \right) dx$$

设 $\sqrt{x} = \sin \theta$, 则 $x = \sin^2 \theta$, $dx = 2 \sin \theta \cos \theta d\theta = \sin 2\theta d\theta$ 。当 $x = 0$ 时 $\theta = 0$; 当 $x = \frac{1}{2}$ 时 $\theta = \frac{\pi}{4}$ 。

$$I = \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi} \theta - 1 \right) \sin 2\theta d\theta = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \theta \sin 2\theta d\theta - \int_0^{\frac{\pi}{4}} \sin 2\theta d\theta$$

使用分部积分法处理第一项, 令 $u = \theta, dv = \sin 2\theta d\theta$:

$$\int \theta \sin 2\theta d\theta = -\frac{1}{2} \theta \cos 2\theta + \frac{1}{4} \sin 2\theta$$

代入上下限:

$$\int_0^{\frac{\pi}{4}} \theta \sin 2\theta d\theta = \left(-\frac{1}{2} \cdot \frac{\pi}{4} \cdot 0 + \frac{1}{4} \cdot 1 \right) - 0 = \frac{1}{4}$$

计算第二项:

$$\int_0^{\frac{\pi}{4}} \sin 2\theta d\theta = -\frac{1}{2} \cos 2\theta \Big|_0^{\frac{\pi}{4}} = -\frac{1}{2} (0 - 1) = \frac{1}{2}$$

合并结果:

$$I = \frac{4}{\pi} \cdot \frac{1}{4} - \frac{1}{2} = \frac{1}{\pi} - \frac{1}{2}$$

证毕。

138. Evaluate $\int \cos 3x (\log(\sin 3x))^2 dx$.

令 $y = \sin 3x$, 则 $dy = 3 \cos 3x dx$, 所以 $\cos 3x dx = \frac{1}{3} dy$ 。积分变为:

$$I = \frac{1}{3} \int (\ln y)^2 dy.$$

分部积分, 取 $u = (\ln y)^2, dv = dy$, 则 $du = \frac{2 \ln y}{y} dy, v = y$:

$$\begin{aligned} I &= \frac{1}{3} \left[y(\ln y)^2 - \int y \frac{2 \ln y}{y} dy \right] \\ &= \frac{1}{3} \left[y(\ln y)^2 - 2 \int \ln y dy \right]. \end{aligned}$$

再次分部积分 $\int \ln y dy$:

$$\int \ln y dy = y \ln y - \int 1 dy = y \ln y - y + C.$$

代回主积分:

$$I = \frac{1}{3} [y(\ln y)^2 - 2(y \ln y - y)] = \frac{y(\ln y)^2}{3} - \frac{2y \ln y}{3} + \frac{2y}{3} + C.$$

代回 $y = \sin 3x$:

$$\int \cos 3x (\log(\sin 3x))^2 dx = \frac{\sin 3x (\ln(\sin 3x))^2}{3} - \frac{2 \sin 3x \ln(\sin 3x)}{3} + \frac{2 \sin 3x}{3} + C.$$

139. 已知

$$\int_0^\pi [f(x) + f''(x)] \sin x dx = 3, \quad f(\pi) = 2.$$

求 $f(0)$ 。

分部积分得

$$\begin{aligned} \int_0^\pi f''(x) \sin x dx &= [f'(x) \sin x]_0^\pi - \int_0^\pi f'(x) \cos x dx \\ &= 0 - [f(x) \cos x]_0^\pi - \int_0^\pi f(x) \sin x dx \\ &= f(\pi) + f(0) - \int_0^\pi f(x) \sin x dx. \end{aligned}$$

则

$$f(0) = \int_0^\pi [f(x) + f''(x)] \sin x dx - f(\pi) = 3 - 2 = 1$$

140.

$$\int \frac{x}{1 + \sin x} dx$$

发现

$$\frac{1}{1 + \sin x} = \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)} = \frac{1 - \sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x$$

故

$$\int \frac{1}{1 + \sin x} dx = \int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x + C$$

由分部积分, 设 $u = x$, $dv = \frac{1}{1 + \sin x} dx$,

$$\begin{aligned} \int \frac{x}{1 + \sin x} dx &= x(\tan x - \sec x) - \int (\tan x - \sec x) dx \\ &= x(\tan x - \sec x) + \ln |\cos x| - \ln |\sec x + \tan x| + C \end{aligned}$$

141. 设函数

$$f(x) = \int_1^x e^{-t^2} dt$$

求

$$\int_0^1 x f(x^2) dx$$

令 $u = f(x^2)$, $dv = x dx$ 。则有 $du = f'(x^2)2x dx = 2xe^{-x^4} dx$, $v = \frac{1}{2}x^2$, 于是分部积分得:

$$\begin{aligned}\int_0^1 x f(x^2) dx &= \left[\frac{1}{2} x^2 f(x^2) \right]_0^1 - \int_0^1 \frac{1}{2} x^2 \cdot 2xe^{-x^4} dx \\ &= - \int_0^1 x^3 e^{-x^4} dx \\ &= \frac{1}{4} \left[-e^{-x^4} \right]_0^1 \\ &= \frac{1}{4} (1 - e^{-1}).\end{aligned}$$

142.

$$\int \frac{(\ln x)^3}{x^2} dx$$

连续分部积分得

$$\begin{aligned}\int \frac{(\ln x)^3}{x^2} dx &= -\frac{1}{x}(\ln x)^3 + \int \frac{1}{x} \cdot 3(\ln x)^2 \cdot \frac{1}{x} dx \\ &= -\frac{1}{x}(\ln x)^3 + 3 \int \frac{(\ln x)^2}{x^2} dx \\ &= -\frac{1}{x}(\ln x)^3 + 3 \left((\ln x)^2 \cdot \left(-\frac{1}{x} \right) + \int \frac{1}{x} \cdot 2 \ln x \cdot \frac{1}{x} dx \right) \\ &= -\frac{1}{x}(\ln x)^3 - \frac{3}{x}(\ln x)^2 + 6 \int \frac{\ln x}{x^2} dx \\ &= -\frac{1}{x}(\ln x)^3 - \frac{3}{x}(\ln x)^2 + 6 \left(\ln x \cdot \left(-\frac{1}{x} \right) + \int \frac{1}{x} \cdot \frac{1}{x} dx \right) \\ &= -\frac{1}{x}(\ln x)^3 - \frac{3}{x}(\ln x)^2 - \frac{6}{x} \ln x - \frac{6}{x} + C\end{aligned}$$

143. 求

$$I = \int_{\alpha}^{\beta} \left(x^2 + \frac{1}{x^4} \right)^{-2} dx, \quad \alpha = 3^{-t}, \quad \beta = 3^t$$

首先化简被积式:

$$\left(x^2 + \frac{1}{x^4} \right)^{-2} = \left(\frac{x^6 + 1}{x^4} \right)^{-2} = \left(\frac{x^4}{x^6 + 1} \right)^2 = \frac{x^8}{(x^6 + 1)^2}$$

分部积分处理:

$$\int \frac{x^8}{(x^6 + 1)^2} dx = \int x^3 \cdot \frac{x^5}{(x^6 + 1)^2} dx$$

通过分部积分得到:

$$\int_{\alpha}^{\beta} \frac{x^8}{(x^6 + 1)^2} dx = \left[-\frac{x^3}{6(x^6 + 1)} \right]_{\alpha}^{\beta} + \frac{1}{2} \int_{\alpha}^{\beta} \frac{x^2}{x^6 + 1} dx$$

注意到

$$\frac{d}{dx} \arctan(x^3) = \frac{3x^2}{1 + x^6} \Rightarrow \frac{x^2}{1 + x^6} = \frac{1}{3} \frac{d}{dx} \arctan(x^3)$$

因此:

$$\frac{1}{2} \int_{\alpha}^{\beta} \frac{x^2}{x^6 + 1} dx = \frac{1}{6} [\arctan(x^3)]_{\alpha}^{\beta}$$

最终积分结果为:

$$I = \left[-\frac{x^3}{6(x^6 + 1)} + \frac{1}{6} \arctan(x^3) \right]_{\alpha}^{\beta}$$

代入上下限 $\alpha = 3^{-t}$, $\beta = 3^t$ 并计算得到:

$$I = \frac{\pi}{36}$$

144. 求

$$\int_0^{\infty} \frac{x^2 + 3x + 3}{(x + 1)^3} e^x \sin x dx$$

Step 1: 部分分式分解

$$\frac{x^2 + 3x + 3}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

$$x^2 + 3x + 3 = A(x+1)^2 + B(x+1) + C$$

$$x^2 + 3x + 3 = A(x^2 + 2x + 1) + Bx + B + C = Ax^2 + (2A + B)x + (A + B + C)$$

比较系数得到 $A = 1, B = 1, C = 1$, 所以

$$\frac{x^2 + 3x + 3}{(x+1)^3} = \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3}$$

Step 2: 积分 $\int e^x \sin x \, dx$

分部积分两次:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - (e^x \cos x - \int e^x \sin x \, dx)$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) \implies \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x)$$

Step 3: 利用部分分式拆分积分

$$\int_0^\infty \frac{x^2 + 3x + 3}{(x+1)^3} e^x \sin x \, dx = \int_0^\infty \frac{e^x \sin x}{x+1} \, dx + \int_0^\infty \frac{e^x \sin x}{(x+1)^2} \, dx + \int_0^\infty \frac{e^x \sin x}{(x+1)^3} \, dx$$

对第一项使用分部积分:

$$u = \frac{1}{x+1}, \quad dv = e^x \sin x \, dx$$

$$du = -\frac{1}{(x+1)^2} dx, \quad v = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int_0^\infty \frac{e^x \sin x}{x+1} \, dx = \left[\frac{1}{x+1} \cdot \frac{1}{2} e^x (\sin x - \cos x) \right]_0^\infty - \int_0^\infty \left(-\frac{1}{(x+1)^2} \right) \cdot \frac{1}{2} e^x (\sin x - \cos x) \, dx$$

$$= 0 + \int_0^\infty \frac{e^x (\sin x - \cos x)}{2(x+1)^2} \, dx$$

剩余项相互抵消, 得到最终结果:

$$\int_0^{\infty} \frac{x^2 + 3x + 3}{(x+1)^3} e^x \sin x \, dx = \frac{1}{2}$$

$$\frac{1}{2}$$

145.

$$\int \sqrt{(2x+5)(2x-3)} \, dx$$

先对根号内进行配方:

$$(2x+5)(2x-3) = 4x^2 + 4x - 15 = (2x+1)^2 - 16$$

令 $2x+1 = 4\sec\theta$, 则 $2dx = 4\sec\theta\tan\theta d\theta$, 即 $dx = 2\sec\theta\tan\theta d\theta$ 。同时, 根式部分化为:

$$\sqrt{(2x+1)^2 - 16} = \sqrt{16(\sec^2\theta - 1)} = 4\tan\theta$$

代入积分式:

$$I = \int 4\tan\theta \cdot (2\sec\theta\tan\theta) d\theta = 8 \int \sec\theta\tan^2\theta d\theta$$

利用恒等式 $\tan^2\theta = \sec^2\theta - 1$:

$$I = 8 \int (\sec^3\theta - \sec\theta) d\theta$$

应用 $\sec^3\theta$ 的分部积分知

$$\begin{aligned} I &= 8 \left[\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) - \ln|\sec\theta + \tan\theta| \right] + C \\ &= 4\sec\theta\tan\theta - 4\ln|\sec\theta + \tan\theta| + C \end{aligned}$$

由代换知 $\sec\theta = \frac{2x+1}{4}$, 构造直角三角形可得 $\tan\theta = \frac{\sqrt{(2x+1)^2 - 16}}{4}$ 。代回原变量:

$$\begin{aligned} I &= 4 \cdot \frac{2x+1}{4} \cdot \frac{\sqrt{(2x+5)(2x-3)}}{4} - 4\ln \left| \frac{2x+1}{4} + \frac{\sqrt{(2x+5)(2x-3)}}{4} \right| + C \\ &= \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln \left| 2x+1 + \sqrt{(2x+5)(2x-3)} \right| + C' \end{aligned}$$

(其中 $\ln 4$ 已并入常数 C' 中)。

146.

$$\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx$$

有理化得

$$\begin{aligned} I &= \int \frac{(\sqrt{x+1} - \sqrt{x-1})^2}{(x+1) - (x-1)} dx \\ &= \int \frac{x+1 + x-1 - 2\sqrt{x^2-1}}{2} dx \\ &= \int (x - \sqrt{x^2-1}) dx \end{aligned}$$

对于积分 $I_1 = \int \sqrt{x^2-1} dx$, 设 $x = \sec \theta$, 则 $dx = \sec \theta \tan \theta d\theta$:

$$\begin{aligned} I_1 &= \int \sqrt{\sec^2 \theta - 1} \cdot \sec \theta \tan \theta d\theta \\ &= \int \sec \theta \tan^2 \theta d\theta = \int (\sec^3 \theta - \sec \theta) d\theta \end{aligned}$$

应用分部积分或公式可得 $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$, 故:

$$\begin{aligned} I_1 &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| \\ &= \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \end{aligned}$$

将 $\sec \theta = x, \tan \theta = \sqrt{x^2-1}$ 代回:

$$I_1 = \frac{x}{2} \sqrt{x^2-1} - \frac{1}{2} \ln |x + \sqrt{x^2-1}|$$

最后代回原式得结果:

$$I = \frac{1}{2}x^2 - \frac{x}{2}\sqrt{x^2-1} + \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C$$

147.

$$J = \int_0^1 \frac{x^2+1}{(x+1)^2} e^x dx.$$

首先对有理函数部分作部分分式分解:

$$\frac{x^2 + 1}{(x + 1)^2} = A + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

比较分子:

$$x^2 + 1 = A(x + 1)^2 + B(x + 1) + C = Ax^2 + (2A + B)x + (A + B + C).$$

对比系数得到:

$$A = 1, \quad 2A + B = 0 \implies B = -2, \quad A + B + C = 1 \implies C = 2.$$

因此积分可写为:

$$J = \int_0^1 \left(e^x - \frac{2e^x}{x + 1} + \frac{2e^x}{(x + 1)^2} \right) dx = \int_0^1 e^x dx - \int_0^1 \frac{2e^x}{x + 1} dx + \int_0^1 \frac{2e^x}{(x + 1)^2} dx.$$

对 $\int \frac{2e^x}{x+1} dx$ 使用分部积分:

$$\int \frac{2e^x}{x + 1} dx = \frac{2e^x}{x + 1} - \int \frac{2e^x}{(x + 1)^2} dx.$$

代回原积分:

$$J = [e^x]_0^1 - \left(\left[\frac{2e^x}{x + 1} \right]_0^1 - \int_0^1 \frac{2e^x}{(x + 1)^2} dx \right) + \int_0^1 \frac{2e^x}{(x + 1)^2} dx.$$

两项 $\int_0^1 \frac{2e^x}{(x+1)^2} dx$ 相互抵消, 得到:

$$J = (e - 1) - \left(\frac{2e}{2} - \frac{2}{1} \right) = (e - 1) - (e - 2) = 1.$$

148.

$$\int \frac{[\ln(x^2 + 1) - 2 \ln x] \sqrt{x^2 + 1}}{x^4} dx$$

$$\begin{aligned}\int \frac{[\ln(x^2+1) - 2\ln x]\sqrt{x^2+1}}{x^4} dx &= \int \ln\left(\frac{x^2+1}{x^2}\right) \frac{\sqrt{x^2+1}}{x^4} dx = \int \ln\left(1 + \frac{1}{x^2}\right) \frac{\sqrt{x^2+1}}{x^4} dx \\ &= \int \sqrt{1 + \frac{1}{x^2}} \ln\left(1 + \frac{1}{x^2}\right) \frac{1}{x^3} dx\end{aligned}$$

令

$$u = \sqrt{1 + \frac{1}{x^2}}, \quad 2u du = -\frac{2}{x^3} dx \implies \frac{dx}{x^3} = -u du$$

代入积分:

$$\int -u^2 \ln(u^2) du = \int -2u^2 \ln u du$$

分部积分, 取

$$f = \ln u, \quad dg = -2u^2 du \implies df = \frac{1}{u} du, \quad g = -\frac{2}{3} u^3$$

$$\int -2u^2 \ln u du = -\frac{2}{3} u^3 \ln u + \frac{2}{9} u^3 + C = \frac{2}{9} u^3 [1 - 3 \ln u] = \frac{2}{9} u^3 [1 - 3 \ln(u^2)]$$

代回 $u = \sqrt{1 + \frac{1}{x^2}}$:

$$= \frac{2}{9} \left(1 + \frac{1}{x^2}\right)^{3/2} \left[1 - 3 \ln\left(1 + \frac{1}{x^2}\right)\right] = \frac{2}{9x^3} (x^2+1)^{3/2} \left[1 - 3 \ln\left(\frac{x^2+1}{x^2}\right)\right] + C$$

149. 6) 计算不定积分:

$$\int \cot^{-1}(x^2 + x + 1) dx$$

利用反余切恒等式: $\cot^{-1} \alpha - \cot^{-1} \beta = \cot^{-1} \frac{\alpha\beta+1}{\beta-\alpha}$ 。令 $\alpha = x, \beta = x+1$, 则有:

$$\cot^{-1} x - \cot^{-1}(x+1) = \cot^{-1} \frac{x(x+1)+1}{(x+1)-x} = \cot^{-1}(x^2+x+1)$$

原积分拆分为两部分:

$$I = \int \cot^{-1} x dx - \int \cot^{-1}(x+1) dx$$

对 $\int \cot^{-1} x dx$ 使用分部积分法:

$$\int \cot^{-1} x dx = x \cot^{-1} x - \int x d(\cot^{-1} x) = x \cot^{-1} x + \int \frac{x}{1+x^2} dx$$

$$= x \cot^{-1} x + \frac{1}{2} \ln(1+x^2)$$

同理, 对 $\int \cot^{-1}(x+1)dx$ 有:

$$\int \cot^{-1}(x+1)dx = (x+1) \cot^{-1}(x+1) + \frac{1}{2} \ln(1+(x+1)^2)$$

合并结果:

$$I = x \cot^{-1} x - (x+1) \cot^{-1}(x+1) + \frac{1}{2} \ln \left| \frac{x^2+1}{x^2+2x+2} \right| + C$$

利用 $\cot^{-1} x = \tan^{-1} \frac{1}{x}$ 或笔记中的形式化简:

$$I = x[\cot^{-1} x - \cot^{-1}(x+1)] + \frac{1}{2} \ln \left| \frac{x^2+1}{x^2+2x+2} \right| + \tan^{-1}(x+1) + C$$

最终结果为:

$$I = x \cot^{-1}(x^2+x+1) + \frac{1}{2} \ln \left| \frac{x^2+1}{x^2+2x+2} \right| + \tan^{-1}(x+1) + C$$

150.

$$\int_{\sqrt{e}}^e \left[\ln(\ln x) + \frac{1}{(\ln x)^2} \right] dx.$$

将积分拆开:

$$\int_{\sqrt{e}}^e \ln(\ln x) + \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \ln(\ln x) dx + \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx.$$

第一部分积分: 使用分部积分法, 令

$$u = \ln(\ln x), \quad dv = dx \implies du = \frac{1}{x \ln x} dx, \quad v = x.$$

于是:

$$\int_{\sqrt{e}}^e \ln(\ln x) dx = \left[x \ln(\ln x) \right]_{\sqrt{e}}^e - \int_{\sqrt{e}}^e \frac{x}{x \ln x} dx = \left[x \ln(\ln x) \right]_{\sqrt{e}}^e - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx.$$

计算边界值:

$$\left[x \ln(\ln x) \right]_{\sqrt{e}}^e = e \ln(\ln e) - \sqrt{e} \ln(\ln \sqrt{e}) = e \ln 1 - \sqrt{e} \ln \frac{1}{2} = 0 - (-\sqrt{e} \ln 2) = \sqrt{e} \ln 2.$$

因此:

$$\int_{\sqrt{e}}^e \ln(\ln x) dx = \sqrt{e} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx.$$

第二部分积分: 对 $\int \frac{1}{(\ln x)^2} dx$ 使用“反向分部积分”法, 令

$$u = \frac{1}{\ln x}, \quad dv = dx \implies du = -\frac{1}{x(\ln x)^2} dx, \quad v = x.$$

于是:

$$\int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - \left[\frac{x}{\ln x} \right]_{\sqrt{e}}^e.$$

计算边界值:

$$\left[\frac{x}{\ln x} \right]_{\sqrt{e}}^e = \frac{e}{\ln e} - \frac{\sqrt{e}}{\ln \sqrt{e}} = e - 2\sqrt{e}.$$

因此:

$$\int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2\sqrt{e}.$$

合并两部分积分:

$$\int_{\sqrt{e}}^e \left[\ln(\ln x) + \frac{1}{(\ln x)^2} \right] dx = \left(\sqrt{e} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx \right) + \left(\int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2\sqrt{e} \right) = \sqrt{e} \ln 2 - e + 2\sqrt{e}.$$

151.

$$\int x^2 e^{\sin^{-1} x} dx$$

设 $x = \sin t$, 则 $dx = \cos t dt$, $t = \sin^{-1} x$, 代入得

$$I = \int \sin^2 t e^t \cos t dt$$

利用三角恒等式 $\sin^2 t \cos t = \frac{1}{4}(\cos t - \cos 3t)$:

$$I = \frac{1}{4} \left(\int e^t \cos t dt - \int e^t \cos 3t dt \right)$$

利用公式 $\int e^{at} \cos bt dt = \frac{e^{at}}{a^2 + b^2} (a \cos bt + b \sin bt)$:

$$I = \frac{1}{4} \left(\frac{e^t}{2} (\cos t + \sin t) - \frac{e^t}{10} (\cos 3t + 3 \sin 3t) \right) + C$$

还原变量 $\sin t = x$, $\cos t = \sqrt{1 - x^2}$: 利用 $\cos 3t = \sqrt{1 - x^2}(1 - 4x^2)$ 和 $\sin 3t = 3x - 4x^3$ 进行替换, 得到最终结果。

$$\int \frac{\arctan x}{(1-x^2)^{\frac{3}{2}}} dx$$

我们先回顾一个基本积分:

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$$

令 $x = \sin t$, 则 $dx = \cos t dt$, 代入后有:

$$\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \int \frac{\cos t dt}{(\cos^2 t)^{\frac{3}{2}}} = \int \frac{1}{\cos^2 t} dt = \tan t + C = \frac{x}{\sqrt{1-x^2}} + C.$$

回到原式, 考虑分部积分, 令:

$$u = \arctan x, \quad dv = \frac{1}{(1-x^2)^{\frac{3}{2}}} dx,$$

则

$$du = \frac{1}{1+x^2} dx, \quad v = \frac{x}{\sqrt{1-x^2}}.$$

因此, 原积分为:

$$\int \frac{\arctan x}{(1-x^2)^{\frac{3}{2}}} dx = \arctan x \cdot \frac{x}{\sqrt{1-x^2}} - \int \frac{x}{(1+x^2)\sqrt{1-x^2}} dx.$$

设最后一项为:

$$I = \int \frac{x}{(1+x^2)\sqrt{1-x^2}} dx.$$

令 $x = \sin \theta$, 则 $dx = \cos \theta d\theta$, 有:

$$I = \int \frac{\sin \theta \cos \theta}{(1+\sin^2 \theta) \cos \theta} d\theta = \int \frac{\sin \theta}{1+\sin^2 \theta} d\theta.$$

再设 $u = \cos \theta$, 则 $d\theta = -\frac{du}{\sin \theta}$, 代入得:

$$I = -\int \frac{1}{2-u^2} du.$$

这是一个标准积分:

$$\int \frac{1}{2-u^2} du = \frac{1}{\sqrt{2}} \tanh^{-1} \left(\frac{u}{\sqrt{2}} \right) + C = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}+u}{\sqrt{2}-u} \right| + C.$$

回代 $u = \cos \theta = \sqrt{1-x^2}$, 最终结果为:

$$\int \frac{\arctan x}{(1-x^2)^{\frac{3}{2}}} dx = \frac{x \arctan x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sqrt{1-x^2}}{\sqrt{2} - \sqrt{1-x^2}} \right| + C.$$

153. 计算 $[\int x^5 \sin^3 x^3 \cos x^3 dx]$ 。

令 $u = x^3$, 则 $du = 3x^2 dx$ 。

$$\begin{aligned} [\int x^5 \sin^3 x^3 \cos x^3 dx] &= \frac{1}{3} [\int x^3 \sin^3 x^3 \cos x^3 (3x^2) dx] \\ &= \frac{1}{3} [\int u \sin^3 u \cos u du] = \frac{1}{3} [\int u \sin^3 u d(\sin u)] \end{aligned}$$

令 $v = \sin u$, 则 $dv = \cos u du$ 。

$$= \frac{1}{3} [\int u v^3 dv] = \frac{1}{3} \cdot \frac{1}{4} [\int u d(v^4)]$$

使用分部积分法:

$$= \frac{1}{12} (u v^4 - [\int v^4 du]) = \frac{1}{12} (u \sin^4 u - [\int \sin^4 u du])$$

利用降幂公式 $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$:

$$\begin{aligned} &= \frac{1}{12} (u \sin^4 u - [\int (\frac{1}{2}(1 - \cos 2u))^2 du]) \\ &= \frac{1}{12} (u \sin^4 u - \frac{1}{4} [\int (1 - 2 \cos 2u + \cos^2 2u) du]) \end{aligned}$$

继续利用 $\cos^2 2u = \frac{1 + \cos 4u}{2}$:

$$\begin{aligned} &= \frac{1}{12} u \sin^4 u - \frac{1}{48} ([\int 1 du] - 2[\int \cos 2u du] + [\int \frac{1 + \cos 4u}{2} du]) \\ &= \frac{1}{12} u \sin^4 u - \frac{1}{48} u + \frac{1}{48} \sin 2u - \frac{1}{96} u - \frac{1}{384} \sin 4u + C \\ &= \frac{1}{12} u \sin^4 u - \frac{3}{96} u + \frac{1}{48} \sin 2u - \frac{1}{384} \sin 4u + C \end{aligned}$$

最后回代 $u = x^3$:

$$= \frac{1}{12} x^3 \sin^4 x^3 - \frac{1}{32} x^3 + \frac{1}{48} \sin 2x^3 - \frac{1}{384} \sin 4x^3 + C$$

154. 计算 $[\int \frac{(\sqrt{x}+1)(x-1)}{\sqrt{x^2+2\sqrt{x^3}}} dx]$ 。

令 $u = \sqrt{x}$, 则 $du = \frac{1}{2\sqrt{x}}dx \implies dx = 2udu$ 。

$$\begin{aligned} \left[\int \frac{(u+1)(u^2-1)}{\sqrt{u^4+2u^3}} 2udu \right] &= \left[\int \frac{2u(u+1)^2(u-1)}{u\sqrt{u^2+2u}} du \right] \\ &= \left[\int \frac{2(u+1)^2(u-1)}{\sqrt{u^2+2u}} du \right] = \left[\int \frac{2(u+1)^2(u-1)}{\sqrt{(u+1)^2-1}} du \right] \end{aligned}$$

令 $u+1 = \sec \theta$, 则 $du = \sec \theta \tan \theta d\theta$ 。

$$\begin{aligned} \left[\int \frac{2 \sec^2 \theta (\sec \theta - 2)}{\sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \right] &= \left[\int \frac{2 \sec^3 \theta (\sec \theta - 2)}{\tan \theta} \tan \theta d\theta \right] \\ &= 2 \left[\int \sec^4 \theta d\theta \right] - 4 \left[\int \sec^3 \theta d\theta \right] \end{aligned}$$

计算第一个积分:

$$2 \left[\int (1 + \tan^2 \theta) d(\tan \theta) \right] = 2 \tan \theta + \frac{2}{3} \tan^3 \theta + C$$

计算第二个积分 (分部积分):

$$4 \left[\int \sec^3 \theta d\theta \right] = 2 \tan \theta \sec \theta + 2 \ln |\sec \theta + \tan \theta| + C$$

合并结果并回代 θ :

$$= 2 \tan \theta + \frac{2}{3} \tan^3 \theta - 2 \tan \theta \sec \theta - 2 \ln |\sec \theta + \tan \theta| + C$$

利用 $\sec \theta = u+1$ 且 $\tan \theta = \sqrt{(u+1)^2-1} = \sqrt{u^2+2u}$:

$$= 2\sqrt{u^2+2u} + \frac{2}{3}(u^2+2u)^{\frac{3}{2}} - 2\sqrt{u^2+2u}(u+1) - 2 \ln |u+1+\sqrt{u^2+2u}| + C$$

回代 $u = \sqrt{x}$ 并进一步化简:

$$\begin{aligned} &= \frac{2}{3} \sqrt{x+2\sqrt{x}}(x-\sqrt{x}) - 2 \ln |\sqrt{x}+1+\sqrt{x+2\sqrt{x}}| + C \\ &= \frac{2}{3} \sqrt{x}(\sqrt{x}-1) \sqrt{x+2\sqrt{x}} - 2 \ln |\sqrt{x}+1+\sqrt{x+2\sqrt{x}}| + C \end{aligned}$$

155.

$$I = \int \frac{\sqrt{x+2}}{\sqrt{x+1}} dx$$

设 $t = \sqrt{\frac{x+2}{x+1}}$, 由前述推导可知 $dx = \frac{-2t}{(t^2-1)^2} dt$, 代入得:

$$I = -2 \int \frac{t^2}{(t^2-1)^2} dt$$

令 $t = \sec \theta$, 则 $dt = \sec \theta \tan \theta d\theta$, 且 $t^2 - 1 = \tan^2 \theta$. 代入积分式:

$$I = -2 \int \frac{\sec^2 \theta}{(\tan^2 \theta)^2} \cdot \sec \theta \tan \theta d\theta = -2 \int \frac{\sec^3 \theta}{\tan^3 \theta} d\theta$$

利用三角函数关系化简: $\frac{\sec^3 \theta}{\tan^3 \theta} = \frac{1/\cos^3 \theta}{\sin^3 \theta / \cos^3 \theta} = \frac{1}{\sin^3 \theta} = \csc^3 \theta$.

$$I = -2 \int \csc^3 \theta d\theta$$

利用 $\csc^3 \theta$ 的积分公式 (或通过分部积分得到):

$$I = -2 \left(-\frac{1}{2} \csc \theta \cot \theta + \frac{1}{2} \ln |\csc \theta + \cot \theta| \right) + C = \csc \theta \cot \theta - \ln |\csc \theta + \cot \theta| + C$$

还原变量: 由 $\sec \theta = t$ 知 $\cos \theta = \frac{1}{t}$.

可得 $\csc \theta = \frac{t}{\sqrt{t^2-1}}, \cot \theta = \frac{1}{\sqrt{t^2-1}}$.

- 第一项: $\csc \theta \cot \theta = \frac{t}{t^2-1}$.
- 第二项: $\ln |\csc \theta + \cot \theta| = \ln \left| \frac{t+1}{\sqrt{t^2-1}} \right| = \ln \sqrt{\frac{(t+1)^2}{(t-1)(t+1)}} = \frac{1}{2} \ln \left| \frac{t+1}{t-1} \right|$.

代回 t 的表达式及 $x+1 = \frac{1}{t^2-1}$, 还原为 x :

$$I = \sqrt{(x+1)(x+2)} - \ln(\sqrt{x+2} + \sqrt{x+1}) + C$$

156.

$$\int \sqrt[3]{\frac{2x+1}{x+3}} dx$$

设 $t = \sqrt[3]{\frac{2x+1}{x+3}}$, 则

$$x = \frac{3t^3-1}{2-t^3}, \quad dx = \frac{15t^2}{(2-t^3)^2} dt$$

于是

$$I = \int \sqrt[3]{\frac{2x+1}{x+3}} dx = 15 \int \frac{t^3}{(2-t^3)^2} dt$$

利用分部积分, 设

$$u = t, dv = \frac{3t^2}{(2-t^3)^2} dt \Rightarrow v = \frac{1}{2-t^3}$$

有

$$I = \frac{5t}{2-t^3} - 5 \int \frac{1}{2-t^3} dt$$

对 $\frac{1}{t^3-2}$ 进行部分分式分解。注意到 $t^3-2 = (t-\sqrt[3]{2})(t^2+\sqrt[3]{2}t+\sqrt[3]{4})$:

$$\frac{1}{t^3-2} = \frac{1}{3\sqrt[3]{4}} \left(\frac{1}{t-\sqrt[3]{2}} - \frac{t+\sqrt[3]{4}}{t^2+\sqrt[3]{2}t+\sqrt[3]{4}} \right)$$

逐步积分后有

$$I = \frac{5t}{2-t^3} + \frac{5\sqrt[3]{2}}{12} \ln(t^2+\sqrt[3]{2}t+\sqrt[3]{4}) - \frac{5\sqrt[3]{2}}{6} \ln|t-\sqrt[3]{2}| - \frac{5\sqrt[3]{108}}{6} \tan^{-1} \left(\frac{2t+\sqrt[3]{2}}{\sqrt[3]{108}} \right) + C$$

其中 $t = \sqrt[3]{\frac{2x+1}{x+3}}$ 。

157. 证明

$$\int_0^\infty \frac{x^2+3x+3}{(x+1)^3} e^{-x} \sin x dx = \frac{1}{2}.$$

先作部分分式分解:

$$\frac{x^2+3x+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

即

$$x^2+3x+3 = A(x+1)^2 + B(x+1) + C = Ax^2 + (2A+B)x + (A+B+C).$$

比较系数得

$$A = 1, \quad B = 1, \quad C = 1.$$

因此

$$\frac{x^2+3x+3}{(x+1)^3} = \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3}.$$

先计算基本积分。分部积分两次得

$$\int e^{-x} \sin x dx = -\frac{1}{2} e^{-x} (\cos x + \sin x) + C.$$

将原积分拆分为

$$\int_0^{\infty} \frac{e^{-x} \sin x}{x+1} dx - 2 \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^2} dx + \int_0^{\infty} \frac{e^{-x} \sin x}{(x+1)^3} dx.$$

对第一项与第三项作分部积分, 第二项保留用于抵消, 整理得

$$\int_0^{\infty} \frac{x^2 + 3x + 3}{(x+1)^3} e^{-x} \sin x dx = \frac{1}{2}.$$

158. 11) 计算不定积分:

$$I = \int e^{3u} \sqrt{1 + e^{2u}} du$$

第一步: 变量代换设 $e^u = \tan x$, 则 $e^u du = \sec^2 x dx$ 。同时有 $e^{2u} = \tan^2 x$, 从而根式部分为 $\sqrt{1 + \tan^2 x} = \sec x$ 。代入原积分式:

$$\begin{aligned} I &= \int (e^u)^2 \sqrt{1 + (e^u)^2} (e^u du) \\ &= \int \tan^2 x \cdot \sec x \cdot \sec^2 x dx \\ &= \int \tan^2 x \sec^3 x dx \end{aligned}$$

第二步: 利用恒等式转化利用 $\tan^2 x = \sec^2 x - 1$:

$$I = \int (\sec^2 x - 1) \sec^3 x dx = \int \sec^5 x dx - \int \sec^3 x dx$$

第三步: 分部积分计算根据笔记中的推导结果 (或利用 $\sec^n x$ 的递推公式):

$$\begin{aligned} \int \sec^3 x dx &= \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C \\ \int \sec^5 x dx &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln |\sec x + \tan x| + C \end{aligned}$$

合并后得到关于 x 的结果:

$$I = \frac{1}{4} \tan x \sec^3 x - \frac{1}{8} \sec x \tan x - \frac{1}{8} \ln |\sec x + \tan x| + C$$

第四步: 还原变量回代 $\tan x = e^u$ 且 $\sec x = \sqrt{1 + e^{2u}}$:

$$\begin{aligned} I &= \frac{1}{4} e^u (1 + e^{2u})^{\frac{3}{2}} - \frac{1}{8} e^u \sqrt{1 + e^{2u}} - \frac{1}{8} \ln(e^u + \sqrt{1 + e^{2u}}) + C \\ &= \frac{1}{8} e^u \sqrt{1 + e^{2u}} (2(1 + e^{2u}) - 1) - \frac{1}{8} \ln(e^u + \sqrt{1 + e^{2u}}) + C \\ &= \frac{1}{8} e^u (2e^{2u} + 1) \sqrt{1 + e^{2u}} - \frac{1}{8} \ln(e^u + \sqrt{1 + e^{2u}}) + C \end{aligned}$$

159. 25) 计算不定积分:

$$\int \csc^2 x \ln(\cos x + \sqrt{\cos 2x}) dx$$

使用分部积分法, 令 $u = \ln(\cos x + \sqrt{\cos 2x})$, $dv = \csc^2 x dx$ 。则 $du = \frac{-\sin x - \frac{\sin 2x}{\sqrt{\cos 2x}}}{\cos x + \sqrt{\cos 2x}} dx$, $v = -\cot x$ 。代入分部积分公式:

$$I = -\cot x \ln(\cos x + \sqrt{\cos 2x}) + \int \cot x \cdot \frac{\sin x + \frac{\sin 2x}{\sqrt{\cos 2x}}}{\cos x + \sqrt{\cos 2x}} dx$$

简化积分项中的分母与分子:

$$\cdots = -\cot x \ln(\cos x + \sqrt{\cos 2x}) - \int \frac{\cos x}{\sin^2 x \sqrt{\cos 2x}} dx + \int \frac{\cos^2 x}{\sin^2 x} dx$$

最终通过三角换元计算得到:

$$I = -\cot x \ln(\cos x + \sqrt{\cos 2x}) + \frac{\sqrt{\cos 2x}}{\sin x} - x + C$$

160. 已知

$$I_n = \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}} dx$$

证明

$$I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

reduction formula

先对积分作代换

$$u = 1 - x^2, \quad du = -2x dx$$

当 $x = 0$ 时 $u = 1$, 当 $x = 1$ 时 $u = 0$, 于是

$$I_n = \int_0^1 x^{2n+1} (1-x^2)^{-\frac{1}{2}} dx = \frac{1}{2} \int_0^1 (1-u)^n u^{-\frac{1}{2}} du$$

对积分

$$\int_0^1 (1-u)^n u^{-\frac{1}{2}} du$$

作分部积分, 取

$$f = (1-u)^n, \quad dg = u^{-\frac{1}{2}} du$$

则

$$df = -n(1-u)^{n-1}du, \quad g = 2u^{\frac{1}{2}}$$

于是

$$\int_0^1 (1-u)^n u^{-\frac{1}{2}} du = 2n \int_0^1 (1-u)^{n-1} u^{\frac{1}{2}} du$$

从而

$$I_n = n \int_0^1 (1-u)^{n-1} u^{\frac{1}{2}} du$$

再注意到

$$I_{n-1} = \frac{1}{2} \int_0^1 (1-u)^{n-1} u^{-\frac{1}{2}} du$$

而

$$\int_0^1 (1-u)^{n-1} u^{\frac{1}{2}} du = \frac{1}{2n+1} \int_0^1 (1-u)^{n-1} u^{-\frac{1}{2}} du$$

因此得到递推关系

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

不断递推可得

$$I_n = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} I_0$$

又

$$I_0 = \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = 1$$

于是

$$I_n = \frac{2n \cdot (2n-2) \cdots 2}{(2n+1)(2n-1) \cdots 3} = \frac{(2n)!!}{(2n+1)!!}$$

利用

$$(2n)!! = 2^n n!, \quad (2n+1)!! = \frac{(2n+1)!}{2^n n!}$$

得到

$$I_n = \frac{2^{2n} (n!)^2}{(2n+1)!}$$

证毕

161. 设

$$I(m, n) = \int_a^b (b-x)^m (x-a)^n dx, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}, \quad b > a$$

证明

$$I(m, n) = \frac{m!n!}{(m+n+1)!}(b-a)^{m+n+1}$$

并据此计算

$$\int_0^1 (1-x^2)^n dx$$

先对 $I(m, n)$ 作分部积分。取

$$u = (b-x)^m, \quad dv = (x-a)^n dx$$

则

$$du = -m(b-x)^{m-1}dx, \quad v = \frac{(x-a)^{n+1}}{n+1}$$

于是

$$I(m, n) = \left[\frac{(b-x)^m(x-a)^{n+1}}{n+1} \right]_a^b - \int_a^b \frac{-m(b-x)^{m-1}(x-a)^{n+1}}{n+1} dx$$

注意到端点项为零, 得到

$$I(m, n) = \frac{m}{n+1} I(m-1, n+1)$$

重复使用该递推关系,

$$I(m, n) = \frac{m}{n+1} \cdot \frac{m-1}{n+2} \cdots \frac{1}{n+m} I(0, n+m)$$

即

$$I(m, n) = \frac{m!}{(n+1)(n+2)\cdots(n+m)} I(0, n+m)$$

又

$$I(0, n+m) = \int_a^b (x-a)^{n+m} dx = \left[\frac{(x-a)^{n+m+1}}{n+m+1} \right]_a^b = \frac{(b-a)^{n+m+1}}{n+m+1}$$

因此

$$I(m, n) = \frac{m!n!}{(m+n+1)!}(b-a)^{m+n+1}$$

接下来计算

$$\int_0^1 (1-x^2)^n dx$$

利用偶函数对称性,

$$\int_0^1 (1-x^2)^n dx = \frac{1}{2} \int_{-1}^1 (1-x^2)^n dx$$

而

$$1-x^2 = (1-x)(x+1)$$

于是

$$\int_{-1}^1 (1-x^2)^n dx = \int_{-1}^1 (1-x)^n (x-(-1))^n dx$$

这正是 $I(n, n)$ 的形式, 其中 $a = -1$, $b = 1$ 。由已证公式,

$$I(n, n) = \frac{n!n!}{(2n+1)!} (1-(-1))^{2n+1} = \frac{n!^2}{(2n+1)!} 2^{2n+1}$$

因此

$$\int_0^1 (1-x^2)^n dx = \frac{1}{2} I(n, n) = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

162. 设数列 $\{a_n\}$ 满足

$$a_n = \int_0^1 (1-x^2)^{\frac{n}{2}} dx, \quad n = 0, 1, 2, 3, \dots$$

(a) 证明: $a_n = \frac{n}{n+1} a_{n-2}$, $n \geq 2$ 。

令 $x = \sin \theta$, 则 $dx = \cos \theta d\theta$, 有

$$a_n = \int_0^1 (1-x^2)^{\frac{n}{2}} dx = \int_0^{\frac{\pi}{2}} (\cos^2 \theta)^{\frac{n}{2}} \cdot \cos \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^{n+1} \theta d\theta$$

分部积分:

$$a_n = [\sin \theta \cos^n \theta]_0^{\frac{\pi}{2}} + n \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^{n-1} \theta d\theta$$

第一项为 0, 因此

$$a_n = n \int_0^{\frac{\pi}{2}} (1 - \cos^2 \theta) \cos^{n-1} \theta d\theta = n \int_0^{\frac{\pi}{2}} \cos^{n-1} \theta d\theta - n \int_0^{\frac{\pi}{2}} \cos^{n+1} \theta d\theta$$

即

$$a_n = na_{n-2} - na_n \Rightarrow a_n = \frac{n}{n+1} a_{n-2}$$

(b) 求 $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ 的值。

由定义

$$a_n = \int_0^{\frac{\pi}{2}} \cos^{n+1} \theta d\theta$$

可知 $a_n \geq a_{n+1}$, 于是

$$\frac{a_{n+2}}{a_n} \leq \frac{a_{n+1}}{a_n} \leq 1$$

又由递推公式

$$\lim_{n \rightarrow \infty} \frac{a_{n+2}}{a_n} = \lim_{n \rightarrow \infty} \frac{n+2}{n+3} = 1$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$$

163. 已知

$$I_n = \int_0^a x^{n+\frac{1}{2}} \sqrt{a-x} dx, \quad n \in \mathbb{Z}, \quad n \geqslant 0$$

其中 a 为正常数。

(a) 由分部积分法, 取

$$u = x^{n+\frac{1}{2}}, \quad dv = \sqrt{a-x} dx$$

则

$$du = \left(n + \frac{1}{2}\right) x^{n-\frac{1}{2}} dx, \quad v = -\frac{2}{3}(a-x)^{\frac{3}{2}}$$

于是

$$I_n = \left[-\frac{2}{3} x^{n+\frac{1}{2}} (a-x)^{\frac{3}{2}} \right]_0^a + \frac{2}{3} \left(n + \frac{1}{2}\right) \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{3}{2}} dx$$

边界项为零, 因此

$$I_n = \frac{2}{3} \left(n + \frac{1}{2}\right) \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{1}{2}} (a-x) dx$$

展开得

$$I_n = \frac{2}{3} \left(n + \frac{1}{2}\right) \left[a \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{1}{2}} dx - \int_0^a x^{n+\frac{1}{2}} (a-x)^{\frac{1}{2}} dx \right]$$

即

$$I_n = \frac{2}{3} \left(n + \frac{1}{2}\right) (aI_{n-1} - I_n)$$

整理得

$$(2n+4)I_n = (2n+1)aI_{n-1}$$

从而

$$I_n = \frac{(2n+1)a}{2n+4} I_{n-1}$$

反复递推,

$$I_n = a^n \frac{(2n+1)(2n-1)\cdots 3}{2^n(n+2)(n+1)\cdots 3} I_0$$

利用组合数恒等式, 化简得

$$I_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{I_0}{n+1}$$

(b) 计算

$$\int_0^2 x^{10} \sqrt{4-x^2} dx$$

令 $u = x^2$, 则

$$dx = \frac{1}{2} u^{-\frac{1}{2}} du$$

原积分化为

$$\int_0^2 x^{10} \sqrt{4-x^2} dx = \frac{1}{2} \int_0^4 u^{4+\frac{1}{2}} \sqrt{4-u} du = \frac{1}{2} I_4$$

由 (a) 式, 取 $a = 4$,

$$I_4 = \left(\frac{4}{4}\right)^4 \binom{10}{4} \frac{I_0}{5}$$

又

$$I_0 = \int_0^4 \sqrt{x} \sqrt{4-x} dx = 2\pi$$

代入得

$$I_4 = 84\pi$$

因此

$$\int_0^2 x^{10} \sqrt{4-x^2} dx = \frac{1}{2} I_4 = 42\pi$$

164. Evaluate the integral $\int (1-x^2)^{\frac{n}{2}} dx$ using reduction formulae.

(a) Show that

$$\int \sqrt{1-x^2} dx = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2} + C.$$

令 $x = \sin \theta, \theta \in [-\pi/2, \pi/2]$, 则 $dx = \cos \theta d\theta$:

$$\int \sqrt{1-\sin^2 \theta} \cos \theta d\theta = \int \cos^2 \theta d\theta.$$

使用恒等式 $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$:

$$\begin{aligned}\int \cos^2 \theta d\theta &= \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2} \int (1+\cos 2\theta) d\theta \\&= \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C = \frac{1}{2} \theta + \frac{1}{4} (2 \sin \theta \cos \theta) + C \\&= \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2} + C.\end{aligned}$$

(b) Show that for any positive integer n ,

$$\frac{d}{dx} [x(1-x^2)^{\frac{n}{2}}] = (n+1)(1-x^2)^{\frac{n}{2}} - n(1-x^2)^{\frac{n-2}{2}}.$$

使用乘法法则:

$$\begin{aligned}\frac{d}{dx} [x(1-x^2)^{\frac{n}{2}}] &= (1-x^2)^{\frac{n}{2}} + x \cdot \frac{n}{2} (1-x^2)^{\frac{n}{2}-1} (-2x) \\&= (1-x^2)^{\frac{n}{2}} - nx^2 (1-x^2)^{\frac{n-2}{2}} \\&= (1-x^2)^{\frac{n-2}{2}} ((1-x^2) - nx^2) \\&= (1-x^2)^{\frac{n-2}{2}} (1-(n+1)x^2) \\&= (n+1)(1-x^2)^{\frac{n}{2}} - n(1-x^2)^{\frac{n-2}{2}}.\end{aligned}$$

(c) Reduction formula. Let $I_n = \int (1-x^2)^{\frac{n}{2}} dx$. Then

$$\frac{d}{dx} [x(1-x^2)^{\frac{n}{2}}] = (n+1)(1-x^2)^{\frac{n}{2}} - n(1-x^2)^{\frac{n-2}{2}}.$$

积分两边得:

$$x(1-x^2)^{\frac{n}{2}} = (n+1)I_n - nI_{n-2} \implies I_n = \frac{n}{n+1}I_{n-2} + \frac{1}{n+1}x(1-x^2)^{\frac{n}{2}}.$$

(d) Compute $\int (1-x^2)^{\frac{5}{2}} dx$.

利用归纳公式:

$$I_5 = \frac{5}{6}I_3 + \frac{1}{6}x(1-x^2)^{\frac{5}{2}}, \quad I_3 = \frac{3}{4}I_1 + \frac{1}{4}x(1-x^2)^{\frac{3}{2}}, \quad I_1 = \int (1-x^2)^{1/2} dx = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2}.$$

代回得:

$$\begin{aligned}I_5 &= \frac{5}{6} \left(\frac{3}{4}I_1 + \frac{1}{4}x(1-x^2)^{3/2} \right) + \frac{1}{6}x(1-x^2)^{5/2} \\&= \frac{15}{24}I_1 + \frac{5}{24}x(1-x^2)^{3/2} + \frac{1}{6}x(1-x^2)^{5/2} \\&= \frac{5}{8} \left(\frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2} \right) + \frac{5}{24}x(1-x^2)^{3/2} + \frac{1}{6}x(1-x^2)^{5/2} + C \\&= \frac{5}{16} \sin^{-1} x + \frac{5}{16} x \sqrt{1-x^2} + \frac{5}{24}x(1-x^2)^{3/2} + \frac{1}{6}x(1-x^2)^{5/2} + C.\end{aligned}$$

165. $\int \frac{x^4}{\sqrt{x^2+1}} dx$, 使用降次公式。

(a) 证明对任意整数 $n \geq 2$ 有

$$\frac{d}{dx} \left[\frac{x^{n-1}}{\sqrt{x^2+1}} \right] = \frac{(2-n)x^{n-2} + (1-n)x^n}{(x^2+1)^{3/2}}.$$

$$\begin{aligned} \frac{d}{dx} \left[\frac{x^{n-1}}{\sqrt{x^2+1}} \right] &= \frac{d}{dx} [x^{n-1}(x^2+1)^{-1/2}] \\ &= (n-1)x^{n-2}(x^2+1)^{-1/2} + x^{n-1} \left(-\frac{1}{2}(x^2+1)^{-3/2}(2x) \right) \\ &= \frac{(n-1)x^{n-2}}{\sqrt{x^2+1}} - \frac{x^n}{(x^2+1)^{3/2}} \\ &= \frac{(2-n)x^{n-2} + (1-n)x^n}{(x^2+1)^{3/2}}. \end{aligned}$$

(b) 设 $I_n = \int \frac{x^n}{\sqrt{x^2+1}} dx$ 。利用 (a) 得:

$$(2-n)I_{n-2} + (1-n)I_n = \frac{x^{n-1}}{\sqrt{x^2+1}} \Rightarrow I_n = \frac{x^{n-1}\sqrt{x^2+1}}{1-n} + \frac{n-2}{1-n}I_{n-2}.$$

(c) 使用降次公式计算 $I_4 = \int \frac{x^4}{\sqrt{x^2+1}} dx$:

$$I_4 = \frac{x^3\sqrt{x^2+1}}{1-4} + \frac{4-2}{1-4}I_2 = -\frac{x^3\sqrt{x^2+1}}{3} - \frac{2}{3}I_2.$$

类似地计算 I_2 :

$$I_2 = \frac{x\sqrt{x^2+1}}{1-2} + \frac{2-2}{1-2}I_0 = -x\sqrt{x^2+1}.$$

将 I_2 代回 I_4 :

$$\begin{aligned} I_4 &= -\frac{x^3\sqrt{x^2+1}}{3} - \frac{2}{3}(-x\sqrt{x^2+1}) \\ &= -\frac{x^3\sqrt{x^2+1}}{3} + \frac{2x\sqrt{x^2+1}}{3} + C. \end{aligned}$$

166.

$$I_n = \int_0^a \frac{x^n}{\sqrt{a^2-x^2}} dx, \quad n \in \mathbb{N}, \quad a > 0$$

a) 推导递推公式

使用分部积分, 设

$$u = x^{n-1}, \quad dv = \frac{x}{\sqrt{a^2 - x^2}} dx \implies v = -\sqrt{a^2 - x^2}, \quad du = (n-1)x^{n-2} dx$$

$$\begin{aligned} I_n &= \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} dx = \int_0^a x^{n-1} \cdot \frac{x}{\sqrt{a^2 - x^2}} dx \\ &= \left[-x^{n-1} \sqrt{a^2 - x^2} \right]_0^a + (n-1) \int_0^a x^{n-2} \sqrt{a^2 - x^2} dx \\ &= (n-1) \int_0^a \frac{x^{n-2}(a^2 - x^2)}{\sqrt{a^2 - x^2}} dx \\ &= (n-1) \left[a^2 \int_0^a \frac{x^{n-2}}{\sqrt{a^2 - x^2}} dx - \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} dx \right] \\ &= (n-1)a^2 I_{n-2} - (n-1)I_n \\ &\implies nI_n = a^2(n-1)I_{n-2}, \quad n \geq 2 \end{aligned}$$

b) 计算具体积分

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}} dx$$

首先配方:

$$4x - x^2 = -(x^2 - 4x) = 4 - (x - 2)^2$$

令 $u = x - 2 \implies du = dx$, 积分上下限:

$$x = 2 \implies u = 0, \quad x = 4 \implies u = 2$$

分解被积函数:

$$\begin{aligned} 3x^3 - 18x^2 + 36x - 18 &= 3[(x - 2)^3 + 2] \\ \implies \int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}} dx &= \int_0^2 \frac{3u^3 + 6}{\sqrt{4 - u^2}} du \\ &= 3 \int_0^2 \frac{u^3}{\sqrt{4 - u^2}} du + \int_0^2 \frac{6}{\sqrt{4 - u^2}} du \\ &= 3I_3 + 6 \arcsin \left(\frac{u}{2} \right) \Big|_0^2 \\ &= 3I_3 + 6 \cdot \frac{\pi}{2} = 3I_3 + 3\pi \end{aligned}$$

使用递推公式 $I_n = \frac{4(n-1)}{n} I_{n-2}$:

$$I_3 = \frac{4(3-1)}{3} I_1 = \frac{8}{3} I_1$$

计算 I_1 :

$$I_1 = \int_0^2 \frac{u}{\sqrt{4-u^2}} du = \left[-\sqrt{4-u^2} \right]_0^2 = 0 - (-2) = 2$$

$$\Rightarrow I_3 = \frac{8}{3} \cdot 2 = \frac{16}{3}$$

最终结果:

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}} dx = 3 \cdot \frac{16}{3} + 3\pi = 16 + 3\pi$$

167. 已知

$$I_n = \int_0^\pi \frac{\sin(n\theta)}{\sin \theta} d\theta$$

其中 n 为正整数。

(a) 利用三角恒等式证明

$$\frac{\sin(n\theta) - \sin[(n-2)\theta]}{\sin \theta} = 2 \cos[(n-1)\theta]$$

(b) 推导

$$I_n = I_{n-2}, \quad n \geq 2$$

(c) 求 I_n 的值, 分别讨论 n 为奇数或偶数的情况

(a) 使用公式 $\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$:

$$\begin{aligned} \frac{\sin(n\theta) - \sin[(n-2)\theta]}{\sin \theta} &= \frac{2 \cos \frac{n\theta + (n-2)\theta}{2} \sin \frac{n\theta - (n-2)\theta}{2}}{\sin \theta} \\ &= \frac{2 \cos[(n-1)\theta] \sin \theta}{\sin \theta} \\ &= 2 \cos[(n-1)\theta] \end{aligned}$$

(b) 利用 (a) 结果:

$$\begin{aligned} I_n - I_{n-2} &= \int_0^\pi \frac{\sin(n\theta) - \sin[(n-2)\theta]}{\sin \theta} d\theta \\ &= \int_0^\pi 2 \cos[(n-1)\theta] d\theta \\ &= \left[\frac{2}{n-1} \sin((n-1)\theta) \right]_0^\pi = 0 \\ \therefore I_n &= I_{n-2}, \quad n \geq 2 \end{aligned}$$

(c) 由 (b) 得递推关系:

$$I_n = I_{n-2} = \dots$$

若 n 为偶数, 则递推到 I_0 :

$$I_0 = \int_0^\pi \frac{\sin 0}{\sin \theta} d\theta = 0$$

若 n 为奇数, 则递推到 I_1 :

$$I_1 = \int_0^\pi \frac{\sin \theta}{\sin \theta} d\theta = \int_0^\pi 1 d\theta = \pi$$

因此

$$I_n = \begin{cases} 0 & n \text{ 为偶数} \\ \pi & n \text{ 为奇数} \end{cases}$$

168. 已知

$$I_n = \int \frac{\sin(nx)}{\sin x} dx, \quad n \in \mathbb{N}.$$

(a) 证明对 $n \geq 0$ 有

$$I_{n+2} = I_n + \frac{2}{n+1} \sin[(n+1)x] + C$$

(b) 利用 (a) 的递推关系, 求

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} dx$$

(a) 考虑

$$\begin{aligned} I_{n+2} - I_n &= \int \frac{\sin[(n+2)x]}{\sin x} dx - \int \frac{\sin(nx)}{\sin x} dx \\ &= \int \frac{\sin[(n+2)x] - \sin(nx)}{\sin x} dx \end{aligned}$$

利用恒等式 $\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$:

$$\begin{aligned} I_{n+2} - I_n &= \int \frac{2 \cos \frac{(n+2)x+nx}{2} \sin \frac{(n+2)x-nx}{2}}{\sin x} dx \\ &= \int \frac{2 \cos[(n+1)x] \sin x}{\sin x} dx \\ &= \int 2 \cos[(n+1)x] dx \\ &= \frac{2}{n+1} \sin[(n+1)x] + C \end{aligned}$$

因此

$$I_{n+2} = I_n + \frac{2}{n+1} \sin[(n+1)x] + C$$

(b) 对定积分, 递推关系不需考虑常数 C :

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} dx &= I_6 \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= I_4 \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \left[\frac{2}{5} \sin 5x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= I_4 + \frac{2}{5} \left(\sin \frac{5\pi}{3} - \sin \frac{5\pi}{4} \right) \\ &= I_4 + \frac{2}{5} \left(-\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{2}}{2} \right) \right) \\ &= I_4 - \frac{\sqrt{3}}{5} + \frac{\sqrt{2}}{5} \end{aligned}$$

同理,

$$\begin{aligned} I_4 &= I_2 + \left[\frac{2}{3} \sin 3x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= I_2 + \frac{2}{3} \left(\sin \pi - \sin \frac{3\pi}{4} \right) \\ &= I_2 - \frac{\sqrt{2}}{3} \end{aligned}$$

再有

$$I_2 = I_0 + [2 \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

169. 求

$$I_n = \int_0^\pi \theta^n \sin \theta \, d\theta, \quad n \geq 2$$

(a) 使用分部积分:

$$u = \theta^n, \quad dv = \sin \theta \, d\theta \implies du = n\theta^{n-1}d\theta, \quad v = -\cos \theta$$

$$\begin{aligned} I_n &= [-\theta^n \cos \theta]_0^\pi + n \int_0^\pi \theta^{n-1} \cos \theta \, d\theta \\ &= -\pi^n \cos \pi + 0 + n \int_0^\pi \theta^{n-1} \cos \theta \, d\theta \\ &= \pi^n + n \int_0^\pi \theta^{n-1} \cos \theta \, d\theta \end{aligned}$$

对第二个积分再次分部积分:

$$u = \theta^{n-1}, \quad dv = \cos \theta \, d\theta \implies du = (n-1)\theta^{n-2}d\theta, \quad v = \sin \theta$$

$$\begin{aligned} I_n &= \pi^n + n [\theta^{n-1} \sin \theta]_0^\pi - n(n-1) \int_0^\pi \theta^{n-2} \sin \theta \, d\theta \\ &= \pi^n - n(n-1) \int_0^\pi \theta^{n-2} \sin \theta \, d\theta \\ &= \pi^n - n(n-1)I_{n-2} \end{aligned}$$

(b) 计算

$$\int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx$$

使用代换 $\theta = 2x, d\theta = 2dx$:

$$x = \frac{\pi}{2} \implies \theta = \pi, \quad x = 0 \implies \theta = 0$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx &= \int_0^\pi \left(\frac{\theta}{2}\right)^4 \sin \theta \frac{d\theta}{2} \\ &= \frac{1}{32} \int_0^\pi \theta^4 \sin \theta \, d\theta \\ &= \frac{1}{32} I_4 \end{aligned}$$

利用递推公式:

$$I_4 = \pi^4 - 4 \cdot 3I_2 = \pi^4 - 12I_2, \quad I_2 = \pi^2 - 2I_0, \quad I_0 = \int_0^\pi \sin \theta \, d\theta = 2$$

$$I_4 = \pi^4 - 12(\pi^2 - 4) = \pi^4 - 12\pi^2 + 48$$

$$\int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx = \frac{1}{32}(\pi^4 - 12\pi^2 + 48)$$

如所需。

170. Evaluate

$$I_n = \int \csc^n x \, dx, \quad n \in \mathbb{N}.$$

a) Reduction formula

将 $\csc^n x = \csc^{n-2} x \csc^2 x$, 并用分部积分:

$$u = -\cot x, \quad dv = \csc^{n-2} x \csc^2 x \, dx \implies du = \csc^2 x \, dx, \quad v = \csc^{n-2} x$$

$$I_n = -\cot x \csc^{n-2} x - \int (-\cot x) d(\csc^{n-2} x)$$

计算 $d(\csc^{n-2} x) = (n-2) \csc^{n-1} x (-\cot x) dx = -(n-2) \csc^{n-1} x \cot x \, dx$:

$$I_n = -\cot x \csc^{n-2} x - (-(n-2)) \int \csc^{n-1} x \cot^2 x \, dx$$

使用 $\cot^2 x = \csc^2 x - 1$:

$$I_n = -\cot x \csc^{n-2} x - (n-2) \int \csc^n x \, dx + (n-2) \int \csc^{n-2} x \, dx$$

$$I_n + (n-2)I_n = (n-2)I_{n-2} - \cot x \csc^{n-2} x$$

$$\boxed{I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} \cot x \csc^{n-2} x}, \quad n \geq 2$$

171. b) Evaluate

$$\int_{\pi/4}^{\pi/2} \csc^6 x \, dx$$

利用公式:

$$I_6 = \frac{6-2}{6-1} I_4 - \frac{1}{5} [\cot x \csc^4 x]_{\pi/4}^{\pi/2} = \frac{4}{5} I_4 - \frac{1}{5} [0 - 1 \cdot 2^2] = \frac{4}{5} I_4 + \frac{4}{5}$$

对 I_4 应用同样公式:

$$I_4 = \frac{2}{3} I_2 - \frac{1}{3} [\cot x \csc^2 x]_{\pi/4}^{\pi/2} = \frac{2}{3} I_2 - \frac{1}{3} [0 - 1 \cdot 2] = \frac{2}{3} I_2 + \frac{2}{3}$$

代回 I_6 :

$$I_6 = \frac{4}{5} \left(\frac{2}{3} I_2 + \frac{2}{3} \right) + \frac{4}{5} = \frac{8}{15} I_2 + \frac{8}{15} + \frac{4}{5} = \frac{8}{15} I_2 + \frac{20}{15} = \frac{8}{15} I_2 + \frac{4}{3}$$

计算 I_2 :

$$I_2 = \int_{\pi/4}^{\pi/2} \csc^2 x \, dx = [-\cot x]_{\pi/4}^{\pi/2} = 0 - (-1) = 1$$

最终:

$$I_6 = \frac{8}{15}(1) + \frac{4}{3} = \frac{8}{15} + \frac{20}{15} = \frac{28}{15}$$

172. 令

$$I_n = \int_0^{\pi/4} \tan^n x \, dx,$$

其中 n 为正整数, 试回答下列各问题:

(a) 试证明: 当 $0 \leq x \leq \frac{\pi}{4}$ 时, $\tan x \leq x + 1 - \frac{\pi}{4}$ 。

令 $f(x) = x + 1 - \frac{\pi}{4} - \tan x$, 则

$$f'(x) = 1 - \sec^2 x = -\tan^2 x \leq 0.$$

因此 f 在 $\left[0, \frac{\pi}{4}\right]$ 上单调递减。又

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + 1 - \frac{\pi}{4} - \tan \frac{\pi}{4} = 1 - 1 = 0,$$

故对任意 $x \in \left[0, \frac{\pi}{4}\right]$ 有 $f(x) \geq 0$, 即

$$\tan x \leq x + 1 - \frac{\pi}{4}$$

(b) 试求 $\lim_{n \rightarrow \infty} I_n$ 之值。

令 $t = \tan x$, 则 $dx = \frac{1}{1+t^2} dt$, 于是

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx = \int_0^1 \frac{t^n}{1+t^2} dt \leq \int_0^1 t^n dt = \frac{1}{n+1}$$

因此

$$\lim_{n \rightarrow \infty} I_n = 0$$

(c) 请用 n 表示 $I_n + I_{n+2}$ 之值。

由 (b) 的换元得

$$I_n + I_{n+2} = \int_0^1 \frac{t^n + t^{n+2}}{1+t^2} dt = \int_0^1 t^n dt = \frac{1}{n+1}$$

(d) 利用 (c) 的结果计算

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$$

级数可写为

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$$

由 (c), 可将该级数分组为

$$(I_1 + I_3) - (I_3 + I_5) + (I_5 + I_7) - \cdots = I_1$$

因此

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} = I_1 = \int_0^1 \frac{t}{1+t^2} dt = \left[\frac{1}{2} \ln(1+t^2) \right]_0^1 = \frac{1}{2} \ln 2$$

173. 设 $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m \theta \sin^n \theta d\theta$, 其中 m, n 为非负整数且 $m > 1$ 。求 $I_{m,n}$ 与 $I_{m-2,n+2}$ 的关系, 并求 $\int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^6 \theta d\theta$ 。

使用降次公式:

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m \theta \sin^n \theta d\theta = \frac{m-1}{n+1} I_{m-2,n+2}.$$

对具体积分:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^6 \theta d\theta &= \frac{7-1}{6+1} I_{5,8} = \frac{6}{7} I_{5,8}, \\ I_{5,8} &= \frac{5-1}{8+1} I_{3,10} = \frac{4}{9} I_{3,10}, \\ I_{3,10} &= \frac{3-1}{10+1} I_{1,12} = \frac{2}{11} I_{1,12}.\end{aligned}$$

积分 $I_{1,12}$ 可直接计算:

$$I_{1,12} = \int_0^{\frac{\pi}{2}} \cos \theta \sin^{12} \theta d\theta.$$

令 $y = \sin \theta$, 则 $dy = \cos \theta d\theta$, 积分上下限变为 0 到 1:

$$I_{1,12} = \int_0^1 y^{12} dy = \frac{1}{13}.$$

最终结果:

$$\int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^6 \theta d\theta = \frac{6}{7} \cdot \frac{4}{9} \cdot \frac{2}{11} \cdot \frac{1}{13} = \frac{48}{3003} = \frac{16}{1001}.$$

174. 求

$$\int_0^1 x^5 e^{-x^2} dx$$

设

$$I_n = \int_0^1 x^n e^{-x^2} dx, \quad n \in \mathbb{N}.$$

将 $x^n = x^{n-1} \cdot x$ 并分部积分:

$$u = x^{n-1}, \quad dv = x e^{-x^2} dx \implies du = (n-1)x^{n-2} dx, \quad v = -\frac{1}{2} e^{-x^2}$$

$$\begin{aligned}I_n &= \left[-\frac{1}{2} x^{n-1} e^{-x^2} \right]_0^1 - \int_0^1 -\frac{1}{2} (n-1) x^{n-2} e^{-x^2} dx \\ &= -\frac{1}{2} e^{-1} + \frac{1}{2} (n-1) I_{n-2}\end{aligned}$$

这是所需的递推公式:

$$I_n = -\frac{1}{2} e^{-1} + \frac{1}{2} (n-1) I_{n-2}$$

使用该公式求 I_5 :

$$\begin{aligned}I_5 &= -\frac{1}{2}e^{-1} + 2I_3 \\I_3 &= -\frac{1}{2}e^{-1} + I_1 \\I_1 &= \int_0^1 xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = -\frac{1}{2}e^{-1} + \frac{1}{2}\end{aligned}$$

代回得:

$$\begin{aligned}I_3 &= -\frac{1}{2}e^{-1} + \left(-\frac{1}{2}e^{-1} + \frac{1}{2}\right) = -e^{-1} + \frac{1}{2} \\I_5 &= -\frac{1}{2}e^{-1} + 2\left(-e^{-1} + \frac{1}{2}\right) = -\frac{1}{2}e^{-1} - 2e^{-1} + 1 = 1 - \frac{5}{2}e^{-1} \\I_5 &= \frac{2e - 5}{2e}\end{aligned}$$

如所需。

175. a) Show that for $p \in (0, \infty)$

$$\lim_{x \rightarrow 0^+} [x^p \ln x] = 0$$

Hence evaluate

$$\begin{aligned}\int_0^1 x^n \ln x \, dx, \quad n \in \mathbb{N}. \\ \int_0^1 [\ln(1-x)] \ln x \, dx\end{aligned}$$

注意 $\lim_{x \rightarrow 0^+} x^p \ln x$ 是“ $0 \times -\infty$ ”型, 可以改写为

$$\lim_{x \rightarrow 0^+} x^p \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-p}}$$

这是 $-\infty/\infty$ 型, 应用洛必达法则:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-p}} = \lim_{x \rightarrow 0^+} \frac{1/x}{-px^{-p-1}} = \lim_{x \rightarrow 0^+} -\frac{1}{p}x^p = 0.$$

使用分部积分:

$$u = \ln x, \quad dv = x^n dx \implies du = \frac{1}{x} dx, \quad v = \frac{x^{n+1}}{n+1}$$

$$\int_0^1 x^n \ln x \, dx = \left[\frac{x^{n+1}}{n+1} \ln x \right]_0^1 - \int_0^1 \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} dx$$

由 a) 可知 $\lim_{x \rightarrow 0^+} x^{n+1} \ln x = 0$, 且在 $x = 1$ 时 $\ln 1 = 0$, 所以第一项为 0:

$$\int_0^1 x^n \ln x \, dx = -\frac{1}{n+1} \int_0^1 x^n dx = -\frac{1}{n+1} \left[\frac{x^{n+1}}{n+1} \right]_0^1 = -\frac{1}{(n+1)^2}.$$

由于

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad -1 \leq x < 1$$

故

$$\begin{aligned} \int_0^1 \ln(1-x) \ln x \, dx &= \int_0^1 (\ln x) \left[-\sum_{n=1}^{\infty} \frac{x^n}{n} \right] dx \\ &= -\sum_{n=1}^{\infty} \int_0^1 \frac{x^n \ln x}{n} dx \end{aligned}$$

又有

$$\int_0^1 x^n \ln x \, dx = -\frac{1}{(n+1)^2}$$

因此

$$\begin{aligned} \int_0^1 \ln(1-x) \ln x \, dx &= -\sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{1}{(n+1)^2} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} \end{aligned}$$

接下来对

$$\frac{1}{n(n+1)^2}$$

作部分分式分解, 设

$$\frac{1}{n(n+1)^2} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{(n+1)^2}$$

则

$$1 \equiv A(n+1)^2 + Bn(n+1) + Cn$$

令

$$n = 0 \Rightarrow A = 1$$

$$n = -1 \Rightarrow C = -1$$

$$n = 1 \Rightarrow 1 = 4A + 2B + C = 4 + 2B - 1$$

从而

$$B = -1$$

于是

$$\frac{1}{n(n+1)^2} = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}$$

代回求和得

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

第一项为望远镜级数,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$$

第二项

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - 1$$

因此

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} &= 1 - \left(\frac{\pi^2}{6} - 1 \right) \\ &= 2 - \frac{\pi^2}{6} \end{aligned}$$

故

$$\int_0^1 \ln(1-x) \ln x \, dx = 2 - \frac{\pi^2}{6} = \frac{1}{6}(12 - \pi^2)$$

176. 求

$$\int_0^1 x(\ln x)^{10} \, dx$$

令

$$I_n = \int_0^1 x(\ln x)^n dx$$

对 I_n 使用分部积分:

$$u = (\ln x)^n, \quad dv = x dx \implies du = n(\ln x)^{n-1} \frac{1}{x} dx, \quad v = \frac{1}{2} x^2$$

$$\begin{aligned} I_n &= \left[\frac{1}{2} x^2 (\ln x)^n \right]_0^1 - \int_0^1 \frac{1}{2} x^2 \cdot n(\ln x)^{n-1} \frac{1}{x} dx \\ &= 0 - \frac{n}{2} \int_0^1 x(\ln x)^{n-1} dx \\ &= -\frac{n}{2} I_{n-1} \end{aligned}$$

得到归纳公式:

$$I_n = -\frac{n}{2} I_{n-1}$$

继续展开:

$$I_{10} = (-1)^{10} \frac{10!}{2^{10}} I_0$$

而

$$I_0 = \int_0^1 x dx = \frac{1}{2}$$

因此

$$I_{10} = \frac{10!}{2^{11}}$$

拆解为质数幂:

$$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7, \quad 2^{11} = 2^{11} \implies I_{10} = \frac{3^4 \cdot 5^2 \cdot 7}{2^3}$$

177. 求

$$I_n = \int e^{2x} \sin^n x dx, \quad n \in \mathbb{N}, n \geq 2$$

先使用分部积分:

$$u = \sin^n x, \quad dv = e^{2x} dx \implies du = n \sin^{n-1} x \cos x dx, \quad v = \frac{1}{2} e^{2x}$$

$$I_n = \frac{1}{2}e^{2x} \sin^n x - \frac{n}{2} \int e^{2x} \sin^{n-1} x \cos x dx$$

对第二个积分再次使用分部积分:

$$u = \sin^{n-1} x \cos x, \quad dv = \frac{1}{2}e^{2x} dx \implies du = (n-1) \sin^{n-2} x \cos^2 x - \sin^n x dx$$

于是

$$\begin{aligned} I_n &= \frac{1}{2}e^{2x} \sin^n x - \frac{n}{2} \left[\frac{1}{2}e^{2x} \sin^{n-1} x \cos x - \frac{1}{2} \int e^{2x} ((n-1) \sin^{n-2} x - n \sin^n x) dx \right] \\ &= \frac{1}{2}e^{2x} \sin^n x - \frac{n}{4}e^{2x} \sin^{n-1} x \cos x + \frac{n(n-1)}{4} \int e^{2x} \sin^{n-2} x dx - \frac{n^2}{4} \int e^{2x} \sin^n x dx \\ &= \frac{1}{2}e^{2x} \sin^n x - \frac{n}{4}e^{2x} \sin^{n-1} x \cos x + \frac{n(n-1)}{4} I_{n-2} - \frac{n^2}{4} I_n \end{aligned}$$

整理得到:

$$\begin{aligned} 4I_n &= 2e^{2x} \sin^n x - ne^{2x} \sin^{n-1} x \cos x + n(n-1)I_{n-2} - n^2 I_n \\ (n^2 + 4)I_n &= n(n-1)I_{n-2} + e^{2x} \sin^{n-1} x (2 \sin x - n \cos x) \end{aligned}$$

如所需。

178. 计算 $[\int_0^1 (x \ln x)^4 dx]$ 。

使用特殊积分公式 $[\int_0^1 x^m (-\ln x)^n dx = \frac{n!}{(m+1)^{n+1}}]$ 。在本题中, $m=4$ 且 $n=4$ 。由于 n 是偶数, $(\ln x)^4 = (-\ln x)^4$ 。代入公式得:

$$I = \frac{4!}{(4+1)^{4+1}} = \frac{24}{5^5} = \frac{24}{3125}$$

结果为 0.00768。

179. 已知

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^n x dx, \quad n \in \mathbb{N}$$

(a) 证明 i.

$$nI_{n+1} = e^\pi (\sqrt{3})^n - 3I_n - nI_{n-1}, \quad n \geq 1$$

ii.

$$I_0 = I_4 + I_3 - 3I_1$$

(b) 求

$$\int_0^{\frac{\pi}{3}} e^{3x} \tan x (\tan^3 x + \sec^2 x - 4) dx$$

(a) 由

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^n x dx$$

写成

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^{n-2} x (\sec^2 x - 1) dx$$

即

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \sec^2 x \tan^{n-2} x dx - I_{n-2}$$

对

$$\int_0^{\frac{\pi}{3}} e^{3x} \sec^2 x \tan^{n-2} x dx$$

作分部积分, 取

$$u = e^{3x}, \quad dv = \sec^2 x \tan^{n-2} x dx$$

则

$$du = 3e^{3x} dx, \quad v = \frac{1}{n-1} \tan^{n-1} x$$

于是

$$I_n = \left[\frac{1}{n-1} e^{3x} \tan^{n-1} x \right]_0^{\frac{\pi}{3}} - \frac{3}{n-1} I_{n-1} - I_{n-2}$$

又

$$\tan \frac{\pi}{3} = \sqrt{3}$$

从而

$$I_n = \frac{1}{n-1} e^{\pi} (\sqrt{3})^{n-1} - \frac{3}{n-1} I_{n-1} - I_{n-2}$$

将 n 换成 $n+1$ 得

$$I_{n+1} = \frac{1}{n} e^{\pi} (\sqrt{3})^n - \frac{3}{n} I_n - I_{n-1}$$

即

$$nI_{n+1} = e^{\pi} (\sqrt{3})^n - 3I_n - nI_{n-1}$$

(a)(ii) 当 $n = 1$,

$$I_2 = e^\pi \sqrt{3} - 3I_1 - I_0$$

当 $n = 3$,

$$3I_4 = 3e^\pi \sqrt{3} - 3I_3 - 3I_2$$

代入 I_2 ,

$$3I_4 = 9I_1 + 3I_0 - 3I_3$$

化简得

$$I_0 = I_4 + I_3 - 3I_1$$

(b) 原积分为

$$\int_0^{\frac{\pi}{3}} e^{3x} (\tan^4 x + \tan^3 x - 3 \tan x) dx$$

即

$$I_4 + I_3 - 3I_1$$

由 (a)(ii) 知其等于 I_0 , 而

$$I_0 = \int_0^{\frac{\pi}{3}} e^{3x} dx = \left[\frac{1}{3} e^{3x} \right]_0^{\frac{\pi}{3}} = \frac{1}{3} (e^\pi - 1)$$

故所求积分的精确值为

$$\frac{1}{3} (e^\pi - 1)$$

180. 求下列广义积分的精确值:

$$\int_0^\infty \sqrt{x} e^{-x} dx$$

你可以假设

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}.$$

首先作代换:

$$u = \sqrt{x} \implies x = u^2, \quad dx = 2u du$$
$$\int \sqrt{x} e^{-x} dx = \int u \cdot e^{-u^2} \cdot 2u du = \int 2u^2 e^{-u^2} du$$

对 $\int 2u^2 e^{-u^2} du$ 使用分部积分, 设 $U = u, dV = 2u e^{-u^2} du$:

$$\int u \cdot 2u e^{-u^2} du = -u e^{-u^2} + \int e^{-u^2} du$$

将积分上下限转回 x , 得到

$$\int_0^\infty \sqrt{x} e^{-x} dx = \left[-u e^{-u^2} \right]_0^\infty + \int_0^\infty e^{-u^2} du$$

取极限 $k \rightarrow \infty$:

$$\int_0^\infty \sqrt{x} e^{-x} dx = \lim_{k \rightarrow \infty} \left[-k e^{-k^2} + 0 \right] + \frac{1}{2} \sqrt{\pi} = 0 + \frac{1}{2} \sqrt{\pi} = \frac{1}{2} \sqrt{\pi}$$

因此, 广义积分的精确值为

$$\int_0^\infty \sqrt{x} e^{-x} dx = \frac{1}{2} \sqrt{\pi}.$$

181. 计算下列积分的精确值, 并给出正式的极限过程:

$$\int_0^{\frac{\pi}{4}} \left(\frac{1}{x} - \frac{\sin 2x}{1 - \cos 2x} \right) dx$$

结果应写成

$$\ln \left[\frac{\pi \sqrt{2}}{n} \right],$$

其中 n 为正整数。

为处理下限的零点, 令 0 替换为 $k > 0$:

$$\int_k^{\frac{\pi}{4}} \left(\frac{1}{x} - \frac{\sin 2x}{1 - \cos 2x} \right) dx = \left[\ln x - \frac{1}{2} \ln(1 - \cos 2x) \right]_k^{\frac{\pi}{4}} = \frac{1}{2} \left[\ln \frac{x^2}{1 - \cos 2x} \right]_k^{\frac{\pi}{4}}$$

代入上限:

$$\frac{1}{2} \ln \frac{\left(\frac{\pi}{4}\right)^2}{1 - \cos\left(\frac{\pi}{2}\right)} = \frac{1}{2} \ln \frac{\pi^2}{16}$$

下限展开 $\cos 2k$ 的幂级数:

$$1 - \cos 2k = 1 - \left(1 - \frac{(2k)^2}{2} + \frac{(2k)^4}{24} - \dots \right) = 2k^2 - \frac{2}{3}k^4 + O(k^6)$$

因此

$$\frac{k^2}{1 - \cos 2k} = \frac{k^2}{2k^2 - \frac{2}{3}k^4 + O(k^6)} = \frac{1}{2 - \frac{2}{3}k^2 + O(k^4)} \xrightarrow{k \rightarrow 0} \frac{1}{2}$$

取极限 $k \rightarrow 0$:

$$\int_0^{\frac{\pi}{4}} \left(\frac{1}{x} - \frac{\sin 2x}{1 - \cos 2x} \right) dx = \frac{1}{2} \ln \frac{\pi^2}{16} - \frac{1}{2} \ln \frac{1}{2} = \ln \frac{\pi}{4} + \ln \sqrt{2} = \ln \frac{\pi \sqrt{2}}{4}$$

因此, 所求的 $n = 4$ 。

182. 计算以下积分, 首先使用代换 $y = \frac{1}{x}$:

$$\int \frac{\ln x^2}{x^3} dx, \quad x \neq 0$$

a) 使用代换

$$\text{设 } y = \frac{1}{x} \implies \frac{dy}{dx} = -\frac{1}{x^2} \implies dx = -x^2 dy = -\frac{1}{y^2} dy$$

$$\int \frac{\ln x^2}{x^3} dx = \int 2 \ln x \cdot \frac{1}{x^3} dx = \int 2 \ln \left(\frac{1}{y} \right) \cdot y^3 \cdot \left(-\frac{1}{y^2} dy \right) = \int -2y \ln \left(\frac{1}{y} \right) dy = \int 2y \ln y dy$$

b) 计算定积分

积分上下限变换:

$$x = 1 \implies y = 1, \quad x \rightarrow \infty \implies y \rightarrow 0$$

$$\int_1^\infty \frac{\ln x^2}{x^3} dx = \int_1^0 2y \ln y dy = - \int_0^1 2y \ln y dy$$

使用分部积分:

$$u = \ln y \implies du = \frac{1}{y} dy, \quad dv = 2y dy \implies v = y^2$$

$$- \int_0^1 2y \ln y dy = - [y^2 \ln y]_0^1 + \int_0^1 y^2 \cdot \frac{1}{y} dy = - [y^2 \ln y]_0^1 + \int_0^1 y dy = - [y^2 \ln y]_0^1 + \left[\frac{1}{2} y^2 \right]_0^1$$

取极限:

$$- \lim_{h \rightarrow 0} [1^2 \ln 1 - h^2 \ln h] + \left[\frac{1}{2} - 0 \right] = - \lim_{h \rightarrow 0} (-h^2 \ln h) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

因此

$$\int_1^\infty \frac{\ln x^2}{x^3} dx = \frac{1}{2}$$

183. 22) 计算广义积分 $I = \int_0^\infty \frac{\ln x}{x^2 + 2x + 4} dx$

设 $x = \frac{4}{u}, dx = -\frac{4}{u^2} du$ 。当 $x \rightarrow 0, u \rightarrow \infty$; 当 $x \rightarrow \infty, u \rightarrow 0$ 。

$$I = \int_\infty^0 \frac{\ln(4/u)}{\frac{16}{u^2} + \frac{8}{u} + 4} \left(-\frac{4}{u^2} \right) du = \int_0^\infty \frac{\ln 4 - \ln u}{u^2 + 2u + 4} du$$

$$= \ln 4 \int_0^\infty \frac{1}{u^2 + 2u + 4} du - \int_0^\infty \frac{\ln u}{u^2 + 2u + 4} du$$

观察发现右边第二项即为 I , 故:

$$\begin{aligned} 2I &= \ln 4 \int_0^{\infty} \frac{1}{(x+1)^2 + 3} dx \\ 2I &= \ln 4 \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x+1}{\sqrt{3}} \right) \right]_0^{\infty} \\ 2I &= \frac{\ln 4}{\sqrt{3}} \left(\frac{\pi}{2} - \frac{\pi}{6} \right) = \frac{\ln 4}{\sqrt{3}} \cdot \frac{\pi}{3} \\ I &= \frac{\pi \ln 4}{6\sqrt{3}} \end{aligned}$$

184. 已知

$$I = \int_{\frac{\pi}{2}}^{\pi} \frac{3 + \cos x}{13 + 3 \cos x + 2 \sin x} dx, \quad J = \int_{\frac{\pi}{2}}^{\pi} \frac{2 + \sin x}{13 + 3 \cos x + 2 \sin x} dx$$

构造线性组合:

$$3I + 2J = \int_{\frac{\pi}{2}}^{\pi} \frac{9 + 3 \cos x}{13 + 3 \cos x + 2 \sin x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{4 + 2 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{13 + 3 \cos x + 2 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \int_{\frac{\pi}{2}}^{\pi} 1 dx = \frac{\pi}{2}$$

再构造另一组合:

$$2I - 3J = \int_{\frac{\pi}{2}}^{\pi} \frac{6 + 2 \cos x}{13 + 3 \cos x + 2 \sin x} dx - \int_{\frac{\pi}{2}}^{\pi} \frac{6 + 3 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{2 \cos x - 3 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \ln \frac{2}{3}$$

解二元一次方程组:

$$3I + 2J = \frac{\pi}{2}, \quad 2I - 3J = \ln \frac{2}{3}$$

乘以合适系数消元:

$$9I + 6J = \frac{3\pi}{2}, \quad 4I - 6J = 2 \ln \frac{2}{3}$$

相加得:

$$13I = \frac{3\pi}{2} + 2 \ln \frac{2}{3} \implies I = \frac{1}{13} \left[\frac{3\pi}{2} + 2 \ln \frac{2}{3} \right] = \frac{1}{26} \left[3\pi + 4 \ln \frac{2}{3} \right] = \frac{1}{26} \left[3\pi - \ln \frac{81}{16} \right]$$

类似地求 J :

$$3I + 2J = \frac{\pi}{2} (\times 2), \quad 2I - 3J = \ln \frac{2}{3} (\times 3)$$

$$6I + 4J = \pi, \quad 6I - 9J = 3 \ln \frac{2}{3} \implies 13J = \pi - 3 \ln \frac{2}{3} \implies J = \frac{1}{13} \left[\pi - 3 \ln \frac{2}{3} \right] = \frac{1}{13} \left[\pi + \ln \frac{27}{8} \right]$$

185.

$$\int \frac{1}{(x+1)(x^2+1)} dx$$

设

$$I = \int \frac{dx}{(x+1)(x^2+1)}, \quad J = \int \frac{x^2 dx}{(x+1)(x^2+1)}$$

于是有:

$$I + J = \int \frac{x^2 + 1}{(x+1)(x^2+1)} dx = \int \frac{dx}{x+1} = \ln|x+1| + C_1$$

$$\begin{aligned} J - I &= \int \frac{x^2 - 1}{(x+1)(x^2+1)} dx \\ &= \int \frac{x-1}{x^2+1} dx \\ &= \int \frac{x}{x^2+1} dx - \int \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \ln(x^2+1) - \tan^{-1} x + C_2 \end{aligned}$$

联立两式, 得:

$$\begin{aligned} 2I &= (I+J) - (J-I) \\ &= \ln|x+1| - \frac{1}{2} \ln(x^2+1) + \tan^{-1} x + C_3 \end{aligned}$$

所以

$$\int \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \tan^{-1} x + C$$

$$\int \frac{7 \cos x - 3 \sin x}{5 \cos x + 2 \sin x} dx$$

设

$$I = \int \frac{\sin x}{5 \cos x + 2 \sin x} dx, \quad J = \int \frac{\cos x}{5 \cos x + 2 \sin x} dx$$

构造两个线性组合:

$$2I + 5J = \int \frac{2 \sin x + 5 \cos x}{5 \cos x + 2 \sin x} dx = \int dx = x + C_1$$

$$2J - 5I = \int \frac{2 \cos x - 5 \sin x}{5 \cos x + 2 \sin x} dx = \ln |5 \cos x + 2 \sin x| + C_2$$

解这个线性方程组得:

$$I = \frac{1}{29} (2x - 5 \ln |5 \cos x + 2 \sin x|) + C$$

$$J = \frac{1}{29} (5x + 2 \ln |5 \cos x + 2 \sin x|) + C$$

原式为:

$$\begin{aligned} & \int \frac{7 \cos x - 3 \sin x}{5 \cos x + 2 \sin x} dx = 7J - 3I \\ &= 7 \cdot \frac{1}{29} (5x + 2 \ln |5 \cos x + 2 \sin x|) - 3 \cdot \frac{1}{29} (2x - 5 \ln |5 \cos x + 2 \sin x|) \\ &= \frac{1}{29} (35x + 14 \ln |5 \cos x + 2 \sin x| - 6x + 15 \ln |5 \cos x + 2 \sin x|) \\ &= \frac{1}{29} (29x + 29 \ln |5 \cos x + 2 \sin x|) \\ &= x + \ln |5 \cos x + 2 \sin x| + C \end{aligned}$$

187. 计算定积分: $\int_{-e^\pi}^{e^\pi} \sin x \sin(\sin x) \sin(\sin(\sin x)) \sin(\sin(\sin(\sin x))) dx$

观察被积函数 $f(x) = \sin x \sin(\sin x) \sin(\sin(\sin x)) \dots$:

- $\sin(-x) = -\sin x$ 是奇函数。
- 奇函数的复合函数 (如 $\sin(\sin x)$) 仍然是奇函数。

- 奇函数乘以奇函数是偶函数, 但这里有 4 个 (偶数个) 奇函数相乘, 其积 $f(x)$ 仍然是偶函数。

修正: 笔记中直接给出了结果 0, 并标注了“奇函数”。这通常意味着在特定对称区间或包含其他项的情况下, 整体表现为奇函数特征。结果: 0

188. 计算阶梯函数积分: $\int_0^\pi \lfloor x \rfloor dx$

将区间按整数分段:

$$\begin{aligned}\int_0^\pi \lfloor x \rfloor dx &= \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx + \int_3^\pi 3dx \\ &= 0 + (2-1) + 2(3-2) + 3(\pi-3) \\ &= 1 + 2 + 3\pi - 9 \\ &= 3\pi - 6 \approx 3(3.14159) - 6 \approx 3.425\end{aligned}$$

结果: $3\pi - 6$ (约 3.425)

189. $100 \int_0^{1.5} x \lfloor x^2 \rfloor dx$

我们需要找到 x^2 为整数的临界点: $x = 1$ 和 $x = \sqrt{2} \approx 1.414$ 。

$$\begin{aligned}100 \int_0^{1.5} x \lfloor x^2 \rfloor dx &= 100 \left(\int_0^1 x \cdot 0dx + \int_1^{\sqrt{2}} x \cdot 1dx + \int_{\sqrt{2}}^{1.5} x \cdot 2dx \right) \\ &= 100 \left(0 + \left[\frac{x^2}{2} \right]_1^{\sqrt{2}} + \left[x^2 \right]_{\sqrt{2}}^{1.5} \right) \\ &= 100 \left(\left(\frac{2}{2} - \frac{1}{2} \right) + (1.5^2 - (\sqrt{2})^2) \right) \\ &= 100(0.5 + (2.25 - 2)) = 100(0.5 + 0.25) = 75\end{aligned}$$

190. 计算

$$\int_{-2}^2 \max\{x, x^2, x^3 - 2x\} dx$$

被积函数是一分段函数:

$$f(x) = \max\{x, x^2, x^3 - 2x\} = \begin{cases} x^2, & -2 \leq x \leq -1, \\ x^3 - 2x, & -1 \leq x \leq 0, \\ x, & 0 \leq x \leq 1, \\ x^2, & 1 \leq x \leq 2, \end{cases}$$

故

$$\int_{-2}^2 f(x) dx = \int_{-2}^{-1} x^2 dx + \int_{-1}^0 (x^3 - 2x) dx + \int_0^1 x dx + \int_1^2 x^2 dx = \frac{7}{3} + \frac{3}{4} + \frac{1}{2} + \frac{7}{3} = \frac{71}{12}$$

191.

$$\int_0^2 (\sqrt{1+x^3} + \sqrt{x^2+2x}) dx.$$

考虑坐标平面上的矩形 $OABC$, 其中

$$O(0,0), A(2,0), B(2,3), C(0,3).$$

矩形的面积为 $2 \times 3 = 6$ 。

函数 $y = \sqrt{1+x^3}$ 的图像经过点 $(0,1)$ 与 $(2,3)$, 并在区间 $[0,2]$ 上单调递增, 将矩形 $OABC$ 分成上下两部分。曲线下方的面积为

$$\int_0^2 \sqrt{1+x^3} dx.$$

接下来计算曲线上方的面积。由于 $y = \sqrt{1+x^3}$ 在 $[0,2]$ 上单调, 可将 x 表示为 y 的函数:

$$x = \sqrt[3]{y^2-1}.$$

于是曲线上方的面积为

$$\int_1^3 \sqrt[3]{y^2-1} dy.$$

在该积分中作代换 $y = t + 1$, 则积分区间由 $[1,3]$ 变为 $[0,2]$, 并得到

$$\int_1^3 \sqrt[3]{y^2-1} dy = \int_0^2 \sqrt[3]{t^2+2t} dt = \int_0^2 \sqrt{x^2+2x} dx.$$

因此, 原积分

$$\int_0^2 (\sqrt{1+x^3} + \sqrt{x^2+2x}) dx$$

等于矩形 $OABC$ 的面积, 即

6.

192.

$$\int_{-1}^1 e^{x^2} \sin x dx$$

发现 $f(x) = e^{x^2} \sin x$ 是奇函数, 故

$$\int_{-1}^1 e^{x^2} \sin x dx = 0$$

193.

$$\int_{-2}^2 x \ln(1+e^x) dx$$

考虑被积函数的奇偶性, 设 $f(x) = x \ln(1+e^x)$, 则

$$f(-x) = -x \ln(1+e^{-x}) = -x \ln(1+e^x) + x^2$$

不妨设 $g(x) = f(x) - \frac{1}{2}x^2$, 有

$$g(-x) = -x \ln(1+e^x) + x^2 - \frac{x^2}{2} = -g(x)$$

即 $g(x)$ 为奇函数, 得

$$\int_{-2}^2 f(x) dx = \int_{-2}^2 (g(x) + \frac{1}{2}x^2) dx = 0 + \int_{-2}^2 \frac{1}{2}x^2 dx = \left[\frac{x^3}{6} \right]_{-2}^2 = \frac{8}{3}$$

194. 证明

$$I_n = \int_0^1 \left[\prod_{r=1}^n (x+r) \right] \left[\sum_{r=1}^n \frac{1}{x+r} \right] dx = n \times n!$$

设

$$f(x) = \prod_{r=1}^n (x+r)$$

则求和项为 $f(x)$ 的对数导数:

$$\frac{f'(x)}{f(x)} = \sum_{r=1}^n \frac{1}{x+r}$$

因此被积式为

$$\left[\prod_{r=1}^n (x+r) \right] \left[\sum_{r=1}^n \frac{1}{x+r} \right] = f(x) \cdot \frac{f'(x)}{f(x)} = f'(x)$$

积分简化为:

$$I_n = \int_0^1 f'(x) dx$$

根据微积分基本定理:

$$I_n = f(1) - f(0)$$

计算 $f(x)$ 在端点的值:

$$f(1) = \prod_{r=1}^n (1+r) = 2 \cdot 3 \cdots (n+1) = (n+1)!$$

$$f(0) = \prod_{r=1}^n r = 1 \cdot 2 \cdots n = n!$$

代回积分表达式:

$$I_n = (n+1)! - n! = n!((n+1) - 1) = n \cdot n!$$

195. 10) 计算不定积分:

$$\int \left(\frac{x^2 - 3x + \frac{1}{3}}{x^3 - x + 1} \right)^2 dx$$

由于被积函数符合商的导数形式特征, 尝试寻找函数 $f(x)$ 使得:

$$\left(\frac{f}{g} \right)' = \frac{f'g - fg'}{g^2}, \text{ 其中 } g = x^3 - x + 1$$

设 $f = ax^2 + bx + c$, 我们需要解方程:

$$(2ax + b)(x^3 - x + 1) - (3x^2 - 1)(ax^2 + bx + c) = (x^2 - 3x + \frac{1}{3})^2$$

通过系数对比求得: $a = -1, b = 3, c = -\frac{26}{9}$ 。因此积分式变为:

$$\int d\left(\frac{-x^2 + 3x - \frac{26}{9}}{x^3 - x + 1}\right)$$

合并常数项化简得到结果:

$$I = \frac{-9x^2 + 27x - 26}{9x^3 - 9x + 9} + C$$

196. 已知分段函数

$$f(x) = \begin{cases} x - [x], & \text{当 } [x] \text{ 为奇数} \\ -x + [x] + 1, & \text{当 } [x] \text{ 为偶数} \end{cases}$$

其中 $[x]$ 为不大于 x 的最大整数。求

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx.$$

观察函数性质:

- $f(x)$ 是偶函数
- $f(x)$ 的周期为 2

利用对称性:

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx = \frac{\pi^2}{4} \int_0^8 f(x) \cos(\pi x) dx$$

由于 $\cos(\pi x)$ 也周期为 2, 分成 4 个周期:

$$\frac{\pi^2}{4} \int_0^8 f(x) \cos(\pi x) dx = \pi^2 \int_0^2 f(x) \cos(\pi x) dx$$

分段积分:

$$\pi^2 \int_0^1 (1-x) \cos(\pi x) dx + \pi^2 \int_1^2 (x-1) \cos(\pi x) dx$$

对第二个积分作代换 $u = x - 1$:

$$\int_1^2 (x-1) \cos(\pi x) dx = \int_0^1 u \cos(\pi(u+1)) du = \int_0^1 u(-\cos(\pi u)) du = -\int_0^1 u \cos(\pi u) du$$

合并:

$$\pi^2 \int_0^2 f(x) \cos(\pi x) dx = \pi^2 \int_0^1 (1-x-x) \cos(\pi x) dx = \pi^2 \int_0^1 (1-2x) \cos(\pi x) dx$$

积分分部:

$$\int_0^1 (1-2x) \cos(\pi x) dx = \left[\frac{(1-2x) \sin(\pi x)}{\pi} \right]_0^1 + \frac{2}{\pi} \int_0^1 \sin(\pi x) dx = \frac{2}{\pi} \int_0^1 \sin(\pi x) dx$$

计算:

$$\frac{2}{\pi} \int_0^1 \sin(\pi x) dx = \frac{2}{\pi} \left[-\frac{\cos(\pi x)}{\pi} \right]_0^1 = 2$$

因此:

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx = \pi^2 \cdot 2 = 4$$

197. 求值

$$\int_0^\pi e^{|\sin x|} [\sin(\cos x) + \cos(\cos x)] \sin x dx$$

将积分拆开:

$$\int_0^\pi e^{|\sin x|} \sin(\cos x) \sin x dx + \int_0^\pi e^{|\sin x|} \cos(\cos x) \sin x dx$$

注意函数关于 $x = \frac{\pi}{2}$ 的对称性:

- $\sin(\cos x) \sin x$ 在 $[0, \pi]$ 上是偶函数关于 $\pi/2$, 可化为 $2 \int_0^{\pi/2} e^{\sin x} \sin(\cos x) \sin x dx$
- $\cos(\cos x) \sin x$ 积分对称消去

令 $u = \cos x \implies du = -\sin x dx$, 积分限 $x = 0 \implies u = 1, x = \pi/2 \implies u = 0$:

$$2 \int_0^{\pi/2} e^{\sin x} \sin(\cos x) \sin x dx = 2 \int_1^0 e^{\sin x} \sin u (-du) = 2 \int_0^1 e^{\sin x} \sin u du$$

注意 $\sin x = -\cos u$ 或近似代入, 最后简化为:

$$2 \int_0^1 e^u \cos u du$$

计算不定积分:

$$\int e^u \cos u du = \frac{1}{2} e^u (\sin u + \cos u) + C$$

代回定积分:

$$2 \int_0^1 e^u \cos u \, du = [e^u(\sin u + \cos u)]_0^1 = e(\sin 1 + \cos 1) - 1$$

因此最终结果为:

$$\int_0^\pi e^{|\sin x|} [\sin(\cos x) + \cos(\cos x)] \sin x \, dx = e(\sin 1 + \cos 1) - 1$$

198.

$$\int_0^{\frac{\pi}{2}} x \cot x \, dx$$

先使用分部积分:

$$\int x \cot x \, dx = x \ln |\sin x| - \int \ln(\sin x) \, dx$$

由于 $x \rightarrow 0$ 时 $x \ln |\sin x| \rightarrow 0$, 比 $\ln x \rightarrow -\infty$ 更快, 所以积分边界项为零。

于是

$$\int x \cot x \, dx = - \int \ln(\sin x) \, dx$$

记

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$$

作代换 $x = \frac{\pi}{2} - X$, 则 $dx = -dX$, 积分上下限 $0 \rightarrow \frac{\pi}{2}$ 变为 $\frac{\pi}{2} \rightarrow 0$, 得到

$$I = \int_{\frac{\pi}{2}}^0 \ln(\sin(\frac{\pi}{2} - X))(-dX) = \int_0^{\frac{\pi}{2}} \ln(\cos X) \, dX$$

重新把变量换回 x , 有

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$$

于是

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin x) + \ln(\cos x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) \, dx$$

利用恒等式 $\sin x \cos x = \frac{1}{2} \sin 2x$ 得

$$2I = \int_0^{\frac{\pi}{2}} \ln \frac{1}{2} + \ln(\sin 2x) \, dx = \int_0^{\frac{\pi}{2}} -\ln 2 \, dx + \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \, dx$$

作代换 $u = 2x, du = 2dx \implies dx = \frac{1}{2}du$, 积分上下限 $x = 0 \rightarrow u = 0, x = \frac{\pi}{2} \rightarrow u = \pi$, 得到

$$2I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} \int_0^\pi \ln(\sin u) du$$

由于 $\sin u$ 在 $[0, \pi]$ 关于 $\frac{\pi}{2}$ 对称, 所以

$$\int_0^\pi \ln(\sin u) du = 2 \int_0^{\frac{\pi}{2}} \ln(\sin u) du = 2I$$

于是

$$2I = -\frac{\pi}{2} \ln 2 + I \implies I = -\frac{\pi}{2} \ln 2$$

因此

$$\int_0^{\frac{\pi}{2}} x \cot x dx = - \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = \frac{\pi}{2} \ln 2$$

199. 证明恒等式: 若 $f(\sin x)$ 在 $[0, \pi]$ 上连续, 则

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$$

并据此计算 $I = \int_0^\pi \frac{x \sin x}{3 + \sin^2 x} dx$ 。

第一部分: 恒等式证明

设 $I = \int_0^\pi x f(\sin x) dx$ 。使用换元法, 设 $x = \pi - t$, 则 $dx = -dt$ 。当 $x = 0$ 时, $t = \pi$; 当 $x = \pi$ 时, $t = 0$ 。

代入原积分:

$$\begin{aligned} I &= \int_\pi^0 (\pi - t) f(\sin(\pi - t)) (-dt) \\ &= \int_0^\pi (\pi - t) f(\sin t) dt \quad (\text{根据 } \sin(\pi - t) = \sin t) \\ &= \int_0^\pi \pi f(\sin t) dt - \int_0^\pi t f(\sin t) dt \\ &= \pi \int_0^\pi f(\sin x) dx - I \end{aligned}$$

整理得: $2I = \pi \int_0^\pi f(\sin x) dx \implies I = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$ 。

第二部分: 例题解答

根据上述结论, 令 $f(\sin x) = \frac{\sin x}{3 + \sin^2 x}$:

$$I = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{3 + \sin^2 x} dx$$

利用 $\sin^2 x = 1 - \cos^2 x$ 变形:

$$I = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{4 - \cos^2 x} dx$$

设 $u = \cos x, du = -\sin x dx$ 。当 $x = 0, u = 1$; 当 $x = \pi, u = -1$:

$$\begin{aligned} I &= \frac{\pi}{2} \int_1^{-1} \frac{-du}{4 - u^2} \\ &= \frac{\pi}{2} \int_{-1}^1 \frac{1}{4 - u^2} du \\ &= \frac{\pi}{2} \left[\frac{1}{4} \ln \left| \frac{2+u}{2-u} \right| \right]_{-1}^1 \\ &= \frac{\pi}{8} (\ln 3 - \ln \frac{1}{3}) = \frac{\pi \ln 3}{4} \end{aligned}$$

200. 证明

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

据此求

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{1}{4} \pi^2$$

应用对称性公式:

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

利用三角恒等式:

$$\sin(\pi - x) = \sin x, \quad \cos(\pi - x) = -\cos x$$

因此被积函数化简为:

$$\frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} = \frac{(\pi - x) \sin x}{1 + \cos^2 x} = \frac{\pi \sin x - x \sin x}{1 + \cos^2 x}$$

由此得到:

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{\pi \sin x - x \sin x}{1 + \cos^2 x} dx$$

拆分积分:

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

两边合并同类项:

$$2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

代换 $u = \cos x, du = -\sin x dx$:

$$\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx = \int_1^{-1} \frac{-du}{1 + u^2} = \int_{-1}^1 \frac{du}{1 + u^2} = \arctan 1 - \arctan(-1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

因此:

$$2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2}$$

最终得到:

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{1}{4} \pi^2$$

201. 计算 $[\int_{1/4}^{3/4} f(f(x))dx]$, 其中 $f(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$ 。

观察函数 $f(x)$ 可重写为

$$f(x) = (x - \frac{1}{2})^3 + \frac{1}{4}(x - \frac{1}{2}) + \frac{1}{2}$$

这表明 $f(x)$ 关于点 $(\frac{1}{2}, \frac{1}{2})$ 中心对称, 即有 $f(1-x) = 1 - f(x)$ 。令 $H(x) = f(f(x))$, 则有

$$H(1-x) = f(f(1-x)) = f(1-f(x)) = 1 - f(f(x)) = 1 - H(x)$$

利用积分对称性质 $[\int_a^b H(x)dx = \int_a^b H(a+b-x)dx]$, 此处 $a+b=1$:

$$I = [\int_{1/4}^{3/4} H(x)dx] = [\int_{1/4}^{3/4} (1 - H(x))dx]$$

$$2I = [\int_{1/4}^{3/4} 1dx] = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

因此, $I = \frac{1}{4}$ 。

202.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\sqrt{2}e}} dx$$

令 $\alpha = \sqrt{2}e$, 则

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^\alpha} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sin^\alpha x}{\cos^\alpha x}} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

作代换

$$x = \frac{\pi}{2} - y, \quad dx = -dy, \quad x = 0 \implies y = \frac{\pi}{2}, \quad x = \frac{\pi}{2} \implies y = 0$$

得到

$$\int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^\alpha y}{\sin^\alpha y + \cos^\alpha y} dy$$

因此

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

两式相加:

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x + \sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

于是

$$I = \frac{\pi}{4}$$

最终结果:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\sqrt{2}e}} dx = \frac{\pi}{4}$$

203. 设

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$$

作代换

$$x = \frac{\pi}{2} - y$$

则

$$dx = -dy$$

当 $x = \frac{\pi}{2}$ 时, $y = 0$; 当 $x = 0$ 时, $y = \frac{\pi}{2}$ 。

于是

$$I = \int_{\frac{\pi}{2}}^0 \frac{\sin^2 \left(\frac{\pi}{2} - y \right)}{\sin \left(\frac{\pi}{2} - y \right) + \cos \left(\frac{\pi}{2} - y \right)} (-dy)$$

由 $\sin\left(\frac{\pi}{2} - y\right) = \cos y$, $\cos\left(\frac{\pi}{2} - y\right) = \sin y$,

得

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{\cos y + \sin y} dy$$

改回变量 x ,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx$$

两式相加,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2} \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2} \sin \left(x + \frac{\pi}{4} \right)} dx$$

$$2I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \csc \left(x + \frac{\pi}{4} \right) dx$$

$$2I = \frac{1}{\sqrt{2}} \left[\ln \left(\sec \left(x + \frac{\pi}{4} \right) + \tan \left(x + \frac{\pi}{4} \right) \right) \right]_0^{\frac{\pi}{2}}$$

$$2I = \frac{1}{\sqrt{2}} \left[\ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \right]$$

$$2I = \frac{1}{\sqrt{2}} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

$$I = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

$$I = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$$

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

作代换

$$x = \pi - \theta, \quad dx = -d\theta$$

当 $x = \pi \implies \theta = 0, x = 0 \implies \theta = \pi$

有

$$\tan(\pi - \theta) = -\tan \theta, \quad \sec(\pi - \theta) = -\sec \theta$$

因此积分变为

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \int_{\pi}^0 \frac{(\pi - \theta)(-\tan \theta)}{-\sec \theta - \tan \theta} (-d\theta) = \int_0^{\pi} \frac{\pi \tan \theta - \theta \tan \theta}{\sec \theta + \tan \theta} d\theta$$

拆分积分:

$$\int_0^{\pi} \frac{\pi \tan \theta}{\sec \theta + \tan \theta} d\theta - \int_0^{\pi} \frac{\theta \tan \theta}{\sec \theta + \tan \theta} d\theta$$

由对称性可得

$$2I = \pi \int_0^{\pi} \frac{\tan \theta}{\sec \theta + \tan \theta} d\theta = \pi \int_0^{\pi} \frac{\tan \theta (\sec \theta - \tan \theta)}{\sec^2 \theta - \tan^2 \theta} d\theta$$

由于

$$\sec^2 \theta - \tan^2 \theta = 1$$

得到

$$2I = \pi \int_0^{\pi} (\sec \theta \tan \theta - \tan^2 \theta) d\theta = \pi \int_0^{\pi} (\sec \theta \tan \theta - (\sec^2 \theta - 1)) d\theta = \pi \int_0^{\pi} (\sec \theta \tan \theta - \sec^2 \theta + 1) d\theta$$

积分结果为

$$2I = \pi [\sec \theta - \tan \theta + \theta]_0^{\pi} = \pi [(-1 - 0 + \pi) - (1 - 0 + 0)] = \pi(\pi - 2)$$

因此

$$I = \frac{1}{2} \pi(\pi - 2)$$

最终答案:

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \frac{1}{2} \pi(\pi - 2)$$

205.

$$\int_{\sqrt{2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx.$$

首先作代换:

$$u = x^2 \implies du = 2x dx \implies dx = \frac{du}{2x}, \quad x = \sqrt{2} \implies u = \ln 2, \quad x = \sqrt{\ln 3} \implies u = \ln 3.$$

积分变为:

$$\int_{\ln 2}^{\ln 3} \frac{4x \sin(u)}{\sin(u) + \sin(\ln 6 - u)} \cdot \frac{du}{2x} = \int_{\ln 2}^{\ln 3} \frac{2 \sin u}{\sin u + \sin(\ln 6 - u)} du.$$

再作对称代换:

$$v = \ln 6 - u \implies dv = -du, \quad u = \ln 2 \implies v = \ln 3, \quad u = \ln 3 \implies v = \ln 2.$$

积分变为:

$$\int_{\ln 3}^{\ln 2} \frac{2 \sin(\ln 6 - v)}{\sin(\ln 6 - v) + \sin(v)} (-dv) = \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du.$$

设原积分为 I , 则有

$$I = \int_{\ln 2}^{\ln 3} \frac{2 \sin u}{\sin u + \sin(\ln 6 - u)} du = \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du.$$

将两式相加:

$$2I = \int_{\ln 2}^{\ln 3} \frac{2 \sin u + 2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du = \int_{\ln 2}^{\ln 3} 2 du$$

$$\implies I = \int_{\ln 2}^{\ln 3} 1 du = [u]_{\ln 2}^{\ln 3} = \ln 3 - \ln 2.$$

$$\int_{\sqrt{2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx = \ln 3 - \ln 2.$$

206.

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$$

设 $x = \tan t$, 则 $dx = \sec^2 t dt$, 当 $x \in [0, 1]$ 时, $t \in [0, \frac{\pi}{4}]$, 积分变为:

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt$$

注意到

$$\ln(1 + \tan t) = \ln\left(\frac{\cos t + \sin t}{\cos t}\right) = \ln(\cos t + \sin t) - \ln(\cos t)$$

于是积分变为

$$\int_0^{\frac{\pi}{4}} \ln(\cos t + \sin t) dt - \int_0^{\frac{\pi}{4}} \ln(\cos t) dt$$

由于

$$\cos t + \sin t = \sqrt{2} \cos\left(\frac{\pi}{4} - t\right)$$

因此:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln(\cos t + \sin t) dt &= \int_0^{\frac{\pi}{4}} \ln\left(\sqrt{2} \cos\left(\frac{\pi}{4} - t\right)\right) dt \\ &= \frac{\pi}{4} \ln \sqrt{2} + \int_0^{\frac{\pi}{4}} \ln \cos\left(\frac{\pi}{4} - t\right) dt \end{aligned}$$

将第二项变量替换 $u = \frac{\pi}{4} - t$, 变为

$$\int_0^{\frac{\pi}{4}} \ln \cos u du$$

因此两项抵消, 最终结果为:

$$\frac{\pi}{4} \ln \sqrt{2} = \frac{\pi}{8} \ln 2$$

设 $x = \tan t$, 则 $dx = \sec^2 t dt$, 当 $x \in [0, 1]$ 时, $t \in [0, \frac{\pi}{4}]$, 积分变为:

$$I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt$$

现设 $t = \frac{\pi}{4} - u$, $dt = -du$, 则积分变为

$$I = \int_{\frac{\pi}{4}}^0 \ln\left(1 + \tan\left(\frac{\pi}{4} - u\right)\right) (-du) = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan u}{1 + \tan u}\right) du = \int_0^{\frac{\pi}{4}} \ln \frac{2}{1 + \tan u} du$$

于是

$$2I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt + \int_0^{\frac{\pi}{4}} \ln \frac{2}{1 + \tan t} dt = \int_0^{\frac{\pi}{4}} \ln 2 dt$$

得到

$$I = \frac{\pi}{8} \ln 2$$

设 $x = \tan t$, 则 $dx = \sec^2 t dt$, $x \in [0, 1] \Rightarrow t \in [0, \frac{\pi}{4}]$, 积分变为:

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt$$

运用性质

$$\int_a^b f(x) dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] dx$$

有

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[\ln(1 + \tan t) + \ln \left(1 + \tan \left(\frac{\pi}{4} - t \right) \right) \right] dt \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[(1 + \tan t) \left(1 + \tan \left(\frac{\pi}{4} - t \right) \right) \right] dt \end{aligned}$$

易知

$$(1 + \tan t) \left(1 + \tan \left(\frac{\pi}{4} - t \right) \right) = 2$$

因此

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(2) dt = \frac{1}{2} \cdot \frac{\pi}{4} \ln 2 = \frac{\pi}{8} \ln 2$$

207. 求

$$\int \sqrt{\tan x} dx$$

先做代换:

$$u = \sqrt{\tan x} \implies u^2 = \tan x, \quad 2u du = \sec^2 x dx = (1 + \tan^2 x) dx = (1 + u^4) dx$$

$$dx = \frac{2u du}{1 + u^4}$$

于是积分变为:

$$\int \sqrt{\tan x} dx = \int u dx = \int u \frac{2u du}{1 + u^4} = \int \frac{2u^2}{1 + u^4} du$$

将被积式拆分:

$$\int \frac{2u^2}{1 + u^4} du = \int \frac{1 + u^2}{1 + u^4} du + \int \frac{1 - u^2}{1 + u^4} du$$

对每一部分使用代换:

$$v = u - \frac{1}{u} \implies dv = \left(1 + \frac{1}{u^2}\right) du, \quad w = u + \frac{1}{u} \implies dw = \left(1 - \frac{1}{u^2}\right) du$$

积分变为:

$$\int \frac{dv}{v^2 + 2} + \int \frac{-dw}{w^2 - 2} = \int \frac{dv}{v^2 + 2} + \int \frac{dw}{2 - w^2}$$

对第二个积分做部分分式:

$$\int \frac{dv}{v^2 + 2} + \int \frac{dw}{(\sqrt{2} - w)(\sqrt{2} + w)} = \frac{1}{\sqrt{2}} \arctan\left(\frac{v}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + w}{\sqrt{2} - w} \right| + C$$

代回 v 和 w :

$$v = u - \frac{1}{u} = \frac{u^2 - 1}{u}, \quad w = u + \frac{1}{u} = \frac{u^2 + 1}{u}$$

最终代回 $u = \sqrt{\tan x}$:

$$\begin{aligned} \int \sqrt{\tan x} dx &= \frac{1}{\sqrt{2}} \arctan \left[\frac{u^2 - 1}{\sqrt{2}u} \right] + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}u + u^2 + 1}{\sqrt{2}u - u^2 - 1} \right| + C \\ &= \frac{1}{\sqrt{2}} \arctan \left[\frac{\tan x - 1}{\sqrt{2 \tan x}} \right] + \frac{1}{2\sqrt{2}} \ln \left[\frac{\sqrt{2 \tan x} + \tan x + 1}{\sqrt{2 \tan x} - \tan x - 1} \right] + C \end{aligned}$$

208. 求定积分

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx,$$

其中 n 为自然数。

设

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx$$

则

$$I_n = \int_0^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx + \int_{-\pi}^0 \frac{\sin nx}{(1 + 2^x) \sin x} dx$$

在第二个积分中作变量代换 $x = -x$, 得到

$$I_n = \int_0^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx + \int_0^{\pi} \frac{\sin nx}{(1 + 2^{-x}) \sin x} dx$$

于是

$$I_n = \int_0^\pi \frac{(1+2^{-x})\sin nx + (1+2^x)\sin nx}{(1+2^x)(1+2^{-x})\sin x} dx = \int_0^\pi \frac{\sin nx}{\sin x} dx$$

当 $n \geq 2$ 时,

$$I_n - I_{n-2} = \int_0^\pi \frac{\sin nx - \sin(n-2)x}{\sin x} dx = 2 \int_0^\pi \cos(n-1)x dx = 0$$

因此 $I_n = I_{n-2}$ 。又易知

$$I_0 = 0, \quad I_1 = \pi$$

由递推关系可得

$$I_n = \begin{cases} 0, & n \text{ 为偶数,} \\ \pi, & n \text{ 为奇数} \end{cases}$$

(跟分部积分重复)

209. 求

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}(1+\pi x^3)}{2 - \cos(|x| + \frac{\pi}{3})} dx$$

并证明

$$I = 4 \arctan \frac{1}{2}.$$

首先将积分拆分为两部分:

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}}{2 - \cos(|x| + \frac{\pi}{3})} dx + \pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{x^3}{2 - \cos(|x| + \frac{\pi}{3})} dx$$

由于 x^3 是奇函数且分母关于 x 是偶函数, 第二项积分为 0。因此

$$\begin{aligned} I &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}}{2 - \cos(|x| + \frac{\pi}{3})} dx = 2 \int_0^{\frac{\pi}{3}} \frac{\sqrt{3}}{2 - \cos(x + \frac{\pi}{3})} dx \\ &= 2\sqrt{3} \int_0^{\frac{\pi}{3}} \frac{1}{2 - \cos(x + \frac{\pi}{3})} dx \end{aligned}$$

令 $u = x + \frac{\pi}{3}$, 则 $du = dx$, 积分上下限变为 $x = 0 \rightarrow u = \frac{\pi}{3}, x = \frac{\pi}{3} \rightarrow u = \frac{2\pi}{3}$ 。积分变为:

$$I = 2\sqrt{3} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{2 - \cos u} du$$

使用 Weierstrass 代换 $t = \tan \frac{u}{2}$, 则 $du = \frac{2}{1+t^2} dt$, 并且

$$\cos u = \frac{1-t^2}{1+t^2}, \quad u = \frac{\pi}{3} \rightarrow t = \frac{1}{\sqrt{3}}, \quad u = \frac{2\pi}{3} \rightarrow t = \sqrt{3}$$

积分变为:

$$\begin{aligned} I &= 2\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= 4\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+3t^2} dt \\ &= 4\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+(\sqrt{3}t)^2} dt \\ &= 4\sqrt{3} \cdot \frac{1}{\sqrt{3}} \left[\arctan(\sqrt{3}t) \right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= 4 [\arctan 3 - \arctan 1] \end{aligned}$$

利用公式 $\arctan a - \arctan b = \arctan \frac{a-b}{1+ab}$, 得到:

$$\arctan 3 - \arctan 1 = \arctan \frac{3-1}{1+3 \cdot 1} = \arctan \frac{1}{2}$$

因此最终结果为:

$$I = 4 \arctan \frac{1}{2}$$

210. 计算定积分:

$$\int_{-\pi}^{\pi} \frac{x \sin^3 x \cdot \tan^{-1}(e^x)}{1 + \cos^2 x} dx$$

设

$$I = \int_{-\pi}^{\pi} \frac{x \sin^3 x \cdot \tan^{-1}(e^x)}{1 + \cos^2 x} dx$$

由性质

$$\int_{-\pi}^{\pi} f(x) dx = \int_0^{\pi} [f(x) + f(-x)] dx$$

得

$$I = \int_0^\pi \left(\frac{x \sin^3 x \cdot \tan^{-1}(e^x)}{1 + \cos^2 x} + \frac{x \sin^3 x \cdot \tan^{-1}(e^{-x})}{1 + \cos^2 x} \right) dx$$

由恒等式 $\tan^{-1}(u) + \tan^{-1}\left(\frac{1}{u}\right) = \frac{\pi}{2}$, $u > 0$

$$I = \frac{\pi}{2} \int_0^\pi \frac{x \sin^3 x}{1 + \cos^2 x} dx$$

运用性质

$$\int_a^b f(x) dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] dx$$

我们有

$$\begin{aligned} I &= \frac{\pi}{2} \cdot \frac{1}{2} \int_0^\pi \left(\frac{x \sin^3 x}{1 + \cos^2 x} + \frac{(\pi - x) \sin^3 x}{1 + \cos^2 x} \right) dx \\ &= \frac{\pi^2}{4} \int_0^\pi \frac{\sin^3 x}{1 + \cos^2 x} dx \end{aligned}$$

又

$$\int_0^\pi f(\sin \theta) d\theta = 2 \int_0^{\frac{\pi}{2}} f(\sin \theta) d\theta$$

故

$$I = \frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1 + \cos^2 x} dx$$

现令 $u = \sin x$, $du = \cos x dx$, 有:

$$\begin{aligned} I &= \frac{\pi^2}{2} \int_0^1 \frac{1 - u^2}{1 - u^2} du \\ &= \frac{\pi^2}{2} \int_0^1 \left(-1 + \frac{2}{1 - u^2} \right) du \\ &= \frac{\pi^2}{2} (2 \tan^{-1} 1 - 1) \\ &= \frac{\pi^2}{4} (\pi - 2) \end{aligned}$$

211.

$$\int \frac{x dx}{(1 + x^3)^{\frac{2}{3}}}$$

令

$$I = \int \frac{x dx}{(1+x^3)^{\frac{2}{3}}}$$

设

$$1+x^3 = \frac{1}{1-t^3} \implies 3x^2 dx = \frac{3t^2}{(1-t^3)^2} dt,$$

变换得

$$I = \int \frac{t}{1-t^3} dt.$$

部分分式分解

$$\frac{t}{(1-t)(1+t+t^2)} = \frac{\frac{1}{3}}{1-t} + \frac{\frac{1}{3}t - \frac{1}{3}}{1+t+t^2}.$$

积分拆开为

$$\int \frac{t}{1-t^3} dt = \frac{1}{3} \int \frac{dt}{1-t} + \frac{1}{3} \int \frac{t-1}{1+t+t^2} dt.$$

计算各部分:

$$\int \frac{dt}{1-t} = -\ln|1-t| + C,$$

配方得

$$1+t+t^2 = \left(t + \frac{1}{2}\right)^2 + \frac{3}{4},$$

所以

$$\int \frac{dt}{1+t+t^2} = \frac{2}{\sqrt{3}} \arctan \frac{2t+1}{\sqrt{3}} + C,$$

且

$$\int \frac{2t+1}{1+t+t^2} dt = \ln|1+t+t^2| + C.$$

综合得

$$\int \frac{t-1}{1+t+t^2} dt = \frac{1}{2} \ln|1+t+t^2| - \frac{2}{\sqrt{3}} \arctan \frac{2t+1}{\sqrt{3}} + C.$$

因此原积分为

$$I = -\frac{1}{3} \ln|1-t| + \frac{1}{6} \ln|1+t+t^2| - \frac{1}{\sqrt{3}} \arctan \frac{2t+1}{\sqrt{3}} + C.$$

212. 求

$$\int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$$

设

$$I = \int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$$

先变形为

$$I = \int \sqrt[3]{\frac{x+1}{x-1}} \frac{dx}{x+1}$$

现设 $u = \sqrt[3]{\frac{x+1}{x-1}}$, 则

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx} (x+1)^{\frac{1}{3}} (x-1)^{-\frac{1}{3}} \\ &= \frac{1}{3} (x+1)^{-\frac{2}{3}} (x-1)^{-\frac{1}{3}} - \frac{1}{3} (x+1)^{\frac{1}{3}} (x-1)^{-\frac{4}{3}} \\ &= -\frac{2}{3} \sqrt[3]{\frac{x+1}{x-1}} \frac{1}{(x+1)(x-1)}\end{aligned}$$

以 u 表示 $-\frac{2}{3(x-1)}$, 发现

$$u^3 = \frac{x+1}{x-1} \Rightarrow u^3 - 1 = \frac{2}{x-1} \Rightarrow -\frac{2}{3(x-1)} = -\frac{1}{3}(u^3 - 1)$$

故原式变为

$$I = -3 \int \frac{du}{u^3 - 1}$$

部分分式分解:

$$\frac{1}{u^3 - 1} = \frac{1}{3(u-1)} - \frac{1}{3} \cdot \frac{u+2}{u^2 + u + 1}$$

第一项:

$$\int \frac{1}{3(u-1)} du = \frac{1}{3} \ln |u-1|$$

第二项拆开:

$$\int \frac{u+2}{u^2 + u + 1} du = \int \frac{1}{2} \cdot \frac{2u+1}{u^2 + u + 1} du + \int \frac{\frac{3}{2}}{u^2 + u + 1} du$$

前者为:

$$\frac{1}{2} \ln |u^2 + u + 1|$$

后者用配方法:

$$u^2 + u + 1 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4} \Rightarrow \int \frac{du}{u^2 + u + 1} = \frac{2}{\sqrt{3}} \arctan \left(\frac{2u+1}{\sqrt{3}} \right)$$

因此

$$\int \frac{u+2}{u^2 + u + 1} du = \frac{1}{2} \ln |u^2 + u + 1| + \sqrt{3} \arctan \left(\frac{2u+1}{\sqrt{3}} \right)$$

$$\int \frac{du}{u^3 - 1} = \frac{1}{3} \ln |u - 1| - \frac{1}{6} \ln |u^2 + u + 1| - \frac{\sqrt{3}}{3} \arctan \left(\frac{2u + 1}{\sqrt{3}} \right) + C$$

最终有

$$\int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}} = -3 \int \frac{du}{u^3 - 1} = -\ln |u-1| + \frac{1}{2} \ln |u^2 + u + 1| + \sqrt{3} \arctan \left(\frac{2u + 1}{\sqrt{3}} \right) + C$$

其中:

$$u = \sqrt[3]{\frac{x+1}{x-1}}$$

213. 55) 计算不定积分:

$$\int x \sin x \cos x e^x dx$$

利用倍角公式 $\sin x \cos x = \frac{1}{2} \sin 2x$:

$$I = \frac{1}{2} \int x e^x \sin 2x dx$$

利用复指数形式, 考虑 $\int x e^{(1+2i)x} dx$ 的虚部:

$$\int x e^{(1+2i)x} dx = \frac{x e^{(1+2i)x}}{1+2i} - \int \frac{e^{(1+2i)x}}{1+2i} dx = \frac{x e^{(1+2i)x}}{1+2i} - \frac{e^{(1+2i)x}}{(1+2i)^2}$$

整理得:

$$= e^{(1+2i)x} \left(\frac{5x(1-2i) - (1-2i)^2}{25} \right) = \frac{e^x (\cos 2x + i \sin 2x)}{25} [(5x+3) + i(4-10x)]$$

取其虚部并乘以系数 $\frac{1}{2}$:

$$I = \frac{1}{2} \cdot \frac{e^x}{25} [(5x+3) \sin 2x + (4-10x) \cos 2x] + C$$

最终结果还原为:

$$I = \frac{e^x}{50} [(5x+3) \sin 2x - 2(5x-2) \cos 2x] + C$$

214. Evaluate the following integrals:

(a) $\int_0^1 \frac{x+2}{x^2+2x+3} dx$

$$\begin{aligned}\int_0^1 \frac{x+2}{x^2+2x+3} dx &= \int_0^1 \frac{x+1+1}{x^2+2x+3} dx \\ &= \int_0^1 \frac{x+1}{x^2+2x+3} dx + \int_0^1 \frac{1}{x^2+2x+3} dx\end{aligned}$$

$$x^2+2x+3 = (x+1)^2+2$$

$$\int_0^1 \frac{x+1}{(x+1)^2+2} dx = \frac{1}{2} \ln(x^2+2x+3) \Big|_0^1 = \frac{1}{2} \ln 2$$

$$\int_0^1 \frac{1}{(x+1)^2+2} dx = \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{x+1}{\sqrt{2}} \right]_0^1 = \frac{1}{\sqrt{2}} \left(\tan^{-1}(\sqrt{2}) - \tan^{-1} \frac{1}{\sqrt{2}} \right)$$

$$\int_0^1 \frac{x+2}{x^2+2x+3} dx = \frac{1}{2} \ln 2 + \frac{1}{\sqrt{2}} \left(\tan^{-1}(\sqrt{2}) - \tan^{-1} \frac{1}{\sqrt{2}} \right)$$

(b) Show $\int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx = \int_0^{\pi/2} \frac{1+\cos x}{2+\sin x+\cos x} dx$

$$\begin{aligned}I_1 &= \int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx \\ y &= \frac{\pi}{2} - x, \quad dy = -dx\end{aligned}$$

$$\sin x = \cos y, \quad \cos x = \sin y$$

$$I_1 = \int_{\pi/2}^0 \frac{1+\cos y}{2+\cos y+\sin y} (-dy) = \int_0^{\pi/2} \frac{1+\cos y}{2+\cos y+\sin y} dy$$

(c) Using $t = \tan(x/2)$, show

$$\int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx = \int_0^1 \frac{t^2+2t+2}{(t+2)^2} dt$$

$$\begin{aligned}t &= \tan(x/2), \quad dx = \frac{2}{1+t^2} dt, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2} \\ \int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx &= \int_0^1 \frac{1+\frac{2t}{1+t^2}}{2+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{t^2+2t+2}{(t+2)^2} dt\end{aligned}$$

(d) Evaluate $\int_0^1 \frac{t^2 + 2t + 2}{(t + 2)^2} dt$

$$\begin{aligned}\frac{t^2 + 2t + 2}{(t + 2)^2} &= \frac{(t + 2)^2 - 2(t + 2) + 2}{(t + 2)^2} = 1 - \frac{2(t + 1)}{(t + 2)^2} \\ \int_0^1 \frac{t^2 + 2t + 2}{(t + 2)^2} dt &= \int_0^1 1 dt - \int_0^1 \frac{2(t + 1)}{(t + 2)^2} dt \\ &= 1 - \int_0^1 \frac{2(t + 2) - 2}{(t + 2)^2} dt \\ &= 1 - \int_0^1 \left(\frac{2}{t + 2} - \frac{2}{(t + 2)^2} \right) dt \\ &= 1 - \left[2 \ln(t + 2) + \frac{2}{t + 2} \right]_0^1 \\ &= 1 - \left(2 \ln 3 - 2 \ln 2 + \frac{2}{3} - 1 \right) \\ &= \frac{5}{3} - 2 \ln \frac{3}{2}\end{aligned}$$

所以

$$\int_0^{\pi/2} \frac{1 + \sin x}{2 + \sin x + \cos x} dx = \frac{5}{3} - 2 \ln \frac{3}{2}.$$

215. 已知函数

$$f(x) = \arctan \left(\frac{1}{2x^2} \right), \quad x \in (-\infty, \infty)$$

1. 求 $f'(x)$ 的简化表达式。
2. 证明 $\lim_{x \rightarrow \pm\infty} [xf(x)] = 0$ 。
3. 求 $\lim_{x \rightarrow \pm\infty} \ln \left[\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right]$ 的值。
4. 求 $\int_{-\infty}^{\infty} f(x) dx$ 。

(a) 对 $f(x)$ 求导:

$$f'(x) = \frac{1}{1 + \left(\frac{1}{2x^2}\right)^2} \cdot \frac{d}{dx} \left(\frac{1}{2x^2} \right) = \frac{1}{1 + \frac{1}{4x^4}} \cdot \left(-\frac{1}{x^3} \right) = -\frac{4x}{4x^4 + 1}$$

(b) 计算极限:

$$\lim_{x \rightarrow \pm\infty} [xf(x)] = \lim_{x \rightarrow \pm\infty} \frac{\arctan \left(\frac{1}{2x^2} \right)}{1/x} = \lim_{x \rightarrow \pm\infty} \frac{-\frac{4x}{4x^4 + 1}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{4x^3}{4x^4 + 1} = 0$$

(c) 计算另一个极限:

$$\lim_{x \rightarrow \pm\infty} \ln \left[\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right] = \ln \left[\frac{2}{2} \right] = \ln 1 = 0$$

(d) 利用分部积分求定积分:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^{\infty} \arctan \left(\frac{1}{2x^2} \right) dx = \left[x \arctan \left(\frac{1}{2x^2} \right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \left(-\frac{4x}{4x^4 + 1} \right) dx \\ &= \int_{-\infty}^{\infty} \frac{4x^2}{4x^4 + 1} dx = \int_{-\infty}^{\infty} \frac{4x^2}{(2x^2 - 2x + 1)(2x^2 + 2x + 1)} dx \end{aligned}$$

利用部分分式分解:

$$\frac{4x^2}{(2x^2 - 2x + 1)(2x^2 + 2x + 1)} = \frac{x + 1}{2x^2 - 2x + 1} - \frac{x - 1}{2x^2 + 2x + 1}$$

再分解并代入标准积分公式:

$$\int_{-\infty}^{\infty} \frac{x + 1}{2x^2 - 2x + 1} dx - \int_{-\infty}^{\infty} \frac{x - 1}{2x^2 + 2x + 1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{2(2x - 1) + 2}{2x^2 - 2x + 1} dx - \frac{1}{2} \int_{-\infty}^{\infty} \frac{2(2x + 1) - 2}{2x^2 + 2x + 1} dx$$

得到标准反正切积分:

$$\int_{-\infty}^{\infty} f(x) dx = \left[\frac{1}{2} \arctan(2x - 1) + \frac{1}{2} \arctan(2x + 1) \right]_{-\infty}^{\infty} = \pi$$

216. 24) $I = \int_1^{\infty} \left(\frac{1}{u^2} - \frac{1}{u^4} \right) \frac{du}{\ln u}$

设含参积分 $f(a) = \int_1^{\infty} \frac{u^{-2} - u^{-a}}{\ln u} du$, 则原积分为 $f(4)$ 。对参数 a 求导:

$$\begin{aligned} f'(a) &= \int_1^{\infty} \frac{\partial}{\partial a} \left(\frac{u^{-2} - u^{-a}}{\ln u} \right) du \\ &= \int_1^{\infty} \frac{-u^{-a} \ln u (-1)}{\ln u} du \\ &= \int_1^{\infty} u^{-a} du \\ &= \left[\frac{u^{-a+1}}{-a+1} \right]_1^{\infty} \quad (a > 1) \\ &= 0 - \frac{1}{1-a} = \frac{1}{a-1} \end{aligned}$$

对 a 进行积分:

$$f(a) = \int \frac{1}{a-1} da = \ln|a-1| + C$$

当 $a=2$ 时, $f(2) = \int_1^\infty \frac{u^{-2}-u^{-2}}{\ln u} du = 0$:

$$0 = \ln|2-1| + C \implies C = 0$$

所以 $f(a) = \ln(a-1)$ 。代入 $a=4$ 得到最终结果:

$$I = f(4) = \ln(4-1) = \ln 3$$

217. 求以下积分, 并使用复指数方法验证结果:

$$I = \int \cos(\ln x) dx, \quad J = \int \sin(\ln x) dx, \quad \int_1^{e^{\pi/2}} 2x^i dx.$$

a) 使用换元法求 I 和 J

令 $u = \ln x \implies x = e^u, dx = e^u du$, 则

$$I = \int \cos(\ln x) dx = \int e^u \cos(u) du.$$

设 $I = \int e^u (P \cos u + Q \sin u)' du$, 则

$$\begin{aligned} \frac{d}{du} [e^u (P \cos u + Q \sin u)] &= e^u (P \cos u + Q \sin u) + e^u (-P \sin u + Q \cos u) \\ &= e^u [(P+Q) \cos u + (Q-P) \sin u]. \end{aligned}$$

对比 $\int e^u \cos u du$, 得到

$$P+Q=1, \quad Q-P=0 \implies P=Q=\frac{1}{2}.$$

所以

$$I = \frac{1}{2} e^u (\cos u + \sin u) = \frac{x}{2} [\cos(\ln x) + \sin(\ln x)].$$

同理, 对于 J :

$$J = \int \sin(\ln x) dx = \int e^u \sin u du.$$

设 $P + Q = 0, Q - P = 1 \implies P = -\frac{1}{2}, Q = \frac{1}{2}$, 所以

$$J = \frac{1}{2}e^u(\sin u - \cos u) = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)].$$

b) 使用 x^i 验证结果

$$x^i = e^{i \ln x} = \cos(\ln x) + i \sin(\ln x)$$

$$\int x^i dx = \frac{x^{1+i}}{1+i} + C = \frac{1-i}{2}x^{1+i} + C = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + i\frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

因此

$$I = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)], \quad J = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)].$$

c) 求定积分 $\int_1^{e^{\pi/2}} 2x^i dx$

$$\begin{aligned} \int_1^{e^{\pi/2}} 2x^i dx &= 2 \left[\frac{x^{1+i}}{1+i} \right]_1^{e^{\pi/2}} = 2 \left[\frac{1-i}{2} x^{1+i} \right]_1^{e^{\pi/2}} \\ &= [x(\cos(\ln x) + \sin(\ln x)) + ix(\sin(\ln x) - \cos(\ln x))]_1^{e^{\pi/2}} \\ &= e^{\pi/2}[1+i] - [1-i] \\ &= (e^{\pi/2} - 1) + i(e^{\pi/2} + 1). \end{aligned}$$

218. 23) 利用含参变量积分求导法 (Leibniz Rule) 证明 $I(a) = \int_0^\infty \frac{e^{-ax} \sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} a$ 。

首先, 设 $I(a) = \int_0^\infty \frac{e^{-ax} \sin x}{x} dx$ 。对 a 求导:

$$\begin{aligned} \frac{\partial I(a)}{\partial a} &= \int_0^\infty \frac{\partial}{\partial a} \left(\frac{e^{-ax} \sin x}{x} \right) dx \\ &= \int_0^\infty -e^{-ax} \sin x dx \end{aligned}$$

利用分部积分法或公式可知 $\int e^{-ax} \sin x dx = -\frac{e^{-ax}(a \sin x + \cos x)}{1+a^2}$:

$$\begin{aligned}\frac{\partial I(a)}{\partial a} &= \left[\frac{e^{-ax}(a \sin x + \cos x)}{1+a^2} \right]_0^\infty \\ &= 0 - \frac{1}{1+a^2} = -\frac{1}{1+a^2}\end{aligned}$$

对 a 进行积分:

$$I(a) = \int -\frac{1}{1+a^2} da = -\tan^{-1} a + C$$

注意到当 $a \rightarrow \infty$ 时, $I(a) \rightarrow 0$:

$$0 = -\frac{\pi}{2} + C \implies C = \frac{\pi}{2}$$

所以, $I(a) = \frac{\pi}{2} - \tan^{-1} a$ 。特别地, 当 $a = 1$ 时, $I(1) = \frac{\pi}{4}$ 。

219. $\int_{-1}^1 \frac{\ln(1+x^2)}{\sqrt{1-x^2}} dx$

令 $x = \sin \theta$, 则 $dx = \cos \theta d\theta$ 。

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\ln(1+\sin^2 \theta)}{\cos \theta} \cos \theta d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln(1+\sin^2 \theta) d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \ln(1+\sin^2 \theta) d\theta\end{aligned}$$

利用参数积分法 $I(a) = \int_0^{\frac{\pi}{2}} \ln(1+a \sin^2 \theta) d\theta$, 最终结果为:

$$\pi \ln\left(\frac{1+\sqrt{2}}{2}\right)$$

220. 计算广义积分:

$$I = \int_0^\infty \frac{1}{x^n + 1} dx$$

使用换元法, 设 $t = \frac{1}{x^n+1}$, 则 $x^n + 1 = \frac{1}{t}$, 从而 $x^n = \frac{1-t}{t}$ 。由此可得变量 x 关于 t 的表达式:

$$x = (1-t)^{\frac{1}{n}} t^{-\frac{1}{n}}$$

对两边微分:

$$dx = \frac{1}{n}(1-t)^{\frac{1}{n}-1}(-1)t^{-\frac{1}{n}}dt + (1-t)^{\frac{1}{n}}\left(-\frac{1}{n}\right)t^{-\frac{1}{n}-1}dt$$

通过代换并简化, 积分限变为从 1 到 0:

$$I = \frac{1}{n} \int_0^1 (1-t)^{\frac{1}{n}-1} t^{-\frac{1}{n}} dt$$

第一步: 引入 **Beta** 函数注意到 Beta 函数的定义为 $B(x, y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt$ 。对应上述积分形式, 令 $x = \frac{1}{n}, y = 1 - \frac{1}{n}$, 则有:

$$I = \frac{1}{n} B\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$$

第二步: 利用 **Gamma** 函数转换利用公式 $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$:

$$I = \frac{1}{n} \cdot \frac{\Gamma(\frac{1}{n})\Gamma(1 - \frac{1}{n})}{\Gamma(1)}$$

由于 $\Gamma(1) = 0! = 1$:

$$I = \frac{1}{n} \Gamma\left(\frac{1}{n}\right) \Gamma\left(1 - \frac{1}{n}\right)$$

第三步: 利用余元公式化简根据 Gamma 函数的余元公式 $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$:

$$I = \frac{1}{n} \cdot \frac{\pi}{\sin(\frac{\pi}{n})}$$

最终结果:

$$\int_0^\infty \frac{1}{x^n + 1} dx = \frac{\pi/n}{\sin(\pi/n)}$$

221. $\int_0^\pi e^{\cos \theta} \cos(\sin \theta) d\theta$

利用欧拉公式 $\cos(\sin \theta) = \operatorname{Re}(e^{i \sin \theta})$:

$$\begin{aligned} \int_0^\pi \operatorname{Re}(e^{\cos \theta} e^{i \sin \theta}) d\theta &= \operatorname{Re} \int_0^\pi e^{\cos \theta + i \sin \theta} d\theta \\ &= \operatorname{Re} \int_0^\pi e^{e^{i\theta}} d\theta \end{aligned}$$

利用级数展开 $e^{e^{i\theta}} = \sum_{n=0}^\infty \frac{(e^{i\theta})^n}{n!}$ 。除了 $n=0$ 项外, 其余项 $\int_0^\pi e^{in\theta} d\theta$ 的实部在对称性下抵消或为零 (对于单位圆路径)。其结果为

$$\pi$$

222. $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx$

对指数部分配方: $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$ 。

$$\int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2} e^{-(c-\frac{b^2}{4a})} dx = e^{\frac{b^2-4ac}{4a}} \int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2} dx$$

令 $u = \sqrt{a}(x + \frac{b}{2a})$, 利用标准高斯积分 $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$:

$$e^{\frac{b^2-4ac}{4a}} \frac{1}{\sqrt{a}} \sqrt{\pi} = e^{\frac{b^2-4ac}{4a}} \sqrt{\frac{\pi}{a}}$$

223. $\int_0^1 (-1)^x dx$ 。

利用欧拉恒等式, 将 $(-1)^x$ 写成指数形式:

$$(-1)^x = (e^{i(2n+1)\pi})^x = e^{i(2n+1)\pi x}$$

其中 $n \in \mathbb{Z}$ 代表不同的分支。进行积分:

$$\int_0^1 e^{i(2n+1)\pi x} dx = \left[\frac{1}{i(2n+1)\pi} e^{i(2n+1)\pi x} \right]_0^1$$

代入边界值:

$$\frac{1}{i(2n+1)\pi} (e^{i(2n+1)\pi} - e^0) = \frac{1}{i(2n+1)\pi} (-1 - 1) = -\frac{2}{i(2n+1)\pi}$$

化简得:

$$\int_0^1 (-1)^x dx = \frac{2i}{(2n+1)\pi}$$

当取主值分支 ($n = 0$) 时, 结果为 $\frac{2i}{\pi}$ 。

224. 计算定积分 $[\int_0^1 \frac{\ln(x+1)}{x} dx]$ 。

将被积函数展开为幂级数。当 $|x| < 1$ 时:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

因此:

$$\frac{\ln(1+x)}{x} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n-1}}{n}$$

在积分区间 $[0, 1]$ 上进行逐项积分:

$$\begin{aligned}\int_0^1 \frac{\ln(1+x)}{x} dx &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 x^{n-1} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\end{aligned}$$

该级数的和为 $\frac{\pi^2}{12}$ 。所以, $[\int_0^1 \frac{\ln(x+1)}{x} dx] = \frac{\pi^2}{12}$ 。

225. 计算定积分 (用到幂级数):

$$\int_0^1 (\ln x) \ln(1-x) dx$$

已知 $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

226.

$$\int_0^{\infty} \frac{\ln\left(\frac{1+x^{11}}{1+x^3}\right)}{(1+x^2)\ln x} dx = 2\pi.$$

(待解)

227.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3 x}{1+e^x} dx = \frac{2}{3}.$$

(待解)

228.

$$\int_0^1 \frac{2xe^x - 1}{2x^2e^x + 2} dx = 0.156631$$

(待解)

229.

$$\int_0^4 \frac{x^4 - 4x + 4}{1 + 2017^{x-2}} dx = 7.22255$$

(待解)

微分方程

考点：一阶微分方程式、变量可分离微分方程式、一阶线性微分方程、伯努利方程、正合微分方程、以积分因子转化非正合微分方程、齐次微分方程、降阶法、二阶线性常系数微分方程、待定系数法、变分法、高阶线性微分方程、一阶微分方程组

1. 解方程

$$yy' = 3y + 2$$

观察得

$$y = -\frac{2}{3}$$

是一解。若 $3y + 2 \neq 0$, 此方程可分离变量, 有

$$y \frac{dy}{dx} = 3y + 2 \Rightarrow \int \frac{y}{3y + 2} dy = \int \left(\frac{1}{3} - \frac{2}{3(3y + 2)} \right) dy = \int dx + C$$

解得

$$\frac{y}{3} - \frac{2}{9} \ln |3y + 2| = x + C$$

由于不存在任何常数 C 使得 $y = -\frac{2}{3}$, 因此原方程的解为

$$\frac{y}{3} - \frac{2}{9} \ln |3y + 2| = x + C \quad \text{或} \quad y = -\frac{2}{3}$$

2. 解微分方程

$$y' = y^2 - 4$$

可验证 $y = \pm 2$ 是原方程的解。若 $y^2 \neq 4$, 分离变量得

$$\frac{dy}{dx} = y^2 - 4 \Rightarrow \int \frac{dy}{y^2 - 4} \Rightarrow \int \left(\frac{-\frac{1}{4}}{y + 2} + \frac{\frac{1}{4}}{y - 2} \right) dy = x + C$$

积分得到:

$$-\frac{1}{4} \ln |y + 2| + \frac{1}{4} \ln |y - 2| = x + C \Rightarrow \left| \frac{y - 2}{y + 2} \right| = e^{4C} e^{4x}$$

令 $C_1 = \pm e^{4C}$, 可写为

$$\frac{y-2}{y+2} = C_1 e^{4x} \Rightarrow y = \frac{2(1+C_1 e^{4x})}{1-C_1 e^{4x}}$$

其中 $y=2$ 可由令 $C_1=0$ 得到, 因此通解为

$$y = \frac{2(1+C_1 e^{4x})}{1-C_1 e^{4x}} \quad \text{或} \quad y = -2$$

3. 由换元法 $y = xV$, 其中 $V = V(x)$, 解常微分方程

$$2xyy' = y^2 - x^2$$

设 $y = xV$, 则 $y' = V + xV'$, 代入原方程得

$$2x \cdot xV \cdot (V + xV') = (xV)^2 - x^2 \Rightarrow 2xVV' = -(V^2 + 1)$$

其中 $x \neq 0$, 分离变量得

$$\frac{2V}{V^2 + 1} dV = -\frac{1}{x} dx$$

两边积分得

$$\ln(V^2 + 1) = -\ln|x| + C \Rightarrow V^2 + 1 = \frac{C_1}{x}$$

由 $V = \frac{y}{x}$ 得

$$y^2 + x^2 = C_1 x$$

4. 已知非零函数 $f(x)$ 满足

$$\sqrt{\int f(x) dx} = \int \sqrt{f(x)} dx, \quad f(0) = \frac{1}{4},$$

用代换 $f(x) = \left(\frac{dy}{dx}\right)^2$, 求 $f(x)$ 的简化表达式。

设

$$f(x) = \left(\frac{dy}{dx}\right)^2.$$

原方程变为

$$\sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = \int \frac{dy}{dx} dx = y + k$$

两边对 x 求导:

$$\frac{1}{2\sqrt{\int (dy/dx)^2 dx}} \cdot \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \implies \left(\frac{dy}{dx}\right)^2 = 2(y+k)\frac{dy}{dx}.$$

由 $dy/dx \neq 0$, 得到

$$\frac{dy}{dx} = 2(y+k).$$

分离变量并积分:

$$\frac{1}{y+k} dy = 2dx \implies \int \frac{1}{y+k} dy = \int 2dx \implies \ln|y+k| = 2x + C.$$

指数化:

$$y+k = Ae^{2x} \implies y = Ae^{2x} - k.$$

由初值条件 $x=0, f(0) = (dy/dx)^2 = \frac{1}{4}$, 得到

$$\left.\frac{dy}{dx}\right|_{x=0} = 2Ae^0 = 2A = \frac{1}{2} \implies A = \frac{1}{4}.$$

因此

$$\frac{dy}{dx} = 2Ae^{2x} = \frac{1}{2}e^{2x} \implies f(x) = \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}e^{4x}.$$

5. 已知 $f: [0, \infty) \rightarrow [0, \infty)$ 可微, 且满足从 $x=a$ 到 $x=b$ 的曲线 $y=f(x)$ 下的面积等于曲线弧长. 已知 $f(0) = 5/4$, 且 $f(x)$ 在 $(0, \infty)$ 上有最小值, 求该最小值.

从 $x=a$ 到 $x=b$ 的面积为

$$\int_a^b f(t) dt,$$

曲线弧长为

$$\int_a^b \sqrt{1 + (f'(t))^2} dt.$$

因此对于所有非负的 a, b 有

$$\int_a^b f(t) dt = \int_a^b \sqrt{1 + (f'(t))^2} dt.$$

取 $a=0$, 得

$$\int_0^x f(t) dt = \int_0^x \sqrt{1 + (f'(t))^2} dt.$$

两边对 x 求导, 利用微积分基本定理, 得

$$f(x) = \sqrt{1 + (f'(x))^2}.$$

设 $y = f(x)$, 则得到微分方程

$$y = \sqrt{1 + (y')^2} \Rightarrow y^2 = 1 + (y')^2 \Rightarrow (y')^2 = y^2 - 1 \Rightarrow y' = \sqrt{y^2 - 1}.$$

分离变量:

$$\frac{dy}{\sqrt{y^2 - 1}} = dx.$$

两边积分:

$$\int \frac{dy}{\sqrt{y^2 - 1}} = \int dx \Rightarrow \ln |y + \sqrt{y^2 - 1}| = x + C.$$

由于 $f(0) = 5/4 > 0$, 可去绝对值, 并解得

$$y = \frac{A}{2}e^x + \frac{1}{2A}e^{-x},$$

其中 $A > 0$ 为常数.

利用初值 $y(0) = 5/4$:

$$\frac{A}{2} + \frac{1}{2A} = \frac{5}{4} \Rightarrow 2A^2 - 5A + 2 = 0 \Rightarrow A = 1 \text{ 或 } A = \frac{1}{2}.$$

函数 $y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ 在 $x > 0$ 的最小值发生在 $x = 0$, 值为 1, 而 $y = e^x/2 + e^{-x}/1$ 的最小值在 $x < 0$, 不符合要求.

因此 $f(x)$ 在 $(0, \infty)$ 上的最小值为

1.

6. 非零函数 $u(x)$ 和 $v(x)$ 满足积分方程

$$\int u(x) dx = x^2 u(x), \quad \int u(x)v(x) dx = \left[\int u(x) dx \right] \left[\int v(x) dx \right].$$

求 $u(x)$ 的一般表达式, 以及 $[v(x)]^2$ 的简化表达式。

步骤 1: 求 $u(x)$

$$\int u \, dx = ux^2$$

对 x 求导:

$$\begin{aligned}u &= 2xu + x^2 \frac{du}{dx} \\x^2 \frac{du}{dx} &= u - 2xu = u(1 - 2x) \\ \frac{1}{u} \frac{du}{dx} &= \frac{1 - 2x}{x^2} \\ \int \frac{1}{u} du &= \int \left(\frac{1}{x^2} - \frac{2}{x} \right) dx \\ \ln |u| &= -\frac{1}{x} - 2 \ln |x| + C \\ u &= A \frac{e^{-\frac{1}{x}}}{x^2}\end{aligned}$$

步骤 2: 利用第二个积分方程求 $v(x)$

$$\int uv \, dx = \left[\int u \, dx \right] \left[\int v \, dx \right]$$

对 x 求导:

$$\begin{aligned}uv &= \left[\int u \, dx \right] v + u \left[\int v \, dx \right] \\ uv &= x^2 uv + u \int v \, dx \\ v - x^2 v &= \int v \, dx \\ v(1 - x^2) &= \int v \, dx\end{aligned}$$

对 x 再求导:

$$v'(1 - x^2) - 2xv = v \implies v'(1 - x^2) = v(1 + 2x)$$

步骤 3: 分离变量并积分

$$\frac{dv}{v} = \frac{1 + 2x}{1 - x^2} dx$$

部分分式分解:

$$\frac{1 + 2x}{1 - x^2} = \frac{2x + 1}{(1 - x)(1 + x)} = -\frac{3}{2} \frac{1}{1 - x} + \frac{1}{2} \frac{1}{1 + x}$$

积分:

$$\begin{aligned}\ln |v| &= -\frac{3}{2} \ln |1-x| + \frac{1}{2} \ln |1+x| + B \\ \ln v^2 &= \ln \left| \frac{B(1+x)}{(1-x)^3} \right| \\ v^2 &= \frac{B(1+x)}{(1-x)^3}\end{aligned}$$

最终结果:

$$u(x) = A \frac{e^{-1/x}}{x^2}, \quad v^2(x) = \frac{B(1+x)}{(1-x)^3}.$$

7. 证明方程

$$y' + p(x)y = q(x)$$

的一般解为

$$y = \frac{\int I(x) q(x) dx + C}{I(x)},$$

其中积分因子为

$$I(x) = \exp \left(\int p(x) dx \right).$$

对积分因子求导,

$$I'(x) = \frac{d}{dx} \exp \left(\int p(x) dx \right) = I(x)p(x)$$

由原方程可得

$$I(x)(y' + p(x)y) = \frac{d}{dx}(yI(x)) = I(x)q(x)$$

对两边积分,

$$yI(x) = \int I(x)q(x)dx + C \Rightarrow y = \frac{\int I(x)q(x)dx + C}{I(x)}$$

8. 求微分方程

$$y' = 4y + x$$

的通解。

积分因子为

$$I(x) = \exp\left(\int -4 dx\right) = e^{-4x}$$

由公式得

$$y = \frac{1}{I(x)} \left(\int I(x) q(x) dx + C \right) = e^{4x} \left(\int x e^{-4x} dx + C \right) = -\frac{x}{4} - \frac{1}{16} + C e^{4x}$$

其中由分部积分,

$$\int x e^{-4x} dx = -\frac{1}{4} x e^{-4x} + \frac{1}{4} \int e^{-4x} dx = -\frac{1}{4} x e^{-4x} - \frac{1}{16} e^{-4x}$$

9. 求解微分方程并满足初始条件

$$x^2 y' + 3xy = \frac{1}{x}, \quad x > 0, \quad y(1) = -1.$$

原方程即

$$y' + \frac{3}{x}y = \frac{1}{x^3}.$$

其中积分因子为

$$I(x) = \exp\left(\int \frac{3}{x} dx\right) = x^3,$$

故

$$y = \frac{1}{I(x)} \left(\int I(x) q(x) dx + C \right) = \frac{1}{x^3} \left(\int x^3 \cdot \frac{1}{x^3} dx + C \right) = \frac{1}{x^2} + \frac{C}{x^3}$$

由 $y(1) = -1$ 知

$$-1 = 1 + C \Rightarrow C = -2$$

因此解为

$$y = \frac{1}{x^2} - \frac{2}{x^3}$$

10. 设 $f(x)$ 为整系数多项式, 若

$$g(x) = \int x f(x) dx,$$

且

$$\frac{d}{dx}[f(x) + g(x)] = x^4 - 4x^2 + x - 7,$$

求 $f(x)$ 。

由条件得 $g'(x) = xf(x)$, 故

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + xf(x) = x^4 - 4x^2 + x - 7$$

为一阶微分方程; 设积分因子 $I(x) = \exp\left(\int x dx\right) = e^{\frac{x^2}{2}}$, 则

$$\left(e^{\frac{x^2}{2}} f(x)\right)' = e^{\frac{x^2}{2}} (x^4 - 4x^2 + x - 7)$$

两边积分得

$$e^{\frac{x^2}{2}} f(x) = \int e^{\frac{x^2}{2}} (x^4 - 4x^2 + x - 7) dx = e^{\frac{x^2}{2}} (x^3 - 7x + 1) + C$$

即

$$f(x) = x^3 - 7x + 1 + Ce^{-\frac{x^2}{2}}$$

因为 $f(x)$ 是整系数多项式, 所以 $C = 0$, 于是

$$f(x) = x^3 - 7x + 1$$

11. 求解伯努利方程

$$2x(\ln x)y' - y = -9x^3y^3 \ln x$$

化为标准形式,

$$y' - \frac{1}{2x(\ln x)}y = -\frac{9}{2x^2}y^3,$$

这是阶数为 $n = 3$ 的伯努利方程, 两边除以 y^3 ,

$$y^{-3} \frac{dy}{dx} - \frac{1}{2x(\ln x)}y^{-2} = -\frac{9}{2}x^2$$

设 $u = y^{1-n} = y^{-2}$, 则

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{du}{dx}$$

代入前式得到

$$-\frac{1}{2} \frac{du}{dx} - \frac{1}{2x(\ln x)}u = -\frac{9}{2}x^2 \Rightarrow \frac{du}{dx} + \frac{1}{x(\ln x)}u = 9x^2$$

积分因子为

$$I(x) = \exp\left(\int \frac{1}{x(\ln x)} dx\right) = e^{\ln(\ln x)} = \ln x$$

于是

$$u = y^{-2} = \frac{1}{\ln x} \left(\int 9x^2 \ln x \, dx + C \right) = \frac{1}{\ln x} (3x^3 \ln x - x^3 + C)$$

即

$$y^2 = \frac{\ln x}{x^3(3 \ln x - 1) + C}$$

12. 判断函数 $I(x, y) = \cos(xy)$ 是否为微分方程

$$[\tan(xy) + xy] \, dx + x^2 \, dy = 0$$

的积分因子。若是, 求其通解。

将方程乘以 $I(x, y)$ 得

$$[\sin(xy) + xy \cos(xy)] \, dx + [x^2 \cos(xy)] \, dy = 0$$

设

$$P(x, y) = \sin(xy) + xy \cos(xy), \quad Q(x, y) = x^2 \cos(xy).$$

观察得偏导数

$$\frac{\partial P}{\partial y} = 2x \cos(xy) - x^2 y \sin(xy) = \frac{\partial Q}{\partial x}$$

由此可得 $\cos(xy)$ 是给定方程的积分因子。注意到

$$d(x \sin(xy)) = (\sin(xy) + xy \cos(xy)) \, dx + x^2 \cos(xy) \, dy$$

因此通解为

$$x \sin(xy) = C$$

13. 求方程

$$y \, dx - (2x + y^4) \, dy = 0$$

的积分因子, 并由此求通解。

设 $I(x, y)$ 是该积分因子, 将方程乘以 $I(x, y)$ 得

$$Iy \, dx - (2x + y^4)I \, dy = 0$$

记

$$P(x, y) = Iy, \quad Q(x, y) = -(2x + y^4)I$$

方程的正合条件为

$$P_y = yI_y + I = Q_x = -2I - (2x + y^4)I_x$$

若 I 只与 y 有关, 即 $I_x = 0$, 则有

$$yI_y + I = -2I \Rightarrow yI_y = -3I \Rightarrow I(y) = \frac{1}{y^3}$$

将其代入方程得到

$$\frac{1}{y^2}dx - \frac{2x + y^4}{y^3}dy = 0$$

此时方程正合。设函数 $\phi(x, y)$ 满足

$$\frac{\partial \phi}{\partial x} = \frac{1}{y^2}, \quad \frac{\partial \phi}{\partial y} = -\frac{2x + y^4}{y^3}$$

由第一式积分得

$$\phi(x, y) = \frac{x}{y^2} + h(y) \Rightarrow \frac{\partial \phi}{\partial y} = -\frac{2x}{y^3} + h'(y)$$

由第二式可得

$$h'(y) = -y \Rightarrow h(y) = -\frac{y^2}{2}$$

因此通解为

$$\frac{x}{y^2} - \frac{y^2}{2} = C \Rightarrow 2x - y^4 = Cy^2$$

14. 求解齐次微分方程

$$y' - x^{-1}y = x^{-1}\sqrt{x^2 - y^2}, \quad x > 0$$

考虑齐次性, 将方程改写为

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$$

作变量代换,

$$y = xV(x) \Rightarrow \frac{dy}{dx} = x\frac{dV}{dx} + V$$

代入方程得

$$x\frac{dV}{dx} + V = V + \sqrt{1 - V^2} \Rightarrow \frac{dV}{\sqrt{1 - V^2}} = \frac{dx}{x}$$

两边积分得

$$\int \frac{dV}{\sqrt{1-V^2}} = \int \frac{dx}{x} + C \Rightarrow \sin^{-1} V = \ln |x| + C$$

因此通解为

$$y = xV = x \sin(C + \ln x), \quad x > 0$$

15. 解微分方程

$$(x^2 + y^2) dx + (x^2 - xy) dy = 0$$

若 $x \neq 0$, 有

$$\left(1 + \frac{y^2}{x^2}\right) + \left(1 - \frac{y}{x}\right) \frac{dy}{dx} = 0$$

设 $\frac{y}{x} = u \Rightarrow y = xu$, 则 $\frac{dy}{dx} = x \frac{du}{dx} + u$,

$$(1 + u^2) + (1 - u) \left(x \frac{du}{dx} + u\right) = 0 \Rightarrow x \frac{du}{dx} = \frac{u + 1}{u - 1}$$

分离变量得

$$\int \frac{u-1}{u+1} du = \int \frac{1}{x} dx \Rightarrow u - 2 \ln |u+1| = \ln x + C$$

即

$$\frac{y}{x} - \ln \left(1 + \frac{y}{x}\right)^2 = \ln x + C$$

16. 解

$$xy'' + y' = 8x, \quad x > 0$$

令 $v = y'$, 则 $v' = y''$ 。方程变为一阶方程

$$xv' + v = 8x \Rightarrow v' + \frac{1}{x}v = 8$$

这是线性一阶方程, 积分因子为

$$I(x) = \exp\left(\int \frac{1}{x} dx\right) = x$$

所以

$$v(x) = \frac{1}{x} \left(\int 8x \, dx + C_1 \right) = \frac{1}{x} (4x^2 + C_1)$$

积分得到

$$y = \int v(x) \, dx + C_2 = \int \frac{1}{x} (4x^2 + C_1) \, dx + C_2 = 2x^2 + C_1 \ln x + C_2$$

17. 解微分方程

$$y' + e^{y'} - x = 0$$

令 $u = y' \Rightarrow u + e^u = x$, 由链导法,

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = u \frac{dx}{du}$$

又因为

$$x = u + e^u \Rightarrow \frac{dx}{du} = 1 + e^u$$

所以

$$\frac{dy}{du} = u(1 + e^u) \Rightarrow y = \int u(1 + e^u) \, du = \frac{u^2}{2} + (u - 1)e^u + C$$

故解可写为参数形式:

$$\begin{cases} x = u + e^u \\ y = \frac{u^2}{2} + (u - 1)e^u + C \end{cases}$$

18. 解微分方程

$$y' + \frac{y}{x} = e^{xy}$$

令 $u = e^{xy}$, 则

$$\frac{du}{dx} = e^{xy} \left(y + x \frac{dy}{dx} \right) \Rightarrow y' + \frac{y}{x} = \frac{1}{xu} \frac{du}{dx}$$

由原方程可得

$$\frac{1}{xu} \frac{du}{dx} = u \Rightarrow \frac{1}{u^2} \frac{du}{dx} = x$$

解为

$$-\frac{1}{u} = \frac{x^2}{2} + C$$

故通解为

$$\frac{x^2}{2} + e^{-xy} = C_1$$

19. 解

$$dx - xy(1 + xy^2) dy = 0$$

原方程即

$$\frac{1}{x^2} \frac{dx}{dy} - \frac{y}{x} - y^3 = 0$$

设 $u = -\frac{1}{x}$, 则 $\frac{du}{dy} = \frac{1}{x^2} \frac{dx}{dy}$, 化为线性一阶微分方程

$$\frac{du}{dy} + yu = y^3$$

积分因子为 $\exp\left(\int y dy\right) = e^{\frac{y^2}{2}}$, 于是

$$ue^{\frac{y^2}{2}} = \int y^3 e^{\frac{y^2}{2}} dy + C$$

作代换 $t = \frac{y^2}{2}$, 则 $dt = y dy$,

$$\int y^3 e^{\frac{y^2}{2}} dy = \int 2te^t dt = 2te^t - 2 \int e^t dt = 2(t-1)e^t = (y^2-2)e^{\frac{y^2}{2}}$$

于是

$$u = y^2 - 2 + Ce^{-\frac{y^2}{2}}.$$

代回 $u = -\frac{1}{x}$ 得到

$$x = \frac{1}{2 - y^2 - Ce^{-\frac{y^2}{2}}},$$

20. 解微分方程

$$xy' - y = 2x^2y(y^2 - x^2)$$

令 $u = x^2y$, 对 x 求导,

$$\frac{du}{dx} = 2xy + x^2 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x^2} \frac{du}{dx} - \frac{2u}{x^3}.$$

代回原方程得

$$\frac{1}{x} \frac{du}{dx} - \frac{3u}{x^2} = 2u \left(\frac{u^2}{x^4} - x^2 \right) \Rightarrow \frac{du}{dx} + \left(2x^3 - \frac{3}{x} \right) u = \frac{2}{x^3} u^3$$

这是阶数为 3 的伯努利方程。令 $v = u^{-2}$, 则 $\frac{dv}{dx} = -2u^{-3} \frac{du}{dx}$, 于是

$$v' + \left(-4x^3 + \frac{6}{x} \right) v = -\frac{4}{x^3}.$$

积分因子为

$$I(x) = \exp \left(\int \left(-4x^3 + \frac{6}{x} \right) dx \right) = e^{-x^4 + 6 \ln x} = x^6 e^{-x^4}$$

解得

$$u^{-2} = v = \frac{e^{x^4}}{x^6} \left(\int x^6 e^{-x^4} \cdot \left(-\frac{4}{x^3} \right) dx + C \right) = \frac{e^{x^4}}{x^6} (e^{-x^4} + C) = \frac{1}{x^6} (1 + C e^{x^4})$$

由 $u = x^2y$, 通解即

$$x^2 - y^2 = c y^2 e^{x^4}$$

21. 解

$$yy'' + (y')^2 = yy'$$

设 $u = y'$, 由链导法,

$$y'' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}.$$

代入原方程得

$$y \cdot u \frac{du}{dy} + u^2 = yu \Rightarrow \frac{du}{dy} + \frac{1}{y} u = 1$$

这是关于 u 的线性一阶方程。积分因子为 $e^{\int \frac{1}{y} dy} = y$, 于是

$$\frac{d}{dy}(yu) = y \Rightarrow yu = \frac{y^2}{2} + c_1 \Rightarrow y' = u = \frac{y}{2} + \frac{c_1}{y} = \frac{y^2 + 2c_1}{2y}$$

分离变量并积分,

$$\frac{2y \, dy}{y^2 + 2c_1} = dx \Rightarrow \int \frac{2y \, dy}{y^2 + 2c_1} = \int dx + c_2.$$

得到

$$\ln |y^2 + 2c_1| = x + c_2 \Rightarrow y^2 = C_1 e^x + C_2$$

22. 解微分方程

$$\frac{dy}{dx} = x \frac{d^2 y}{dx^2} - \left(\frac{dy}{dx} \right)^3$$

令 $u = \frac{dy}{dx}$, 则 $\frac{d^2 y}{dx^2} = \frac{du}{dx}$, 整理为

$$x \frac{du}{dx} = u + u^3$$

分离变量得

$$\frac{du}{u(1+u^2)} = \frac{dx}{x} \Rightarrow \int \left(\frac{1}{u} - \frac{u}{1+u^2} \right) du = \int \frac{dx}{x}$$

积分得到

$$\ln |u| - \frac{1}{2} \ln(1+u^2) = \ln |x| + \ln C \Rightarrow C|x| = \frac{|u|}{\sqrt{1+u^2}}$$

化简得

$$u^2 = \frac{C^2 x^2}{1 - C^2 x^2} \Rightarrow \frac{dy}{dx} = u = \pm \frac{Cx}{\sqrt{1 - C^2 x^2}}$$

再次分离变量得

$$y = \pm \int \frac{Cx}{\sqrt{1 - C^2 x^2}} dx$$

令 $t = Cx, dt = C dx$, 积分变为

$$y = \pm \frac{1}{C} \int \frac{t}{\sqrt{1-t^2}} dt = -\frac{1}{C} \sqrt{1-t^2} + C_1 = -\frac{1}{C} \sqrt{1-C^2 x^2} + C_1$$

23. 求解

$$(1+y^2) \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^3 + \frac{dy}{dx} = 0$$

令 $u = \frac{dy}{dx}$, 由链导法,

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u \frac{du}{dy}$$

代入原方程得

$$u \left((1+y^2) \frac{du}{dy} + u^2 + 1 \right) = 0$$

若 $u = \frac{dy}{dx} = 0$, 解为 $y = C_1$ 。若 $u \neq 0$, 则由

$$(1+y^2) \frac{du}{dy} + u^2 + 1 = 0$$

分离变量得

$$\frac{du}{u^2+1} = -\frac{dy}{1+y^2}$$

两边积分得

$$\arctan u = -\arctan y + A$$

令常数 $A = \arctan C$, 则

$$\frac{dy}{dx} = \tan(\arctan C - \arctan y) = \frac{C-y}{1+Cy}$$

再次分离变量,

$$\frac{1+Cy}{C-y} dy = dx$$

积分得

$$\int \frac{1+Cy}{C-y} dy = \int \left(-C + \frac{1+C^2}{C-y} \right) dy = -Cy - (1+C^2) \ln |C-y| = x + C_2$$

综上, 除 $y = C_1$ 外, 通解为

$$-Cy - (1+C^2) \ln |C-y| = x + C_2$$

24. 求解

$$\frac{d^2y}{dx^2} = \frac{1}{1-y^2} \left(\frac{dy}{dx} \right)^2$$

设 $u = \frac{dy}{dx}$, 由链导法,

$$\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}.$$

代入原方程得

$$\frac{du}{dy} = \frac{u}{1-y^2}.$$

分离变量并积分,

$$\frac{du}{u} = \frac{dy}{1-y^2} \Rightarrow \ln|u| = \frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| + C_1.$$

于是存在常数 C 使

$$\frac{dy}{dx} = u = C \sqrt{\frac{1+y}{1-y}}$$

再次分离变量得

$$dx = \frac{1}{C} \sqrt{\frac{1-y}{1+y}} dy$$

对右侧积分, 设 $y = \cos 2\theta$, 则 $dy = -2 \sin 2\theta d\theta$, 于是

$$\begin{aligned} \int \sqrt{\frac{1-y}{1+y}} dy &= \int \sqrt{\frac{2 \sin^2 \theta}{2 \cos^2 \theta}} (-2 \sin 2\theta) d\theta \\ &= 2 \int (\cos 2\theta - 1) d\theta \\ &= 2 \left(\frac{1}{2} \sin 2\theta - \theta \right) \\ &= \sqrt{1-y^2} - \arccos y \end{aligned}$$

因此通解为

$$x = \frac{1}{C_1} \left(\sqrt{1-y^2} - \arccos y \right) + C_2$$

25. 求解

$$\frac{1}{2} \left(\frac{dy}{dx} \right)^2 = 4y^2 + y \frac{d^2y}{dx^2}$$

令 $u = \frac{dy}{dx}$, 由链导法,

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u \frac{du}{dy}$$

代入原方程得

$$y u \frac{du}{dy} = \frac{1}{2} u^2 - 4y^2$$

若 $u = 0$, 则 $y = 0$ 为一解。若 $u \neq 0$ 即 $y \neq 0$, 则

$$\frac{dv}{dy} - \frac{1}{y}v = -8y$$

这是关于 $v = u^2$ 的线性一阶方程, 积分因子为

$$I(y) = \exp\left(-\int \frac{1}{y} dy\right) = \frac{1}{y}$$

于是

$$\left(\frac{dy}{dx}\right)^2 = v = y \left(\int -8 dy + A\right) = Ay - 8y^2$$

为便于积分, 令 $B = \frac{A}{16}$, 则配方得 $Ay - 8y^2 = 8(B^2 - (y - B)^2)$, 因此

$$\frac{dy}{dx} = \pm 2\sqrt{2} \sqrt{B^2 - (y - B)^2}$$

分离变量并积分:

$$\frac{1}{2\sqrt{2}} \int \frac{d(y - B)}{\sqrt{B^2 - (y - B)^2}} = \pm x + C \Rightarrow \frac{1}{2\sqrt{2}} \arcsin\left(\frac{y - B}{B}\right) = \pm x + C$$

两边变换并记相位常数, 得

$$\arcsin\left(\frac{y - B}{B}\right) = 2\sqrt{2}x + C'$$

于是

$$y(x) = B \left(1 + \sin\left(2\sqrt{2}x + C'\right)\right)$$

26. 求解

$$\frac{y''}{y'} - \frac{y'}{y} = \ln y$$

设 $u = \frac{dy}{dx}$, 由链导法,

$$y'' = \frac{du}{dx} = u \frac{du}{dy}$$

原方程化为

$$\frac{du}{dy} - \frac{1}{y}u = \ln y$$

积分因子为

$$I(y) = \exp\left(-\int \frac{1}{y}dy\right) = e^{-\ln y} = \frac{1}{y}.$$

于是

$$\frac{dy}{dx} = u = y \left(\int \frac{\ln y}{y} dy + C_1 \right) = y \left(\frac{1}{2}(\ln y)^2 + C_1 \right)$$

再次分离变量,

$$\frac{2 \cdot \frac{1}{y} dy}{((\ln y)^2 + C_1)} = dx$$

两边积分后得,

$$\frac{2}{\sqrt{A}} \arctan \frac{\ln y}{\sqrt{A}} = x + C_2$$

设 $C_1 = \sqrt{A}$, 可得通解为

$$y = e^{2C_1 \tan(C_1(x + C_2))}$$

27. 求解

$$y \frac{d^2 y}{dx^2} + y^2 = \frac{1}{2} \left(\frac{dy}{dx} \right)^2$$

令 $z = \sqrt{y}$, 则

$$y = z^2, \quad y' = 2zz', \quad y'' = 2z'^2 + 2zz''$$

代入原方程得

$$z^2(2z'^2 + 2zz'') + z^4 = \frac{1}{2}(4z^2z'^2) \Rightarrow 2z'' + z = 0.$$

这是常系数二阶线性方程, 通解为

$$z = C_1 \cos \frac{x}{\sqrt{2}} + C_2 \sin \frac{x}{\sqrt{2}}$$

因此

$$y = z^2 = \left(C_1 \cos \frac{x}{\sqrt{2}} + C_2 \sin \frac{x}{\sqrt{2}} \right)^2$$

28. 求解

$$y''' = y'y''$$

令 $u = \frac{dy}{dx}$, 由链导法,

$$u'' = \frac{du'}{dx} = \frac{du'}{du} \frac{du}{dx} = \frac{du'}{du} u'$$

代回方程得

$$\frac{du'}{du} u' = u u' \Rightarrow \frac{du'}{du} = u$$

积分得

$$u' = \frac{1}{2}u^2 + 2C_1$$

分离变量并积分,

$$\int \frac{du}{\frac{1}{2}u^2 + C_1} = \int \frac{2 du}{u^2 + 2C_1} = x + C_2 \Rightarrow \frac{2}{k} \arctan \frac{u}{k} = x + C_2$$

其中 $k^2 = 2C_1$, 代回 $u = y'$ 得

$$\arctan \frac{y'}{\sqrt{2C_1}} = \frac{\sqrt{2C_1}}{2}(x + C_2).$$

再积分得到 y ,

$$y = \int y' dx = \int \sqrt{2C_1} \tan \left(\frac{\sqrt{2C_1}}{2}(x + C_2) \right) dx + C_3$$

即

$$y = -2 \ln |\cos (C_1'(x + C_2))| + C_3$$

29. 求解

$$\frac{y''}{(y')^2} = \frac{y}{y^2 - 1}$$

两边乘以 y' 得

$$\frac{y''}{y'} = \frac{yy'}{y^2 - 1}.$$

因此对 x 积分得

$$\ln y' = \frac{1}{2} \ln(y^2 - 1) + \ln A \Rightarrow y' = A \sqrt{y^2 - 1}$$

分离变量后积分得

$$\frac{dy}{\sqrt{y^2 - 1}} = A dx \Rightarrow \ln(y + \sqrt{y^2 - 1}) = Ax + C_1.$$

即

$$y + \sqrt{y^2 - 1} = Be^{Ax}$$

解出 y , 得通解为

$$y = \frac{Be^{Ax} + \frac{1}{B}e^{-Ax}}{2}$$

30. 解

$$y'' - y' \tan x + 2y = 0$$

观察到 $y_1 = \sin x$ 为原方程的解, 由降阶法, $y_2 = u(x)y_1$, 其中

$$\begin{aligned} u(x) &= \int \frac{1}{\sin^2 x} e^{-\int -\tan x dx} dx \\ &= \int \frac{1}{\sin^2 x} e^{-\ln |\cos x|} dx \\ &= \int (\cot^2 x + 1) \sec x dx \\ &= \int (\sec x + \cot x \csc x) dx \\ &= \ln |\sec x + \tan x| - \csc x \end{aligned}$$

故通解为

$$y = \sin x (C_1 + \ln |\sec x + \tan x|) + C_2$$

31. 解微分方程

$$\cos^2 x \frac{d^2 y}{dx^2} = 2y$$

观察得 $y_1 = \tan x$ 满足方程

$$y'' = 2y \sec^2 x$$

现设 $y = u(x) \tan x$, 则 $y' = u' \tan x + u \sec^2 x$, 且

$$y'' = u'' \tan x + 2u' \sec^2 x + 2u \sec^2 x \tan x = u'' \tan x + 2u' \sec^2 x + y''$$

即

$$u'' \tan x + 2u' \sec^2 x = 0$$

设 $u' = z$, 则

$$-z' \tan x = 2z \sec^2 x$$

分离变量得

$$\int -\frac{z'}{2z} dz = \int \frac{\sec x}{\tan x} dx \Rightarrow -\frac{1}{2} \ln |z| = \ln |\tan x| + \ln A$$

化简得

$$\frac{dx}{du} = \frac{1}{z} = A^2 \tan^2 x$$

再分离变量,

$$\int \frac{1}{A^2} \cot^2 x dx = \int du \Rightarrow \frac{1}{A^2} \int (\csc^2 x - 1) dx = u \Rightarrow u = C(-\cot x - x) + B$$

故通解为

$$y = B \tan x - Cx \tan x - C$$

32. 解微分方程

$$\frac{dy}{dx} = 2 \left(\frac{y+2}{x+y+1} \right)^2$$

设 $u = y + 2, v = x + y + 1$, 则

$$\frac{du}{dx} = \frac{dy}{dx} = 2 \left(\frac{u}{v} \right)^2, \quad \frac{dv}{dx} = 1 + \frac{dy}{dx} = 1 + 2 \left(\frac{u}{v} \right)^2.$$

于是有

$$\frac{dv}{du} = \frac{1 + 2 \left(\frac{u}{v} \right)^2}{2 \left(\frac{u}{v} \right)^2} = \frac{v^2 + 2u^2}{2u^2} = 1 + \frac{v^2}{2u^2}.$$

令 $w = \frac{v}{u}$, 则 $v = wu$, 并且 $\frac{dv}{du} = w + u \frac{dw}{du}$, 代入上式得

$$u \frac{dw}{du} = \frac{w^2}{2} - w + 1$$

分离变量再积分得

$$\frac{du}{u} = \frac{dw}{\frac{1}{2}((w-1)^2 + 1)} \Rightarrow \ln |u| = 2 \arctan(w-1) + C$$

把 u, w 代回原变量, 因此得到

$$\ln |y+2| = 2 \arctan \left(\frac{x-1}{y+2} \right) + C$$

33. 解

$$f(x) + f'(-x) = x^2 + \alpha, \quad \alpha \in \mathbb{R}$$

由

$$f(x) + f'(-x) = x^2 + \alpha \quad (1)$$

令 $x \rightarrow -x$, 得 $f(-x) + f'(x) = x^2 + \alpha$, 两边求导,

$$-f'(-x) + f''(x) = 2x \quad (2)$$

由 (1) + (2),

$$f''(x) + f(x) = x^2 + 2x + \alpha$$

齐次解为

$$f_c = C_1 \sin x + C_2 \cos x$$

且发现特解为

$$f_p = x^2 + 2x + \alpha - 2$$

于是通解为

$$f(x) = C_1 \sin x + C_2 \cos x + x^2 + 2x + \alpha - 2$$

尚未结束, 与原方程比较系数得

$$f'(-x) + f(x) = (C_1 + C_2)(\cos x + \sin x) + \sqrt{x^2 + \alpha} \Rightarrow C_2 = -C_1$$

故通解为

$$f(x) = C_1(\sin x - \cos x) + x^2 + 2x + \alpha - 2$$

34. 解

$$y = xy' + \sqrt{(y')^2 + 1}$$

此方程属于克莱罗方程 (Clairaut's equation)。对 x 求导得

$$y' = y' + xy'' + \frac{1}{x} \frac{x \cdot 2y'y''}{\sqrt{(y')^2 + 1}}$$

整理得

$$y'' \left(x + \frac{y'}{\sqrt{(y')^2 + 1}} \right) = 0$$

若 $y'' = 0$, 则 $y' = C \Rightarrow y = Cx + \sqrt{C^2 + 1}$ 。对于

$$\left(x + \frac{y'}{\sqrt{(y')^2 + 1}} \right) = 0$$

化简得

$$y' = \pm \frac{x}{\sqrt{1 - x^2}}$$

照常分离系数后积分得

$$\int dy = \pm \int \frac{x}{\sqrt{1 - x^2}} dx \Rightarrow y = \pm \sqrt{1 - x^2} + A$$

将 $y = \pm \sqrt{1 - x^2} + A$ 代入原方程解得 $A = 0$, 故解为

$$y = \pm \sqrt{1 - x^2} \Rightarrow x^2 + y^2 = 1$$

35. 求解

$$\sqrt{\tan x} \frac{dy}{dx} = x$$

分离系数得,

$$\frac{1}{y} dy = \sqrt{\cot x} dx \Rightarrow \ln y = \int \sqrt{\cot x} dx$$

设 $\cot x = u^2$, 则 $-\csc^2 x dx = 2u du \Rightarrow dx = -\frac{2u du}{1 + u^4}$,

$$I = \int \sqrt{\cot x} dx = - \int \frac{2u du}{1 + u^4} = - \int \frac{2 du}{u^2 + \frac{1}{u^2}} = - \int \frac{1 + \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du - \int \frac{1 - \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du = -(I_1 + I_2)$$

其中设 $u - \frac{1}{u} = t$, $\left(1 + \frac{1}{u^2}\right) du = dt$, 则 $u^2 + \frac{1}{u^2} = t^2 + 2$,

$$I_1 = \int \frac{1 + \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du = \int \frac{dt}{t^2 + 2} = \frac{1}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}} \arctan \left(\frac{u^2 - 1}{u\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \arctan \left(\frac{\cot x - 1}{\sqrt{2} \cot x} \right)$$

且设 $u + \frac{1}{u} = \phi$, $\left(1 - \frac{1}{u^2}\right) du = d\phi$, 则 $u^2 + \frac{1}{u^2} = \phi^2 - 2$,

$$\begin{aligned} I_2 &= \int \frac{1 - \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du I_2 = \int \frac{d\phi}{\phi^2 - 2} = \frac{1}{2\sqrt{2}} \int \left(\frac{1}{\phi - \sqrt{2}} - \frac{1}{\phi + \sqrt{2}} \right) d\phi = \frac{1}{2\sqrt{2}} \ln \left| \frac{\phi - \sqrt{2}}{\phi + \sqrt{2}} \right| \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{u^2 - u\sqrt{2} + 1}{u^2 + u\sqrt{2} + 1} \right| = \frac{1}{2\sqrt{2}} \ln \left| \frac{\cot x - \sqrt{2} \cot x + 1}{\cot x + \sqrt{2} \cot x + 1} \right| \end{aligned}$$

于是 $\ln y = I$ 给出

$$y = C e^{-\frac{1}{2\sqrt{2}} \arctan \left(\frac{\cot x - 1}{\sqrt{2} \cot x} \right)} \left(\frac{\cot x + \sqrt{2} \cot x + 1}{\cot x - \sqrt{2} \cot x + 1} \right)^{\frac{1}{2\sqrt{2}}}$$

36. 解

$$y'' - y' - 2y = 10 \sin x, \quad y(0) = 0, \quad y'(0) = 1.$$

对应齐次方程的特征方程为

$$r^2 - r - 2 = (r - 2)(r + 1) = 0 \Rightarrow r = 2, -1$$

因此齐次通解为

$$y_c(x) = C_1 e^{2x} + C_2 e^{-x}.$$

取特解

$$y_p(x) = A \cos x + B \sin x$$

代入方程

$$y_p'' - y_p' - 2y_p = (B - 3A) \cos x + (-A - 3B) \sin x = 0$$

比较系数解得 $A = -1, B = -3$, 因此特解为

$$y_p(x) = -\cos x - 3 \sin x$$

通解为

$$y(x) = y_c + y_p = C_1 e^{2x} + C_2 e^{-x} - \cos x - 3 \sin x.$$

使用初值条件 $y(0) = 0, y'(0) = 1$,

$$C_1 + C_2 - 1 = 0 \Rightarrow C_1 + C_2 = 1.$$

$$y'(x) = 2C_1e^{2x} - C_2e^{-x} + \sin x - 3\cos x \implies y'(0) = 2C_1 - C_2 - 3 = 1.$$

解得 $C_1 = 1, C_2 = 0$, 因此初值问题的解为

$$y(x) = e^{2x} - \cos x - 3\sin x$$

37. 设初值条件为 $y(0) = 2, y'(0) = 0$, 求解方程

$$y'' + 4y = 16x \cos 2x$$

对应的齐次方程为 $y'' + 4y = 0$, 其通解为

$$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$$

非齐次项为 $16x \cos 2x$, 原方程的特解包含了 $A_0 \cos 2x + B_0 \sin 2x$, 将与 $y_c(x)$ 重叠。因此特解需取

$$y_p(x) = x((A_0 + A_1x) \cos 2x + (B_0 + B_1x) \sin 2x).$$

将 y_p 代入原方程, 比较系数可得 $A_0 = 1, A_1 = 0, B_0 = 0, B_1 = 2$, 因此特解为

$$y_p(x) = x \cos 2x + 2x^2 \sin 2x.$$

所以通解为

$$y(x) = y_c + y_p = (C_1 + x) \cos 2x + (C_2 + 2x^2) \sin 2x$$

由 $y(0) = 2$ 得 $C_1 = 2$, 求导得

$$y'(x) = (2x - 2C_1) \sin 2x + (4x^2 + 2C_2 + 1) \cos 2x$$

代入 $x = 0, y'(0) = 0$, 得到 $C_2 = -\frac{1}{2}$, 因此满足初值条件的解为

$$y(x) = (x + 2) \cos 2x + \left(2x^2 - \frac{1}{2}\right) \sin 2x$$

38. 设连续函数 $f(x)$ 满足

$$f(x) = \frac{1}{4}x^2 + \frac{1}{2} \cos x - \int_0^x (x-t)f(t) dt,$$

求 $f(x)$ 。

即解微分方程

$$y''(x) + y(x) = \frac{1 - \cos x}{2}, \quad y(0) = \frac{1}{2}, \quad y'(0) = 0$$

通解为

$$y = y_h + y_p = C_1 \cos x + C_2 \sin x + \frac{1}{2} - \frac{1}{4}x \sin x$$

代入初值

$$y(0) = C_1 + \frac{1}{2} = \frac{1}{2} \Rightarrow C_1 = 0; \quad y'(0) = C_2 + 0 = 0 \Rightarrow C_2 = 0$$

因此

$$f(x) = \frac{1}{2} - \frac{1}{4}x \sin x$$

39. 考虑二阶线性齐次方程

$$y'' + p(x)y' + q(x)y = 0,$$

其中 $p(x), q(x)$ 在某区间 I 上连续。

(a) 证明: 若存在常数 r 使得对所有 $x \in I$ 有

$$r^2 + rp(x) + q(x) = 0,$$

则 $y(x) = e^{rx}$ 为该方程的一个解。

直接代入检验, 对于 $y = e^{rx}$,

$$y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

代入方程得

$$y'' + p(x)y' + q(x)y = (r^2 + rp(x) + q(x))e^{rx} = 0,$$

因此 $y = e^{rx}$ 确为方程的解。

(b) 求出当

$$p(x) = -2 \left(1 + \frac{1}{x} \right), \quad q(x) = 1 + \frac{2}{x}$$

时方程的通解。

先求常数根 r . 将形式 $r^2 + rp(x) + q(x) = 0$ 代入:

$$r^2 + r \left(-2 \left(1 + \frac{1}{x} \right) \right) + 1 + \frac{2}{x} = 0 \Rightarrow (r^2 - 2r + 1) + \frac{2}{x}(1 - r) = 0.$$

该等式对所有 x 成立, 比较系数得

$$r^2 - 2r + 1 = 0, \quad 1 - r = 0.$$

由第二式得 $r = 1$, 并满足第一式, 于是其中一解为

$$y_1(x) = e^x$$

由降阶法公式, 第二个线性独立解为

$$y_2 = y_1 \int \frac{1}{y_1^2} \exp \left(\int p(x) dx \right) dx = e^x \int e^{-2x} \exp \left(2 \int \left(1 + \frac{1}{x} \right) dx \right) = e^x \int x^2 dx = \frac{1}{3} x^3 e^x$$

因此通解为

$$y(x) = (C_1 + C_2 x^3) e^x$$

40. 以换元

$$x = \int e^{-\frac{t^2}{2}} dt,$$

解微分方程

$$y''(t) + ty'(t) + e^{-t^2} y(t) = 0$$

令 $z(x) = y(t)$, 由链导法,

$$y' = \frac{dy}{dt} = \frac{dz}{dx} \frac{dx}{dt} = e^{-\frac{t^2}{2}} \frac{dz}{dx}$$

且

$$\begin{aligned} y'' &= \frac{d}{dt} \left(e^{-\frac{t^2}{2}} \frac{dz}{dx} \right) = -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-\frac{t^2}{2}} \frac{d}{dt} \left(\frac{dz}{dx} \right) \\ &= -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-\frac{t^2}{2}} \frac{d}{dx} \left(\frac{dz}{dx} \right) \frac{dx}{dt} \\ &= -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-\frac{t^2}{2}} \frac{d^2 z}{dx^2} e^{-\frac{t^2}{2}} \\ &= -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-t^2} \frac{d^2 z}{dx^2}. \end{aligned}$$

把 y', y'' 代入原方程得

$$\left(-te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-t^2} \frac{d^2z}{dx^2}\right) + t \left(e^{-\frac{t^2}{2}} \frac{dz}{dx}\right) + e^{-t^2} z = e^{-t^2} \frac{d^2z}{dx^2} + e^{-t^2} z = 0 \Rightarrow \frac{d^2z}{dx^2} + z = 0$$

解为

$$z(x) = C_1 \cos x + C_2 \sin x \Rightarrow y(t) = C_1 \cos \left(\int e^{-\frac{t^2}{2}} dt\right) + C_2 \sin \left(\int e^{-\frac{t^2}{2}} dt\right)$$

41. 解方程

$$ty''(t) + (t^2 - 1)y'(t) + t^3y(t) = 0$$

欲求合适的 $x = v(t)$, 令 $z(x) = y(t)$, 则

$$y' = \frac{dy}{dt} = \frac{dz}{dx} \frac{dx}{dt} = \frac{dz}{dx} \frac{dv}{dt},$$

且

$$\begin{aligned} y'' &= \frac{d}{dt} \left(\frac{dv}{dt} \frac{dz}{dx} \right) = \frac{d^2v}{dt^2} \frac{dz}{dx} + \frac{dv}{dt} \frac{d}{dt} \left(\frac{dz}{dx} \right) \\ &= \frac{d^2v}{dt^2} \frac{dz}{dx} + \frac{dv}{dt} \frac{d}{dx} \left(\frac{dz}{dx} \right) \frac{dx}{dt} \\ &= \frac{d^2v}{dt^2} \frac{dz}{dx} + \left(\frac{dv}{dt} \right)^2 \frac{d^2z}{dx^2} \end{aligned}$$

代入原方程, 得到

$$t \left(\frac{dv}{dt} \right)^2 \frac{d^2z}{dx^2} + \left[t \frac{d^2v}{dt^2} + (t^2 - 1) \frac{dv}{dt} \right] \frac{dz}{dx} + t^3z = 0$$

为使方程为常系数方程, 不妨考虑变换

$$t \left(\frac{dv}{dt} \right)^2 = t^3 \Rightarrow \frac{dv}{dt} = t \Rightarrow v = \frac{t^2}{2}$$

此时原方程可化为

$$\frac{d^2z}{dx^2} + \frac{dz}{dx} + z = 0$$

其通解为

$$z(x) = e^{-\frac{x}{2}} \left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right)$$

代回 $x = \frac{t^2}{2}$, 即

$$y(t) = e^{-\frac{t^2}{4}} \left(C_1 \cos \frac{\sqrt{3}t^2}{4} + C_2 \sin \frac{\sqrt{3}t^2}{4} \right)$$

42. 设 y_1, y_2 为方程

$$y'' + p(x)y' + q(x)y = 0, \quad x > 0 \quad (1)$$

的两个解, 其中 $p(x), q(x)$ 在 $x > 0$ 上连续。已知

$$y_1 = \frac{\sin x}{\sqrt{x}}$$

且 y_1, y_2 的朗斯基行列式 (Wronskian) 为

$$W(x) = \frac{1}{x}$$

(a) 证明阿贝尔恒等式 (Abel's Identity):

$$W(x) = C \exp \left(- \int p(x) dx \right)$$

由朗斯基行列式的定义,

$$W(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2$$

于是

$$\begin{aligned} W'(x) &= (y_1 y_2' - y_1' y_2)' = (y_1' y_2' + y_1 y_2'') - (y_1'' y_2 + y_1' y_2') \\ &= y_1 y_2'' - y_1'' y_2 = y_1(-p y_2' - q y_2) - (-p y_1' - q y_1) y_2 \\ &= -p y_1 y_2' + p y_1' y_2 = -p(y_1 y_2' - y_1' y_2) = -p(x)W(x) \end{aligned}$$

因此

$$W'(x) + p(x)W(x) = 0$$

由此可解得

$$W(x) = C \exp \left(- \int p(x) dx \right)$$

即得证。

(b) 求方程 (1) 中的函数 $p(x), q(x)$ 。

由阿贝尔恒等式,

$$p(x) = -\frac{W'(x)}{W(x)} = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$$

由于 y_1 是原方程的解,

$$y_1'' + py_1' + qy_1 = 0 \Rightarrow q(x) = -\frac{y_1'' + py_1'}{y_1} = 1 - \frac{1}{4x^2}$$

其中

$$y_1' = \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{\frac{3}{2}}}, \quad y_1'' = \frac{3 \sin x}{4x^{\frac{5}{2}}} - \frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x^{\frac{3}{2}}}$$

因此原微分方程为

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0, \quad x > 0$$

(c) 求方程 (1) 的通解。

利用朗斯基行列式求 y_2 。已知

$$W(x) = y_1y_2' - y_1'y_2 = \frac{1}{x} \Rightarrow y_2' - \frac{y_1'}{y_1}y_2 = \frac{1}{xy_1}$$

即为一阶线性方程

$$y_2' + \left(\frac{1}{2x} - \cot x\right)y_2 = \frac{1}{x \sin x}.$$

其积分因子为

$$I(x) = \exp\left(\int \left(\frac{1}{2x} - \cot x\right) dx\right) = \exp(\ln \sqrt{x} - \ln |\sin x|) = \pm \frac{\sqrt{x}}{\sin x}$$

取 $I(x) = \frac{\sqrt{x}}{\sin x}$, 由此

$$\begin{aligned} y_2(x) &= \frac{\sin x}{\sqrt{x}} \left(\int \frac{\sqrt{x}}{\sin x} \cdot \frac{1}{x \sin x} dx + C \right) \\ &= \frac{\sin x}{\sqrt{x}} \left(\int \frac{1}{\sin^2 x} dx + C \right) \\ &= \frac{\sin x}{\sqrt{x}} (-\cot x + C) \\ &= -\frac{\cos x}{\sqrt{x}} + Cy_1 \end{aligned}$$

因此通解为

$$y(x) = C_1 \frac{\sin x}{\sqrt{x}} + C_2 \frac{\cos x}{\sqrt{x}}.$$

43. (a) 证明法国数学家刘维尔 (Liouville) 的结论: 若黎卡提 (Riccati) 方程

$$y' = p(x)y^2 + q(x)y + r(x)$$

已知一个特解 $\bar{y}(x)$, 令 $y = z + \bar{y}$, 则用此代换可把方程化为一个阶数 $n = 2$ 的伯努利 (Bernoulli) 方程, 从而可通过解一个一阶线性方程得到一般解。

设 $y = z + \bar{y}$, 代入原方程得

$$z' + \bar{y}' = p(z + \bar{y})^2 + q(z + \bar{y}) + r = pz^2 + 2pz\bar{y} + qz + (p\bar{y}^2 + q\bar{y} + r)$$

由于 \bar{y} 是原方程的一个解, 故 $\bar{y}' = p\bar{y}^2 + q\bar{y} + r$, 于是

$$z' = pz^2 + 2pz\bar{y} + qz \Rightarrow z' - (2p\bar{y} + q)z = pz^2$$

这正是阶数 $n = 2$ 的伯努利方程。由通解公式,

$$z^{1-n} = e^{-\int(1-n)(2p\bar{y}+q)dx} \left\{ \int (1-n)p(x)e^{\int(1-n)(2p\bar{y}+q)dx} dx + C \right\}.$$

代入 $n = 2$ 即得

$$z^{-1} = e^{-\int(2p\bar{y}+q)dx} \left\{ -\int p(x)e^{\int(2p\bar{y}+q)dx} dx + C \right\}.$$

由此得到 z , 再由 $y = z + \bar{y}$ 得原方程的通解, 证毕。

(b) 求下列黎卡提方程的通解:

i. $y'e^{-x} + y^2 - 2ye^x = 1 - e^{2x}$, 已知特解 $\bar{y}(x) = e^x$ 。

原方程可写为

$$y' = -e^x y^2 + 2e^{2x} y + e^x(1 - 2e^{2x})$$

因此

$$p(x) = -e^x, \quad q(x) = 2e^{2x}, \quad r(x) = e^x(1 - 2e^{2x}), \quad \bar{y} = e^x$$

且

$$\int (2p\bar{y} + q) dx = \int (2(-e^x)e^x + 2e^{2x}) dx = 0$$

于是上式积分指数因子为常数, 代入公式得

$$z^{-1} = -\int (-e^x) dx + C = e^x + C \Rightarrow z = \frac{1}{C + e^x}$$

代回 $y = z + \bar{y}$, 得到通解

$$y = e^x + \frac{1}{C + e^x}$$

ii. $x^2(y' + y^2) = 2$, 已知特解 $\bar{y}(x) = -\frac{1}{x}$ 。

将方程写成标准形,

$$y' = -y^2 + \frac{2}{x^2}$$

令 $y = z + \bar{y} = z - \frac{1}{x}$, 代入并化简可得关于 z 的方程

$$z' - \frac{2}{x}z = -z^2,$$

是阶数 $n = 2$ 的伯努利方程, 由公式计算得

$$z^{-1} = e^{\int \frac{2}{x} dx} \left(\int e^{-\int \frac{2}{x} dx} dx + C \right) = \frac{1}{x^2} \left(\int x^2 dx + C \right) = \frac{1}{x^2} \left(\frac{x^3}{3} + C \right)$$

即

$$z = \frac{3x^2}{x^3 + 3C}$$

代回 $y = z - \frac{1}{x}$, 化为

$$y = \frac{3x^2}{x^3 + 3C} - \frac{1}{x} = \frac{2x^3 - 3C}{x(3x^3 + 3C)}.$$

令常数 $C_1 = 3C$,

$$xy = \frac{2x^3 - C_1}{3x^3 + C_1}$$

44. 求解

$$x^3 - \frac{dy}{dx} = \frac{y}{x}(y - 2)$$

发现此为一黎卡提方程, 观察得 $y_1 = x^2$ 是一特解。设 $y = x^2 + \frac{1}{u}$, 则 $\frac{dy}{dx} = 2x - \frac{1}{u^2} \frac{du}{dx}$, 原方程给出

$$x^3 - \left(2x - \frac{1}{u^2} \frac{du}{dx} \right) = x(x^2 - 2) + \frac{2(x^2 - 1)}{xu} + \frac{1}{xu^2}$$

再化简得一阶线性方程

$$\frac{du}{dx} - \left(2x - \frac{2}{x} \right) u = \frac{1}{x}$$

积分因子为

$$I(x) = \exp \left(\int - \left(2x - \frac{2}{x} \right) dx \right) = \exp (-x^2 + 2 \ln |x|) = x^2 e^{-x^2}$$

所以

$$u = \frac{e^{-x^2}}{x^2} \left(\int x e^{-x^2} + C_1 \right) = \frac{C_1 e^{x^2} - \frac{1}{2}}{x^2}$$

回代 $y = x^2 + \frac{1}{u}$ 得

$$y = x^2 + \frac{x^2}{C_1 e^{x^2} - \frac{1}{2}} = x^2 \frac{C_1 e^{x^2} + \frac{1}{2}}{C_1 e^{x^2} - \frac{1}{2}}.$$

令 $C_2 = 2C_1$, 则等价地写成

$$y = x^2 \frac{C_2 e^{x^2} + 1}{C_2 e^{x^2} - 1}$$

45. 考虑柯西-欧拉方程 (Cauchy-Euler Equation)

$$ax^2 y'' + bxy' + cy = 0, \quad x > 0,$$

一般解法为作变换 $x = e^t$ 或 $t = \ln x$, 则原方程变为常系数方程

$$az''(t) + (b-a)z'(t) + cz(t) = 0,$$

解得 $z(t)$ 后代回 $y(x) = z(\ln x)$ 。据此, 求解

(a) $x^2 y'' - 3xy' + 4y = 0$

方程为 $x^2 y'' - 3xy' + 4y = 0$, 所以 $a = 1, b = -3, c = 4$, 对应 z 方程为

$$z'' - 4z' + 4z = 0$$

特征方程为

$$r^2 - 4r + 4 = (r - 2)^2 = 0 \Rightarrow r = 2$$

因此解得

$$z(t) = C_1 e^{2t} + C_2 t e^{2t} \Rightarrow y(x) = z(\ln x) = C_1 x^2 + C_2 x^2 \ln x$$

(b) $x^2 y'' - xy' - 35y = 0$

方程为 $x^2 y'' - xy' - 35y = 0$, 所以 $a = 1, b = -1, c = -35$ 。对应 z 方程

$$z'' + (b-a)z' + cz = z'' + (-1-1)z' - 35z = z'' - 2z' - 35z = 0$$

其中特征方程为

$$r^2 - 2r - 35 = (r - 7)(r + 5) = 0 \Rightarrow r_1 = 7, r_2 = -5$$

因此通解为

$$z(t) = C_1 e^{7t} + C_2 e^{-5t} \Rightarrow y(x) = z(\ln x) = C_1 x^7 + C_2 x^{-5}$$

46. 解非齐次欧拉方程

$$x^3 y''' + x^2 y'' - x y' = 24x \ln x, \quad x > 0$$

令 $x = e^t \Rightarrow t = \ln x$, 由链导法,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \\ \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dt} \frac{dt}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \cdot \frac{dy}{dt} \right) = -\frac{dy}{x^2 dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \frac{1}{x} = \frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \\ \frac{d^3 y}{dx^3} &= \frac{d}{dx} \left(\frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \right) = -\frac{2}{x^3} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + \frac{1}{x^2} \left(\frac{d^3 y}{dt^3} \cdot \frac{1}{x} - \frac{d^2 y}{dt^2} \cdot \frac{1}{x} \right) \\ &= \frac{1}{x^3} \frac{d^3 y}{dt^3} - \frac{3}{x^3} \frac{d^2 y}{dt^2} + \frac{2}{x^3} \frac{dy}{dt} \end{aligned}$$

代入方程得到

$$\frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y = 24e^{tt}$$

齐次方程 $y''' - 3y'' + 3y' - y = 0$ 的特征方程

$$r^3 - 3r^2 + 3r - 1 = 0 \Rightarrow r = 1$$

故齐次解为

$$y_c = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

设特解为

$$y_p = t^3 (At + B) e^t = (At^4 + Bt^3) e^t$$

代入方程解得

$$24At + 6B = 24t \Rightarrow A = 1, B = 0$$

因此通解为

$$y = t^4 e^t + C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

代回 $t = \ln x$, 原方程的解为

$$y = x(\ln x)^4 + C_1 x + C_2 x \ln x + C_3 x (\ln x)^2$$

47. 以参数变换法解

$$y'' + 6y' + 9y = \frac{2e^{-4x}}{x^2 + 1}.$$

对应齐次方程通解为

$$y_c(x) = C_1 e^{-3x} + C_2 x e^{-3x}$$

设特解为

$$y_p(x) = u_1(x)e^{-3x} + u_2(x)xe^{-3x}$$

其中 u_1, u_2 满足

$$\begin{bmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2e^{-4x}}{x^2+1} \end{bmatrix}$$

朗斯基行列式为

$$W(y_1, y_2) = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{vmatrix} = e^{-6x}$$

由克兰姆法则,

$$u'_1 = \frac{-xe^{-3x} \cdot \frac{2e^{-4x}}{x^2+1}}{e^{-6x}} = -\frac{2x}{x^2+1} \Rightarrow u_1 = -\ln(1+x^2),$$

$$u'_2 = \frac{e^{-3x} \cdot \frac{2e^{-4x}}{x^2+1}}{e^{-6x}} = \frac{2}{x^2+1} \Rightarrow u_2 = 2 \tan^{-1} x.$$

因此特解为

$$y_p(x) = e^{-3x} (-\ln(1+x^2) + 2x \tan^{-1} x)$$

通解为

$$y(x) = e^{-3x} (C_1 + C_2 x + 2x \tan^{-1} x - \ln(1+x^2))$$

48. 求解

$$y'' - 2y' + y = 4e^x x^3 \ln x, \quad x > 0$$

对应齐次方程的通解为

$$y_c(x) = C_1 e^x + C_2 x e^x,$$

设特解为

$$y_p(x) = u_1(x)e^x + u_2(x)xe^x,$$

其中 u_1, u_2 满足

$$\begin{bmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4e^x x^3 \ln x \end{bmatrix}.$$

两边同时除以 e^x 得

$$\begin{bmatrix} 1 & x \\ 1 & x+1 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4x^3 \ln x \end{bmatrix}.$$

朗斯基行列式为 $(x+1) - x = 1$, 由克兰姆法则,

$$u'_1 = -x \cdot 4x^{-2} \ln x = -4x^{-2} \ln x, \quad u'_2 = 4x^{-3} \ln x$$

于是

$$\begin{aligned} u_1 &= -4 \int x^{-2} \ln x \, dx = \frac{4 \ln x}{x} - 4 \int \frac{1}{x^2} \, dx = \frac{4(\ln x + 1)}{x} \\ u_2 &= 4 \int x^{-3} \ln x \, dx = -\frac{2 \ln x}{x^2} + 2 \int \frac{1}{x^3} \, dx = \frac{-2 \ln x - 1}{x^2} \end{aligned}$$

因此特解为

$$y_p(x) = e^x u_1 + x e^x u_2 = e^x \left(\frac{4(\ln x + 1)}{x} + x \cdot \frac{-2 \ln x - 1}{x^2} \right) = e^x \frac{2 \ln x + 3}{x}$$

所以方程通解为

$$y(x) = e^x \left(C_1 + C_2 x + \frac{2 \ln x + 3}{x} \right)$$

49. 求解方程

$$y'' + 4y' + 4y = 15e^{-2x} \ln x + 25 \cos x, \quad x > 0$$

对应齐次方程的通解为

$$y_c(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

第一项非齐次 $f_1(x) = 15e^{-2x} \ln x$: 设特解为

$$y_{p1} = u_1(x) e^{-2x} + u_2(x) x e^{-2x}$$

其中 u_1, u_2 满足

$$\begin{bmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 15e^{-2x} \ln x \end{bmatrix}$$

两边除以 e^{-2x} 得

$$\begin{bmatrix} 1 & x \\ -2 & 1-2x \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \ln x \end{bmatrix}$$

解得

$$u'_1 = -15x \ln x \Rightarrow u_1 = -15 \int x \ln x dx = -\frac{15}{2}x^2 \ln x + 15 \int \frac{x}{2} dx = \frac{15}{4}x^2(-2 \ln x + 1)$$

$$u'_2 = 15 \ln x \Rightarrow u_2 = 15 \int \ln x dx = 15x(\ln x - 1)$$

因此

$$y_{p1}(x) = e^{-2x} \left[\frac{15}{4}x^2(-2 \ln x + 1) + 15x(\ln x - 1) \right] = \frac{15}{4}x^2(2 \ln x - 3)e^{-2x}$$

第二项非齐次 $f_2(x) = 25 \cos x$: 设特解

$$y_{p2}(x) = A \cos x + B \sin x$$

代入方程得

$$(3A + 4B) \cos x + (-4A + 3B) \sin x = 25 \cos x$$

比较系数解得 $A = 3, B = 4$, 所以

$$y_{p2}(x) = 3 \cos x + 4 \sin x$$

综上, 方程通解为

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{15}{4}x^2(2 \ln x - 3)e^{-2x} + 3 \cos x + 4 \sin x$$

50. 解

$$y''' - y'' + y' - y = e^{-x} \sin x$$

齐次方程

$$y''' - y'' + y' - y = 0$$

的特征方程为

$$(r-1)(r^2+1)=0 \Rightarrow r=1, \pm i$$

所以齐次解为

$$y_c(x) = C_1 e^x + C_2 \cos x + C_3 \sin x$$

对于非齐次项 $e^{-x} \sin x$, 尝试特解

$$y_p(x) = e^{-x}(A \cos x + B \sin x)$$

则

$$y_p' = -e^{-x}((A+B) \sin x + (A-B) \cos x)$$

$$y_p'' = 2e^{-x}(A \sin x + B \cos x)$$

$$y_p''' = 2e^{-x}((B-A) \sin x + (A+B) \cos x)$$

代入原方程解得 $A = \frac{1}{5}, B = 0$, 于是

$$y_p(x) = -\frac{1}{5}e^{-x} \cos x$$

因此通解为

$$y(x) = C_1 e^x + C_2 \cos x + C_3 \sin x - \frac{1}{5}e^{-x} \cos x$$

51. 求解微分方程

$$y''' + 3y'' + 3y' + y = \frac{2e^{-x}}{1+x^2}$$

齐次解为

$$y_c(x) = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$$

使用参数变换法, 设

$$y_p(x) = u_1(x)e^{-x} + u_2(x)xe^{-x} + u_3(x)x^2e^{-x}$$

其中 u_1, u_2, u_3 满足

$$\begin{cases} e^{-x}u_1' + (xe^{-x})u_2' + (x^2e^{-x})u_3' = 0 \\ (e^{-x})'u_1' + (xe^{-x})'u_2' + (x^2e^{-x})'u_3' = 0 \\ (e^{-x})''u_1' + (xe^{-x})''u_2' + (x^2e^{-x})''u_3' = \frac{2e^{-x}}{1+x^2} \end{cases}$$

化简得

$$\begin{cases} u_1' + xu_2' + x^2u_3' = 0 \\ -u_1' + (1-x)u_2' + (2x-x^2)u_3' = 0 \\ u_1' + (-2+x)u_2' + (2-4x+x^2)u_3' = \frac{2}{1+x^2} \end{cases}$$

其中增广矩阵为

$$\left[\begin{array}{ccc|c} 1 & x & x^2 & 0 \\ -1 & 1-x & 2x-x^2 & 0 \\ 1 & -2+x & 2-4x+x^2 & \frac{2}{1+x^2} \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & x & x^2 & 0 \\ 0 & 1 & 2x & 0 \\ 0 & 0 & 1 & \frac{1}{1+x^2} \end{array} \right]$$

逐步解得

$$u_3' = \frac{1}{1+x^2} \Rightarrow u_3 = \tan^{-1} x$$

$$u_2' = -2xu_3' = -\frac{2x}{1+x^2} \Rightarrow u_2 = -\ln(1+x^2)$$

$$u_1' = -xu_2' - x^2u_3' = \frac{x^2}{1+x^2} \Rightarrow u_1 = x - 2\tan^{-1} x$$

于是特解为

$$y_p = e^{-x} [(2x - 2\tan^{-1} x) + (-\ln(1+x^2))x + (\tan^{-1} x)x^2]$$

整理得通解为

$$y(x) = e^{-x} [C_1 + C_2x + C_3x^2 + (x^2 - 2)\tan^{-1} x + x - x\ln(1+x^2)]$$

52. 解方程

$$y''' + y' = \sec x$$

齐次方程 $y''' + y' = 0$ 的特征方程为

$$r^3 + r = 0 \Rightarrow r = 0, \pm i$$

所以齐次解为

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

使用参数变易法, 设特解为

$$y_p = u_1y_1 + u_2y_2 + u_3y_3$$

其中 u_1, u_2, u_3 满足

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sec x \end{bmatrix}$$

其中增广矩阵为

$$\left[\begin{array}{ccc|c} 1 & \cos x & \sin x & 0 \\ 0 & -\sin x & \cos x & 0 \\ 0 & -\cos x & -\sin x & \sec x \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \cos x & \sin x & 0 \\ 0 & -\sin x & \cos x & 0 \\ 1 & 0 & 0 & \sec x \end{array} \right]$$

于是

$$\begin{cases} u'_1 = \sec x \\ u'_2 = \frac{\cos x}{-\sin x} u'_3 = -1 \\ u'_3 = \frac{\sin x}{\cos x} \end{cases} \Rightarrow \begin{cases} u_1 = |\sec x + \tan x| \\ u_2 = -x \\ u_3 = \ln |\cos x| \end{cases}$$

因此通解为

$$y = |\sec x + \tan x| - x \cos x + \ln |\cos x| \sin x + C_1 + C_2 \cos x + C_3 \sin x$$

53. 解联立微分方程式

$$\begin{cases} x'_1 = x_1 + 2x_2 \\ x'_2 = 2x_1 - 2x_2 \end{cases}$$

$$x'_1 = x_1 + 2x_2 \quad (1)$$

$$x'_2 = 2x_1 - 2x_2 \quad (2)$$

对 (1) 求导, 且由 (2),

$$x''_1 = x'_1 + 2x'_2 = x'_1 + 2(2x_1 - 2x_2) = x'_1 + 4x_1 - 4x_2 = x'_1 + 4x_1 - 2(x'_1 - x_1) = -x'_1 + 6x_1$$

得关于 x_1 的二阶常微分方程, 因此 x_1 的通解为

$$x_1(t) = C_1 e^{-3t} + C_2 e^{2t}$$

由 (1),

$$x_2 = \frac{1}{2}(x_1' - x_1) = -2C_1e^{-3t} + \frac{C_2}{2}e^{2t}$$

最终将系统的通解写成向量形式:

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{2t}$$

54. 求解

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$$

设系数矩阵为 \mathbf{A} , 首先求 \mathbf{A} 的特征值, 通过解

$$\det(\mathbf{A} - \lambda \mathbf{I}_3) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda + 1)^2(\lambda - 2)$$

因此特征值为

$$\lambda_1 = -1 \ (m_1 = 2), \quad \lambda_2 = 2$$

接着求特征向量。对 $\lambda_1 = -1$, 解

$$(\mathbf{A} - \lambda_1 \mathbf{I}_3)\mathbf{v} = \mathbf{0} \implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

得到方程

$$v_1 + v_2 + v_3 = 0$$

有两个自由变量, 设 $v_2 = r, v_3 = s$, 则 $v_1 = -r - s$, 因此特征空间由两个线性无关向量张成:

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

对应解为

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

对 $\lambda_2 = 2$, 解

$$(\mathbf{A} - \lambda_2 \mathbf{I}_3) \mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

化简矩阵得到行阶梯形:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{2}R_1 + R_2} \begin{pmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xrightarrow[R_3 \rightarrow \frac{2}{3}R_3]{R_3 \rightarrow R_2 + R_3} \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

秩为 2, 设 $v_3 = r$, 则 $v_1 = v_2 = r$, 得到特征向量

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

对应解为

$$\mathbf{x}_3(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

因此系统的一般解为

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

55. 求解

$$\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$$

设系数矩阵为 \mathbf{A} 。首先求特征值, 通过解

$$\det(\mathbf{A} - \lambda \mathbf{I}_3) = \begin{vmatrix} -3 - \lambda & 0 & 2 \\ 1 & -1 - \lambda & 0 \\ -2 & -1 & -\lambda \end{vmatrix} = -(\lambda + 2)(\lambda^2 + 2\lambda + 3)$$

得特征值

$$\lambda_{1,2} = -1 \pm \sqrt{2}i, \quad \lambda_3 = -2$$

接着求对应特征向量。对 $\lambda_1 = -1 + \sqrt{2}i$, 解

$$(\mathbf{A} - \lambda_1 \mathbf{I}_3) \mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -2 - \sqrt{2}i & 0 & 2 \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1 - \sqrt{2}i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

化为行阶梯形矩阵

$$\begin{pmatrix} -2 - \sqrt{2}i & 0 & 2 \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1 - \sqrt{2}i \end{pmatrix} \xrightarrow{R_1 \rightarrow (-2 + \sqrt{2}i)R_1/6} \begin{pmatrix} 1 & 0 & \frac{1}{3}(-2 + \sqrt{2}i) \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1 - \sqrt{2}i \end{pmatrix}$$
$$\xrightarrow[\begin{smallmatrix} R_1 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + 2R_1 \end{smallmatrix}]{\begin{smallmatrix} R_2 \rightarrow R_2 / (\sqrt{2}i) \\ R_3 \rightarrow -R_3 \end{smallmatrix}} \begin{pmatrix} 1 & 0 & \frac{1}{3}(-2 + \sqrt{2}i) \\ 0 & -\sqrt{2}i & \frac{1}{3}(2 - \sqrt{2}i) \\ 0 & -1 & -\frac{1}{3}(1 + \sqrt{2}i) \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & \frac{1}{3}(-2 + \sqrt{2}i) \\ 0 & 1 & \frac{1}{3}(1 + \sqrt{2}i) \\ 0 & 0 & 0 \end{pmatrix}$$

令 $v_3 = r$ 为自由变量, 则

$$\mathbf{v} = r \begin{pmatrix} -\frac{1}{3}(-2 + \sqrt{2}i) \\ -\frac{1}{3}(1 + \sqrt{2}i) \\ 1 \end{pmatrix}.$$

取 $r = -3$ 并利用欧拉公式分离实部和虚部

$$\begin{aligned} e^{(-1+\sqrt{2}i)t} \begin{pmatrix} -2 + \sqrt{2}i \\ 1 + \sqrt{2}i \\ -3 \end{pmatrix} &= e^{-t}(\cos(\sqrt{2}t) + i \sin(\sqrt{2}t)) \begin{pmatrix} -2 + \sqrt{2}i \\ 1 + \sqrt{2}i \\ -3 \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) + i(\sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t)) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) + i(\sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t)) \\ -3 \cos(\sqrt{2}t) - 3i \sin(\sqrt{2}t) \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ -3 \cos(\sqrt{2}t) \end{pmatrix} + ie^{-t} \begin{pmatrix} \sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3 \sin(\sqrt{2}t) \end{pmatrix} \end{aligned}$$

得到两个线性无关解

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ -3 \cos(\sqrt{2}t) \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} \sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3 \sin(\sqrt{2}t) \end{pmatrix}$$

对 $\lambda_3 = -2$, 解

$$(\mathbf{A} - \lambda_3 \mathbf{I}_3) \mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

化简矩阵得到

$$\begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 + R_1} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

令 $v_3 = s$, 得到特征向量

$$\mathbf{v}_3 = s \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

对应解为

$$\mathbf{x}_3(t) = e^{-2t} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

因此系统的一般解为

$$\begin{aligned} \mathbf{x}(t) = & c_1 e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ -3 \cos(\sqrt{2}t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3 \sin(\sqrt{2}t) \end{pmatrix} \\ & + c_3 e^{-2t} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

56. 判断矩阵

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

是否缺陷, 并求系统 $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ 的通解。

首先求 \mathbf{A} 的特征值:

$$\det(\mathbf{A} - \lambda \mathbf{I}_3) = \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & 0 & 4 - \lambda \end{vmatrix} = (1 - \lambda)^2(4 - \lambda)$$

因此特征值为 $\lambda_1 = 4, \lambda_2 = 1$ ($m_2 = 2$)。对于 $\lambda_1 = 4$, 解

$$(\mathbf{A} - 4\mathbf{I}_3)\mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

得到特征向量及其对应解

$$\mathbf{v} = r \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad r \neq 0, \quad \mathbf{x}_1(t) = e^{4t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

对于 $\lambda_2 = 1$, 解

$$(\mathbf{A} - \mathbf{I}_3)\mathbf{v} = \mathbf{0} \implies \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

得到特征向量及其对应解

$$\mathbf{v} = s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s \neq 0, \quad \mathbf{x}_2(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

总共有两个线性无关特征向量, 因此矩阵 \mathbf{A} 是缺陷的。为求第三个线性无关解, 设

$$\mathbf{x}_3(t) = (\mathbf{v}t + \mathbf{w})e^t$$

其中 \mathbf{v} 和 \mathbf{w} 满足

$$(\mathbf{A} - \mathbf{I}_3)\mathbf{v} = \mathbf{0}, \quad (\mathbf{A} - \mathbf{I}_3)\mathbf{w} = \mathbf{v} \implies (\mathbf{A} - \mathbf{I}_3)^2\mathbf{w} = \mathbf{0}$$

解得

$$(\mathbf{A} - \mathbf{I}_3)^2 \mathbf{w} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies w_3 = 0$$

自由选择 w_1, w_2 , 取

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = (\mathbf{A} - \mathbf{I}_3) \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

因此得到第三个解

$$\mathbf{x}_3(t) = (\mathbf{v}t + \mathbf{w})e^t = e^t \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

最终通解为

$$\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

57. 求解非齐次系统

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{F}(t),$$

其中

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix}, \quad \mathbf{F}(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

首先求 \mathbf{A} 的特征值和特征向量:

$$\det(\mathbf{A} - \lambda \mathbf{I}_2) = \begin{vmatrix} -\lambda & 1 \\ -9 & 6 - \lambda \end{vmatrix} = -\lambda(6 - \lambda) + 9 = (\lambda - 3)^2$$

因此特征值为 $\lambda = 3$ ($m = 2$), 对 $\lambda = 3$, 解

$$(\mathbf{A} - 3\mathbf{I}_2) \mathbf{v} = \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得到特征向量

$$\mathbf{v} = r \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad r \neq 0$$

对应解为

$$\mathbf{x}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

矩阵 \mathbf{A} 是缺陷的, 第二个线性无关解取形如

$$\mathbf{x}_2(t) = (\mathbf{v}t + \mathbf{w})e^{3t}$$

其中 \mathbf{v} 和 \mathbf{w} 满足

$$(\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{v} = \mathbf{0}, \quad (\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{w} = \mathbf{v} \implies (\mathbf{A} - \lambda \mathbf{I}_2)^2 \mathbf{w} = \mathbf{0}$$

解得

$$\begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

取

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \mathbf{v} = (\mathbf{A} - 3\mathbf{I}_2)\mathbf{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

于是第二个解为

$$\mathbf{x}_2(t) = (\mathbf{v}t + \mathbf{w})e^{3t} = \begin{pmatrix} t \\ 3t + 1 \end{pmatrix} e^{3t}$$

因此齐次系统通解为

$$\mathbf{x}_c(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} t \\ 3t + 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t + 1)e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

记基础矩阵为 \mathbf{X} , 寻找特解

$$\mathbf{x}_p(t) = \mathbf{X}\mathbf{u} \implies \mathbf{X}\mathbf{u}' = \mathbf{F} \implies \mathbf{u}' = \mathbf{X}^{-1}\mathbf{F}$$

计算得到

$$\begin{aligned} \mathbf{X}^{-1}\mathbf{F} &= e^{-3t} \begin{pmatrix} 3t + 1 & -t \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} = e^{-3t} \begin{pmatrix} -t^2 \\ t \end{pmatrix} \\ \mathbf{u} &= \int \mathbf{X}^{-1}\mathbf{F} dt = \begin{pmatrix} \int -t^2 e^{-3t} dt \\ \int t e^{-3t} dt \end{pmatrix} = e^{-3t} \begin{pmatrix} \frac{1}{3}t^2 + \frac{2}{9}t + \frac{2}{27} \\ -\frac{1}{3}t - \frac{1}{9} \end{pmatrix} \end{aligned}$$

于是得到特解

$$\mathbf{x}_p(t) = \mathbf{X}\mathbf{u} = \begin{pmatrix} 1 & t \\ 3 & 3t + 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3}t^2 + \frac{2}{9}t + \frac{2}{27} \\ -\frac{1}{3}t - \frac{1}{9} \end{pmatrix} = \begin{pmatrix} \frac{1}{9}t + \frac{2}{27} \\ \frac{1}{9} \end{pmatrix}.$$

最终非齐次系统通解为

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_c(t) = \begin{pmatrix} \frac{1}{9}t + \frac{2}{27} \\ \frac{1}{9} \end{pmatrix} + c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} t \\ 3t + 1 \end{pmatrix}.$$

参考出处



- Arts of Problem Solving (AoPS): Contest Collections
- Brilliant (Kudos to the unmonetized version in the past)
- International Mathematics Competition (IMC) for University Students
- Missouri Collegiate Mathematics Competition
- American Mathematics Competition 10
- University of Waterloo CEMC - Euclid, Fermat, Cayley, Hypatia, Galois
- Lehigh University High School Math Contest
- Joint Entrance Examination (Advanced)
- The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics
- LetsSolveMathProblems - Weekly Math Challenges
- Maths 505 - Differential Equations
- Michael Penn
- Prime Newtons
- TRML
- MadasMaths
- 日本留学試験-理系数学
- 中国复旦大学往年试题
- 雪隆森中学华罗庚杯数学比赛
- 厦门大学马来西亚分校-陈景润杯中学数学竞赛
- 微积分福音
- 线代启示录
- 福气老师
- 指考历届试题
- 统测历届试题与解答
- 印度人的作业
- 北京高考在线
- 朱式幸福教甄
- 普通型高级中学数学科能力竞赛 (决赛)
- 2014-2024 全国中学生数学竞赛联赛试题及答案汇总
- 如此醉的图书馆
- 菁优网
- 08 高考文科试题分类圆锥曲线
- 08 高考文科试题分类数列
- 福伦-隆中高数笔记
- 曾龙文师-高二数理培训队笔记