

2025 年循人中学高三期末考兼统考预试

高中组

高级数学 (III)

(SC007)

# 训练集

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# 前言

此训练集的雏形要追溯到 2019 年,依稀记得为当时的备赛爬了大量的 AMC 历届考题,创了“AMC 10 精选”的文档,其中我认为是属于统考范围的难题,便收录于另一个文档。

尘封已久,直到今年 5 月,我在整理电脑资料时偶然翻出,看着当年的内容,心想:读了一个数学系,难免觉得发展空间实属不少。在这几个月内,我随性地在网络上寻找、采集、整理更有趣(恶心)、更值得深思(烧脑)的题目,于是诞生了——高级数学 (III) 训练集。

起初,我并没想过要为每一题整理出解析,但发现到不少当年的题目只附答案没附解法。可惜的是,我的脑袋算是停止运作了一年,发现有一些题目对于现在的我实在是难以下咽。好吧,此乃下下策,但我决定用 LaTeX 开始动手为每一题补上解法、重写排版、增加新的内容。也好,既是备忘录,也是为了防止未来的我看到这些题目时会再次怀疑人生:「咦?是不是百尺竿头,更蠢一步?」

依据统考范围,我把训练集大致分为代数、组合数学、几何及微积分。若你想自行动手尝试,可以在 preamble 中将 `\printanswers` 这行注释掉。

绝大部分解法都由官方或我提供,只有某些是 ChatGpt 生成后由我再校对,之中一定会有纰漏,欢迎批评指正,也希望这份训练集能助你一路披荆斩棘、越战越勇!

——冬恒

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总题数: 1613



# 微积分

# 极限

1.

$$\lim_{x \rightarrow 0} \lfloor x \rfloor$$

由于

$$\lim_{x \rightarrow 0^+} \lfloor x \rfloor = 0, \quad \lim_{x \rightarrow 0^-} \lfloor x \rfloor = -1$$

极限  $\lim_{x \rightarrow 0} \lfloor x \rfloor$  不存在。

2.

$$\lim_{x \rightarrow 3} (\lceil x \rceil - \lfloor x \rfloor)$$

由于

$$\lim_{x \rightarrow 3^+} (\lceil x \rceil - \lfloor x \rfloor) = \lim_{x \rightarrow 3^+} \lceil x \rceil - \lim_{x \rightarrow 3^+} \lfloor x \rfloor = 4 - 3 = 1$$

且

$$\lim_{x \rightarrow 3^-} (\lceil x \rceil - \lfloor x \rfloor) = \lim_{x \rightarrow 3^-} \lceil x \rceil - \lim_{x \rightarrow 3^-} \lfloor x \rfloor = 3 - 2 = 1$$

故

$$\lim_{x \rightarrow 3} (\lceil x \rceil - \lfloor x \rfloor) = 1$$

3.

$$\lim_{x \rightarrow 0} \left\lfloor \frac{\sin x}{x} \right\rfloor$$

由于

$$\lim_{x \rightarrow 0^+} \left\lfloor \frac{\sin x}{x} \right\rfloor = \lim_{x \rightarrow 0^-} \left\lfloor \frac{\sin x}{x} \right\rfloor = 0$$

故

$$\lim_{x \rightarrow 0} \left\lfloor \frac{\sin x}{x} \right\rfloor = 0$$

4.

$$\lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x}$$

有

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x} &= \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

5.

$$\lim_{\theta \rightarrow 0} \frac{\theta - \theta \cos \theta}{\sin \theta \tan \theta}$$

有

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\theta - \theta \cos \theta}{\sin \theta \tan \theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\frac{\sin \theta}{\theta}} \cdot \frac{1}{\tan \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} \cdot \frac{1}{\frac{\sin \theta}{\theta}} \cdot \frac{\theta}{\tan \theta} \cdot \theta \\ &= \frac{1}{2} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot 0 \\ &= 0 \end{aligned}$$

6.

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \left( \frac{1 - \cos \theta}{2} \right)}{\theta^4}$$

有

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos \left( \frac{1 - \cos \theta}{2} \right)}{\theta^4} &= \lim_{\theta \rightarrow 0} \left[ \frac{1 - \cos \left( \frac{1 - \cos \theta}{2} \right)}{\left( \frac{1 - \cos \theta}{2} \right)^2} \cdot \left( \frac{1 - \cos \theta}{\theta^2} \right)^2 \cdot \frac{1}{4} \right] \\ &= \frac{1}{2} \cdot \left( \frac{1}{2} \right)^2 \cdot \frac{1}{4} = \frac{1}{32} \end{aligned}$$

其中

$$\lim_{x \rightarrow 0} \frac{1 + \cos x}{x^2} = \frac{1}{4} \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} = \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}$$

7.

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^{-1} x \tan x}$$

有

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^{-1} x \tan x} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^{-1} x \cdot \frac{\sin x}{\cos x}} \\ &= \lim_{x \rightarrow 0} \left(-\frac{1 - \cos x}{x^2}\right) \cdot \frac{x}{\sin^{-1} x} \cdot \frac{x}{\sin x} \cdot \cos x \\ &= -\frac{1}{2} \cdot 1 \cdot 1 \cdot 1 \\ &= -\frac{1}{2} \end{aligned}$$

8. 求极限

$$\lim_{x \rightarrow 1} \left[ \cos\left(\frac{\pi}{2}x\right) + x \cos\left(\frac{\pi}{2}x\right) + x^2 \cos\left(\frac{\pi}{2}x\right) + \dots \right]$$

注意到

$$\begin{aligned} &\lim_{x \rightarrow 1} \left[ \cos\left(\frac{\pi}{2}x\right) + x \cos\left(\frac{\pi}{2}x\right) + x^2 \cos\left(\frac{\pi}{2}x\right) + \dots \right] \\ &= \lim_{x \rightarrow 1} \frac{\cos\left(\frac{\pi}{2}x\right)}{1 - x} \\ &\stackrel{H}{=} \lim_{x \rightarrow 1} \frac{-\frac{\pi}{2} \sin\left(\frac{\pi}{2}x\right)}{-1} \\ &= \frac{\pi}{2} \sin \frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

9. 计算极限

$$\lim_{x \rightarrow 0} \left( \sqrt{\frac{1}{x(x-1)} + \frac{1}{4x^2}} - \frac{1}{2x} \right)$$

有理化得

$$\begin{aligned} \lim_{x \rightarrow 0} \left( \sqrt{\frac{1}{x(x-1)} + \frac{1}{4x^2}} - \frac{1}{2x} \right) &= \lim_{x \rightarrow 0} \frac{\left( \frac{1}{x(x-1)} + \frac{1}{4x^2} \right) - \frac{1}{4x^2}}{\sqrt{\frac{1}{x(x-1)} + \frac{1}{4x^2}} + \frac{1}{2x}} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{x-1}}{\sqrt{\frac{x}{x-1} + \frac{1}{4}} + \frac{1}{2}} \\ &= \frac{-1}{\frac{1}{2} + \frac{1}{2}} \\ &= -1 \end{aligned}$$

10. 试求

$$\lim_{x \rightarrow \infty} \left( \sqrt[5]{x^5 + 3x^4 + 4x^3 + 3x} - \sqrt[3]{x^3 + 3x^2 + 4x + 1} \right).$$

由洛必达法则,

$$\begin{aligned} &\lim_{x \rightarrow \infty} \left( \sqrt[5]{x^5 + 3x^4 + 4x^3 + 3x} - \sqrt[3]{x^3 + 3x^2 + 4x + 1} \right) \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt[5]{1 + \frac{3}{x} + \frac{4}{x^2} + \frac{3}{x^4}} - \sqrt[3]{1 + \frac{3}{x} + \frac{4}{x^2} + \frac{1}{x^3}}}{\frac{1}{x}} \\ &\stackrel{H}{=} - \lim_{x \rightarrow \infty} x^2 \left( \frac{1}{5} \left( 1 + \frac{3}{x} + \frac{4}{x^2} + \frac{3}{x^4} \right)^{-\frac{4}{5}} \left( -\frac{3}{x^2} - \frac{8}{x^3} - \frac{12}{x^5} \right) \right. \\ &\quad \left. - \frac{1}{3} \left( 1 + \frac{3}{x} + \frac{4}{x^2} + \frac{1}{x^3} \right)^{-\frac{2}{3}} \left( -\frac{3}{x^2} - \frac{8}{x^3} - \frac{3}{x^4} \right) \right) \\ &= \frac{3}{5} - 1 = -\frac{2}{5} \end{aligned}$$

11. 已知等差数列  $\{a_n\}, \{b_n\}$  公差均不为 0, 且  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 5$ , 求

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_n}{n \cdot b_{2n}}.$$

设  $\{a_n\}, \{b_n\}$  公差为  $d_1, d_2$ , 则

$$a_n = a_1 + (n-1)d_1, \quad b_n = b_1 + (n-1)d_2.$$

由已知极限得

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{a_1 + (n-1)d_1}{b_1 + (n-1)d_2} = \frac{d_1}{d_2} = 5.$$

所以  $d_1 = 5d_2$ , 因此所求极限为

$$\lim_{n \rightarrow \infty} \frac{\frac{n}{2}(2a_1 + (n-1)d_1)}{n(b_1 + (2n-1)d_2)} = \lim_{n \rightarrow \infty} \frac{2a_1 + (n-1)d_1}{2(b_1 + (2n-1)d_2)} = \frac{d_1}{4d_2} = \frac{5}{4}$$

12. 设  $a_1 + a_2 + \cdots + a_n = n^3 - 2n$ , 试求

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{a_3 + a_6 + \cdots + a_{3n}} - \sqrt[3]{a_2 + a_4 + \cdots + a_{2n}}}{n}$$

由

$$S(n) = \sum_{k=1}^n a_k = n^3 - 2n$$

得

$$a_n = S(n) - S(n-1) = n^3 - 2n - [(n-1)^3 - 2(n-1)] = 3n^2 - 3n - 1$$

于是

$$a_{3k} = 27k^2 - 9k - 1, \quad a_{2k} = 12k^2 - 6k - 1,$$

且

$$\sum_{k=1}^n a_{3k} = \frac{27}{6}n(n+1)(2n+1) - \frac{9}{2}n(n+1) - n = 9n^3 + 9n^2 - n,$$

$$\sum_{k=1}^n a_{2k} = 2n(n+1)(2n+1) - 3n(n+1) - n = 4n^3 + 3n^2 - 2n$$

故极限为

$$\lim_{n \rightarrow \infty} \left( \sqrt[3]{9 + \frac{9}{n} - \frac{1}{n^2}} - \sqrt[3]{4 + \frac{3}{n} - \frac{2}{n^2}} \right) = \sqrt[3]{9} - \sqrt[3]{4}$$

13.

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1+3^{2n}} + \sqrt[n]{3^{2n}+5^{2n}} + \sqrt[n]{5^{2n}+7^{2n}} + \cdots + \sqrt[n]{(2m-1)^{2n}+(2m+1)^{2n}}}{m^3}$$

首先有

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left( \sqrt[n]{1+3^{2n}} + \sqrt[n]{3^{2n}+5^{2n}} + \cdots + \sqrt[n]{(2m-1)^{2n}+(2m+1)^{2n}} \right) \\ &= \lim_{n \rightarrow \infty} \left( 3^2 \sqrt[n]{\left(\frac{1}{3}\right)^{2n}} + 1 + 5^2 \sqrt[n]{\left(\frac{3}{5}\right)^{2n}} + 1 + \cdots + (2m+1)^2 \sqrt[n]{\left(\frac{2m-1}{2m+1}\right)^{2n}} + 1 \right) \\ &= \sum_{k=1}^m (2k+1)^2 \\ &= \sum_{k=1}^m (4k^2 + 4k + 1) \\ &= \frac{2}{3}m(m+1)(2m+1) + 2m(m+1) + m \end{aligned}$$

因此, 原式

$$\lim_{m \rightarrow \infty} \frac{\frac{2}{3}m(m+1)(2m+1) + 2m(m+1) + m}{m^3} = \frac{4}{3}$$

14.

$$\lim_{x \rightarrow 0} \frac{\int_{\cos x}^1 e^{-y^2} dy}{x \sin x}$$

由洛必达法则,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\int_{\cos x}^1 e^{-y^2} dy}{x \sin x} &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin x \cdot e^{-(\cos x)^2}}{\sin x + x \cos x} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x (1 + 2 \sin^2 x) \cdot e^{-(\cos x)^2}}{2 \cos x - x \sin x} \\ &= \frac{1}{2e} \end{aligned}$$

15.

$$\lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 - \sqrt[3]{x})}{1 + \cos \pi x}$$

消去根号, 令  $x = t^6$ , 当  $x \rightarrow 1$  时,  $t \rightarrow 1$ , 由洛必达法则,

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 - \sqrt[3]{x})}{1 + \cos \pi x} &= \lim_{t \rightarrow 1} \frac{(1 - t^3)(1 - t^2)}{1 + \cos \pi t^6} \\&= \lim_{t \rightarrow 1} \frac{1 - t^2 - t^3 + t^5}{1 + \cos \pi t^6} \\&\stackrel{H}{=} \lim_{t \rightarrow 1} \frac{-2t - 3t^2 + 5t^4}{-6\pi t^5 \sin \pi t^6} \\&\stackrel{H}{=} \lim_{t \rightarrow 1} \frac{-2 - 6t + 20t^3}{-30\pi t^4 \sin \pi t^6 - 6\pi t^5 \cdot \cos \pi t^6 \cdot 6\pi t^5} \\&= \frac{-2 - 6(1) + 20(1)^3}{-6\pi \cdot (-1) \cdot 6\pi} \\&= \frac{1}{3\pi^2}\end{aligned}$$

16.

$$\lim_{x \rightarrow 0} \frac{\left(\int_0^x t \cos t^2 dt\right)^2}{\int_0^x \sin t^2 dt}$$

首先发现到

$$\int_0^x t \cos t^2 dt = \left[ \frac{1}{2} \sin t^2 \right]_0^x = \frac{1}{2} \sin x^2$$

将积分结果代入原极限式:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\left(\int_0^x t \cos t^2 dt\right)^2}{\int_0^x \sin t^2 dt} &= \lim_{x \rightarrow 0} \frac{\frac{1}{4} \sin^2 x^2}{\int_0^x \sin t^2 dt} \\&\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{4} \cdot 2 \sin x^2 \cdot \cos x^2 \cdot 2x}{\sin x^2} \\&= \lim_{x \rightarrow 0} (x \cos x^2) \\&= 0 \cdot 1 = 0\end{aligned}$$

17. 求实数  $a, b$ , 使得

$$\lim_{x \rightarrow 0} \frac{\sqrt{ax + b} - 2}{x} = 1.$$

有理化得

$$\lim_{x \rightarrow 0} \frac{\sqrt{ax+b}-2}{x} \cdot \frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x \rightarrow 0} \frac{ax+b-4}{x(\sqrt{ax+b}+2)}$$

当  $x \rightarrow 0$  时分母趋于 0, 欲使极限存在, 分子也必须趋于 0, 因此

$$a \cdot 0 + b - 4 = 0,$$

从而  $b = 4$ , 于是

$$\lim_{x \rightarrow 0} \frac{a}{\sqrt{ax+4}+2} = \frac{a}{\sqrt{4}+2} = 1,$$

解得

$$a = 4$$

18. 已知

$$\lim_{x \rightarrow -1} \frac{ax^2 + 3x + b}{x + 1} = 2, \quad a, b \in \mathbb{R}$$

试求  $(a, b)$ 。

当  $x \rightarrow -1, x + 1 \rightarrow 0$ ; 欲使该极限存在且等于 2, 分子  $ax^2 + 3x + b$  在  $x \rightarrow -1$  也应趋于 0, 即

$$\lim_{x \rightarrow -1} (ax^2 + 3x + b) = 0 \Rightarrow a(-1)^2 + 3(-1) + b = 0 \Rightarrow a + b = 3 \quad (1)$$

又原极限中分子分母处处可导, 且极限

$$\lim_{x \rightarrow -1} \frac{(ax^2 + 3x + b)'}{(x + 1)'} = \lim_{x \rightarrow -1} \frac{2ax + 3}{1} = -2a + 3$$

存在, 则由洛必达法则,

$$\lim_{x \rightarrow -1} \frac{ax^2 + 3x + b}{x + 1} \stackrel{H}{=} -2a + 3 = 2 \Rightarrow (a, b) = \left(\frac{1}{5}, \frac{14}{5}\right)$$

19. 已知  $a, b$  为正整数, 设函数

$$f(x) = \lim_{n \rightarrow \infty} \frac{2x^{2n+1} + ax^2 + bx - 1}{2x^{2n} + 3},$$

若  $\forall x \in \mathbb{R}, f(x)$  为连续函数, 求序对  $(a, b)$ 。

首先注意到

$$f(x) = \lim_{n \rightarrow \infty} \frac{2x^{2n+1} + ax^2 + bx - 1}{2x^{2n} + 3} = \lim_{n \rightarrow \infty} \frac{x + \frac{ax^2+bx-1}{2x^{2n}}}{1 + \frac{3}{2x^{2n}}} = \begin{cases} x, & |x| > 1, \\ \frac{ax^2 + bx - 1}{3}, & |x| < 1. \end{cases}$$

由连续性,

$$\lim_{x \rightarrow 1^+} f(x) = 1, \quad \lim_{x \rightarrow 1^-} f(x) = \frac{a+b-1}{3}, \quad f(1) = \lim_{n \rightarrow \infty} \frac{2+a+b-1}{2+3} = \frac{a+b+1}{5}$$

可得

$$\frac{a+b+1}{5} = 1 = \frac{a+b-1}{3} \Rightarrow a+b=4 \quad (1)$$

同理,

$$\lim_{x \rightarrow -1^+} f(x) = \frac{a-b-1}{3}, \quad \lim_{x \rightarrow -1^-} f(x) = -1, \quad f(-1) = \frac{-2+a-b-1}{5} = \frac{a-b-3}{5}$$

可得

$$\frac{a-b-3}{5} = -1 = \frac{a-b-1}{3} \Rightarrow a-b=-2 \quad (2)$$

由 (1), (2) 得

$$(a, b) = (1, 3)$$

20. 若

$$\lim_{x \rightarrow 0} \frac{\sqrt{f(x) \sin x + 1} - 1}{e^{4x} - 1} = 2,$$

求

$$\lim_{x \rightarrow 0} f(x)$$

当  $x \rightarrow 0$  时, 原式为不定型  $\frac{0}{0}$ , 可用洛必达法则:

$$\lim_{x \rightarrow 0} \frac{\sqrt{f(x) \sin x + 1} - 1}{e^{4x} - 1} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{f(x) \sin x + 1}} \cdot [f'(x) \sin x + f(x) \cos x]}{4e^{4x}}.$$

当  $x \rightarrow 0$ ,

$$\sin x \rightarrow 0, \cos x \rightarrow 1, e^{4x} \rightarrow 1, \sqrt{f(x) \sin x + 1} \rightarrow 1$$

上式化简为

$$\lim_{x \rightarrow 0} \frac{f(x)}{8} = 2 \Rightarrow \lim_{x \rightarrow 0} f(x) = 16$$

21. 定义序列  $\{x_n\}_{n=2}^{\infty}$  如下:

$$(n + x_n)[\sqrt{2} - 1] = \ln 2.$$

求  $\lim_{n \rightarrow \infty} x_n$ .

由方程可解得

$$x_n = \frac{\ln 2}{2^{\frac{1}{n}} - 1} - n,$$

极限形式为  $\infty - \infty$ 。令  $u = \frac{1}{n}$ , 并使用洛必达法则两次, 有

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \frac{\ln 2 - n \cdot 2^{\frac{1}{n}} + n}{2^{\frac{1}{n}} - 1} \\ &= \lim_{u \rightarrow 0} \frac{u \ln 2 - 2^u + 1}{u 2^u - u} \\ &\stackrel{H}{=} \lim_{u \rightarrow 0} \frac{\ln 2 - 2^u \ln 2}{2^u - 1 + u 2^u \ln 2} \\ &\stackrel{H}{=} \lim_{u \rightarrow 0} \frac{-2^u (\ln 2)^2}{2^u \ln 2 + 2^u \ln 2 + u 2^u (\ln 2)^2} \\ &= \frac{-(\ln 2)^2}{2 \ln 2} \\ &= -\frac{1}{2} \ln 2 \end{aligned}$$

22. 计算极限

$$\lim_{x \rightarrow 0} \left[ \frac{1}{x^3} \int_0^x \frac{t \ln(t+1)}{t^4 + \frac{1}{6}} dt \right]$$

使用两次洛必达法则,

$$\begin{aligned} \lim_{x \rightarrow 0} \left[ \frac{1}{x^3} \int_0^x \frac{t \ln(t+1)}{t^4 + \frac{1}{6}} dt \right] &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{x \ln(x+1)}{x^4 + \frac{1}{6}}}{3x^2} \\ &= \lim_{x \rightarrow 0} \frac{\ln(x+1)}{3x^5 + \frac{x}{2}} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{x+1}}{15x^4 + \frac{1}{2}} \\ &= \frac{1}{0 + \frac{1}{2}} \\ &= 2 \end{aligned}$$

23. 求极限

$$\lim_{x \rightarrow 0^+} (\sin x)^{\frac{1}{\ln x}}$$

此极限属于  $0^0$  型。利用对数恒等式将原式重写为以  $e$  为底的形式:

$$(\sin x)^{\frac{1}{\ln x}} = \exp\left(\frac{\ln(\sin x)}{\ln x}\right)$$

由于  $e^x$  是连续函数, 只需计算指数部分的极限

$$L = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\ln x}$$

属于  $\frac{\infty}{\infty}$  型, 由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{x}{\sin x} \cdot \cos x = 1 \cdot 1 = 1$$

因此原极限为

$$e^1 = e$$

24. 求极限

$$\lim_{x \rightarrow \infty} (x + e^x)^{\frac{1}{x}}$$

此极限属于  $\infty^0$  型, 由

$$(x + e^x)^{\frac{1}{x}} = \exp\left(\frac{\ln(x + e^x)}{x}\right)$$

设

$$L = \lim_{x \rightarrow \infty} \frac{\ln(x + e^x)}{x}$$

此极限属于  $\frac{\infty}{\infty}$  型, 由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1+e^x}{x+e^x}}{1} = \lim_{x \rightarrow \infty} \frac{1+e^x}{x+e^x}$$

再由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{1+e^x} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{e^x} + 1} = \frac{1}{0+1} = 1$$

因此, 原极限为

$$e^1 = e$$

25. 求

$$\lim_{x \rightarrow \infty} \left[ \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} \right]^x$$

首先注意到

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{(x^2 + 2x - 1) - (x^2 - 1)}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1}} \\ &= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 + \frac{2}{x} - \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}} \\ &= \frac{2}{1 + 1} = 1 \end{aligned}$$

故原极限为  $1^\infty$  型, 于是

$$\lim_{x \rightarrow \infty} \left[ \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} \right]^x = \exp \left[ \lim_{x \rightarrow \infty} x \left( \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} - 1 \right) \right]$$

其中

$$\begin{aligned} \lim_{x \rightarrow \infty} x \left( \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} - 1 \right) &= \lim_{x \rightarrow \infty} x \cdot \frac{(x^2 + 2x - 1) - (\sqrt{x^2 - 1} + 1)^2}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1} + 1} \\ &= \lim_{x \rightarrow \infty} x \cdot \frac{2x - 2\sqrt{x^2 - 1}}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1} + 1} \\ &= \lim_{x \rightarrow \infty} x \cdot \frac{\frac{2}{x + \sqrt{x^2 - 1}} - 1}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - 1} + 1} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2}{x + \sqrt{x^2 - 1}} - 1}{\sqrt{1 + \frac{2}{x} - \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}} + \frac{1}{x^2}} \\ &= \frac{0 - 1}{1 + 1} \\ &= -\frac{1}{2} \end{aligned}$$

故原极限为

$$\lim_{x \rightarrow \infty} \left[ \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - 1} \right]^x = \frac{1}{\sqrt{e}}$$

26. 设

$$\lim_{x \rightarrow 0} \frac{\sin 6x + xf(x)}{x^3} = 0,$$

求

$$\lim_{x \rightarrow 0} \frac{6 + f(x)}{x^2}$$

原极限为

$$\lim_{x \rightarrow 0} \frac{\sin 6x + xf(x)}{x^3} = 0.$$

利用幂级数展开

$$\sin 6x = 6x - \frac{(6x)^3}{6} + o(x^3) = 6x - 36x^3 + o(x^3),$$

代入得

$$\frac{6x - 36x^3 + xf(x)}{x^3} = \frac{6x + xf(x)}{x^3} - 36 + o(1) = \frac{6 + f(x)}{x^2} - 36 + o(1).$$

极限存在且为 0, 则

$$\lim_{x \rightarrow 0} \left( \frac{6 + f(x)}{x^2} - 36 \right) = 0 \implies \lim_{x \rightarrow 0} \frac{6 + f(x)}{x^2} = 36.$$

27. 若  $f(x)$  为满足

$$\lim_{x \rightarrow 1} \frac{f(x)}{x-1} = 36, \quad \lim_{x \rightarrow -1} \frac{f(x)}{x+1} = -36, \quad \lim_{x \rightarrow 2} \frac{f(x)}{x-2} = 0, \quad \lim_{x \rightarrow -2} \frac{f(x)}{x+2} = 0$$

的最低次多项式, 求  $f(3)$ 。

设

$$f(x) = (ax + b)(x-1)(x+1)(x-2)^2(x+2)^2$$

由

$$\lim_{x \rightarrow 1} \frac{f(x)}{x-1} = 36, \quad \lim_{x \rightarrow -1} \frac{f(x)}{x+1} = -36$$

得方程式

$$\begin{cases} (a+b) \cdot 2 \cdot 1^2 \cdot 3^2 = 36 \\ (-a+b) \cdot (-2) \cdot (-3)^2 \cdot 1^2 = -36 \end{cases} \implies a=0, b=2$$

因此

$$f(x) = 2(x^2-1)(x-2)^2(x+2)^2 \implies f(3) = 400$$

28. 若

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^4+3} - [A + B(x-1) + C(x-1)^2]}{(x-1)^2} = 0,$$

求常数  $A, B, C$ 。

设

$$L = \lim_{x \rightarrow 1} \frac{\sqrt{x^4 + 3} - [A + B(x - 1) + C(x - 1)^2]}{(x - 1)^2}$$

由于分母当  $x \rightarrow 1$  时趋于 0, 而极限存在且为 0, 故分子必趋于 0:

$$\lim_{x \rightarrow 1} \left( \sqrt{x^4 + 3} - [A + B(x - 1) + C(x - 1)^2] \right) = \sqrt{1 + 3} - A = 2 - A = 0$$

解得

$$A = 2$$

此时原式变为  $\frac{0}{0}$  型, 由洛必达法则,

$$L \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{\frac{2x^3}{\sqrt{x^4 + 3}} - B - 2C(x - 1)}{2(x - 1)} = 0$$

同样地, 由于分母趋于 0, 分子必趋于 0:

$$\lim_{x \rightarrow 1} \left( \frac{2x^3}{\sqrt{x^4 + 3}} - B - 2C(x - 1) \right) = \frac{2}{2} - B = 1 - B = 0$$

解得

$$B = 1$$

再次应用洛必达法则处理该  $\frac{0}{0}$  型极限:

$$L \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \left( \frac{2x^3}{\sqrt{x^4 + 3}} - 1 - 2C(x - 1) \right)}{\frac{d}{dx} 2(x - 1)} = 0$$

计算分子导数:

$$\frac{d}{dx} \left( \frac{2x^3}{\sqrt{x^4 + 3}} \right) = \frac{6x^2\sqrt{x^4 + 3} - 2x^3 \cdot \frac{4x^3}{2\sqrt{x^4 + 3}}}{x^4 + 3} = \frac{6x^2(x^4 + 3) - 4x^6}{(x^4 + 3)\sqrt{x^4 + 3}}$$

代入  $x = 1$  得到

$$\frac{6(4) - 4}{4\sqrt{4}} = \frac{20}{8} = \frac{5}{2}$$

因此解得

$$\frac{\frac{5}{2} - 2C}{2} = 0 \Rightarrow C = \frac{5}{4}$$

综上, 常数分别为

$$A = 2, \quad B = 1, \quad C = \frac{5}{4}$$

由于极限式

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^4+3} - [A + B(x-1) + C(x-1)^2]}{(x-1)^2} = 0$$

符合函数  $f(x)$  在  $x=1$  处的二阶泰勒展开定义, 其中多项式

$$P_2(x) = A + B(x-1) + C(x-1)^2$$

必须等于  $f(x)$  的二阶泰勒多项式。设  $f(x) = \sqrt{x^4+3}$ , 根据泰勒公式可知:

$$A = f(1), \quad B = f'(1), \quad C = \frac{f''(1)}{2!}$$

由

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x^4+3}} \cdot 4x^3 = \frac{2x^3}{\sqrt{x^4+3}} \\ f''(x) &= \frac{6x^2\sqrt{x^4+3} - 2x^3 \cdot \frac{2x^3}{\sqrt{x^4+3}}}{x^4+3} = \frac{6x^2(x^4+3) - 4x^6}{(x^4+3)\sqrt{x^4+3}} \end{aligned}$$

代入得

$$f(1) = 2, \quad f'(1) = 1, \quad f''(1) = \frac{5}{2}$$

因此

$$A = 2, \quad B = 1, \quad C = \frac{5}{4}$$

29.

$$\lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{3n^2+1}} + \frac{1}{\sqrt{3n^2+2}} + \cdots + \frac{1}{\sqrt{3n^2+2n}} \right)$$

令

$$L = \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{3n^2+1}} + \frac{1}{\sqrt{3n^2+2}} + \cdots + \frac{1}{\sqrt{3n^2+2n}} \right)$$

由夹挤定理,

$$\frac{2}{\sqrt{3}} = \lim_{n \rightarrow \infty} \left( \frac{2n}{\sqrt{3n^2}} \right) < L < \lim_{n \rightarrow \infty} \left( \frac{2n}{\sqrt{3n^2+2n}} \right) = \frac{2}{\sqrt{3}} \Rightarrow L = \frac{2\sqrt{3}}{3}$$

30. 设  $a_n = \sqrt{1 \cdot 2} + \sqrt{2 \cdot 3} + \cdots + \sqrt{n(n+1)}$ , 求  $\lim_{n \rightarrow \infty} \frac{a_n}{n^2}$  的值。

发现到

$$\sum_{k=1}^n \sqrt{k^2} < a_n < \sum_{k=1}^n \sqrt{(k+1)^2} \Rightarrow \frac{n(n+1)}{2} < a_n < \frac{n(n+3)}{2}$$

于是有

$$\lim_{n \rightarrow \infty} \frac{n(n+1)}{2n^2} < \lim_{n \rightarrow \infty} \frac{a_n}{n^2} < \lim_{n \rightarrow \infty} \frac{n(n+3)}{2n^2} \Rightarrow \frac{1}{2} < \lim_{n \rightarrow \infty} \frac{a_n}{n^2} < \frac{1}{2}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{a_n}{n^2} = \frac{1}{2}$$

31. 设  $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$ , 求  $\lim_{n \rightarrow \infty} a_n$  之值。

由

$$(2n-1)(2n+1) < (2n)^2$$

令  $S = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ , 则

$$(2n+1)S^2 = (1 \cdot 3)(3 \cdot 5)(5 \cdot 7) \cdots ((2n-3)(2n-1))((2n-1)(2n+1)) < 2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2$$

即

$$\sqrt{2n+1}S < 2 \cdot 4 \cdot 6 \cdots (2n)$$

于是

$$\Rightarrow 0 < a_n = \frac{S}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{2n+1}}$$

由夹挤定理,

$$0 < \lim_{n \rightarrow \infty} a_n < \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n+1}} = 0 \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

32. 求极限

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{kn}}$$

由放缩法,

$$\sqrt{k-1} + \sqrt{k} < 2\sqrt{k} < \sqrt{k} + \sqrt{k+1}$$

有理化得

$$2(\sqrt{k+1} - \sqrt{k}) < \frac{2}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1})$$

于是有

$$\frac{2}{\sqrt{n}} \sum_{k=1}^n (\sqrt{k+1} - \sqrt{k}) < \frac{2}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{k}} < \frac{2}{\sqrt{n}} \sum_{k=1}^n (\sqrt{k} - \sqrt{k-1})$$

累加得

$$\frac{2\sqrt{n+1} - 2}{\sqrt{n}} < \frac{2}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{k}} < \frac{2\sqrt{n}}{\sqrt{n}} = 2$$

因为

$$\lim_{n \rightarrow \infty} \frac{2\sqrt{n+1} - 2}{\sqrt{n}} = 2$$

由夹挤定理得,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{kn}} = 2$$

33. 求极限

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n^2 + n + 1} + \frac{2}{n^2 + n + 2} + \cdots + \frac{n}{n^2 + n + n} \right)$$

由放缩法,

$$\frac{1 + 2 + \cdots + n}{n^2 + n + n} < \sum_{k=1}^n \frac{k}{n^2 + n + k} < \frac{1 + 2 + \cdots + n}{n^2 + n + 1}$$

即

$$\frac{n(n+1)}{2(n^2 + 2n)} < \sum_{k=1}^n \frac{k}{n^2 + n + k} < \frac{n(n+1)}{2(n^2 + n + 1)}$$

由于

$$\lim_{n \rightarrow \infty} \frac{n(n+1)}{2(n^2 + 2n)} = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2(n^2 + n + 1)} = \frac{1}{2}$$

由夹挤定理得

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2 + n + k} = \frac{1}{2}$$

34. 求极限

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \sqrt{k^2 + n + k}$$

对于整数  $1 \leq k \leq n$ , 有

$$\sqrt{k^2} < \sqrt{k^2 + n + k} < \sqrt{k^2 + 2n}$$

于是

$$\frac{1}{n^2} \sum_{k=1}^n k < \frac{1}{n^2} \sum_{k=1}^n \sqrt{k^2 + n + k} \leq \frac{1}{n^2} \sum_{k=1}^n (k + \sqrt{2n})$$

由于

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}$$

且

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n (k + \sqrt{2n}) &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k + \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \sqrt{2n} \\ &= \frac{1}{2} + \lim_{n \rightarrow \infty} \frac{n\sqrt{2n}}{n^2} \\ &= \frac{1}{2} \end{aligned}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \sqrt{k^2 + n + k} = \frac{1}{2}$$

35. 求极限

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+\sqrt{2}} + \frac{1}{n+\sqrt{3}} + \cdots + \frac{1}{n+\sqrt{n}} \right)$$

由放缩法得

$$\frac{n}{n+\sqrt{n}} < \sum_{k=1}^n \frac{1}{n+\sqrt{k}} < \frac{n}{n+1}$$

由于

$$\lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{1+1/n} = 1$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+\sqrt{2}} + \cdots + \frac{1}{n+\sqrt{n}} \right) = 1$$

36. 求极限

$$\lim_{n \rightarrow \infty} \frac{n^k \sin^2(n!)}{n+2}, \quad 0 < k < 1$$

由于

$$0 \leq \sin^2 x \leq 1, \quad \forall x \in \mathbb{R}$$

得到不等式

$$0 \leq \frac{n^k \sin^2(n!)}{n+2} \leq \frac{n^k}{n+2}$$

当  $0 < k < 1$  时, 上界的极限为

$$\lim_{n \rightarrow \infty} \frac{n^k}{n+2} = 0$$

故由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{n^k \sin^2(n!)}{n+2} = 0$$

37. 求极限

$$\lim_{n \rightarrow \infty} \sqrt[n]{3^n + (2|\sin(n^n)|)^n}$$

由于

$$0 \leq |\sin(n^n)| \leq 1$$

得到上下界:

$$\sqrt[n]{3^n} \leq \sqrt[n]{3^n + (2|\sin(n^n)|)^n} \leq \sqrt[n]{3^n + 2^n} \leq \sqrt[n]{2 \cdot 3^n}$$

由于

$$\lim_{n \rightarrow \infty} \sqrt[n]{3^n} = \lim_{n \rightarrow \infty} \sqrt[n]{2 \cdot 3^n} = 3$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \sqrt[n]{3^n + (2|\sin(n^n)|)^n} = 3$$

38. 已知数列  $\{a_n\}$  满足

$$\frac{2n^2 - 7}{4n + 5} < a_n < \frac{3n^2 + 8}{6n - 1}, \quad n \in \mathbb{Z}^+$$

求极限

$$\lim_{n \rightarrow \infty} \frac{3na_n}{(n+1)^2}$$

将不等式两边同时乘以  $\frac{3n}{(n+1)^2}$ :

$$\frac{3n}{(n+1)^2} \cdot \frac{2n^2-7}{4n+5} < \frac{3na_n}{(n+1)^2} < \frac{3n}{(n+1)^2} \cdot \frac{3n^2+8}{6n-1}$$

取极限得

$$\frac{3}{2} = \lim_{n \rightarrow \infty} \frac{3n(2n^2-7)}{(4n+5)(n+1)^2} \leq \lim_{n \rightarrow \infty} \frac{3na_n}{(n+1)^2} \leq \lim_{n \rightarrow \infty} \frac{3n(3n^2+8)}{(6n-1)(n+1)^2} = \frac{3}{2}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{3na_n}{(n+1)^2} = \frac{3}{2}$$

39. 已知  $x_1 = 1$ , 且对每个正整数  $n \geq 2$ ,

$$n(x_n)^2 - x_{n-1} - n = 0, \quad x_n \geq 0.$$

求  $\lim_{n \rightarrow \infty} x_n$  或证明  $\{x_n\}$  发散。

由等式得对所有  $n \geq 2$ , 有

$$x_n = \sqrt{1 + \frac{x_{n-1}}{n}}$$

首先证明  $x_n \leq 2$ : 有  $x_1 = 1 \leq 2$ 。假设  $n \geq 2$  有  $x_n \leq 2$  成立, 则

$$x_n = \sqrt{1 + \frac{x_n}{n+1}} \leq \sqrt{1+2} < 2$$

故由数学归纳法得对所有  $n \geq 2$  都皆有  $x_n \leq 2$ , 故

$$1 \leq x_n = \sqrt{1 + \frac{x_{n-1}}{n}} \leq \sqrt{1 + \frac{2}{n}}$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} x_n = 1$$

40. (a) 证明

$$\int_0^1 \left(1 + \sin \frac{\pi}{2} x\right)^n dx > \frac{2^{n+1} - 1}{n+1} \quad (n = 1, 2, \dots)$$

由

$$\sin x \geq \frac{2}{\pi}x, \quad 0 \leq x \leq \frac{\pi}{2}$$

得

$$\sin\left(\frac{\pi}{2}x\right) \geq x, \quad 0 \leq x \leq 1,$$

从而

$$\int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \geq \int_0^1 (1+x)^n dx = \frac{(1+x)^{n+1}}{n+1} \Big|_0^1 = \frac{2^{n+1}-1}{n+1}.$$

因此原不等式成立。

(b) 求极限:

$$\lim_{n \rightarrow \infty} \left[ \int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \right]^{\frac{1}{n}}$$

注意到:

$$\frac{2^n}{n+1} < \frac{2^{n+1}-1}{n+1} \leq \int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \leq \int_0^1 2^n dx = 2^n.$$

对上述不等式取  $n$  次方根, 得:

$$2(n+1)^{-\frac{1}{n}} < \left[ \int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \right]^{\frac{1}{n}} \leq 2.$$

因为  $(n+1)^{-\frac{1}{n}} \rightarrow 1$ , 由夹挤定理可得:

$$\lim_{n \rightarrow \infty} \left[ \int_0^1 \left(1 + \sin \frac{\pi}{2}x\right)^n dx \right]^{\frac{1}{n}} = 2$$

41. (a) 设  $n$  是正整数, 计算

$$\int_0^{n\pi} x \sin^2 x dx$$

对非负整数  $k$ ,

$$\int_{k\pi}^{(k+1)\pi} x \sin^2 x dx = \int_0^\pi (k\pi + x) \sin^2 x dx$$

利用恒等式  $\sin^2 x = \frac{1-\cos 2x}{2}$ , 得:

$$= \frac{1}{2} \int_0^\pi (k\pi + x)(1 - \cos 2x) dx = \frac{1}{2} \left( (k\pi + x) \Big|_0^\pi - \int_0^\pi (k\pi + x) \cos 2x dx \right)$$

由于  $\int_0^\pi \cos 2x \, dx = 0$  且  $\int_0^\pi x \cos 2x \, dx = 0$ , 所以最后只剩:

$$\int_{k\pi}^{(k+1)\pi} x \sin^2 x \, dx = \frac{1}{2} \int_0^\pi (k\pi + x) \, dx = \frac{1}{2} \left( k\pi^2 + \frac{\pi^2}{2} \right) = \frac{\pi^2(2k+1)}{4}$$

所以,

$$\int_0^{n\pi} x \sin^2 x \, dx = \sum_{k=0}^{n-1} \frac{\pi^2(2k+1)}{4} = \frac{\pi^2}{4} \sum_{k=0}^{n-1} (2k+1) = \frac{\pi^2}{4} \cdot n^2$$

因为  $\sum_{k=0}^{n-1} (2k+1) = n^2$ , 所以原式为

$$\frac{\pi^2 n^2}{4}$$

(b) 证明对任何正实数  $p$ , 函数极限

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} \int_0^x t |\sin t|^p \, dt$$

存在.

记

$$S_0 = \int_0^\pi |\sin t|^p \, dt, \quad S_1 = \int_0^\pi t |\sin t|^p \, dt.$$

对任意非负整数  $k$ , 有

$$\int_{k\pi}^{(k+1)\pi} t |\sin t|^p \, dt = \int_0^\pi (k\pi + x) |\sin x|^p \, dx = k\pi S_0 + S_1.$$

若设  $x = (n + \alpha)\pi$  且  $0 \leq \alpha < 1$ , 则

$$\int_0^x t |\sin t|^p \, dt = \sum_{k=0}^{n-1} (k\pi S_0 + S_1) + \int_{n\pi}^x t |\sin t|^p \, dt.$$

利用估计 (注意  $\int_{n\pi}^x t |\sin t|^p \, dt \leq (n+1)\pi \cdot S_0$ ), 得上下界:

$$\frac{\frac{1}{2}n(n-1)\pi S_0 + nS_1}{(n+1)^2\pi^2} \leq \frac{1}{x^2} \int_0^x t |\sin t|^p \, dt \leq \frac{\frac{1}{2}n(n+1)\pi S_0 + (n+1)S_1}{n^2\pi^2}.$$

随着  $n \rightarrow \infty$ , 上下界极限均趋于  $\frac{S_0}{2\pi}$ , 所以由夹挤定理知极限存在, 且为

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} \int_0^x t |\sin t|^p \, dt = \frac{S_0}{2\pi}$$

42. (a) 求

$$\int_0^{\pi} \frac{1}{1 + \cos^2 x} dx$$

及

$$\int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx$$

设

$$I_1 = \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx, \quad I_2 = \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx.$$

发现

$$I_1 = \frac{1}{2} \int_0^{\pi} \frac{1 + \sin^2 x + \cos^2 x}{1 + \cos^2 x} dx = \frac{1}{2} (I_2 + \pi) \Rightarrow I_2 = 2I_1 - \pi$$

所以只需计算  $I_1$  即可得出两个积分。发现被积函数在  $x = \frac{\pi}{2}$  处不连续, 将  $I_1$  写成

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos^2 x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{1 + \cos^2 x} dx$$

其中第一个积分变为

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{2 + \tan^2 x} dx = \int_0^{\infty} \frac{1}{2 + u^2} du = \left[ \frac{1}{\sqrt{2}} \arctan \left( \frac{u}{\sqrt{2}} \right) \right]_0^{\infty} = \frac{1}{\sqrt{2}} \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{2\sqrt{2}}.$$

对于第二个积分如下, 设  $x = \pi - y$ ,  $dx = -dy$ ,

$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{1 + \cos^2 x} dx = \int_{\frac{\pi}{2}}^0 \frac{1}{1 + \cos^2(\pi - y)} (-dy) = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos^2 y} dy = \frac{\pi}{2\sqrt{2}}$$

所以

$$I_1 = \frac{\pi}{2\sqrt{2}} + \frac{\pi}{2\sqrt{2}} = \frac{\pi\sqrt{2}}{2}$$

$$I_2 = 2I_1 - \pi = 2 \cdot \frac{\pi\sqrt{2}}{2} - \pi = \pi(\sqrt{2} - 1)$$

(b) 证明

$$\lim_{x \rightarrow \infty} \frac{\int_0^x \frac{\sin^2 t}{1 + \cos^2 t} dt}{\int_0^x \frac{1}{1 + \cos^2 t} dt} = 2 - \sqrt{2}$$

取  $n = \left\lfloor \frac{x}{\pi} \right\rfloor$ , 则

$$\int_0^x \frac{1}{1 + \cos^2 t} dt = \sum_{k=1}^n \int_{(k-1)\pi}^{k\pi} \frac{1}{1 + \cos^2 t} dt + \int_{n\pi}^x \frac{1}{1 + \cos^2 t} dt$$

于是

$$\frac{n\pi}{\sqrt{2}} \leq \int_0^x \frac{1}{1 + \cos^2 t} dt \leq \frac{(n+1)\pi}{\sqrt{2}}$$

同理

$$n\pi(\sqrt{2} - 1) \leq \int_0^x \frac{1}{1 + \cos^2 t} dt \leq (n+1)\pi(\sqrt{2} - 1)$$

于是

$$\frac{n}{n+1}(2 - \sqrt{2}) \leq \frac{\int_0^x \frac{\sin^2 t}{1 + \cos^2 t} dt}{\int_0^x \frac{1}{1 + \cos^2 t} dt} \leq \frac{n+1}{n}(2 - \sqrt{2}),$$

令  $x \rightarrow \infty$  便得结论。

43. 对每个正整数  $n$ , 设

$$R_n = \{(x, y) \mid 0 \leq x \leq n, 0 \leq y \leq \sqrt{x}\},$$

记  $N(n)$  为  $R_n$  中坐标都是整数的点的个数。求

$$\lim_{n \rightarrow \infty} \frac{N(n)}{n^{\frac{3}{2}}}.$$

对每个正整数  $n$ , 满足  $0 \leq y \leq \sqrt{k}$  的整数  $y$  个数为  $[\sqrt{k}] + 1$ , 所以

$$N(n) = \sum_{k=0}^n ([\sqrt{k}] + 1) = (n+1) + \sum_{k=1}^n [\sqrt{k}]$$

由于  $\sqrt{k} - 1 < [\sqrt{k}] \leq \sqrt{k}$ , 并且

$$\int_0^n \sqrt{x} dx \leq \sum_{k=1}^n [\sqrt{k}] \leq \int_0^{n+1} \sqrt{x} dx$$

所以

$$1 + \int_0^n \sqrt{x} dx \leq N(n) \leq (n+1) + \int_0^{n+1} \sqrt{x} dx$$

即

$$1 + \frac{2}{3}n^{\frac{3}{2}} \leq N(n) \leq (n+1) + \frac{2}{3}(n+1)^{\frac{3}{2}}$$

由夹挤定理, 得到

$$\lim_{n \rightarrow \infty} \frac{N(n)}{n^{\frac{3}{2}}} = \frac{2}{3}$$

# 微分

1. 已知  $f(x) = x^{2013} - x^{2012} + x^{2011} - x^{2010} + 1$ , 求

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1-h)}{h}$$

计算导数:

$$f'(x) = 2013x^{2012} - 2012x^{2011} + 2011x^{2010} - 2010x^{2009}$$

于是

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(1+h) - f(1-h)}{h} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} + \lim_{h \rightarrow 0} \frac{f(1) - f(1-h)}{h} \\ &= f'(1) + f'(1) \\ &= 2 \cdot (2013 - 2012 + 2011 - 2010) \\ &= 4 \end{aligned}$$

2. 已知  $f'(1) = 12$ , 求

$$\lim_{h \rightarrow 0} \frac{f(1+4h) - f(1-2h)}{3h}$$

改写

$$\begin{aligned} \frac{f(1+4h) - f(1-2h)}{3h} &= \frac{f(1+4h) - f(1)}{3h} + \frac{f(1) - f(1-2h)}{3h} \\ &= \frac{4}{3}f'(1) + \frac{2}{3}f'(1) \\ &= 24 \end{aligned}$$

3. 求

$$\lim_{h \rightarrow 0} \left[ \int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \frac{2\sqrt{\sin x}}{\pi h} dx \right]$$

先提取常数

$$\lim_{h \rightarrow 0} \left[ \int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \frac{2\sqrt{\sin x}}{\pi h} dx \right] = \frac{2}{\pi} \lim_{h \rightarrow 0} \left[ \frac{1}{h} \int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \sqrt{\sin x} dx \right]$$

令

$$F(x) = \int \sqrt{\sin x} dx \Rightarrow F'(x) = \sqrt{\sin x}$$

则

$$\frac{1}{h} \int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \sqrt{\sin x} dx = \frac{F\left(\frac{\pi}{6}+h\right) - F\left(\frac{\pi}{6}\right)}{h}$$

当  $h \rightarrow 0$  时, 上式为导数定义:

$$\lim_{h \rightarrow 0} \frac{F\left(\frac{\pi}{6}+h\right) - F\left(\frac{\pi}{6}\right)}{h} = F'\left(\frac{\pi}{6}\right) = \sqrt{\sin \frac{\pi}{6}}$$

因此原极限为

$$\frac{2}{\pi} \cdot \sqrt{\sin \frac{\pi}{6}} = \frac{2}{\pi} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\pi}$$

4. 由导数定义, 证明

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

由导数的定义,

$$\begin{aligned} \frac{d}{dx}(\sec x) &= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x - \cos(x+h)}{h \cos x \cos(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}}{h \cos x \cos(x+h)} \\ &= \lim_{h \rightarrow 0} \left[ \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \frac{\sin\left(x + \frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\ &= 1 \cdot \frac{\sin x}{\cos^2 x} \\ &= \sec x \tan x \end{aligned}$$

5. 通过导数的定义, 求

$$f(x) = \frac{1}{\sqrt{x^2 - 1}}, \quad |x| > 1$$

的导数  $f'(x)$ 。

由导数的定义,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{(x+h)^2-1}} - \frac{1}{\sqrt{x^2-1}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x^2-1} - \sqrt{(x+h)^2-1}}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}} \\ &= \lim_{h \rightarrow 0} \frac{(x^2-1) - ((x+h)^2-1)}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}(\sqrt{x^2-1} + \sqrt{(x+h)^2-1})} \\ &= \lim_{h \rightarrow 0} \frac{-2xh - h^2}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}(\sqrt{x^2-1} + \sqrt{(x+h)^2-1})} \\ &= \lim_{h \rightarrow 0} \frac{-2x - h}{\sqrt{x^2-1}\sqrt{(x+h)^2-1}(\sqrt{x^2-1} + \sqrt{(x+h)^2-1})} \\ &= \frac{-2x}{\sqrt{x^2-1}\sqrt{x^2-1}(2\sqrt{x^2-1})} \\ &= -\frac{x}{(x^2-1)^{\frac{3}{2}}} \end{aligned}$$

6. 设

$$y = \frac{(x^4+2)(x^3-2)}{(x+2)(x^2-2)},$$

求  $\frac{dy}{dx}$ 。

对等式两边取自然对数,

$$\ln y = \ln(x^4+2) + \ln(x^3-2) - \ln(x+2) - \ln(x^2-2)$$

对  $x$  求导,

$$\frac{1}{y} \frac{dy}{dx} = \frac{4x^3}{x^4+2} + \frac{3x^2}{x^3-2} - \frac{1}{x+2} - \frac{2x}{x^2-2}$$

整理得

$$\frac{dy}{dx} = \frac{4x^3(x^3-2)}{(x+2)(x^2-2)} + \frac{3x^2(x^4+2)}{(x+2)(x^2-2)} - \frac{(x^4+2)(x^3-2)}{(x+2)^2(x^2-2)} - \frac{2x(x^4+2)(x^3-2)}{(x+2)(x^2-2)^2}$$

7. 已知曲线方程

$$y = 2^{3e^{2x}}, \quad x \in \mathbb{R}.$$

以  $y$  表示  $\frac{dy}{dx}$ 。

由链导法,

$$\frac{dy}{dx} = 2^{3e^{2x}} \cdot \ln 2 \cdot 6e^{2x} == y \cdot \ln 2 \cdot 6e^{2x}$$

又由原式取自然对数得

$$\ln y = 3e^{2x} \ln 2 \Rightarrow 2 \ln y = 6e^{2x} \ln 2$$

因此得到

$$\frac{dy}{dx} = 2y \ln y$$

不如一开始就取  $\ln$ ,

$$\ln y = 3e^{2x} \ln 2.$$

对  $x$  求导,

$$\frac{1}{y} \frac{dy}{dx} = \ln 2 \cdot 6e^{2x}$$

同样代入  $\ln y = 3e^{2x} \ln 2$  得

$$\frac{dy}{dx} == y \cdot (\ln 2) \cdot 6e^{2x} = 2y \ln y$$

8. 已知曲线由方程

$$x^m y^n = (x+y)^{m+n}, \quad x \neq 0, y \neq 0, x+y \neq 0, my - nx \neq 0$$

隐式定义, 其中  $m, n$  为有理数。证明

$$\frac{dy}{dx} = \frac{y}{x}$$

两边取对数,

$$m \ln x + n \ln y = (m+n) \ln(x+y)$$

对  $x$  求导,

$$\frac{m}{x} + \frac{n}{y} \frac{dy}{dx} = \frac{m+n}{x+y} \left( 1 + \frac{dy}{dx} \right)$$

整理得

$$\frac{my - nx}{x(x+y)} = \frac{my - nx}{y(x+y)} \frac{dy}{dx}$$

由于  $x+y \neq 0, my - nx \neq 0$ , 故

$$\frac{dy}{dx} = \frac{y}{x}$$

9. 已知曲线  $C$  的隐函数方程为

$$y = xe^y, \quad x \neq 0, y \neq 1, y \neq 2.$$

证明

$$(1-y) \frac{d^2y}{dx^2} = (2-y) \left( \frac{dy}{dx} \right)^2.$$

对  $y = xe^y$  两边求导,

$$\frac{dy}{dx} = e^y + xe^y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} (1-y) = e^y$$

再次求导,

$$\frac{d^2y}{dx^2} (1-y) - \left( \frac{dy}{dx} \right)^2 = e^y \frac{dy}{dx}$$

代入  $e^y = \frac{dy}{dx} (1-y)$ ,

$$\frac{d^2y}{dx^2} (1-y) - \left( \frac{dy}{dx} \right)^2 = \left( \frac{dy}{dx} \right)^2 (1-y)$$

即

$$\frac{d^2y}{dx^2} (1-y) = \left( \frac{dy}{dx} \right)^2 (2-y)$$

10. 求曲线  $C$

$$x^2 + 3xy - 2y^2 + 17 = 0$$

的极值。

设  $F(x, y) = x^2 + 3xy - 2y^2 + 17$ 。对方程  $F(x, y) = 0$  两边求导,

$$2x + 3y + 3xy' - 4yy' = 0$$

整理得:

$$y' = -\frac{2x+3y}{3x-4y}$$

令  $y' = 0$ , 得到

$$y = -\frac{2}{3}x$$

代入原方程解得

$$x^2 + 3x \left(-\frac{2}{3}x\right) - 2 \left(-\frac{2}{3}x\right)^2 + 17 = 0 \Rightarrow x_1 = 3, x_2 = -3$$

得临界点  $P_1(3, -2), P_2(-3, 2)$ 。再求导得,

$$y''(3x-4y) + y'(3-4y') = -(2+3y')$$

在临界点处  $y' = 0$ , 化简得

$$y''(3x-4y) = -2 \Rightarrow y'' = \frac{-2}{3x-4y}$$

由于

$$y''(3, -2) = \frac{-2}{3(3)-4(-2)} = -\frac{2}{17} < 0, \quad y''(-3, 2) = \frac{-2}{3(-3)-4(2)} = \frac{2}{17} > 0$$

故  $y_1 = -2$  是极大值,  $y_2 = 2$  是极小值。该曲线在  $x = 3$  处取得极大值  $-2$ , 在  $x = -3$  处取得极小值  $2$ 。

11. 设函数  $f(x)$  处处可导, 且  $f'(0) = 1$ , 并对任意实数  $x, h$  恒有

$$f(x+h) = f(x) + f(h) + 2hx,$$

求  $f'(x)$ 。

已知  $\forall x, h \in \mathbb{R}$ ,

$$f(x+h) = f(x) + f(h) + 2hx$$

对  $h$  微分,

$$f'(x+h) = f'(h) + 2x$$

令  $h = 0$ , 则

$$f'(x) = f'(0) + 2x = 1 + 2x$$

12. 设  $f(x) = x(x-1)(x-2)\cdots(x-2023)$ ,  $g(x) = f(f(x))$ , 求  $g'(1)$ 。

求导得

$$f'(x) = (x-1)P(x) + x(x-2)(x-3)\cdots(x-2023) = xQ(x) + (x-1)(x-2)\cdots(x-2023)$$

所以有

$$f(1) = 0, \quad f'(1) = 2022!, \quad f'(0) = -2023!$$

由于  $g(x) = f(f(x))$ , 有

$$g'(x) = f'(f(x)) \cdot f'(x) \Rightarrow g'(1) = f'(f(1)) \cdot f'(1) = f'(0) \cdot 2022! = -2023! \cdot 2022!$$

13. 设  $f(x)$  为正实函数且为可微分函数, 对任意实数  $x, y$ , 满足  $f(x+y) = 2f(x)f(y)$ , 若  $f'(0) = 2$ , 试求  $\frac{f''(x)}{f(x)}$ 。

由  $f(x+y) = 2f(x)f(y)$  代入  $x = y = 0$  得

$$f(0) = 2f(0)f(0) \Rightarrow f(0) = \frac{1}{2}$$

考虑  $f'(x)$ ,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2f(x)f(h) - f(x)}{h} \\ &= 2f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = 2f(x)f'(0) = 2f(x) \cdot 2 = 4f(x) \end{aligned}$$

对两边求导得

$$f''(x) = 4f'(x) = 4 \cdot 4f(x) \Rightarrow \frac{f''(x)}{f(x)} = 16$$

14. 已知  $f(x)$  在  $x = 0$  连续, 且

$$\lim_{x \rightarrow 0} \frac{\ln(f(x) + 2)}{x - \sin x} = 1,$$

求  $f'(0)$ 。

因为  $f(x)$  在  $x = 0$  连续, 所以

$$\lim_{x \rightarrow 0} \ln(f(x) + 2) = \ln(f(0) + 2).$$

又因  $x - \sin x \rightarrow 0$ , 分母趋于 0, 极限存在, 需有分子也趋于 0, 即

$$\ln(f(0) + 2) = 0 \Rightarrow f(0) = -1.$$

所以

$$\lim_{x \rightarrow 0} \frac{\ln(f(x) + 2)}{x - \sin x} = \lim_{x \rightarrow 0} \frac{f(x) + 1}{x - \sin x} \quad (\text{因 } \ln(1 + y) \sim y).$$

又  $x - \sin x \sim \frac{x^3}{6}$ , 于是

$$\lim_{x \rightarrow 0} \frac{f(x) + 1}{x^3} = \frac{1}{1/6} = 6.$$

因此

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(x) + 1}{x} = \lim_{x \rightarrow 0} \frac{f(x) + 1}{x^3} \cdot x^2 = 6 \cdot 0 = 0.$$

故  $f'(0) = \boxed{0}$ 。待验证

15. 已知  $a, b$  为实数, 若函数  $f(x) = 2ax^3 - 3ax^2 + b$  在  $0 \leq x \leq 2$  时有最大值 7, 最小值 -3, 求数对  $(a, b)$ 。

导数为

$$f'(x) = 6ax^2 - 6ax = 6ax(x - 1)$$

临界点为  $x = 0, 1$ , 二阶导为

$$f''(x) = 12ax - 6a = 6a(2x - 1)$$

当  $a > 0$ , 则

$$f(x) \begin{cases} \text{递增} & x \geq 1 \\ \text{递减} & 0 \leq x \leq 1 \\ \text{递增} & x \leq 0 \end{cases}$$

极小值在  $x = 1$ , 极大值在  $x = 2$  或  $x = 0$ , 解得

$$\begin{cases} f(2) = 4a + b = 7 \\ f(1) = -a + b = -3 \end{cases} \Rightarrow a = 2, b = -1$$

当  $a < 0$ , 则

$$f(x) \begin{cases} \text{递减} & x \geq 1 \\ \text{递增} & 0 \leq x \leq 1 \\ \text{递减} & x \leq 0 \end{cases}$$

极大值在  $x = 1$ , 极小值在  $x = 0$  或  $x = 2$ , 同理解得

$$\begin{cases} f(1) = -a + b = 7 \\ f(2) = 4a + b = -3 \end{cases} \Rightarrow a = -2, b = 5$$

$\therefore (a, b) = (2, -1)$  或  $(-2, 5)$

16. 函数  $f$  定义为

$$f(n, y) = \sum_{x=1}^n \frac{x^2 y^x}{k}, \quad n \in \mathbb{N}, \quad y \in \mathbb{R}$$

其中

$$k = \sum_{r=1}^n r^2,$$

若已知数列求和公式

$$\sum_{r=1}^n r^4 = \frac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1),$$

证明

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[ \left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n}{20(2n+1)^2}.$$

首先, 利用标准求和公式:

$$k = \sum_{x=1}^n x^2 = \frac{n(n+1)(2n+1)}{6}.$$

既然  $k$  与  $x$  无关, 可以将其作为常数因子提取到求和号外

$$f(n, y) = \frac{1}{k} \sum_{x=1}^n x^2 y^x.$$

对  $y$  求导,

$$\frac{df}{dy} = \frac{1}{k} \sum_{x=1}^n x^3 y^{x-1}, \quad \left. \frac{df}{dy} \right|_{y=1} = \frac{1}{k} \sum_{x=1}^n x^3.$$

再次对  $y$  求导,

$$\frac{d^2 f}{dy^2} = \frac{1}{k} \sum_{x=1}^n x^3 (x-1) y^{x-2} = \frac{1}{k} \sum_{x=1}^n x^4 y^{x-2} - \frac{1}{k} \sum_{x=1}^n x^3 y^{x-2}.$$

在  $y = 1$  时,

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[ \left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{1}{k} \sum_{x=1}^n x^4 - \frac{1}{k^2} \left( \sum_{x=1}^n x^3 \right)^2.$$

代入求和公式得

$$\begin{aligned}& \frac{6}{n(n+1)(2n+1)} \cdot \frac{1}{30} n(n+1)(6n^3+9n^2+n-1) - \frac{36}{n^2(n+1)^2(2n+1)^2} \cdot \left[ \frac{n^2(n+1)^2}{4} \right] \\&= \frac{6n^3+9n^2+n-1}{5(2n+1)} - \frac{9}{4(2n+1)^2} \\&= \frac{3n^4+6n^3-n^2-4n}{20(2n+1)^2}.\end{aligned}$$

因此,

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[ \left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4+6n^3-n^2-4n}{20(2n+1)^2},$$

得证。

17. (a) 设

$$f(x) = x^3 + x,$$

且  $g(x)$  为  $f(x)$  的反函数, 求  $g'(10)$ 。

由于  $f$  与  $g$  为反函数,  $f(g(x)) = x$ , 对两边求导得到

$$f'(g(x))g'(x) = 1$$

又因为  $f(2) = 10$ , 所以  $g(10) = 2$ 。令  $x = 10$ , 则

$$f'(2)g'(10) = 1$$

而  $f'(x) = 3x^2 + 1$ , 所以  $f'(2) = 13$ , 因此

$$g'(10) = \frac{1}{13}$$

(b) 对  $x > 0$ , 定义

$$h(x) = \frac{1}{f(x)},$$

证明函数  $f(x) + h(x)$  在  $x = g(1)$  处取得最小值。

令

$$F(x) = f(x) + h(x) = f(x) + \frac{1}{f(x)}$$

则

$$F'(x) = f'(x) - \frac{f'(x)}{[f(x)]^2} = f'(x) \frac{[f(x)]^2 - 1}{[f(x)]^2}$$

因为  $f'(x) > 1$ , 所以在临界点有  $f(x_0)^2 - 1 = 0$ , 即  $f(x_0) = 1$ , 因此  $x_0 = g(1)$ , 再次求导,

$$F''(x) = f''(x) - \frac{[f(x)]^2 f''(x) - 2f(x)[f'(x)]^2}{[f(x)]^4}$$

在  $x = x_0$  时,  $f(x_0) = 1, f''(x_0) = 6g(1), f'(x_0) = 3(g(1))^2 + 1$ , 所以

$$F''(x_0) = 6g(1) - (6g(1) - 2[3(g(1))^2 + 1]^2) = 2[3(g(1))^2 + 1]^2 > 0$$

因此  $x = x_0$  为极小值。由于

$$\lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow \infty} F(x) = \infty,$$

故  $x = g(1)$  为  $F(x)$  的最小值。

18. 已知对任意正整数  $k$ , 方程

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$$

恰有一个正实根, 设该方程唯一正实根为  $\alpha_k$ , 试证该无穷数列  $\langle \alpha_k \rangle$  收敛, 且

$$\lim_{k \rightarrow \infty} \alpha_k = 2$$

原方程可化为

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = x^k - (x^{k-1} + x^{k-2} + \cdots + 1) = x^k - \frac{x^k - 1}{x - 1}$$

令

$$f_k(x) = x^{k+1} - 2x^k + 1$$

则  $\alpha_k$  为  $f_k(x) = 0$  在  $(1, 2)$  内的唯一正实根 (注意  $\alpha_k \neq 1$ , 当  $k > 1$ ), 考虑

$$f_k(0) = 1, \quad f_k(1) = 0, \quad f_k(2) = 1$$

对  $f_k(x)$  求导得

$$f'_k(x) = (k+1)x^k - 2kx^{k-1} = x^{k-1}[(k+1)x - 2k]$$

导函数零点为

$$x = 0 \quad \text{或} \quad x = \frac{2k}{k+1} = 2 - \frac{2}{k+1}$$

因此

$f_k(x)$  在  $\left(0, 2 - \frac{2}{k+1}\right)$  上递减,  $\left(2 - \frac{2}{k+1}, \infty\right)$  上递增

由于  $f_k(1) = 0$ ,  $f_k(2) = 1$  且  $\alpha_k$  是唯一正实根, 结合单调性可得:

$$2 - \frac{2}{k+1} < \alpha_k < 2$$

故由夹挤定理,

$$\lim_{k \rightarrow \infty} \alpha_k = 2$$

因此  $\alpha_k$  收敛, 且收敛至 2, 故得证。

19. 已知  $a < b < c$ ,  $f'(x)$  在  $(a, c)$  上严格递增, 且  $f(x)$  在  $[a, c]$  上连续, 证明

$$(b-a)f(c) + (c-b)f(a) > (c-a)f(b).$$

由拉格朗日中值定理, 因此存在  $\alpha$  和  $\beta$  使得

$$\frac{f(c) - f(b)}{c - b} = f'(\beta), \quad b < \beta < c$$

以及

$$\frac{f(b) - f(a)}{b - a} = f'(\alpha), \quad a < \alpha < b$$

由于  $f'$  严格递增, 我们有  $f'(\beta) > f'(\alpha)$ , 因此

$$\frac{f(c) - f(b)}{c - b} > \frac{f(b) - f(a)}{b - a}$$

即

$$(b-a)f(c) + (c-b)f(a) > (c-a)f(b)$$

20. 设  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  是连续函数, 且  $g$  可导, 若

$$(f(0) - g'(0))(g'(1) - f(1)) > 0,$$

证明存在实数  $c \in (0, 1)$ , 使得  $f(c) = g'(c)$ 。

令

$$F(x) = \int_0^x f(t) dt, \quad h(x) = F(x) - g(x).$$

由  $f$  的连续性,  $F$  可导且  $F' = f$ , 因此

$$h'(x) = F'(x) - g'(x) = f(x) - g'(x).$$

故由  $(f(0) - g'(0))(g'(1) - f(1)) > 0$  知

$$h'(0)(-h'(1)) > 0,$$

因此  $h'(0)$  和  $h'(1)$  异号。由达布定理, 存在  $c \in (0, 1)$  使得

$$h'(c) = 0 \Rightarrow f(c) = g'(c).$$

21. 设  $f: (0, \infty) \rightarrow \mathbb{R}$  为连续可微函数,  $b > a > 0$  且  $f(a) = f(b) = k$ 。证明存在  $\xi \in (a, b)$  使得

$$f(\xi) - \xi f'(\xi) = k.$$

若考虑函数  $g(x) = \frac{f(x)}{x}$ , 则

$$g'(x) = \frac{xf'(x) - f(x)}{x^2},$$

其中分子与已知正好对应。考虑在  $[a, b]$  上的函数

$$g(x) = \frac{f(x)}{x}, \quad h(x) = \frac{1}{x}$$

由柯西中值定理, 存在  $\xi \in (a, b)$  使得

$$\frac{g(a) - g(b)}{h(a) - h(b)} = \frac{g'(\xi)}{h'(\xi)}$$

注意到

$$\frac{g'(\xi)}{h'(\xi)} = \frac{\frac{\xi f'(\xi) - f(\xi)}{\xi^2}}{-\frac{1}{\xi^2}} = f(\xi) - \xi f'(\xi)$$

而

$$\frac{g(a) - g(b)}{h(a) - h(b)} = \frac{\frac{f(a)}{a} - \frac{f(b)}{b}}{\frac{1}{a} - \frac{1}{b}} = k$$

故原命题得证。

22. 设  $f: \mathbb{R} \rightarrow \mathbb{R}$  为二阶可导函数, 且  $f(0) = 0$ . 证明: 存在  $\xi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  使得

$$f''(\xi) = f(\xi)(1 + 2 \tan^2 \xi).$$

设  $g(x) = f(x) \cos x$ , 由于

$$g\left(-\frac{\pi}{2}\right) = g(0) = g\left(\frac{\pi}{2}\right) = 0$$

由罗尔定理, 存在

$$\xi_1 \in \left(-\frac{\pi}{2}, 0\right), \quad \xi_2 \in \left(0, \frac{\pi}{2}\right)$$

使得

$$g'(\xi_1) = g'(\xi_2) = 0$$

考虑函数

$$h(x) = \frac{g'(x)}{\cos^2 x} = \frac{f'(x) \cos x - f(x) \sin x}{\cos^2 x}$$

显然有

$$h(\xi_1) = h(\xi_2) = 0$$

再由罗尔定理, 存在

$$\xi \in (\xi_1, \xi_2) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

使得  $h'(\xi) = 0$ ,  $h$  导数为

$$\begin{aligned} 0 = h'(\xi) &= \frac{g''(\xi) \cos^2 \xi + 2 \cos \xi \sin \xi g'(\xi)}{\cos^4 \xi} \\ &= \frac{(f''(\xi) \cos \xi - 2f'(\xi) \sin \xi - f(\xi) \cos \xi) \cos \xi + 2 \sin \xi (f'(\xi) \cos \xi - f(\xi) \sin \xi)}{\cos^3 \xi} \\ &= \frac{f''(\xi) \cos^2 \xi - f(\xi) (\cos^2 \xi + 2 \sin^2 \xi)}{\cos^3 \xi} \end{aligned}$$

整理得

$$0 = \frac{1}{\cos \xi} \left( f''(\xi) - f(\xi) (1 + 2 \tan^2 \xi) \right)$$

因此得证

$$f''(\xi) = f(\xi) (1 + 2 \tan^2 \xi)$$

23. 证明当  $0 < \theta < \frac{\pi}{4}$  时, 有

$$\tan \theta < \frac{4\theta}{\pi}.$$

设

$$f(x) = \tan x, \quad x = \theta \in \left(0, \frac{\pi}{4}\right)$$

因为  $f$  在  $\left[0, \frac{\pi}{4}\right]$  上连续,  $\left(0, \frac{\pi}{4}\right)$  上可导, 由拉格朗日中值定理, 存在

$$0 < \xi < \theta, \theta < \xi' < \frac{\pi}{4}$$

使得

$$\sec^2 \xi = \frac{\tan \theta - \tan 0}{\theta - 0}, \quad \sec^2 \xi' = \frac{\tan \frac{\pi}{4} - \tan \theta}{\frac{\pi}{4} - \theta}$$

由于  $\sec^2 x$  在  $\left(0, \frac{\pi}{4}\right)$  上单调递增, 因此  $\sec^2 \xi' > \sec^2 \xi$ , 即

$$\frac{1 - \tan \theta}{\frac{\pi}{4} - \theta} > \frac{\tan \theta}{\theta} \Rightarrow \frac{4\theta}{\pi} > \tan \theta$$

令

$$f(\theta) = \frac{4\theta}{\pi} - \tan \theta.$$

则

$$f'(\theta) = \frac{4}{\pi} - \sec^2 \theta$$

因此可解得唯一的  $\xi \in \left(0, \frac{\pi}{4}\right)$  使得  $f'(\xi) = 0$ , 又

$$f'(\theta) < 0, \forall \theta \in (0, \xi), \quad f'(\theta) > 0, \forall \theta \in \left(\xi, \frac{\pi}{4}\right)$$

因此  $f$  在  $(0, \xi)$  上严格递减, 在  $\left(\xi, \frac{\pi}{4}\right)$  上严格递增. 又  $f(0) = f\left(\frac{\pi}{4}\right) = 0$ , 所以对所有  $\left(0, \frac{\pi}{4}\right)$  皆有  $f(\theta) < 0$ , 即

$$\tan \theta < \frac{4\theta}{\pi}, \quad 0 < \theta < \frac{\pi}{4}$$

24. 证明对于所有非负整数  $n$ , 有

$$2(3n-1)^n \geq (3n+1)^n$$

当  $n = 0$  时, 不等式显然成立, 因为

$$2(3 \cdot 0 - 1)^0 = 2 \cdot 1 = 2 \geq 1 = (3 \cdot 0 + 1)^0.$$

考虑  $n \geq 1$ , 命题等价于

$$2(3n - 1)^n \geq (3n + 1)^n \iff \left(\frac{3n - 1}{3n + 1}\right)^n \geq \frac{1}{2} \iff n \ln \frac{3n - 1}{3n + 1} \geq -\ln 2$$

定义函数

$$f(x) = x \ln \frac{3x - 1}{3x + 1}, \quad x \geq 1.$$

对  $f(x)$  求导,

$$f'(x) = \ln \frac{3x - 1}{3x + 1} + \frac{6x}{9x^2 - 1}, \quad f''(x) = \frac{6(3x^2 - 1)}{(9x^2 - 1)^2}$$

当  $x \geq 1$  时,  $f''(x) > 0$ , 故  $f'(x)$  在  $[1, +\infty)$  上严格递增。又因为

$$\lim_{x \rightarrow +\infty} f'(x) = \lim_{x \rightarrow +\infty} \left( \ln \frac{3 - 1/x}{3 + 1/x} + \frac{6/x}{9 - 1/x^2} \right) = 0$$

由  $f'(x)$  严格递增且趋于 0 可知, 在  $[1, +\infty)$  上  $f'(x) < 0$ 。因此  $f(x)$  在  $[1, +\infty)$  上严格递减。根据其极限:

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \ln \left( 1 - \frac{2}{3x + 1} \right)^x = -\frac{2}{3}$$

由于  $f(x)$  严格递减且下界为  $-\frac{2}{3}$ , 而  $-\frac{2}{3} > -0.693 \approx -\ln 2$ 。故对于所有  $n \geq 1$ , 有  $f(n) > -\frac{2}{3} > -\ln 2$ , 原不等式成立。(待验证)

25. 当  $x \in (0, \frac{\pi}{2})$ , 比较  $\tan(\sin x)$  及  $\sin(\tan x)$  的大小。

设

$$f(x) = \tan(\sin x) - \sin(\tan x)$$

求导得,

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan x)}{\cos^2 x} = \frac{\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x)}{\cos^2 x \cdot \cos^2(\tan x)}$$

对  $0 < x < \arctan \frac{\pi}{2}$ , 由与  $\cos$  在  $(0, \frac{\pi}{2})$  上凸, 由琴生不等式及 AM-GM 不等式,

$$\sqrt[3]{\cos(\tan x) \cdot \cos^2(\sin x)} < \frac{1}{3}[\cos(\tan x) + 2\cos(\sin x)] \leq \cos\left(\frac{\tan x + 2\sin x}{3}\right) < \cos x$$

其中最后一个不等式由

$$\left(\frac{\tan x + 2\sin x}{3}\right)' = \frac{1}{3}\left(\frac{1}{\cos^2 x} + 2\cos x\right) \geq 1$$

得到, 据此有

$$\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x) > 0 \Rightarrow f'(x) > 0$$

因此  $f$  在  $\left[0, \arctan \frac{\pi}{2}\right]$  上单调递增。又注意到

$$\tan\left(\sin\left(\arctan \frac{\pi}{2}\right)\right) = \tan \frac{\frac{\pi}{2}}{\sqrt{1 + \frac{\pi^2}{4}}} > \tan \frac{\pi}{4} = 1$$

因此对于  $x \in \left[\arctan \frac{\pi}{2}, \frac{\pi}{2}\right]$ , 有  $\tan(\sin x) > 1$ , 所以  $f(x) > 0$ 。综上, 对于所有  $x \in (0, \frac{\pi}{2})$ ,

$$\tan(\sin x) > \sin(\tan x).$$

26. 证明: 对半开区间  $\left(0, \frac{\pi}{2}\right]$  内的任意  $x$ , 有

$$\left(\frac{\sin x}{x}\right)^3 > \cos x.$$

在  $\left(0, \frac{\pi}{2}\right]$  上, 有泰勒展开的不等式:

$$\sin x > x - \frac{x^3}{6}, \quad \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

因此

$$\left(\frac{\sin x}{x}\right)^3 > \left(1 - \frac{x^2}{6}\right)^3$$

展开右边并与  $\cos x$  比较:

$$\left(1 - \frac{x^2}{6}\right)^3 = 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{216} > 1 - \frac{x^2}{2} + \frac{x^4}{24} > \cos x$$

其中

$$0 < x \leq \frac{\pi}{2} < 3 \Rightarrow x^2 < 9 \Rightarrow \frac{x^4}{12} - \frac{x^6}{216} > \frac{x^4}{24}$$

故原不等式得证。

27. 证明不等式: 当  $0 < x \leq 1$  时, 有

$$\sin x + \arcsin x > 2x.$$

$\sin x$  与  $\arcsin x$  的麦克劳林展开式分别为

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \cdots$$

因此, 对任意  $0 < x \leq 1$ , 存在  $\theta_1, \theta_2$ , 满足  $0 < \theta_1 < x, 0 < \theta_2 < x$ , 使得

$$\sin x = x - \frac{x^3}{3!} + \frac{\theta_1^5}{5!}, \quad \arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\theta_2^5}{5}$$

于是

$$\sin x + \arcsin x = 2x + \frac{\theta_1^5}{5!} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\theta_2^5}{5} > 2x$$

故得证。

设

$$f(x) = \sin x + \arcsin x,$$

则  $f(0) = 0$ , 且

$$f'(x) = \cos x + \frac{1}{\sqrt{1-x^2}},$$

从而

$$f'(0) = 2,$$

故直线  $y = 2x$  是曲线  $y = f(x)$  在 origin 处的切线。又

$$f''(x) = -\sin x + \frac{x}{(1-x^2)^{\frac{3}{2}}}.$$

当  $0 < x < 1$  时, 有

$$\frac{x}{(1-x^2)^{\frac{3}{2}}} > \sin x,$$

因此  $f''(x) > 0$ , 函数  $f(x)$  在区间  $(0, 1)$  上上凸, 即  $f(x)$  在  $(0, 1]$  上位于其在 origin 处的切线  $y = 2x$  之上, 故得证

$$\sin x + \arcsin x > 2x$$

28. 设  $f: \mathbb{R} \rightarrow \mathbb{R}$  为二次可微函数, 满足  $f(0) = 1, f'(0) = 0$ , 且对所有  $x \in [0, \infty)$  有

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

证明对所有  $x \in [0, \infty)$  有

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

设  $x \in [0, \infty)$ , 令  $g(x) = f'(x) - 2f(x)$ . 则

$$g'(x) - 3g(x) \geq 0$$

即

$$(g(x)e^{-3x})' \geq 0$$

因此

$$g(x)e^{-3x} \geq g(0) = f'(0) - 2f(0) = -2$$

即

$$f'(x) - 2f(x) \geq -2e^{3x}$$

两边同乘  $e^{-2x}$  并整理得

$$(f(x)e^{-2x})' \geq -2e^x$$

即

$$(f(x)e^{-2x} + 2e^x)' \geq 0$$

于是

$$f(x)e^{-2x} + 2e^x \geq f(0) + 2 = 3$$

即

$$f(x) \geq 3e^{2x} - 2e^{3x}$$

29. 设函数

$$F(x) = e^{-x} - \left(1 - \frac{x}{n}\right)^n,$$

证明: 当  $n \geq 2$  且  $x \in [0, n]$  时,

$$0 \leq F(x) \leq \frac{e^{-1}}{n}.$$

函数  $F(x)$  在区间  $[0, n]$  上连续且可微, 因此在该区间内必有最大值与最小值, 且只能出现在端点或临界点处。有  $F(0) = 1 - 1 = 0$ ,  $F(n) = e^{-n}$ , 且

$$F'(x) = -e^{-x} + \left(1 - \frac{x}{n}\right)^{n-1}$$

先证明  $F(x) \geq 0$ 。只需证明

$$\left(1 - \frac{x}{n}\right)^n \leq e^{-x}, \quad x \in [0, n]$$

等价于

$$0 \leq 1 - \frac{x}{n} \leq e^{-x/n}, \quad x \in [0, n]$$

令  $t = \frac{x}{n}$ , 则  $t \in [0, 1]$ , 上述不等式化为

$$1 - t \leq e^{-t}, \quad t \in [0, 1]$$

这是显然成立的, 因为直线  $y = 1 - t$  是曲线  $y = e^{-t}$  在  $(0, 1)$  处的切线, 而  $y = e^{-t}$  在  $[0, 1]$  上向下凸, 因此图像始终位于其切线之上, 故  $F(x) \geq 0$ , 最小值在  $x = 0$  处取得。接下来确定最大值, 由于

$$F'(n) = -e^{-n} < 0,$$

最大值不可能出现在  $x = n$ , 因此必存在  $x_0 \in (0, n)$  使得  $F'(x_0) = 0$ , 这意味

$$e^{-x_0} = \left(1 - \frac{x_0}{n}\right)^{n-1}$$

于是

$$\begin{aligned} F(x_0) &= e^{-x_0} - \left(1 - \frac{x_0}{n}\right)^n \\ &= e^{-x_0} - \left(1 - \frac{x_0}{n}\right)^{n-1} \left(1 - \frac{x_0}{n}\right) \\ &= e^{-x_0} - e^{-x_0} \left(1 - \frac{x_0}{n}\right) \\ &= e^{-x_0} \frac{x_0}{n} \end{aligned}$$

设  $g(x) = xe^{-x}$ , 可推导得知  $g(x)$  在  $x = 1$  处取得最大值  $e^{-1}$ , 因此

$$F(x_0) \leq \frac{e^{-1}}{n}$$

综上所述  $n \geq 2$  且  $x \in [0, n]$  时,

$$0 \leq F(x) \leq \frac{e^{-1}}{n}$$

30. 已知在  $(-\infty, +\infty)$  上具有二阶连续导数的函数  $f(x)$  满足方程

$$x^2 f''(x) - 2x \sin x f'(x) = e^x + e^{-x} - 2,$$

若  $f(x)$  在  $x = a$  处取极值, 问  $f(a)$  是函数  $f(x)$  的极大值还是极小值? 请说明理由。

因为  $f(x)$  在  $x = a$  处取极值, 所以

$$f'(a) = 0.$$

代入原方程得

$$a^2 f''(a) = e^a + e^{-a} - 2.$$

由 AM-GM 不等式,

$$e^a + e^{-a} \geqslant 2,$$

等号成立当且仅当  $a = 0$ , 所以

$$a^2 f''(a) = e^a + e^{-a} - 2 \geqslant 0,$$

若  $a \neq 0$ , 则  $a^2 > 0$ , 因此

$$f''(a) = \frac{e^a + e^{-a} - 2}{a^2} > 0,$$

所以  $f(x)$  在  $x = a$  处为极小值; 若  $a = 0$ , 则

$$f''(0) = \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{x^2}$$

由洛必达法则,

$$f''(0) = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{2} = 1 > 0.$$

因此  $f''(0) > 0$  仍为极小值, 故  $f(a)$  是函数  $f(x)$  的极小值。

31. 已知函数  $f(t)$  在  $[a, x]$  上可微, 且  $f'(t)$  可微。对所有  $x > a$ , 存在  $c_x$  满足  $a < c_x < x$  且

$$\int_a^x f(t) dt = f(c_x)(x - a).$$

假设  $f'(a) \neq 0$ , 证明

$$\lim_{x \rightarrow a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

设

$$F(x) = \int_a^x f(t) dt$$

利用  $F(x)$  的泰勒展开, 有

$$F(x) = F(a) + (x - a)F'(a) + \frac{(x - a)^2}{2}F''(\theta_x)$$

其中  $a < \theta_x < x$ , 且当  $x \rightarrow a$  时,  $\theta_x \rightarrow a$ 。又  $F(a) = 0, F'(x) = f(x), F''(x) = f'(x)$ , 所以

$$F(x) = 0 + (x-a)f(a) + \frac{(x-a)^2}{2}f'(\theta_x)$$

由定义有

$$f(c_x) = \frac{F(x)}{x-a} = f(a) + \frac{x-a}{2}f'(\theta_x)$$

因此

$$\frac{f(c_x) - f(a)}{x-a} = \frac{1}{2}f'(\theta_x)$$

另一方面可以写成

$$\frac{f(c_x) - f(a)}{x-a} = \frac{f(c_x) - f(a)}{c_x - a} \cdot \frac{c_x - a}{x-a}$$

取极限得到

$$\lim_{x \rightarrow a} \frac{1}{2}f'(\theta_x) = \lim_{x \rightarrow a} \frac{f(c_x) - f(a)}{c_x - a} \cdot \lim_{x \rightarrow a} \frac{c_x - a}{x-a}$$

由此可得

$$\frac{1}{2}f'(a) = f'(a) \cdot \lim_{x \rightarrow a} \frac{c_x - a}{x-a}$$

即

$$\lim_{x \rightarrow a} \frac{c_x - a}{x-a} = \frac{1}{2}$$

32. (a) 已知  $f(x)$  在  $[0, +\infty)$  上连续且单调增加, 且  $f(0) \geq 0$ , 证明:

$$F(x) = \begin{cases} \frac{1}{x^n} \int_0^x t^{n-1} f(t) dt, & x > 0, \\ 0, & x = 0 \end{cases}$$

在  $[0, +\infty)$  上连续且单调增加, 其中  $n > 0$ 。

先证连续性。由于  $f$  在  $[0, +\infty)$  上连续, 且  $f(0) \geq 0$ , 有:

$$\lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow 0^+} \frac{1}{x^n} \int_0^x t^{n-1} f(t) dt = 0 = F(0),$$

所以  $F(x)$  在  $x=0$  处连续。

再证单调性。对于  $x > 0$ , 由积分中值定理可得:

$$\int_0^x t^{n-1} f(t) dt = f(\xi) \int_0^x t^{n-1} dt = f(\xi) \cdot \frac{x^n}{n}, \quad \text{其中 } \xi \in (0, x).$$

因此,

$$F(x) = \frac{1}{x^n} \cdot \int_0^x t^{n-1} f(t) dt = \frac{1}{x^n} \cdot f(\xi) \cdot \frac{x^n}{n} = \frac{f(\xi)}{n}.$$

因为  $f$  单调递增,  $\xi \in (0, x)$ , 所以  $x \uparrow \Rightarrow \xi \uparrow \Rightarrow f(\xi) \uparrow$ , 故  $F(x)$  单调递增。

(b) 设  $f(x)$  在  $[0, 1]$  上二阶可导, 且满足  $|f''(x)| \leq 1$ 。已知  $f(x)$  在  $(0, 1)$  内取最大值为  $\frac{1}{4}$ , 证明:

$$|f(0)| + |f(1)| \leq 1.$$

设  $x_0 \in (0, 1)$ , 使得  $f(x_0) = \frac{1}{4}$ , 且  $f'(x_0) = 0$ 。对  $f(0)$  做 Taylor 展开 (Lagrange 余项形式) 得:

$$f(0) = f(x_0) + f'(x_0)(0 - x_0) + \frac{1}{2}f''(\xi)(0 - x_0)^2 = \frac{1}{4} + \frac{1}{2}f''(\xi)x_0^2,$$

其中  $\xi \in (0, x_0)$ 。同理,

$$f(1) = \frac{1}{4} + \frac{1}{2}f''(\eta)(1 - x_0)^2, \quad \eta \in (x_0, 1).$$

因为  $|f''(x)| \leq 1$ , 故

$$|f(0)| \leq \left| \frac{1}{4} + \frac{1}{2}f''(\xi)x_0^2 \right| \leq \frac{1}{4} + \frac{1}{2}x_0^2,$$

$$|f(1)| \leq \left| \frac{1}{4} + \frac{1}{2}f''(\eta)(1 - x_0)^2 \right| \leq \frac{1}{4} + \frac{1}{2}(1 - x_0)^2.$$

相加得:

$$|f(0)| + |f(1)| \leq \frac{1}{2} + \frac{1}{2}(x_0^2 + (1 - x_0)^2).$$

注意  $x_0^2 + (1 - x_0)^2 = 1 - 2x_0(1 - x_0) \leq 1$ , 因此:

$$|f(0)| + |f(1)| \leq \frac{1}{2} + \frac{1}{2} \cdot 1 = 1.$$

33. 设函数  $f(x)$  在  $x = 0$  处可导, 且  $f(0) = 0, f'(0) = 1$ , 设

$$F(x) = \int_0^x t^{n-1} f(x^n - t^n) dt,$$

求

$$\lim_{x \rightarrow 0} \frac{F(x)}{x^{2n}}.$$

由题设  $f(0) = 0$ , 且  $f$  在 0 可导, 考虑换元  $t = xu$ , 则

$$F(x) = \int_0^x t^{n-1} f(x^n - t^n) dt = x^n \int_0^1 u^{n-1} f(x^n - x^n u^n) du.$$

注意  $x^n - x^n u^n = x^n(1 - u^n)$ , 故

$$F(x) = x^n \int_0^1 u^{n-1} f(x^n(1 - u^n)) du.$$

因  $f(0) = 0$ , 且  $f$  在 0 处可导, 令  $h = x^n(1 - u^n)$ , 有

$$f(h) = f(0) + f'(0)h + o(h) = f'(0)x^n(1 - u^n) + o(x^n).$$

所以

$$F(x) = x^n \int_0^1 u^{n-1} (f'(0)x^n(1 - u^n) + o(x^n)) du = f'(0)x^{2n} \int_0^1 u^{n-1}(1 - u^n) du + o(x^{2n}).$$

计算积分:

$$\int_0^1 u^{n-1}(1 - u^n) du = \int_0^1 u^{n-1} du - \int_0^1 u^{2n-1} du = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

故

$$\lim_{x \rightarrow 0} \frac{F(x)}{x^{2n}} = \frac{1}{2n}$$

# 积分

1. 已知  $p > 0$ , 证明

$$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \frac{1}{p+1}.$$

设

$$S_n = \sum_{k=1}^n k^p$$

则

$$\frac{S_n}{n^{p+1}} = \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^p.$$

这是函数  $f(x) = x^p$  在  $[0, 1]$  的黎曼和, 故

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{p+1}} = \int_0^1 x^p dx = \frac{1}{p+1}.$$

2. 求

$$\lim_{k \rightarrow \infty} \left( \frac{2017^{\frac{1}{k}}}{k+1} + \frac{2017^{\frac{2}{k}}}{k+\frac{1}{2}} + \cdots + \frac{2017^{\frac{k}{k}}}{k+\frac{1}{k}} \right)$$

写成黎曼和:

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k 2017^{\frac{j}{k}} = \int_0^1 2017^x dx = \left[ \frac{2017^x}{\ln 2017} \right]_0^1 = \frac{2016}{\ln 2017}.$$

(待验证,  $2017^{\frac{j}{k}}/(1+0?)$ )

3. 设

$$a_n = \frac{2}{n} \left[ (2^2 + 1) + \left(2 + \frac{2}{n}\right)^2 + 1 + \cdots + \left(2 + \frac{2n-2}{n}\right)^2 + 1 \right]$$

求  $\lim_{n \rightarrow \infty} a_n$

发现

$$a_n = \sum_{k=1}^n \frac{2}{n} \left[ \left( 2 + \frac{2k-2}{n} \right)^2 + 1 \right].$$

是函数  $f(x) = (2+x)^2 + 1$  在  $[0, 2]$  上的黎曼和, 故

$$\lim_{n \rightarrow \infty} a_n = \int_0^2 [(2+x)^2 + 1] dx = \left[ \frac{x^3}{3} + 2x^2 + 5x \right]_0^2 = \frac{62}{3}.$$

4. 试求

$$\lim_{n \rightarrow \infty} \frac{5}{n^2} \left[ \sqrt{4n^2 - 2 \cdot 1^2} + \sqrt{4n^2 - 2 \cdot 2^2} + \cdots + \sqrt{4n^2 - 2 \cdot n^2} \right]$$

将其写成黎曼和, 得

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{5}{n} \sqrt{4 - 2 \left( \frac{k}{n} \right)^2} = \int_0^1 5\sqrt{4 - 2x^2} dx.$$

可得原极限为

$$\left[ \frac{5}{\sqrt{2}} \left( x\sqrt{2 - x^2} + 2 \sin^{-1} \frac{x}{\sqrt{2}} \right) \right]_0^1 = \frac{5\sqrt{2}(\pi + 2)}{4}.$$

5. 设函数  $f: (0, \infty) \rightarrow \mathbb{R}$ , 且当  $x > 0$  时, 恒有

$$\lim_{n \rightarrow \infty} \frac{1}{3n-1} \left( f\left(\frac{x^2}{n}\right) + f\left(\frac{2x^2}{n}\right) + \cdots + f\left(\frac{nx^2}{n}\right) \right) = \frac{\sqrt[3]{7+x}}{x},$$

求  $f(1)$ 。

先写成黎曼和:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{3n-1} \left( f\left(\frac{x^2}{n}\right) + f\left(\frac{2x^2}{n}\right) + \cdots + f\left(\frac{nx^2}{n}\right) \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{3n-1} \sum_{k=1}^n f\left(\frac{kx^2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{n}{3n-1} \sum_{k=1}^n \frac{1}{n} f\left(\frac{kx^2}{n}\right) \\ &= \frac{1}{3} \int_0^1 f(x^2 t) dt = \frac{\sqrt[3]{7+x}}{x}. \end{aligned}$$

于是

$$\int_0^1 x^2 f(x^2 t) dt = 3x \sqrt[3]{7+x}.$$

取  $u = x^2 t \Rightarrow du = x^2 dt$ , 则

$$\begin{aligned} \int_0^{x^2} f(u) du &= 3x \sqrt[3]{7+x}, \\ \frac{d}{dx} \left( \int_0^{x^2} f(u) du \right) &= \frac{d}{dx} \left( 3x \sqrt[3]{7+x} \right), \\ 2x f(x^2) &= 3 \sqrt[3]{7+x} + x(7+x)^{-\frac{2}{3}}. \end{aligned}$$

令  $x = 1$ , 则

$$2f(1) = 3 \cdot 2 + \frac{1}{4} \Rightarrow f(1) = \frac{25}{8}.$$

6.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sin \frac{\pi}{12n} + \sin \frac{3\pi}{12n} + \sin \frac{5\pi}{12n} + \cdots + \sin \frac{(2n-1)\pi}{12n} \right)$$

写成黎曼和,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sin \frac{(2k-1)\pi}{12n} &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^{2n} \sin \frac{k\pi}{12n} - \sum_{k=1}^n \sin \frac{2k\pi}{12n} \right) \\ &= \int_0^2 \sin \frac{\pi}{12} x dx - \int_0^1 \sin \frac{\pi}{6} x dx \end{aligned}$$

可得

$$\left[ -\frac{12}{\pi} \cos \frac{\pi}{12} x \right]_0^2 - \left[ -\frac{6}{\pi} \cos \frac{\pi}{6} x \right]_0^1 = \frac{6-3\sqrt{3}}{\pi}$$

7. 求

$$f(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{x(1-x)}{k + (n-k)x}, \quad x \in [0, 1]$$

显然  $f(0) = f(1) = 0$ , 对于  $x \in [0, 1]$ ,

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{x(1-x)}{k + (n-k)x} &= x(1-x) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\left(\frac{k}{n}\right)(1-x) + x} \cdot \frac{1}{n} \\ &= x(1-x) \int_0^1 \frac{1}{(1-x)y + x} dy \\ &= x \cdot [\ln |(1-x)y + x|]_0^1 \\ &= -x \ln(x)\end{aligned}$$

## 8. 求极限

$$\lim_{n \rightarrow \infty} \left( \sum_{k=10}^{n+9} \frac{2^{11(k-9)/n}}{\log_2 e^{n/11}} - \sum_{k=0}^{n-1} \frac{58}{\pi \sqrt{(n-k)(n+k)}} \right)$$

首先有

$$\lim_{n \rightarrow \infty} \sum_{k=10}^{n+9} \frac{2^{11(k-9)/n}}{\log_2 e^{n/11}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n 2^{11\left(\frac{k}{n}\right)} \ln \left( \frac{11}{n} \right) \ln 2 = \int_0^{11} 2^x \ln 2 \, dx$$

且

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{58}{\pi \sqrt{(n-k)(n+k)}} = \frac{58}{\pi} \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{1}{n \sqrt{1 - \left(\frac{k}{n}\right)^2}} = \frac{58}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx$$

故所求为

$$\int_0^{11} 2^x \ln 2 \, dx - \frac{58}{\pi} \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = [2^x]_0^{11} - \frac{58}{\pi} [\arcsin x]_0^1 = 2047 - 29 = 2018$$

## 9. 求极限

$$\lim_{t \rightarrow 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n},$$

注意到

$$\lim_{t \rightarrow 1^-} \frac{1-t}{-\ln t} = 1$$

因此

$$\lim_{t \rightarrow 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n} = \lim_{t \rightarrow 1^-} (-\ln t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n} = (-\ln t) \sum_{n=1}^{\infty} \frac{1}{1+e^{-n \ln t}}$$

设  $h = -\ln t$ , 则当  $t \rightarrow 1^-$  时  $h \rightarrow 0^+$ , 于是上述极限化为

$$\lim_{h \rightarrow 0^+} h \sum_{n=1}^{\infty} \frac{1}{1 + e^{nh}}$$

这是一个黎曼和, 对应的积分为

$$\int_0^{\infty} \frac{dx}{1 + e^x} = \ln 2$$

10. 对于  $n = 1, 2, \dots$ , 定义

$$S_n = \log \left( \sqrt[n^2]{1^1 \cdot 2^2 \cdot \dots \cdot n^n} \right) - \log(\sqrt{n}),$$

其中  $\log$  表示自然对数。求  $\lim_{n \rightarrow \infty} S_n$ 。

将  $S_n$  变形如下:

$$\begin{aligned} S_n &= \frac{1}{n^2} \sum_{k=1}^n k \log k - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \left( \frac{k}{n} \left( \log \frac{k}{n} + \log n \right) \right) - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} + \frac{\log n}{n^2} \sum_{k=1}^n k - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} + \frac{\log n}{n^2} \cdot \frac{n(n+1)}{2} - \frac{1}{2} \log n \\ &= \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} + \frac{\log n}{2n} \end{aligned}$$

当  $n \rightarrow \infty$  时, 最后一项  $\frac{\log n}{2n}$  趋于 0, 而第一项是函数  $f(x) = x \log x$  在区间  $[0, 1]$  上的黎曼和, 因此

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \log \frac{k}{n} = \int_0^1 x \log x \, dx$$

利用分部积分法计算得

$$\int_0^1 x \log x \, dx = \left[ \frac{x^2}{2} \log x - \frac{x^2}{4} \right]_0^1 = -\frac{1}{4}$$

所以极限  $\lim_{n \rightarrow \infty} S_n$  存在, 且

$$\lim_{n \rightarrow \infty} S_n = -\frac{1}{4}$$

### 11. 求极限

$$\lim_{n \rightarrow \infty} \frac{(1^2 + 2^2 + \cdots + n^2)(1^5 + 2^5 + \cdots + n^5)}{(1^3 + 2^3 + \cdots + n^3)(1^4 + 2^4 + \cdots + n^4)}$$

首先看到

$$a_n = \frac{\left(\sum_{k=1}^n \left(\frac{k}{n}\right)^2\right) \left(\sum_{k=1}^n \left(\frac{k}{n}\right)^5\right)}{\left(\sum_{k=1}^n \left(\frac{k}{n}\right)^3\right) \left(\sum_{k=1}^n \left(\frac{k}{n}\right)^4\right)} = \frac{\left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^2\right) \left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^5\right)}{\left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^3\right) \left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^4\right)}$$

故

$$\lim_{n \rightarrow \infty} a_n = \frac{\left(\int_0^1 x^2 dx\right) \left(\int_0^1 x^5 dx\right)}{\left(\int_0^1 x^3 dx\right) \left(\int_0^1 x^4 dx\right)} = \frac{\frac{1}{3} \cdot \frac{1}{6}}{\frac{1}{4} \cdot \frac{1}{5}} = \frac{10}{9}$$

### 12. 计算极限

$$\lim_{n \rightarrow \infty} \frac{(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n})^2 (1^3 + 2^3 + \cdots + n^3)}{(\sqrt[3]{1} + \sqrt[3]{2} + \cdots + \sqrt[3]{n})^3 (1^2 + 2^2 + \cdots + n^2)}$$

由

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

可得

$$f(n) = \frac{n^2 \left(\sum_{k=1}^n \frac{1}{n} \sqrt{\frac{k}{n}}\right)^2}{n^3 \left(\sum_{k=1}^n \frac{1}{n} \sqrt[3]{\frac{k}{n}}\right)^3} \cdot \frac{3n(n+1)}{2(2n+1)}$$

由积分定义,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sqrt{\frac{k}{n}} = \int_0^1 \sqrt{x} dx = \frac{2}{3}, \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sqrt[3]{\frac{k}{n}} = \int_0^1 x^{\frac{1}{3}} dx = \frac{3}{4}$$

因此

$$\lim_{n \rightarrow \infty} f(n) = \frac{\left(\frac{2}{3}\right)^2}{\left(\frac{3}{4}\right)^3} \cdot \frac{3}{4} = \frac{64}{81}$$

13. 求极限

$$L = \lim_{n \rightarrow \infty} \left( \frac{(2n)!}{n! n^n} \right)^{\frac{1}{n}}$$

首先有

$$\ln L = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \ln(2n)! - \ln n! - n \ln n \right)$$

其中

$$\ln(2n)! - \ln n! = \sum_{k=n+1}^{2n} \ln k = \sum_{k=1}^n \ln(n+k)$$

因此

$$\ln L = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n \ln(n+k) - n \ln n \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left( 1 + \frac{k}{n} \right)$$

这是在  $[0, 1]$  上的黎曼和, 得

$$\ln L = \int_0^1 \ln(1+x) dx = \left[ (1+x) \ln(1+x) - x \right]_0^1 = 2 \ln 2 - 1$$

所以

$$L = e^{2 \ln 2 - 1} = \frac{4}{e}$$

14. 设

$$A_n = \{\sin(\ln 1), \sin(\ln 2), \dots, \sin(\ln n)\}, \quad B_n = \{\cos(\ln 1), \cos(\ln 2), \dots, \cos(\ln n)\}$$

且  $\bar{a}_n, \bar{b}_n$  分别为  $A_n, B_n$  的算术平均值. 求

$$\lim_{n \rightarrow \infty} (\bar{a}_n \cos(\ln n) - \bar{b}_n \sin(\ln n)).$$

由积分定义,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{i=1}^n (\sin(\ln i) \cos(\ln n) - \cos(\ln i) \sin(\ln n)) \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{i=1}^n \sin(\ln i - \ln n) \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \sin \left( \ln \left( \frac{i}{n} \right) \right) \\
 &= \int_0^1 \sin(\ln x) dx \\
 &= \Im \int_0^1 (e^{i \ln x}) dx \\
 &= \Im \int_0^1 x^i dx \\
 &= \Im \left( \frac{1}{i+1} \right) = -\frac{1}{2}
 \end{aligned}$$

15. 设函数  $f(x)$  在闭区间  $[0, 1]$  上具有连续导数,  $f(0) = 0, f(1) = 1$ 。证明:

$$\lim_{n \rightarrow \infty} n \left( \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right) = -\frac{1}{2}.$$

考察括号内的表达式:

$$\begin{aligned}
 & \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \\
 &= \sum_{k=1}^n \left( \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x) dx - \frac{1}{n} f\left(\frac{k}{n}\right) \right) \\
 &= \sum_{k=1}^n \left( \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( f\left(\frac{k}{n}\right) + \left(x - \frac{k}{n}\right) f'\left(\frac{k}{n}\right) + o\left(x - \frac{k}{n}\right) \right) dx - \frac{1}{n} f\left(\frac{k}{n}\right) \right) \\
 &= \sum_{k=1}^n \left( \left[ f\left(\frac{k}{n}\right) x + \frac{f'\left(\frac{k}{n}\right)}{2} \left(x - \frac{k}{n}\right)^2 + o\left(\left(x - \frac{k}{n}\right)^2\right) \right]_{\frac{k-1}{n}}^{\frac{k}{n}} - \frac{1}{n} f\left(\frac{k}{n}\right) \right) \\
 &= \sum_{k=1}^n \left( -\frac{f'\left(\frac{k}{n}\right)}{2n^2} + o\left(\frac{1}{n^2}\right) \right)
 \end{aligned}$$

将上述结果乘以  $n$  并代入原极限中, 利用黎曼和定义可得原式为

$$\begin{aligned}\lim_{n \rightarrow \infty} \left( -\frac{1}{2} \cdot \frac{1}{n} \sum_{k=1}^n f' \left( \frac{k}{n} \right) + o \left( \frac{1}{n} \right) \right) &= -\frac{1}{2} \int_0^1 f'(x) dx \\ &= -\frac{1}{2} [f(x)]_0^1 \\ &= -\frac{1}{2} (1 - 0) = -\frac{1}{2}\end{aligned}$$

证毕。

16. 证明

$$\frac{\pi}{4} < \int_0^1 \frac{1}{1+x^8} dx < 1$$

在  $x \in (0, 1)$  上, 有  $0 < x^8 < 1$ , 于是

$$\frac{1}{1+x^8} < 1 \implies \int_0^1 \frac{1}{1+x^8} dx < \int_0^1 1 dx = 1$$

又  $x^8 < x^2$ , 所以

$$\frac{1}{1+x^8} > \frac{1}{1+x^2} \implies \int_0^1 \frac{dx}{1+x^8} > \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

17. 若  $f(x)$  为一实系数多项式函数, 已知

$$\int_0^1 f(x)f'(x) dx = 1, \quad \int_0^1 (f(x))^3 f'(x) dx = 2,$$

求

$$\int_0^1 (f(x))^5 f'(x) dx$$

的值。

有

$$1 = \int_0^1 f(x)f'(x) dx = \left[ \frac{1}{2}(f(x))^2 \right]_0^1 = \frac{1}{2}([f(1)]^2 - [f(0)]^2) \quad (1)$$

同理,

$$2 = \int_0^1 (f(x))^3 f'(x) dx = \left[ \frac{1}{4}(f(x))^4 \right]_0^1 = \frac{1}{4}([f(1)]^4 - [f(0)]^4) \quad (2)$$

由 (1), (2) 解得

$$[f(0)]^2 = 1, \quad [f(1)]^2 = 3$$

于是

$$\int_0^1 (f(x))^5 f'(x) dx = \left[ \frac{1}{6} (f(x))^6 \right]_0^1 = \frac{1}{6} (f(1)^6 - f(0)^6) = \frac{13}{3}$$

18. 设  $f$  为实数系上的连续函数, 若

$$\frac{d}{dx} \int_{-x}^x f(t) dt = x^2 + 1,$$

求

$$\int_{-1}^1 f(x) dx$$

两边积分得

$$g(x) = \int_{-x}^x f(t) dt = \int (x^2 + 1) dx = \frac{1}{3}x^3 + x + C$$

令  $x = 0$ , 得

$$g(0) = \int_0^0 f(t) dt = 0 \Rightarrow C = 0 \Rightarrow g(x) = \frac{1}{3}x^3 + x$$

所以

$$\int_{-1}^1 f(x) dx = g(1) = \frac{4}{3}$$

19. 已知一连续实函数  $f(x)$  满足  $f(2x) = 3f(x), \forall x \in \mathbb{R}$ , 若  $\int_0^1 f(x) dx = 1$ , 求

$$\int_0^2 f(x) dx$$

由  $f(2x) = 3f(x)$  得

$$\int_0^1 f(x) dx = \frac{1}{3} \int_0^1 f(2x) dx = 1$$

令  $u = 2x$ , 则  $du = 2dx$ , 于是

$$\int_0^2 f(u) du = 6$$

故

$$\int_0^2 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx = 1 + \int_1^2 f(x) dx = 6 \Rightarrow \int_1^2 f(x) dx = 5$$

20. 设多项式函数  $y = f(x)$  满足

$$f(x) = 4x^3 - 12x^2 + 8x + 20 - \int_1^x f(t) dt,$$

求  $f(x)$ 。

对两边求导得

$$f'(x) + f(x) = 12x^2 - 24x + 8$$

故  $f(x)$  为二次多项式, 设

$$f(x) = ax^2 + bx + c, f'(x) = 2ax + b$$

代入得

$$f(x) + f'(x) = ax^2 + bx + c + 2ax + b = ax^2 + (b + 2a)x + (b + c)$$

比较系数, 解得

$$a = 12, b = -48, c = 56 \Rightarrow f(x) = 12x^2 - 48x + 56$$

21. 设

$$f(x) = \int_0^x g(t) dt + 1, \quad g(x) = 12x^2 - 6x + \int_0^1 [f(t) + g'(t)] dt,$$

求  $g(0)$ 。

由题意有

$$f(0) = 1, \quad g(0) = \int_0^1 [f(t) + g'(t)] dt,$$

又

$$f(x) = \int_0^x [12t^2 - 6t + g(0)] dt + 1 = 4x^3 - 3x^2 + xg(0) + 1$$

代入  $g(x)$  得

$$g(x) = 12x^2 - 6x + \int_0^1 (4t^3 - 3t^2 + tg(0) + 1) dt + g(1) - g(0)$$

其中

$$\int_0^1 (4t^3 - 3t^2 + tg(0) + 1)dt = \left[ t^4 - t^3 + \frac{1}{2}g(0)t^2 + t \right]_0^1 = \frac{1}{2}g(0) + 1$$

因此

$$g(x) = 12x^2 - 6x + 1 + g(1) - \frac{1}{2}g(0)$$

令  $x = 1$ , 得

$$g(0) = 14$$

22. 设

$$f(x) = x + 3 + \int_0^x g(t) dt, \quad g(x) = 2x - 9 + \int_0^x f(t) dt,$$

试求  $f(3)$ 。

设

$$a = \int_0^1 g(x) dx, \quad b = \int_0^2 f(x) dx.$$

由题得

$$f(x) = x + 3 + a, \quad g(x) = 2x - 9 + b.$$

因此

$$b = \int_0^2 f(x) dx = \int_0^2 (x + 3 + a) dx = \left[ \frac{1}{2}x^2 + (3 + a)x \right]_0^2 = 8 + 2a,$$

$$a = \int_0^1 g(x) dx = \int_0^1 (2x - 9 + b) dx = \left[ x^2 + (b - 9)x \right]_0^1 = b - 8.$$

联立得

$$b = 8 + 2a, \quad a = b - 8.$$

代入消元得

$$b = 8 + 2(b - 8) \Rightarrow b = 8, \quad a = b - 8 = 0.$$

所以

$$f(x) = x + 3 + a = x + 3.$$

因此

$$f(3) = 3 + 3 = 6.$$

(待验证, 为什么设  $a, b$ ?)

23. 设  $0 < a < b$ , 证明

$$\int_a^b (x^2 + 1)e^{-x^2} dx > e^{-a^2} - e^{-b^2}.$$

令

$$f(x) = \int_0^x (t^2 + 1)e^{-t^2} dt, \quad g(x) = -e^{-x^2}$$

两函数在  $(0, \infty)$  上均递增。由柯西中值定理, 存在  $x \in (a, b)$  使得

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)} = \frac{(x^2 + 1)e^{-x^2}}{2xe^{-x^2}} = \frac{1}{2} \left( x + \frac{1}{x} \right) \geq \sqrt{x \cdot \frac{1}{x}} = 1$$

于是

$$\int_a^b (x^2 + 1)e^{-x^2} dx = f(b) - f(a) \geq g(b) - g(a) = e^{-a^2} - e^{-b^2}$$

由不等式  $(x - 1)^2 \geq 0 \Rightarrow x^2 + 1 \geq 2x$ ,

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geq \int_a^b 2xe^{-x^2} dx = [-e^{-x^2}]_a^b = e^{-a^2} - e^{-b^2}$$

24. 设  $f$  是定义在区间  $[0, 1]$  上的连续实值函数。证明存在  $\xi \in [0, 1]$ , 使得

$$\int_0^1 x^2 f(x) dx = \frac{1}{3} f(\xi).$$

由于  $f$  连续, 由最值定理,  $f$  必在  $[0, 1]$  上取得最小值和最大值, 分别设之为  $f(a), f(b)$ , 其中  $a, b \in [0, 1]$ , 于是

$$f(a) \int_0^1 x^2 dx \leq \int_0^1 x^2 f(x) dx \leq f(b) \int_0^1 x^2 dx,$$

即

$$f(a) \leq 3 \int_0^1 x^2 f(x) dx \leq f(b).$$

由拉格朗日中值定理, 存在  $\xi \in [0, 1]$ , 使得

$$f(\xi) = 3 \int_0^1 x^2 f(x) dx.$$

即得证命题。

25. 设  $f: [0, 1] \rightarrow (0, \infty)$  是可积函数, 且对所有  $x \in [0, 1]$  有

$$f(x) \cdot f(1-x) = 1,$$

证明

$$\int_0^1 f(x) dx \geq 1.$$

对任意  $x \in [0, 1]$ , 由 AM-GM 不等式,

$$f(x) + f(1-x) \geq 2\sqrt{f(x)f(1-x)} = 2$$

在区间  $[0, \frac{1}{2}]$  上积分得

$$\int_0^1 f(x) dx = \int_0^{\frac{1}{2}} f(x) dx + \int_0^{\frac{1}{2}} f(1-x) dx = \int_0^{\frac{1}{2}} (f(x) + f(1-x)) dx \geq \int_0^{\frac{1}{2}} 2 dx = 1$$

由条件可得

$$\int_0^1 f(x) dx = \int_0^1 f(1-x) dx = \int_0^1 \frac{1}{f(x)} dx$$

于是由柯西不等式,

$$\left( \int_0^1 f(x) dx \right)^2 = \int_0^1 f(x) dx \cdot \int_0^1 \frac{1}{f(x)} dx \geq \left( \int_0^1 1 dx \right)^2 = 1$$

因此

$$\int_0^1 f(x) dx \geq 1$$

26. 设  $f: (-1, 1) \rightarrow \mathbb{R}$  是二阶可导函数, 满足

$$2f'(x) + xf''(x) \geq 1, \quad x \in (-1, 1).$$

证明

$$\int_{-1}^1 xf(x) dx \geq \frac{1}{3}.$$

令

$$g(x) = xf(x) - \frac{x^2}{2}$$

则

$$g''(x) = 2f'(x) + xf''(x) - 1 \geq 0$$

因此  $g$  是凸函数, 以  $g$  在  $x=0$  处的切线估计  $g$ 。设  $g'(0) = a$ , 则由凸性知

$$g(x) \geq g(0) + g'(0)x = ax$$

于是

$$\int_{-1}^1 xf(x) dx = \int_{-1}^1 \left( g(x) + \frac{x^2}{2} \right) dx \geq \int_{-1}^1 \left( ax + \frac{x^2}{2} \right) dx = \frac{1}{3}$$

这就证明了结论。

27. 设  $f$  为闭区间  $[0, 1]$  上的实值连续函数。已知

$$\int_0^1 f(x) dx = \frac{\pi}{4},$$

证明存在  $y$  满足  $0 < y < 1$  且

$$\frac{1}{1+y} < f(y) < \frac{1}{2y}.$$

注意到

$$\int_0^1 \frac{1}{1+x^2} dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

令

$$g(x) = f(x) - \frac{1}{1+x^2}$$

若  $g(x) \equiv 0 \quad \forall x \in (0, 1)$ , 则  $f(y) = \frac{1}{1+y^2}$  对所有  $y \in (0, 1)$  都成立; 反之, 由于

$$\int_0^1 g(x) dx = 0$$

存在  $x_0, x_1 \in (0, 1)$  使得  $g(x_0) < 0$  且  $g(x_1) > 0$ 。由  $g$  在  $(0, 1)$  上连续, 由介值定理, 存在  $y$  满足  $x_0 < y < x_1$  使得  $g(y) = 0$ , 于是  $f(y) = \frac{1}{1+y^2}$ 。

综上, 皆存在  $y \in (0, 1)$  使得  $f(y) = \frac{1}{1+y^2}$ 。由于  $0 < y < 1$ , 有

$$2y < 1 + y^2 < 1 + y$$

从而得到

$$\frac{1}{1+y} < f(y) < \frac{1}{2y}$$

证毕。

28. 设函数  $f$  在区间  $[a, b]$  上具有连续导数, 且满足  $f(a) = f(b) = 0$ 。证明

$$\max_{a \leq x \leq b} |f'(x)| \geq \frac{4}{(b-a)^2} \left| \int_a^b f(x) dx \right|.$$

由拉格朗日中值定理, 对任意  $x \in (a, b)$ , 存在  $\xi_1 \in (a, x), \xi_2 \in (x, b)$  使得

$$f(x) - f(a) = f'(\xi_1)(x-a), \quad f(x) - f(b) = f'(\xi_2)(x-b)$$

设

$$M = \max_{a \leq x \leq b} |f'(x)|$$

则

$$|f(x)| \leq M(x-a), \quad |f(x)| \leq M(b-x).$$

因此

$$\begin{aligned} \frac{4}{(b-a)^2} \int_a^b |f(x)| dx &\leq \frac{4}{(b-a)^2} \left( \int_a^{(a+b)/2} M(x-a) dx + \int_{(a+b)/2}^b M(b-x) dx \right) \\ &= \frac{4}{(b-a)^2} \left( \frac{(b-a)^2}{8} M + \frac{(b-a)^2}{8} M \right) = M \end{aligned}$$

又因为

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx,$$

于是得证

$$\max_{a \leq x \leq b} |f'(x)| \geq \frac{4}{(b-a)^2} \left| \int_a^b f(x) dx \right|$$

29. 设  $a$  为实数, 已知函数

$$f(x) = 4x^2 - 3ax + 4 \int_0^1 (tf(t)) dt,$$

及

$$g(x) = x^2 + 4x + a - \int_0^x ((t+1)g'(t)) dt.$$

若方程  $f(x) - x \cdot g(x) = 0$  有两相异实根  $\alpha, \beta$ , 且  $\alpha < \beta$ , 求

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} (3x^2 - 2ax + a^2) dx$$

的最小值。

设常数

$$C = \int_0^1 t f(t) dt,$$

则

$$f(x) = 4x^2 - 3ax + 4C.$$

计算  $C$ :

$$C = \int_0^1 t(4t^2 - 3at + 4C) dt = \int_0^1 (4t^3 - 3at^2 + 4Ct) dt = [t^4 - at^3 + 2Ct^2]_0^1 = 1 - a + 2C.$$

整理得

$$C = 1 - a + 2C \implies C = a - 1.$$

因此

$$f(x) = 4x^2 - 3ax + 4a - 4.$$

对  $g(x)$ , 由条件有

$$g'(x) = 2x + 4 - (x + 1)g'(x),$$

整理得

$$g'(x) + (x + 1)g'(x) = 2x + 4 \implies (x + 2)g'(x) = 2x + 4,$$

即

$$g'(x) = \frac{2x + 4}{x + 2} = 2.$$

计算积分

$$\int_0^x (t + 1)g'(t) dt = \int_0^x 2(t + 1) dt = x^2 + 2x,$$

代入  $g(x)$ :

$$g(x) = x^2 + 4x + a - (x^2 + 2x) = 2x + a.$$

因此

$$f(x) - xg(x) = (4x^2 - 3ax + 4a - 4) - x(2x + a) = 4x^2 - 3ax + 4a - 4 - 2x^2 - ax = 2x^2 - 4ax + 4a - 4.$$

设两根为  $\alpha, \beta$ , 满足

$$\alpha + \beta = 2a, \quad \alpha\beta = 2a - 2.$$

计算

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} (3x^2 - 2ax + a^2) dx = \frac{1}{\beta - \alpha} [x^3 - ax^2 + a^2x]_{\alpha}^{\beta} = \frac{1}{\beta - \alpha} (\beta^3 - \alpha^3 - a(\beta^2 - \alpha^2) + a^2(\beta - \alpha)).$$

利用因式分解:

$$\beta^3 - \alpha^3 = (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2), \quad \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha),$$

因此表达式为

$$(\beta^2 + \alpha\beta + \alpha^2) - a(\beta + \alpha) + a^2.$$

利用已知根的和与积:

$$\beta + \alpha = 2a, \quad \alpha\beta = 2a - 2,$$

以及

$$\beta^2 + \alpha^2 = (\beta + \alpha)^2 - 2\alpha\beta = (2a)^2 - 2(2a - 2) = 4a^2 - 4a + 4.$$

所以

$$\beta^2 + \alpha\beta + \alpha^2 = (\beta^2 + \alpha^2) + \alpha\beta = (4a^2 - 4a + 4) + (2a - 2) = 4a^2 - 2a + 2.$$

代入原式:

$$4a^2 - 2a + 2 - a \cdot 2a + a^2 = 4a^2 - 2a + 2 - 2a^2 + a^2 = 3a^2 - 2a + 2.$$

令

$$h(a) = 3a^2 - 2a + 2 = 3\left(a - \frac{1}{3}\right)^2 + \frac{5}{3}.$$

因此最小值为

$$\frac{5}{3}.$$

(待验证)

30. 已知

$$\int_0^2 f(x) dx = 1, \quad f(2) = \frac{1}{2}, \quad f'(2) = 0,$$

计算

$$\int_0^1 x^2 f''(2x) dx$$

进行变量代换, 令  $t = 2x$ , 则  $dt = 2dx$ , 于是

$$I = \int_0^1 x^2 f''(2x) dx = \frac{1}{2} \int_0^2 \left(\frac{t}{2}\right)^2 f''(t) dt = \frac{1}{8} \int_0^2 t^2 f''(t) dt$$

由分部积分法,

$$I = \frac{1}{8} \left( [t^2 f'(t)]_0^2 - \int_0^2 2t f'(t) dt \right) = -\frac{1}{4} \int_0^2 t f'(t) dt$$

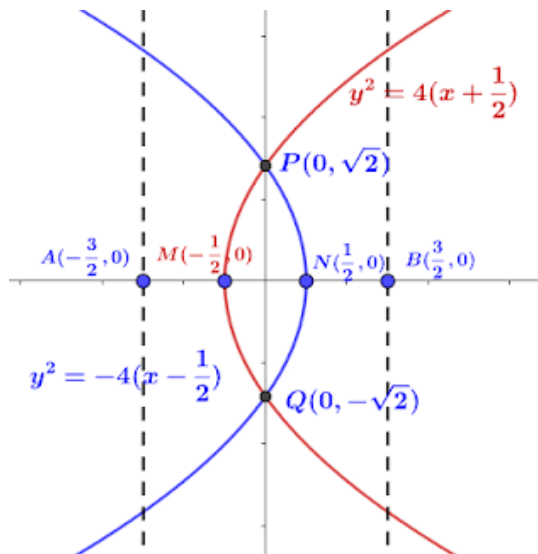
再由分部积分法,

$$I = -\frac{1}{4} ([t f(t)]_0^2 - \int_0^2 f(t) dt) = -\frac{1}{4} (2 \cdot \frac{1}{2} - 1) = 0$$

31. 坐标平面上,  $A, B$  两点分别在直线

$$L_1 : x = -\frac{3}{2}, \quad L_2 : x = \frac{3}{2}$$

上。已知  $AB \perp L_1$ , 且  $M, N$  为  $AB$  的三等分点, 并满足  $AM = MN = NB$ 。设  $\Gamma_1$  为以  $L_1$  为准线,  $N$  为焦点的抛物线;  $\Gamma_2$  为以  $L_2$  为准线,  $N$  为顶点的抛物线, 求  $\Gamma_1$  与  $\Gamma_2$  所围区域的面积。



所围面积为

$$I = 4 \int_0^{\frac{1}{2}} \sqrt{-4 \left( x - \frac{1}{2} \right)} dx = 8 \int_0^{\frac{1}{2}} \sqrt{\frac{1}{2} - x} dx$$

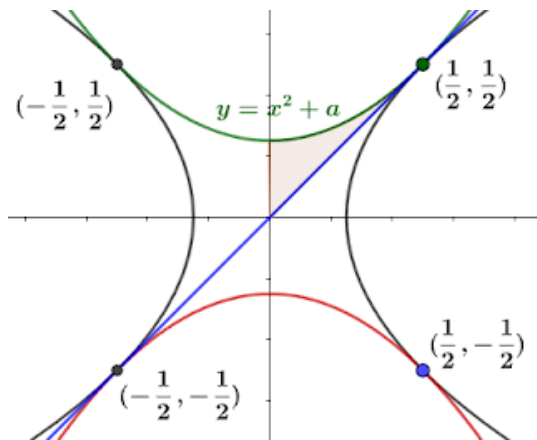
令  $u = \frac{1}{2} - x, du = -dx$ , 则

$$I = 8 \int_0^{\frac{1}{2}} \sqrt{u} du = 8 \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_0^{\frac{1}{2}} = \frac{4\sqrt{2}}{3}$$

32. 已知四条抛物线

$$\Gamma_1: y = x^2 + a, \quad \Gamma_2: y = -x^2 - a, \quad \Gamma_3: y^2 = x - a, \quad \Gamma_4: y^2 = -x - a,$$

其中  $a$  为正实数, 若任相邻两条抛物线均相切, 试求这四条抛物线所围成之区域面积。



$\Gamma_1, \Gamma_3$  的共切点在  $x = y$  上, 故知

$$x^2 - x + a = 0$$

恰有一根, 其中判别式为

$$1 - 4a = 0 \Rightarrow a = \frac{1}{4}$$

故四个切点为

$$\left( \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, -\frac{1}{2} \right), \left( -\frac{1}{2}, \frac{1}{2} \right), \left( -\frac{1}{2}, -\frac{1}{2} \right),$$

所围面积为

$$8 \int_0^{\frac{1}{2}} \left( x^2 - x + \frac{1}{4} \right) dx = 8 \left[ \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4} \right]_0^{\frac{1}{2}} = \frac{1}{3}$$

## 33. 已知曲线

$$y^2 + 2xy + 2x^2 = 50,$$

求被  $x$  轴和曲线上  $y \geq 0$  部分围成的区域面积。

将曲线改写为

$$y^2 + 2xy + 2x^2 = 50 \Rightarrow y = -x \pm \sqrt{50 - x^2}$$

可知曲线与坐标轴交于  $(0, \pm 5\sqrt{2})$ ,  $(\pm 5, 0)$ , 设曲线与某垂直线切于点  $P$ , 对曲线求导,

$$4x + 2y + 2x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x + y}{x + y}$$

此时  $\frac{dy}{dx}$  不存在, 故有

$$x + y = 0 \Rightarrow y = -x$$

代入曲线方程解得

$$2x^2 + 2x(-x) + (-x)^2 = 50 \Rightarrow x = -5\sqrt{2}$$

所以垂直切线点为  $P = (-5\sqrt{2}, 5\sqrt{2})$ , 故所围面积为

$$A = \int_{-5\sqrt{2}}^5 (-x + \sqrt{50 - x^2}) dx - \int_{-5\sqrt{2}}^{-5} (-x - \sqrt{50 - x^2}) dx$$

其中由换元  $x = \sqrt{50} \sin \theta$  知不定积分

$$\begin{aligned} \int \sqrt{50 - x^2} dx &= \int \sqrt{50 - \sin^2 \theta} \cdot \sqrt{50} \cos \theta d\theta \\ &= 50 \int \cos^2 \theta d\theta \\ &= 25 \int (1 + \cos 2\theta) d\theta \\ &= 25 \left( \theta + \frac{1}{2} \sin 2\theta \right) + C \end{aligned}$$

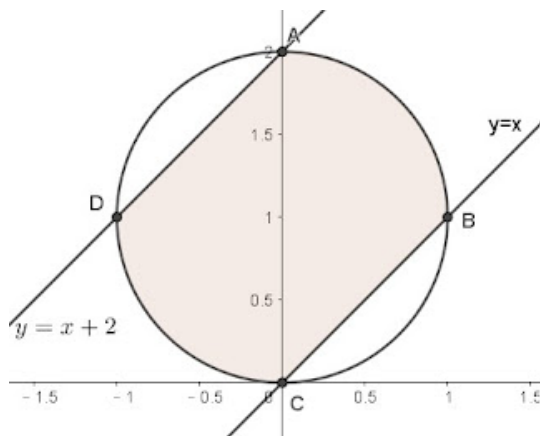
于是

$$\begin{aligned} A &= \left[ -\frac{1}{2}x^2 \right]_{-5\sqrt{2}}^5 + \left[ \frac{1}{2}x^2 \right]_{-5\sqrt{2}}^{-5} + \left[ 25\theta + \frac{25 \sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + \left[ 25\theta + \frac{25 \sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \\ &= 25\pi \end{aligned}$$

34. 坐标平面上, 满足联立不等式

$$\begin{cases} x^2 + (y-1)^2 \leq 1 \\ x-y \leq 0 \\ x-y \geq -2 \end{cases}$$

的解区域为  $S$ , 求区域  $S$  绕  $x$  轴旋转一圈所得立体的体积。



联立  $x^2 + (y-1)^2 = 1$  与  $y = x+2$  得

$$A(0, 2), D(-1, 1)$$

联立  $x^2 + (y-1)^2 = 1$  与  $y = x$  得

$$B(1, 1), C(0, 0)$$

如上图, 左半部绕  $x$  轴旋转体的体积为

$$\begin{aligned} V_1 &= \pi \int_{-1}^0 \left[ (x+2)^2 - (1 - \sqrt{1-x^2})^2 \right] dx \\ &= \pi \int_{-1}^0 \left[ 2x^2 + 4x + 2 + 2\sqrt{1-x^2} \right] dx \\ &= \pi \left[ \frac{2}{3}x^3 + 2x^2 + 2x + (\sqrt{1-x^2} + \arcsin x) \right]_{-1}^0 = \frac{1}{2}\pi^2 + \frac{5}{3}\pi \end{aligned}$$

上图右半部绕  $x$  轴旋转体的体积为

$$\begin{aligned} V_2 &= \pi \int_0^1 \left[ (1 + \sqrt{1-x^2})^2 - x^2 \right] dx \\ &= \pi \int_0^1 \left[ 2 - 2x^2 + 2\sqrt{1-x^2} \right] dx \\ &= \pi \left[ 2x - \frac{2}{3}x^3 + \sqrt{1-x^2} + \arcsin x \right]_0^1 = \frac{1}{2}\pi^2 + \frac{1}{3}\pi \end{aligned}$$

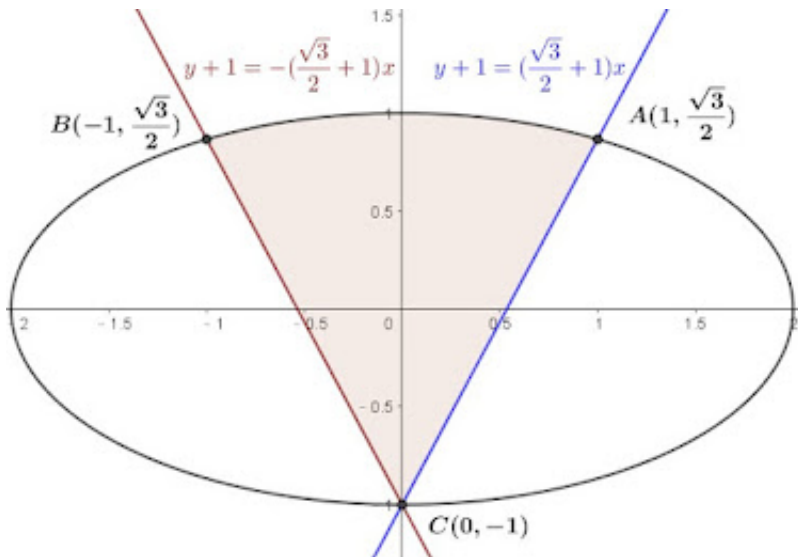
因此总体积为

$$V = V_1 + V_2 = \pi^2 + 2\pi$$

35. 在坐标平面上, 求由不等式

$$\frac{x^2}{4} + y^2 \leq 1, \quad y + 1 \geq \left(\frac{\sqrt{3}}{2} + 1\right)x, \quad y + 1 \geq -\left(\frac{\sqrt{3}}{2} + 1\right)x$$

所围成的图形面积。



三方程的交点为

$$A\left(1, \frac{\sqrt{3}}{2}\right), \quad B\left(-1, \frac{\sqrt{3}}{2}\right), \quad C(0, -1)$$

图形关于  $y$  轴对称, 于是所求面积为

$$S = 2 \left[ \int_0^1 \sqrt{1 - \frac{x^2}{4}} dx - \int_0^1 \left( \frac{\sqrt{3} + 2}{2}x - 1 \right) dx \right]$$

令  $x = 2 \sin \theta, dx = 2 \cos \theta d\theta$ , 则

$$\int_0^1 \sqrt{1 - \frac{x^2}{4}} dx = \int_0^{\frac{\pi}{6}} \sqrt{1 - \sin^2 \theta} \cdot 2 \cos \theta d\theta = \int_0^{\frac{\pi}{6}} (1 + \cos 2\theta) d\theta = \frac{\sqrt{3}}{4} + \frac{\pi}{6}$$

且

$$\int_0^1 \left( \frac{\sqrt{3} + 2}{2}x - 1 \right) dx = \left[ \frac{\sqrt{3} + 2}{4}x^2 - x \right]_0^1 = \frac{\sqrt{3} - 2}{4}$$

所以

$$S = 2 \left( \frac{\pi}{6} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}-2}{4} \right) = 1 + \frac{\pi}{3}$$

36. 计算

$$\lim_{x \rightarrow 0^+} \int_x^{2x} \frac{\sin^m t}{t^n} dt, \quad m, n \in \mathbb{N}.$$

注意到当  $t \rightarrow 0^+$  时  $\frac{\sin t}{t} \rightarrow 1$ , 且函数  $\frac{\sin t}{t}$  在  $(0, \pi)$  上单调递减。因此, 对于  $x \in (0, \frac{\pi}{2})$  且  $t \in [x, 2x]$ , 有

$$\frac{\sin 2x}{2x} < \frac{\sin t}{t} < 1.$$

于是

$$\left( \frac{\sin 2x}{2x} \right)^m \int_x^{2x} t^{m-n} dt < \int_x^{2x} \frac{\sin^m t}{t^n} dt < \int_x^{2x} t^{m-n} dt.$$

对积分做换元  $t = xu$  得

$$\int_x^{2x} t^{m-n} dt = x^{m-n+1} \int_1^2 u^{m-n} du.$$

注意到  $\left( \frac{\sin 2x}{2x} \right)^m \rightarrow 1$ , 于是极限取决于  $x^{m-n+1}$  的幂:

$$\lim_{x \rightarrow 0^+} \int_x^{2x} \frac{\sin^m t}{t^n} dt = \begin{cases} 0, & m - n + 1 > 0 \\ \ln 2, & m - n + 1 = 0 \\ +\infty, & m - n + 1 < 0 \end{cases}$$

37. (a) 已知  $a, b$  是相异实数,  $\lambda$  为非零实参数。证明

$$\left[ \int_a^b f(x)g(x) dx \right]^2 \leq \left[ \int_a^b [f(x)]^2 dx \right] \left[ \int_a^b [g(x)]^2 dx \right]$$

由关系  $[\lambda f(x) + g(x)]^2 \geq 0$ , 展开得

$$\lambda^2 [f(x)]^2 + 2\lambda f(x)g(x) + [g(x)]^2 \geq 0$$

对  $x$  从  $a$  到  $b$  积分知

$$\lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x) dx + \int_a^b [g(x)]^2 dx \geq 0$$

即关于  $\lambda$  的二次不等式, 要求判别式为非正:

$$4\lambda^2 \left[ \int_a^b f(x)g(x) dx \right]^2 - \left[ 2\lambda \int_a^b [f(x)]^2 dx \right] \left[ 2\lambda \int_a^b [g(x)]^2 dx \right] \leq 0$$

消去  $\lambda$  即得证。

(b) 试证明

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \leq \sqrt{\frac{\pi}{2}}$$

令  $f(x) = \sqrt{\sin x}$ ,  $g(x) = 1$ , 则

$$\left[ \int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \right]^2 \leq \left[ \int_0^{\frac{\pi}{2}} \sin x dx \right] \left[ \int_0^{\frac{\pi}{2}} 1 dx \right] = [-\cos x]_0^{\frac{\pi}{2}} \cdot [x]_0^{\frac{\pi}{2}} = 1 \cdot \frac{\pi}{2}$$

故

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \leq \sqrt{\frac{\pi}{2}}$$

(c) 试证明

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \geq \frac{16}{25} \cdot \frac{4}{\pi} = \frac{64}{25\pi}$$

令  $f(x) = (\sin x)^{\frac{1}{4}}$ ,  $g(x) = \cos x$ , 则

$$\left[ \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{4}} \cos x dx \right]^2 \leq \left[ \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{2}} dx \right] \left[ \int_0^{\frac{\pi}{2}} \cos^2 x dx \right]$$

左式积分可用代换  $u = \sin x$ ,  $du = \cos x dx$ :

$$\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{4}} \cos x dx = \frac{4}{5}$$

而右式第二个积分为

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx = \frac{\pi}{4}$$

代入不等式得

$$\left(\frac{4}{5}\right)^2 \leq \left[ \int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \right] \cdot \frac{\pi}{4}$$

即

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \geq \frac{64}{25\pi}$$

# 积分技巧

1. 计算  $[\int_2^4 (\log_x 2 - \frac{\log_x^2 2}{\ln 2}) dx]$ 。

利用换底公式简化被积函数:

$$\log_x 2 - \frac{(\frac{\ln 2}{\ln x})^2}{\ln 2} = \frac{\ln 2}{\ln x} - \frac{\ln 2}{(\ln x)^2} = \ln 2 \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right)$$

由于  $\frac{d}{dx}(\frac{x}{\ln x}) = \frac{\ln x - 1}{\ln^2 x} = \frac{1}{\ln x} - \frac{1}{\ln^2 x}$ , 该积分的原始函数为:

$$I = \ln 2 \left[ \frac{x}{\ln x} \right]_2^4 = \ln 2 \left( \frac{4}{\ln 4} - \frac{2}{\ln 2} \right)$$

已知  $\ln 4 = 2 \ln 2$ , 代入得:

$$I = \ln 2 \left( \frac{4}{2 \ln 2} - \frac{2}{\ln 2} \right) = \ln 2 \left( \frac{2}{\ln 2} - \frac{2}{\ln 2} \right) = 0$$

2. 计算积分 substitution methods

$$\int_{80^4}^{15^4} \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx$$

因为积分区间为正, 可将被积式中的  $x^4$  提出:

$$\int_{80^4}^{15^4} \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx = \int_{80^4}^{15^4} \frac{1}{x^2(x^4)^{\frac{3}{4}}(x^{-4}+1)^{\frac{3}{4}}} dx = \int_{80^4}^{15^4} \frac{1}{x^2 x^3 (x^{-4}+1)^{\frac{3}{4}}} dx = \int_{80^4}^{15^4} x^{-5} (1+x^{-4})^{-\frac{3}{4}} dx$$

注意到

$$\frac{d}{dx} \left[ (1+x^{-4})^{\frac{1}{4}} \right] = \frac{1}{4} (1+x^{-4})^{-\frac{3}{4}} \cdot (-4x^{-5}) = -x^{-5} (1+x^{-4})^{-\frac{3}{4}}$$

因此

$$\int_{80^4}^{15^4} x^{-5} (1+x^{-4})^{-\frac{3}{4}} dx = - \left[ (1+x^{-4})^{\frac{1}{4}} \right]_{80^4}^{15^4} = - \left[ (1+15^{-4})^{\frac{1}{4}} - (1+80^{-4})^{\frac{1}{4}} \right] = -(16^{\frac{1}{4}} - 81^{\frac{1}{4}}) = -(2-3) = 1$$

最终结果:

$$\int_{80^4}^{15^4} \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx = 1$$

### 3. 计算积分

$$\int_0^1 \frac{1}{(x^2 + 4x^3)^{\frac{2}{4}}} dx$$

先将分母中的根式因式分解:

$$x^2 + 4x^3 = x^2(1 + 4x) = [x^{\frac{1}{2}}(x^{\frac{1}{2}} + 4)^{\frac{3}{4}}]^{\frac{2}{4}} \quad (\text{整理成适合代换的形式})$$

更简便地处理:

$$\int_0^1 \frac{1}{(x^2 + 4x^3)^{\frac{1}{2}}} dx = \int_0^1 \frac{1}{x^{\frac{1}{2}}(x^{\frac{1}{2}} + 4)^{\frac{3}{4}}} dx$$

作代换:

$$u = x^{\frac{1}{2}} \implies x = u^2, \quad dx = 2u du$$

积分上下限不变 ( $x = 0 \rightarrow u = 0, x = 1 \rightarrow u = 1$ ):

$$\int_0^1 \frac{1}{u(u+4)^{\frac{3}{4}}} \cdot 2u du = \int_0^1 \frac{2}{(u+4)^{\frac{3}{4}}} du$$

直接积分:

$$\begin{aligned} \int_0^1 2(u+4)^{-\frac{3}{4}} du &= \left[ 8(u+4)^{\frac{1}{4}} \right]_0^1 = 8 \left( 5^{\frac{1}{4}} - 4^{\frac{1}{4}} \right) \\ &= 8 \left( \sqrt[4]{5} - \sqrt{2} \right) \end{aligned}$$

### 4. 7) 计算定积分: $\int_0^4 x^3 \sqrt{9+x^2} dx$

设  $u = 9 + x^2, du = 2x dx$ :

$$\begin{aligned} I &= \frac{1}{2} \int_9^{25} (u-9) \sqrt{u} du = \frac{1}{2} \left[ \frac{2}{5} u^{5/2} - 6u^{3/2} \right]_9^{25} \\ &= \left[ \frac{1}{5} u^{5/2} - 3u^{3/2} \right]_9^{25} = (625 - 375) - \left( \frac{243}{5} - 81 \right) = 282.4 \end{aligned}$$

结果: 282.4

### 5. 1) 计算不定积分:

$$\int \frac{1}{1+x^4} dx$$

首先将被积函数拆分:

$$\int \frac{1}{1+x^4} dx = \frac{1}{2} \int \frac{x^2+1}{x^4+1} dx - \frac{1}{2} \int \frac{x^2-1}{x^4+1} dx$$

对于第一部分, 分子分母同除以  $x^2$ :

$$\begin{aligned} \frac{1}{2} \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx &= \frac{1}{2} \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} \\ &= \frac{1}{2\sqrt{2}} \tan^{-1} \left( \frac{x-\frac{1}{x}}{\sqrt{2}} \right) = \frac{\sqrt{2}}{4} \tan^{-1} \left( \frac{x^2-1}{\sqrt{2}x} \right) \end{aligned}$$

对于第二部分, 同理可得:

$$\frac{1}{2} \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \frac{1}{2} \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}$$

利用对数积分公式:

$$= \frac{1}{4\sqrt{2}} \ln \left| \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right| = \frac{\sqrt{2}}{8} \ln \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right|$$

合并结果得到:

$$\int \frac{1}{1+x^4} dx = \frac{\sqrt{2}}{4} \tan^{-1} \left( \frac{x^2-1}{\sqrt{2}x} \right) - \frac{\sqrt{2}}{8} \ln \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| + C$$

## 6. 计算积分

$$\int \frac{4}{e^{3x}\sqrt{e^{2x}+4}} dx$$

作代换:

$$t = \frac{1}{e^x} \implies dx = -\frac{dt}{t}$$

代入积分:

$$\int \frac{4}{e^{3x}\sqrt{e^{2x}+4}} dx = \int \frac{4}{\frac{1}{t^3}\sqrt{\frac{1}{t^2}+4}} \left(-\frac{dt}{t}\right) = \int \frac{-4t^3}{\sqrt{1+4t^2}} dt = \int \frac{-4t^3}{\frac{\sqrt{1+4t^2}}{t}} dt = \int \frac{-4t^4}{\sqrt{1+4t^2}} dt$$

整理:

$$\frac{-4t^4}{\sqrt{1+4t^2}} = -4t(1+4t^2)^{-\frac{1}{2}} dt$$

直接使用反链式法则或作代换  $u = 1 + 4t^2$ :

$$\int -4t(1 + 4t^2)^{-\frac{1}{2}} dt = -(1 + 4t^2)^{\frac{1}{2}} + C$$

代回  $t = \frac{1}{e^x}$ :

$$-(1 + 4t^2)^{\frac{1}{2}} + C = -\left(1 + \frac{4}{e^{2x}}\right)^{\frac{1}{2}} + C = -\left(\frac{e^{2x} + 4}{e^{2x}}\right)^{\frac{1}{2}} + C = -\frac{\sqrt{e^{2x} + 4}}{e^x} + C$$

7. 求

$$\int_0^\pi \frac{1 + x \cos x}{x + e^{\sin x}} dx$$

作代换

$$u = 1 + xe^{\sin x}$$

$$\frac{du}{dx} = e^{\sin x} + xe^{\sin x} \cos x = e^{\sin x}(1 + x \cos x)$$

$$dx = \frac{du}{e^{\sin x}(1 + x \cos x)}$$

积分的上下限变换

$$x = 0 \implies u = 1$$

$$x = \pi \implies u = 1 + \pi$$

代入积分

$$\int_0^\pi \frac{1 + x \cos x}{x + e^{\sin x}} dx = \int_1^{1+\pi} \frac{1 + x \cos x}{x + e^{\sin x}} \cdot \frac{du}{e^{\sin x}(1 + x \cos x)} = \int_1^{1+\pi} \frac{1}{u} du$$

计算积分

$$[\ln |u|]_1^{1+\pi} = \ln(1 + \pi) - \ln 1 = \ln(1 + \pi)$$

8. 计算积分

$$\int_0^{\ln 2} \sqrt{e^x - 1} dx$$

令

$$u = \sqrt{e^x - 1} \implies u^2 = e^x - 1, \quad 2u \, du = e^x \, dx, \quad dx = \frac{2u}{u^2 + 1} \, du.$$

积分上下限变为:

$$x = 0 \implies u = 0, \quad x = \ln 2 \implies u = \sqrt{2 - 1} = 1.$$

原积分变为:

$$\int_0^1 u \cdot \frac{2u}{u^2 + 1} \, du = \int_0^1 \frac{2u^2}{u^2 + 1} \, du.$$

通过拆分:

$$\frac{2u^2}{u^2 + 1} = 2 - \frac{2}{u^2 + 1}.$$

因此积分为:

$$\int_0^1 \left( 2 - \frac{2}{u^2 + 1} \right) \, du = [2u - 2 \arctan u]_0^1 = 2 - 2 \cdot \frac{\pi}{4} = 2 - \frac{\pi}{2}.$$

$$\therefore \int_0^{\ln 2} \sqrt{e^x - 1} \, dx = 2 - \frac{\pi}{2}.$$

## 9. 计算积分

$$\int_0^\infty \frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} \, dx$$

先通过部分分式分解 (观察得出):

$$\frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} = -\frac{1}{e^{8x} + 3} + \frac{1}{e^{2x} + 3}$$

于是积分可写为:

$$\int_0^\infty \frac{e^{8x} - e^{2x}}{(e^{8x} + 3)(e^{2x} + 3)} \, dx = \int_0^\infty \frac{1}{e^{2x} + 3} \, dx - \int_0^\infty \frac{1}{e^{8x} + 3} \, dx$$

先计算第一个积分:

$$\int_0^\infty \frac{1}{e^{2x} + 3} \, dx = \int_0^\infty \frac{e^{-2x}}{1 + 3e^{-2x}} \, dx = \left[ -\frac{1}{6} \ln(1 + 3e^{-2x}) \right]_0^\infty = \frac{1}{6} \ln 4$$

第二个积分:

$$-\int_0^{\infty} \frac{1}{e^{8x}+3} dx = -\int_0^{\infty} \frac{e^{-8x}}{1+3e^{-8x}} dx = \left[ -\frac{1}{24} \ln(1+3e^{-8x}) \right]_0^{\infty} = \frac{1}{24} \ln 4$$

合并结果:

$$\int_0^{\infty} \frac{e^{8x} - e^{2x}}{(e^{8x}+3)(e^{2x}+3)} dx = \frac{1}{6} \ln 4 - \frac{1}{24} \ln 4 = \frac{3}{24} \ln 4 = \frac{1}{8} \ln 4 = \frac{1}{8} \ln(2^2) = \frac{1}{4} \ln 2$$

10. 计算

$$\int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} dx$$

首先观察积分形式, 尝试将分母调整为  $1 + (xe^{-x})^2$  的形式:

$$\int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} dx = \int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} \cdot \frac{e^{-2x}}{e^{-2x}} dx = \int_0^1 \frac{e^{-x}(1-x)}{(xe^{-x})^2 + 1} dx$$

设

$$u = xe^{-x} \implies du = (1-x)e^{-x} dx$$

积分上下限对应:

$$x = 0 \rightarrow u = 0, \quad x = 1 \rightarrow u = e^{-1}$$

因此积分变为

$$\int_0^1 \frac{e^{-x}(1-x)}{(xe^{-x})^2 + 1} dx = \int_0^{e^{-1}} \frac{1}{u^2 + 1} du = [\arctan u]_0^{e^{-1}} = \arctan e^{-1} - \arctan 0$$

最终得到

$$\int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} dx = \arctan \frac{1}{e}$$

11. 已知

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

求

$$\int_e^{\infty} e^{-x^2} \ln(x^2) x^{(\ln(x^{-x^2})+2x^2)} dx$$

发现

$$\begin{aligned} I &= \int_e^\infty e^{-x^2} \ln(x^2) x^{(\ln(x^{-x^2})+2x^2)} dx \\ &= \int_e^\infty e^{-x^2} \ln(x^2) e^{\ln x(-x^2 \ln x + 2x^2)} dx \\ &= 2 \int_e^\infty e^{-(x \ln x - x)^2} \ln(x) dx \end{aligned}$$

设  $u = x \ln x - x$ , 则  $du = \left( \ln x + x \cdot \frac{1}{x} - 1 \right) dx = \ln x dx$ , 于是

$$I = 2 \int_0^\infty e^{-u^2} du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

12. 计算不定积分:

$$\int \left( \frac{\ln x - 1}{1 + (\ln x)^2} \right)^2 dx$$

这道题可以通过换元法简化。设  $u = \ln x$ , 则  $x = e^u, dx = e^u du$ 。

代入原式:

$$\begin{aligned} I &= \int \left( \frac{u - 1}{1 + u^2} \right)^2 e^u du \\ &= \int \frac{u^2 - 2u + 1}{(1 + u^2)^2} e^u du \\ &= \int \left[ \frac{1 + u^2}{(1 + u^2)^2} - \frac{2u}{(1 + u^2)^2} \right] e^u du \\ &= \int \left[ \frac{1}{1 + u^2} - \frac{2u}{(1 + u^2)^2} \right] e^u du \end{aligned}$$

注意到被积函数符合分部积分的特征  $\int [f(u) + f'(u)]e^u du = f(u)e^u + C$ 。令  $f(u) = \frac{1}{1+u^2}$ , 则  $f'(u) = -\frac{2u}{(1+u^2)^2}$ 。

因此:

$$\begin{aligned} I &= \frac{1}{1 + u^2} e^u + C \\ &= \frac{x}{1 + (\ln x)^2} + C \end{aligned}$$

结果:  $\frac{x}{(\ln x)^2 + 1} + C$

13. 18) 计算不定积分  $\int x\sqrt{x+19} dx$

本题展示了解两种正确的解法。

解法一：利用凑微分与分配律

$$\begin{aligned}\int x\sqrt{x+19} dx &= \int (x+19-19)(x+19)^{\frac{1}{2}} dx \\&= \int (x+19)^{\frac{3}{2}} dx - 19 \int (x+19)^{\frac{1}{2}} dx \\&= \frac{2}{5}(x+19)^{\frac{5}{2}} - \frac{38}{3}(x+19)^{\frac{3}{2}} + C\end{aligned}$$

解法二：分部积分法设  $u = x, dv = (x+19)^{\frac{1}{2}} dx \implies du = dx, v = \frac{2}{3}(x+19)^{\frac{3}{2}}$ :

$$\begin{aligned}\int x\sqrt{x+19} dx &= \frac{2}{3}x(x+19)^{\frac{3}{2}} - \frac{2}{3} \int (x+19)^{\frac{3}{2}} dx \\&= \frac{2}{3}x(x+19)^{\frac{3}{2}} - \frac{4}{15}(x+19)^{\frac{5}{2}} + C\end{aligned}$$

注：两结果形式不同但通过化简后实质相等。

14. 20) 计算定积分  $I = \int_3^6 (\sqrt{x+\sqrt{12x-36}} + \sqrt{x-\sqrt{12x-36}}) dx$

设  $x = 3+t, dx = dt$ 。当  $x = 3, t = 0$ ；当  $x = 6, t = 3$ 。注意到  $\sqrt{12x-36} = \sqrt{12(x-3)} = \sqrt{12t} = 2\sqrt{3t}$ 。

$$I = \int_0^3 (\sqrt{3+t+2\sqrt{3t}} + \sqrt{3+t-2\sqrt{3t}}) dt$$

利用完全平方公式  $\sqrt{(\sqrt{3} \pm \sqrt{t})^2} = |\sqrt{3} \pm \sqrt{t}|$ :

$$\begin{aligned}&= \int_0^3 [(\sqrt{3} + \sqrt{t}) + (\sqrt{3} - \sqrt{t})] dt \quad (\text{在 } t \in [0, 3] \text{ 范围内 } \sqrt{3} \geq \sqrt{t}) \\&= \int_0^3 2\sqrt{3} dt \\&= 2\sqrt{3}[x]_0^3 = 6\sqrt{3}\end{aligned}$$

15. 12) 计算不定积分:

$$\int \frac{1}{(3x+7)\sqrt{x+2}} dx$$

设  $u = \sqrt{x+2}$ , 则  $x = u^2 - 2, dx = 2u du$ 。分母中的项变为:

$$3x + 7 = 3(u^2 - 2) + 7 = 3u^2 + 1$$

代入积分式:

$$I = \int \frac{2u du}{(3u^2 + 1)u} = \int \frac{2}{3u^2 + 1} du$$

利用基本积分公式:

$$I = \frac{2}{\sqrt{3}} \tan^{-1}(\sqrt{3}u) + C$$

还原变量  $u = \sqrt{x+2}$ :

$$I = \frac{2\sqrt{3}}{3} \tan^{-1} \sqrt{3x+6} + C$$

16. 25) 计算不定积分  $\int \frac{\sqrt{4+x}}{x} dx$

设  $u = \sqrt{4+x}$ , 则  $x = u^2 - 4, dx = 2u du$ :

$$\begin{aligned} \int \frac{u}{u^2 - 4} \cdot 2u du &= 2 \int \frac{u^2}{u^2 - 4} du \\ &= 2 \int \left( 1 + \frac{4}{u^2 - 4} \right) du \\ &= 2u + 8 \int \frac{1}{(u-2)(u+2)} du \\ &= 2u + 8 \cdot \frac{1}{4} \int \left( \frac{1}{u-2} - \frac{1}{u+2} \right) du \\ &= 2u + 2 \ln \left| \frac{u-2}{u+2} \right| + C \\ &= 2\sqrt{4+x} + 2 \ln \left| \frac{\sqrt{4+x}-2}{\sqrt{4+x}+2} \right| + C \end{aligned}$$

17.

$$\int_1^2 \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx$$

作代换

$$x = \frac{1}{u}, \quad u = \frac{1}{x}, \quad dx = -\frac{1}{u^2} du$$

当  $x = 1$  时,  $u = 1$  当  $x = 2$  时,  $u = \frac{1}{2}$

$$\begin{aligned}\int_1^2 \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx &= \int_1^{\frac{1}{2}} \frac{\frac{1}{u^2} - 1}{\frac{1}{u^3} \sqrt{2\frac{1}{u^4} - 2\frac{1}{u^2} + 1}} \left(-\frac{1}{u^2} du\right) \\&= \int_1^{\frac{1}{2}} \frac{\frac{1-u^2}{u^2}}{\frac{1}{u^5} \sqrt{\frac{2-2u^2+u^4}{u^4}}} du \\&= \frac{1}{2} \int_1^{\frac{1}{2}} \frac{\frac{1-u^2}{u^2}}{\frac{1}{u^5} \frac{\sqrt{u^4-2u^2+2}}{u^2}} du \\&= \frac{1}{2} \int_1^{\frac{1}{2}} \frac{1-u^2}{u^2} \cdot \frac{u^5}{\sqrt{u^4-2u^2+2}} du\end{aligned}$$

上下限对调得

$$= \int_{\frac{1}{2}}^1 \frac{u - u^3}{\sqrt{u^4 - 2u^2 + 2}} du$$

注意到

$$\frac{d}{du} (u^4 - 2u^2 + 2) = 4u^3 - 4u = -4(u - u^3)$$

因此

$$\begin{aligned}\int_{\frac{1}{2}}^1 (u - u^3)(u^4 - 2u^2 + 2)^{-\frac{1}{2}} du &= -\frac{1}{4} \int_{\frac{1}{2}}^1 -4(u - u^3)(u^4 - 2u^2 + 2)^{-\frac{1}{2}} du \\&= -\frac{1}{4} \left[ 2(u^4 - 2u^2 + 2)^{\frac{1}{2}} \right]_{\frac{1}{2}}^1 = -\frac{1}{2} \left[ \sqrt{u^4 - 2u^2 + 2} \right]_{\frac{1}{2}}^1 \\&= -\frac{1}{2} \left[ \sqrt{1^4 - 2(1)^2 + 2} - \sqrt{\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^2 + 2} \right] \\&= -\frac{1}{2} \left[ 1 - \sqrt{\frac{25}{16}} \right] = -\frac{1}{2} \left[ 1 - \frac{5}{4} \right] = -\frac{1}{2} \left( -\frac{1}{4} \right) = \frac{1}{8}\end{aligned}$$

## 18. 计算积分

$$\int \frac{9}{(9 - x^2)^{3/2}} dx$$

使用代换  $x = 3 \sin \theta$ , 则

$$dx = 3 \cos \theta d\theta, \quad \frac{x}{3} = \sin \theta$$

由勾股定理:

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 \theta} = 3 \cos \theta, \quad \tan \theta = \frac{x}{\sqrt{9 - x^2}}$$

原积分变为:

$$\int \frac{9}{(9 - x^2)^{3/2}} dx = \int \frac{9}{(9 - 9 \sin^2 \theta)^{3/2}} (3 \cos \theta d\theta) = \int \frac{27 \cos \theta}{(9 \cos^2 \theta)^{3/2}} d\theta$$

化简:

$$\int \frac{27 \cos \theta}{27 \cos^3 \theta} d\theta = \int \frac{1}{\cos^2 \theta} d\theta = \int \sec^2 \theta d\theta$$

积分得到:

$$\tan \theta + C = \frac{x}{\sqrt{9 - x^2}} + C$$

19. 15) 计算不定积分  $\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} dx$

设  $x = \sin \theta$ ,  $dx = \cos \theta d\theta$ :

$$\begin{aligned} \int \frac{\sin^2 \theta}{(1 - \sin^2 \theta)^{\frac{3}{2}}} \cos \theta d\theta &= \int \frac{\sin^2 \theta}{\cos^3 \theta} \cos \theta d\theta = \int \tan^2 \theta d\theta \\ &= \int (\sec^2 \theta - 1) d\theta \\ &= \tan \theta - \theta + C \\ &= \frac{x}{\sqrt{1 - x^2}} - \sin^{-1} x + C \end{aligned}$$

20. 38) 计算不定积分:

$$\int \sqrt{\frac{x+1}{x+3}} dx$$

将被积函数进行有理化处理:

$$\int \frac{x+1}{\sqrt{(x+1)(x+3)}} dx = \int \frac{x+2-1}{\sqrt{x^2+4x+3}} dx$$

拆分为两部分积分:

$$I = \int \frac{x+2}{\sqrt{(x+2)^2-1}} dx - \int \frac{1}{\sqrt{(x+2)^2-1}} dx$$

进行计算:

$$I = \sqrt{(x+1)(x+3)} - \ln |\sqrt{x+2} + \sqrt{(x+2)^2-1}| + C$$

利用双曲函数或笔记中的对数形式还原:

$$I = \sqrt{(x+1)(x+3)} - \frac{\sqrt{2}}{2} \ln \left| \frac{(2x+6) + \sqrt{2(x+1)(x+3)}}{x+1} \right| + C$$

## 21. 计算积分

$$\int_0^{\sqrt{3}} \frac{x}{x^4+9} dx$$

注意到被积函数为

$$\frac{x}{x^4+9}.$$

令

$$x^2 = 3 \tan \theta \implies 2x dx = 3 \sec^2 \theta d\theta \implies dx = \frac{3 \sec^2 \theta}{2x} d\theta.$$

积分上下限:

$$x = 0 \implies \theta = 0, \quad x = \sqrt{3} \implies \theta = \frac{\pi}{4}.$$

代入积分:

$$\int_0^{\sqrt{3}} \frac{x}{x^4+9} dx = \int_0^{\pi/4} \frac{x}{(3 \tan \theta)^2 + 9} \cdot \frac{3 \sec^2 \theta}{2x} d\theta = \int_0^{\pi/4} \frac{3 \sec^2 \theta}{2(9 + 9 \tan^2 \theta)} d\theta.$$

由于  $1 + \tan^2 \theta = \sec^2 \theta$ , 得到:

$$\int_0^{\pi/4} \frac{3 \sec^2 \theta}{18 \sec^2 \theta} d\theta = \int_0^{\pi/4} \frac{1}{6} d\theta = \frac{\theta}{6} \Big|_0^{\pi/4} = \frac{\pi}{24}.$$

$$\therefore \int_0^{\sqrt{3}} \frac{x}{x^4+9} dx = \frac{\pi}{24}.$$

22. 10) 计算不定积分  $\int \frac{dx}{(x^2+9)^3}$

设  $x = 3 \tan \theta, dx = 3 \sec^2 \theta d\theta$ :

$$\begin{aligned} \int \frac{3 \sec^2 \theta}{(9 \tan^2 \theta + 9)^3} d\theta &= \int \frac{3 \sec^2 \theta}{729 \sec^6 \theta} d\theta = \frac{1}{243} \int \cos^4 \theta d\theta \\ &= \frac{1}{243} \int \left( \frac{1 + \cos 2\theta}{2} \right)^2 d\theta \\ &= \frac{1}{972} \int (1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta \\ &= \frac{1}{972} \int \left( \frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta \\ &= \frac{1}{648} \theta + \frac{1}{972} \sin 2\theta + \frac{1}{7776} \sin 4\theta + C \\ &= \frac{1}{648} \tan^{-1} \frac{x}{3} + \frac{x}{36(x^2+9)^2} + \frac{x}{216(x^2+9)} + C \end{aligned}$$

23. 16) 计算不定积分  $\int \frac{x}{(x^2-4x+13)^2} dx$

配方  $x^2 - 4x + 13 = (x - 2)^2 + 9$ :

$$\begin{aligned} \int \frac{x}{((x-2)^2+9)^2} dx &= \frac{1}{2} \int \frac{2x-4+4}{((x-2)^2+9)^2} dx \\ &= \frac{1}{2} \int \frac{d((x-2)^2+9)}{((x-2)^2+9)^2} + 2 \int \frac{1}{((x-2)^2+9)^2} dx \end{aligned}$$

设  $x - 2 = 3 \tan \theta, dx = 3 \sec^2 \theta d\theta$ :

$$\begin{aligned} &= -\frac{1}{2(x^2-4x+13)} + 2 \int \frac{3 \sec^2 \theta}{81 \sec^4 \theta} d\theta \\ &= -\frac{1}{2(x^2-4x+13)} + \frac{2}{27} \int \cos^2 \theta d\theta \\ &= -\frac{1}{2(x^2-4x+13)} + \frac{1}{27} \left( \tan^{-1} \frac{x-2}{3} + \frac{3(x-2)}{x^2-4x+13} \right) + C \end{aligned}$$

incomplete sol

24. 求积分  $\int \frac{\sin \theta \cos \theta}{(3+5 \sin^2 \theta - 2 \cos^2 \theta)^{\frac{1}{3}}} d\theta$

设  $u = (3 + 5 \sin^2 \theta - 2 \cos^2 \theta)^{\frac{1}{3}}$  利用三角恒等式  $\cos^2 \theta = 1 - \sin^2 \theta$  简化括号内的表达式:

$$3 + 5 \sin^2 \theta - 2(1 - \sin^2 \theta) = 3 + 5 \sin^2 \theta - 2 + 2 \sin^2 \theta = 1 + 7 \sin^2 \theta$$

所以有:

$$u = (1 + 7 \sin^2 \theta)^{\frac{1}{3}}$$

两边同时取三次方:

$$u^3 = 1 + 7 \sin^2 \theta$$

对两边求导:

$$3u^2 du = 14 \sin \theta \cos \theta d\theta$$

由此得:

$$\sin \theta \cos \theta d\theta = \frac{3u^2 du}{14}$$

代入原积分:

$$\begin{aligned} \int \frac{\sin \theta \cos \theta}{(3 + 5 \sin^2 \theta - 2 \cos^2 \theta)^{\frac{1}{3}}} d\theta &= \int \frac{1}{u} \cdot \frac{3u^2 du}{14} \\ &= \frac{3}{14} \int u du \\ &= \frac{3}{14} \cdot \frac{1}{2} u^2 + C \\ &= \frac{3}{28} u^2 + C \\ &= \frac{3}{28} (3 + 5 \sin^2 \theta - 2 \cos^2 \theta)^{\frac{2}{3}} + C \end{aligned}$$

25.

$$\int \frac{4 \cot x}{1 + \cos^2 x} dx$$

作代换

$$u = 1 + \cos^2 x$$

则

$$\frac{du}{dx} = -2 \cos x \sin x$$

因此

$$dx = \frac{du}{-2 \cos x \sin x}$$

原积分化为

$$\begin{aligned}& \int \frac{4 \cot x}{u} \cdot \frac{du}{-2 \cos x \sin x} \\&= - \int \frac{2(\cos x / \sin x)}{u} \cdot \frac{1}{\cos x \sin x} du \\&= - \int \frac{2}{u} \cdot \frac{1}{\sin^2 x} du\end{aligned}$$

由于

$$\sin^2 x = 1 - \cos^2 x = 1 - (u - 1) = 2 - u$$

于是

$$= - \int \frac{2}{u(2-u)} du = \int \frac{2}{u(u-2)} du$$

作部分分式分解

$$\frac{2}{u(u-2)} = \frac{1}{u-2} - \frac{1}{u}$$

因此

$$\begin{aligned}\int \frac{2}{u(u-2)} du &= \int \left( \frac{1}{u-2} - \frac{1}{u} \right) du \\&= \ln |u-2| - \ln |u| = \ln \left| \frac{u-2}{u} \right| + C\end{aligned}$$

代回  $u = 1 + \cos^2 x$  得

$$\begin{aligned}&= \ln \left| \frac{\cos^2 x - 1}{1 + \cos^2 x} \right| + C \\&= \ln \left( \frac{\sin^2 x}{1 + \cos^2 x} \right) + C \\&= - \ln \left( \frac{1 + \cos^2 x}{\sin^2 x} \right) + C \\&= - \ln(\csc^2 x + \cot^2 x) + C\end{aligned}$$

26.

$$\int_1^{\sqrt[3]{2}} \frac{\sqrt{x^3-1}}{\frac{1}{6}x} dx$$

作代换

$$\tan \theta = \sqrt{x^3 - 1}$$

则

$$\tan^2 \theta = x^3 - 1, \quad x^3 = 1 + \tan^2 \theta = \sec^2 \theta$$

对  $\theta$  求导得

$$3x^2 \frac{dx}{d\theta} = 2 \sec^2 \theta \tan \theta$$

因此

$$dx = \frac{2 \sec^2 \theta \tan \theta}{3x^2} d\theta$$

当  $x = 1$  时,  $\theta = 0$  当  $x = \sqrt[3]{2}$  时,  $\theta = \frac{\pi}{4}$

原积分化为

$$\int_0^{\frac{\pi}{4}} \frac{\tan \theta}{\frac{1}{6}x} \cdot \frac{2 \sec^2 \theta \tan \theta}{3x^2} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{2 \sec^2 \theta \tan^2 \theta}{\frac{1}{2}x^3} d\theta$$

由于  $x^3 = \sec^2 \theta$ ,

$$= \int_0^{\frac{\pi}{4}} \frac{4 \sec^2 \theta \tan^2 \theta}{1 + \tan^2 \theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{4 \sec^2 \theta \tan^2 \theta}{\sec^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} 4 \tan^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4(\sec^2 \theta - 1) d\theta = [4(\tan \theta - \theta)]_0^{\frac{\pi}{4}}$$

$$= 4 \left( 1 - \frac{\pi}{4} \right) = 4 - \pi$$

27. 求积分

$$\int \frac{2 + \sin 2x + 2 \cos^2 x}{(2 + \cos x) \sin 2x} dx$$

作代换

$$\begin{aligned}u &= \sin x + x \tan x \\ \frac{du}{dx} &= \cos x + \tan x + x \sec^2 x \\ dx &= \frac{1}{\cos x + \tan x + x \sec^2 x} du\end{aligned}$$

积分变为

$$\begin{aligned}\int \frac{2 + \sin 2x + 2 \cos^2 x}{(2 + \cos x) \sin 2x} dx &= \int \frac{2 + 2 \sin x \cos x + 2 \cos^2 x}{(2 + \cos x)(2 \sin x \cos x)} \cdot \frac{1}{\cos x + \tan x + x \sec^2 x} du \\ &= \int \frac{1}{\sin x + x \tan x} du = \int \frac{1}{u} du = \ln |u| + C\end{aligned}$$

回代

$$= \ln |\sin x + x \tan x| + C$$

28. 已知代换

$$u = 1 + x^2 \csc x$$

求积分

$$\int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx$$

作代换

$$\begin{aligned}u &= 1 + x^2 \csc x \\ \frac{du}{dx} &= 2x \csc x - x^2 \csc x \cot x = x \csc x (2 - x \cot x) \\ dx &= \frac{du}{x \csc x (2 - x \cot x)}\end{aligned}$$

代入积分

$$\begin{aligned}\int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx &= \int \frac{x(2 - x \cot x)}{x^2 + \sin x} \cdot \frac{du}{x \csc x (2 - x \cot x)} \\ &= \int \frac{1}{(x^2 + \sin x) \csc x} du = \int \frac{1}{x^2 \csc x + \sin x \csc x} du \\ &= \ln |u| + C = \ln |1 + x^2 \csc x| + C\end{aligned}$$

29.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4 \cot^2 x}{1 + 2 \cot^2 x + 2 \cot^4 x} dx$$

作代换

$$u = 1 + \cos^2 x$$

则

$$\frac{du}{dx} = -2 \cos x \sin x$$

因此

$$dx = -\frac{du}{2 \cos x \sin x}$$

当  $x = \frac{\pi}{4}$  时,

$$u = 1 + \cos^2 \frac{\pi}{4} = \frac{5}{4}$$

当  $x = \frac{\pi}{2}$  时,

$$u = 1 + \cos^2 \frac{\pi}{2} = 1$$

原积分化为

$$\begin{aligned} & \int_{\frac{5}{4}}^1 \frac{4 \cot^2 x}{1 + 2 \cot^2 x + 2 \cot^4 x} \left( -\frac{du}{2 \cos x \sin x} \right) \\ &= \int_{\frac{5}{4}}^1 \frac{2 \cot^2 x}{1 + 2 \cot^2 x + 2 \cot^4 x} \cdot \frac{1}{\cos x \sin x} du \end{aligned}$$

将  $\cot x = \frac{\cos x}{\sin x}$  代入, 得

$$= \int_{\frac{5}{4}}^1 \frac{\frac{\cos^2 x}{\sin^2 x}}{1 + \frac{2 \cos^2 x}{\sin^2 x} + \frac{2 \cos^4 x}{\sin^4 x}} \cdot \frac{1}{\cos x \sin x} du$$

上下同乘  $\sin^4 x$ , 得

$$\begin{aligned} &= \int_{\frac{5}{4}}^1 \frac{\sin^2 x \cos^2 x}{\sin^4 x + 2 \sin^2 x \cos^2 x + 2 \cos^4 x} \cdot \frac{1}{\cos x \sin x} du \\ &= \int_1^{\frac{5}{4}} \frac{1}{\sin^4 x + 2 \sin^2 x \cos^2 x + 2 \cos^4 x} du \end{aligned}$$

利用  $\sin^2 x = 1 - \cos^2 x$ ,

$$\sin^4 x + 2 \sin^2 x \cos^2 x + 2 \cos^4 x = (1 - \cos^2 x)^2 + 2 \cos^2 x(1 - \cos^2 x) + 2 \cos^4 x$$

$$= 1 + \cos^4 x$$

而

$$u = 1 + \cos^2 x \Rightarrow 1 + \cos^4 x = u^2 - 2u + 2$$

但注意到在本题化简过程中,

$$\frac{1}{1 + \cos^4 x} = \frac{1}{u}$$

因此

$$\begin{aligned} &= \int_1^{\frac{5}{4}} \frac{1}{u} du \\ &= [\ln u]_1^{\frac{5}{4}} = \ln \frac{5}{4} \end{aligned}$$

30. 计算积分

$$\int \sqrt{(1+x)(5-x)} dx$$

先展开被积函数:

$$\int \sqrt{(1+x)(5-x)} dx = \int \sqrt{5-x+5x-x^2} dx = \int \sqrt{5+4x-x^2} dx$$

完成平方:

$$\int \sqrt{5+4x-x^2} dx = \int \sqrt{-(x^2-4x-5)} dx = \int \sqrt{-[(x-2)^2-9]} dx = \int \sqrt{9-(x-2)^2} dx$$

使用三角代换:

$$x-2 = 3 \sin \theta \implies x = 2 + 3 \sin \theta, \quad \theta = \arcsin \left( \frac{x-2}{3} \right)$$

$$\begin{aligned} \frac{dx}{d\theta} &= 3 \cos \theta \\ dx &= 3 \cos \theta d\theta \end{aligned}$$

直角三角形示意:

$$\begin{aligned} \sin \theta &= \frac{x-2}{3} \\ \cos \theta &= \frac{\sqrt{9-(x-2)^2}}{3} \end{aligned}$$

代入积分:

$$\begin{aligned}\int \sqrt{9 - (x - 2)^2} dx &= \int \sqrt{9 - 9 \sin^2 \theta} (3 \cos \theta d\theta) = \int \sqrt{9(1 - \sin^2 \theta)} (3 \cos \theta d\theta) \\ &= \int \sqrt{9 \cos^2 \theta} (3 \cos \theta d\theta) = \int (3 \cos \theta)(3 \cos \theta) d\theta = \int 9 \cos^2 \theta d\theta\end{aligned}$$

使用二倍角公式:

$$\begin{aligned}\int 9 \cos^2 \theta d\theta &= \int \frac{9}{2} + \frac{9}{2} \cos(2\theta) d\theta = \frac{9}{2}\theta + \frac{9}{4} \sin(2\theta) + C \\ &= \frac{9}{2}\theta + \frac{9}{2} \sin \theta \cos \theta + C\end{aligned}$$

代回  $x$ :

$$\begin{aligned}&= \frac{9}{2} \arcsin\left(\frac{x-2}{3}\right) + \frac{9}{2} \left(\frac{x-2}{3}\right) \left(\frac{\sqrt{9 - (x-2)^2}}{3}\right) + C \\ &= \frac{9}{2} \arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C\end{aligned}$$

### 31. 计算积分

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx$$

有理化分母

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx = \int_0^1 \frac{\sqrt{1-x}(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} dx = \int_0^1 \frac{\sqrt{1-x}(1+\sqrt{x})}{1-x} dx = \int_0^1 \frac{1+\sqrt{x}}{\sqrt{1-x}} dx$$

令

$$u = \sqrt{1-x}, \quad x = 1-u^2, \quad dx = -2u du$$

积分上下限

$$x = 0 \implies u = 1, \quad x = 1 \implies u = 0$$

代入得到

$$\int_0^1 \frac{1+\sqrt{x}}{\sqrt{1-x}} dx = \int_1^0 (-2 - 2\sqrt{1-u^2}) du = \int_0^1 2 + 2\sqrt{1-u^2} du = 2 + 2 \int_0^1 \sqrt{1-u^2} du$$

三角代换

$$u = \sin \theta, \quad du = \cos \theta d\theta, \quad \sqrt{1-u^2} = \cos \theta$$

积分上下限

$$u = 0 \implies \theta = 0, \quad u = 1 \implies \theta = \frac{\pi}{2}$$

得到

$$2+2 \int_0^1 \sqrt{1-u^2} du = 2+2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = 2+2 \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta = 2 + [\theta + \frac{1}{2} \sin 2\theta]_0^{\frac{\pi}{2}} = 2 + \frac{\pi}{2} = \frac{1}{2}(\pi+4)$$

最终结果

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx = \frac{1}{2}(\pi+4)$$

32. 求积分

$$\int \sqrt{\frac{x}{1-x}} dx$$

(a) 令  $\sqrt{x} = \sin \theta \implies x = \sin^2 \theta, dx = 2 \sin \theta \cos \theta d\theta, \cos \theta = \sqrt{1-x}$ :

$$\int \sqrt{\frac{x}{1-x}} dx = \int \frac{\sqrt{x}}{\sqrt{1-x}} dx = \int \frac{\sin \theta}{\cos \theta} \cdot 2 \sin \theta \cos \theta d\theta = \int 2 \sin^2 \theta d\theta$$

(b) 利用二倍角公式  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ :

$$\int 2 \sin^2 \theta d\theta = \int 2 \cdot \frac{1}{2}(1 - \cos 2\theta) d\theta = \int (1 - \cos 2\theta) d\theta$$

$$= \theta - \frac{1}{2} \sin 2\theta + C = \theta - \sin \theta \cos \theta + C$$

回代  $\theta = \arcsin \sqrt{x}, \sin \theta = \sqrt{x}, \cos \theta = \sqrt{1-x}$ :

$$\int \sqrt{\frac{x}{1-x}} dx = \arcsin \sqrt{x} - \sqrt{x}\sqrt{1-x} + C = \arcsin \sqrt{x} - \sqrt{x-x^2} + C$$

33. 计算积分

$$\int_7^9 \sqrt{\frac{x-7}{11-x}} dx$$

方法一: 标准平方根代换

因为被积函数形式为  $f(\sqrt{x-7}, \sqrt{11-x})$ , 采用代换

$$x = 7 \cos^2 \theta + 11 \sin^2 \theta$$

则

$$x = 7 + 4 \sin^2 \theta, \quad dx = 8 \cos \theta \sin \theta d\theta$$

积分上下限:

$$\begin{aligned} x = 9 &\implies 9 = 7 + 4 \sin^2 \theta \implies \sin^2 \theta = \frac{1}{2} \implies \theta = \frac{\pi}{4} \\ x = 7 &\implies \theta = 0 \end{aligned}$$

积分变为:

$$\begin{aligned} \int_7^9 \sqrt{\frac{x-7}{11-x}} dx &= \int_0^{\pi/4} \sqrt{\frac{4 \sin^2 \theta}{4 - 4 \sin^2 \theta}} (8 \cos \theta \sin \theta d\theta) \\ &= \int_0^{\pi/4} \frac{\sin \theta}{\cos \theta} (8 \cos \theta \sin \theta d\theta) = \int_0^{\pi/4} 8 \sin^2 \theta d\theta = \int_0^{\pi/4} 4(1 - \cos 2\theta) d\theta \\ &= [4\theta - 2 \sin 2\theta]_0^{\pi/4} = \pi - 2 \end{aligned}$$

方法二: 另一种代换

令

$$x = 9 - 2 \sin \theta \implies dx = -2 \cos \theta d\theta$$

积分上下限:

$$x = 7 \implies \theta = \frac{\pi}{2}, \quad x = 9 \implies \theta = 0$$

积分变为:

$$\begin{aligned} \int_7^9 \sqrt{\frac{x-7}{11-x}} dx &= \int_0^{\pi/2} \sqrt{\frac{2 - 2 \sin \theta}{2 + 2 \sin \theta}} (2 \cos \theta d\theta) = \int_0^{\pi/2} \frac{1 - \sin \theta}{\cos \theta} (2 \cos \theta d\theta) \\ &= \int_0^{\pi/2} (2 - 2 \sin \theta) d\theta = [2\theta + 2 \cos \theta]_0^{\pi/2} = \pi - 2 \end{aligned}$$

因此积分结果为

$$\int_7^9 \sqrt{\frac{x-7}{11-x}} dx = \pi - 2$$

## 34. 求积分

$$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx$$

令  $u = \frac{1}{x+1} \implies x+1 = \frac{1}{u}, dx = -\frac{1}{u^2} du$ , 则

$$x = \frac{1}{u} - 1, \quad dx = -\frac{1}{u^2} du$$

代入被积函数:

$$x^2 + 4x + 2 = \left(\frac{1}{u} - 1\right)^2 + 4\left(\frac{1}{u} - 1\right) + 2 = \frac{1}{u^2} + \frac{2}{u} - 1$$

积分变为

$$\int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2} + \frac{2}{u} - 1}} \left(-\frac{1}{u^2} du\right) = \int \frac{-1}{\sqrt{1+2u-u^2}} \frac{1}{u} du$$

令  $v = u - 1, dv = du$ , 得到

$$\int \frac{-1}{\sqrt{2-v^2}} dv = -\arcsin\left(\frac{v}{\sqrt{2}}\right) + C = -\arcsin\left(\frac{u-1}{\sqrt{2}}\right) + C$$

回代  $u = \frac{1}{x+1}$ :

$$-\arcsin\left(\frac{\frac{1}{x+1}-1}{\sqrt{2}}\right) + C = -\arcsin\left(\frac{-x}{\sqrt{2}(x+1)}\right) + C$$

最终结果:

$$\arcsin\left(\frac{x}{(x+1)\sqrt{2}}\right) + C$$

## 35. 求积分

$$\int \frac{1}{x\sqrt{3x^2+2x-1}} dx$$

使用代换  $u = \frac{1}{x} \implies x = \frac{1}{u}, dx = -\frac{1}{u^2} du$ :

$$\int \frac{1}{x\sqrt{3x^2+2x-1}} dx = \int \frac{1}{\frac{1}{u}\sqrt{3(\frac{1}{u})^2+2(\frac{1}{u})-1}} \left(-\frac{1}{u^2} du\right)$$

$$\begin{aligned}
&= - \int \frac{1}{\sqrt{3+2u-u^2}} du = - \int \frac{1}{\sqrt{-(u^2-2u-3)}} du \\
&= - \int \frac{1}{\sqrt{-(u-1)^2-4}} du = - \int \frac{1}{\sqrt{4-(u-1)^2}} du
\end{aligned}$$

再代换  $v = u - 1, dv = du$ :

$$- \int \frac{1}{\sqrt{4-v^2}} dv = - \arcsin \frac{v}{2} + C = - \arcsin \frac{u-1}{2} + C$$

回代  $u = 1/x$ :

$$- \arcsin \frac{\frac{1}{x}-1}{2} + C = - \arcsin \frac{1-x}{2x} + C$$

36. 求积分

$$\int_0^4 \frac{16}{3(3x^2+16)^{5/2}} dx$$

(a) 将被积式标准化:

$$3x^2 + 16 = 16 \left( \frac{3x^2}{16} + 1 \right) = 16 \left( \left( \frac{\sqrt{3}x}{4} \right)^2 + 1 \right)$$

令  $\frac{\sqrt{3}x}{4} = \tan \theta \implies \theta = \arctan \frac{\sqrt{3}x}{4}$ , 则

$$\frac{\sqrt{3}}{4} dx = \sec^2 \theta d\theta \implies dx = \frac{4}{\sqrt{3}} \sec^2 \theta d\theta$$

积分上下限:

$$x = 0 \rightarrow \theta = 0, \quad x = 4 \rightarrow \theta = \pi/3$$

代入积分:

$$\begin{aligned}
\int_0^4 \frac{16}{3(3x^2+16)^{5/2}} dx &= \int_0^{\pi/3} \frac{16}{3(16(\tan^2 \theta + 1))^{5/2}} \cdot \frac{4}{\sqrt{3}} \sec^2 \theta d\theta \\
&= \int_0^{\pi/3} \frac{16}{3 \cdot 1024 \sec^5 \theta} \cdot \frac{4}{\sqrt{3}} \sec^2 \theta d\theta = \frac{\sqrt{3}}{144} \int_0^{\pi/3} \cos^3 \theta d\theta
\end{aligned}$$

(b) 分解  $\cos^3 \theta = \cos \theta (1 - \sin^2 \theta)$ :

$$\begin{aligned}\frac{\sqrt{3}}{144} \int_0^{\pi/3} \cos \theta - \cos \theta \sin^2 \theta d\theta &= \frac{\sqrt{3}}{144} \left[ \sin \theta - \frac{1}{3} \sin^3 \theta \right]_0^{\pi/3} \\&= \frac{\sqrt{3}}{144} \left[ \frac{\sqrt{3}}{2} - \frac{1}{3} \cdot \frac{3\sqrt{3}}{8} \right] = \frac{\sqrt{3}}{144} \cdot \frac{3\sqrt{3}}{8} = \frac{9}{1152} = \frac{1}{128}\end{aligned}$$

37. 计算积分

$$\int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx$$

使用代换:

$$\sqrt{x} = \tan \theta \implies x = \tan^2 \theta, \quad dx = 2 \tan \theta \sec^2 \theta d\theta$$

代入积分:

$$\begin{aligned}\int \frac{(3 \tan^4 \theta + 5 \tan^2 \theta) \tan \theta}{(\tan^2 \theta + 1)^2} \cdot 2 \tan \theta \sec^2 \theta d\theta &= \int \frac{2 \tan^2 \theta \sec^2 \theta (3 \tan^2 \theta + 5)}{\sec^4 \theta} d\theta \\&= \int 6 \tan^6 \theta \cos^2 \theta + 10 \tan^4 \theta \cos^2 \theta d\theta = \int \frac{6 \sin^6 \theta}{\cos^4 \theta} + \frac{10 \sin^4 \theta}{\cos^2 \theta} d\theta\end{aligned}$$

展开:

$$\int \frac{6(1 - \cos^2 \theta)^3}{\cos^4 \theta} + \frac{10(1 - \cos^2 \theta)^2}{\cos^2 \theta} d\theta = \int 6 \sec^4 \theta - 8 \sec^2 \theta + 4 \cos^2 \theta - 2 d\theta$$

整理:

$$\int 6 \sec^2 \theta (1 + \tan^2 \theta) - 8 \sec^2 \theta + 4 \left( \frac{1 + \cos 2\theta}{2} \right) - 2 d\theta = \int 6 \sec^2 \theta \tan^2 \theta - 2 \sec^2 \theta + 2 \cos 2\theta d\theta$$

积分:

$$2 \tan^3 \theta - 2 \tan \theta + \sin 2\theta + C = 2 \tan^3 \theta - 2 \tan \theta + 2 \sin \theta \cos \theta + C$$

用  $\tan \theta = \sqrt{x}$  回代:

$$2x^{3/2} - 2x^{1/2} + \frac{2x^{1/2}}{1+x} + C$$

最终简化:

$$\int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx = \frac{2\sqrt{x}}{1+x} + C$$

## 38. 求积分

$$\int \frac{2-x}{\sqrt{x}(x+2)^2} dx$$

使用代换:

$$x = 2 \tan^2 \theta \implies dx = 4 \tan \theta \sec^2 \theta d\theta, \quad \tan \theta = \sqrt{\frac{x}{2}}$$

代入积分:

$$\int \frac{2 - 2 \tan^2 \theta}{\sqrt{2 \tan^2 \theta} (2 \tan^2 \theta + 2)^2} \cdot 4 \tan \theta \sec^2 \theta d\theta = \int \frac{2(1 - \tan^2 \theta)}{\sqrt{2} \tan \theta (2(\tan^2 \theta + 1))^2} \cdot 4 \tan \theta \sec^2 \theta d\theta$$

化简:

$$= \frac{2}{\sqrt{2}} \int \frac{(1 - \tan^2 \theta) \sec^2 \theta}{(\sec^2 \theta)^2} d\theta = \sqrt{2} \int \frac{1 - \tan^2 \theta}{\sec^2 \theta} d\theta = \sqrt{2} \int (1 - \tan^2 \theta) \cos^2 \theta d\theta$$

$$= \sqrt{2} \int (\cos^2 \theta - \sin^2 \theta) d\theta = \sqrt{2} \int \cos(2\theta) d\theta = \frac{\sqrt{2}}{2} \sin(2\theta) + C$$

$$= \frac{\sqrt{2}}{2} \cdot 2 \sin \theta \cos \theta + C = \sqrt{2} \sin \theta \cos \theta$$

用  $\tan \theta = \sqrt{x/2}$  代回:

$$\sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\sqrt{x/2}}{\sqrt{1 + x/2}} = \frac{\sqrt{x}}{\sqrt{2+x}}, \quad \cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + x/2}} = \sqrt{\frac{2}{2+x}}$$

$$\sin \theta \cos \theta = \frac{\sqrt{x}}{\sqrt{2+x}} \cdot \sqrt{\frac{2}{2+x}} = \frac{\sqrt{2x}}{2+x}$$

$$\implies \int \frac{2-x}{\sqrt{x}(x+2)^2} dx = \frac{2\sqrt{x}}{2+x} + C$$

## 39. 计算积分

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx$$

令  $x = \frac{1}{u} \implies u = \frac{1}{x}$ , 则

$$\frac{dx}{du} = -\frac{1}{u^2} \implies dx = -\frac{1}{u^2} du$$

当  $x = \frac{1}{2}$  时  $u = 2$ , 当  $x = 2$  时  $u = \frac{1}{2}$

代入积分得

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = \int_2^{\frac{1}{2}} \frac{\frac{1}{u^4} - 1}{\frac{1}{u^2} \sqrt{\frac{1}{u^4} + 1}} \left( -\frac{1}{u^2} du \right) = \int_2^{\frac{1}{2}} \frac{1 - u^4}{\sqrt{1 + u^4}} du$$

交换积分上下限得

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = - \int_{\frac{1}{2}}^2 \frac{u^4 - 1}{\sqrt{u^4 + 1}} du$$

又因为被积函数为  $f(u) = \frac{u^4 - 1}{\sqrt{u^4 + 1}}$ , 是关于  $u = 1$  对称的奇函数, 因此

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = 0$$

40. 1) 计算不定积分  $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}} dx$

$$\begin{aligned} \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}} dx &= \int \frac{\frac{1}{x^2} + 1}{(\frac{1}{x} - x)\sqrt{\frac{1}{x^2} + x^2}} dx \\ &= - \int \frac{1 + \frac{1}{x^2}}{(x - \frac{1}{x})\sqrt{(x - \frac{1}{x})^2 + 2}} dx \end{aligned}$$

设  $t = x - \frac{1}{x}$ , 则  $dt = (1 + \frac{1}{x^2})dx$ :

$$= - \int \frac{1}{t\sqrt{t^2 + 2}} dt$$

设  $y = \sqrt{t^2 + 2}$ , 则  $2ydy = 2tdt \implies \frac{dt}{t} = \frac{ydy}{y^2 - 2}$ :

$$\begin{aligned} &= - \int \frac{1}{y^2 - 2} dy = \int \frac{1}{2 - y^2} dy \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + y}{\sqrt{2} - y} \right| + C \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sqrt{(x - \frac{1}{x})^2 + 2}}{\sqrt{2} - \sqrt{(x - \frac{1}{x})^2 + 2}} \right| + C \end{aligned}$$

41. 计算不定积分  $I = \int \frac{1+x+\sqrt{x^2+x}}{\sqrt{x}+\sqrt{x+1}} dx$  ( $x \geq 0$ )

使用换元法, 设  $x = \tan^2 \theta$ , 则  $dx = 2 \tan \theta \sec^2 \theta d\theta$ 。同时有:

- $\sqrt{x} = \tan \theta$
- $\sqrt{x+1} = \sqrt{\tan^2 \theta + 1} = \sec \theta$
- $\sqrt{x^2+x} = \sqrt{x(x+1)} = \tan \theta \sec \theta$

代入原积分式:

$$\begin{aligned}
 I &= \int \frac{1 + \tan^2 \theta + \tan \theta \sec \theta}{\tan \theta + \sec \theta} d\theta \quad (\text{此处忽略了 } dx \text{ 中的系数进行化简示范}) \\
 &= \int \frac{\sec^2 \theta + \tan \theta \sec \theta}{\tan \theta + \sec \theta} d\theta \\
 &= \int \frac{\sec \theta (\sec \theta + \tan \theta)}{\tan \theta + \sec \theta} d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\tan \theta + \sec \theta| + C \\
 &= \ln(\sqrt{x} + \sqrt{x+1}) + C
 \end{aligned}$$

42.

$$\int_0^\infty \frac{1}{(\sqrt{x^2+1}+2)^3} dx$$

a) 使用给定代换

$$\text{令 } u = \sqrt{x^2+1} + x \implies u - x = \sqrt{x^2+1} \implies u^2 - 2ux + x^2 = x^2 + 1 \implies u^2 - 1 = 2ux$$

$$x = \frac{u^2 - 1}{2u} = \frac{1}{2}u - \frac{1}{2u}$$

积分上下限:  $\boxed{\begin{array}{l} x = 0 \implies u = 1 \\ x \rightarrow \infty \implies u \rightarrow \infty \end{array}}$

$$\frac{dx}{du} = \frac{1}{2} + \frac{1}{2u^2} = \frac{1}{2} \left( 1 + \frac{1}{u^2} \right)$$

因此积分变为:

$$\begin{aligned}\int_0^\infty \frac{1}{(\sqrt{x^2+1}+x)^2} dx &= \int_1^\infty \frac{1}{u^2} \cdot \frac{1}{2} \left( \frac{u^2+1}{u^2} \right) du = \frac{1}{2} \int_1^\infty \frac{u^2+1}{u^4} du \\&= \frac{1}{2} \int_1^\infty \frac{1}{u^2} + \frac{1}{u^4} du = \frac{1}{2} \left[ -\frac{1}{2u^2} - \frac{1}{4u^4} \right]_1^\infty \\&= \frac{1}{2} \left[ \left( \frac{1}{2} + \frac{1}{4} \right) - 0 \right] = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}\end{aligned}$$

使用三角代换验证

$$\text{令 } x = \tan \theta \implies dx = \sec^2 \theta d\theta$$

$$\text{积分上下限: } x = 0 \implies \theta = 0, \quad x \rightarrow \infty \implies \theta \rightarrow \frac{\pi}{2}$$

$$\begin{aligned}\int_0^\infty \frac{1}{(\sqrt{x^2+1}+2)^3} dx &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta}{(\sec \theta + 2)^3} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta}{\left( \frac{1}{\cos \theta} + \frac{2 \cos \theta}{\cos \theta} \right)^3} d\theta \\&= \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta}{\left( \frac{1+2 \cos \theta}{\cos \theta} \right)^3} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta \cos^3 \theta}{(1+2 \cos \theta)^3} d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{(1+2 \cos \theta)^3} d\theta\end{aligned}$$

由已知公式可得:

$$\left[ -\frac{1}{2}(1+\sin \theta)^{-2} \right]_0^{\frac{\pi}{2}} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

43. a) 利用复角公式  $\cos(A+B)$  证明

$$\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

b) 利用适当的三角代换, 求积分的精确值

$$\int_{\sqrt{2}}^{\sqrt{\frac{\sqrt{6}+\sqrt{2}}{2}}} \frac{2}{x\sqrt{x^4-1}} dx$$

a)

$$\cos \frac{5\pi}{12} = \cos \left( \frac{\pi}{4} + \frac{\pi}{6} \right)$$

$$= \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

b) 设

$$x^2 = \sec \theta$$

则

$$2x \frac{dx}{d\theta} = \sec \theta \tan \theta$$

$$dx = \frac{\sec \theta \tan \theta}{2x} d\theta$$

当  $x = \sqrt{2}$  时,

$$x^2 = 2, \quad \sec \theta = 2, \quad \theta = \frac{\pi}{3}$$

当  $x = \sqrt{\sqrt{6} + \sqrt{2}}$  时,

$$x^2 = \sqrt{6} + \sqrt{2}, \quad \sec \theta = \sqrt{6} + \sqrt{2}$$

$$\cos \theta = \frac{1}{\sqrt{6} + \sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\theta = \frac{5\pi}{12}$$

原积分化为

$$\begin{aligned}
 & \int_{\sqrt{2}}^{\sqrt{\sqrt{6}+\sqrt{2}}} \frac{2}{x\sqrt{x^4-1}} dx \\
 &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{1}{x\sqrt{\sec^2\theta-1}} \cdot \frac{\sec\theta\tan\theta}{2x} d\theta \\
 &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{\sec\theta\tan\theta}{2x^2\sqrt{\tan^2\theta}} d\theta \\
 &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{\sec\theta\tan\theta}{2\sec\theta\tan\theta} d\theta \\
 &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \frac{1}{2} d\theta \\
 &= \left[ \frac{1}{2}\theta \right]_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \\
 &= \frac{1}{2} \left( \frac{5\pi}{12} - \frac{\pi}{3} \right) = \frac{1}{2} \cdot \frac{\pi}{12} = \frac{\pi}{24}
 \end{aligned}$$

44.

$$\int_{-1}^1 \frac{1}{(e^x+1)(1+x^2)} dx$$

巧妙换元  $u = -x$ ,

$$\begin{aligned}
 \int_{-1}^1 \frac{dx}{(e^x+1)(1+x^2)} &= \int_{-1}^0 \frac{dx}{(e^x+1)(1+x^2)} + \int_0^1 \frac{dx}{(e^x+1)(1+x^2)} \\
 &= -\int_1^0 \frac{du}{(e^{-u}+1)(1+u^2)} + \int_0^1 \frac{dx}{(e^x+1)(1+x^2)} \\
 &= \int_0^1 \frac{e^u du}{(1+e^u)(1+u^2)} + \int_0^1 \frac{dx}{(e^x+1)(1+x^2)} \\
 &= \int_0^1 \frac{du}{1+u^2} = \frac{\pi}{4}
 \end{aligned}$$

45. 使用代换  $u = \sqrt{\frac{1+x}{1-x}}$  计算积分

$$\int_0^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2}-3(1-x^2)} dx$$

先整理给定代换。

$$u^2 = \frac{1+x}{1-x}$$

于是

$$u^2(1-x) = 1+x$$

整理得

$$u^2 - 1 = x(1+u^2)$$

从而

$$x = \frac{u^2 - 1}{u^2 + 1}$$

对  $x$  求导,

$$\begin{aligned} \frac{dx}{du} &= \frac{(u^2 + 1)(2u) - (u^2 - 1)(2u)}{(u^2 + 1)^2} \\ &= \frac{4u}{(u^2 + 1)^2} \end{aligned}$$

因此

$$dx = \frac{4u}{(u^2 + 1)^2} du$$

再分别化简被积式中的各项。

$$\begin{aligned} 1 - x^2 &= 1 - \left( \frac{u^2 - 1}{u^2 + 1} \right)^2 \\ &= \frac{(u^2 + 1)^2 - (u^2 - 1)^2}{(u^2 + 1)^2} \\ &= \frac{4u^2}{(u^2 + 1)^2} \end{aligned}$$

从而

$$\sqrt{1-x^2} = \frac{2u}{u^2 + 1}$$

同时

$$\begin{aligned} 4x + 5 &= 4 \left( \frac{u^2 - 1}{u^2 + 1} \right) + 5 \\ &= \frac{9u^2 + 1}{u^2 + 1} \end{aligned}$$

积分上下限变为

$$x = 0 \Rightarrow u = 1, \quad x = \frac{1}{4} \Rightarrow u = \sqrt{\frac{5}{3}}$$

代入原积分得

$$\int_1^{\sqrt{5/3}} \frac{3}{\left(\frac{9u^2+1}{u^2+1}\right)\left(\frac{2u}{u^2+1}\right) - 3\left(\frac{4u^2}{(u^2+1)^2}\right)} \frac{4u}{(u^2+1)^2} du$$

化简后得到

$$\begin{aligned} &= \int_1^{\sqrt{5/3}} \frac{12u}{2u(9u^2+1) - 12u^2} du \\ &= \int_1^{\sqrt{5/3}} \frac{6u}{9u^3 - 6u^2 + u} du \\ &= \int_1^{\sqrt{5/3}} \frac{6}{9u^2 - 6u + 1} du \\ &= \int_1^{\sqrt{5/3}} \frac{6}{(3u-1)^2} du \end{aligned}$$

直接积分,

$$\int \frac{6}{(3u-1)^2} du = -\frac{2}{3u-1}$$

代入上下限,

$$\begin{aligned} &= \left[ -\frac{2}{3u-1} \right]_1^{\sqrt{5/3}} \\ &= -\frac{2}{3\sqrt{5/3}-1} + \frac{2}{2} \end{aligned}$$

继续化简,

$$\begin{aligned} &= 1 - \frac{2}{\sqrt{15}-1} \\ &= 1 - \frac{2(\sqrt{15}+1)}{14} \\ &= \frac{6-\sqrt{15}}{7} \end{aligned}$$

因此积分结果为

$$\frac{6-\sqrt{15}}{7}$$

46. 使用代换  $x = \frac{ab}{t}$  求积分

$$\int_0^{\infty} \frac{\ln x}{(x+a)(x+b)} dx$$

其中  $a > b > 0$ 。

设

$$I = \int_0^{\infty} \frac{\ln x}{(x+a)(x+b)} dx$$

作代换

$$x = \frac{ab}{t}, \quad dx = -\frac{ab}{t^2} dt$$

当  $x = 0$  时,  $t = \infty$ ; 当  $x = \infty$  时,  $t = 0$ 。

则

$$I = \int_{\infty}^0 \frac{\ln\left(\frac{ab}{t}\right)}{\left(\frac{ab}{t}+a\right)\left(\frac{ab}{t}+b\right)} \left(-\frac{ab}{t^2}\right) dt$$

整理得

$$\begin{aligned} I &= \int_0^{\infty} \frac{\ln(ab) - \ln t}{\frac{ab(b+t)(a+t)}{t^2}} \frac{ab}{t^2} dt \\ &= \int_0^{\infty} \frac{\ln(ab) - \ln t}{(t+a)(t+b)} dt \end{aligned}$$

因此

$$\begin{aligned} I &= \ln(ab) \int_0^{\infty} \frac{1}{(t+a)(t+b)} dt - \int_0^{\infty} \frac{\ln t}{(t+a)(t+b)} dt \\ &= \ln(ab) \int_0^{\infty} \frac{1}{(t+a)(t+b)} dt - I \end{aligned}$$

移项得

$$2I = \ln(ab) \int_0^{\infty} \frac{1}{(t+a)(t+b)} dt$$

对分式作部分分式分解

$$\frac{1}{(t+a)(t+b)} = \frac{1}{a-b} \left( \frac{1}{t+b} - \frac{1}{t+a} \right)$$

于是

$$\begin{aligned} 2I &= \frac{\ln(ab)}{a-b} \int_0^{\infty} \left( \frac{1}{t+b} - \frac{1}{t+a} \right) dt \\ &= \frac{\ln(ab)}{a-b} \left[ \ln \left( \frac{t+b}{t+a} \right) \right]_0^{\infty} \end{aligned}$$

计算上下限

$$\left[ \ln \left( \frac{t+b}{t+a} \right) \right]_0^\infty = 0 - \ln \left( \frac{b}{a} \right) = \ln \left( \frac{a}{b} \right)$$

因此

$$2I = \frac{\ln(ab)}{a-b} \ln \left( \frac{a}{b} \right)$$

从而

$$I = \frac{\ln(ab)}{2(a-b)} \ln \left( \frac{a}{b} \right)$$

47. 使用代换求

$$\int \frac{2 \sin x \sqrt{\tan x} + 1}{2 \cos x \sqrt{\tan x} (\cos x \sqrt{\tan x} + 1)} dx$$

代换:

$$u = \sec x + \sqrt{\tan x} = \sec x + (\tan x)^{\frac{1}{2}}$$

求导:

$$\frac{du}{dx} = \sec x \tan x + \frac{1}{2} (\tan x)^{-\frac{1}{2}} \sec^2 x = \frac{\sin x}{\cos^3 x} + \frac{\cos^{1/2} x}{2 \sin^{1/2} x \cos^2 x} = \frac{1}{\cos^2 x} \left( \frac{2 \sin^{3/2} x + \cos^{3/2} x}{2 \sin^{1/2} x \cos^{1/2} x} \right)$$

所以:

$$dx = \frac{2 \cos^3 x \sin^{1/2} x \cos^{1/2} x}{2 \sin^{3/2} x + \cos^{3/2} x} du$$

原积分:

$$\int \frac{2 \sin x \sqrt{\tan x} + 1}{2 \cos x \sqrt{\tan x} (\cos x \sqrt{\tan x} + 1)} dx$$

代入  $dx$ :

$$= \int \frac{2 \sin^{3/2} x + \cos^{1/2} x}{2 \cos^{1/2} x \sin^{1/2} x (\cos^{1/2} x \sin^{1/2} x + 1)} \times \frac{2 \cos^{3/2} x \sin^{1/2} x}{2 \sin^{3/2} x + \cos^{3/2} x} du$$

化简:

$$= \int \frac{\cos x}{\cos x \sin x + 1} du$$

两边同时乘  $\sec x$ :

$$= \int \frac{\sec x}{\sec x (\cos x \sin x + 1)} du = \int \frac{1}{\sin x + \sec x} du = \int \frac{1}{\sqrt{\tan x} + \sec x} du$$

最终积分:

$$\int \frac{1}{u} du = \ln |u| + C = \ln |\sqrt{\tan x} + \sec x| + C$$

48. 使用代换求积分

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx$$

使用代换  $u = \sqrt{x+2} \implies x = u^2 - 2, dx = 2u du$ :

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx = \int \frac{16}{(u^2+4)(u^2-4)u} \cdot 2u du = \int \frac{32}{(u^2+4)(u^2-4)} du$$

分解为部分分式:

$$\frac{32}{(u^2+4)(u^2-4)} = \frac{4}{u^2-4} - \frac{4}{u^2+4} = \frac{1}{u-2} - \frac{1}{u+2} - \frac{4}{u^2+4}$$

积分得到:

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx = \ln |u-2| - \ln |u+2| - 2 \arctan \left( \frac{u}{2} \right) + C$$

回代  $u = \sqrt{x+2}$ :

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx = \ln \left| \frac{\sqrt{x+2}-2}{\sqrt{x+2}+2} \right| - 2 \arctan \left( \frac{\sqrt{x+2}}{2} \right) + C$$

49. 11) 计算不定积分  $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}} dx$

分子有理化:

$$\begin{aligned} \int \frac{(\sqrt{x+1}-\sqrt{x-1})^2}{(x+1)-(x-1)} dx &= \int \frac{x+1+x-1-2\sqrt{x^2-1}}{2} dx \\ &= \int (x - \sqrt{x^2-1}) dx \\ &= \frac{1}{2}x^2 - \left( \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| \right) + C \\ &= \frac{1}{2}x^2 - \frac{x}{2}\sqrt{x^2-1} + \frac{1}{2}\ln|x+\sqrt{x^2-1}| + C \end{aligned}$$

50. 使用代换  $x = e^{-u/2}$  求

$$\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx$$

使用代换  $x = e^{-u/2} \implies dx = -\frac{1}{2}e^{-u/2} du$ :

$$\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = \int \frac{e^{-2u} - 1}{e^{-u} \sqrt{e^{-4u} + 1}} \left( -\frac{1}{2}e^{-u/2} du \right)$$

化简得到:

$$= \frac{1}{\sqrt{2}} \int \sinh u (\cosh u)^{-1/2} du = \frac{1}{\sqrt{2}} \cdot 2(\cosh u)^{1/2} + C = \sqrt{2 \cosh u} + C$$

回代  $x = e^{-u/2} \implies \cosh u = \frac{1}{2}(e^u + e^{-u}) = \frac{1}{2} \left( \frac{1}{x^2} + x^2 \right)$ :

$$\sqrt{2 \cosh u} + C = \sqrt{\frac{1}{x^2} + x^2} + C = \frac{\sqrt{x^4 + 1}}{x} + C$$

因此积分的简化形式为:

$$\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = \frac{\sqrt{x^4 + 1}}{x} + C$$

51. 已知

$$x^2 + x + 2 = (u - x)^2$$

求

$$\int \frac{1}{x \sqrt{x^2 + x + 2}} dx$$

a) (i) 解  $x$  关于  $u$  的关系:

$$x^2 + x + 2 = u^2 - 2ux + x^2$$

$$x + 2 = u^2 - 2ux$$

$$x(2u + 1) = u^2 - 2$$

$$x = \frac{u^2 - 2}{2u + 1}$$

(ii) 对  $x$  求导:

$$\frac{dx}{du} = \frac{(2u+1)(2u) - 2(u^2-2)}{(2u+1)^2} = \frac{4u^2 + 2u - 2u^2 + 4}{(2u+1)^2} = \frac{2u^2 + 2u + 4}{(2u+1)^2} = \frac{2(u^2 + u + 2)}{(2u+1)^2}$$

b) 积分变换:

$$\int \frac{1}{x\sqrt{x^2+x+2}} dx = \int \frac{1}{x(u-x)} \cdot \frac{2(u^2+u+2)}{(2u+1)^2} du$$

代入  $x = \frac{u^2-2}{2u+1}$  和  $u-x = \sqrt{x^2+x+2}$ :

$$\int \frac{1}{\frac{u^2-2}{2u+1} \cdot \frac{u^2+u+2}{2u+1}} \cdot \frac{2(u^2+u+2)}{(2u+1)^2} du = \int \frac{(2u+1)^2}{(u^2-2)(u^2+u+2)} \cdot \frac{2(u^2+u+2)}{(2u+1)^2} du = \int \frac{2}{u^2-2} du$$

使用标准积分公式:

$$\int \frac{1}{u^2 - a^2} du = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C$$

得:

$$\int \frac{2}{u^2-2} du = \frac{1}{\sqrt{2}} \ln \left| \frac{u-\sqrt{2}}{u+\sqrt{2}} \right| + C$$

最后代回  $u = x + \sqrt{x^2+x+2}$ :

$$\int \frac{1}{x\sqrt{x^2+x+2}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{x + \sqrt{x^2+x+2} - \sqrt{2}}{x + \sqrt{x^2+x+2} + \sqrt{2}} \right| + C$$

## 52. 计算积分

$$\int_{0.2}^{0.5} \frac{\sqrt{x-x^2}}{x^4} dx$$

作代换:

$$x = \frac{1}{u^2+1} \implies dx = -\frac{2u}{(u^2+1)^2} du$$

积分上下限变化:

$$x = 0.2 \implies u = 2, \quad x = 0.5 \implies u = 1$$

代入积分:

$$\int_{0.2}^{0.5} \frac{\sqrt{x-x^2}}{x^4} dx = \int_2^1 \frac{\sqrt{\frac{1}{u^2+1} - \frac{1}{(u^2+1)^2}}}{\left(\frac{1}{u^2+1}\right)^4} \left(-\frac{2u}{(u^2+1)^2} du\right)$$

整理被积函数:

$$\sqrt{\frac{1}{u^2+1} - \frac{1}{(u^2+1)^2}} = \sqrt{\frac{u^2}{(u^2+1)^2}} = \frac{u}{u^2+1}$$

$$\frac{1}{\left(\frac{1}{u^2+1}\right)^4} = (u^2+1)^4$$

因此积分变为:

$$\int_1^2 \frac{u}{u^2+1} (u^2+1)^4 \cdot \frac{2u}{(u^2+1)^2} du = \int_1^2 2u^2(u^2+1) du$$

展开:

$$\int_1^2 2u^4 + 2u^2 du = \left[ \frac{2}{5}u^5 + \frac{2}{3}u^3 \right]_1^2$$

代入上下限:

$$\left(\frac{64}{5} + \frac{16}{3}\right) - \left(\frac{2}{5} + \frac{2}{3}\right) = \frac{256}{15}$$

$$\therefore \int_{0.2}^{0.5} \frac{\sqrt{x-x^2}}{x^4} dx = \frac{256}{15}$$

53.

$$\int \frac{dx}{x^2(a+bx)^2}, \quad a, b \in \mathbb{R}$$

设  $u = \frac{a+bx}{x}$ , 则

$$x = \frac{a}{u-b}, \quad dx = \frac{-a du}{(u-b)^2}, \quad a+bx = xu = \frac{au}{u-b}$$

代入原式:

$$\begin{aligned}\int \frac{dx}{x^2(a+bx)^2} &= \int \frac{-a du}{(u-b)^2} \cdot \frac{1}{x^2(a+bx)^2} \\&= \int \frac{-a du}{(u-b)^2} \cdot \frac{(u-b)^4}{a^4 u^2} \\&= -\frac{1}{a^3} \int \frac{(u-b)^2}{u^2} du \\&= -\frac{1}{a^3} \int \left(1 - \frac{2b}{u} + \frac{b^2}{u^2}\right) du \\&= -\frac{1}{a^3} \left(u - 2b \ln |u| - \frac{b^2}{u}\right) + C\end{aligned}$$

代回  $u = \frac{a+bx}{x}$ , 得:

$$\int \frac{dx}{x^2(a+bx)^2} = -\frac{1}{a^3} \left(\frac{a+bx}{x} - 2b \ln \left|\frac{a+bx}{x}\right| - \frac{b^2 x}{a+bx}\right) + C$$

54. 使用代换  $2x+1=4\cosh\theta$ , 求不定积分

$$\int \sqrt{(2x+5)(2x-3)} dx$$

先化简被积式,

$$(2x+5)(2x-3) = 4x^2 + 4x - 15 = (2x+1)^2 - 16$$

因此原积分化为

$$\int \sqrt{(2x+1)^2 - 16} dx$$

采用双曲函数换元, 令

$$2x+1 = 4\cosh\theta$$

则

$$2dx = 4\sinh\theta d\theta, \quad dx = 2\sinh\theta d\theta$$

利用恒等式

$$\cosh^2\theta - \sinh^2\theta = 1$$

可得

$$\sqrt{(2x+1)^2 - 16} = \sqrt{16(\cosh^2\theta - 1)} = 4\sinh\theta$$

代入积分,

$$\int \sqrt{(2x+1)^2 - 16} dx = \int 4 \sinh \theta (2 \sinh \theta d\theta) = 8 \int \sinh^2 \theta d\theta$$

又有

$$\sinh^2 \theta = \frac{1}{2} \cosh(2\theta) - \frac{1}{2}$$

于是

$$\begin{aligned} 8 \int \sinh^2 \theta d\theta &= 8 \int \left( \frac{1}{2} \cosh(2\theta) - \frac{1}{2} \right) d\theta \\ &= 4 \int \cosh(2\theta) d\theta - 4 \int d\theta \\ &= 2 \sinh(2\theta) - 4\theta + C \end{aligned}$$

利用

$$\sinh(2\theta) = 2 \sinh \theta \cosh \theta$$

得

$$2 \sinh(2\theta) = 4 \sinh \theta \cosh \theta$$

又因为

$$\sinh \theta = \frac{\sqrt{(2x+1)^2 - 16}}{4}, \quad \cosh \theta = \frac{2x+1}{4}$$

且

$$\theta = \operatorname{arccosh} \frac{2x+1}{4}$$

代回原变量,

$$\int \sqrt{(2x+5)(2x-3)} dx = \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4 \operatorname{arccosh} \frac{2x+1}{4} + C$$

利用

$$\operatorname{arccosh} u = \ln \left( u + \sqrt{u^2 - 1} \right)$$

可进一步写为

$$\int \sqrt{(2x+5)(2x-3)} dx = \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4 \ln \left( 2x+1 + \sqrt{(2x+5)(2x-3)} \right) + C$$

55. 已知

$$\sqrt{5-4x-x^2} = (1-x)u, \quad x \neq 1, x \neq -5$$

a) 代换与求导

i) 两边平方

$$5 - 4x - x^2 = (1 - x)^2 u^2$$

移项整理

$$-(x^2 + 4x - 5) = u^2(x - 1)^2 \implies -(x - 1)(x + 5) = u^2(x - 1)^2$$

两边除以  $x - 1$

$$-(x + 5) = u^2(x - 1) \implies u^2 - 5 = x(u^2 + 1) \implies x = \frac{u^2 - 5}{u^2 + 1}$$

ii) 对  $x$  求导

$$dx = \frac{d}{du} \frac{u^2 - 5}{u^2 + 1} du = \frac{(u^2 + 1)(2u) - (u^2 - 5)(2u)}{(u^2 + 1)^2} du = \frac{12u}{(u^2 + 1)^2} du$$

b) 积分化简

由 a) 可得

$$\int \frac{x}{(5 - 4x - x^2)^{3/2}} dx = \int \frac{x}{((1 - x)u)^3} \cdot \frac{12u}{(u^2 + 1)^2} du$$

$$x = \frac{u^2 - 5}{u^2 + 1}, \quad (1 - x)^3 = \left( \frac{6}{u^2 + 1} \right)^3$$

$$\implies \int \frac{x}{(5 - 4x - x^2)^{3/2}} dx = \int \frac{u^2 - 5}{u^2 + 1} \cdot \frac{12u}{216u^3} du = \int \frac{u^2 - 5}{18u^2} du$$

c) 积分求解

分拆积分

$$\int \frac{u^2 - 5}{18u^2} du = \int \frac{1}{18} - \frac{5}{18u^2} du = \frac{1}{18}u + \frac{5}{18u} + C$$

代回  $u$

$$u = \frac{\sqrt{5 - 4x - x^2}}{1 - x} \implies \frac{1}{u} = \frac{1 - x}{\sqrt{5 - 4x - x^2}}$$

$$\implies \int \frac{x}{(5 - 4x - x^2)^{3/2}} dx = \frac{1}{18} \frac{\sqrt{5 - 4x - x^2}}{1 - x} + \frac{5}{18} \frac{1 - x}{\sqrt{5 - 4x - x^2}} + C$$

整理为

$$\int \frac{x}{(5 - 4x - x^2)^{3/2}} dx = \frac{5 - 2x}{9\sqrt{5 - 4x - x^2}} + C$$

56. 28) 计算不定积分:

$$\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$$

设  $u = \sqrt[3]{x}$ , 则  $du = \frac{1}{3}x^{-\frac{2}{3}}dx$ , 即  $dx = 3\sqrt[3]{x^2}du$ 。代入积分式:

$$I = 3 \int \sqrt{1+u} du$$

进行积分计算:

$$I = 2(1+u)^{\frac{3}{2}} + C$$

还原变量  $u = \sqrt[3]{x}$ :

$$I = 2(1+\sqrt[3]{x})^{\frac{3}{2}} + C$$

57. 16) 计算不定积分:

$$\int \frac{1}{x(1+x^{119})} dx$$

分子分母同乘  $x^{118}$ :

$$\int \frac{x^{118}}{x^{119}(1+x^{119})} dx$$

设  $t = x^{119}$ , 则  $dt = 119x^{118}dx$ :

$$\begin{aligned} I &= \frac{1}{119} \int \frac{1}{t(1+t)} dt = \frac{1}{119} \int \left( \frac{1}{t} - \frac{1}{1+t} \right) dt \\ &= \frac{1}{119} \ln \left| \frac{t}{1+t} \right| + C = \frac{1}{119} \ln \left| \frac{x^{119}}{1+x^{119}} \right| + C \end{aligned}$$

58.

$$\int_{-1}^1 \frac{x^2}{1+2^x} dx$$

设  $u = -x$ , 则

$$I = \int_{-1}^1 \frac{u^2}{1+2^{-u}} du = \int_{-1}^1 \frac{u^2 2^u}{1+2^u} du = \int_{-1}^1 \frac{x^2 2^x}{1+2^x} dx$$

于是有

$$2I = \int_{-1}^1 \frac{x^2(2^x+1)}{1+2^x} dx = \int_{-1}^1 x^2 dx = \frac{2}{3} \Rightarrow I = \frac{1}{3}$$

## 59. 计算积分

$$I = \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx.$$

将积分分成两部分:

$$I = \int_{\frac{1}{2}}^1 \frac{\ln x}{1+x^2} dx + \int_1^2 \frac{\ln x}{1+x^2} dx.$$

在第二个积分中作代换  $x = 1/u$ , 则  $dx = -\frac{1}{u^2} du$ , 积分限变为  $x = 1 \rightarrow u = 1, x = 2 \rightarrow u = 1/2$ :

$$\int_1^2 \frac{\ln x}{1+x^2} dx = \int_1^{1/2} \frac{\ln(1/u)}{1+(1/u)^2} \left(-\frac{1}{u^2} du\right) = -\int_{1/2}^1 \frac{-\ln u}{1+u^2} du = -\int_{1/2}^1 \frac{\ln u}{1+u^2} du.$$

因此

$$I = \int_{1/2}^1 \frac{\ln x}{1+x^2} dx - \int_{1/2}^1 \frac{\ln u}{1+u^2} du = 0.$$

60. 30) 计算定积分  $I = \int_0^{\sqrt{3}} \frac{1}{1+x^2} \sin^{-1} \frac{2x}{1+x^2} dx$ 

使用换元法, 设  $x = \tan \theta$ , 则  $dx = \sec^2 \theta d\theta$  且  $\frac{1}{1+x^2} = \cos^2 \theta$ 。同时利用二倍角公式可知  $\sin^{-1} \frac{2x}{1+x^2} = \sin^{-1}(\sin 2\theta) = 2\theta$ 。

更换积分上下限:

- 当  $x = 0$  时,  $\theta = 0$
- 当  $x = \sqrt{3}$  时,  $\theta = \frac{\pi}{3}$

代入原积分式:

$$\begin{aligned} I &= \int_0^{\frac{\pi}{3}} \cos^2 \theta \cdot (2\theta) \cdot \sec^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{3}} 2\theta d\theta \\ &= [\theta^2]_0^{\frac{\pi}{3}} \\ &= \left(\frac{\pi}{3}\right)^2 - 0 \\ &= \frac{\pi^2}{9} \end{aligned}$$

61. 计算  $[\int_{1/2}^2 \frac{x^{2012}-1}{x^{2014}+1} dx]$ 。

令  $x = \frac{1}{t}$ , 则  $dx = -\frac{1}{t^2} dt$ 。积分上下限从  $1/2 \rightarrow 2$  变为  $2 \rightarrow 1/2$ :

$$I = [\int_2^{1/2} \frac{(1/t)^{2012} - 1}{(1/t)^{2014} + 1} (-\frac{1}{t^2}) dt]$$

$$I = [\int_{1/2}^2 \frac{\frac{1-t^{2012}}{t^{2012}}}{\frac{1+t^{2014}}{t^{2014}}} \frac{1}{t^2} dt] = [\int_{1/2}^2 \frac{1-t^{2012}}{1+t^{2014}} dt]$$

注意到  $[\int_{1/2}^2 \frac{1-x^{2012}}{1+x^{2014}} dx] = -[\int_{1/2}^2 \frac{x^{2012}-1}{x^{2014}+1} dx] = -I$ 。即  $I = -I$ , 故  $I = 0$ 。

62. 若正整数  $k$  满足

$$\left( \int_0^1 x^{2018} (2019 + kx^{10}) \sqrt{1+x^{10}} dx \right)^2 = 8,$$

求  $k$  的最大值。

由于被积函数随  $k$  严格递增, 则当

$$\int_0^1 x^{2018} (2019 + kx^{10}) \sqrt{1+x^{10}} dx = \sqrt{8}$$

时  $k$  取最大。欲写成形如

$$\int_0^1 (2019x^{2018-n} + kx^{2028-n}) \sqrt{x^{2n} + x^{2n+10}} dx$$

的积分, 考虑强迫

$$2n - 1 = 2018 - n,$$

可得  $n = 673$ , 此时设  $u = x^{1346} + x^{1356}$ , 则  $du = 1346x^{1345} + 1356x^{1355} dx$ , 于是

$$\begin{aligned} I &= \int_0^1 (2019x^{1345} + kx^{1355}) \sqrt{x^{1346} + x^{1356}} dx \\ &= \frac{3}{2} \int_0^1 (1346^{1345} + \frac{2}{3} kx^{1355}) \sqrt{x^{1346} + x^{1356}} dx \end{aligned}$$

当  $\frac{2}{3}k = 1356$  即  $k = 2034$  时, 有

$$I = \frac{3}{2} \int_0^2 \sqrt{u} du = \frac{3}{2} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_0^2 = \sqrt{8}$$

63.

$$\int \frac{dx}{x^3 + 1}$$

$$\int \frac{dx}{x^3 + 1} = \int \frac{dx}{(x + 1)(x^2 - x + 1)}$$

部分分式分解得

$$\frac{1}{x^3 + 1} = \frac{1}{3(x + 1)} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2 - x + 1}$$

于是

$$\int \frac{dx}{x^3 + 1} = \frac{1}{3} \int \frac{dx}{x + 1} - \frac{1}{3} \int \frac{x}{x^2 - x + 1} dx + \frac{2}{3} \int \frac{dx}{x^2 - x + 1}$$

其中

$$\begin{aligned} \int \frac{dx}{x + 1} &= \ln|x + 1|, & \int \frac{x}{x^2 - x + 1} dx &= \frac{1}{2} \ln(x^2 - x + 1) \\ \int \frac{dx}{x^2 - x + 1} &= \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \arctan\left(\frac{2x - 1}{\sqrt{3}}\right) + C \end{aligned}$$

故

$$\int \frac{dx}{x^3 + 1} = \frac{1}{3} \ln|x + 1| - \frac{1}{6} \ln(x^2 - x + 1) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x - 1}{\sqrt{3}}\right) + C$$

64. 17) 计算不定积分:

$$\int \frac{x^4 + 1}{x^6 + 1} dx$$

由于  $x^6 + 1 = (x^2 + 1)(x^4 - x^2 + 1)$ , 拆分分子:

$$\int \frac{(x^4 - x^2 + 1) + x^2}{x^6 + 1} dx = \int \frac{1}{x^2 + 1} dx + \int \frac{x^2}{x^6 + 1} dx$$

第一部分:  $\int \frac{1}{x^2 + 1} dx = \tan^{-1} x$ . 第二部分: 设  $u = x^3$ ,  $du = 3x^2 dx$ :

$$\frac{1}{3} \int \frac{1}{u^2 + 1} du = \frac{1}{3} \tan^{-1} u = \frac{1}{3} \tan^{-1} x^3$$

合并结果:

$$I = \tan^{-1} x + \frac{1}{3} \tan^{-1} x^3 + C$$

65. 计算积分

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx$$

首先展开分子  $(1-x)^4$ :

$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$

因此积分变为:

$$\int_0^1 \frac{x^4(1-4x+6x^2-4x^3+x^4)}{x^2+1} dx = \int_0^1 \frac{x^8-4x^7+6x^6-4x^5+x^4}{x^2+1} dx$$

进行多项式除法, 将分子按  $x^2+1$  除:

$$\frac{x^8-4x^7+6x^6-4x^5+x^4}{x^2+1} = x^6-4x^5+5x^4-4x^2+4-\frac{4}{x^2+1}$$

于是积分拆分为:

$$\int_0^1 x^6-4x^5+5x^4-4x^2+4-\frac{4}{x^2+1} dx$$

分别积分各项:

$$\begin{aligned} \int x^6 dx &= \frac{1}{7}x^7, & \int -4x^5 dx &= -\frac{2}{3}x^6, & \int 5x^4 dx &= x^5, \\ \int -4x^2 dx &= -\frac{4}{3}x^3, & \int 4 dx &= 4x, & \int -\frac{4}{x^2+1} dx &= -4 \arctan x \end{aligned}$$

组合结果:

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx = \left[ \frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4 \arctan x \right]_0^1$$

代入上下限:

$$\left( \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4 \cdot \frac{\pi}{4} \right) - 0 = \frac{1}{7} - \frac{6}{3} + 5 - \pi = \frac{1}{7} - 2 + 5 - \pi = 3 + \frac{1}{7} - \pi = \frac{22}{7} - \pi$$

因此最终结果为:

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx = \frac{22}{7} - \pi$$

66.

$$\int_0^1 \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} dx$$

$$\begin{aligned}
\int_0^1 \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} dx &= \int_0^1 \left[ 1 + \frac{2}{7} \left[ \frac{1}{x^2+3} + \frac{5}{x-2} - \frac{5}{x+2} \right] \right] dx \\
&= \left[ x + \frac{2}{7} \left[ \frac{1}{\sqrt{3}} \arctan \left( \frac{x}{\sqrt{3}} \right) + 5 \ln |x-2| - 5 \ln |x+2| \right] \right]_0^1 \\
&= \left[ 1 + \frac{2}{7} \left[ \frac{1}{\sqrt{3}} \times \frac{\pi}{6} + 5 \ln |-1| - 5 \ln |3| \right] \right] - \left[ \frac{2}{7} [5 \ln |-2| - 5 \ln |2|] \right] \\
&= 1 + \frac{2}{7} \left[ \frac{\pi}{6\sqrt{3}} - 5 \ln 3 \right] \\
&= 1 + \frac{1}{7} \left[ \frac{\pi}{3\sqrt{3}} - 10 \ln 3 \right]
\end{aligned}$$

67. 计算积分

$$\int_0^{\frac{1}{3}} \frac{32x^2}{(x^2-1)(x+1)^3} dx$$

先化简被积函数

$$\int_0^{\frac{1}{3}} \frac{32x^2}{(x^2-1)(x+1)^3} dx = \int_0^{\frac{1}{3}} \frac{32x^2}{(x-1)(x+1)^4} dx$$

作代换

$$u = x + 1, \quad x = u - 1, \quad du = dx$$

当  $x = 0$  时,  $u = 1$  当  $x = \frac{1}{3}$  时,  $u = \frac{4}{3}$

则积分化为

$$\int_1^{\frac{4}{3}} \frac{32(u-1)^2}{(u-2)u^4} du = \int_1^{\frac{4}{3}} \frac{32}{u^4} \cdot \frac{u^2 - 2u + 1}{u - 2} du$$

将有理函数分解后得

$$\int_1^{\frac{4}{3}} \left( -\frac{16}{u^4} + \frac{24}{u^3} - \frac{4}{u^2} - \frac{2}{u} + \frac{2}{u-2} \right) du$$

逐项积分

$$= \left[ \frac{16}{3u^3} - \frac{12}{u^2} + \frac{4}{u} - 2 \ln |u| + 2 \ln |u-2| \right]_1^{\frac{4}{3}}$$

代入上、下限

$$= \left( \frac{16}{3 \left( \frac{4}{3} \right)^3} - \frac{12}{\left( \frac{4}{3} \right)^2} + \frac{4}{\frac{4}{3}} - 2 \ln \frac{4}{3} + 2 \ln \frac{2}{3} \right) - \left( \frac{16}{3} - 12 + 4 \right)$$

化简得

$$= 11 - \frac{16}{3} - \frac{9}{2} - 2 \ln 2$$

整理为

$$\frac{7}{6} - 2 \ln 2$$

因此

$$\int_0^{\frac{1}{3}} \frac{32x^2}{(x^2-1)(x+1)^3} dx = \frac{7}{6} - 2 \ln 2$$

68. 24) 计算不定积分:

$$\int \frac{1}{\sqrt[4]{1+x^4}} dx$$

使用换元法, 设  $t^4 x^4 = 1 + x^4$ , 则  $x^4 = \frac{1}{t^4-1}$ 。两边求导得  $4x^3 dx = -\frac{4t^3}{(t^4-1)^2} dt$ , 故  $dx = -\frac{t^3}{(t^4-1)^2 x^3} dt$ 。代入积分式:

$$I = - \int \frac{1}{t \cdot x} \cdot \frac{t^3}{(t^4-1)^2 x^2} dt = - \int \frac{t^2}{t^4-1} dt$$

将被积函数进行部分分式拆分:

$$- \int \frac{t^2}{t^4-1} dt = -\frac{1}{2} \int \left( \frac{1}{t^2-1} + \frac{1}{t^2+1} \right) dt$$

进行积分计算:

$$I = -\frac{1}{4} \ln \left| \frac{t-1}{t+1} \right| - \frac{1}{2} \tan^{-1} t + C$$

其中还原变量为  $t = \frac{\sqrt[4]{1+x^4}}{x}$ 。

69. 20) 计算不定积分:

$$\int \frac{2}{\sqrt{x} - 3\sqrt[4]{x} + 2} dx$$

使用换元法, 设  $\sqrt[4]{x} = t$ , 则  $x = t^4$ ,  $dx = 4t^3 dt$ 。代入原积分式:

$$I = \int \frac{8t^3}{t^2 - 3t + 2} dt$$

对被积函数进行多项式除法与部分分式拆分:

$$\frac{8t^3}{t^2 - 3t + 2} = 8t + 24 + \frac{56t - 48}{t^2 - 3t + 2} = 8t + 24 + \frac{64}{t-2} - \frac{8}{t-1}$$

进行积分计算:

$$\begin{aligned} I &= \int (8t + 24)dt + \int \frac{64}{t-2}dt - \int \frac{8}{t-1}dt \\ &= 4t^2 + 24t + 64 \ln |t-2| - 8 \ln |t-1| + C \end{aligned}$$

还原变量  $t = \sqrt[4]{x}$ :

$$I = 4\sqrt[4]{x} + 24\sqrt[4]{x} + 64 \ln |\sqrt[4]{x} - 2| - 8 \ln |\sqrt[4]{x} - 1| + C$$

70. 22) 计算不定积分:

$$\int \frac{3x^3}{x^4 + x^3 + x + 1} dx$$

首先对分母进行因式分解:

$$x^4 + x^3 + x + 1 = x^3(x+1) + (x+1) = (x^3+1)(x+1) = (x^2-x+1)(x+1)^2$$

将被积函数拆分为部分分式:

$$\frac{3x^3}{(x^2-x+1)(x+1)^2} = \frac{x-1}{x^2-x+1} + \frac{2}{x+1} - \frac{1}{(x+1)^2}$$

逐步进行积分:

$$\begin{aligned} I &= \int \frac{x-1}{x^2-x+1} dx + \int \frac{2}{x+1} dx - \int \frac{1}{(x+1)^2} dx \\ &= \frac{1}{2} \int \frac{2x-1-1}{x^2-x+1} dx + 2 \ln |x+1| + \frac{1}{x+1} \\ &= \frac{1}{2} \ln(x^2-x+1) - \frac{1}{2} \int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx + 2 \ln |x+1| + \frac{1}{x+1} \end{aligned}$$

计算得到最终结果:

$$I = \ln \left| (x+1)^2 \sqrt{x^2-x+1} \right| - \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2x-1}{\sqrt{3}} \right) + \frac{1}{x+1} + C$$

71.

$$\int \frac{x^2(x^4+1)}{\sqrt{x^4+2}} dx$$

先作代数变形

$$\int \frac{x^2(x^4+1)}{\sqrt{x^4+2}} dx = \int x^2(x^4+1)(x^4+2)^{-\frac{1}{2}} dx$$

将分子整理

$$= \int (x^7+x^3)(x^4)^{-\frac{1}{2}}(x^4+2)^{-\frac{1}{2}} dx$$

合并为一个整体

$$= \int (x^7+x^3)[x^4(x^4+2)]^{-\frac{1}{2}} dx$$

即

$$= \int (x^7+x^3)(x^8+2x^4)^{-\frac{1}{2}} dx$$

注意到

$$\frac{d}{dx}(x^8+2x^4) = 8x^7+8x^3 = 8(x^7+x^3)$$

于是

$$= \frac{1}{8} \int 8(x^7+x^3)(x^8+2x^4)^{-\frac{1}{2}} dx$$

积分得

$$= \frac{1}{8} \cdot \frac{1}{\frac{1}{2}} (x^8+2x^4)^{\frac{1}{2}} + C$$

化简

$$= \frac{1}{4} (x^8+2x^4)^{\frac{1}{2}} + C$$

72. 14) 计算不定积分:

$$\int \frac{x^2+1}{(x^2-2x+2)^3} dx$$

首先对分子进行拆分, 凑出分母的导数项与基本项:

$$\int \frac{(x^2-2x+2) + (2x-2) + 1}{(x^2-2x+2)^3} dx$$

将其拆分为三个部分进行积分:

$$= \int \frac{dx}{(x^2-2x+2)^2} + \int \frac{(2x-2)dx}{(x^2-2x+2)^3} + \int \frac{dx}{(x^2-2x+2)^3}$$

对第一项和第三项进行配方:

$$x^2-2x+2 = (x-1)^2+1$$

设  $x - 1 = \tan t$ , 则  $dx = \sec^2 t dt$ 。代入后积分式变为:

$$\int \cos^2 t dt + \int \cos^4 t dt - \frac{1}{2(x^2 - 2x + 2)^2}$$

利用降幂公式计算三角积分:

$$\int \cos^2 t dt = \frac{1}{2} \int (1 + \cos 2t) dt = \frac{t}{2} + \frac{\sin 2t}{4}$$

$$\int \cos^4 t dt = \frac{1}{4} \int (1 + \cos 2t)^2 dt = \cdots = \frac{3t}{8} + \frac{\sin 2t}{4} + \frac{\sin 4t}{32}$$

将各部分结果合并:

$$I = \frac{7t}{8} + \frac{\sin 2t}{2} + \frac{\sin 4t}{32} - \frac{1}{2(x^2 - 2x + 2)^2} + C$$

利用三角恒等式将  $t$  还原为  $x$ :

$$\sin 2t = \frac{2(x-1)}{x^2 - 2x + 2}, \quad \sin 4t = \frac{4(x-1)(1 - 2\sin^2 t)}{(x^2 - 2x + 2)^2}$$

最终整理得到原文结果:

$$I = \frac{7}{8} \tan^{-1}(x-1) + \frac{x-1}{x^2 - 2x + 2} + \frac{(x-1)(1 - 2\sin^2 t)}{8(x^2 - 2x + 2)^2} - \cdots + C$$

$$I = \frac{7}{8} \tan^{-1}(x-1) + \frac{7x^3 - 21x^2 + 30x - 20}{8(x^2 - 2x + 2)^2} + C$$

73. 40) 计算不定积分:

$$\int \sqrt[3]{\frac{2x+1}{x+3}} dx$$

使用换元法, 设  $t^3 = \frac{2x+1}{x+3}$ , 则可求得  $x = \frac{3t^3-1}{2-t^3}$ 。对  $x$  求导得  $dx = \frac{15t^2}{(2-t^3)^2} dt$ 。代入原积分式:

$$I = 15 \int \frac{t^3}{(2-t^3)^2} dt$$

根据原文推导过程, 利用分部积分或拆分项处理:

$$= 5 \int \left( -\frac{1}{2-t^3} + \frac{2(t^3+1)}{(2-t^3)^2} \right) dt = -5 \int \frac{1}{2-t^3} dt + \frac{5t}{2-t^3}$$

进一步整理为:

$$= 5 \int \frac{1}{t^3 - 2} dt - \frac{5t}{t^3 - 2}$$

利用部分分式进行拆分:

$$= 5 \int \left( \frac{-\frac{\sqrt[3]{2}}{6}t - \frac{\sqrt[3]{4}}{3}}{t^2 + \sqrt[3]{2}t + \sqrt[3]{4}} + \frac{\frac{\sqrt[3]{2}}{6}}{t - \sqrt[3]{2}} \right) dt - \frac{5t}{t^3 - 2}$$

进行积分计算并还原变量  $t = \sqrt[3]{\frac{2x+1}{x+3}}$ :

$$I = \frac{5\sqrt[3]{108}}{6} \tan^{-1} \frac{\sqrt{3}(\sqrt[3]{4}t + 1)}{3} - \frac{5\sqrt[3]{2}}{12} \ln(t^2 + \sqrt[3]{2}t + \sqrt[3]{4}) + \frac{5\sqrt[3]{2}}{6} \ln|t - \sqrt[3]{2}| - \frac{5t}{t^3 - 2} + C$$

74. 42) 计算不定积分:

$$\int \frac{dx}{\sqrt{x+1} + \sqrt[4]{x+1}}$$

设  $t = x + 1$ , 则  $dt = dx$ 。积分变为:

$$\int \frac{dt}{\sqrt{t} + \sqrt[4]{t}} = \int \frac{dt}{\sqrt{t}(\sqrt[4]{t} + 1)}$$

再设  $u = \sqrt[4]{t}$ , 则  $t = u^4, dt = 4u^3 du$ 。代入积分式:

$$\int \frac{4u^3}{u^2 + u} du = \int \frac{4u^2}{u + 1} du$$

对被积函数进行多项式除法:

$$\int \left( 4u - 4 + \frac{4}{u + 1} \right) du = 2u^2 - 4u + 4 \ln|u + 1| + C$$

还原变量  $u = \sqrt[4]{x+1}$ , 得到最终结果:

$$I = 2\sqrt{x+1} - 4\sqrt[4]{x+1} + 4 \ln(\sqrt[4]{x+1} + 1) + C$$

75. 43) 计算不定积分:

$$\int \frac{1 + x^2}{x\sqrt{x^4 - x^2 + 1}} dx$$

分子分母同时除以  $x^2$ :

$$I = \int \frac{1 + \frac{1}{x^2}}{\sqrt{x^2 - 1 + \frac{1}{x^2}}} dx$$

设  $t = x - \frac{1}{x}$ , 则  $dt = (1 + \frac{1}{x^2})dx$ , 且  $x^2 - 1 + \frac{1}{x^2} = t^2 + 1$ 。代入积分式:

$$I = \int \frac{dt}{\sqrt{t^2 + 1}} = \ln |t + \sqrt{t^2 + 1}| + C$$

还原变量:

$$I = \ln \left| x - \frac{1}{x} + \sqrt{x^2 - 1 + \frac{1}{x^2}} \right| + C$$

76. 44) 计算不定积分:

$$\int \frac{1}{(2x^2 + 3)\sqrt{5x^2 + 4}} dx$$

将被积函数改写, 提取  $x^3$ :

$$I = \int \frac{dx}{x^3(2 + \frac{3}{x^2})\sqrt{5 + \frac{4}{x^2}}}$$

设  $t^2 = 5 + \frac{4}{x^2}$ , 则  $2t dt = -\frac{8}{x^3} dx$ , 即  $\frac{1}{x^3} dx = -\frac{1}{4} t dt$ 。同时  $\frac{1}{x^2} = \frac{t^2 - 5}{4}$ , 代入分母项  $2 + \frac{3}{x^2} = \frac{3t^2 - 7}{4}$ 。代入积分式:

$$I = - \int \frac{\frac{1}{4} t dt}{\frac{3t^2 - 7}{4} \cdot t} = - \int \frac{dt}{3t^2 - 7} = -\frac{1}{3} \int \frac{dt}{t^2 - \frac{7}{3}}$$

进行积分计算:

$$I = -\frac{\sqrt{21}}{42} \ln \left| \frac{\sqrt{3}t - \sqrt{7}}{\sqrt{3}t + \sqrt{7}} \right| + C$$

还原变量  $t = \sqrt{5 + \frac{4}{x^2}}$ 。

77. 45) 计算不定积分:

$$\int \frac{x + x^{\frac{2}{3}} + x^{\frac{1}{6}}}{x(1 + x^{\frac{1}{3}})} dx$$

使用换元法, 设  $x = u^6$ , 则  $dx = 6u^5 du$ 。代入后简化:

$$I = 6 \int \frac{u^6 + u^4 + u}{u^6(1 + u^2)} \cdot u^5 du = 6 \int \frac{u^6 + u^4 + u}{1 + u^2} du$$

利用多项式除法拆分:

$$I = 6 \int \left( u^4 + \frac{u}{1+u^2} \right) du = \frac{6}{5} u^5 + 3 \ln(1+u^2) + C$$

还原变量  $u = x^{\frac{1}{6}}$ :

$$I = \frac{6}{5} x^{\frac{5}{6}} + 3 \ln(1+x^{\frac{1}{3}}) + C$$

78. 47) 计算不定积分:

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}} dx$$

设  $1+x^{\frac{1}{4}}=t^3$ , 则  $x=(t^3-1)^4, dx=12t^2(t^3-1)^3 dt$ 。代入积分式:

$$I = \int \frac{t}{(t^3-1)^2} \cdot 12t^2(t^3-1)^3 dt = 12 \int (t^6-t^3) dt$$

进行积分计算:

$$I = 12 \left( \frac{t^7}{7} - \frac{t^4}{4} \right) + C = \frac{12}{7} t^7 - 3t^4 + C$$

还原变量  $t = (1+x^{\frac{1}{4}})^{\frac{1}{3}}$ 。

79. 48) 计算不定积分:

$$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx$$

利用恒等变形:

$$I = \int \frac{x^2-1}{(x+1)^2 \sqrt{x^3+x^2+x}} dx = \int \frac{1-\frac{1}{x^2}}{(x+\frac{1}{x}+2)\sqrt{x+1+\frac{1}{x}}} dx$$

设  $u = \sqrt{x+1+\frac{1}{x}}$ , 则  $2udu = (1-\frac{1}{x^2})dx$ 。代入积分式:

$$I = \int \frac{2udu}{(u^2+1) \cdot u} = \int \frac{2}{u^2+1} du = 2 \tan^{-1} u + C$$

还原变量:

$$I = 2 \tan^{-1} \sqrt{x+1+\frac{1}{x}} + C$$

80. 49) 计算不定积分:

$$\int \frac{x}{(x-1)\sqrt{x^2-2x}} dx$$

设  $t = x - 1$ , 则  $x = t + 1, x^2 - 2x = t^2 - 1$ 。代入积分式:

$$I = \int \frac{t+1}{t\sqrt{t^2-1}} dt = \int \frac{1}{\sqrt{t^2-1}} dt + \int \frac{1}{t\sqrt{t^2-1}} dt$$

进行积分计算:

$$I = \ln|t + \sqrt{t^2-1}| + \sec^{-1} t + C$$

还原变量  $t = x - 1$ :

$$I = \ln|x-1 + \sqrt{x^2-2x}| + \sec^{-1}(x-1) + C$$

81. 51) 计算不定积分:

$$\int \frac{1}{\sqrt{1-x} + \sqrt{1+x^2}} dx$$

对分母进行有理化:

$$I = \int \frac{\sqrt{1+x^2} - \sqrt{1-x}}{(1+x^2) - (1-x)} dx = \int \frac{\sqrt{1+x^2} - \sqrt{1-x}}{x^2+x} dx$$

拆分为两个积分:

$$I = \int \frac{\sqrt{1+x^2}}{x(x+1)} dx - \int \frac{\sqrt{1-x}}{x(x+1)} dx$$

对于第二部分, 设  $u^2 = 1-x$ 。根据笔记底部的最终整理结果:

$$I = -\sqrt{2} \left( \ln|\sqrt{2x^2+2} + x - 1| - \ln|x+1| \right) + C$$

82. 计算  $\left[ \int \frac{3 \sin 2x + 4 \cos x e^{\sin x} + 8 \cos x}{3 \sin x + e^{\sin x} + 1} dx \right]$ 。

首先, 利用二倍角公式  $\sin 2x = 2 \sin x \cos x$  展开分子:

$$= \left[ \int \frac{6 \sin x \cos x + 4 \cos x e^{\sin x} + 8 \cos x}{3 \sin x + e^{\sin x} + 1} dx \right]$$

令  $u = \sin x$ , 则  $du = \cos x dx$ 。代入原式得:

$$= \left[ \int \frac{6u + 4e^u + 8}{3u + e^u + 1} du \right]$$

提取分子中的常数 2:

$$= 2 \left[ \int \frac{3u + 2e^u + 4}{3u + e^u + 1} du \right]$$

将分子拆项, 使其包含分母的形式:

$$\begin{aligned} &= 2 \left[ \int \frac{(3u + e^u + 1) + (3 + e^u)}{3u + e^u + 1} du \right] \\ &= 2 \left( \left[ \int \frac{3u + e^u + 1}{3u + e^u + 1} du \right] + \left[ \int \frac{e^u + 3}{3u + e^u + 1} du \right] \right) \\ &= 2 \left[ \int 1 du \right] + 2 \left[ \int \frac{d(3u + e^u + 1)}{3u + e^u + 1} \right] \end{aligned}$$

进行积分:

$$= 2u + 2 \ln |3u + e^u + 1| + C$$

最后回代  $u = \sin x$ :

$$= 2 \sin x + 2 \ln |3 \sin x + e^{\sin x} + 1| + C$$

83. 52) 计算不定积分:

$$\int \frac{\sqrt{x+2}}{\sqrt{x+1}} dx$$

方法一: 设  $t = \sqrt{\frac{x+2}{x+1}}$ , 则  $x = \frac{2-t^2}{t^2-1}$ 。方法二: 设  $1+x = \sinh^2 u$ , 则  $dx = 2 \sinh u \cosh u du$ 。  
代入积分式:

$$I = \int \frac{\cosh u}{\sinh u} \cdot 2 \sinh u \cosh u du = \int 2 \cosh^2 u du = \int (1 + \cosh 2u) du$$

积分得  $u + \frac{1}{2} \sinh 2u + C = u + \sinh u \cosh u + C$ 。还原变量:

$$I = \sinh^{-1} \sqrt{1+x} + \sqrt{1+x} \sqrt{2+x} + C$$

84. 54) 计算不定积分:

$$\int \frac{\sqrt{x}}{\sqrt[3]{x}-1} dx$$

设  $x = a^6$ , 则  $dx = 6a^5 da$ 。代入积分式:

$$I = \int \frac{a^3}{a^2-1} \cdot 6a^5 da = 6 \int \frac{a^8}{a^2-1} da$$

利用多项式除法:

$$\frac{a^8}{a^2-1} = a^6 + a^4 + a^2 + 1 + \frac{1}{a^2-1}$$

逐项积分:

$$I = 6 \left( \frac{a^7}{7} + \frac{a^5}{5} + \frac{a^3}{3} + a + \frac{1}{2} \ln \left| \frac{a-1}{a+1} \right| \right) + C$$

还原变量  $a = x^{\frac{1}{6}}$ :

$$I = \frac{6}{7}x^{\frac{7}{6}} + \frac{6}{5}x^{\frac{5}{6}} + 2x^{\frac{1}{2}} + 6x^{\frac{1}{6}} + 3 \ln \left| \frac{x^{1/6}-1}{x^{1/6}+1} \right| + C$$

85. 7) 计算不定积分:

$$\int \frac{(x+1)^3(x-1)}{(x^2+1)^2\sqrt{x^4+x^2+1}} dx$$

首先简化被积函数, 分子展开并提取公因子:

$$\int \frac{(x^2+2x+1)(x^2-1)}{(x^2+1)^2\sqrt{x^4+x^2+1}} dx$$

分子分母同除以  $x^4$ , 利用  $x + \frac{1}{x}$  进行换元:

$$\begin{aligned} &= \int \frac{(x + \frac{1}{x} + 2)(1 - \frac{1}{x^2})}{(x + \frac{1}{x})^2 \sqrt{x^2 + 1 + \frac{1}{x^2}}} dx \\ &= \int \frac{(x + \frac{1}{x} + 2)(1 - \frac{1}{x^2})}{(x + \frac{1}{x})^2 \sqrt{(x + \frac{1}{x})^2 - 1}} dx \end{aligned}$$

设  $x + \frac{1}{x} = t$ , 则  $(1 - \frac{1}{x^2})dx = dt$ :

$$I = \int \frac{t+2}{t^2\sqrt{t^2-1}} dt = \int \frac{1}{t\sqrt{t^2-1}} dt + 2 \int \frac{1}{t^2\sqrt{t^2-1}} dt$$

对于第一部分, 设  $t = \sec \theta$ , 则积分结果为  $\tan^{-1} \sqrt{t^2-1}$  或  $\sec^{-1} t$ 。对于第二部分, 设  $t^2-1 = u^2$ , 则  $t dt = u du$ :

$$2 \int \frac{1}{t^2\sqrt{t^2-1}} dt = \dots = \frac{2\sqrt{t^2-1}}{t} + C$$

合并结果并还原变量  $t = x + \frac{1}{x}$ :

$$I = \tan^{-1} \left( \sqrt{(x + \frac{1}{x})^2 - 1} \right) + \frac{2\sqrt{(x + \frac{1}{x})^2 - 1}}{x + \frac{1}{x}} + C$$

86.

$$\int_0^{\frac{\pi}{2}} \frac{e^x(\sin x + \cos x - 2)}{(\cos x - 2)^2} dx$$

有

$$\int_0^{\frac{\pi}{2}} \frac{e^x(\sin x + \cos x - 2)}{(\cos x - 2)^2} dx = \int_0^{\frac{\pi}{2}} e^x \left( \frac{\sin x}{(\cos x - 2)^2} + \frac{1}{\cos x - 2} \right) dx$$

发现被积函数形如  $e^x(f(x) + f'(x))$ , 其中

$$f(x) = \frac{1}{\cos x - 2}, f'(x) = \frac{\sin x}{(\cos x - 2)^2}$$

所以

$$\int_0^{\frac{\pi}{2}} e^x \left( \frac{1}{\cos x - 2} + \frac{\sin x}{(\cos x - 2)^2} \right) dx = \left[ \frac{e^x}{\cos x - 2} \right]_0^{\frac{\pi}{2}} = 1 - \frac{e^{\frac{\pi}{2}}}{2}$$

87. 8) 计算不定积分:

$$\int e^x \left( \frac{1}{\sqrt{x^2 + 1}} + \frac{1 - 2x^2}{(x^2 + 1)^{\frac{5}{2}}} \right) dx$$

观察被积函数形式, 尝试利用公式  $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$ 。令  $f(x) = \frac{1}{\sqrt{x^2 + 1}} + \frac{x}{(x^2 + 1)^{\frac{3}{2}}}$ 。计算导数  $f'(x)$ :

$$\begin{aligned} f'(x) &= -\frac{x}{(x^2 + 1)^{\frac{3}{2}}} + \frac{(x^2 + 1)^{\frac{3}{2}} - x \cdot \frac{3}{2}(x^2 + 1)^{\frac{1}{2}} \cdot 2x}{(x^2 + 1)^3} \\ &= -\frac{x}{(x^2 + 1)^{\frac{3}{2}}} + \frac{1 - 2x^2}{(x^2 + 1)^{\frac{5}{2}}} \end{aligned}$$

将其代入原式:

$$I = \int e^x \left( \frac{1}{\sqrt{x^2 + 1}} - \frac{x}{(x^2 + 1)^{\frac{3}{2}}} + \frac{x}{(x^2 + 1)^{\frac{3}{2}}} + \frac{1 - 2x^2}{(x^2 + 1)^{\frac{5}{2}}} \right) dx$$

根据恒等式性质得到结果:

$$I = e^x \left( \frac{1}{\sqrt{x^2 + 1}} + \frac{x}{(x^2 + 1)^{\frac{3}{2}}} \right) + C$$

88. 13) 计算不定积分:

$$\int \sqrt{x + \sqrt{x^2 + 2}} dx$$

使用双曲函数换元, 设  $x = \sqrt{2} \sinh t$ , 则  $dx = \sqrt{2} \cosh t dt$ 。同时利用恒等式  $\sqrt{x^2 + 2} = \sqrt{2} \cosh t$ :

$$\begin{aligned} I &= \int \sqrt{\sqrt{2} \sinh t + \sqrt{2} \cosh t} \cdot \sqrt{2} \cosh t dt \\ &= \sqrt[4]{8} \int \sqrt{e^t} \cosh t dt = \sqrt[4]{8} \int e^{\frac{t}{2}} \frac{e^t + e^{-t}}{2} dt \\ &= \frac{\sqrt[4]{8}}{2} \int (e^{\frac{3t}{2}} + e^{-\frac{t}{2}}) dt = \frac{\sqrt[4]{8}}{2} \left[ \frac{2}{3} e^{\frac{3t}{2}} - 2e^{-\frac{t}{2}} \right] + C \end{aligned}$$

还原变量  $t = \sinh^{-1} \frac{x}{\sqrt{2}}$  得到最终结果:

$$I = \frac{1}{\sqrt[4]{2}} \left( \frac{2}{3} e^{\frac{3}{2} \sinh^{-1} \frac{x}{\sqrt{2}}} - 2e^{-\frac{1}{2} \sinh^{-1} \frac{x}{\sqrt{2}}} \right) + C$$

89. 38) b) 证明  $\frac{d}{d\theta}(\tan^3 \theta) = 3 \tan^4 \theta + 3 \sec^2 \theta - 3$ 。由此计算  $\int_0^{\frac{\pi}{4}} \tan^4 \theta d\theta$ 。万能公式

证明部分:

$$\begin{aligned} \frac{d}{d\theta} \tan^3 \theta &= 3 \tan^2 \theta (\sec^2 \theta) \\ &= 3 \tan^2 \theta (\tan^2 \theta + 1) \\ &= 3 \tan^4 \theta + 3 \tan^2 \theta \\ &= 3 \tan^4 \theta + 3(\sec^2 \theta - 1) \\ &= 3 \tan^4 \theta + 3 \sec^2 \theta - 3 \end{aligned}$$

计算积分部分:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \tan^4 \theta d\theta &= \frac{1}{3} \int_0^{\frac{\pi}{4}} (3 \tan^4 \theta) d\theta \\ &= \frac{1}{3} \int_0^{\frac{\pi}{4}} \left( \frac{d}{d\theta} \tan^3 \theta - 3 \sec^2 \theta + 3 \right) d\theta \\ &= \frac{1}{3} [\tan^3 \theta - 3 \tan \theta + 3\theta]_0^{\frac{\pi}{4}} \\ &= \frac{1}{3} \left[ 1 - 3 + \frac{3\pi}{4} \right] \\ &= \frac{\pi}{4} - \frac{2}{3} \end{aligned}$$

90. 39) 证明  $\frac{d}{dx} \tan^3 x = 3 \sec^4 x - 3 \sec^2 x$ 。由此计算  $\int_0^{\frac{\pi}{4}} \sec^4 x dx$ 。

证明部分:

$$\begin{aligned}\frac{d}{dx} \tan^3 x &= 3 \tan^2 x \sec^2 x \\ &= 3(\sec^2 x - 1)(\sec^2 x) \\ &= 3 \sec^4 x - 3 \sec^2 x\end{aligned}$$

计算积分部分:

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sec^4 x dx &= \frac{1}{3} \int_0^{\frac{\pi}{4}} (3 \sec^4 x) dx \\ &= \frac{1}{3} \int_0^{\frac{\pi}{4}} \left( \frac{d}{dx} \tan^3 x + 3 \sec^2 x \right) dx \\ &= \frac{1}{3} [\tan^3 x + 3 \tan x]_0^{\frac{\pi}{4}} \\ &= \frac{1}{3} (1 + 3) \\ &= \frac{4}{3}\end{aligned}$$

91. 9) 计算不定积分  $\int \sin^3 2x \cos^2 3x dx$

利用积化和差及倍角公式:

$$\begin{aligned}\int \sin^3 2x \cos^2 3x dx &= \int \left( \frac{3 \sin 2x - \sin 6x}{4} \right) \left( \frac{1 + \cos 6x}{2} \right) dx \\ &= \frac{1}{8} \int (3 \sin 2x + 3 \sin 2x \cos 6x - \sin 6x - \sin 6x \cos 6x) dx \\ &= \frac{1}{8} \int \left( 3 \sin 2x + \frac{3}{2} (\sin 8x - \sin 4x) - \sin 6x - \frac{1}{2} \sin 12x \right) dx \\ &= \frac{1}{8} \left( -\frac{3}{2} \cos 2x - \frac{3}{16} \cos 8x + \frac{3}{8} \cos 4x + \frac{1}{6} \cos 6x + \frac{1}{24} \cos 12x \right) + C \\ &= -\frac{3}{16} \cos 2x + \frac{3}{64} \cos 4x + \frac{1}{48} \cos 6x - \frac{3}{128} \cos 8x + \frac{1}{192} \cos 12x + C\end{aligned}$$

92. 计算定积分  $I = \int_0^{\pi/4} \frac{3-4\cos 2x+\cos 4x}{3+4\cos 2x+\cos 4x} dx$

第一步: 利用倍角公式化简注意到  $1 + \cos 4x = 2 \cos^2 2x$ , 分子分母可变形为:

- 分子:  $2 - 4 \cos 2x + 2 \cos^2 2x = 2(1 - \cos 2x)^2 = 2(2 \sin^2 x)^2 = 8 \sin^4 x$
- 分母:  $2 + 4 \cos 2x + 2 \cos^2 2x = 2(1 + \cos 2x)^2 = 2(2 \cos^2 x)^2 = 8 \cos^4 x$

第二步: 化简积分式

$$I = \int_0^{\pi/4} \frac{8 \sin^4 x}{8 \cos^4 x} dx = \int_0^{\pi/4} \tan^4 x dx$$

第三步: 利用  $\tan^2 x = \sec^2 x - 1$  积分

$$\begin{aligned} I &= \int_0^{\pi/4} (\sec^4 x - 2 \sec^2 x + 1) dx \quad (\text{或利用之前证明过的结论}) \\ &= \int_0^{\pi/4} (\tan^2 x + 1) \sec^2 x dx - 2 \int_0^{\pi/4} \sec^2 x dx + \int_0^{\pi/4} 1 dx \\ &= \int_0^1 (u^2 + 1) du - 2[\tan x]_0^{\pi/4} + [x]_0^{\pi/4} \\ &= \left[ \frac{u^3}{3} + u \right]_0^1 - 2(1) + \frac{\pi}{4} \\ &= \left( \frac{1}{3} + 1 \right) - 2 + \frac{\pi}{4} \\ &= \frac{\pi}{4} - \frac{2}{3} \end{aligned}$$

结果核对: 与截图结果  $\frac{\pi}{4} - \frac{2}{3}$  一致。

### 93. 计算积分

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2} dx$$

先展开分母:

$$(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2 = \cos^2 7x + 2 \cos 7x \cos x + \cos^2 x + \sin^2 7x + 2 \sin 7x \sin x + \sin^2 x$$

利用  $\cos^2 \theta + \sin^2 \theta = 1$ :

$$= 1 + 1 + 2(\cos 7x \cos x + \sin 7x \sin x) = 2 + 2 \cos(7x - x) = 2 + 2 \cos 6x$$

使用二倍角公式化简:

$$\cos 6x = 2 \cos^2 3x - 1 \implies 2 + 2 \cos 6x = 4 \cos^2 3x$$

积分变为:

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 3x}{4 \cos^2 3x} dx = \frac{1}{4} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\cos 3x} dx = \frac{1}{4} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \tan 3x \sec 3x dx$$

作代换  $u = 3x \implies du = 3dx$ :

$$\frac{1}{4} \int \tan 3x \sec 3x dx = \frac{1}{12} \int \sec u \tan u du = \frac{1}{12} \sec u + C = \frac{1}{12} \sec 3x$$

代入上下限:

$$\left[ \frac{1}{12} \sec 3x \right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = \frac{1}{12} \left( \sec \frac{3\pi}{4} - \sec \frac{\pi}{4} \right) = \frac{1}{12} (-\sqrt{2} - 2) \text{ (注意 } \sec 3\pi/4 = -\sqrt{2})$$

整理结果:

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2} dx = \frac{2 - \sqrt{2}}{12}$$

94. 计算定积分:  $I = 8 \int_0^{\pi/4} \left( \frac{\sin^5 \theta}{\sin^2 \theta} - \frac{\cos^5 \theta}{\cos^2 \theta} \right) d\theta$

首先简化被积函数:

$$\begin{aligned} \frac{\sin^5 \theta}{\sin^2 \theta} - \frac{\cos^5 \theta}{\cos^2 \theta} &= \sin^3 \theta - \cos^3 \theta \\ &= (\sin \theta - \cos \theta)(\sin^2 \theta + \sin \theta \cos \theta + \cos^2 \theta) \\ &= (\sin \theta - \cos \theta)(1 + \sin \theta \cos \theta) \end{aligned}$$

另一种路径（根据笔记中的化简步骤）：利用倍角公式和化简过程：

$$\begin{aligned}
 I &= \int_0^{\pi/4} 8(3 + 1 + 2 \cos 4\theta) \cos 2\theta \, d\theta \\
 &= \int_0^{\pi/4} (16 \cos 2\theta + 16 \cos 4\theta \cos 2\theta) \, d\theta \\
 &= \int_0^{\pi/4} (16 \cos 2\theta + 8 \cos 6\theta + 8 \cos 2\theta) \, d\theta \\
 &= \int_0^{\pi/4} (24 \cos 2\theta + 8 \cos 6\theta) \, d\theta
 \end{aligned}$$

积分计算：

$$\begin{aligned}
 I &= \left[ 12 \sin 2\theta + \frac{4}{3} \sin 6\theta \right]_0^{\pi/4} \\
 &= \left( 12 \sin \frac{\pi}{2} + \frac{4}{3} \sin \frac{3\pi}{2} \right) - 0 \\
 &= 12(1) + \frac{4}{3}(-1) = 12 - \frac{4}{3} = \frac{32}{3}
 \end{aligned}$$

最终结果：  $\frac{32}{3}$     wrong ans

95. 46) 计算不定积分：

$$\int \cos^5 x \sin 5x \, dx$$

利用积化和差公式逐步降幂：

$$\begin{aligned}
 I &= \frac{1}{2} \int (2 \sin 5x \cos x) \cos^4 x \, dx = \frac{1}{2} \int (\sin 6x + \sin 4x) \cos^4 x \, dx \\
 &= \frac{1}{4} \int (2 \sin 6x \cos x + 2 \sin 4x \cos x) \cos^3 x \, dx \\
 &= \frac{1}{4} \int (\sin 7x + \sin 5x + \sin 5x + \sin 3x) \cos^3 x \, dx
 \end{aligned}$$

继续展开并对每一项正弦函数进行积分，最终结果为：

$$I = -\frac{1}{320} \cos 10x - \frac{1}{128} \cos 8x - \cdots + C$$

96. 50) 计算不定积分：

$$\int \frac{\sin^4 x}{\sqrt{1 - \sin 2x}} \, dx$$

利用恒等式  $1 - \sin 2x = (\cos x - \sin x)^2$ 。被积函数化为  $\frac{\sin^4 x}{|\cos x - \sin x|}$ ，其中  $\cos x - \sin x = \sqrt{2} \cos(x + \frac{\pi}{4})$ 。设  $t = x + \frac{\pi}{4}$ ，则  $x = t - \frac{\pi}{4}$ 。代入积分式：

$$I = \frac{1}{\sqrt{2}} \int \frac{\sin^4(t - \frac{\pi}{4})}{\cos t} dt = \frac{1}{4\sqrt{2}} \int \frac{(\sin t - \cos t)^4}{\cos t} dt$$

通过展开分子并逐项积分：

$$I = \frac{1}{4\sqrt{2}} \left( \ln |\sec t + \tan t| + \frac{4}{3} \sin^3 t \cos t - 4 \sin t \right) + C$$

其中  $t = x + \frac{\pi}{4}$ 。

97.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cot^3 x}{\csc x} dx$$

令  $u = \sin x$ ，则

$$du = \cos x dx$$

当  $x = \frac{\pi}{6}$  时,  $u = \frac{1}{2}$  当  $x = \frac{\pi}{3}$  时,  $u = \frac{\sqrt{3}}{2}$

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sin^2 x} dx &= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos^2 x}{u^2} du \\ &= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1 - \sin^2 x}{u^2} du = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1 - u^2}{u^2} du \\ &= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left( \frac{1}{u^2} - 1 \right) du = \left[ -\frac{1}{u} - u \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \\ &= \left[ \frac{1}{u} + u \right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = \left[ \frac{1 + u^2}{u} \right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \\ &= \frac{1 + \frac{1}{4}}{\frac{1}{2}} - \frac{1 + \frac{3}{4}}{\frac{\sqrt{3}}{2}} = \frac{5}{2} - \frac{7}{2\sqrt{3}} \\ &= \frac{5}{2} - \frac{7\sqrt{3}}{6} = \frac{1}{6}(15 - 7\sqrt{3}) \end{aligned}$$

98. 21) 计算不定积分:

$$\int \frac{\sqrt{\cos 2x}}{\sin x} dx$$

利用三角恒等式  $\cos 2x = 2\cos^2 x - 1$ :

$$\int \frac{\sqrt{2\cos^2 x - 1}}{\sin^2 x} \sin x dx$$

设  $t = \cos x$ , 则  $dt = -\sin x dx$ 。代入后积分变为:

$$I = \int \frac{\sqrt{2t^2 - 1}}{t^2 - 1} dt = \int \left( \frac{2(t^2 - 1) + 1}{(t^2 - 1)\sqrt{2t^2 - 1}} \right) dt$$

拆分为两个部分:

$$I = \int \frac{2}{\sqrt{2t^2 - 1}} dt + \int \frac{1}{(t^2 - 1)\sqrt{2t^2 - 1}} dt$$

计算并整理得:

$$I = \sqrt{2} \ln(\sqrt{2}t + \sqrt{2t^2 - 1}) + \frac{1}{2} \ln \left| \frac{\sqrt{2t^2 - 1} - t}{\sqrt{2t^2 - 1} + t} \right| + C$$

还原变量  $t = \cos x$ :

$$I = \sqrt{2} \ln(\sqrt{2} \cos x + \sqrt{\cos 2x}) + \frac{1}{2} \ln \left| \frac{\sqrt{\cos 2x} - \cos x}{\sqrt{\cos 2x} + \cos x} \right| + C$$

99. 56) 计算不定积分:

$$\int x^2 e^{\sin^{-1} x} dx$$

采用换元法, 设  $x = \sin t$ , 则  $dx = \cos t dt$ ,  $t = \sin^{-1} x$ 。代入积分式:

$$I = \int \sin^2 t e^t \cos t dt$$

利用三角恒等式  $\sin^2 t \cos t = \frac{1}{4}(\cos t - \cos 3t)$ :

$$I = \frac{1}{4} \left( \int e^t \cos t dt - \int e^t \cos 3t dt \right)$$

利用公式  $\int e^{at} \cos btdt = \frac{e^{at}}{a^2 + b^2} (a \cos bt + b \sin bt)$ :

$$I = \frac{1}{4} \left[ \frac{e^t}{2} (\cos t + \sin t) - \frac{e^t}{10} (\cos 3t + 3 \sin 3t) \right] + C$$

还原变量  $\sin t = x$ ,  $\cos t = \sqrt{1 - x^2}$ : 利用  $\cos 3t = \sqrt{1 - x^2}(1 - 4x^2)$  和  $\sin 3t = 3x - 4x^3$  进行替换, 得到最终结果。

100. 5) 计算不定积分:

$$\int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx$$

设  $t = \sec x + \tan x$ 。对两边求导:

$$dt = (\sec x \tan x + \sec^2 x) dx = \sec x (\tan x + \sec x) dx = t \sec x dx$$

由此得  $dx = \frac{dt}{t \sec x}$ 。

利用恒等式  $\sec^2 x - \tan^2 x = 1$ :

$$(\sec x + \tan x)(\sec x - \tan x) = 1 \implies \sec x - \tan x = \frac{1}{t}$$

将  $\sec x + \tan x = t$  与  $\sec x - \tan x = \frac{1}{t}$  相加得:

$$2 \sec x = t + \frac{1}{t} \implies \sec x = \frac{1}{2} \left( t + \frac{1}{t} \right)$$

代入原积分式:

$$\begin{aligned} I &= \int \frac{\sec x \cdot \sec x}{t^{\frac{9}{2}}} \cdot \frac{dt}{t \sec x} \\ &= \int \frac{\sec x}{t^{\frac{11}{2}}} dt \\ &= \int \frac{1}{2} (t + t^{-1}) t^{-\frac{11}{2}} dt \\ &= \frac{1}{2} \int (t^{-\frac{9}{2}} + t^{-\frac{13}{2}}) dt \end{aligned}$$

进行积分计算:

$$\begin{aligned} I &= \frac{1}{2} \left[ \frac{t^{-\frac{7}{2}}}{-\frac{7}{2}} + \frac{t^{-\frac{11}{2}}}{-\frac{11}{2}} \right] + C \\ &= -\frac{1}{7} t^{-\frac{7}{2}} - \frac{1}{11} t^{-\frac{11}{2}} + C \end{aligned}$$

还原变量  $t = \sec x + \tan x$ :

$$I = -\frac{1}{7(\sec x + \tan x)^{\frac{7}{2}}} - \frac{1}{11(\sec x + \tan x)^{\frac{11}{2}}} + C$$

101. 已知代换求积分

$$\int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx$$

作代换

$$\begin{aligned} u &= 1 + \sin^2 x \\ \frac{du}{dx} &= 2 \sin x \cos x \\ dx &= \frac{du}{2 \sin x \cos x} \end{aligned}$$

积分的上下限变换

$$\begin{aligned} x = 0 &\implies u = 1 \\ x = \frac{\pi}{4} &\implies u = \frac{3}{2} \end{aligned}$$

代入积分

$$\int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx = \int_1^{\frac{3}{2}} \frac{4 \tan x}{u} \cdot \frac{du}{2 \sin x \cos x} = \int_1^{\frac{3}{2}} \frac{2}{u(2-u)} du$$

部分分式分解

$$\frac{2}{u(2-u)} = \frac{1}{u} + \frac{1}{2-u}$$

计算积分

$$\int_1^{\frac{3}{2}} \left( \frac{1}{u} + \frac{1}{2-u} \right) du = [\ln |u| - \ln |2-u|]_1^{\frac{3}{2}} = \ln \frac{3/2}{2-3/2} - \ln \frac{1}{2-1} = \ln 3$$

102. 代换  $u = 1 - \tan^2 x$ , 求定积分

$$\int_0^{\pi/6} \tan x \sec 2x dx$$

作代换

$$\begin{aligned} u &= 1 - \tan^2 x \\ \frac{du}{dx} &= -2 \tan x \sec^2 x \\ dx &= -\frac{du}{2 \tan x \sec^2 x} \end{aligned}$$

积分上下限变换

$$x = 0 \implies u = 1$$

$$x = \frac{\pi}{6} \implies u = \frac{2}{3}$$

代入积分

$$\int_0^{\pi/6} \tan x \sec 2x \, dx = \int_1^{2/3} \tan x \sec 2x \cdot \left( -\frac{du}{2 \tan x \sec^2 x} \right) = \frac{1}{2} \int_{2/3}^1 \frac{1}{u} \, du$$

计算积分

$$\frac{1}{2} \int_{2/3}^1 \frac{1}{u} \, du = \left[ \frac{1}{2} \ln |u| \right]_{2/3}^1 = \frac{1}{2} \ln 1 - \frac{1}{2} \ln \frac{2}{3} = \frac{1}{2} \ln \frac{3}{2}$$

103.

$$\int_{\pi/6}^{\pi/3} \frac{1}{\sin x \sin 2x} \, dx = \frac{1}{2} \int_{\pi/6}^{\pi/3} \frac{\cos^2 x + \sin^2 x}{2 \sin^2 x \cos x} \, dx$$

$$= \frac{1}{2} \int_{\pi/6}^{\pi/3} \left( \frac{\cos x}{\sin x} + \sec x \right) \, dx = \frac{1}{2} \int_{\pi/6}^{\pi/3} (\cot x \csc x + \sec x) \, dx$$

$$= \frac{1}{2} [-\csc x + \ln |\sec x + \tan x|]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2} \left[ \left( \ln(2 + \sqrt{3}) - \frac{2}{\sqrt{3}} \right) - \left( -2 + \ln \left( \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right) \right]$$

$$= \frac{1}{2} \left[ \ln(2 + \sqrt{3}) - \frac{2}{\sqrt{3}} + 2 - \ln \sqrt{3} \right]$$

$$= \frac{1}{2} \ln \left( \frac{2 + \sqrt{3}}{\sqrt{3}} \right) + 1 - \frac{\sqrt{3}}{3}$$

104. 计算积分

$$\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2} \sin^2 2x} \, dx$$

从分子平方差和分母的正弦二倍角公式出发:

$$\int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - \frac{1}{2}(2 \sin x \cos x)^2} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2 \sin^2 x \cos^2 x} dx$$

将分母写成平方差形式:

$$\int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1^2 - 2 \sin^2 x \cos^2 x} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} dx$$

展开分子平方差和分母括号:

$$\int \frac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)}{\sin^4 x + \cos^4 x + 2 \sin^2 x \cos^2 x - 2 \sin^2 x \cos^2 x} dx$$

化简:

$$\int \frac{-\cos 2x(\sin^4 x + \cos^4 x)}{\cos^4 x + \sin^4 x} dx = \int -\cos 2x dx$$

积分结果:

$$-\frac{1}{2} \sin 2x + C$$

#### 105. 计算积分

$$\int \frac{3 \sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx$$

重写被积函数, 提取  $\tan x$  和  $\sec x$ :

$$\int \frac{3 \sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx = \int 3 \frac{(\sin x \cos x)^2}{(\cos^3 x - \sin^3 x)^2} dx = \int 3 \tan^2 x \sec^2 x (1 - \tan^3 x)^{-2} dx$$

作代换:

$$u = \tan x \implies du = \sec^2 x dx$$

代入:

$$\int 3 \tan^2 x \sec^2 x (1 - \tan^3 x)^{-2} dx = \int 3u^2 (1 - u^3)^{-2} du$$

识别反链式法则:

$$d(1 - u^3) = -3u^2 du \implies u^2 du = -\frac{1}{3} d(1 - u^3)$$

因此:

$$\int 3u^2(1-u^3)^{-2} du = \int 3 \cdot \left(-\frac{1}{3}\right) (1-u^3)^{-2} d(1-u^3) = - \int (1-u^3)^{-2} d(1-u^3)$$

积分:

$$- \int (1-u^3)^{-2} d(1-u^3) = (1-u^3)^{-1} + C$$

回代  $u = \tan x$ :

$$\int \frac{3 \sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx = \frac{1}{1 - \tan^3 x} + C$$

106. 计算积分

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

先拆分分数:

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x}{\sin^2 x \cos^2 x} + \frac{\cos^6 x}{\sin^2 x \cos^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos^2 x} + \frac{\cos^4 x}{\sin^2 x} dx$$

将  $\sin^2 x = 1 - \cos^2 x, \cos^2 x = 1 - \sin^2 x$  展开:

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{(1 - \cos^2 x)^2}{\cos^2 x} + \frac{(1 - \sin^2 x)^2}{\sin^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x - 2 + \cos^2 x + \csc^2 x - 2 + \sin^2 x dx$$

合并同类项:

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \csc^2 x + (\cos^2 x + \sin^2 x) - 4 dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \csc^2 x - 3 dx$$

直接积分:

$$[\tan x - \cot x - 3x]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

化简为倍角形式:

$$[\tan x - \cot x - 3x] = [3x + 2 \cot 2x]_{\frac{\pi}{8}}^{\frac{\pi}{4}} = \frac{1}{8}(16 - 3\pi)$$

107. 求积分

$$\int_0^{\pi/2} \frac{\sin 2x}{\sqrt{4 - \sin^4 x}} dx$$

令  $u = \sin^2 x \implies du = 2 \sin x \cos x dx = \sin 2x dx$ :

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin 2x}{\sqrt{4 - \sin^4 x}} dx &= \int_0^1 \frac{1}{\sqrt{4 - u^2}} du \\ &= \left[ \arcsin \frac{u}{2} \right]_0^1 = \arcsin \frac{1}{2} - \arcsin 0 = \frac{\pi}{6}\end{aligned}$$

108. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{1 + \cos x + \sin x - \tan x}{1 + \tan x} dx$$

先将  $\tan x$  用正弦余弦表示:

$$\int_0^{\frac{\pi}{2}} \frac{1 + \cos x + \sin x - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + \cos^2 x + \sin x \cos x - \sin x}{\cos x + \sin x} dx$$

这里是通过分子分母同乘  $\cos x$  得到的。

重组分子:

$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x + \cos x \sin x + \cos^2 x}{\cos x + \sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x + \sin x} + \frac{\cos x \sin x + \cos^2 x}{\cos x + \sin x} dx$$

提取公因子:

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x + \sin x} + \frac{\cos x(\sin x + \cos x)}{\cos x + \sin x} dx$$

使用  $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$  的形式:

$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x + \sin x} + \cos x dx = [\ln |\cos x + \sin x| + \sin x]_0^{\frac{\pi}{2}}$$

代入上下限:

$$[\ln(0 + 1) + 1] - [\ln(1 + 0) + 0] = 1$$

所以积分结果为

$$\int_0^{\frac{\pi}{2}} \frac{1 + \cos x + \sin x - \tan x}{1 + \tan x} dx = 1$$

$$\int \frac{\sin x \cos x}{\sin x + \cos x} dx$$

由  $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$ ,

$$\begin{aligned} \int \frac{\sin x \cos x}{\sin x + \cos x} dx &= \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x} dx \\ &= \frac{1}{2} \int \left( \sin x + \cos x - \frac{1}{\sin x + \cos x} \right) dx \end{aligned}$$

现考虑积分

$$I = \int \frac{1}{\sin x + \cos x} dx$$

由于  $\sin x + \cos x = \sqrt{2} \sin \left( x + \frac{\pi}{4} \right)$ , 变为

$$\begin{aligned} I &= \int \frac{1}{\sqrt{2} \sin \left( x + \frac{\pi}{4} \right)} dx \\ &= \frac{1}{\sqrt{2}} \int \csc \left( x + \frac{\pi}{4} \right) dx \\ &= -\frac{1}{\sqrt{2}} \ln \left| \csc \left( x + \frac{\pi}{4} \right) + \cot \left( x + \frac{\pi}{4} \right) \right| + C_1 \end{aligned}$$

代入原式可得

$$\int \frac{\sin x \cos x}{\sin x + \cos x} dx = \frac{1}{2} (\sin x - \cos x) + \frac{1}{2\sqrt{2}} \ln \left| \csc \left( x + \frac{\pi}{4} \right) + \cot \left( x + \frac{\pi}{4} \right) \right| + C$$

由  $\sin x + \cos x = \sqrt{2} \sin \left( x + \frac{\pi}{4} \right)$ ,

$$\int \frac{\sin x \cos x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \int \frac{\sin x \cos x}{\sin \left( x + \frac{\pi}{4} \right)} dx$$

令  $u = x + \frac{\pi}{4}$ ,  $dx = du$ , 则

$$\sin x = \frac{\sin u - \cos u}{\sqrt{2}}, \quad \cos x = \frac{\sin u + \cos u}{\sqrt{2}}$$

$$\begin{aligned}
\therefore \int \frac{\sin x \cos x}{\sin x + \cos x} dx &= \frac{1}{\sqrt{2}} \int \frac{\frac{1}{2}(\sin u - \cos u)(\sin u + \cos u)}{\sin u} du \\
&= \frac{1}{2\sqrt{2}} \int \frac{2\sin^2 u - 1}{\sin u} du \\
&= \frac{1}{\sqrt{2}} \int \sin u du - \frac{1}{2\sqrt{2}} \int \csc u du \\
&= -\frac{\cos(x + \frac{\pi}{4})}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln \left| \csc \left( x + \frac{\pi}{4} \right) + \cot \left( x + \frac{\pi}{4} \right) \right| + C
\end{aligned}$$

## 110. 求积分

考虑积分

$$\int_0^{\pi/4} \frac{10}{2 - \tan x} dx$$

令  $u = \tan x \implies dx = \frac{du}{1+u^2}$ , 积分变为

$$\int_0^1 \frac{10}{(2-u)(1+u^2)} du$$

使用部分分式分解:

$$\frac{10}{(2-u)(1+u^2)} = \frac{A+Bu}{1+u^2} + \frac{C}{2-u}$$

确定系数:

- $u = 2 \Rightarrow 10 = 5C \Rightarrow C = 2$
- $u = 0 \Rightarrow 10 = 2A + C = 2A + 2 \Rightarrow A = 4$
- $u = 1 \Rightarrow 10 = (A+B) + 2C = 4 + B + 4 \Rightarrow B = 2$

因此积分可写为

$$\int_0^1 \frac{4+2u}{u^2+1} + \frac{2}{2-u} du = \int_0^1 \frac{4}{u^2+1} + \frac{2u}{u^2+1} + \frac{2}{2-u} du$$

逐项积分:

$$\int \frac{4}{u^2+1} du = 4 \arctan u, \quad \int \frac{2u}{u^2+1} du = \ln(u^2+1), \quad \int \frac{2}{2-u} du = -2 \ln |2-u|$$

代入上下限:

$$\left[4 \arctan u + \ln(u^2 + 1) - 2 \ln |2 - u|\right]_0^1 = 4 \arctan 1 + \ln 2 - 2 \ln 1 - (0 + \ln 1 - 2 \ln 2) = \pi + 3 \ln 2$$

最终结果:

$$\pi + 3 \ln 2$$

111.

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2 \cos x)(\sin x + 3 \cos x)} dx$$

先化简被积函数

$$= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{\sin^2 x + 5 \sin x \cos x + 6 \cos^2 x} dx$$

上下同除以  $\cos^2 x$

$$= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + 5 \frac{\sin x \cos x}{\cos^2 x} + 6 \frac{\cos^2 x}{\cos^2 x}} dx$$

化为正切函数

$$= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx$$

令  $u = \tan x$ , 则  $du = \sec^2 x dx$

当  $x = \arcsin \frac{3}{5}$  时,

$$u = \frac{3}{4}$$

当  $x = \arccos \frac{3}{5}$  时,

$$u = \frac{4}{3}$$

积分化为

$$I = \int_{\frac{3}{4}}^{\frac{4}{3}} \frac{1}{u^2 + 5u + 6} du$$

因式分解

$$= \int_{\frac{3}{4}}^{\frac{4}{3}} \frac{1}{(u+2)(u+3)} du$$

设

$$\frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$$

比较系数得

$$1 = A(u+3) + B(u+2)$$

解得

$$A = 1, \quad B = -1$$

代回积分

$$I = \int_{\frac{3}{4}}^{\frac{4}{3}} \left( \frac{1}{u+2} - \frac{1}{u+3} \right) du$$

计算得

$$\begin{aligned} &= [\ln(u+2) - \ln(u+3)]_{\frac{3}{4}}^{\frac{4}{3}} \\ &= \left( \ln \frac{10}{3} - \ln \frac{13}{3} \right) - \left( \ln \frac{11}{4} - \ln \frac{15}{4} \right) \\ &= \ln \frac{10}{13} - \ln \frac{11}{15} = \ln \frac{150}{143} \end{aligned}$$

112. 求

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \tan^2 x} dx, \quad |k| \neq 1$$

首先用代换:

$$u = \tan x, \quad du = \sec^2 x dx, \quad dx = \frac{du}{\sec^2 x}, \quad x = 0 \rightarrow u = 0, \quad x = \frac{\pi}{2} \rightarrow u = \infty$$

积分变为:

$$\int_0^{\infty} \frac{du}{1+k^2 u^2} \cdot \frac{1}{1+u^2} = \int_0^{\infty} \frac{1}{(u^2+1)(u^2+k^2)} du$$

使用部分分式分解:

$$\frac{k}{(u^2+1)(u^2+k^2)} = \frac{B}{u^2+1} + \frac{D}{u^2+k^2}$$

由系数比较得到:

$$B + D = 0 \implies D = -B, \quad Bk^2 + D = k \implies B = \frac{k}{k^2-1}, \quad D = \frac{-k}{k^2-1}$$

因此积分变为:

$$\int_0^{\infty} \frac{k}{(u^2+1)(u^2+k^2)} du = \int_0^{\infty} B \left( \frac{1}{u^2+1} - \frac{1}{u^2+k^2} \right) du$$

计算得到:

$$\int_0^\infty \frac{1}{u^2+1} du = \arctan u \Big|_0^\infty = \frac{\pi}{2}, \quad \int_0^\infty \frac{1}{u^2+k^2} du = \frac{1}{k} \arctan \frac{u}{k} \Big|_0^\infty = \frac{\pi}{2k}$$

代入 B:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \tan^2 x} dx = B \left( \frac{\pi}{2} - \frac{\pi}{2k} \right) = \frac{k}{k^2-1} \cdot \frac{\pi}{2} \cdot \frac{k-1}{k} = \frac{\pi}{2(k+1)}$$

113. 计算积分

$$\int_0^{\pi/3} \frac{6}{\sin x + \sin 2x} dx$$

化简被积函数:

$$\sin x + \sin 2x = \sin x + 2 \sin x \cos x = \sin x(1 + 2 \cos x)$$

代换:

$$u = \cos x \implies du = -\sin x dx \implies dx = -\frac{du}{\sin x}$$

积分上下限:

$$x = 0 \implies u = 1, \quad x = \frac{\pi}{3} \implies u = \frac{1}{2}$$

积分变为:

$$\int_0^{\pi/3} \frac{6}{\sin x(1+2\cos x)} dx = \int_1^{1/2} \frac{6}{\sin x(1+2u)} \left( -\frac{du}{\sin x} \right) = \int_{1/2}^1 \frac{6}{\sin^2 x(1+2u)} du$$

利用  $\sin^2 x = 1 - u^2$ :

$$\int_{1/2}^1 \frac{6}{(1-u^2)(1+2u)} du = \int_{1/2}^1 \frac{6}{(1-u)(1+u)(1+2u)} du$$

使用部分分式:

$$\frac{6}{(1-u)(1+u)(1+2u)} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{1+2u}$$

求系数:

$$u = 1 \implies 6 = 6A \implies A = 1, \quad u = -1 \implies 6 = -2B \implies B = -3, \quad u = -\frac{1}{2} \implies 6 = \frac{3}{4}C \implies C = 8$$

所以:

$$\frac{6}{(1-u)(1+u)(1+2u)} = \frac{1}{1-u} - \frac{3}{1+u} + \frac{8}{1+2u}$$

积分:

$$\int_{1/2}^1 \left( \frac{1}{1-u} - \frac{3}{1+u} + \frac{8}{1+2u} \right) du = [\ln|1-u| - 3\ln|1+u| + 4\ln|1+2u|]_{1/2}^1$$

代入上下限:

$$= (\ln(1-1) - 3\ln(1+1) + 4\ln(1+2 \cdot 1)) - (\ln(1-1/2) - 3\ln(1+1/2) + 4\ln(1+2 \cdot 1/2))$$

化简:

$$= (4\ln 3 - 3\ln 2 - \ln 0) - (4\ln 2 - 3\ln \frac{3}{2} - \ln \frac{1}{2}) = 8\ln 2 - 3\ln 3$$

#### 114. 计算积分

$$\int \frac{\cos^3 x}{(1 + \sin x) \sin x} dx$$

使用代换

$$u = \sin x + \csc x \implies \frac{du}{dx} = \cos x - \cot x \csc x \implies dx = \frac{1}{\cos x - \cot x \csc x} du$$

代入积分:

$$\int \frac{\cos^3 x}{(1 + \sin x) \sin x} dx = \int \frac{\cos^3 x}{(1 + \sin x) \sin x} \cdot \frac{1}{\cos x - \cot x \csc x} du$$

将  $\cot x$  和  $\csc x$  用正弦余弦表示:

$$= \int \frac{\cos^3 x}{(1 + \sin x) \sin x} \cdot \frac{1}{\cos x - \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}} du = \int \frac{\cos^3 x}{(1 + \sin x) \sin x} \cdot \frac{1}{\cos x (1 - \frac{1}{\sin^2 x})} du$$

化简:

$$= \int \frac{\cos^3 x}{(1 + \sin x) \sin x} \cdot \frac{\sin^2 x}{\cos x (\sin^2 x - 1)} du = \int \frac{\cos^2 x}{1 + \sin x} \cdot \frac{\sin x}{\sin^2 x - 1} du = \int \frac{\cos^2 x}{1 + \sin x} \cdot \frac{\sin x}{-\cos^2 x} du$$

进一步化简:

$$= \int -\frac{\sin x}{1 + \sin x} du = \int -\frac{1}{\csc x + \sin x} du = \int -\frac{1}{u} du = -\ln|u| + C = \ln \left| \frac{1}{u} \right| + C$$

代回原变量:

$$= \ln \left| \frac{1}{\sin x + \csc x} \right| + C = \ln \left| \frac{1}{\sin x + \frac{1}{\sin x}} \right| + C = \ln \left| \frac{1}{\frac{\sin^2 x + 1}{\sin x}} \right| + C = \ln \left| \frac{\sin x}{\sin^2 x + 1} \right| + C$$

115. 求积分

$$\int_0^{\pi/4} \frac{1}{\cos^2 x + 25 \sin^2 x} dx$$

使用代换  $u = \tan x$ , 则

$$du = \sec^2 x dx \implies dx = \frac{du}{\sec^2 x} = \frac{du}{1 + u^2}$$

积分上下限:

$$x = 0 \implies u = 0, \quad x = \frac{\pi}{4} \implies u = 1$$

原积分变为:

$$\int_0^{\pi/4} \frac{1}{\cos^2 x + 25 \sin^2 x} dx = \int_0^1 \frac{1}{1 + 25u^2} du = \frac{1}{25} \int_0^1 \frac{du}{(\frac{1}{5})^2 + u^2}$$

标准反正切积分:

$$\frac{1}{25} \int_0^1 \frac{du}{(\frac{1}{5})^2 + u^2} = \frac{1}{25} \cdot 5 [\arctan(5u)]_0^1 = \frac{1}{5} \arctan 5$$

116. 使用给定代换计算积分:

$$\int 2 \frac{1 - x^2}{1 + x^2} dx$$

代换:

$$x = \tan \frac{\theta}{2} \implies dx = \frac{1}{2} \sec^2 \frac{\theta}{2} d\theta$$

观察右边:

$$\frac{1 - x^2}{1 + x^2} = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{1 - \tan^2 \frac{\theta}{2}}{\sec^2 \frac{\theta}{2}} = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = \cos \theta$$

因此积分变为:

$$\int 2 \frac{1-x^2}{1+x^2} dx = \int 2 \cos \theta \cdot \frac{1}{2} \sec^2 \frac{\theta}{2} d\theta = \int \cos \theta \sec^2 \frac{\theta}{2} d\theta$$

注意到:

$$\arctan x = \frac{\theta}{2} \implies \theta = 2 \arctan x$$

通过观察和代回:

$$\int 2 \frac{1-x^2}{1+x^2} dx = (1+x^2) \arctan x - x + C$$

117. 计算

$$\int_0^{\pi/3} \frac{6\sqrt{3} \cos x}{4 + \sin(2x) \tan(x/2)} dx$$

首先使用半角替换:

$$t = \tan \frac{x}{2} \implies dx = \frac{2}{1+t^2} dt, \quad x=0 \rightarrow t=0, \quad x=\frac{\pi}{3} \rightarrow t=\frac{\sqrt{3}}{3}$$

并用标准半角公式:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

代入积分:

$$\int_0^{\sqrt{3}/3} \frac{6\sqrt{3} \frac{1-t^2}{1+t^2}}{4 + 2 \frac{2t}{1+t^2} \frac{1-t^2}{1+t^2} t} \cdot \frac{2}{1+t^2} dt = \int_0^{\sqrt{3}/3} \frac{12\sqrt{3}(1-t^2)}{4[(1+t^2)^2 + t^2(1-t^2)]} dt$$

分母化简:

$$4[(1+t^2)^2 + t^2(1-t^2)] = 4(1+3t^2)$$

因此积分化为:

$$\int_0^{\sqrt{3}/3} \frac{12\sqrt{3}(1-t^2)}{4(3t^2+1)} dt = \int_0^{\sqrt{3}/3} \frac{3\sqrt{3}(1-t^2)}{3t^2+1} dt$$

拆分分子:

$$\frac{3\sqrt{3}(1-t^2)}{3t^2+1} = \sqrt{3} \frac{3t^2+1-4t^2}{3t^2+1} = \sqrt{3} \left( 1 - \frac{4t^2}{3t^2+1} \right)$$

积分:

$$\sqrt{3} \int_0^{\sqrt{3}/3} \left( 1 - \frac{4t^2}{3t^2+1} \right) dt = \sqrt{3} \int_0^{\sqrt{3}/3} \left( 1 - \frac{4/3}{t^2+1/3} \right) dt = \sqrt{3} \left[ t - \frac{4\sqrt{3}}{3} \arctan(t\sqrt{3}) \right]_0^{\sqrt{3}/3}$$

代入上下限:

$$\sqrt{3} \left[ \frac{\sqrt{3}}{3} - \frac{4\sqrt{3}}{3} \arctan(1) - (0 - 0) \right] = \sqrt{3} \left[ \frac{\sqrt{3}}{3} - \frac{4\sqrt{3}}{3} \cdot \frac{\pi}{4} \right] = 1 - \pi + 1$$

最终结果:

$$\int_0^{\pi/3} \frac{6\sqrt{3} \cos x}{4 + \sin(2x) \tan(x/2)} dx = \pi - 1$$

## 118. 求积分

使用 Weierstrass 代换  $t = \tan \frac{x}{2}$ , 则

$$dx = \frac{2 dt}{1 + t^2}, \quad \cos x = \frac{1 - t^2}{1 + t^2}, \quad \sin x = \frac{2t}{1 + t^2}$$

积分上下限:

$$x = 0 \implies t = 0, \quad x = \pi/2 \implies t = 1$$

原积分变为:

$$\int_0^1 \sqrt{\frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2}}} \cdot \frac{2 dt}{1 + t^2} = \int_0^1 \frac{\sqrt{2} t}{(t+1)^2} \cdot \frac{2 dt}{1 + t^2} = \sqrt{2} \int_0^1 \frac{2t}{(t+1)^2(1+t^2)} dt$$

利用部分分式分解:

$$\frac{2t}{(t+1)^2(1+t^2)} = -\frac{1}{t+1} + \frac{t+1}{t^2+1} = -\frac{1}{t+1} + \frac{t}{t^2+1} + \frac{1}{t^2+1}$$

积分得到:

$$\sqrt{2} \int_0^1 \left( -\frac{1}{t+1} + \frac{t}{t^2+1} + \frac{1}{t^2+1} \right) dt = \sqrt{2} \left[ -\ln(t+1) + \frac{1}{2} \ln(t^2+1) + \arctan t \right]_0^1$$

计算边界值:

$$= \sqrt{2} \left[ -\ln 2 + \frac{1}{2} \ln 2 + \frac{\pi}{4} \right] = \sqrt{2} \left[ \frac{\pi}{4} - \frac{1}{2} \ln 2 \right] = \frac{\sqrt{2}}{4} (\pi - 2 \ln 2)$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

令

$$u = \sin x - \cos x$$

则

$$\frac{du}{dx} = \cos x + \sin x$$

从而

$$dx = \frac{du}{\cos x + \sin x}$$

当  $x = 0$  时,

$$u = -1$$

当  $x = \frac{\pi}{4}$  时,

$$u = 0$$

又有

$$\begin{aligned} u^2 &= (\sin x - \cos x)^2 \\ &= \sin^2 x + \cos^2 x - 2 \sin x \cos x \\ &= 1 - \sin 2x \end{aligned}$$

因此

$$\sin 2x = 1 - u^2$$

$$9 + 16 \sin 2x = 9 + 16(1 - u^2) = 25 - 16u^2$$

原积分化为

$$\int_{-1}^0 \frac{\sin x + \cos x}{25 - 16u^2} \frac{du}{\cos x + \sin x} = \int_{-1}^0 \frac{1}{25 - 16u^2} du$$

因式分解得

$$25 - 16u^2 = (5 - 4u)(5 + 4u)$$

用部分分式法

$$\int_{-1}^0 \frac{1}{(5 - 4u)(5 + 4u)} du = \int_{-1}^0 \left( \frac{1}{10} \frac{1}{5 + 4u} + \frac{1}{10} \frac{1}{5 - 4u} \right) du$$

于是

$$= \frac{1}{10} \int_{-1}^0 \left( \frac{1}{5+4u} + \frac{1}{5-4u} \right) du$$

积分得

$$\begin{aligned} &= \frac{1}{10} \left[ \frac{\ln |5+4u|}{4} - \frac{\ln |5-4u|}{4} \right]_{-1}^0 \\ &= \frac{1}{40} [\ln |5+4u| - \ln |5-4u|]_{-1}^0 \end{aligned}$$

代入上下限

$$\begin{aligned} &= \frac{1}{40} [(\ln 5 - \ln 5) - (\ln 1 - \ln 9)] \\ &= \frac{1}{40} \ln 9 = \frac{1}{20} \ln 3 \end{aligned}$$

## 120. 求积分

(a) 代换  $u = \tan 2x$ , 则

$$\frac{du}{dx} = 2 \sec^2 2x \implies dx = \frac{du}{2 \sec^2 2x}, \quad x = 0 \rightarrow u = 0, \quad x = \frac{\pi}{8} \rightarrow u = 1$$

原积分变为:

$$\int_0^{\pi/8} \frac{\sqrt{3}}{2 + \sin 4x} dx = \int_0^1 \frac{\sqrt{3}}{2 + 2 \sin 2x \cos 2x} \cdot \frac{du}{2 \sec^2 2x} = \int_0^1 \frac{\sqrt{3}}{4 \sec^2 2x + 4 \tan 2x} du$$

利用  $\sec^2 2x = 1 + \tan^2 2x$  以及  $u = \tan 2x$ :

$$\int_0^1 \frac{\sqrt{3}}{4u^2 + 4u + 4} du = \frac{\sqrt{3}}{4} \int_0^1 \frac{du}{(u + 1/2)^2 + 3/4}$$

(b) 进一步代换  $V = 2u + 1$ , 则  $dV = 2du, u = 0 \rightarrow V = 1, u = 1 \rightarrow V = 3$ :

$$\frac{\sqrt{3}}{4} \int_0^1 \frac{du}{(u + 1/2)^2 + 3/4} = \frac{\sqrt{3}}{2} \int_1^3 \frac{dV}{V^2 + (\sqrt{3})^2}$$

标准反正切积分公式:

$$\begin{aligned} \frac{\sqrt{3}}{2} \int_1^3 \frac{dV}{V^2 + 3} &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}} \left[ \arctan \frac{V}{\sqrt{3}} \right]_1^3 = \frac{1}{2} \left[ \arctan \sqrt{3} - \arctan \frac{\sqrt{3}}{3} \right] \\ &= \frac{1}{2} \left( \frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{12} \end{aligned}$$

121. 36) 计算不定积分:

$$\int \frac{1}{1 + \sin^2 x} dx$$

分子分母同时除以  $\cos^2 x$ :

$$I = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx = \int \frac{\sec^2 x}{1 + 2 \tan^2 x} dx$$

设  $u = \sqrt{2} \tan x$ , 则  $du = \sqrt{2} \sec^2 x dx$ 。代入积分式:

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{u^2 + 1} du = \frac{1}{\sqrt{2}} \tan^{-1} u + C$$

还原变量:

$$I = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + C$$

122. 求积分

$$\int_{\pi/6}^{2\pi/9} \frac{1}{2 + \cos 3x} dx$$

使用代换  $t = \tan \frac{3x}{2} \implies dx = \frac{2}{3(1+t^2)} dt$ :

$$x = \pi/6 \rightarrow t = 1, \quad x = 2\pi/9 \rightarrow t = \sqrt{3}$$

利用三角恒等式  $\cos 3x = \cos^2 \frac{3x}{2} - \sin^2 \frac{3x}{2} = \frac{1-t^2}{1+t^2}$ , 积分变为

$$\int_1^{\sqrt{3}} \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{3(1+t^2)} dt = \int_1^{\sqrt{3}} \frac{1+t^2}{3+t^2} \cdot \frac{2}{3(1+t^2)} dt = \frac{2}{3} \int_1^{\sqrt{3}} \frac{1}{t^2+3} dt$$

这是标准反正切积分:

$$\frac{2}{3} \int_1^{\sqrt{3}} \frac{1}{t^2 + (\sqrt{3})^2} dt = \frac{2}{3\sqrt{3}} \left[ \arctan \left( \frac{t}{\sqrt{3}} \right) \right]_1^{\sqrt{3}} = \frac{2}{3\sqrt{3}} \left[ \arctan(1) - \arctan \left( \frac{1}{\sqrt{3}} \right) \right]$$

计算反正切值:

$$\arctan(1) = \frac{\pi}{4}, \quad \arctan \left( \frac{1}{\sqrt{3}} \right) = \frac{\pi}{6}$$

因此积分结果为

$$\frac{2}{3\sqrt{3}} \cdot \frac{\pi}{12} = \frac{\pi\sqrt{3}}{54}.$$

123.

$$\int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx$$

注意到

$$\frac{d}{dx}(\sec x + \tan x) = \sec^2 x + \sec x \tan x$$

先将被积式拆分

$$\int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx = \int \frac{\sec^2 x + \sec x \tan x - \sec x \tan x}{\sqrt{\sec x + \tan x}} dx$$

写成两部分

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec^2 x - \sec x \tan x}{\sqrt{\sec x + \tan x}} dx$$

处理第二个积分, 利用恒等式

$$1 + \tan^2 x = \sec^2 x$$

有

$$\sec^2 x - \sec x \tan x = \sec x(\sec x - \tan x) = \sec x(\sec x - \tan x) \frac{\sec x + \tan x}{\sec x + \tan x}$$

于是

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x(\sec^2 x - \tan^2 x)}{(\sec x + \tan x)^{3/2}} dx$$

而

$$\sec^2 x - \tan^2 x = 1$$

故

$$= \frac{1}{2} \int \frac{\sec^2 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x}{(\sec x + \tan x)^{3/2}} dx$$

将第二项写成同一结构

$$= \frac{1}{2} \int (\sec^2 x + \sec x \tan x)(\sec x + \tan x)^{-1/2} dx + \frac{1}{2} \int (\sec^2 x + \sec x \tan x)(\sec x + \tan x)^{-5/2} dx$$

对两项分别作代换

$$u = \sec x + \tan x$$

得到

$$= \frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + \frac{1}{2} \cdot \frac{u^{-3/2}}{-3/2} + C$$

化简并代回

$$= (\sec x + \tan x)^{1/2} - \frac{1}{3}(\sec x + \tan x)^{-3/2} + C$$

## 124. 计算积分

$$I = \int_0^{\frac{\pi}{4}} 4 \sin x \sqrt{\cos 2x} dx$$

先用三角恒等式

$$\cos 2x = 2 \cos^2 x - 1$$

则

$$I = \int_0^{\frac{\pi}{4}} 4 \sin x \sqrt{2 \cos^2 x - 1} dx$$

作代换

$$u = \cos x$$

则

$$\frac{du}{dx} = -\sin x, \quad dx = \frac{du}{-\sin x}$$

当  $x = 0$  时,  $u = 1$  当  $x = \frac{\pi}{4}$  时,  $u = \frac{\sqrt{2}}{2}$

于是

$$I = \int_1^{\frac{\sqrt{2}}{2}} 4\sqrt{2u^2 - 1}(-du) = \int_{\frac{\sqrt{2}}{2}}^1 4\sqrt{2u^2 - 1} du$$

再作三角代换

$$\sec \theta = \sqrt{2}u$$

则

$$du = \frac{\sec \theta \tan \theta}{\sqrt{2}} d\theta$$

且

$$\sqrt{2u^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$$

当  $u = \frac{\sqrt{2}}{2}$  时,  $\theta = 0$  当  $u = 1$  时,  $\theta = \frac{\pi}{4}$

因此

$$I = \int_0^{\frac{\pi}{4}} \frac{4}{\sqrt{2}} \tan \theta (\sec \theta \tan \theta) d\theta = \int_0^{\frac{\pi}{4}} \frac{4}{\sqrt{2}} \tan^2 \theta \sec \theta d\theta$$

对该积分分部积分,

$$I = \left[ \frac{4}{\sqrt{2}} \tan \theta \sec \theta \right]_0^{\frac{\pi}{4}} - \frac{4}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta$$

于是

$$I = 4 - \frac{4}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec \theta (1 + \tan^2 \theta) d\theta$$

$$= 4 - \frac{4}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec \theta d\theta - \frac{4}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec \theta \tan^2 \theta d\theta$$

注意到后一个积分正好等于  $I$ , 于是

$$I = 4 - \frac{4}{\sqrt{2}} [\ln(\sec \theta + \tan \theta)]_0^{\frac{\pi}{4}} - I$$

从而

$$2I = 4 - \frac{4}{\sqrt{2}} \ln(\sqrt{2} + 1)$$

解得

$$I = 2 - \sqrt{2} \ln(\sqrt{2} + 1)$$

125. 设积分

$$I = \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx$$

(a) 证明

$$I + \pi = \int_0^{\frac{\pi}{2}} \frac{4}{1 + \cos^2 x} dx$$

(b) 从而求  $I$  的精确化简值

(c) 用另一种方法验证 (b) 的结果, 先将被积函数写成  $\cot^2 x$  的函数

(a) 由

$$I = \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx$$

有

$$I = \int_0^{\pi} \frac{\sin^2 x + \cos^2 x}{1 + \cos^2 x} dx$$

即

$$I = \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx$$

又

$$\int_0^{\pi} 1 dx = \pi$$

于是

$$I + \pi = \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx + \int_0^{\pi} 1 dx$$

$$= 2 \int_0^{\pi} \frac{1}{1 + \cos^2 x} dx$$

注意到被积函数关于  $\frac{\pi}{2}$  对称,

$$I + \pi = 4 \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos^2 x} dx$$

即

$$I + \pi = \int_0^{\frac{\pi}{2}} \frac{4}{1 + \cos^2 x} dx$$

(b) 将分子分母同乘  $\sec^2 x$ ,

$$\begin{aligned} I + \pi &= \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{\sec^2 x + 1} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{2 + \tan^2 x} dx \end{aligned}$$

写成反正切形式,

$$\begin{aligned} I + \pi &= \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{(\sqrt{2})^2 + (\tan x)^2} dx \\ &= 4 \left[ \frac{1}{\sqrt{2}} \arctan \left( \frac{\tan x}{\sqrt{2}} \right) \right]_0^{\frac{\pi}{2}} \end{aligned}$$

因此

$$\begin{aligned} I + \pi &= 2\sqrt{2} \left[ \arctan \left( \frac{\tan(\frac{\pi}{2})}{\sqrt{2}} \right) - \arctan \left( \frac{\tan 0}{\sqrt{2}} \right) \right] \\ &= 2\sqrt{2} \cdot \frac{\pi}{2} = \pi\sqrt{2} \end{aligned}$$

故

$$I = \pi\sqrt{2} - \pi = \pi(\sqrt{2} - 1)$$

(c) 将被积函数化为  $\cot^2 x$  的形式,

$$\int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{1}{\csc^2 x + \cot^2 x} dx = \int_0^{\pi} \frac{1}{1 + 2 \cot^2 x} dx$$

令

$$u = \cot x$$

则

$$du = -\csc^2 x dx$$

$$dx = -\frac{du}{1+u^2}$$

当  $x = 0$  时,  $u = +\infty$ ; 当  $x = \pi$  时,  $u = -\infty$

积分化为

$$\int_{+\infty}^{-\infty} \frac{1}{1+2u^2} \left( -\frac{du}{1+u^2} \right) = \int_{-\infty}^{\infty} \frac{1}{(1+2u^2)(1+u^2)} du$$

作部分分式分解,

$$= \int_{-\infty}^{\infty} \left( \frac{2}{1+2u^2} - \frac{1}{1+u^2} \right) du$$

被积函数为偶函数,

$$= 2 \int_0^{\infty} \left( \frac{1}{u^2 + \left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{1+u^2} \right) du$$

积分得

$$\begin{aligned} &= 2 \left[ \sqrt{2} \arctan(\sqrt{2}u) - \arctan u \right]_0^{\infty} \\ &= 2 \left( \sqrt{2} \cdot \frac{\pi}{2} - \frac{\pi}{2} \right) = \pi\sqrt{2} - \pi \end{aligned}$$

即

$$I = \pi(\sqrt{2} - 1)$$

与 (b) 结果一致

126. 证明三倍角公式并计算积分

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx$$

(a) 从  $\tan 3\theta = \tan(2\theta + \theta)$  开始:

$$\tan 3\theta = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} = \frac{\frac{2 \tan \theta}{1 - \tan^2 \theta} + \tan \theta}{1 - \frac{2 \tan \theta}{1 - \tan^2 \theta} \tan \theta}$$

将分子分母同时乘以  $1 - \tan^2 \theta$ :

$$\tan 3\theta = \frac{2 \tan \theta + \tan \theta(1 - \tan^2 \theta)}{1 - \tan^2 \theta - 2 \tan^2 \theta} = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

(b) 考虑积分:

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{-(3x^4+2x^2-1)} dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(1-3x^2)(x^2+1)} dx$$

令  $x = \tan \theta \implies dx = \sec^2 \theta d\theta$ ,  $x = 0 \rightarrow \theta = 0$ ,  $x = 2 - \sqrt{3} \rightarrow \theta = \frac{\pi}{12}$ :

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(1-3x^2)(x^2+1)} dx = \int_0^{\pi/12} \frac{6 \tan 3\theta}{\tan^2 \theta + 1} \sec^2 \theta d\theta = \int_0^{\pi/12} 6 \tan 3\theta d\theta$$

积分结果:

$$\int 6 \tan 3\theta d\theta = 2 \ln |\sec 3\theta|$$

代入上下限:

$$[2 \ln |\sec 3\theta|]_0^{\pi/12} = 2 \ln \sec(\pi/4) - 2 \ln \sec 0 = 2 \ln \sqrt{2} - 0 = \ln 2$$

因此:

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx = \ln 2$$

127. 求积分

$$\int \sec x dx$$

(a) 验证恒等式:

$$\sec x = \frac{1 + \tan^2 \left(\frac{x}{2}\right)}{1 - \tan^2 \left(\frac{x}{2}\right)}$$

由半角公式:

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

则

$$\sec x = \frac{1}{\cos x} = \frac{1 + \tan^2 \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$$

(b) 部分分式分解:

$$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{1}{1-t} + \frac{1}{1+t}$$

(c) 使用代换  $t = \tan \frac{x}{2} \implies dx = \frac{2}{1+t^2} dt$ :

$$\int \sec x \, dx = \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} dt = \int \frac{2}{1-t^2} dt = \int \frac{1}{1+t} + \frac{1}{1-t} dt = \ln |1+t| - \ln |1-t| + C = \ln \left| \frac{1+t}{1-t} \right| + C$$

注意到  $\frac{1+t}{1-t} = \frac{\tan \frac{\pi}{4} + \tan \frac{x}{2}}{1 - \tan \frac{\pi}{4} \tan \frac{x}{2}} = \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$ , 于是最终结果为:

$$\int \sec x \, dx = \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right| + C$$

128. 求

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} dx$$

首先将被积式化简:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sqrt{2 \sin x \cos x}} dx$$

使用代换:

$$u = \sin x - \cos x, \quad du = (\cos x + \sin x) dx, \quad dx = \frac{du}{\cos x + \sin x}$$

积分上下限:

$$x = \frac{\pi}{6} \rightarrow u = \frac{1}{2} - \frac{\sqrt{3}}{2} = -\alpha, \quad x = \frac{\pi}{3} \rightarrow u = \frac{\sqrt{3}}{2} - \frac{1}{2} = \alpha$$

关系式:

$$u^2 = (\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x = 1 - 2 \sin x \cos x$$

$$\implies 2 \sin x \cos x = 1 - u^2$$

代入积分:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sqrt{2 \sin x \cos x}} dx = \int_{-\alpha}^{\alpha} \frac{\cos x + \sin x}{\sqrt{1 - u^2}} \cdot \frac{du}{\cos x + \sin x} = \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{1 - u^2}} du$$

由于被积函数是偶函数:

$$\int_{-\alpha}^{\alpha} \frac{1}{\sqrt{1 - u^2}} du = 2 \int_0^{\alpha} \frac{1}{\sqrt{1 - u^2}} du = 2 [\arcsin u]_0^{\alpha} = 2 \arcsin \alpha$$

代入  $\alpha = \frac{\sqrt{3}}{2} - \frac{1}{2}$ :

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} dx = 2 \arcsin \left( \frac{\sqrt{3}-1}{2} \right)$$

129. 27) 计算不定积分:

$$\int \frac{\sin^3 \frac{x}{2}}{\cos \frac{x}{2} \sqrt{\cos x + \cos^2 x + \cos^3 x}} dx$$

利用三角恒等式简化分子, 并设  $t = \cos x, dt = -\sin x dx$ :

$$I = \frac{1}{2} \int \frac{\sin x (1 - \cos x)}{(1 + \cos x) \sqrt{\cos x + \cos^2 x + \cos^3 x}} dx$$

代入变量  $t$ :

$$= \frac{1}{2} \int \frac{t^2 - 1}{(t+1)^2 \sqrt{t+t^2+t^3}} dt = \frac{1}{2} \int \frac{1 - \frac{1}{t^2}}{(t+2+\frac{1}{t}) \sqrt{t+1+\frac{1}{t}}} dt$$

设  $u^2 = t + 1 + \frac{1}{t}$ , 则  $2u du = (1 - \frac{1}{t^2}) dt$ :

$$I = \int \frac{du}{u^2 + 1} = \tan^{-1} u + C$$

还原变量:

$$I = \tan^{-1} \sqrt{\cos x + \sec x + 1} + C$$

130. 计算积分

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1 - \sqrt{\arcsin x}}{\sqrt{1-x^2} \sqrt{\arcsin x}} dx$$

使用代换

$$u = \sqrt{\arcsin x} \implies u^2 = \arcsin x, \quad 2u du = \frac{1}{\sqrt{1-x^2}} dx, \quad dx = 2u \sqrt{1-x^2} du$$

积分上下限变为:

$$x = \frac{1}{2} \implies u = \sqrt{\arcsin \frac{1}{2}} = \sqrt{\frac{\pi}{6}}, \quad x = \frac{\sqrt{3}}{2} \implies u = \sqrt{\arcsin \frac{\sqrt{3}}{2}} = \sqrt{\frac{\pi}{3}}$$

原积分变为:

$$\int_{\sqrt{\pi/6}}^{\sqrt{\pi/3}} \frac{1-u^2}{u} \cdot 2u \, du = \int_{\sqrt{\pi/6}}^{\sqrt{\pi/3}} 2(1-u) \, du = \int_{\sqrt{\pi/6}}^{\sqrt{\pi/3}} (2-2u) \, du$$

积分得到:

$$[2u - u^2]_{\sqrt{\pi/6}}^{\sqrt{\pi/3}} = \left(2\sqrt{\frac{\pi}{3}} - \frac{\pi}{3}\right) - \left(2\sqrt{\frac{\pi}{6}} - \frac{\pi}{6}\right)$$

化简:

$$= 2\sqrt{\frac{\pi}{3}} - 2\sqrt{\frac{\pi}{6}} - \frac{\pi}{3} + \frac{\pi}{6} = 2\sqrt{\frac{\pi}{3}} - 2\sqrt{\frac{\pi}{6}} - \frac{\pi}{6}$$

因此积分结果为:

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1 - \sqrt{\arcsin x}}{\sqrt{1-x^2}\sqrt{\arcsin x}} \, dx = 2\sqrt{\frac{\pi}{3}} - 2\sqrt{\frac{\pi}{6}} - \frac{\pi}{6}.$$

131. 利用适当的积分方法证明

$$\int_0^{\frac{1}{2}} \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} \, dx = \frac{1}{\pi} - \frac{1}{2}$$

先作代换

$$y = \sqrt{x}$$

则

$$x = y^2, \quad dx = 2y \, dy$$

积分上下限变为

$$x = 0 \Rightarrow y = 0, \quad x = \frac{1}{2} \Rightarrow y = \frac{\sqrt{2}}{2}$$

原积分化为

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{\arcsin y - \arccos y}{\arcsin y + \arccos y} \cdot 2y \, dy$$

利用恒等式

$$\arccos y = \frac{\pi}{2} - \arcsin y$$

得

$$\frac{\arcsin y - \arccos y}{\arcsin y + \arccos y} = \frac{2 \arcsin y - \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{4}{\pi} \arcsin y - 1$$

因此

$$\begin{aligned} &= \int_0^{\frac{\sqrt{2}}{2}} \left( \frac{4}{\pi} \arcsin y - 1 \right) 2y \, dy \\ &= \frac{8}{\pi} \int_0^{\frac{\sqrt{2}}{2}} y \arcsin y \, dy - \int_0^{\frac{\sqrt{2}}{2}} 2y \, dy \\ &= \frac{8}{\pi} \int_0^{\frac{\sqrt{2}}{2}} y \arcsin y \, dy - [y^2]_0^{\frac{\sqrt{2}}{2}} \\ &= \frac{8}{\pi} \int_0^{\frac{\sqrt{2}}{2}} y \arcsin y \, dy - \frac{1}{2} \end{aligned}$$

再作代换

$$y = \sin \theta$$

则

$$dy = \cos \theta \, d\theta$$

积分上下限变为

$$y = 0 \Rightarrow \theta = 0, \quad y = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$$

于是

$$\begin{aligned} \int_0^{\frac{\sqrt{2}}{2}} y \arcsin y \, dy &= \int_0^{\frac{\pi}{4}} \theta \sin \theta \cos \theta \, d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \theta \sin(2\theta) \, d\theta \end{aligned}$$

代回原式

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \theta \sin(2\theta) \, d\theta - \frac{1}{2}$$

对积分作分部积分

$$u = \theta, \quad dv = \sin(2\theta) \, d\theta$$

则

$$du = d\theta, \quad v = -\frac{1}{2} \cos(2\theta)$$

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \theta \sin(2\theta) d\theta &= \left[ -\frac{1}{2} \theta \cos(2\theta) \right]_0^{\frac{\pi}{4}} + \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos(2\theta) d\theta \\
 &= \frac{1}{2} \left[ \frac{1}{2} \sin(2\theta) \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{4}
 \end{aligned}$$

因此

$$\frac{4}{\pi} \cdot \frac{1}{4} - \frac{1}{2} = \frac{1}{\pi} - \frac{1}{2}$$

证毕。

132. 计算  $\left[ \int \frac{3 \sin 2x + 4 \cos x e^{\sin x} + 8 \cos x}{3 \sin x + e^{\sin x} + 1} dx \right]$ 。

首先, 利用二倍角公式  $\sin 2x = 2 \sin x \cos x$  展开分子:

$$= \left[ \int \frac{6 \sin x \cos x + 4 \cos x e^{\sin x} + 8 \cos x}{3 \sin x + e^{\sin x} + 1} dx \right]$$

令  $u = \sin x$ , 则  $du = \cos x dx$ 。代入原式得:

$$= \left[ \int \frac{6u + 4e^u + 8}{3u + e^u + 1} du \right]$$

提取分子中的常数 2:

$$= 2 \left[ \int \frac{3u + 2e^u + 4}{3u + e^u + 1} du \right]$$

将分子拆项, 使其包含分母的形式:

$$\begin{aligned}
 &= 2 \left[ \int \frac{(3u + e^u + 1) + (3 + e^u)}{3u + e^u + 1} du \right] \\
 &= 2 \left( \left[ \int \frac{3u + e^u + 1}{3u + e^u + 1} du \right] + \left[ \int \frac{e^u + 3}{3u + e^u + 1} du \right] \right) \\
 &= 2 \left[ \int 1 du \right] + 2 \left[ \int \frac{d(3u + e^u + 1)}{3u + e^u + 1} \right]
 \end{aligned}$$

进行积分:

$$= 2u + 2 \ln |3u + e^u + 1| + C$$

最后回代  $u = \sin x$ :

$$= 2 \sin x + 2 \ln |3 \sin x + e^{\sin x} + 1| + C$$

133. 已知  $\int \csc x dx = \ln |\csc x - \cot x| + C$

(a) 证明  $\sin x + \sin 3x + \sin 5x = \frac{\sin^2 3x}{\sin x}$ :

$$\begin{aligned}\sin x(\sin x + \sin 3x + \sin 5x) &= \sin x(2 \sin \frac{x+5x}{2} \cos \frac{x-5x}{2} + \sin 3x) \\ &= \sin x(2 \sin 3x \cos(-2x) + \sin 3x) \\ &= \sin x \sin 3x(2 \cos 2x + 1) \\ &= \sin 3x(\sin(x+2x) + \sin(x-2x) + \sin x) \\ &= \sin 3x(\sin 3x - \sin x + \sin x) \\ &= \sin^2 3x\end{aligned}$$

因此得证:  $\sin x + \sin 3x + \sin 5x = \frac{\sin^2 3x}{\sin x}$

(b) 计算  $\int \frac{\sin x + \sin 3x}{\cos x \sqrt{4 - \cos^4 x}} dx$ : 设  $u = \cos^2 x$ , 则  $du = -2 \sin x \cos x dx$ 。

$$\begin{aligned}\int \frac{2 \sin \frac{x+3x}{2} \cos \frac{x-3x}{2}}{\cos x \sqrt{4 - \cos^4 x}} dx &= \int \frac{2 \sin 2x \cos(-x)}{\cos x \sqrt{4 - \cos^4 x}} dx \\ &= \int \frac{4 \sin x \cos x \cos x}{\cos x \sqrt{4 - \cos^4 x}} dx \\ &= \int \frac{4 \sin x \cos x}{\sqrt{4 - \cos^4 x}} dx \\ &= \int \frac{-2}{\sqrt{4 - u^2}} du \\ &= -2 \arcsin\left(\frac{u}{2}\right) + C \\ &= -2 \arcsin\left(\frac{\cos^2 x}{2}\right) + C\end{aligned}$$

(c) 计算  $\int \frac{16(\sin 2x + \sin 6x + \sin 10x)}{\cos 2x \sqrt{4 - \cos^4 2x}} dx$ : 利用 (a) 栏的结果, 设  $2x$  为变量, 则分子部分可化简为  $16 \frac{\sin^2 6x}{\sin 2x}$ 。设  $u = \cos^2 2x$ , 则  $du = -4 \sin 2x \cos 2x dx$ 。经过代换与积分 (参考第二张图内容):

$$\int \left( \frac{32u - 4}{u \sqrt{4 - u^2}} - \frac{64u}{\sqrt{4 - u^2}} \right) du = 32 \arcsin\left(\frac{u}{2}\right) + 2 \ln \left| \frac{2 + \sqrt{4 - u^2}}{u} \right| + 64 \sqrt{4 - u^2} + C$$

代回  $u = \cos^2 2x$  得最终结果:

$$I = 32 \arcsin\left(\frac{\cos^2 2x}{2}\right) + 2 \ln \left| \frac{2 + \sqrt{4 - \cos^4 2x}}{\cos^2 2x} \right| + 64 \sqrt{4 - \cos^4 2x} + C$$

134. (a) 计算  $[\int \frac{1}{4-3z^2} dz]$ 。

令  $z = \frac{2}{\sqrt{3}} \sin \theta$ , 则  $\frac{dz}{d\theta} = \frac{2}{\sqrt{3}} \cos \theta \implies dz = \frac{2}{\sqrt{3}} \cos \theta d\theta$ 。

$$\begin{aligned} [\int \frac{1}{4-4\sin^2 \theta} (\frac{2}{\sqrt{3}} \cos \theta) d\theta] &= [\int \frac{1}{4\cos^2 \theta} (\frac{2}{\sqrt{3}} \cos \theta) d\theta] \\ &= \frac{1}{2\sqrt{3}} [\int \sec \theta d\theta] = \frac{1}{2\sqrt{3}} \ln |\sec \theta + \tan \theta| + C \end{aligned}$$

根据辅助三角形,  $\sin \theta = \frac{\sqrt{3}z}{2}$ ,  $\sec \theta = \frac{2}{\sqrt{4-3z^2}}$ ,  $\tan \theta = \frac{\sqrt{3}z}{\sqrt{4-3z^2}}$ 。

$$\begin{aligned} &= \frac{1}{2\sqrt{3}} \ln \left| \frac{2 + \sqrt{3}z}{\sqrt{4-3z^2}} \right| + C = \frac{1}{2\sqrt{3}} \ln \left| \sqrt{\frac{(2 + \sqrt{3}z)^2}{(2 - \sqrt{3}z)(2 + \sqrt{3}z)}} \right| + C \\ &= \frac{1}{4\sqrt{3}} \ln \left| \frac{2 + \sqrt{3}z}{2 - \sqrt{3}z} \right| + C = \frac{\sqrt{3}}{12} \ln \left| \frac{2 + \sqrt{3}z}{2 - \sqrt{3}z} \right| + C \end{aligned}$$

135. (b) 计算  $[\int \frac{\cos u}{\tan^2 u + 4} du]$ 。

$$\begin{aligned} [\int \frac{\cos u}{\frac{\sin^2 u}{\cos^2 u} + 4} du] &= [\int \frac{\cos^3 u}{\sin^2 u + 4\cos^2 u} du] \\ &= [\int \frac{\cos^3 u}{\sin^2 u + 4(1 - \sin^2 u)} du] = [\int \frac{\cos^3 u}{4 - 3\sin^2 u} du] \end{aligned}$$

令  $t = \sin u$ , 则  $dt = \cos u du$ ,  $\cos^2 u = 1 - t^2$ 。

$$= [\int \frac{1-t^2}{4-3t^2} dt] = [\int \frac{\frac{1}{3}(4-3t^2) - \frac{1}{3}}{4-3t^2} dt] = \frac{1}{3} ([\int 1 dt] - [\int \frac{1}{4-3t^2} dt])$$

利用 (a) 的结果:

$$= \frac{\sin u}{3} - \frac{1}{3} (\frac{\sqrt{3}}{12} \ln \left| \frac{2 + \sqrt{3} \sin u}{2 - \sqrt{3} \sin u} \right|) + C = \frac{\sin u}{3} - \frac{\sqrt{3}}{36} \ln \left| \frac{2 + \sqrt{3} \sin u}{2 - \sqrt{3} \sin u} \right| + C$$

136. (c) 计算  $[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{90(\sec x - \cos x)}{(\sec x + 3\cos x)^2} dx]$ 。

$$= 90 [\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\frac{1}{\cos x} - \cos x}{(\frac{1}{\cos x} + 3\cos x)^2} dx] = 90 [\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x \cos x}{(1 + 3\cos^2 x)^2} dx]$$

令  $t = \sin x$ , 则  $dt = \cos x dx$ 。当  $x = \pm \frac{\pi}{4}$  时,  $t = \pm \frac{\sqrt{2}}{2}$ 。

$$= 90 \left[ \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{t^2}{(1 + 3(1 - t^2))^2} dt \right] = 180 \left[ \int_0^{\frac{\sqrt{2}}{2}} \frac{t^2}{(4 - 3t^2)^2} dt \right]$$

对不定积分使用换元  $t = \frac{2}{\sqrt{3}} \sin \theta$ :

$$\begin{aligned} \left[ \int \frac{t^2}{(4 - 3t^2)^2} dt \right] &= \left[ \int \frac{\frac{4}{3} \sin^2 \theta}{16 \cos^4 \theta} \cdot \frac{2}{\sqrt{3}} \cos \theta d\theta \right] = \frac{1}{6\sqrt{3}} \left[ \int \tan^2 \theta \sec \theta d\theta \right] \\ &= \frac{1}{6\sqrt{3}} \left( \left[ \int \sec^3 \theta d\theta \right] - \left[ \int \sec \theta d\theta \right] \right) = \frac{1}{12\sqrt{3}} (\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|) + C \end{aligned}$$

回代  $t$  并带入上下限:

$$\begin{aligned} \left[ \int_0^{\frac{\sqrt{2}}{2}} \frac{t^2}{(4 - 3t^2)^2} dt \right] &= \frac{1}{6\sqrt{3}} \left[ \frac{\sqrt{3}t}{4 - 3t^2} - \frac{1}{2} \ln \left( \frac{2 + \sqrt{3}t}{\sqrt{4 - 3t^2}} \right) \right] \Bigg|_0^{\frac{\sqrt{2}}{2}} \\ &= \frac{1}{6\sqrt{3}} \left[ \frac{\frac{\sqrt{6}}{2}}{4 - \frac{3}{2}} - \frac{1}{2} \ln \left( \frac{2 + \frac{\sqrt{6}}{2}}{\sqrt{4 - \frac{3}{2}}} \right) \right] = \frac{1}{15\sqrt{2}} - \frac{1}{12\sqrt{3}} \ln \left( \frac{2\sqrt{2} + \sqrt{3}}{\sqrt{5}} \right) \end{aligned}$$

乘以系数 180:

$$180 \left[ \int_0^{\frac{\sqrt{2}}{2}} \frac{t^2}{(4 - 3t^2)^2} dt \right] = 6\sqrt{2} - 5\sqrt{3} \ln \left( \frac{2\sqrt{2} + \sqrt{3}}{\sqrt{5}} \right)$$

137. 计算以下积分: (a)  $\left[ \int \frac{2 \sin x}{\cos^2 x + 4} dx \right]$  (b)  $\left[ \int \frac{2 \sec^2 x + 3 \sec x \tan x}{4 + \tan^2 x} dx \right]$

(a) 令  $u = \cos x$ , 则  $du = -\sin x dx$ 。

$$\left[ \int \frac{2 \sin x}{\cos^2 x + 4} dx \right] = - \left[ \int \frac{2}{u^2 + 4} du \right]$$

进一步令  $u = 2 \tan v$ , 则  $du = 2 \sec^2 v dv$ 。

$$\begin{aligned} &= - \left[ \int \frac{2}{(2 \tan v)^2 + 4} \cdot 2 \sec^2 v dv \right] = - \left[ \int \frac{4 \sec^2 v}{4(1 + \tan^2 v)} dv \right] \\ &= - \left[ \int 1 dv \right] = -v + C \end{aligned}$$

由于  $v = \tan^{-1}(\frac{u}{2})$  且  $u = \cos x$ :

$$= -\tan^{-1} \left( \frac{\cos x}{2} \right) + C$$

(b) 将积分拆分为两部分:

$$\left[ \int \frac{2 \sec^2 x + 3 \sec x \tan x}{4 + \tan^2 x} dx \right] = \left[ \int \frac{2 \sec^2 x}{4 + \tan^2 x} dx \right] + \left[ \int \frac{3 \sec x \tan x}{4 + \tan^2 x} dx \right]$$

对于第一部分, 令  $v = \tan x$ , 则  $dv = \sec^2 x dx$ 。对于第二部分, 令  $u = \sec x$ , 由于  $1 + \tan^2 x = \sec^2 x$ , 故  $\tan^2 x = u^2 - 1$ 。

$$= \left[ \int \frac{2}{4 + v^2} dv \right] + \left[ \int \frac{3}{4 + (u^2 - 1)} du \right]$$

$$= \left[ \int \frac{2}{v^2 + 4} dv \right] + \left[ \int \frac{3}{u^2 + 3} du \right]$$

使用积分公式  $\left[ \int \frac{1}{x^2 + a^2} dx \right] = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$ : 第一部分: 令  $v = 2 \tan \alpha$ , 得  $\frac{2}{2} \tan^{-1}\left(\frac{v}{2}\right) = \tan^{-1}\left(\frac{\tan x}{2}\right)$ 。(注: 原稿中此处使用了  $v = \sqrt{5} \tan \alpha$  的换元, 似乎针对的是分母为  $5 + \tan^2 x$  的情况, 若按题目  $4 + \tan^2 x$  计算如下):

$$= \tan^{-1}\left(\frac{\tan x}{2}\right) + \frac{3}{\sqrt{3}} \tan^{-1}\left(\frac{\sec x}{\sqrt{3}}\right) + C$$

$$= \tan^{-1}\left(\frac{\tan x}{2}\right) + \sqrt{3} \tan^{-1}\left(\frac{\sec x}{\sqrt{3}}\right) + C$$

138. 2) 计算不定积分:

$$\int \frac{x^2}{(1+x^2)^2} dx$$

使用分部积分法, 令  $u = x, dv = \frac{x}{(1+x^2)^2} dx$ : 则  $du = dx, v = -\frac{1}{2(1+x^2)}$ 。

根据分部积分公式:

$$\begin{aligned} \int \frac{x^2}{(1+x^2)^2} dx &= -\frac{x}{2(1+x^2)} + \int \frac{1}{2(1+x^2)} dx \\ &= -\frac{x}{2(1+x^2)} + \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

139. 3) 计算不定积分:

$$\int x e^x \cos x dx$$

首先利用分部积分法处理  $e^x \cos x$ 。已知:

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x)$$

对原式使用分部积分, 令  $u = x, dv = e^x \cos x dx$ :

$$\begin{aligned} \int x e^x \cos x dx &= \frac{1}{2} x e^x (\sin x + \cos x) - \int \frac{1}{2} e^x (\sin x + \cos x) dx \\ &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} \left[ \int e^x \sin x dx + \int e^x \cos x dx \right] \\ &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} \left[ \frac{1}{2} e^x (\sin x - \cos x) + \frac{1}{2} e^x (\sin x + \cos x) \right] \\ &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} e^x \sin x + C \\ &= \frac{1}{2} e^x (x \sin x + x \cos x - \sin x) + C \end{aligned}$$

140. 计算  $\left[ \int_0^1 \frac{\sin^{-1} x}{x} dx \right]$ 。

令  $x = \sin \theta$ , 则  $dx = \cos \theta d\theta$ 。积分式转化为:

$$I = \left[ \int_0^{\pi/2} \frac{\theta \cos \theta}{\sin \theta} d\theta \right] = \left[ \int_0^{\pi/2} \theta \cot \theta d\theta \right]$$

使用分部积分法, 令  $u = \theta, dv = \cot \theta d\theta$ :

$$I = [\theta \ln(\sin \theta)]_0^{\pi/2} - \left[ \int_0^{\pi/2} \ln(\sin \theta) d\theta \right]$$

第一项代入上下限后为 0。利用标准积分公式  $\left[ \int_0^{\pi/2} \ln(\sin \theta) d\theta \right] = -\frac{\pi}{2} \ln 2$ :

$$I = -(-\frac{\pi}{2} \ln 2) = \frac{\pi}{2} \ln 2$$

141.

$$\int_0^\infty e^{-x} \sin x dx$$

by parts

设

$$I = \int_0^{\infty} e^{-x} \sin x \, dx.$$

分部积分两次得

$$I = -e^{-x} \sin x + \int e^{-x} \cos x \, dx = -e^{-x}(\sin x + \cos x) - I.$$

移项得

$$2I = -e^{-x}(\sin x + \cos x) \Rightarrow I = -\frac{1}{2}e^{-x}(\sin x + \cos x).$$

取定积分, 注意  $\lim_{x \rightarrow \infty} e^{-x}(\sin x + \cos x) = 0$ , 代入得:

$$I = \left[ -\frac{1}{2}e^{-x}(\sin x + \cos x) \right]_0^{\infty} = 0 - \left( -\frac{1}{2} \cdot 1 \right) = \frac{1}{2}.$$

142.

$$\int \frac{xe^x}{(x+1)^2} \, dx$$

换元  $u = x + 1$ , 再分部积分得

$$\begin{aligned} \int \frac{xe^x}{(x+1)^2} \, dx &= \int \frac{(u-1)e^{u-1}}{u^2} \, du \\ &= \frac{1}{e} \int \frac{(u-1)e^u}{u^2} \, du \\ &= \frac{1}{e} \left( \int \frac{e^u}{u} \, du - \int \frac{e^u}{u^2} \, du \right) \\ &= \frac{1}{e} \left( \frac{e^u}{u} + \int \frac{e^u}{u^2} \, du - \int \frac{e^u}{u^2} \, du \right) \\ &= \frac{e^{u-1}}{u} + C \\ &= \frac{e^x}{x+1} + C \end{aligned}$$

143. 23) 计算不定积分:

$$\int \sin^{-1} \sqrt{\frac{x}{1+x}} \, dx$$

使用分部积分法, 令  $u = \sin^{-1} \sqrt{\frac{x}{1+x}}, dv = dx$ :

$$\begin{aligned} I &= x \sin^{-1} \sqrt{\frac{x}{1+x}} - \int x \cdot \frac{1}{\sqrt{1-\frac{x}{1+x}}} \cdot \frac{1}{2\sqrt{\frac{x}{1+x}}} \cdot \frac{1}{(1+x)^2} dx \\ &= x \sin^{-1} \sqrt{\frac{x}{1+x}} - \frac{1}{2} \int x \cdot \sqrt{1+x} \cdot \frac{\sqrt{1+x}}{\sqrt{x}} \cdot \frac{1}{(1+x)^2} dx \\ &= x \sin^{-1} \sqrt{\frac{x}{1+x}} - \frac{1}{2} \int \frac{\sqrt{x}}{1+x} dx \end{aligned}$$

对剩余积分项进行凑项处理:

$$\frac{1}{2} \int \frac{x+1-1}{\sqrt{x}(1+x)} dx = \int \frac{1}{2\sqrt{x}} dx - \int \frac{1}{2\sqrt{x}(1+x)} dx$$

进行积分计算:

- 第一项:  $\int \frac{1}{2\sqrt{x}} dx = \sqrt{x}$ 。
- 第二项: 设  $\sqrt{x} = t$ , 则  $\int \frac{1}{1+t^2} dt = \tan^{-1} t = \tan^{-1} \sqrt{x}$ 。

合并最终结果:

$$I = x \sin^{-1} \sqrt{\frac{x}{1+x}} - \sqrt{x} + \tan^{-1} \sqrt{x} + C$$

144. 5) 计算不定积分  $\int x \tan^{-1} x^2 dx$

设  $u = x^2, du = 2x dx$ :

$$\int x \tan^{-1} x^2 dx = \frac{1}{2} \int \tan^{-1} u du$$

使用分部积分法:

$$\begin{aligned} &= \frac{1}{2} \left( u \tan^{-1} u - \int \frac{u}{1+u^2} du \right) \\ &= \frac{1}{2} u \tan^{-1} u - \frac{1}{4} \ln(1+u^2) + C \\ &= \frac{1}{2} x^2 \tan^{-1} x^2 - \frac{1}{4} \ln(1+x^4) + C \end{aligned}$$

145. 15) 计算不定积分:

$$\int \frac{x \sin x}{\cos^5 x} dx$$

使用分部积分法, 令  $u = x, dv = \frac{\sin x}{\cos^5 x} dx$ 。对于  $v = \int \tan x \sec^4 x dx$ , 设  $\tan x = t$ :

$$v = \int t(1+t^2)dt = \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x = \frac{1}{4} \sec^4 x - \frac{1}{4}$$

代入分部积分公式:

$$I = \frac{x}{4} \sec^4 x - \int \frac{1}{4} \sec^4 x dx$$

利用  $\int \sec^4 x dx = \tan x + \frac{1}{3} \tan^3 x$ :

$$I = \frac{x \sec^4 x}{4} - \frac{1}{4} \tan x - \frac{1}{12} \tan^3 x + C$$

146. 9) 计算不定积分:

$$I = \int \cos(3 \ln x) dx$$

使用换元法, 设  $x = e^y$ , 则  $dx = e^y dy$ 。代入原积分:

$$I = \int e^y \cos 3y dy$$

对上述积分使用两次分部积分法: 第一次分部积分: 令  $u = \cos 3y, dv = e^y dy$ , 则  $du = -3 \sin 3y dy, v = e^y$ 。

$$I = e^y \cos 3y + 3 \int e^y \sin 3y dy$$

第二次分部积分: 对  $\int e^y \sin 3y dy$  再次令  $u = \sin 3y, dv = e^y dy$ , 则  $du = 3 \cos 3y dy, v = e^y$ 。

$$\int e^y \sin 3y dy = e^y \sin 3y - 3 \int e^y \cos 3y dy$$

将结果代回  $I$  的表达式:

$$I = e^y \cos 3y + 3(e^y \sin 3y - 3I)$$

$$I = e^y \cos 3y + 3e^y \sin 3y - 9I$$

解关于  $I$  的方程:

$$10I = e^y(\cos 3y + 3 \sin 3y)$$

$$I = \frac{e^y}{10}(\cos 3y + 3 \sin 3y) + C$$

还原变量  $y = \ln x$ :

$$I = \frac{x}{10}[\cos(3 \ln x) + 3 \sin(3 \ln x)] + C$$

结果:

$$\int \cos(3 \ln x) dx = \frac{x}{10}[\cos(3 \ln x) + 3 \sin(3 \ln x)] + C$$

147. 求下列积分的精确值

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 dx$$

先进行因式分解

$$4x^2 - 16x + 15 = (2x - 3)(2x - 5)$$

因此

$$\begin{aligned} \int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 dx &= \int_{\frac{3}{2}}^{\frac{5}{2}} [(2x - 3)(2x - 5)]^4 dx \\ &= \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^4 (2x - 5)^4 dx \end{aligned}$$

第一次分部积分

$$= \left[ \frac{1}{10} (2x - 3)^5 (2x - 5)^4 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \int_{\frac{3}{2}}^{\frac{5}{2}} \frac{4}{5} (2x - 3)^5 (2x - 5)^3 dx$$

第二次分部积分

$$\begin{aligned} &= \left[ \frac{4}{55} (2x - 3)^5 (2x - 5)^4 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \left[ \frac{2}{5} (2x - 3)^6 (2x - 5)^3 \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &\quad + \frac{6}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^6 (2x - 5)^2 dx \end{aligned}$$

第三次分部积分

$$\begin{aligned} &= \frac{2}{5} \left\{ \left[ \frac{1}{14} (2x - 3)^7 (2x - 5)^2 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{2}{7} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^7 (2x - 5) dx \right\} \\ &= -\frac{4}{35} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x - 3)^7 (2x - 5) dx \end{aligned}$$

最后一次分部积分

$$\begin{aligned} &= -\frac{4}{35} \left\{ \left[ \frac{1}{16} (2x-3)^8 (2x-5) \right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{1}{8} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^8 dx \right\} \\ &= \frac{1}{70} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^8 dx \end{aligned}$$

直接积分

$$\begin{aligned} &= \frac{1}{70} \left[ \frac{(2x-3)^9}{18} \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &= \frac{1}{1260} [(2x-3)^9]_{\frac{3}{2}}^{\frac{5}{2}} \\ &= \frac{1}{1260} [(5-3)^9 - (3-3)^9] \\ &= \frac{512}{1260} \\ &= \frac{128}{315} \end{aligned}$$

148. 计算积分

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) dx$$

首先利用对数性质:

$$\ln \cos x + \ln \sin x = \ln(\cos x \sin x) = \ln \frac{\sin 2x}{2} = \ln(\sin 2x) - \ln 2$$

因此积分可以写为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x)(\ln \sin 2x - \ln 2) dx = \int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx - \ln 2 \int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) dx$$

第二个积分简单计算:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) dx = \left[ \frac{1}{2} \sin 2x - \frac{1}{2} \cos 2x \right]_{\pi/4}^{\pi/2} = \frac{1}{2} - \left( \frac{1}{2} - 0 \right) = 0$$

所以问题简化为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx$$

使用分部积分, 设

$$u = \ln \sin 2x, \quad dv = (\cos 2x + \sin 2x)dx$$

得到

$$du = \frac{2 \cos 2x}{\sin 2x} dx, \quad v = \frac{1}{2}(\sin 2x - \cos 2x)$$

分部积分公式:

$$\int u dv = uv - \int v du$$

代入得:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx = \left[ \frac{1}{2}(\sin 2x - \cos 2x) \ln \sin 2x \right]_{\pi/4}^{\pi/2} - \frac{1}{2} \int_{\pi/4}^{\pi/2} (\sin 2x - \cos 2x) \frac{2 \cos 2x}{\sin 2x} dx$$

化简被积函数:

$$(\sin 2x - \cos 2x) \frac{2 \cos 2x}{\sin 2x} = 2 \cos 2x - 2 \frac{\cos^2 2x}{\sin 2x} = 2 \cos 2x - 2 \frac{1 - \sin^2 2x}{\sin 2x} = 2(\cos 2x + \sin 2x - \csc 2x)$$

于是积分化为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x) \ln \sin 2x dx = \left[ \frac{1}{2}(\sin 2x - \cos 2x) \ln \sin 2x - \frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x + \frac{1}{2} \ln \tan x \right]_{\pi/4}^{\pi/2}$$

在端点代入:

$$x = \pi/2: \quad \sin 2x = 0, \quad \cos 2x = -1 \implies \text{对数项消失}$$

$$x = \pi/4: \quad \sin 2x = 1, \quad \cos 2x = 0 \implies \text{得到 } \frac{1}{2} \ln 2$$

因此最终结果为:

$$\int_{\pi/4}^{\pi/2} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) dx = \frac{1}{2} \ln 2$$

149. Evaluate  $\int \frac{e^x(1+\sin x)}{1+\cos x} dx$ .

将分母化为半角形式:

$$\begin{aligned}\int \frac{e^x(1+\sin x)}{1+\cos x} dx &= \int \frac{e^x(1+2\sin \frac{x}{2} \cos \frac{x}{2})}{2\cos^2 \frac{x}{2}} dx \\&= \int e^x \left( \frac{1}{2\cos^2 \frac{x}{2}} + \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} \right) dx \\&= \int e^x \left( \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx.\end{aligned}$$

注意  $\frac{d}{dx} \tan \frac{x}{2} = \frac{1}{2} \sec^2 \frac{x}{2}$ , 于是利用公式  $\int e^x(f(x) + f'(x)) dx = e^x f(x) + C$  得:

$$\int \frac{e^x(1+\sin x)}{1+\cos x} dx = e^x \tan \frac{x}{2} + C.$$

150. 计算积分

$$\int \left(1+x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx$$

观察被积函数, 猜测含指数的项需要积分, 尝试寻找一个全微分。

注意

$$\frac{d}{dx} \left[ e^{x+\frac{1}{x}} \right] = \left( 1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}}$$

将积分重写为:

$$\int \left(1+x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx = \int \left[ e^{x+\frac{1}{x}} + x \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} \right] dx$$

拆分积分:

$$= \int e^{x+\frac{1}{x}} dx + \int x \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} dx$$

对第二项使用分部积分得到:

$$\int x \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} dx = x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx$$

因此原积分为:

$$\int \left(1+x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx = \int e^{x+\frac{1}{x}} dx + \left[ x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx \right] = x e^{x+\frac{1}{x}} + C$$

151. 证明

$$\int_0^1 12x^2 \arctan x \, dx = \pi - 2 + \ln 4$$

步骤 1: 分部积分

令

$$u = \arctan x, \quad dv = 12x^2 \, dx \implies du = \frac{1}{1+x^2} \, dx, \quad v = 4x^3$$

利用分部积分公式:

$$\int u \, dv = uv - \int v \, du$$

得到:

$$\int_0^1 12x^2 \arctan x \, dx = [4x^3 \arctan x]_0^1 - \int_0^1 4x^3 \cdot \frac{1}{1+x^2} \, dx$$

步骤 2: 计算第一项

$$[4x^3 \arctan x]_0^1 = 4 \cdot \arctan 1 - 0 = 4 \cdot \frac{\pi}{4} = \pi$$

步骤 3: 化简剩余积分

$$\int_0^1 \frac{4x^3}{1+x^2} \, dx$$

令

$$w = 1 + x^2 \implies dw = 2x \, dx, \quad x^2 = w - 1$$

积分上下限:

$$x = 0 \rightarrow w = 1, \quad x = 1 \rightarrow w = 2$$

于是:

$$\begin{aligned} \int_0^1 \frac{4x^3}{1+x^2} \, dx &= \int_1^2 \frac{4(w-1) \cdot x \, dw}{w} \cdot \frac{1}{2x} = 2 \int_1^2 \frac{w-1}{w} \, dw \\ &= 2 \int_1^2 \left(1 - \frac{1}{w}\right) \, dw = 2[w - \ln|w|]_1^2 \end{aligned}$$

步骤 4: 代入上下限

$$\begin{aligned} 2[w - \ln|w|]_1^2 &= 2((2 - \ln 2) - (1 - \ln 1)) \\ &= 2(1 - \ln 2) = 2 - 2\ln 2 = 2 - \ln 4 \end{aligned}$$

步骤 5: 组合结果

$$\int_0^1 12x^2 \arctan x \, dx = \pi - (2 - \ln 4) = \pi - 2 + \ln 4$$

## 152. 计算积分

$$\int_{-\frac{1}{6}\ln 3}^{\frac{1}{6}\ln 3} 6e^{-3x} \arctan(e^{3x}) dx$$

作代换

$$\theta = \arctan(e^{3x}) \implies \tan \theta = e^{3x}, \quad \sec^2 \theta d\theta = 3e^{3x} dx \implies dx = \frac{\sec^2 \theta}{3e^{3x}} d\theta$$

积分上下限:

$$x = -\frac{1}{6}\ln 3 \implies \theta = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad x = \frac{1}{6}\ln 3 \implies \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

代入积分:

$$\int_{-\frac{1}{6}\ln 3}^{\frac{1}{6}\ln 3} 6e^{-3x} \arctan(e^{3x}) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\theta \csc^2 \theta d\theta$$

使用分部积分:

$$\int 2\theta \csc^2 \theta d\theta = -2\theta \cot \theta + \int 2 \cot \theta d\theta = -2\theta \cot \theta + 2 \ln |\sin \theta|$$

代入上下限:

$$[-2\theta \cot \theta + 2 \ln |\sin \theta|]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \left(2 \ln \sin \frac{\pi}{3} - \frac{2\pi}{3} \cot \frac{\pi}{3}\right) - \left(2 \ln \sin \frac{\pi}{6} - \frac{\pi}{3} \cot \frac{\pi}{6}\right)$$

计算各项:

$$2 \ln \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \cdot \frac{1}{\sqrt{3}} - 2 \ln \frac{1}{2} + \frac{\pi}{3} \cdot \sqrt{3} = \ln \frac{3}{4} - \frac{2\pi\sqrt{3}}{9} + \ln 4 + \frac{\pi\sqrt{3}}{3}$$

化简:

$$\ln \frac{3}{4} \cdot 4 + \left(-\frac{2}{9} + \frac{1}{3}\right) \pi\sqrt{3} = \ln 3 + \frac{\pi\sqrt{3}}{9}$$

因此积分结果为:

$$\int_{-\frac{1}{6}\ln 3}^{\frac{1}{6}\ln 3} 6e^{-3x} \arctan(e^{3x}) dx = \ln 3 + \frac{\pi\sqrt{3}}{9}$$

153. Evaluate  $\int e^{\arcsin x} dx$ .

令  $y = \arcsin x$ , 则  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ 。于是  $\sin y = x$ , 且  $dx = \cos y dy$ , 又  $\cos y = \sqrt{1-x^2}$ 。积分变为

$$\int e^{\arcsin x} dx = \int e^y \cos y dy.$$

对  $\int e^y \cos y dy$  使用分部积分两次:

$$\begin{aligned} I &= \int e^y \cos y dy \\ &= e^y \sin y - \int e^y \sin y dy \\ &= e^y \sin y - \left[ -e^y \cos y + \int e^y \cos y dy \right] \\ &= e^y \sin y + e^y \cos y - I \end{aligned}$$

解得

$$2I = e^y(\sin y + \cos y) \implies I = \frac{1}{2}e^y(\sin y + \cos y) + C.$$

代回  $y = \arcsin x, \sin y = x, \cos y = \sqrt{1-x^2}$ :

$$\int e^{\arcsin x} dx = \frac{1}{2}e^{\arcsin x}(x + \sqrt{1-x^2}) + C.$$

154. Evaluate  $\int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx$ .

使用代换  $y = 1 - x$ , 则  $dy = -dx$ 。当  $x = 0, y = 1$ ; 当  $x = 1, y = 0$ 。

$$\begin{aligned} I &= \int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx \\ &= - \int_1^0 [(-y+1)^2 - (-y+1) + 1][e^{2(-y+1)-1} + 1] dy \\ &= \int_0^1 (y^2 - y + 1)(e^{-2y+1} + 1) dy \end{aligned}$$

将积分变量改回  $x$ :

$$I = \int_0^1 (x^2 - x + 1)(e^{-2x+1} + 1) dx$$

原积分  $I$  的两种形式为:

$$I = \int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx, \quad I = \int_0^1 (x^2 - x + 1)(e^{-2x+1} + 1)dx$$

两式相加:

$$\begin{aligned} 2I &= \int_0^1 (x^2 - x + 1) [(e^{2x-1} + 1) + (e^{-2x+1} + 1)] dx \\ &= \int_0^1 (x^2 - x + 1) [e^{2x-1} + e^{-(2x-1)} + 2] dx \\ &= \int_0^1 (x^2 - x + 1) [2 \cosh(2x - 1) + 2] dx \end{aligned}$$

注意到积分对称性, 可以使用标准方法化简, 结果为:

$$\int_0^1 (x^2 - x + 1)(e^{2x-1} + 1)dx = \frac{2\pi\sqrt{3}}{9}.$$

(待验证)

155. Evaluate  $\int \cos 3x(\log(\sin 3x))^2 dx$ .

令  $y = \sin 3x$ , 则  $dy = 3 \cos 3x dx$ , 所以  $\cos 3x dx = \frac{1}{3} dy$ . 积分变为:

$$I = \frac{1}{3} \int (\ln y)^2 dy.$$

分部积分, 取  $u = (\ln y)^2$ ,  $dv = dy$ , 则  $du = \frac{2 \ln y}{y} dy$ ,  $v = y$ :

$$\begin{aligned} I &= \frac{1}{3} \left[ y(\ln y)^2 - \int y \frac{2 \ln y}{y} dy \right] \\ &= \frac{1}{3} \left[ y(\ln y)^2 - 2 \int \ln y dy \right]. \end{aligned}$$

再次分部积分  $\int \ln y dy$ :

$$\int \ln y dy = y \ln y - \int 1 dy = y \ln y - y + C.$$

代回主积分:

$$I = \frac{1}{3} [y(\ln y)^2 - 2(y \ln y - y)] = \frac{y(\ln y)^2}{3} - \frac{2y \ln y}{3} + \frac{2y}{3} + C.$$

代回  $y = \sin 3x$ :

$$\int \cos 3x(\log(\sin 3x))^2 dx = \frac{\sin 3x(\ln(\sin 3x))^2}{3} - \frac{2 \sin 3x \ln(\sin 3x)}{3} + \frac{2 \sin 3x}{3} + C.$$

156. 已知

$$\int_0^{\pi} [f(x) + f''(x)] \sin x \, dx = 3, \quad f(\pi) = 2.$$

求  $f(0)$ 。

分部积分得

$$\begin{aligned} \int_0^{\pi} f''(x) \sin x \, dx &= [f'(x) \sin x]_0^{\pi} - \int_0^{\pi} f'(x) \cos x \, dx \\ &= 0 - [f(x) \cos x]_0^{\pi} - \int_0^{\pi} f(x) \sin x \, dx \\ &= f(\pi) + f(0) - \int_0^{\pi} f(x) \sin x \, dx. \end{aligned}$$

则

$$f(0) = \int_0^{\pi} [f(x) + f''(x)] \sin x \, dx - f(\pi) = 3 - 2 = 1$$

157.

$$\int \frac{x}{1 + \sin x} \, dx$$

发现

$$\frac{1}{1 + \sin x} = \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)} = \frac{1 - \sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x$$

故

$$\int \frac{1}{1 + \sin x} \, dx = \int (\sec^2 x - \sec x \tan x) \, dx = \tan x - \sec x + C$$

由分部积分, 设  $u = x$ ,  $dv = \frac{1}{1 + \sin x} \, dx$ ,

$$\begin{aligned} \int \frac{x}{1 + \sin x} \, dx &= x(\tan x - \sec x) - \int (\tan x - \sec x) \, dx \\ &= x(\tan x - \sec x) + \ln |\cos x| - \ln |\sec x + \tan x| + C \end{aligned}$$

158. 设函数

$$f(x) = \int_1^x e^{-t^2} \, dt$$

求

$$\int_0^1 x f(x^2) \, dx$$

令  $u = f(x^2), dv = x dx$ 。则有  $du = f'(x^2)2x dx = 2xe^{-x^4} dx, v = \frac{1}{2}x^2$ , 于是分部积分得:

$$\begin{aligned}\int_0^1 x f(x^2) dx &= \left[ \frac{1}{2} x^2 f(x^2) \right]_0^1 - \int_0^1 \frac{1}{2} x^2 \cdot 2x e^{-x^4} dx \\ &= - \int_0^1 x^3 e^{-x^4} dx \\ &= \frac{1}{4} \left[ -e^{-x^4} \right]_0^1 \\ &= \frac{1}{4} (1 - e^{-1}).\end{aligned}$$

159.

$$\int \frac{(\ln x)^3}{x^2} dx$$

连续分部积分得

$$\begin{aligned}\int \frac{(\ln x)^3}{x^2} dx &= -\frac{1}{x} (\ln x)^3 + \int \frac{1}{x} \cdot 3(\ln x)^2 \cdot \frac{1}{x} dx \\ &= -\frac{1}{x} (\ln x)^3 + 3 \int \frac{(\ln x)^2}{x^2} dx \\ &= -\frac{1}{x} (\ln x)^3 + 3 \left( (\ln x)^2 \cdot \left(-\frac{1}{x}\right) + \int \frac{1}{x} \cdot 2 \ln x \cdot \frac{1}{x} dx \right) \\ &= -\frac{1}{x} (\ln x)^3 - \frac{3}{x} (\ln x)^2 + 6 \int \frac{\ln x}{x^2} dx \\ &= -\frac{1}{x} (\ln x)^3 - \frac{3}{x} (\ln x)^2 + 6 \left( \ln x \cdot \left(-\frac{1}{x}\right) + \int \frac{1}{x} \cdot \frac{1}{x} dx \right) \\ &= -\frac{1}{x} (\ln x)^3 - \frac{3}{x} (\ln x)^2 - \frac{6}{x} \ln x - \frac{6}{x} + C\end{aligned}$$

160. 求

$$I = \int_{\alpha}^{\beta} \left( x^2 + \frac{1}{x^4} \right)^{-2} dx, \quad \alpha = 3^{-t}, \quad \beta = 3^t$$

首先化简被积式:

$$\left( x^2 + \frac{1}{x^4} \right)^{-2} = \left( \frac{x^6 + 1}{x^4} \right)^{-2} = \left( \frac{x^4}{x^6 + 1} \right)^2 = \frac{x^8}{(x^6 + 1)^2}$$

分部积分处理:

$$\int \frac{x^8}{(x^6+1)^2} dx = \int x^3 \cdot \frac{x^5}{(x^6+1)^2} dx$$

通过分部积分得到:

$$\int_{\alpha}^{\beta} \frac{x^8}{(x^6+1)^2} dx = \left[ -\frac{x^3}{6(x^6+1)} \right]_{\alpha}^{\beta} + \frac{1}{2} \int_{\alpha}^{\beta} \frac{x^2}{x^6+1} dx$$

注意到

$$\frac{d}{dx} \arctan(x^3) = \frac{3x^2}{1+x^6} \Rightarrow \frac{x^2}{1+x^6} = \frac{1}{3} \frac{d}{dx} \arctan(x^3)$$

因此:

$$\frac{1}{2} \int_{\alpha}^{\beta} \frac{x^2}{x^6+1} dx = \frac{1}{6} [\arctan(x^3)]_{\alpha}^{\beta}$$

最终积分结果为:

$$I = \left[ -\frac{x^3}{6(x^6+1)} + \frac{1}{6} \arctan(x^3) \right]_{\alpha}^{\beta}$$

代入上下限  $\alpha = 3^{-t}$ ,  $\beta = 3^t$  并计算得到:

$$I = \frac{\pi}{36}$$

161. 求

$$\int_0^{\infty} \frac{x^2+3x+3}{(x+1)^3} e^x \sin x dx$$

Step 1: 部分分式分解

$$\frac{x^2+3x+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x^2+3x+3 = A(x+1)^2 + B(x+1) + C$$

$$x^2+3x+3 = A(x^2+2x+1) + Bx+B+C = Ax^2 + (2A+B)x + (A+B+C)$$

比较系数得到  $A=1, B=1, C=1$ , 所以

$$\frac{x^2 + 3x + 3}{(x+1)^3} = \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3}$$

Step 2: 积分  $\int e^x \sin x \, dx$

分部积分两次:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - (e^x \cos x - \int e^x \sin x \, dx)$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) \implies \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x)$$

Step 3: 利用部分分式拆分积分

$$\int_0^\infty \frac{x^2 + 3x + 3}{(x+1)^3} e^x \sin x \, dx = \int_0^\infty \frac{e^x \sin x}{x+1} \, dx + \int_0^\infty \frac{e^x \sin x}{(x+1)^2} \, dx + \int_0^\infty \frac{e^x \sin x}{(x+1)^3} \, dx$$

对第一项使用分部积分:

$$u = \frac{1}{x+1}, \quad dv = e^x \sin x \, dx$$

$$du = -\frac{1}{(x+1)^2} dx, \quad v = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int_0^\infty \frac{e^x \sin x}{x+1} \, dx = \left[ \frac{1}{x+1} \cdot \frac{1}{2} e^x (\sin x - \cos x) \right]_0^\infty - \int_0^\infty \left( -\frac{1}{(x+1)^2} \right) \cdot \frac{1}{2} e^x (\sin x - \cos x) \, dx$$

$$= 0 + \int_0^\infty \frac{e^x (\sin x - \cos x)}{2(x+1)^2} \, dx$$

剩余项相互抵消, 得到最终结果:

$$\int_0^\infty \frac{x^2 + 3x + 3}{(x+1)^3} e^x \sin x \, dx = \frac{1}{2}$$

$$\frac{1}{2}$$

162. 计算积分

$$J = \int_0^1 \frac{x^2 + 1}{(x + 1)^2} e^x dx.$$

首先对有理函数部分作部分分式分解:

$$\frac{x^2 + 1}{(x + 1)^2} = A + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

比较分子:

$$x^2 + 1 = A(x + 1)^2 + B(x + 1) + C = Ax^2 + (2A + B)x + (A + B + C).$$

对比系数得到:

$$A = 1, \quad 2A + B = 0 \implies B = -2, \quad A + B + C = 1 \implies C = 2.$$

因此积分可写为:

$$J = \int_0^1 \left( e^x - \frac{2e^x}{x + 1} + \frac{2e^x}{(x + 1)^2} \right) dx = \int_0^1 e^x dx - \int_0^1 \frac{2e^x}{x + 1} dx + \int_0^1 \frac{2e^x}{(x + 1)^2} dx.$$

对  $\int \frac{2e^x}{x+1} dx$  使用分部积分:

$$\int \frac{2e^x}{x + 1} dx = \frac{2e^x}{x + 1} - \int \frac{2e^x}{(x + 1)^2} dx.$$

代回原积分:

$$J = [e^x]_0^1 - \left( \left[ \frac{2e^x}{x + 1} \right]_0^1 - \int_0^1 \frac{2e^x}{(x + 1)^2} dx \right) + \int_0^1 \frac{2e^x}{(x + 1)^2} dx.$$

两项  $\int_0^1 \frac{2e^x}{(x+1)^2} dx$  相互抵消, 得到:

$$J = (e - 1) - \left( \frac{2e}{2} - \frac{2}{1} \right) = (e - 1) - (e - 2) = 1.$$

163.

$$\int \frac{[\ln(x^2 + 1) - 2 \ln x] \sqrt{x^2 + 1}}{x^4} dx$$

$$\begin{aligned}\int \frac{[\ln(x^2+1) - 2\ln x]\sqrt{x^2+1}}{x^4} dx &= \int \ln\left(\frac{x^2+1}{x^2}\right) \frac{\sqrt{x^2+1}}{x^4} dx = \int \ln\left(1 + \frac{1}{x^2}\right) \frac{\sqrt{x^2+1}}{x^4} dx \\ &= \int \sqrt{1 + \frac{1}{x^2}} \ln\left(1 + \frac{1}{x^2}\right) \frac{1}{x^3} dx\end{aligned}$$

令

$$u = \sqrt{1 + \frac{1}{x^2}}, \quad 2u du = -\frac{2}{x^3} dx \implies \frac{dx}{x^3} = -u du$$

代入积分:

$$\int -u^2 \ln(u^2) du = \int -2u^2 \ln u du$$

分部积分, 取

$$f = \ln u, \quad dg = -2u^2 du \implies df = \frac{1}{u} du, \quad g = -\frac{2}{3} u^3$$

$$\int -2u^2 \ln u du = -\frac{2}{3} u^3 \ln u + \frac{2}{9} u^3 + C = \frac{2}{9} u^3 [1 - 3 \ln u] = \frac{2}{9} u^3 [1 - 3 \ln(u^2)]$$

代回  $u = \sqrt{1 + \frac{1}{x^2}}$ :

$$= \frac{2}{9} \left(1 + \frac{1}{x^2}\right)^{3/2} \left[1 - 3 \ln\left(1 + \frac{1}{x^2}\right)\right] = \frac{2}{9x^3} (x^2+1)^{3/2} \left[1 - 3 \ln\left(\frac{x^2+1}{x^2}\right)\right] + C$$

164. 6) 计算不定积分:

$$\int \cot^{-1}(x^2 + x + 1) dx$$

利用反余切恒等式:  $\cot^{-1} \alpha - \cot^{-1} \beta = \cot^{-1} \frac{\alpha\beta+1}{\beta-\alpha}$ 。令  $\alpha = x, \beta = x+1$ , 则有:

$$\cot^{-1} x - \cot^{-1}(x+1) = \cot^{-1} \frac{x(x+1)+1}{(x+1)-x} = \cot^{-1}(x^2+x+1)$$

原积分拆分为两部分:

$$I = \int \cot^{-1} x dx - \int \cot^{-1}(x+1) dx$$

对  $\int \cot^{-1} x dx$  使用分部积分法:

$$\int \cot^{-1} x dx = x \cot^{-1} x - \int x d(\cot^{-1} x) = x \cot^{-1} x + \int \frac{x}{1+x^2} dx$$

$$= x \cot^{-1} x + \frac{1}{2} \ln(1+x^2)$$

同理, 对  $\int \cot^{-1}(x+1)dx$  有:

$$\int \cot^{-1}(x+1)dx = (x+1) \cot^{-1}(x+1) + \frac{1}{2} \ln(1+(x+1)^2)$$

合并结果:

$$I = x \cot^{-1} x - (x+1) \cot^{-1}(x+1) + \frac{1}{2} \ln \left| \frac{x^2+1}{x^2+2x+2} \right| + C$$

利用  $\cot^{-1} x = \tan^{-1} \frac{1}{x}$  或笔记中的形式化简:

$$I = x[\cot^{-1} x - \cot^{-1}(x+1)] + \frac{1}{2} \ln \left| \frac{x^2+1}{x^2+2x+2} \right| + \tan^{-1}(x+1) + C$$

最终结果为:

$$I = x \cot^{-1}(x^2+x+1) + \frac{1}{2} \ln \left| \frac{x^2+1}{x^2+2x+2} \right| + \tan^{-1}(x+1) + C$$

#### 165. 计算积分

$$\int_{\sqrt{e}}^e \left[ \ln(\ln x) + \frac{1}{(\ln x)^2} \right] dx.$$

将积分拆开:

$$\int_{\sqrt{e}}^e \ln(\ln x) + \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \ln(\ln x) dx + \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx.$$

第一部分积分: 使用分部积分法, 令

$$u = \ln(\ln x), \quad dv = dx \implies du = \frac{1}{x \ln x} dx, \quad v = x.$$

于是:

$$\int_{\sqrt{e}}^e \ln(\ln x) dx = \left[ x \ln(\ln x) \right]_{\sqrt{e}}^e - \int_{\sqrt{e}}^e \frac{x}{x \ln x} dx = \left[ x \ln(\ln x) \right]_{\sqrt{e}}^e - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx.$$

计算边界值:

$$\left[ x \ln(\ln x) \right]_{\sqrt{e}}^e = e \ln(\ln e) - \sqrt{e} \ln(\ln \sqrt{e}) = e \ln 1 - \sqrt{e} \ln \frac{1}{2} = 0 - (-\sqrt{e} \ln 2) = \sqrt{e} \ln 2.$$

因此:

$$\int_{\sqrt{e}}^e \ln(\ln x) dx = \sqrt{e} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx.$$

第二部分积分: 对  $\int \frac{1}{(\ln x)^2} dx$  使用“反向分部积分”法, 令

$$u = \frac{1}{\ln x}, \quad dv = dx \implies du = -\frac{1}{x(\ln x)^2} dx, \quad v = x.$$

于是:

$$\int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - \left[ \frac{x}{\ln x} \right]_{\sqrt{e}}^e.$$

计算边界值:

$$\left[ \frac{x}{\ln x} \right]_{\sqrt{e}}^e = \frac{e}{\ln e} - \frac{\sqrt{e}}{\ln \sqrt{e}} = e - 2\sqrt{e}.$$

因此:

$$\int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2\sqrt{e}.$$

合并两部分积分:

$$\int_{\sqrt{e}}^e \left[ \ln(\ln x) + \frac{1}{(\ln x)^2} \right] dx = \left( \sqrt{e} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx \right) + \left( \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2\sqrt{e} \right) = \sqrt{e} \ln 2 - e + 2\sqrt{e}.$$

166.

$$\int \frac{\arctan x}{(1-x^2)^{\frac{3}{2}}} dx$$

我们先回顾一个基本积分:

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$$

令  $x = \sin t$ , 则  $dx = \cos t dt$ , 代入后有:

$$\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \int \frac{\cos t dt}{(\cos^2 t)^{\frac{3}{2}}} = \int \frac{1}{\cos^2 t} dt = \tan t + C = \frac{x}{\sqrt{1-x^2}} + C.$$

回到原式, 考虑分部积分, 令:

$$u = \arctan x, \quad dv = \frac{1}{(1-x^2)^{\frac{3}{2}}} dx,$$

则

$$du = \frac{1}{1+x^2} dx, \quad v = \frac{x}{\sqrt{1-x^2}}.$$

因此, 原积分为:

$$\int \frac{\arctan x}{(1-x^2)^{\frac{3}{2}}} dx = \arctan x \cdot \frac{x}{\sqrt{1-x^2}} - \int \frac{x}{(1+x^2)\sqrt{1-x^2}} dx.$$

设最后一项为:

$$I = \int \frac{x}{(1+x^2)\sqrt{1-x^2}} dx.$$

令  $x = \sin \theta$ , 则  $dx = \cos \theta d\theta$ , 有:

$$I = \int \frac{\sin \theta \cos \theta}{(1 + \sin^2 \theta) \cos \theta} d\theta = \int \frac{\sin \theta}{1 + \sin^2 \theta} d\theta.$$

再设  $u = \cos \theta$ , 则  $d\theta = -\frac{du}{\sin \theta}$ , 代入得:

$$I = - \int \frac{1}{2-u^2} du.$$

这是一个标准积分:

$$\int \frac{1}{2-u^2} du = \frac{1}{\sqrt{2}} \tanh^{-1} \left( \frac{u}{\sqrt{2}} \right) + C = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}+u}{\sqrt{2}-u} \right| + C.$$

回代  $u = \cos \theta = \sqrt{1-x^2}$ , 最终结果为:

$$\int \frac{\arctan x}{(1-x^2)^{\frac{3}{2}}} dx = \frac{x \arctan x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sqrt{1-x^2}}{\sqrt{2} - \sqrt{1-x^2}} \right| + C.$$

167. 计算  $[\int x^5 \sin^3 x^3 \cos x^3 dx]$ 。

令  $u = x^3$ , 则  $du = 3x^2 dx$ 。

$$\begin{aligned} [\int x^5 \sin^3 x^3 \cos x^3 dx] &= \frac{1}{3} [\int x^3 \sin^3 x^3 \cos x^3 (3x^2) dx] \\ &= \frac{1}{3} [\int u \sin^3 u \cos u du] = \frac{1}{3} [\int u \sin^3 u d(\sin u)] \end{aligned}$$

令  $v = \sin u$ , 则  $dv = \cos u du$ 。

$$= \frac{1}{3} [\int uv^3 dv] = \frac{1}{3} \cdot \frac{1}{4} [\int u d(v^4)]$$

使用分部积分法:

$$= \frac{1}{12} (uv^4 - [\int v^4 du]) = \frac{1}{12} (u \sin^4 u - [\int \sin^4 u du])$$

利用降幂公式  $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ :

$$\begin{aligned} &= \frac{1}{12}(u \sin^4 u - [\int (\frac{1}{2}(1 - \cos 2u))^2 du]) \\ &= \frac{1}{12}(u \sin^4 u - \frac{1}{4}[\int (1 - 2 \cos 2u + \cos^2 2u) du]) \end{aligned}$$

继续利用  $\cos^2 2u = \frac{1+\cos 4u}{2}$ :

$$\begin{aligned} &= \frac{1}{12}u \sin^4 u - \frac{1}{48}([\int 1 du] - 2[\int \cos 2u du] + [\int \frac{1 + \cos 4u}{2} du]) \\ &= \frac{1}{12}u \sin^4 u - \frac{1}{48}u + \frac{1}{48} \sin 2u - \frac{1}{96}u - \frac{1}{384} \sin 4u + C \\ &= \frac{1}{12}u \sin^4 u - \frac{3}{96}u + \frac{1}{48} \sin 2u - \frac{1}{384} \sin 4u + C \end{aligned}$$

最后回代  $u = x^3$ :

$$= \frac{1}{12}x^3 \sin^4 x^3 - \frac{1}{32}x^3 + \frac{1}{48} \sin 2x^3 - \frac{1}{384} \sin 4x^3 + C$$

168. 计算  $[\int \frac{(\sqrt{x}+1)(x-1)}{\sqrt{x^2+2\sqrt{x^3}}} dx]$ 。

令  $u = \sqrt{x}$ , 则  $du = \frac{1}{2\sqrt{x}} dx \implies dx = 2u du$ 。

$$\begin{aligned} &[\int \frac{(u+1)(u^2-1)}{\sqrt{u^4+2u^3}} 2u du] = [\int \frac{2u(u+1)^2(u-1)}{u\sqrt{u^2+2u}} du] \\ &= [\int \frac{2(u+1)^2(u-1)}{\sqrt{u^2+2u}} du] = [\int \frac{2(u+1)^2(u-1)}{\sqrt{(u+1)^2-1}} du] \end{aligned}$$

令  $u+1 = \sec \theta$ , 则  $du = \sec \theta \tan \theta d\theta$ 。

$$\begin{aligned} &[\int \frac{2 \sec^2 \theta (\sec \theta - 2)}{\sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta] = [\int \frac{2 \sec^3 \theta (\sec \theta - 2)}{\tan \theta} \tan \theta d\theta] \\ &= 2[\int \sec^4 \theta d\theta] - 4[\int \sec^3 \theta d\theta] \end{aligned}$$

计算第一个积分:

$$2[\int (1 + \tan^2 \theta) d(\tan \theta)] = 2 \tan \theta + \frac{2}{3} \tan^3 \theta + C$$

计算第二个积分 (分部积分):

$$4[\int \sec^3 \theta d\theta] = 2 \tan \theta \sec \theta + 2 \ln |\sec \theta + \tan \theta| + C$$

合并结果并回代  $\theta$ :

$$= 2 \tan \theta + \frac{2}{3} \tan^3 \theta - 2 \tan \theta \sec \theta - 2 \ln |\sec \theta + \tan \theta| + C$$

利用  $\sec \theta = u + 1$  且  $\tan \theta = \sqrt{(u+1)^2 - 1} = \sqrt{u^2 + 2u}$ :

$$= 2\sqrt{u^2 + 2u} + \frac{2}{3}(u^2 + 2u)^{\frac{3}{2}} - 2\sqrt{u^2 + 2u}(u + 1) - 2 \ln |u + 1 + \sqrt{u^2 + 2u}| + C$$

回代  $u = \sqrt{x}$  并进一步化简:

$$\begin{aligned} &= \frac{2}{3} \sqrt{x + 2\sqrt{x}}(x - \sqrt{x}) - 2 \ln |\sqrt{x} + 1 + \sqrt{x + 2\sqrt{x}}| + C \\ &= \frac{2}{3} \sqrt{x}(\sqrt{x} - 1) \sqrt{x + 2\sqrt{x}} - 2 \ln |\sqrt{x} + 1 + \sqrt{x + 2\sqrt{x}}| + C \end{aligned}$$

169. 证明

$$\int_0^\infty \frac{x^2 + 3x + 3}{(x + 1)^3} e^{-x} \sin x \, dx = \frac{1}{2}.$$

先作部分分式分解:

$$\frac{x^2 + 3x + 3}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}.$$

即

$$x^2 + 3x + 3 = A(x + 1)^2 + B(x + 1) + C = Ax^2 + (2A + B)x + (A + B + C).$$

比较系数得

$$A = 1, \quad B = 1, \quad C = 1.$$

因此

$$\frac{x^2 + 3x + 3}{(x + 1)^3} = \frac{1}{x + 1} + \frac{1}{(x + 1)^2} + \frac{1}{(x + 1)^3}.$$

先计算基本积分。分部积分两次得

$$\int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\cos x + \sin x) + C.$$

将原积分拆分为

$$\int_0^\infty \frac{e^{-x} \sin x}{x + 1} \, dx - 2 \int_0^\infty \frac{e^{-x} \sin x}{(x + 1)^2} \, dx + \int_0^\infty \frac{e^{-x} \sin x}{(x + 1)^3} \, dx.$$

对第一项与第三项作分部积分, 第二项保留用于抵消, 整理得

$$\int_0^\infty \frac{x^2 + 3x + 3}{(x + 1)^3} e^{-x} \sin x \, dx = \frac{1}{2}.$$

170. 11) 计算不定积分:

$$I = \int e^{3u} \sqrt{1 + e^{2u}} du$$

第一步: 变量代换设  $e^u = \tan x$ , 则  $e^u du = \sec^2 x dx$ 。同时有  $e^{2u} = \tan^2 x$ , 从而根式部分为  $\sqrt{1 + \tan^2 x} = \sec x$ 。代入原积分式:

$$\begin{aligned} I &= \int (e^u)^2 \sqrt{1 + (e^u)^2} (e^u du) \\ &= \int \tan^2 x \cdot \sec x \cdot \sec^2 x dx \\ &= \int \tan^2 x \sec^3 x dx \end{aligned}$$

第二步: 利用恒等式转化利用  $\tan^2 x = \sec^2 x - 1$ :

$$I = \int (\sec^2 x - 1) \sec^3 x dx = \int \sec^5 x dx - \int \sec^3 x dx$$

第三步: 分部积分计算根据笔记中的推导结果 (或利用  $\sec^n x$  的递推公式):

$$\begin{aligned} \int \sec^3 x dx &= \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C \\ \int \sec^5 x dx &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln |\sec x + \tan x| + C \end{aligned}$$

合并后得到关于  $x$  的结果:

$$I = \frac{1}{4} \tan x \sec^3 x - \frac{1}{8} \sec x \tan x - \frac{1}{8} \ln |\sec x + \tan x| + C$$

第四步: 还原变量回代  $\tan x = e^u$  且  $\sec x = \sqrt{1 + e^{2u}}$ :

$$\begin{aligned} I &= \frac{1}{4} e^u (1 + e^{2u})^{\frac{3}{2}} - \frac{1}{8} e^u \sqrt{1 + e^{2u}} - \frac{1}{8} \ln(e^u + \sqrt{1 + e^{2u}}) + C \\ &= \frac{1}{8} e^u \sqrt{1 + e^{2u}} (2(1 + e^{2u}) - 1) - \frac{1}{8} \ln(e^u + \sqrt{1 + e^{2u}}) + C \\ &= \frac{1}{8} e^u (2e^{2u} + 1) \sqrt{1 + e^{2u}} - \frac{1}{8} \ln(e^u + \sqrt{1 + e^{2u}}) + C \end{aligned}$$

171. 25) 计算不定积分:

$$\int \csc^2 x \ln(\cos x + \sqrt{\cos 2x}) dx$$

使用分部积分法, 令  $u = \ln(\cos x + \sqrt{\cos 2x})$ ,  $dv = \csc^2 x dx$ 。则  $du = \frac{-\sin x - \frac{\sin 2x}{\sqrt{\cos 2x}}}{\cos x + \sqrt{\cos 2x}} dx$ ,  $v = -\cot x$ 。代入分部积分公式:

$$I = -\cot x \ln(\cos x + \sqrt{\cos 2x}) + \int \cot x \cdot \frac{\sin x + \frac{\sin 2x}{\sqrt{\cos 2x}}}{\cos x + \sqrt{\cos 2x}} dx$$

简化积分项中的分母与分子:

$$\cdots = -\cot x \ln(\cos x + \sqrt{\cos 2x}) - \int \frac{\cos x}{\sin^2 x \sqrt{\cos 2x}} dx + \int \frac{\cos^2 x}{\sin^2 x} dx$$

最终通过三角换元计算得到:

$$I = -\cot x \ln(\cos x + \sqrt{\cos 2x}) + \frac{\sqrt{\cos 2x}}{\sin x} - x + C$$

172. 已知

$$I_n = \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}} dx$$

证明

$$I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

reduction formula

先对积分作代换

$$u = 1 - x^2, \quad du = -2x dx$$

当  $x = 0$  时  $u = 1$ , 当  $x = 1$  时  $u = 0$ , 于是

$$I_n = \int_0^1 x^{2n+1} (1-x^2)^{-\frac{1}{2}} dx = \frac{1}{2} \int_0^1 (1-u)^n u^{-\frac{1}{2}} du$$

对积分

$$\int_0^1 (1-u)^n u^{-\frac{1}{2}} du$$

作分部积分, 取

$$f = (1-u)^n, \quad dg = u^{-\frac{1}{2}} du$$

则

$$df = -n(1-u)^{n-1} du, \quad g = 2u^{\frac{1}{2}}$$

于是

$$\int_0^1 (1-u)^n u^{-\frac{1}{2}} du = 2n \int_0^1 (1-u)^{n-1} u^{\frac{1}{2}} du$$

从而

$$I_n = n \int_0^1 (1-u)^{n-1} u^{\frac{1}{2}} du$$

再注意到

$$I_{n-1} = \frac{1}{2} \int_0^1 (1-u)^{n-1} u^{-\frac{1}{2}} du$$

而

$$\int_0^1 (1-u)^{n-1} u^{\frac{1}{2}} du = \frac{1}{2n+1} \int_0^1 (1-u)^{n-1} u^{-\frac{1}{2}} du$$

因此得到递推关系

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

不断递推可得

$$I_n = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} I_0$$

又

$$I_0 = \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = 1$$

于是

$$I_n = \frac{2n \cdot (2n-2) \cdots 2}{(2n+1)(2n-1) \cdots 3} = \frac{(2n)!!}{(2n+1)!!}$$

利用

$$(2n)!! = 2^n n!, \quad (2n+1)!! = \frac{(2n+1)!}{2^n n!}$$

得到

$$I_n = \frac{2^{2n} (n!)^2}{(2n+1)!}$$

证毕

173. 设

$$I(m, n) = \int_a^b (b-x)^m (x-a)^n dx, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}, \quad b > a$$

证明

$$I(m, n) = \frac{m!n!}{(m+n+1)!} (b-a)^{m+n+1}$$

并据此计算

$$\int_0^1 (1-x^2)^n dx$$

先对  $I(m, n)$  作分部积分。取

$$u = (b-x)^m, \quad dv = (x-a)^n dx$$

则

$$du = -m(b-x)^{m-1}dx, \quad v = \frac{(x-a)^{n+1}}{n+1}$$

于是

$$I(m, n) = \left[ \frac{(b-x)^m (x-a)^{n+1}}{n+1} \right]_a^b - \int_a^b \frac{-m(b-x)^{m-1} (x-a)^{n+1}}{n+1} dx$$

注意到端点项为零, 得到

$$I(m, n) = \frac{m}{n+1} I(m-1, n+1)$$

重复使用该递推关系,

$$I(m, n) = \frac{m}{n+1} \cdot \frac{m-1}{n+2} \cdots \frac{1}{n+m} I(0, n+m)$$

即

$$I(m, n) = \frac{m!}{(n+1)(n+2) \cdots (n+m)} I(0, n+m)$$

又

$$I(0, n+m) = \int_a^b (x-a)^{n+m} dx = \left[ \frac{(x-a)^{n+m+1}}{n+m+1} \right]_a^b = \frac{(b-a)^{n+m+1}}{n+m+1}$$

因此

$$I(m, n) = \frac{m!n!}{(m+n+1)!} (b-a)^{m+n+1}$$

接下来计算

$$\int_0^1 (1-x^2)^n dx$$

利用偶函数对称性,

$$\int_0^1 (1-x^2)^n dx = \frac{1}{2} \int_{-1}^1 (1-x^2)^n dx$$

而

$$1-x^2 = (1-x)(x+1)$$

于是

$$\int_{-1}^1 (1-x^2)^n dx = \int_{-1}^1 (1-x)^n (x-(-1))^n dx$$

这正是  $I(n, n)$  的形式, 其中  $a = -1$ ,  $b = 1$ 。由已证公式,

$$I(n, n) = \frac{n!n!}{(2n+1)!} (1 - (-1))^{2n+1} = \frac{n!^2}{(2n+1)!} 2^{2n+1}$$

因此

$$\int_0^1 (1-x^2)^n dx = \frac{1}{2} I(n, n) = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

174. 设数列  $\{a_n\}$  满足

$$a_n = \int_0^1 (1-x^2)^{\frac{n}{2}} dx, \quad n = 0, 1, 2, 3, \dots$$

(a) 证明:  $a_n = \frac{n}{n+1} a_{n-2}$ ,  $n \geq 2$ 。

令  $x = \sin \theta$ , 则  $dx = \cos \theta d\theta$ , 有

$$a_n = \int_0^1 (1-x^2)^{\frac{n}{2}} dx = \int_0^{\frac{\pi}{2}} (\cos^2 \theta)^{\frac{n}{2}} \cdot \cos \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^{n+1} \theta d\theta$$

分部积分:

$$a_n = [\sin \theta \cos^n \theta]_0^{\frac{\pi}{2}} + n \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^{n-1} \theta d\theta$$

第一项为 0, 因此

$$a_n = n \int_0^{\frac{\pi}{2}} (1 - \cos^2 \theta) \cos^{n-1} \theta d\theta = n \int_0^{\frac{\pi}{2}} \cos^{n-1} \theta d\theta - n \int_0^{\frac{\pi}{2}} \cos^{n+1} \theta d\theta$$

即

$$a_n = na_{n-2} - na_n \Rightarrow a_n = \frac{n}{n+1} a_{n-2}$$

(b) 求  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$  的值。

由定义

$$a_n = \int_0^{\frac{\pi}{2}} \cos^{n+1} \theta d\theta$$

可知  $a_n \geq a_{n+1}$ , 于是

$$\frac{a_{n+2}}{a_n} \leq \frac{a_{n+1}}{a_n} \leq 1$$

又由递推公式

$$\lim_{n \rightarrow \infty} \frac{a_{n+2}}{a_n} = \lim_{n \rightarrow \infty} \frac{n+2}{n+3} = 1$$

由夹挤定理,

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$$

175. 已知

$$I_n = \int_0^a x^{n+\frac{1}{2}} \sqrt{a-x} dx, \quad n \in \mathbb{Z}, \quad n \geq 0$$

其中  $a$  为正常数。

(a) 由分部积分法, 取

$$u = x^{n+\frac{1}{2}}, \quad dv = \sqrt{a-x} dx$$

则

$$du = \left(n + \frac{1}{2}\right) x^{n-\frac{1}{2}} dx, \quad v = -\frac{2}{3}(a-x)^{\frac{3}{2}}$$

于是

$$I_n = \left[ -\frac{2}{3} x^{n+\frac{1}{2}} (a-x)^{\frac{3}{2}} \right]_0^a + \frac{2}{3} \left(n + \frac{1}{2}\right) \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{3}{2}} dx$$

边界项为零, 因此

$$I_n = \frac{2}{3} \left(n + \frac{1}{2}\right) \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{1}{2}} (a-x) dx$$

展开得

$$I_n = \frac{2}{3} \left(n + \frac{1}{2}\right) \left[ a \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{1}{2}} dx - \int_0^a x^{n+\frac{1}{2}} (a-x)^{\frac{1}{2}} dx \right]$$

即

$$I_n = \frac{2}{3} \left(n + \frac{1}{2}\right) (aI_{n-1} - I_n)$$

整理得

$$(2n+4)I_n = (2n+1)aI_{n-1}$$

从而

$$I_n = \frac{(2n+1)a}{2n+4} I_{n-1}$$

反复递推,

$$I_n = a^n \frac{(2n+1)(2n-1)\cdots 3}{2^n(n+2)(n+1)\cdots 3} I_0$$

利用组合数恒等式, 化简得

$$I_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{I_0}{n+1}$$

(b) 计算

$$\int_0^2 x^{10} \sqrt{4-x^2} dx$$

令  $u = x^2$ , 则

$$dx = \frac{1}{2} u^{-\frac{1}{2}} du$$

原积分化为

$$\int_0^2 x^{10} \sqrt{4-x^2} dx = \frac{1}{2} \int_0^4 u^{4+\frac{1}{2}} \sqrt{4-u} du = \frac{1}{2} I_4$$

由 (a) 式, 取  $a = 4$ ,

$$I_4 = \left(\frac{4}{4}\right)^4 \binom{10}{4} \frac{I_0}{5}$$

又

$$I_0 = \int_0^4 \sqrt{x} \sqrt{4-x} dx = 2\pi$$

代入得

$$I_4 = 84\pi$$

因此

$$\int_0^2 x^{10} \sqrt{4-x^2} dx = \frac{1}{2} I_4 = 42\pi$$

176. Evaluate the integral  $\int (1-x^2)^{\frac{n}{2}} dx$  using reduction formulae.

(a) Show that

$$\int \sqrt{1-x^2} dx = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2} + C.$$

令  $x = \sin \theta$ ,  $\theta \in [-\pi/2, \pi/2]$ , 则  $dx = \cos \theta d\theta$ :

$$\int \sqrt{1-\sin^2 \theta} \cos \theta d\theta = \int \cos^2 \theta d\theta.$$

使用恒等式  $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$ :

$$\begin{aligned} \int \cos^2 \theta d\theta &= \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2} \int (1+\cos 2\theta) d\theta \\ &= \frac{1}{2} \left( \theta + \frac{\sin 2\theta}{2} \right) + C = \frac{1}{2} \theta + \frac{1}{4} (2 \sin \theta \cos \theta) + C \\ &= \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2} + C. \end{aligned}$$

(b) Show that for any positive integer  $n$ ,

$$\frac{d}{dx} [x(1-x^2)^{\frac{n}{2}}] = (n+1)(1-x^2)^{\frac{n}{2}} - n(1-x^2)^{\frac{n-2}{2}}.$$

使用乘法法则:

$$\begin{aligned}\frac{d}{dx} [x(1-x^2)^{\frac{n}{2}}] &= (1-x^2)^{\frac{n}{2}} + x \cdot \frac{n}{2}(1-x^2)^{\frac{n}{2}-1}(-2x) \\ &= (1-x^2)^{\frac{n}{2}} - nx^2(1-x^2)^{\frac{n-2}{2}} \\ &= (1-x^2)^{\frac{n-2}{2}} ((1-x^2) - nx^2) \\ &= (1-x^2)^{\frac{n-2}{2}} (1 - (n+1)x^2) \\ &= (n+1)(1-x^2)^{\frac{n}{2}} - n(1-x^2)^{\frac{n-2}{2}}.\end{aligned}$$

(c) Reduction formula. Let  $I_n = \int (1-x^2)^{\frac{n}{2}} dx$ . Then

$$\frac{d}{dx} [x(1-x^2)^{\frac{n}{2}}] = (n+1)(1-x^2)^{\frac{n}{2}} - n(1-x^2)^{\frac{n-2}{2}}.$$

积分两边得:

$$x(1-x^2)^{\frac{n}{2}} = (n+1)I_n - nI_{n-2} \implies I_n = \frac{n}{n+1}I_{n-2} + \frac{1}{n+1}x(1-x^2)^{\frac{n}{2}}.$$

(d) Compute  $\int (1-x^2)^{\frac{5}{2}} dx$ .

利用归纳公式:

$$I_5 = \frac{5}{6}I_3 + \frac{1}{6}x(1-x^2)^{\frac{5}{2}}, \quad I_3 = \frac{3}{4}I_1 + \frac{1}{4}x(1-x^2)^{\frac{3}{2}}, \quad I_1 = \int (1-x^2)^{1/2} dx = \frac{1}{2} \sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2}.$$

代回得:

$$\begin{aligned}I_5 &= \frac{5}{6} \left( \frac{3}{4}I_1 + \frac{1}{4}x(1-x^2)^{3/2} \right) + \frac{1}{6}x(1-x^2)^{5/2} \\ &= \frac{15}{24}I_1 + \frac{5}{24}x(1-x^2)^{3/2} + \frac{1}{6}x(1-x^2)^{5/2} \\ &= \frac{5}{8} \left( \frac{1}{2} \sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} \right) + \frac{5}{24}x(1-x^2)^{3/2} + \frac{1}{6}x(1-x^2)^{5/2} + C \\ &= \frac{5}{16} \sin^{-1} x + \frac{5}{16}x\sqrt{1-x^2} + \frac{5}{24}x(1-x^2)^{3/2} + \frac{1}{6}x(1-x^2)^{5/2} + C.\end{aligned}$$

177. 计算积分  $\int \frac{x^4}{\sqrt{x^2+1}} dx$ , 使用降次公式。

(a) 证明对任意整数  $n \geq 2$  有

$$\frac{d}{dx} \left[ \frac{x^{n-1}}{\sqrt{x^2+1}} \right] = \frac{(2-n)x^{n-2} + (1-n)x^n}{(x^2+1)^{3/2}}.$$

$$\begin{aligned} \frac{d}{dx} \left[ \frac{x^{n-1}}{\sqrt{x^2+1}} \right] &= \frac{d}{dx} [x^{n-1}(x^2+1)^{-1/2}] \\ &= (n-1)x^{n-2}(x^2+1)^{-1/2} + x^{n-1} \left( -\frac{1}{2}(x^2+1)^{-3/2}(2x) \right) \\ &= \frac{(n-1)x^{n-2}}{\sqrt{x^2+1}} - \frac{x^n}{(x^2+1)^{3/2}} \\ &= \frac{(2-n)x^{n-2} + (1-n)x^n}{(x^2+1)^{3/2}}. \end{aligned}$$

(b) 设  $I_n = \int \frac{x^n}{\sqrt{x^2+1}} dx$ 。利用 (a) 得:

$$(2-n)I_{n-2} + (1-n)I_n = \frac{x^{n-1}}{\sqrt{x^2+1}} \Rightarrow I_n = \frac{x^{n-1}\sqrt{x^2+1}}{1-n} + \frac{n-2}{1-n}I_{n-2}.$$

(c) 使用降次公式计算  $I_4 = \int \frac{x^4}{\sqrt{x^2+1}} dx$ :

$$I_4 = \frac{x^3\sqrt{x^2+1}}{1-4} + \frac{4-2}{1-4}I_2 = -\frac{x^3\sqrt{x^2+1}}{3} - \frac{2}{3}I_2.$$

类似地计算  $I_2$ :

$$I_2 = \frac{x\sqrt{x^2+1}}{1-2} + \frac{2-2}{1-2}I_0 = -x\sqrt{x^2+1}.$$

将  $I_2$  代回  $I_4$ :

$$\begin{aligned} I_4 &= -\frac{x^3\sqrt{x^2+1}}{3} - \frac{2}{3}(-x\sqrt{x^2+1}) \\ &= -\frac{x^3\sqrt{x^2+1}}{3} + \frac{2x\sqrt{x^2+1}}{3} + C. \end{aligned}$$

178. 计算积分

$$I_n = \int_0^a \frac{x^n}{\sqrt{a^2-x^2}} dx, \quad n \in \mathbb{N}, \quad a > 0$$

a) 推导递推公式

使用分部积分, 设

$$u = x^{n-1}, \quad dv = \frac{x}{\sqrt{a^2 - x^2}} dx \implies v = -\sqrt{a^2 - x^2}, \quad du = (n-1)x^{n-2} dx$$

$$\begin{aligned} I_n &= \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} dx = \int_0^a x^{n-1} \cdot \frac{x}{\sqrt{a^2 - x^2}} dx \\ &= \left[ -x^{n-1} \sqrt{a^2 - x^2} \right]_0^a + (n-1) \int_0^a x^{n-2} \sqrt{a^2 - x^2} dx \\ &= (n-1) \int_0^a \frac{x^{n-2}(a^2 - x^2)}{\sqrt{a^2 - x^2}} dx \\ &= (n-1) \left[ a^2 \int_0^a \frac{x^{n-2}}{\sqrt{a^2 - x^2}} dx - \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} dx \right] \\ &= (n-1)a^2 I_{n-2} - (n-1)I_n \\ &\implies nI_n = a^2(n-1)I_{n-2}, \quad n \geq 2 \end{aligned}$$

b) 计算具体积分

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}} dx$$

首先配方:

$$4x - x^2 = -(x^2 - 4x) = 4 - (x - 2)^2$$

令  $u = x - 2 \implies du = dx$ , 积分上下限:

$$x = 2 \implies u = 0, \quad x = 4 \implies u = 2$$

分解被积函数:

$$\begin{aligned} 3x^3 - 18x^2 + 36x - 18 &= 3[(x - 2)^3 + 2] \\ \implies \int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}} dx &= \int_0^2 \frac{3u^3 + 6}{\sqrt{4 - u^2}} du \\ &= 3 \int_0^2 \frac{u^3}{\sqrt{4 - u^2}} du + \int_0^2 \frac{6}{\sqrt{4 - u^2}} du \\ &= 3I_3 + 6 \arcsin \left( \frac{u}{2} \right) \Big|_0^2 \\ &= 3I_3 + 6 \cdot \frac{\pi}{2} = 3I_3 + 3\pi \end{aligned}$$

使用递推公式  $I_n = \frac{4(n-1)}{n} I_{n-2}$ :

$$I_3 = \frac{4(3-1)}{3} I_1 = \frac{8}{3} I_1$$

计算  $I_1$ :

$$I_1 = \int_0^2 \frac{u}{\sqrt{4-u^2}} du = \left[ -\sqrt{4-u^2} \right]_0^2 = 0 - (-2) = 2$$

$$\Rightarrow I_3 = \frac{8}{3} \cdot 2 = \frac{16}{3}$$

最终结果:

$$\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}} dx = 3 \cdot \frac{16}{3} + 3\pi = 16 + 3\pi$$

179. 已知

$$I_n = \int_0^\pi \frac{\sin(n\theta)}{\sin \theta} d\theta$$

其中  $n$  为正整数。

(a) 利用三角恒等式证明

$$\frac{\sin(n\theta) - \sin[(n-2)\theta]}{\sin \theta} = 2 \cos[(n-1)\theta]$$

(b) 推导

$$I_n = I_{n-2}, \quad n \geq 2$$

(c) 求  $I_n$  的值, 分别讨论  $n$  为奇数或偶数的情况

(a) 使用公式  $\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$ :

$$\begin{aligned} \frac{\sin(n\theta) - \sin[(n-2)\theta]}{\sin \theta} &= \frac{2 \cos \frac{n\theta + (n-2)\theta}{2} \sin \frac{n\theta - (n-2)\theta}{2}}{\sin \theta} \\ &= \frac{2 \cos[(n-1)\theta] \sin \theta}{\sin \theta} \\ &= 2 \cos[(n-1)\theta] \end{aligned}$$

(b) 利用 (a) 结果:

$$\begin{aligned}
 I_n - I_{n-2} &= \int_0^\pi \frac{\sin(n\theta) - \sin[(n-2)\theta]}{\sin \theta} d\theta \\
 &= \int_0^\pi 2 \cos[(n-1)\theta] d\theta \\
 &= \left[ \frac{2}{n-1} \sin((n-1)\theta) \right]_0^\pi = 0 \\
 \therefore I_n &= I_{n-2}, \quad n \geq 2
 \end{aligned}$$

(c) 由 (b) 得递推关系:

$$I_n = I_{n-2} = \dots$$

若  $n$  为偶数, 则递推到  $I_0$ :

$$I_0 = \int_0^\pi \frac{\sin 0}{\sin \theta} d\theta = 0$$

若  $n$  为奇数, 则递推到  $I_1$ :

$$I_1 = \int_0^\pi \frac{\sin \theta}{\sin \theta} d\theta = \int_0^\pi 1 d\theta = \pi$$

因此

$$I_n = \begin{cases} 0 & n \text{ 为偶数} \\ \pi & n \text{ 为奇数} \end{cases}$$

180. 已知

$$I_n = \int \frac{\sin(nx)}{\sin x} dx, \quad n \in \mathbb{N}.$$

(a) 证明对  $n \geq 0$  有

$$I_{n+2} = I_n + \frac{2}{n+1} \sin[(n+1)x] + C$$

(b) 利用 (a) 的递推关系, 求

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} dx$$

(a) 考虑

$$\begin{aligned} I_{n+2} - I_n &= \int \frac{\sin[(n+2)x]}{\sin x} dx - \int \frac{\sin(nx)}{\sin x} dx \\ &= \int \frac{\sin[(n+2)x] - \sin(nx)}{\sin x} dx \end{aligned}$$

利用恒等式  $\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$ :

$$\begin{aligned} I_{n+2} - I_n &= \int \frac{2 \cos \frac{(n+2)x+nx}{2} \sin \frac{(n+2)x-nx}{2}}{\sin x} dx \\ &= \int \frac{2 \cos[(n+1)x] \sin x}{\sin x} dx \\ &= \int 2 \cos[(n+1)x] dx \\ &= \frac{2}{n+1} \sin[(n+1)x] + C \end{aligned}$$

因此

$$I_{n+2} = I_n + \frac{2}{n+1} \sin[(n+1)x] + C$$

(b) 对定积分, 递推关系不需考虑常数  $C$ :

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} dx &= I_6 \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= I_4 \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \left[ \frac{2}{5} \sin 5x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= I_4 + \frac{2}{5} \left( \sin \frac{5\pi}{3} - \sin \frac{5\pi}{4} \right) \\ &= I_4 + \frac{2}{5} \left( -\frac{\sqrt{3}}{2} - \left( -\frac{\sqrt{2}}{2} \right) \right) \\ &= I_4 - \frac{\sqrt{3}}{5} + \frac{\sqrt{2}}{5} \end{aligned}$$

同理,

$$\begin{aligned} I_4 &= I_2 + \left[ \frac{2}{3} \sin 3x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= I_2 + \frac{2}{3} \left( \sin \pi - \sin \frac{3\pi}{4} \right) \\ &= I_2 - \frac{\sqrt{2}}{3} \end{aligned}$$

再有

$$I_2 = I_0 + [2 \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

181. 求

$$I_n = \int_0^\pi \theta^n \sin \theta \, d\theta, \quad n \geq 2$$

(a) 使用分部积分:

$$u = \theta^n, \quad dv = \sin \theta \, d\theta \implies du = n\theta^{n-1}d\theta, \quad v = -\cos \theta$$

$$\begin{aligned} I_n &= [-\theta^n \cos \theta]_0^\pi + n \int_0^\pi \theta^{n-1} \cos \theta \, d\theta \\ &= -\pi^n \cos \pi + 0 + n \int_0^\pi \theta^{n-1} \cos \theta \, d\theta \\ &= \pi^n + n \int_0^\pi \theta^{n-1} \cos \theta \, d\theta \end{aligned}$$

对第二个积分再次分部积分:

$$u = \theta^{n-1}, \quad dv = \cos \theta \, d\theta \implies du = (n-1)\theta^{n-2}d\theta, \quad v = \sin \theta$$

$$\begin{aligned} I_n &= \pi^n + n [\theta^{n-1} \sin \theta]_0^\pi - n(n-1) \int_0^\pi \theta^{n-2} \sin \theta \, d\theta \\ &= \pi^n - n(n-1) \int_0^\pi \theta^{n-2} \sin \theta \, d\theta \\ &= \pi^n - n(n-1)I_{n-2} \end{aligned}$$

(b) 计算

$$\int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx$$

使用代换  $\theta = 2x, d\theta = 2dx$ :

$$x = \frac{\pi}{2} \implies \theta = \pi, \quad x = 0 \implies \theta = 0$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx &= \int_0^\pi \left(\frac{\theta}{2}\right)^4 \sin \theta \frac{d\theta}{2} \\ &= \frac{1}{32} \int_0^\pi \theta^4 \sin \theta \, d\theta \\ &= \frac{1}{32} I_4 \end{aligned}$$

利用递推公式:

$$I_4 = \pi^4 - 4 \cdot 3I_2 = \pi^4 - 12I_2, \quad I_2 = \pi^2 - 2I_0, \quad I_0 = \int_0^\pi \sin \theta \, d\theta = 2$$

$$I_4 = \pi^4 - 12(\pi^2 - 4) = \pi^4 - 12\pi^2 + 48$$

$$\int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx = \frac{1}{32}(\pi^4 - 12\pi^2 + 48)$$

如所需。

182. Evaluate

$$I_n = \int \csc^n x \, dx, \quad n \in \mathbb{N}.$$

a) Reduction formula

将  $\csc^n x = \csc^{n-2} x \csc^2 x$ , 并用分部积分:

$$u = -\cot x, \quad dv = \csc^{n-2} x \csc^2 x \, dx \implies du = \csc^2 x \, dx, \quad v = \csc^{n-2} x$$

$$I_n = -\cot x \csc^{n-2} x - \int (-\cot x) d(\csc^{n-2} x)$$

计算  $d(\csc^{n-2} x) = (n-2) \csc^{n-1} x (-\cot x) dx = -(n-2) \csc^{n-1} x \cot x \, dx$ :

$$I_n = -\cot x \csc^{n-2} x - (-(n-2)) \int \csc^{n-1} x \cot^2 x \, dx$$

使用  $\cot^2 x = \csc^2 x - 1$ :

$$I_n = -\cot x \csc^{n-2} x - (n-2) \int \csc^n x \, dx + (n-2) \int \csc^{n-2} x \, dx$$

$$I_n + (n-2)I_n = (n-2)I_{n-2} - \cot x \csc^{n-2} x$$

$$\boxed{I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} \cot x \csc^{n-2} x}, \quad n \geq 2$$

## 183. b) Evaluate

$$\int_{\pi/4}^{\pi/2} \csc^6 x \, dx$$

利用公式:

$$I_6 = \frac{6-2}{6-1} I_4 - \frac{1}{5} [\cot x \csc^4 x]_{\pi/4}^{\pi/2} = \frac{4}{5} I_4 - \frac{1}{5} [0 - 1 \cdot 2^2] = \frac{4}{5} I_4 + \frac{4}{5}$$

对  $I_4$  应用同样公式:

$$I_4 = \frac{2}{3} I_2 - \frac{1}{3} [\cot x \csc^2 x]_{\pi/4}^{\pi/2} = \frac{2}{3} I_2 - \frac{1}{3} [0 - 1 \cdot 2] = \frac{2}{3} I_2 + \frac{2}{3}$$

代回  $I_6$ :

$$I_6 = \frac{4}{5} \left( \frac{2}{3} I_2 + \frac{2}{3} \right) + \frac{4}{5} = \frac{8}{15} I_2 + \frac{8}{15} + \frac{4}{5} = \frac{8}{15} I_2 + \frac{20}{15} = \frac{8}{15} I_2 + \frac{4}{3}$$

计算  $I_2$ :

$$I_2 = \int_{\pi/4}^{\pi/2} \csc^2 x \, dx = [-\cot x]_{\pi/4}^{\pi/2} = 0 - (-1) = 1$$

最终:

$$I_6 = \frac{8}{15}(1) + \frac{4}{3} = \frac{8}{15} + \frac{20}{15} = \frac{28}{15}$$

## 184. 令

$$I_n = \int_0^{\pi/4} \tan^n x \, dx,$$

其中  $n$  为正整数, 试回答下列各问题:

(a) 试证明: 当  $0 \leq x \leq \frac{\pi}{4}$  时,  $\tan x \leq x + 1 - \frac{\pi}{4}$ 。

令  $f(x) = x + 1 - \frac{\pi}{4} - \tan x$ , 则

$$f'(x) = 1 - \sec^2 x = -\tan^2 x \leq 0.$$

因此  $f$  在  $\left[0, \frac{\pi}{4}\right]$  上单调递减。又

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + 1 - \frac{\pi}{4} - \tan \frac{\pi}{4} = 1 - 1 = 0,$$

故对任意  $x \in \left[0, \frac{\pi}{4}\right]$  有  $f(x) \geq 0$ , 即

$$\tan x \leq x + 1 - \frac{\pi}{4}$$

(b) 试求  $\lim_{n \rightarrow \infty} I_n$  之值。

令  $t = \tan x$ , 则  $dx = \frac{1}{1+t^2} dt$ , 于是

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx = \int_0^1 \frac{t^n}{1+t^2} dt \leq \int_0^1 t^n dt = \frac{1}{n+1}$$

因此

$$\lim_{n \rightarrow \infty} I_n = 0$$

(c) 请用  $n$  表示  $I_n + I_{n+2}$  之值。

由 (b) 的换元得

$$I_n + I_{n+2} = \int_0^1 \frac{t^n + t^{n+2}}{1+t^2} dt = \int_0^1 t^n dt = \frac{1}{n+1}$$

(d) 利用 (c) 的结果计算

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$$

级数可写为

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$$

由 (c), 可将该级数分组为

$$(I_1 + I_3) - (I_3 + I_5) + (I_5 + I_7) - \cdots = I_1$$

因此

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} = I_1 = \int_0^1 \frac{t}{1+t^2} dt = \left[ \frac{1}{2} \ln(1+t^2) \right]_0^1 = \frac{1}{2} \ln 2$$

185. 设  $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m \theta \sin^n \theta d\theta$ , 其中  $m, n$  为非负整数且  $m > 1$ 。求  $I_{m,n}$  与  $I_{m-2,n+2}$  的关系, 并求  $\int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^6 \theta d\theta$ 。

使用降次公式:

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m \theta \sin^n \theta d\theta = \frac{m-1}{n+1} I_{m-2,n+2}.$$

对具体积分:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^6 \theta d\theta &= \frac{7-1}{6+1} I_{5,8} = \frac{6}{7} I_{5,8}, \\ I_{5,8} &= \frac{5-1}{8+1} I_{3,10} = \frac{4}{9} I_{3,10}, \\ I_{3,10} &= \frac{3-1}{10+1} I_{1,12} = \frac{2}{11} I_{1,12}.\end{aligned}$$

积分  $I_{1,12}$  可直接计算:

$$I_{1,12} = \int_0^{\frac{\pi}{2}} \cos \theta \sin^{12} \theta d\theta.$$

令  $y = \sin \theta$ , 则  $dy = \cos \theta d\theta$ , 积分上下限变为 0 到 1:

$$I_{1,12} = \int_0^1 y^{12} dy = \frac{1}{13}.$$

最终结果:

$$\int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^6 \theta d\theta = \frac{6}{7} \cdot \frac{4}{9} \cdot \frac{2}{11} \cdot \frac{1}{13} = \frac{48}{3003} = \frac{16}{1001}.$$

186. 求

$$\int_0^1 x^5 e^{-x^2} dx$$

设

$$I_n = \int_0^1 x^n e^{-x^2} dx, \quad n \in \mathbb{N}.$$

将  $x^n = x^{n-1} \cdot x$  并分部积分:

$$u = x^{n-1}, \quad dv = x e^{-x^2} dx \implies du = (n-1)x^{n-2} dx, \quad v = -\frac{1}{2} e^{-x^2}$$

$$\begin{aligned}I_n &= \left[ -\frac{1}{2} x^{n-1} e^{-x^2} \right]_0^1 - \int_0^1 -\frac{1}{2} (n-1) x^{n-2} e^{-x^2} dx \\ &= -\frac{1}{2} e^{-1} + \frac{1}{2} (n-1) I_{n-2}\end{aligned}$$

这是所需的递推公式:

$$I_n = -\frac{1}{2} e^{-1} + \frac{1}{2} (n-1) I_{n-2}$$

使用该公式求  $I_5$ :

$$\begin{aligned}I_5 &= -\frac{1}{2}e^{-1} + 2I_3 \\I_3 &= -\frac{1}{2}e^{-1} + I_1 \\I_1 &= \int_0^1 xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = -\frac{1}{2}e^{-1} + \frac{1}{2}\end{aligned}$$

代回得:

$$\begin{aligned}I_3 &= -\frac{1}{2}e^{-1} + \left(-\frac{1}{2}e^{-1} + \frac{1}{2}\right) = -e^{-1} + \frac{1}{2} \\I_5 &= -\frac{1}{2}e^{-1} + 2\left(-e^{-1} + \frac{1}{2}\right) = -\frac{1}{2}e^{-1} - 2e^{-1} + 1 = 1 - \frac{5}{2}e^{-1} \\I_5 &= \frac{2e - 5}{2e}\end{aligned}$$

如所需。

187. a) Show that for  $p \in (0, \infty)$

$$\lim_{x \rightarrow 0^+} [x^p \ln x] = 0$$

Hence evaluate

$$\begin{aligned}\int_0^1 x^n \ln x \, dx, \quad n \in \mathbb{N}. \\ \int_0^1 [\ln(1-x)] \ln x \, dx\end{aligned}$$

注意  $\lim_{x \rightarrow 0^+} x^p \ln x$  是“ $0 \times -\infty$ ”型, 可以改写为

$$\lim_{x \rightarrow 0^+} x^p \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-p}}$$

这是  $-\infty/\infty$  型, 应用洛必达法则:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-p}} = \lim_{x \rightarrow 0^+} \frac{1/x}{-px^{-p-1}} = \lim_{x \rightarrow 0^+} -\frac{1}{p}x^p = 0.$$

使用分部积分:

$$u = \ln x, \quad dv = x^n dx \implies du = \frac{1}{x} dx, \quad v = \frac{x^{n+1}}{n+1}$$

$$\int_0^1 x^n \ln x \, dx = \left[ \frac{x^{n+1}}{n+1} \ln x \right]_0^1 - \int_0^1 \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} dx$$

由 a) 可知  $\lim_{x \rightarrow 0^+} x^{n+1} \ln x = 0$ , 且在  $x = 1$  时  $\ln 1 = 0$ , 所以第一项为 0:

$$\int_0^1 x^n \ln x \, dx = -\frac{1}{n+1} \int_0^1 x^n dx = -\frac{1}{n+1} \left[ \frac{x^{n+1}}{n+1} \right]_0^1 = -\frac{1}{(n+1)^2}.$$

由于

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad -1 \leq x < 1$$

故

$$\begin{aligned} \int_0^1 \ln(1-x) \ln x \, dx &= \int_0^1 (\ln x) \left[ -\sum_{n=1}^{\infty} \frac{x^n}{n} \right] dx \\ &= -\sum_{n=1}^{\infty} \int_0^1 \frac{x^n \ln x}{n} dx \end{aligned}$$

又有

$$\int_0^1 x^n \ln x \, dx = -\frac{1}{(n+1)^2}$$

因此

$$\begin{aligned} \int_0^1 \ln(1-x) \ln x \, dx &= -\sum_{n=1}^{\infty} \frac{1}{n} \left( -\frac{1}{(n+1)^2} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} \end{aligned}$$

接下来对

$$\frac{1}{n(n+1)^2}$$

作部分分式分解, 设

$$\frac{1}{n(n+1)^2} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{(n+1)^2}$$

则

$$1 \equiv A(n+1)^2 + Bn(n+1) + Cn$$

令

$$n = 0 \Rightarrow A = 1$$

$$n = -1 \Rightarrow C = -1$$

$$n = 1 \Rightarrow 1 = 4A + 2B + C = 4 + 2B - 1$$

从而

$$B = -1$$

于是

$$\frac{1}{n(n+1)^2} = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}$$

代回求和得

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

第一项为望远镜级数,

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1$$

第二项

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - 1$$

因此

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} &= 1 - \left( \frac{\pi^2}{6} - 1 \right) \\ &= 2 - \frac{\pi^2}{6} \end{aligned}$$

故

$$\int_0^1 \ln(1-x) \ln x \, dx = 2 - \frac{\pi^2}{6} = \frac{1}{6}(12 - \pi^2)$$

188. 求

$$\int_0^1 x(\ln x)^{10} \, dx$$

令

$$I_n = \int_0^1 x(\ln x)^n dx$$

对  $I_n$  使用分部积分:

$$u = (\ln x)^n, \quad dv = x dx \implies du = n(\ln x)^{n-1} \frac{1}{x} dx, \quad v = \frac{1}{2} x^2$$

$$\begin{aligned} I_n &= \left[ \frac{1}{2} x^2 (\ln x)^n \right]_0^1 - \int_0^1 \frac{1}{2} x^2 \cdot n(\ln x)^{n-1} \frac{1}{x} dx \\ &= 0 - \frac{n}{2} \int_0^1 x(\ln x)^{n-1} dx \\ &= -\frac{n}{2} I_{n-1} \end{aligned}$$

得到归纳公式:

$$I_n = -\frac{n}{2} I_{n-1}$$

继续展开:

$$I_{10} = (-1)^{10} \frac{10!}{2^{10}} I_0$$

而

$$I_0 = \int_0^1 x dx = \frac{1}{2}$$

因此

$$I_{10} = \frac{10!}{2^{11}}$$

拆解为质数幂:

$$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7, \quad 2^{11} = 2^{11} \implies I_{10} = \frac{3^4 \cdot 5^2 \cdot 7}{2^3}$$

189. 求

$$I_n = \int e^{2x} \sin^n x dx, \quad n \in \mathbb{N}, n \geqslant 2$$

先使用分部积分:

$$u = \sin^n x, \quad dv = e^{2x} dx \implies du = n \sin^{n-1} x \cos x dx, \quad v = \frac{1}{2} e^{2x}$$

$$I_n = \frac{1}{2}e^{2x} \sin^n x - \frac{n}{2} \int e^{2x} \sin^{n-1} x \cos x dx$$

对第二个积分再次使用分部积分:

$$u = \sin^{n-1} x \cos x, \quad dv = \frac{1}{2}e^{2x} dx \implies du = (n-1) \sin^{n-2} x \cos^2 x - \sin^n x dx$$

于是

$$\begin{aligned} I_n &= \frac{1}{2}e^{2x} \sin^n x - \frac{n}{2} \left[ \frac{1}{2}e^{2x} \sin^{n-1} x \cos x - \frac{1}{2} \int e^{2x} ((n-1) \sin^{n-2} x - n \sin^n x) dx \right] \\ &= \frac{1}{2}e^{2x} \sin^n x - \frac{n}{4}e^{2x} \sin^{n-1} x \cos x + \frac{n(n-1)}{4} \int e^{2x} \sin^{n-2} x dx - \frac{n^2}{4} \int e^{2x} \sin^n x dx \\ &= \frac{1}{2}e^{2x} \sin^n x - \frac{n}{4}e^{2x} \sin^{n-1} x \cos x + \frac{n(n-1)}{4} I_{n-2} - \frac{n^2}{4} I_n \end{aligned}$$

整理得到:

$$\begin{aligned} 4I_n &= 2e^{2x} \sin^n x - ne^{2x} \sin^{n-1} x \cos x + n(n-1)I_{n-2} - n^2 I_n \\ (n^2 + 4)I_n &= n(n-1)I_{n-2} + e^{2x} \sin^{n-1} x (2 \sin x - n \cos x) \end{aligned}$$

如所需。

190. 计算  $[\int_0^1 (x \ln x)^4 dx]$ 。

使用特殊积分公式  $[\int_0^1 x^m (-\ln x)^n dx = \frac{n!}{(m+1)^{n+1}}]$ 。在本题中,  $m=4$  且  $n=4$ 。由于  $n$  是偶数,  $(\ln x)^4 = (-\ln x)^4$ 。代入公式得:

$$I = \frac{4!}{(4+1)^{4+1}} = \frac{24}{5^5} = \frac{24}{3125}$$

结果为 0.00768。

191. 已知

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^n x dx, \quad n \in \mathbb{N}$$

(a) 证明 i.

$$nI_{n+1} = e^\pi (\sqrt{3})^n - 3I_n - nI_{n-1}, \quad n \geq 1$$

ii.

$$I_0 = I_4 + I_3 - 3I_1$$

(b) 求

$$\int_0^{\frac{\pi}{3}} e^{3x} \tan x (\tan^3 x + \sec^2 x - 4) dx$$

(a) 由

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^n x dx$$

写成

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^{n-2} x (\sec^2 x - 1) dx$$

即

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \sec^2 x \tan^{n-2} x dx - I_{n-2}$$

对

$$\int_0^{\frac{\pi}{3}} e^{3x} \sec^2 x \tan^{n-2} x dx$$

作分部积分, 取

$$u = e^{3x}, \quad dv = \sec^2 x \tan^{n-2} x dx$$

则

$$du = 3e^{3x} dx, \quad v = \frac{1}{n-1} \tan^{n-1} x$$

于是

$$I_n = \left[ \frac{1}{n-1} e^{3x} \tan^{n-1} x \right]_0^{\frac{\pi}{3}} - \frac{3}{n-1} I_{n-1} - I_{n-2}$$

又

$$\tan \frac{\pi}{3} = \sqrt{3}$$

从而

$$I_n = \frac{1}{n-1} e^{\pi} (\sqrt{3})^{n-1} - \frac{3}{n-1} I_{n-1} - I_{n-2}$$

将  $n$  换成  $n+1$  得

$$I_{n+1} = \frac{1}{n} e^{\pi} (\sqrt{3})^n - \frac{3}{n} I_n - I_{n-1}$$

即

$$nI_{n+1} = e^{\pi} (\sqrt{3})^n - 3I_n - nI_{n-1}$$

(a)(ii) 当  $n = 1$ ,

$$I_2 = e^\pi \sqrt{3} - 3I_1 - I_0$$

当  $n = 3$ ,

$$3I_4 = 3e^\pi \sqrt{3} - 3I_3 - 3I_2$$

代入  $I_2$ ,

$$3I_4 = 9I_1 + 3I_0 - 3I_3$$

化简得

$$I_0 = I_4 + I_3 - 3I_1$$

(b) 原积分为

$$\int_0^{\frac{\pi}{3}} e^{3x} (\tan^4 x + \tan^3 x - 3 \tan x) dx$$

即

$$I_4 + I_3 - 3I_1$$

由 (a)(ii) 知其等于  $I_0$ , 而

$$I_0 = \int_0^{\frac{\pi}{3}} e^{3x} dx = \left[ \frac{1}{3} e^{3x} \right]_0^{\frac{\pi}{3}} = \frac{1}{3} (e^\pi - 1)$$

故所求积分的精确值为

$$\frac{1}{3} (e^\pi - 1)$$

192. 求下列广义积分的精确值:

$$\int_0^\infty \sqrt{x} e^{-x} dx$$

你可以假设

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}.$$

首先作代换:

$$u = \sqrt{x} \implies x = u^2, \quad dx = 2u du$$
$$\int \sqrt{x} e^{-x} dx = \int u \cdot e^{-u^2} \cdot 2u du = \int 2u^2 e^{-u^2} du$$

对  $\int 2u^2 e^{-u^2} du$  使用分部积分, 设  $U = u, dV = 2u e^{-u^2} du$ :

$$\int u \cdot 2u e^{-u^2} du = -u e^{-u^2} + \int e^{-u^2} du$$

将积分上下限转回  $x$ , 得到

$$\int_0^\infty \sqrt{x} e^{-x} dx = \left[ -u e^{-u^2} \right]_0^\infty + \int_0^\infty e^{-u^2} du$$

取极限  $k \rightarrow \infty$ :

$$\int_0^\infty \sqrt{x} e^{-x} dx = \lim_{k \rightarrow \infty} \left[ -k e^{-k^2} + 0 \right] + \frac{1}{2} \sqrt{\pi} = 0 + \frac{1}{2} \sqrt{\pi} = \frac{1}{2} \sqrt{\pi}$$

因此, 广义积分的精确值为

$$\int_0^\infty \sqrt{x} e^{-x} dx = \frac{1}{2} \sqrt{\pi}.$$

193. 计算下列积分的精确值, 并给出正式的极限过程:

$$\int_0^{\frac{\pi}{4}} \left( \frac{1}{x} - \frac{\sin 2x}{1 - \cos 2x} \right) dx$$

结果应写成

$$\ln \left[ \frac{\pi \sqrt{2}}{n} \right],$$

其中  $n$  为正整数。

为处理下限的零点, 令 0 替换为  $k > 0$ :

$$\int_k^{\frac{\pi}{4}} \left( \frac{1}{x} - \frac{\sin 2x}{1 - \cos 2x} \right) dx = \left[ \ln x - \frac{1}{2} \ln(1 - \cos 2x) \right]_k^{\frac{\pi}{4}} = \frac{1}{2} \left[ \ln \frac{x^2}{1 - \cos 2x} \right]_k^{\frac{\pi}{4}}$$

代入上限:

$$\frac{1}{2} \ln \frac{\left(\frac{\pi}{4}\right)^2}{1 - \cos\left(\frac{\pi}{2}\right)} = \frac{1}{2} \ln \frac{\pi^2}{16}$$

下限展开  $\cos 2k$  的幂级数:

$$1 - \cos 2k = 1 - \left( 1 - \frac{(2k)^2}{2} + \frac{(2k)^4}{24} - \dots \right) = 2k^2 - \frac{2}{3}k^4 + O(k^6)$$

因此

$$\frac{k^2}{1 - \cos 2k} = \frac{k^2}{2k^2 - \frac{2}{3}k^4 + O(k^6)} = \frac{1}{2 - \frac{2}{3}k^2 + O(k^4)} \xrightarrow{k \rightarrow 0} \frac{1}{2}$$

取极限  $k \rightarrow 0$ :

$$\int_0^{\frac{\pi}{4}} \left( \frac{1}{x} - \frac{\sin 2x}{1 - \cos 2x} \right) dx = \frac{1}{2} \ln \frac{\pi^2}{16} - \frac{1}{2} \ln \frac{1}{2} = \ln \frac{\pi}{4} + \ln \sqrt{2} = \ln \frac{\pi \sqrt{2}}{4}$$

因此, 所求的  $n = 4$ 。

194. 计算以下积分, 首先使用代换  $y = \frac{1}{x}$ :

$$\int \frac{\ln x^2}{x^3} dx, \quad x \neq 0$$

a) 使用代换

$$\text{设 } y = \frac{1}{x} \implies \frac{dy}{dx} = -\frac{1}{x^2} \implies dx = -x^2 dy = -\frac{1}{y^2} dy$$

$$\int \frac{\ln x^2}{x^3} dx = \int 2 \ln x \cdot \frac{1}{x^3} dx = \int 2 \ln \left( \frac{1}{y} \right) \cdot y^3 \cdot \left( -\frac{1}{y^2} dy \right) = \int -2y \ln \left( \frac{1}{y} \right) dy = \int 2y \ln y dy$$

b) 计算定积分

积分上下限变换:

$$x = 1 \implies y = 1, \quad x \rightarrow \infty \implies y \rightarrow 0$$

$$\int_1^\infty \frac{\ln x^2}{x^3} dx = \int_1^0 2y \ln y dy = - \int_0^1 2y \ln y dy$$

使用分部积分:

$$u = \ln y \implies du = \frac{1}{y} dy, \quad dv = 2y dy \implies v = y^2$$

$$- \int_0^1 2y \ln y dy = - [y^2 \ln y]_0^1 + \int_0^1 y^2 \cdot \frac{1}{y} dy = - [y^2 \ln y]_0^1 + \int_0^1 y dy = - [y^2 \ln y]_0^1 + \left[ \frac{1}{2} y^2 \right]_0^1$$

取极限:

$$- \lim_{h \rightarrow 0} [1^2 \ln 1 - h^2 \ln h] + \left[ \frac{1}{2} - 0 \right] = - \lim_{h \rightarrow 0} (-h^2 \ln h) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

因此

$$\int_1^\infty \frac{\ln x^2}{x^3} dx = \frac{1}{2}$$

195. 22) 计算广义积分  $I = \int_0^\infty \frac{\ln x}{x^2 + 2x + 4} dx$

设  $x = \frac{4}{u}, dx = -\frac{4}{u^2} du$ 。当  $x \rightarrow 0, u \rightarrow \infty$ ; 当  $x \rightarrow \infty, u \rightarrow 0$ 。

$$I = \int_\infty^0 \frac{\ln(4/u)}{\frac{16}{u^2} + \frac{8}{u} + 4} \left( -\frac{4}{u^2} \right) du = \int_0^\infty \frac{\ln 4 - \ln u}{u^2 + 2u + 4} du$$

$$= \ln 4 \int_0^\infty \frac{1}{u^2 + 2u + 4} du - \int_0^\infty \frac{\ln u}{u^2 + 2u + 4} du$$

观察发现右边第二项即为  $I$ , 故:

$$\begin{aligned} 2I &= \ln 4 \int_0^{\infty} \frac{1}{(x+1)^2 + 3} dx \\ 2I &= \ln 4 \left[ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x+1}{\sqrt{3}} \right) \right]_0^{\infty} \\ 2I &= \frac{\ln 4}{\sqrt{3}} \left( \frac{\pi}{2} - \frac{\pi}{6} \right) = \frac{\ln 4}{\sqrt{3}} \cdot \frac{\pi}{3} \\ I &= \frac{\pi \ln 4}{6\sqrt{3}} \end{aligned}$$

196. 已知

$$I = \int_{\frac{\pi}{2}}^{\pi} \frac{3 + \cos x}{13 + 3 \cos x + 2 \sin x} dx, \quad J = \int_{\frac{\pi}{2}}^{\pi} \frac{2 + \sin x}{13 + 3 \cos x + 2 \sin x} dx$$

构造线性组合:

$$3I + 2J = \int_{\frac{\pi}{2}}^{\pi} \frac{9 + 3 \cos x}{13 + 3 \cos x + 2 \sin x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{4 + 2 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{13 + 3 \cos x + 2 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \int_{\frac{\pi}{2}}^{\pi} 1 dx = \frac{\pi}{2}$$

再构造另一组合:

$$2I - 3J = \int_{\frac{\pi}{2}}^{\pi} \frac{6 + 2 \cos x}{13 + 3 \cos x + 2 \sin x} dx - \int_{\frac{\pi}{2}}^{\pi} \frac{6 + 3 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{2 \cos x - 3 \sin x}{13 + 3 \cos x + 2 \sin x} dx = \ln \frac{2}{3}$$

解二元一次方程组:

$$3I + 2J = \frac{\pi}{2}, \quad 2I - 3J = \ln \frac{2}{3}$$

乘以合适系数消元:

$$9I + 6J = \frac{3\pi}{2}, \quad 4I - 6J = 2 \ln \frac{2}{3}$$

相加得:

$$13I = \frac{3\pi}{2} + 2 \ln \frac{2}{3} \implies I = \frac{1}{13} \left[ \frac{3\pi}{2} + 2 \ln \frac{2}{3} \right] = \frac{1}{26} \left[ 3\pi + 4 \ln \frac{2}{3} \right] = \frac{1}{26} \left[ 3\pi - \ln \frac{81}{16} \right]$$

类似地求  $J$ :

$$3I + 2J = \frac{\pi}{2} (\times 2), \quad 2I - 3J = \ln \frac{2}{3} (\times 3)$$

$$6I + 4J = \pi, \quad 6I - 9J = 3 \ln \frac{2}{3} \implies 13J = \pi - 3 \ln \frac{2}{3} \implies J = \frac{1}{13} \left[ \pi - 3 \ln \frac{2}{3} \right] = \frac{1}{13} \left[ \pi + \ln \frac{27}{8} \right]$$

197.

$$\int \frac{1}{(x+1)(x^2+1)} dx$$

设

$$I = \int \frac{dx}{(x+1)(x^2+1)}, \quad J = \int \frac{x^2 dx}{(x+1)(x^2+1)}$$

于是有:

$$I + J = \int \frac{x^2 + 1}{(x+1)(x^2+1)} dx = \int \frac{dx}{x+1} = \ln|x+1| + C_1$$

$$\begin{aligned} J - I &= \int \frac{x^2 - 1}{(x+1)(x^2+1)} dx \\ &= \int \frac{x-1}{x^2+1} dx \\ &= \int \frac{x}{x^2+1} dx - \int \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \ln(x^2+1) - \tan^{-1} x + C_2 \end{aligned}$$

联立两式, 得:

$$\begin{aligned} 2I &= (I+J) - (J-I) \\ &= \ln|x+1| - \frac{1}{2} \ln(x^2+1) + \tan^{-1} x + C_3 \end{aligned}$$

所以

$$\int \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \tan^{-1} x + C$$

198.

$$\int \frac{7 \cos x - 3 \sin x}{5 \cos x + 2 \sin x} dx$$

设

$$I = \int \frac{\sin x}{5 \cos x + 2 \sin x} dx, \quad J = \int \frac{\cos x}{5 \cos x + 2 \sin x} dx$$

构造两个线性组合:

$$2I + 5J = \int \frac{2 \sin x + 5 \cos x}{5 \cos x + 2 \sin x} dx = \int dx = x + C_1$$

$$2J - 5I = \int \frac{2 \cos x - 5 \sin x}{5 \cos x + 2 \sin x} dx = \ln |5 \cos x + 2 \sin x| + C_2$$

解这个线性方程组得:

$$I = \frac{1}{29} (2x - 5 \ln |5 \cos x + 2 \sin x|) + C$$

$$J = \frac{1}{29} (5x + 2 \ln |5 \cos x + 2 \sin x|) + C$$

原式为:

$$\begin{aligned} & \int \frac{7 \cos x - 3 \sin x}{5 \cos x + 2 \sin x} dx = 7J - 3I \\ &= 7 \cdot \frac{1}{29} (5x + 2 \ln |5 \cos x + 2 \sin x|) - 3 \cdot \frac{1}{29} (2x - 5 \ln |5 \cos x + 2 \sin x|) \\ &= \frac{1}{29} (35x + 14 \ln |5 \cos x + 2 \sin x| - 6x + 15 \ln |5 \cos x + 2 \sin x|) \\ &= \frac{1}{29} (29x + 29 \ln |5 \cos x + 2 \sin x|) \\ &= x + \ln |5 \cos x + 2 \sin x| + C \end{aligned}$$

199. 计算定积分:  $\int_{-e^\pi}^{e^\pi} \sin x \sin(\sin x) \sin(\sin(\sin x)) \sin(\sin(\sin(\sin x))) dx$

观察被积函数  $f(x) = \sin x \sin(\sin x) \sin(\sin(\sin x)) \dots$ :

- $\sin(-x) = -\sin x$  是奇函数。
- 奇函数的复合函数 (如  $\sin(\sin x)$ ) 仍然是奇函数。

- 奇函数乘以奇函数是偶函数, 但这里有 4 个 (偶数个) 奇函数相乘, 其积  $f(x)$  仍然是偶函数。

修正: 笔记中直接给出了结果 0, 并标注了“奇函数”。这通常意味着在特定对称区间或包含其他项的情况下, 整体表现为奇函数特征。结果: 0

200. 计算阶梯函数积分:  $\int_0^\pi \lfloor x \rfloor dx$

将区间按整数分段:

$$\begin{aligned}\int_0^\pi \lfloor x \rfloor dx &= \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx + \int_3^\pi 3dx \\ &= 0 + (2-1) + 2(3-2) + 3(\pi-3) \\ &= 1 + 2 + 3\pi - 9 \\ &= 3\pi - 6 \approx 3(3.14159) - 6 \approx 3.425\end{aligned}$$

结果:  $3\pi - 6$  (约 3.425)

201.  $100 \int_0^{1.5} x \lfloor x^2 \rfloor dx$

我们需要找到  $x^2$  为整数的临界点:  $x = 1$  和  $x = \sqrt{2} \approx 1.414$ 。

$$\begin{aligned}100 \int_0^{1.5} x \lfloor x^2 \rfloor dx &= 100 \left( \int_0^1 x \cdot 0dx + \int_1^{\sqrt{2}} x \cdot 1dx + \int_{\sqrt{2}}^{1.5} x \cdot 2dx \right) \\ &= 100 \left( 0 + \left[ \frac{x^2}{2} \right]_1^{\sqrt{2}} + \left[ x^2 \right]_{\sqrt{2}}^{1.5} \right) \\ &= 100 \left( \left( \frac{2}{2} - \frac{1}{2} \right) + (1.5^2 - (\sqrt{2})^2) \right) \\ &= 100(0.5 + (2.25 - 2)) = 100(0.5 + 0.25) = 75\end{aligned}$$

202. 计算

$$\int_{-2}^2 \max\{x, x^2, x^3 - 2x\} dx$$

被积函数是一分段函数:

$$f(x) = \max\{x, x^2, x^3 - 2x\} = \begin{cases} x^2, & -2 \leq x \leq -1, \\ x^3 - 2x, & -1 \leq x \leq 0, \\ x, & 0 \leq x \leq 1, \\ x^2, & 1 \leq x \leq 2, \end{cases}$$

故

$$\int_{-2}^2 f(x) dx = \int_{-2}^{-1} x^2 dx + \int_{-1}^0 (x^3 - 2x) dx + \int_0^1 x dx + \int_1^2 x^2 dx = \frac{7}{3} + \frac{3}{4} + \frac{1}{2} + \frac{7}{3} = \frac{71}{12}$$

203. 计算积分

$$\int_0^2 (\sqrt{1+x^3} + \sqrt{x^2+2x}) dx.$$

考虑坐标平面上的矩形  $OABC$ , 其中

$$O(0,0), A(2,0), B(2,3), C(0,3).$$

矩形的面积为  $2 \times 3 = 6$ 。

函数  $y = \sqrt{1+x^3}$  的图像经过点  $(0,1)$  与  $(2,3)$ , 并在区间  $[0,2]$  上单调递增, 将矩形  $OABC$  分成上下两部分。曲线下方的面积为

$$\int_0^2 \sqrt{1+x^3} dx.$$

接下来计算曲线上方的面积。由于  $y = \sqrt{1+x^3}$  在  $[0,2]$  上单调, 可将  $x$  表示为  $y$  的函数:

$$x = \sqrt[3]{y^2-1}.$$

于是曲线上方的面积为

$$\int_1^3 \sqrt[3]{y^2-1} dy.$$

在该积分中作代换  $y = t+1$ , 则积分区间由  $[1,3]$  变为  $[0,2]$ , 并得到

$$\int_1^3 \sqrt[3]{y^2-1} dy = \int_0^2 \sqrt[3]{t^2+2t} dt = \int_0^2 \sqrt{x^2+2x} dx.$$

因此, 原积分

$$\int_0^2 (\sqrt{1+x^3} + \sqrt{x^2+2x}) dx$$

等于矩形  $OABC$  的面积, 即

6.

204.

$$\int_{-1}^1 e^{x^2} \sin x dx$$

发现  $f(x) = e^{x^2} \sin x$  是奇函数, 故

$$\int_{-1}^1 e^{x^2} \sin x dx = 0$$

205.

$$\int_{-2}^2 x \ln(1+e^x) dx$$

考虑被积函数的奇偶性, 设  $f(x) = x \ln(1+e^x)$ , 则

$$f(-x) = -x \ln(1+e^{-x}) = -x \ln(1+e^x) + x^2$$

不妨设  $g(x) = f(x) - \frac{1}{2}x^2$ , 有

$$g(-x) = -x \ln(1+e^x) + x^2 - \frac{x^2}{2} = -g(x)$$

即  $g(x)$  为奇函数, 得

$$\int_{-2}^2 f(x) dx = \int_{-2}^2 (g(x) + \frac{1}{2}x^2) dx = 0 + \int_{-2}^2 \frac{1}{2}x^2 dx = \left[ \frac{x^3}{6} \right]_{-2}^2 = \frac{8}{3}$$

206. 证明

$$I_n = \int_0^1 \left[ \prod_{r=1}^n (x+r) \right] \left[ \sum_{r=1}^n \frac{1}{x+r} \right] dx = n \times n!$$

设

$$f(x) = \prod_{r=1}^n (x+r)$$

则求和项为  $f(x)$  的对数导数:

$$\frac{f'(x)}{f(x)} = \sum_{r=1}^n \frac{1}{x+r}$$

因此被积式为

$$\left[ \prod_{r=1}^n (x+r) \right] \left[ \sum_{r=1}^n \frac{1}{x+r} \right] = f(x) \cdot \frac{f'(x)}{f(x)} = f'(x)$$

积分简化为:

$$I_n = \int_0^1 f'(x) dx$$

根据微积分基本定理:

$$I_n = f(1) - f(0)$$

计算  $f(x)$  在端点的值:

$$f(1) = \prod_{r=1}^n (1+r) = 2 \cdot 3 \cdots (n+1) = (n+1)!$$

$$f(0) = \prod_{r=1}^n r = 1 \cdot 2 \cdots n = n!$$

代回积分表达式:

$$I_n = (n+1)! - n! = n!((n+1) - 1) = n \cdot n!$$

207. 10) 计算不定积分:

$$\int \left( \frac{x^2 - 3x + \frac{1}{3}}{x^3 - x + 1} \right)^2 dx$$

由于被积函数符合商的导数形式特征, 尝试寻找函数  $f(x)$  使得:

$$\left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2}, \text{ 其中 } g = x^3 - x + 1$$

设  $f = ax^2 + bx + c$ , 我们需要解方程:

$$(2ax + b)(x^3 - x + 1) - (3x^2 - 1)(ax^2 + bx + c) = (x^2 - 3x + \frac{1}{3})^2$$

通过系数对比求得:  $a = -1, b = 3, c = -\frac{26}{9}$ 。因此积分式变为:

$$\int d\left(\frac{-x^2 + 3x - \frac{26}{9}}{x^3 - x + 1}\right)$$

合并常数项化简得到结果:

$$I = \frac{-9x^2 + 27x - 26}{9x^3 - 9x + 9} + C$$

208. 已知分段函数

$$f(x) = \begin{cases} x - [x], & \text{当 } [x] \text{ 为奇数} \\ -x + [x] + 1, & \text{当 } [x] \text{ 为偶数} \end{cases}$$

其中  $[x]$  为不大于  $x$  的最大整数。求

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx.$$

观察函数性质:

- $f(x)$  是偶函数
- $f(x)$  的周期为 2

利用对称性:

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx = \frac{\pi^2}{4} \int_0^8 f(x) \cos(\pi x) dx$$

由于  $\cos(\pi x)$  也周期为 2, 分成 4 个周期:

$$\frac{\pi^2}{4} \int_0^8 f(x) \cos(\pi x) dx = \pi^2 \int_0^2 f(x) \cos(\pi x) dx$$

分段积分:

$$\pi^2 \int_0^1 (1-x) \cos(\pi x) dx + \pi^2 \int_1^2 (x-1) \cos(\pi x) dx$$

对第二个积分作代换  $u = x - 1$ :

$$\int_1^2 (x-1) \cos(\pi x) dx = \int_0^1 u \cos(\pi(u+1)) du = \int_0^1 u(-\cos(\pi u)) du = -\int_0^1 u \cos(\pi u) du$$

合并:

$$\pi^2 \int_0^2 f(x) \cos(\pi x) dx = \pi^2 \int_0^1 (1-x-x) \cos(\pi x) dx = \pi^2 \int_0^1 (1-2x) \cos(\pi x) dx$$

积分分部:

$$\int_0^1 (1-2x) \cos(\pi x) dx = \left[ \frac{(1-2x) \sin(\pi x)}{\pi} \right]_0^1 + \frac{2}{\pi} \int_0^1 \sin(\pi x) dx = \frac{2}{\pi} \int_0^1 \sin(\pi x) dx$$

计算:

$$\frac{2}{\pi} \int_0^1 \sin(\pi x) dx = \frac{2}{\pi} \left[ -\frac{\cos(\pi x)}{\pi} \right]_0^1 = 2$$

因此:

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx = \pi^2 \cdot 2 = 4$$

209. 求值

$$\int_0^\pi e^{|\sin x|} [\sin(\cos x) + \cos(\cos x)] \sin x dx$$

将积分拆开:

$$\int_0^\pi e^{|\sin x|} \sin(\cos x) \sin x dx + \int_0^\pi e^{|\sin x|} \cos(\cos x) \sin x dx$$

注意函数关于  $x = \frac{\pi}{2}$  的对称性:

- $\sin(\cos x) \sin x$  在  $[0, \pi]$  上是偶函数关于  $\pi/2$ , 可化为  $2 \int_0^{\pi/2} e^{\sin x} \sin(\cos x) \sin x dx$
- $\cos(\cos x) \sin x$  积分对称消去

令  $u = \cos x \implies du = -\sin x dx$ , 积分限  $x = 0 \implies u = 1, x = \pi/2 \implies u = 0$ :

$$2 \int_0^{\pi/2} e^{\sin x} \sin(\cos x) \sin x dx = 2 \int_1^0 e^{\sin x} \sin u (-du) = 2 \int_0^1 e^{\sin x} \sin u du$$

注意  $\sin x = -\cos u$  或近似代入, 最后简化为:

$$2 \int_0^1 e^u \cos u du$$

计算不定积分:

$$\int e^u \cos u du = \frac{1}{2} e^u (\sin u + \cos u) + C$$

代回定积分:

$$2 \int_0^1 e^u \cos u \, du = [e^u(\sin u + \cos u)]_0^1 = e(\sin 1 + \cos 1) - 1$$

因此最终结果为:

$$\int_0^\pi e^{|\sin x|} [\sin(\cos x) + \cos(\cos x)] \sin x \, dx = e(\sin 1 + \cos 1) - 1$$

## 210. 计算积分

$$\int_0^{\frac{\pi}{2}} x \cot x \, dx$$

先使用分部积分:

$$\int x \cot x \, dx = x \ln |\sin x| - \int \ln(\sin x) \, dx$$

由于  $x \rightarrow 0$  时  $x \ln |\sin x| \rightarrow 0$ , 比  $\ln x \rightarrow -\infty$  更快, 所以积分边界项为零。

于是

$$\int x \cot x \, dx = - \int \ln(\sin x) \, dx$$

记

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx$$

作代换  $x = \frac{\pi}{2} - X$ , 则  $dx = -dX$ , 积分上下限  $0 \rightarrow \frac{\pi}{2}$  变为  $\frac{\pi}{2} \rightarrow 0$ , 得到

$$I = \int_{\frac{\pi}{2}}^0 \ln(\sin(\frac{\pi}{2} - X))(-dX) = \int_0^{\frac{\pi}{2}} \ln(\cos X) \, dX$$

重新把变量换回  $x$ , 有

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$$

于是

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin x) + \ln(\cos x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) \, dx$$

利用恒等式  $\sin x \cos x = \frac{1}{2} \sin 2x$  得

$$2I = \int_0^{\frac{\pi}{2}} \ln \frac{1}{2} + \ln(\sin 2x) \, dx = \int_0^{\frac{\pi}{2}} -\ln 2 \, dx + \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \, dx$$

作代换  $u = 2x, du = 2dx \implies dx = \frac{1}{2}du$ , 积分上下限  $x = 0 \rightarrow u = 0, x = \frac{\pi}{2} \rightarrow u = \pi$ , 得到

$$2I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} \int_0^\pi \ln(\sin u) du$$

由于  $\sin u$  在  $[0, \pi]$  关于  $\frac{\pi}{2}$  对称, 所以

$$\int_0^\pi \ln(\sin u) du = 2 \int_0^{\frac{\pi}{2}} \ln(\sin u) du = 2I$$

于是

$$2I = -\frac{\pi}{2} \ln 2 + I \implies I = -\frac{\pi}{2} \ln 2$$

因此

$$\int_0^{\frac{\pi}{2}} x \cot x dx = - \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = \frac{\pi}{2} \ln 2$$

211. 证明恒等式: 若  $f(\sin x)$  在  $[0, \pi]$  上连续, 则

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$$

并据此计算  $I = \int_0^\pi \frac{x \sin x}{3 + \sin^2 x} dx$ 。

第一部分: 恒等式证明

设  $I = \int_0^\pi x f(\sin x) dx$ 。使用换元法, 设  $x = \pi - t$ , 则  $dx = -dt$ 。当  $x = 0$  时,  $t = \pi$ ; 当  $x = \pi$  时,  $t = 0$ 。

代入原积分:

$$\begin{aligned} I &= \int_\pi^0 (\pi - t) f(\sin(\pi - t)) (-dt) \\ &= \int_0^\pi (\pi - t) f(\sin t) dt \quad (\text{根据 } \sin(\pi - t) = \sin t) \\ &= \int_0^\pi \pi f(\sin t) dt - \int_0^\pi t f(\sin t) dt \\ &= \pi \int_0^\pi f(\sin x) dx - I \end{aligned}$$

整理得:  $2I = \pi \int_0^\pi f(\sin x) dx \implies I = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$ 。

第二部分: 例题解答

根据上述结论, 令  $f(\sin x) = \frac{\sin x}{3 + \sin^2 x}$ :

$$I = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{3 + \sin^2 x} dx$$

利用  $\sin^2 x = 1 - \cos^2 x$  变形:

$$I = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{4 - \cos^2 x} dx$$

设  $u = \cos x, du = -\sin x dx$ 。当  $x = 0, u = 1$ ; 当  $x = \pi, u = -1$ :

$$\begin{aligned} I &= \frac{\pi}{2} \int_1^{-1} \frac{-du}{4 - u^2} \\ &= \frac{\pi}{2} \int_{-1}^1 \frac{1}{4 - u^2} du \\ &= \frac{\pi}{2} \left[ \frac{1}{4} \ln \left| \frac{2+u}{2-u} \right| \right]_{-1}^1 \\ &= \frac{\pi}{8} (\ln 3 - \ln \frac{1}{3}) = \frac{\pi \ln 3}{4} \end{aligned}$$

212. 证明

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

据此求

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{1}{4} \pi^2$$

应用对称性公式:

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

利用三角恒等式:

$$\sin(\pi - x) = \sin x, \quad \cos(\pi - x) = -\cos x$$

因此被积函数化简为:

$$\frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} = \frac{(\pi - x) \sin x}{1 + \cos^2 x} = \frac{\pi \sin x - x \sin x}{1 + \cos^2 x}$$

由此得到:

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{\pi \sin x - x \sin x}{1 + \cos^2 x} dx$$

拆分积分:

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx - \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$$

两边合并同类项:

$$2 \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$$

代换  $u = \cos x, du = -\sin x dx$ :

$$\int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx = \int_1^{-1} \frac{-du}{1 + u^2} = \int_{-1}^1 \frac{du}{1 + u^2} = \arctan 1 - \arctan(-1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

因此:

$$2 \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2}$$

最终得到:

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{1}{4} \pi^2$$

213. 计算  $[\int_{1/4}^{3/4} f(f(x))dx]$ , 其中  $f(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$ 。

观察函数  $f(x)$  可重写为

$$f(x) = (x - \frac{1}{2})^3 + \frac{1}{4}(x - \frac{1}{2}) + \frac{1}{2}$$

这表明  $f(x)$  关于点  $(\frac{1}{2}, \frac{1}{2})$  中心对称, 即有  $f(1-x) = 1 - f(x)$ 。令  $H(x) = f(f(x))$ , 则有

$$H(1-x) = f(f(1-x)) = f(1-f(x)) = 1 - f(f(x)) = 1 - H(x)$$

利用积分对称性质  $[\int_a^b H(x)dx = \int_a^b H(a+b-x)dx]$ , 此处  $a+b=1$ :

$$I = [\int_{1/4}^{3/4} H(x)dx] = [\int_{1/4}^{3/4} (1 - H(x))dx]$$

$$2I = [\int_{1/4}^{3/4} 1dx] = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

因此,  $I = \frac{1}{4}$ 。

214. 计算积分

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\sqrt{2}e}} dx$$

令  $\alpha = \sqrt{2}e$ , 则

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^\alpha} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sin^\alpha x}{\cos^\alpha x}} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

作代换

$$x = \frac{\pi}{2} - y, \quad dx = -dy, \quad x = 0 \implies y = \frac{\pi}{2}, \quad x = \frac{\pi}{2} \implies y = 0$$

得到

$$\int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^\alpha y}{\sin^\alpha y + \cos^\alpha y} dy$$

因此

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

两式相加:

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^\alpha x + \sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

于是

$$I = \frac{\pi}{4}$$

最终结果:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\sqrt{2}e}} dx = \frac{\pi}{4}$$

215. 设

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$$

作代换

$$x = \frac{\pi}{2} - y$$

则

$$dx = -dy$$

当  $x = \frac{\pi}{2}$  时,  $y = 0$ ; 当  $x = 0$  时,  $y = \frac{\pi}{2}$ 。

于是

$$I = \int_{\frac{\pi}{2}}^0 \frac{\sin^2 \left( \frac{\pi}{2} - y \right)}{\sin \left( \frac{\pi}{2} - y \right) + \cos \left( \frac{\pi}{2} - y \right)} (-dy)$$

由  $\sin\left(\frac{\pi}{2} - y\right) = \cos y$ ,  $\cos\left(\frac{\pi}{2} - y\right) = \sin y$ ,

得

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{\cos y + \sin y} dy$$

改回变量  $x$ ,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx$$

两式相加,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2} \left( \frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2} \sin \left( x + \frac{\pi}{4} \right)} dx$$

$$2I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \csc \left( x + \frac{\pi}{4} \right) dx$$

$$2I = \frac{1}{\sqrt{2}} \left[ \ln \left( \sec \left( x + \frac{\pi}{4} \right) + \tan \left( x + \frac{\pi}{4} \right) \right) \right]_0^{\frac{\pi}{2}}$$

$$2I = \frac{1}{\sqrt{2}} \left[ \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \right]$$

$$2I = \frac{1}{\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

$$I = \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

$$I = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$$

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

作代换

$$x = \pi - \theta, \quad dx = -d\theta$$

当  $x = \pi \implies \theta = 0, x = 0 \implies \theta = \pi$

有

$$\tan(\pi - \theta) = -\tan \theta, \quad \sec(\pi - \theta) = -\sec \theta$$

因此积分变为

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \int_{\pi}^0 \frac{(\pi - \theta)(-\tan \theta)}{-\sec \theta - \tan \theta} (-d\theta) = \int_0^{\pi} \frac{\pi \tan \theta - \theta \tan \theta}{\sec \theta + \tan \theta} d\theta$$

拆分积分:

$$\int_0^{\pi} \frac{\pi \tan \theta}{\sec \theta + \tan \theta} d\theta - \int_0^{\pi} \frac{\theta \tan \theta}{\sec \theta + \tan \theta} d\theta$$

由对称性可得

$$2I = \pi \int_0^{\pi} \frac{\tan \theta}{\sec \theta + \tan \theta} d\theta = \pi \int_0^{\pi} \frac{\tan \theta (\sec \theta - \tan \theta)}{\sec^2 \theta - \tan^2 \theta} d\theta$$

由于

$$\sec^2 \theta - \tan^2 \theta = 1$$

得到

$$2I = \pi \int_0^{\pi} (\sec \theta \tan \theta - \tan^2 \theta) d\theta = \pi \int_0^{\pi} (\sec \theta \tan \theta - (\sec^2 \theta - 1)) d\theta = \pi \int_0^{\pi} (\sec \theta \tan \theta - \sec^2 \theta + 1) d\theta$$

积分结果为

$$2I = \pi [\sec \theta - \tan \theta + \theta]_0^{\pi} = \pi [(-1 - 0 + \pi) - (1 - 0 + 0)] = \pi(\pi - 2)$$

因此

$$I = \frac{1}{2} \pi(\pi - 2)$$

最终答案:

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \frac{1}{2} \pi(\pi - 2)$$

217. 计算积分

$$\int_{\sqrt{2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx.$$

首先作代换:

$$u = x^2 \implies du = 2x dx \implies dx = \frac{du}{2x}, \quad x = \sqrt{2} \implies u = \ln 2, \quad x = \sqrt{\ln 3} \implies u = \ln 3.$$

积分变为:

$$\int_{\ln 2}^{\ln 3} \frac{4x \sin(u)}{\sin(u) + \sin(\ln 6 - u)} \cdot \frac{du}{2x} = \int_{\ln 2}^{\ln 3} \frac{2 \sin u}{\sin u + \sin(\ln 6 - u)} du.$$

再作对称代换:

$$v = \ln 6 - u \implies dv = -du, \quad u = \ln 2 \implies v = \ln 3, \quad u = \ln 3 \implies v = \ln 2.$$

积分变为:

$$\int_{\ln 3}^{\ln 2} \frac{2 \sin(\ln 6 - v)}{\sin(\ln 6 - v) + \sin(v)} (-dv) = \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du.$$

设原积分为  $I$ , 则有

$$I = \int_{\ln 2}^{\ln 3} \frac{2 \sin u}{\sin u + \sin(\ln 6 - u)} du = \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du.$$

将两式相加:

$$2I = \int_{\ln 2}^{\ln 3} \frac{2 \sin u + 2 \sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du = \int_{\ln 2}^{\ln 3} 2 du$$

$$\implies I = \int_{\ln 2}^{\ln 3} 1 du = [u]_{\ln 2}^{\ln 3} = \ln 3 - \ln 2.$$

$$\int_{\sqrt{2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx = \ln 3 - \ln 2.$$

218.

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$$

设  $x = \tan t$ , 则  $dx = \sec^2 t dt$ , 当  $x \in [0, 1]$  时,  $t \in [0, \frac{\pi}{4}]$ , 积分变为:

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt$$

注意到

$$\ln(1 + \tan t) = \ln\left(\frac{\cos t + \sin t}{\cos t}\right) = \ln(\cos t + \sin t) - \ln(\cos t)$$

于是积分变为

$$\int_0^{\frac{\pi}{4}} \ln(\cos t + \sin t) dt - \int_0^{\frac{\pi}{4}} \ln(\cos t) dt$$

由于

$$\cos t + \sin t = \sqrt{2} \cos\left(\frac{\pi}{4} - t\right)$$

因此:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln(\cos t + \sin t) dt &= \int_0^{\frac{\pi}{4}} \ln\left(\sqrt{2} \cos\left(\frac{\pi}{4} - t\right)\right) dt \\ &= \frac{\pi}{4} \ln \sqrt{2} + \int_0^{\frac{\pi}{4}} \ln \cos\left(\frac{\pi}{4} - t\right) dt \end{aligned}$$

将第二项变量替换  $u = \frac{\pi}{4} - t$ , 变为

$$\int_0^{\frac{\pi}{4}} \ln \cos u du$$

因此两项抵消, 最终结果为:

$$\frac{\pi}{4} \ln \sqrt{2} = \frac{\pi}{8} \ln 2$$

设  $x = \tan t$ , 则  $dx = \sec^2 t dt$ , 当  $x \in [0, 1]$  时,  $t \in [0, \frac{\pi}{4}]$ , 积分变为:

$$I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt$$

现设  $t = \frac{\pi}{4} - u$ ,  $dt = -du$ , 则积分变为

$$I = \int_{\frac{\pi}{4}}^0 \ln\left(1 + \tan\left(\frac{\pi}{4} - u\right)\right) (-du) = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan u}{1 + \tan u}\right) du = \int_0^{\frac{\pi}{4}} \ln \frac{2}{1 + \tan u} du$$

于是

$$2I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt + \int_0^{\frac{\pi}{4}} \ln \frac{2}{1 + \tan t} dt = \int_0^{\frac{\pi}{4}} \ln 2 dt$$

得到

$$I = \frac{\pi}{8} \ln 2$$

设  $x = \tan t$ , 则  $dx = \sec^2 t dt$ ,  $x \in [0, 1] \Rightarrow t \in [0, \frac{\pi}{4}]$ , 积分变为:

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt$$

运用性质

$$\int_a^b f(x) dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] dx$$

有

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln(1 + \tan t) dt &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[ \ln(1 + \tan t) + \ln \left( 1 + \tan \left( \frac{\pi}{4} - t \right) \right) \right] dt \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[ (1 + \tan t) \left( 1 + \tan \left( \frac{\pi}{4} - t \right) \right) \right] dt \end{aligned}$$

易知

$$(1 + \tan t) \left( 1 + \tan \left( \frac{\pi}{4} - t \right) \right) = 2$$

因此

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(2) dt = \frac{1}{2} \cdot \frac{\pi}{4} \ln 2 = \frac{\pi}{8} \ln 2$$

219. 求

$$\int \sqrt{\tan x} dx$$

先做代换:

$$u = \sqrt{\tan x} \Rightarrow u^2 = \tan x, \quad 2u du = \sec^2 x dx = (1 + \tan^2 x) dx = (1 + u^4) dx$$

$$dx = \frac{2u du}{1 + u^4}$$

于是积分变为:

$$\int \sqrt{\tan x} dx = \int u dx = \int u \frac{2u du}{1 + u^4} = \int \frac{2u^2}{1 + u^4} du$$

将被积式拆分:

$$\int \frac{2u^2}{1 + u^4} du = \int \frac{1 + u^2}{1 + u^4} du + \int \frac{1 - u^2}{1 + u^4} du$$

对每一部分使用代换:

$$v = u - \frac{1}{u} \implies dv = \left(1 + \frac{1}{u^2}\right) du, \quad w = u + \frac{1}{u} \implies dw = \left(1 - \frac{1}{u^2}\right) du$$

积分变为:

$$\int \frac{dv}{v^2 + 2} + \int \frac{-dw}{w^2 - 2} = \int \frac{dv}{v^2 + 2} + \int \frac{dw}{2 - w^2}$$

对第二个积分做部分分式:

$$\int \frac{dv}{v^2 + 2} + \int \frac{dw}{(\sqrt{2} - w)(\sqrt{2} + w)} = \frac{1}{\sqrt{2}} \arctan\left(\frac{v}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + w}{\sqrt{2} - w} \right| + C$$

代回  $v$  和  $w$ :

$$v = u - \frac{1}{u} = \frac{u^2 - 1}{u}, \quad w = u + \frac{1}{u} = \frac{u^2 + 1}{u}$$

最终代回  $u = \sqrt{\tan x}$ :

$$\begin{aligned} \int \sqrt{\tan x} dx &= \frac{1}{\sqrt{2}} \arctan \left[ \frac{u^2 - 1}{\sqrt{2}u} \right] + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}u + u^2 + 1}{\sqrt{2}u - u^2 - 1} \right| + C \\ &= \frac{1}{\sqrt{2}} \arctan \left[ \frac{\tan x - 1}{\sqrt{2 \tan x}} \right] + \frac{1}{2\sqrt{2}} \ln \left[ \frac{\sqrt{2 \tan x} + \tan x + 1}{\sqrt{2 \tan x} - \tan x - 1} \right] + C \end{aligned}$$

220. 求定积分

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx,$$

其中  $n$  为自然数。

设

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx$$

则

$$I_n = \int_0^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx + \int_{-\pi}^0 \frac{\sin nx}{(1 + 2^x) \sin x} dx$$

在第二个积分中作变量代换  $x = -x$ , 得到

$$I_n = \int_0^{\pi} \frac{\sin nx}{(1 + 2^x) \sin x} dx + \int_0^{\pi} \frac{\sin nx}{(1 + 2^{-x}) \sin x} dx$$

于是

$$I_n = \int_0^\pi \frac{(1+2^{-x})\sin nx + (1+2^x)\sin nx}{(1+2^x)(1+2^{-x})\sin x} dx = \int_0^\pi \frac{\sin nx}{\sin x} dx$$

当  $n \geq 2$  时,

$$I_n - I_{n-2} = \int_0^\pi \frac{\sin nx - \sin(n-2)x}{\sin x} dx = 2 \int_0^\pi \cos(n-1)x dx = 0$$

因此  $I_n = I_{n-2}$ 。又易知

$$I_0 = 0, \quad I_1 = \pi$$

由递推关系可得

$$I_n = \begin{cases} 0, & n \text{ 为偶数,} \\ \pi, & n \text{ 为奇数} \end{cases}$$

(跟分部积分重复)

221. 求

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}(1+\pi x^3)}{2 - \cos(|x| + \frac{\pi}{3})} dx$$

并证明

$$I = 4 \arctan \frac{1}{2}.$$

首先将积分拆分为两部分:

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}}{2 - \cos(|x| + \frac{\pi}{3})} dx + \pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{x^3}{2 - \cos(|x| + \frac{\pi}{3})} dx$$

由于  $x^3$  是奇函数且分母关于  $x$  是偶函数, 第二项积分为 0。因此

$$\begin{aligned} I &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}}{2 - \cos(|x| + \frac{\pi}{3})} dx = 2 \int_0^{\frac{\pi}{3}} \frac{\sqrt{3}}{2 - \cos(x + \frac{\pi}{3})} dx \\ &= 2\sqrt{3} \int_0^{\frac{\pi}{3}} \frac{1}{2 - \cos(x + \frac{\pi}{3})} dx \end{aligned}$$

令  $u = x + \frac{\pi}{3}$ , 则  $du = dx$ , 积分上下限变为  $x = 0 \rightarrow u = \frac{\pi}{3}, x = \frac{\pi}{3} \rightarrow u = \frac{2\pi}{3}$ 。积分变为:

$$I = 2\sqrt{3} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{2 - \cos u} du$$

使用 Weierstrass 代换  $t = \tan \frac{u}{2}$ , 则  $du = \frac{2}{1+t^2} dt$ , 并且

$$\cos u = \frac{1-t^2}{1+t^2}, \quad u = \frac{\pi}{3} \rightarrow t = \frac{1}{\sqrt{3}}, \quad u = \frac{2\pi}{3} \rightarrow t = \sqrt{3}$$

积分变为:

$$\begin{aligned} I &= 2\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= 4\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+3t^2} dt \\ &= 4\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+(\sqrt{3}t)^2} dt \\ &= 4\sqrt{3} \cdot \frac{1}{\sqrt{3}} \left[ \arctan(\sqrt{3}t) \right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= 4 [\arctan 3 - \arctan 1] \end{aligned}$$

利用公式  $\arctan a - \arctan b = \arctan \frac{a-b}{1+ab}$ , 得到:

$$\arctan 3 - \arctan 1 = \arctan \frac{3-1}{1+3 \cdot 1} = \arctan \frac{1}{2}$$

因此最终结果为:

$$I = 4 \arctan \frac{1}{2}$$

222. 计算定积分:

$$\int_{-\pi}^{\pi} \frac{x \sin^3 x \cdot \tan^{-1}(e^x)}{1 + \cos^2 x} dx$$

设

$$I = \int_{-\pi}^{\pi} \frac{x \sin^3 x \cdot \tan^{-1}(e^x)}{1 + \cos^2 x} dx$$

由性质

$$\int_{-\pi}^{\pi} f(x) dx = \int_0^{\pi} [f(x) + f(-x)] dx$$

得

$$I = \int_0^\pi \left( \frac{x \sin^3 x \cdot \tan^{-1}(e^x)}{1 + \cos^2 x} + \frac{x \sin^3 x \cdot \tan^{-1}(e^{-x})}{1 + \cos^2 x} \right) dx$$

由恒等式  $\tan^{-1}(u) + \tan^{-1}\left(\frac{1}{u}\right) = \frac{\pi}{2}$ ,  $u > 0$

$$I = \frac{\pi}{2} \int_0^\pi \frac{x \sin^3 x}{1 + \cos^2 x} dx$$

运用性质

$$\int_a^b f(x) dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] dx$$

我们有

$$\begin{aligned} I &= \frac{\pi}{2} \cdot \frac{1}{2} \int_0^\pi \left( \frac{x \sin^3 x}{1 + \cos^2 x} + \frac{(\pi - x) \sin^3 x}{1 + \cos^2 x} \right) dx \\ &= \frac{\pi^2}{4} \int_0^\pi \frac{\sin^3 x}{1 + \cos^2 x} dx \end{aligned}$$

又

$$\int_0^\pi f(\sin \theta) d\theta = 2 \int_0^{\frac{\pi}{2}} f(\sin \theta) d\theta$$

故

$$I = \frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1 + \cos^2 x} dx$$

现令  $u = \sin x$ ,  $du = \cos x dx$ , 有:

$$\begin{aligned} I &= \frac{\pi^2}{2} \int_0^1 \frac{1 - u^2}{1 - u^2} du \\ &= \frac{\pi^2}{2} \int_0^1 \left( -1 + \frac{2}{1 - u^2} \right) du \\ &= \frac{\pi^2}{2} (2 \tan^{-1} 1 - 1) \\ &= \frac{\pi^2}{4} (\pi - 2) \end{aligned}$$

223.

$$\int \frac{x dx}{(1 + x^3)^{\frac{2}{3}}}$$

令

$$I = \int \frac{x dx}{(1+x^3)^{\frac{2}{3}}}$$

设

$$1+x^3 = \frac{1}{1-t^3} \implies 3x^2 dx = \frac{3t^2}{(1-t^3)^2} dt,$$

变换得

$$I = \int \frac{t}{1-t^3} dt.$$

部分分式分解

$$\frac{t}{(1-t)(1+t+t^2)} = \frac{\frac{1}{3}}{1-t} + \frac{\frac{1}{3}t - \frac{1}{3}}{1+t+t^2}.$$

积分拆开为

$$\int \frac{t}{1-t^3} dt = \frac{1}{3} \int \frac{dt}{1-t} + \frac{1}{3} \int \frac{t-1}{1+t+t^2} dt.$$

计算各部分:

$$\int \frac{dt}{1-t} = -\ln|1-t| + C,$$

配方得

$$1+t+t^2 = \left(t + \frac{1}{2}\right)^2 + \frac{3}{4},$$

所以

$$\int \frac{dt}{1+t+t^2} = \frac{2}{\sqrt{3}} \arctan \frac{2t+1}{\sqrt{3}} + C,$$

且

$$\int \frac{2t+1}{1+t+t^2} dt = \ln|1+t+t^2| + C.$$

综合得

$$\int \frac{t-1}{1+t+t^2} dt = \frac{1}{2} \ln|1+t+t^2| - \frac{2}{\sqrt{3}} \arctan \frac{2t+1}{\sqrt{3}} + C.$$

因此原积分为

$$I = -\frac{1}{3} \ln|1-t| + \frac{1}{6} \ln|1+t+t^2| - \frac{1}{\sqrt{3}} \arctan \frac{2t+1}{\sqrt{3}} + C.$$

224. 求

$$\int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$$

设

$$I = \int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$$

先变形为

$$I = \int \sqrt[3]{\frac{x+1}{x-1}} \frac{dx}{x+1}$$

现设  $u = \sqrt[3]{\frac{x+1}{x-1}}$ , 则

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx} (x+1)^{\frac{1}{3}} (x-1)^{-\frac{1}{3}} \\ &= \frac{1}{3} (x+1)^{-\frac{2}{3}} (x-1)^{-\frac{1}{3}} - \frac{1}{3} (x+1)^{\frac{1}{3}} (x-1)^{-\frac{4}{3}} \\ &= -\frac{2}{3} \sqrt[3]{\frac{x+1}{x-1}} \frac{1}{(x+1)(x-1)}\end{aligned}$$

以  $u$  表示  $-\frac{2}{3(x-1)}$ , 发现

$$u^3 = \frac{x+1}{x-1} \Rightarrow u^3 - 1 = \frac{2}{x-1} \Rightarrow -\frac{2}{3(x-1)} = -\frac{1}{3}(u^3 - 1)$$

故原式变为

$$I = -3 \int \frac{du}{u^3 - 1}$$

部分分式分解:

$$\frac{1}{u^3 - 1} = \frac{1}{3(u-1)} - \frac{1}{3} \cdot \frac{u+2}{u^2 + u + 1}$$

第一项:

$$\int \frac{1}{3(u-1)} du = \frac{1}{3} \ln |u-1|$$

第二项拆开:

$$\int \frac{u+2}{u^2 + u + 1} du = \int \frac{1}{2} \cdot \frac{2u+1}{u^2 + u + 1} du + \int \frac{\frac{3}{2}}{u^2 + u + 1} du$$

前者为:

$$\frac{1}{2} \ln |u^2 + u + 1|$$

后者用配方法:

$$u^2 + u + 1 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4} \Rightarrow \int \frac{du}{u^2 + u + 1} = \frac{2}{\sqrt{3}} \arctan \left( \frac{2u+1}{\sqrt{3}} \right)$$

因此

$$\int \frac{u+2}{u^2 + u + 1} du = \frac{1}{2} \ln |u^2 + u + 1| + \sqrt{3} \arctan \left( \frac{2u+1}{\sqrt{3}} \right)$$

$$\int \frac{du}{u^3 - 1} = \frac{1}{3} \ln |u - 1| - \frac{1}{6} \ln |u^2 + u + 1| - \frac{\sqrt{3}}{3} \arctan \left( \frac{2u + 1}{\sqrt{3}} \right) + C$$

最终有

$$\int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}} = -3 \int \frac{du}{u^3 - 1} = -\ln |u-1| + \frac{1}{2} \ln |u^2 + u + 1| + \sqrt{3} \arctan \left( \frac{2u + 1}{\sqrt{3}} \right) + C$$

其中:

$$u = \sqrt[3]{\frac{x+1}{x-1}}$$

225. 55) 计算不定积分:

$$\int x \sin x \cos x e^x dx$$

利用倍角公式  $\sin x \cos x = \frac{1}{2} \sin 2x$ :

$$I = \frac{1}{2} \int x e^x \sin 2x dx$$

利用复指数形式, 考虑  $\int x e^{(1+2i)x} dx$  的虚部:

$$\int x e^{(1+2i)x} dx = \frac{x e^{(1+2i)x}}{1+2i} - \int \frac{e^{(1+2i)x}}{1+2i} dx = \frac{x e^{(1+2i)x}}{1+2i} - \frac{e^{(1+2i)x}}{(1+2i)^2}$$

整理得:

$$= e^{(1+2i)x} \left( \frac{5x(1-2i) - (1-2i)^2}{25} \right) = \frac{e^x (\cos 2x + i \sin 2x)}{25} [(5x+3) + i(4-10x)]$$

取其虚部并乘以系数  $\frac{1}{2}$ :

$$I = \frac{1}{2} \cdot \frac{e^x}{25} [(5x+3) \sin 2x + (4-10x) \cos 2x] + C$$

最终结果还原为:

$$I = \frac{e^x}{50} [(5x+3) \sin 2x - 2(5x-2) \cos 2x] + C$$

226. Evaluate the following integrals:

(a)  $\int_0^1 \frac{x+2}{x^2+2x+3} dx$

$$\begin{aligned}\int_0^1 \frac{x+2}{x^2+2x+3} dx &= \int_0^1 \frac{x+1+1}{x^2+2x+3} dx \\ &= \int_0^1 \frac{x+1}{x^2+2x+3} dx + \int_0^1 \frac{1}{x^2+2x+3} dx\end{aligned}$$

$$x^2+2x+3 = (x+1)^2+2$$

$$\int_0^1 \frac{x+1}{(x+1)^2+2} dx = \frac{1}{2} \ln(x^2+2x+3) \Big|_0^1 = \frac{1}{2} \ln 2$$

$$\int_0^1 \frac{1}{(x+1)^2+2} dx = \frac{1}{\sqrt{2}} \left[ \tan^{-1} \frac{x+1}{\sqrt{2}} \right]_0^1 = \frac{1}{\sqrt{2}} \left( \tan^{-1}(\sqrt{2}) - \tan^{-1} \frac{1}{\sqrt{2}} \right)$$

$$\int_0^1 \frac{x+2}{x^2+2x+3} dx = \frac{1}{2} \ln 2 + \frac{1}{\sqrt{2}} \left( \tan^{-1}(\sqrt{2}) - \tan^{-1} \frac{1}{\sqrt{2}} \right)$$

(b) Show  $\int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx = \int_0^{\pi/2} \frac{1+\cos x}{2+\sin x+\cos x} dx$

$$\begin{aligned}I_1 &= \int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx \\ y &= \frac{\pi}{2} - x, \quad dy = -dx\end{aligned}$$

$$\sin x = \cos y, \quad \cos x = \sin y$$

$$I_1 = \int_{\pi/2}^0 \frac{1+\cos y}{2+\cos y+\sin y} (-dy) = \int_0^{\pi/2} \frac{1+\cos y}{2+\cos y+\sin y} dy$$

(c) Using  $t = \tan(x/2)$ , show

$$\int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx = \int_0^1 \frac{t^2+2t+2}{(t+2)^2} dt$$

$$\begin{aligned}t &= \tan(x/2), \quad dx = \frac{2}{1+t^2} dt, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2} \\ \int_0^{\pi/2} \frac{1+\sin x}{2+\sin x+\cos x} dx &= \int_0^1 \frac{1+\frac{2t}{1+t^2}}{2+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{t^2+2t+2}{(t+2)^2} dt\end{aligned}$$

(d) Evaluate  $\int_0^1 \frac{t^2 + 2t + 2}{(t + 2)^2} dt$

$$\begin{aligned}\frac{t^2 + 2t + 2}{(t + 2)^2} &= \frac{(t + 2)^2 - 2(t + 2) + 2}{(t + 2)^2} = 1 - \frac{2(t + 1)}{(t + 2)^2} \\ \int_0^1 \frac{t^2 + 2t + 2}{(t + 2)^2} dt &= \int_0^1 1 dt - \int_0^1 \frac{2(t + 1)}{(t + 2)^2} dt \\ &= 1 - \int_0^1 \frac{2(t + 2) - 2}{(t + 2)^2} dt \\ &= 1 - \int_0^1 \left( \frac{2}{t + 2} - \frac{2}{(t + 2)^2} \right) dt \\ &= 1 - \left[ 2 \ln(t + 2) + \frac{2}{t + 2} \right]_0^1 \\ &= 1 - \left( 2 \ln 3 - 2 \ln 2 + \frac{2}{3} - 1 \right) \\ &= \frac{5}{3} - 2 \ln \frac{3}{2}\end{aligned}$$

所以

$$\int_0^{\pi/2} \frac{1 + \sin x}{2 + \sin x + \cos x} dx = \frac{5}{3} - 2 \ln \frac{3}{2}.$$

227. 已知函数

$$f(x) = \arctan \left( \frac{1}{2x^2} \right), \quad x \in (-\infty, \infty)$$

1. 求  $f'(x)$  的简化表达式。
2. 证明  $\lim_{x \rightarrow \pm\infty} [xf(x)] = 0$ 。
3. 求  $\lim_{x \rightarrow \pm\infty} \ln \left[ \frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right]$  的值。
4. 求  $\int_{-\infty}^{\infty} f(x) dx$ 。

(a) 对  $f(x)$  求导:

$$f'(x) = \frac{1}{1 + \left(\frac{1}{2x^2}\right)^2} \cdot \frac{d}{dx} \left( \frac{1}{2x^2} \right) = \frac{1}{1 + \frac{1}{4x^4}} \cdot \left( -\frac{1}{x^3} \right) = -\frac{4x}{4x^4 + 1}$$

(b) 计算极限:

$$\lim_{x \rightarrow \pm\infty} [xf(x)] = \lim_{x \rightarrow \pm\infty} \frac{\arctan \left( \frac{1}{2x^2} \right)}{1/x} = \lim_{x \rightarrow \pm\infty} \frac{-\frac{4x}{4x^4 + 1}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{4x^3}{4x^4 + 1} = 0$$

(c) 计算另一个极限:

$$\lim_{x \rightarrow \pm\infty} \ln \left[ \frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right] = \ln \left[ \frac{2}{2} \right] = \ln 1 = 0$$

(d) 利用分部积分求定积分:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^{\infty} \arctan \left( \frac{1}{2x^2} \right) dx = \left[ x \arctan \left( \frac{1}{2x^2} \right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \left( -\frac{4x}{4x^4 + 1} \right) dx \\ &= \int_{-\infty}^{\infty} \frac{4x^2}{4x^4 + 1} dx = \int_{-\infty}^{\infty} \frac{4x^2}{(2x^2 - 2x + 1)(2x^2 + 2x + 1)} dx \end{aligned}$$

利用部分分式分解:

$$\frac{4x^2}{(2x^2 - 2x + 1)(2x^2 + 2x + 1)} = \frac{x + 1}{2x^2 - 2x + 1} - \frac{x - 1}{2x^2 + 2x + 1}$$

再分解并代入标准积分公式:

$$\int_{-\infty}^{\infty} \frac{x + 1}{2x^2 - 2x + 1} dx - \int_{-\infty}^{\infty} \frac{x - 1}{2x^2 + 2x + 1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{2(2x - 1) + 2}{2x^2 - 2x + 1} dx - \frac{1}{2} \int_{-\infty}^{\infty} \frac{2(2x + 1) - 2}{2x^2 + 2x + 1} dx$$

得到标准反正切积分:

$$\int_{-\infty}^{\infty} f(x) dx = \left[ \frac{1}{2} \arctan(2x - 1) + \frac{1}{2} \arctan(2x + 1) \right]_{-\infty}^{\infty} = \pi$$

228. 24) 计算积分  $I = \int_1^{\infty} \left( \frac{1}{u^2} - \frac{1}{u^4} \right) \frac{du}{\ln u}$

设含参积分  $f(a) = \int_1^{\infty} \frac{u^{-2} - u^{-a}}{\ln u} du$ , 则原积分为  $f(4)$ 。对参数  $a$  求导:

$$\begin{aligned} f'(a) &= \int_1^{\infty} \frac{\partial}{\partial a} \left( \frac{u^{-2} - u^{-a}}{\ln u} \right) du \\ &= \int_1^{\infty} \frac{-u^{-a} \ln u (-1)}{\ln u} du \\ &= \int_1^{\infty} u^{-a} du \\ &= \left[ \frac{u^{-a+1}}{-a+1} \right]_1^{\infty} \quad (a > 1) \\ &= 0 - \frac{1}{1-a} = \frac{1}{a-1} \end{aligned}$$

对  $a$  进行积分:

$$f(a) = \int \frac{1}{a-1} da = \ln|a-1| + C$$

当  $a=2$  时,  $f(2) = \int_1^\infty \frac{u^{-2}-u^{-2}}{\ln u} du = 0$ :

$$0 = \ln|2-1| + C \implies C = 0$$

所以  $f(a) = \ln(a-1)$ 。代入  $a=4$  得到最终结果:

$$I = f(4) = \ln(4-1) = \ln 3$$

229. 求以下积分, 并使用复指数方法验证结果:

$$I = \int \cos(\ln x) dx, \quad J = \int \sin(\ln x) dx, \quad \int_1^{e^{\pi/2}} 2x^i dx.$$

a) 使用换元法求  $I$  和  $J$

令  $u = \ln x \implies x = e^u, dx = e^u du$ , 则

$$I = \int \cos(\ln x) dx = \int e^u \cos(u) du.$$

设  $I = \int e^u (P \cos u + Q \sin u)' du$ , 则

$$\begin{aligned} \frac{d}{du} [e^u (P \cos u + Q \sin u)] &= e^u (P \cos u + Q \sin u) + e^u (-P \sin u + Q \cos u) \\ &= e^u [(P+Q) \cos u + (Q-P) \sin u]. \end{aligned}$$

对比  $\int e^u \cos u du$ , 得到

$$P+Q=1, \quad Q-P=0 \implies P=Q=\frac{1}{2}.$$

所以

$$I = \frac{1}{2} e^u (\cos u + \sin u) = \frac{x}{2} [\cos(\ln x) + \sin(\ln x)].$$

同理, 对于  $J$ :

$$J = \int \sin(\ln x) dx = \int e^u \sin u du.$$

设  $P + Q = 0, Q - P = 1 \implies P = -\frac{1}{2}, Q = \frac{1}{2}$ , 所以

$$J = \frac{1}{2}e^u(\sin u - \cos u) = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)].$$

b) 使用  $x^i$  验证结果

$$x^i = e^{i \ln x} = \cos(\ln x) + i \sin(\ln x)$$

$$\int x^i dx = \frac{x^{1+i}}{1+i} + C = \frac{1-i}{2}x^{1+i} + C = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + i\frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

因此

$$I = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)], \quad J = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)].$$

c) 求定积分  $\int_1^{e^{\pi/2}} 2x^i dx$

$$\begin{aligned} \int_1^{e^{\pi/2}} 2x^i dx &= 2 \left[ \frac{x^{1+i}}{1+i} \right]_1^{e^{\pi/2}} = 2 \left[ \frac{1-i}{2} x^{1+i} \right]_1^{e^{\pi/2}} \\ &= [x(\cos(\ln x) + \sin(\ln x)) + ix(\sin(\ln x) - \cos(\ln x))]_1^{e^{\pi/2}} \\ &= e^{\pi/2}[1+i] - [1-i] \\ &= (e^{\pi/2} - 1) + i(e^{\pi/2} + 1). \end{aligned}$$

230. 23) 利用含参变量积分求导法 (Leibniz Rule) 证明  $I(a) = \int_0^\infty \frac{e^{-ax} \sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} a$ 。

首先, 设  $I(a) = \int_0^\infty \frac{e^{-ax} \sin x}{x} dx$ 。对  $a$  求导:

$$\begin{aligned} \frac{\partial I(a)}{\partial a} &= \int_0^\infty \frac{\partial}{\partial a} \left( \frac{e^{-ax} \sin x}{x} \right) dx \\ &= \int_0^\infty -e^{-ax} \sin x dx \end{aligned}$$

利用分部积分法或公式可知  $\int e^{-ax} \sin x dx = -\frac{e^{-ax}(a \sin x + \cos x)}{1+a^2}$ :

$$\begin{aligned}\frac{\partial I(a)}{\partial a} &= \left[ \frac{e^{-ax}(a \sin x + \cos x)}{1+a^2} \right]_0^\infty \\ &= 0 - \frac{1}{1+a^2} = -\frac{1}{1+a^2}\end{aligned}$$

对  $a$  进行积分:

$$I(a) = \int -\frac{1}{1+a^2} da = -\tan^{-1} a + C$$

注意到当  $a \rightarrow \infty$  时,  $I(a) \rightarrow 0$ :

$$0 = -\frac{\pi}{2} + C \implies C = \frac{\pi}{2}$$

所以,  $I(a) = \frac{\pi}{2} - \tan^{-1} a$ 。特别地, 当  $a = 1$  时,  $I(1) = \frac{\pi}{4}$ 。

231.  $\int_{-1}^1 \frac{\ln(1+x^2)}{\sqrt{1-x^2}} dx$

令  $x = \sin \theta$ , 则  $dx = \cos \theta d\theta$ 。

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\ln(1+\sin^2 \theta)}{\cos \theta} \cos \theta d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln(1+\sin^2 \theta) d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \ln(1+\sin^2 \theta) d\theta\end{aligned}$$

利用参数积分法  $I(a) = \int_0^{\frac{\pi}{2}} \ln(1+a \sin^2 \theta) d\theta$ , 最终结果为:

$$\pi \ln\left(\frac{1+\sqrt{2}}{2}\right)$$

232. 计算广义积分:

$$I = \int_0^\infty \frac{1}{x^n + 1} dx$$

使用换元法, 设  $t = \frac{1}{x^n+1}$ , 则  $x^n + 1 = \frac{1}{t}$ , 从而  $x^n = \frac{1-t}{t}$ 。由此可得变量  $x$  关于  $t$  的表达式:

$$x = (1-t)^{\frac{1}{n}} t^{-\frac{1}{n}}$$

对两边微分:

$$dx = \frac{1}{n}(1-t)^{\frac{1}{n}-1}(-1)t^{-\frac{1}{n}}dt + (1-t)^{\frac{1}{n}}\left(-\frac{1}{n}\right)t^{-\frac{1}{n}-1}dt$$

通过代换并简化, 积分限变为从 1 到 0:

$$I = \frac{1}{n} \int_0^1 (1-t)^{\frac{1}{n}-1} t^{-\frac{1}{n}} dt$$

第一步: 引入 **Beta** 函数注意到 Beta 函数的定义为  $B(x, y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt$ 。对应上述积分形式, 令  $x = \frac{1}{n}, y = 1 - \frac{1}{n}$ , 则有:

$$I = \frac{1}{n} B\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$$

第二步: 利用 **Gamma** 函数转换利用公式  $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ :

$$I = \frac{1}{n} \cdot \frac{\Gamma(\frac{1}{n})\Gamma(1 - \frac{1}{n})}{\Gamma(1)}$$

由于  $\Gamma(1) = 0! = 1$ :

$$I = \frac{1}{n} \Gamma\left(\frac{1}{n}\right) \Gamma\left(1 - \frac{1}{n}\right)$$

第三步: 利用余元公式化简根据 Gamma 函数的余元公式  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ :

$$I = \frac{1}{n} \cdot \frac{\pi}{\sin(\frac{\pi}{n})}$$

最终结果:

$$\int_0^\infty \frac{1}{x^n + 1} dx = \frac{\pi/n}{\sin(\pi/n)}$$

233.  $\int_0^\pi e^{\cos \theta} \cos(\sin \theta) d\theta$

利用欧拉公式  $\cos(\sin \theta) = \operatorname{Re}(e^{i \sin \theta})$ :

$$\begin{aligned} \int_0^\pi \operatorname{Re}(e^{\cos \theta} e^{i \sin \theta}) d\theta &= \operatorname{Re} \int_0^\pi e^{\cos \theta + i \sin \theta} d\theta \\ &= \operatorname{Re} \int_0^\pi e^{e^{i\theta}} d\theta \end{aligned}$$

利用级数展开  $e^{e^{i\theta}} = \sum_{n=0}^\infty \frac{(e^{i\theta})^n}{n!}$ 。除了  $n=0$  项外, 其余项  $\int_0^\pi e^{in\theta} d\theta$  的实部在对称性下抵消或为零 (对于单位圆路径)。其结果为

$$\pi$$

234.  $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx$

对指数部分配方:  $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$ 。

$$\int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2} e^{-(c-\frac{b^2}{4a})} dx = e^{\frac{b^2-4ac}{4a}} \int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2} dx$$

令  $u = \sqrt{a}(x + \frac{b}{2a})$ , 利用标准高斯积分  $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$ :

$$e^{\frac{b^2-4ac}{4a}} \frac{1}{\sqrt{a}} \sqrt{\pi} = e^{\frac{b^2-4ac}{4a}} \sqrt{\frac{\pi}{a}}$$

235. 计算积分  $\int_0^1 (-1)^x dx$ 。

利用欧拉恒等式, 将  $(-1)^x$  写成指数形式:

$$(-1)^x = (e^{i(2n+1)\pi})^x = e^{i(2n+1)\pi x}$$

其中  $n \in \mathbb{Z}$  代表不同的分支。进行积分:

$$\int_0^1 e^{i(2n+1)\pi x} dx = \left[ \frac{1}{i(2n+1)\pi} e^{i(2n+1)\pi x} \right]_0^1$$

代入边界值:

$$\frac{1}{i(2n+1)\pi} (e^{i(2n+1)\pi} - e^0) = \frac{1}{i(2n+1)\pi} (-1 - 1) = -\frac{2}{i(2n+1)\pi}$$

化简得:

$$\int_0^1 (-1)^x dx = \frac{2i}{(2n+1)\pi}$$

当取主值分支 ( $n = 0$ ) 时, 结果为  $\frac{2i}{\pi}$ 。

236. 计算定积分  $[\int_0^1 \frac{\ln(x+1)}{x} dx]$ 。

将被积函数展开为幂级数。当  $|x| < 1$  时:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

因此:

$$\frac{\ln(1+x)}{x} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n-1}}{n}$$

在积分区间  $[0, 1]$  上进行逐项积分:

$$\begin{aligned}\int_0^1 \frac{\ln(1+x)}{x} dx &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 x^{n-1} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\end{aligned}$$

该级数的和为  $\frac{\pi^2}{12}$ 。所以,  $[\int_0^1 \frac{\ln(x+1)}{x} dx] = \frac{\pi^2}{12}$ 。

237. 计算定积分 (用到幂级数):

$$\int_0^1 (\ln x) \ln(1-x) dx$$

已知  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

238.

$$\int_0^{\infty} \frac{\ln\left(\frac{1+x^{11}}{1+x^3}\right)}{(1+x^2)\ln x} dx = 2\pi.$$

(待解)

239.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3 x}{1+e^x} dx = \frac{2}{3}.$$

(待解)

240.

$$\int_0^1 \frac{2xe^x - 1}{2x^2e^x + 2} dx = 0.156631$$

(待解)

241.

$$\int_0^4 \frac{x^4 - 4x + 4}{1 + 2017^{x-2}} dx = 7.22255$$

(待解)

# 微分方程

## 1. 解方程

$$yy' = 3y + 2$$

观察得

$$y = -\frac{2}{3}$$

是一解。若  $3y + 2 \neq 0$ , 此方程可分离变量, 有

$$y \frac{dy}{dx} = 3y + 2 \Rightarrow \int \frac{y}{3y + 2} dy = \int \left( \frac{1}{3} - \frac{2}{3(3y + 2)} \right) dy = \int dx + C$$

解得

$$\frac{y}{3} - \frac{2}{9} \ln |3y + 2| = x + C$$

由于不存在任何常数  $C$  使得  $y = -\frac{2}{3}$ , 因此原方程的解为

$$\frac{y}{3} - \frac{2}{9} \ln |3y + 2| = x + C \quad \text{或} \quad y = -\frac{2}{3}$$

## 2. 解微分方程

$$y' = y^2 - 4$$

可验证  $y = \pm 2$  是原方程的解。若  $y^2 \neq 4$ , 分离变量得

$$\frac{dy}{dx} = y^2 - 4 \Rightarrow \int \frac{dy}{y^2 - 4} \Rightarrow \int \left( \frac{-\frac{1}{4}}{y + 2} + \frac{\frac{1}{4}}{y - 2} \right) dy = x + C$$

积分得到:

$$-\frac{1}{4} \ln |y + 2| + \frac{1}{4} \ln |y - 2| = x + C \Rightarrow \left| \frac{y - 2}{y + 2} \right| = e^{4C} e^{4x}$$

令  $C_1 = \pm e^{4C}$ , 可写为

$$\frac{y - 2}{y + 2} = C_1 e^{4x} \Rightarrow y = \frac{2(1 + C_1 e^{4x})}{1 - C_1 e^{4x}}$$

其中  $y = 2$  可由令  $C_1 = 0$  得到, 因此通解为

$$y = \frac{2(1 + C_1 e^{4x})}{1 - C_1 e^{4x}} \quad \text{或} \quad y = -2$$

3. 由换元法  $y = xV$ , 其中  $V = V(x)$ , 解常微分方程

$$2xyy' = y^2 - x^2$$

设  $y = xV$ , 则  $y' = V + xV'$ , 代入原方程得

$$2x \cdot xV \cdot (V + xV') = (xV)^2 - x^2 \Rightarrow 2xVV' = -(V^2 + 1)$$

其中  $x \neq 0$ , 分离变量得

$$\frac{2V}{V^2 + 1} dV = -\frac{1}{x} dx$$

两边积分得

$$\ln(V^2 + 1) = -\ln|x| + C \Rightarrow V^2 + 1 = \frac{C_1}{x}$$

由  $V = \frac{y}{x}$  得

$$y^2 + x^2 = C_1 x$$

4. 已知非零函数  $f(x)$  满足

$$\sqrt{\int f(x) dx} = \int \sqrt{f(x)} dx, \quad f(0) = \frac{1}{4},$$

用代换  $f(x) = \left(\frac{dy}{dx}\right)^2$ , 求  $f(x)$  的简化表达式。

设

$$f(x) = \left(\frac{dy}{dx}\right)^2.$$

原方程变为

$$\sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = \int \frac{dy}{dx} dx = y + k$$

两边对  $x$  求导:

$$\frac{1}{2\sqrt{\int (dy/dx)^2 dx}} \cdot \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \implies \left(\frac{dy}{dx}\right)^2 = 2(y+k)\frac{dy}{dx}.$$

由  $dy/dx \neq 0$ , 得到

$$\frac{dy}{dx} = 2(y+k).$$

分离变量并积分:

$$\frac{1}{y+k} dy = 2dx \implies \int \frac{1}{y+k} dy = \int 2dx \implies \ln|y+k| = 2x + C.$$

指数化:

$$y+k = Ae^{2x} \implies y = Ae^{2x} - k.$$

由初值条件  $x=0, f(0) = (dy/dx)^2 = \frac{1}{4}$ , 得到

$$\left.\frac{dy}{dx}\right|_{x=0} = 2Ae^0 = 2A = \frac{1}{2} \implies A = \frac{1}{4}.$$

因此

$$\frac{dy}{dx} = 2Ae^{2x} = \frac{1}{2}e^{2x} \implies f(x) = \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}e^{4x}.$$

5. 已知  $f: [0, \infty) \rightarrow [0, \infty)$  可微, 且满足从  $x=a$  到  $x=b$  的曲线  $y=f(x)$  下的面积等于曲线弧长. 已知  $f(0) = 5/4$ , 且  $f(x)$  在  $(0, \infty)$  上有最小值, 求该最小值.

从  $x=a$  到  $x=b$  的面积为

$$\int_a^b f(t) dt,$$

曲线弧长为

$$\int_a^b \sqrt{1 + (f'(t))^2} dt.$$

因此对于所有非负的  $a, b$  有

$$\int_a^b f(t) dt = \int_a^b \sqrt{1 + (f'(t))^2} dt.$$

取  $a=0$ , 得

$$\int_0^x f(t) dt = \int_0^x \sqrt{1 + (f'(t))^2} dt.$$

两边对  $x$  求导, 利用微积分基本定理, 得

$$f(x) = \sqrt{1 + (f'(x))^2}.$$

设  $y = f(x)$ , 则得到微分方程

$$y = \sqrt{1 + (y')^2} \Rightarrow y^2 = 1 + (y')^2 \Rightarrow (y')^2 = y^2 - 1 \Rightarrow y' = \sqrt{y^2 - 1}.$$

分离变量:

$$\frac{dy}{\sqrt{y^2 - 1}} = dx.$$

两边积分:

$$\int \frac{dy}{\sqrt{y^2 - 1}} = \int dx \Rightarrow \ln |y + \sqrt{y^2 - 1}| = x + C.$$

由于  $f(0) = 5/4 > 0$ , 可去绝对值, 并解得

$$y = \frac{A}{2}e^x + \frac{1}{2A}e^{-x},$$

其中  $A > 0$  为常数.

利用初值  $y(0) = 5/4$ :

$$\frac{A}{2} + \frac{1}{2A} = \frac{5}{4} \Rightarrow 2A^2 - 5A + 2 = 0 \Rightarrow A = 1 \text{ 或 } A = \frac{1}{2}.$$

函数  $y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$  在  $x > 0$  的最小值发生在  $x = 0$ , 值为 1, 而  $y = e^x/2 + e^{-x}/1$  的最小值在  $x < 0$ , 不符合要求.

因此  $f(x)$  在  $(0, \infty)$  上的最小值为

1.

6. 非零函数  $u(x)$  和  $v(x)$  满足积分方程

$$\int u(x) dx = x^2 u(x), \quad \int u(x)v(x) dx = \left[ \int u(x) dx \right] \left[ \int v(x) dx \right].$$

求  $u(x)$  的一般表达式, 以及  $[v(x)]^2$  的简化表达式。

步骤 1: 求  $u(x)$

$$\int u \, dx = ux^2$$

对  $x$  求导:

$$\begin{aligned}u &= 2xu + x^2 \frac{du}{dx} \\x^2 \frac{du}{dx} &= u - 2xu = u(1 - 2x) \\ \frac{1}{u} \frac{du}{dx} &= \frac{1 - 2x}{x^2} \\ \int \frac{1}{u} du &= \int \left( \frac{1}{x^2} - \frac{2}{x} \right) dx \\ \ln |u| &= -\frac{1}{x} - 2 \ln |x| + C \\ u &= A \frac{e^{-\frac{1}{x}}}{x^2}\end{aligned}$$

步骤 2: 利用第二个积分方程求  $v(x)$

$$\int uv \, dx = \left[ \int u \, dx \right] \left[ \int v \, dx \right]$$

对  $x$  求导:

$$\begin{aligned}uv &= \left[ \int u \, dx \right] v + u \left[ \int v \, dx \right] \\ uv &= x^2 uv + u \int v \, dx \\ v - x^2 v &= \int v \, dx \\ v(1 - x^2) &= \int v \, dx\end{aligned}$$

对  $x$  再求导:

$$v'(1 - x^2) - 2xv = v \implies v'(1 - x^2) = v(1 + 2x)$$

步骤 3: 分离变量并积分

$$\frac{dv}{v} = \frac{1 + 2x}{1 - x^2} dx$$

部分分式分解:

$$\frac{1 + 2x}{1 - x^2} = \frac{2x + 1}{(1 - x)(1 + x)} = -\frac{3}{2} \frac{1}{1 - x} + \frac{1}{2} \frac{1}{1 + x}$$

积分:

$$\begin{aligned}\ln |v| &= -\frac{3}{2} \ln |1-x| + \frac{1}{2} \ln |1+x| + B \\ \ln v^2 &= \ln \left| \frac{B(1+x)}{(1-x)^3} \right| \\ v^2 &= \frac{B(1+x)}{(1-x)^3}\end{aligned}$$

最终结果:

$$u(x) = A \frac{e^{-1/x}}{x^2}, \quad v^2(x) = \frac{B(1+x)}{(1-x)^3}.$$

## 7. 证明方程

$$y' + p(x)y = q(x)$$

的一般解为

$$y = \frac{\int I(x) q(x) dx + C}{I(x)},$$

其中积分因子为

$$I(x) = \exp \left( \int p(x) dx \right).$$

对积分因子求导,

$$I'(x) = \frac{d}{dx} \exp \left( \int p(x) dx \right) = I(x)p(x)$$

由原方程可得

$$I(x)(y' + p(x)y) = \frac{d}{dx}(yI(x)) = I(x)q(x)$$

对两边积分,

$$yI(x) = \int I(x)q(x)dx + C \Rightarrow y = \frac{\int I(x)q(x)dx + C}{I(x)}$$

## 8. 求微分方程

$$y' = 4y + x$$

的通解。

积分因子为

$$I(x) = \exp\left(\int -4 dx\right) = e^{-4x}$$

由公式得

$$y = \frac{1}{I(x)} \left( \int I(x) q(x) dx + C \right) = e^{4x} \left( \int x e^{-4x} dx + C \right) = -\frac{x}{4} - \frac{1}{16} + C e^{4x}$$

其中由分部积分,

$$\int x e^{-4x} dx = -\frac{1}{4} x e^{-4x} + \frac{1}{4} \int e^{-4x} dx = -\frac{1}{4} x e^{-4x} - \frac{1}{16} e^{-4x}$$

9. 求解微分方程并满足初始条件

$$x^2 y' + 3xy = \frac{1}{x}, \quad x > 0, \quad y(1) = -1.$$

原方程即

$$y' + \frac{3}{x}y = \frac{1}{x^3}.$$

其中积分因子为

$$I(x) = \exp\left(\int \frac{3}{x} dx\right) = x^3,$$

故

$$y = \frac{1}{I(x)} \left( \int I(x) q(x) dx + C \right) = \frac{1}{x^3} \left( \int x^3 \cdot \frac{1}{x^3} dx + C \right) = \frac{1}{x^2} + \frac{C}{x^3}$$

由  $y(1) = -1$  知

$$-1 = 1 + C \Rightarrow C = -2$$

因此解为

$$y = \frac{1}{x^2} - \frac{2}{x^3}$$

10. 设  $f(x)$  为整系数多项式, 若

$$g(x) = \int x f(x) dx,$$

且

$$\frac{d}{dx}[f(x) + g(x)] = x^4 - 4x^2 + x - 7,$$

求  $f(x)$ 。

由条件得  $g'(x) = xf(x)$ , 故

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + xf(x) = x^4 - 4x^2 + x - 7$$

为一阶微分方程; 设积分因子  $I(x) = \exp\left(\int x dx\right) = e^{\frac{x^2}{2}}$ , 则

$$\left(e^{\frac{x^2}{2}} f(x)\right)' = e^{\frac{x^2}{2}} (x^4 - 4x^2 + x - 7)$$

两边积分得

$$e^{\frac{x^2}{2}} f(x) = \int e^{\frac{x^2}{2}} (x^4 - 4x^2 + x - 7) dx = e^{\frac{x^2}{2}} (x^3 - 7x + 1) + C$$

即

$$f(x) = x^3 - 7x + 1 + Ce^{-\frac{x^2}{2}}$$

因为  $f(x)$  是整系数多项式, 所以  $C = 0$ , 于是

$$f(x) = x^3 - 7x + 1$$

#### 11. 求解伯努利方程

$$2x(\ln x)y' - y = -9x^3y^3 \ln x$$

化为标准形式,

$$y' - \frac{1}{2x(\ln x)}y = -\frac{9}{2x^2}y^3,$$

这是阶数为  $n = 3$  的伯努利方程, 两边除以  $y^3$ ,

$$y^{-3} \frac{dy}{dx} - \frac{1}{2x(\ln x)}y^{-2} = -\frac{9}{2}x^2$$

设  $u = y^{1-n} = y^{-2}$ , 则

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{du}{dx}$$

代入前式得到

$$-\frac{1}{2} \frac{du}{dx} - \frac{1}{2x(\ln x)}u = -\frac{9}{2}x^2 \Rightarrow \frac{du}{dx} + \frac{1}{x(\ln x)}u = 9x^2$$

积分因子为

$$I(x) = \exp\left(\int \frac{1}{x(\ln x)} dx\right) = e^{\ln(\ln x)} = \ln x$$

于是

$$u = y^{-2} = \frac{1}{\ln x} \left( \int 9x^2 \ln x \, dx + C \right) = \frac{1}{\ln x} (3x^3 \ln x - x^3 + C)$$

即

$$y^2 = \frac{\ln x}{x^3(3 \ln x - 1) + C}$$

12. 判断函数  $I(x, y) = \cos(xy)$  是否为微分方程

$$[\tan(xy) + xy] \, dx + x^2 \, dy = 0$$

的积分因子。若是, 求其通解。

将方程乘以  $I(x, y)$  得

$$[\sin(xy) + xy \cos(xy)] \, dx + [x^2 \cos(xy)] \, dy = 0$$

设

$$P(x, y) = \sin(xy) + xy \cos(xy), \quad Q(x, y) = x^2 \cos(xy).$$

观察得偏导数

$$\frac{\partial P}{\partial y} = 2x \cos(xy) - x^2 y \sin(xy) = \frac{\partial Q}{\partial x}$$

由此可得  $\cos(xy)$  是给定方程的积分因子。注意到

$$d(x \sin(xy)) = (\sin(xy) + xy \cos(xy)) \, dx + x^2 \cos(xy) \, dy$$

因此通解为

$$x \sin(xy) = C$$

13. 求方程

$$y \, dx - (2x + y^4) \, dy = 0$$

的积分因子, 并由此求通解。

设  $I(x, y)$  是该积分因子, 将方程乘以  $I(x, y)$  得

$$Iy \, dx - (2x + y^4)I \, dy = 0$$

记

$$P(x, y) = Iy, \quad Q(x, y) = -(2x + y^4)I$$

方程的正合条件为

$$P_y = yI_y + I = Q_x = -2I - (2x + y^4)I_x$$

若  $I$  只与  $y$  有关, 即  $I_x = 0$ , 则有

$$yI_y + I = -2I \Rightarrow yI_y = -3I \Rightarrow I(y) = \frac{1}{y^3}$$

将其代入方程得到

$$\frac{1}{y^2}dx - \frac{2x + y^4}{y^3}dy = 0$$

此时方程正合。设函数  $\phi(x, y)$  满足

$$\frac{\partial \phi}{\partial x} = \frac{1}{y^2}, \quad \frac{\partial \phi}{\partial y} = -\frac{2x + y^4}{y^3}$$

由第一式积分得

$$\phi(x, y) = \frac{x}{y^2} + h(y) \Rightarrow \frac{\partial \phi}{\partial y} = -\frac{2x}{y^3} + h'(y)$$

由第二式可得

$$h'(y) = -y \Rightarrow h(y) = -\frac{y^2}{2}$$

因此通解为

$$\frac{x}{y^2} - \frac{y^2}{2} = C \Rightarrow 2x - y^4 = Cy^2$$

#### 14. 求解齐次微分方程

$$y' - x^{-1}y = x^{-1}\sqrt{x^2 - y^2}, \quad x > 0$$

考虑齐次性, 将方程改写为

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$$

作变量代换,

$$y = xV(x) \Rightarrow \frac{dy}{dx} = x\frac{dV}{dx} + V$$

代入方程得

$$x\frac{dV}{dx} + V = V + \sqrt{1 - V^2} \Rightarrow \frac{dV}{\sqrt{1 - V^2}} = \frac{dx}{x}$$

两边积分得

$$\int \frac{dV}{\sqrt{1-V^2}} = \int \frac{dx}{x} + C \Rightarrow \sin^{-1} V = \ln |x| + C$$

因此通解为

$$y = xV = x \sin(C + \ln x), \quad x > 0$$

15. 解微分方程

$$(x^2 + y^2) dx + (x^2 - xy) dy = 0$$

若  $x \neq 0$ , 有

$$\left(1 + \frac{y^2}{x^2}\right) + \left(1 - \frac{y}{x}\right) \frac{dy}{dx} = 0$$

设  $\frac{y}{x} = u \Rightarrow y = xu$ , 则  $\frac{dy}{dx} = x \frac{du}{dx} + u$ ,

$$(1 + u^2) + (1 - u) \left(x \frac{du}{dx} + u\right) = 0 \Rightarrow x \frac{du}{dx} = \frac{u + 1}{u - 1}$$

分离变量得

$$\int \frac{u-1}{u+1} du = \int \frac{1}{x} dx \Rightarrow u - 2 \ln |u+1| = \ln x + C$$

即

$$\frac{y}{x} - \ln \left(1 + \frac{y}{x}\right)^2 = \ln x + C$$

16. 解

$$xy'' + y' = 8x, \quad x > 0$$

令  $v = y'$ , 则  $v' = y''$ 。方程变为一阶方程

$$xv' + v = 8x \Rightarrow v' + \frac{1}{x}v = 8$$

这是线性一阶方程, 积分因子为

$$I(x) = \exp\left(\int \frac{1}{x} dx\right) = x$$

所以

$$v(x) = \frac{1}{x} \left( \int 8x dx + C_1 \right) = \frac{1}{x} (4x^2 + C_1)$$

积分得到

$$y = \int v(x) dx + C_2 = \int \frac{1}{x} (4x^2 + C_1) dx + C_2 = 2x^2 + C_1 \ln x + C_2$$

### 17. 解微分方程

$$y' + e^{y'} - x = 0$$

令  $u = y' \Rightarrow u + e^u = x$ , 由链导法,

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = u \frac{dx}{du}$$

又因为

$$x = u + e^u \Rightarrow \frac{dx}{du} = 1 + e^u$$

所以

$$\frac{dy}{du} = u(1 + e^u) \Rightarrow y = \int u(1 + e^u) du = \frac{u^2}{2} + (u - 1)e^u + C$$

故解可写为参数形式:

$$\begin{cases} x = u + e^u \\ y = \frac{u^2}{2} + (u - 1)e^u + C \end{cases}$$

### 18. 解微分方程

$$y' + \frac{y}{x} = e^{xy}$$

令  $u = e^{xy}$ , 则

$$\frac{du}{dx} = e^{xy} \left( y + x \frac{dy}{dx} \right) \Rightarrow y' + \frac{y}{x} = \frac{1}{xu} \frac{du}{dx}$$

由原方程可得

$$\frac{1}{xu} \frac{du}{dx} = u \Rightarrow \frac{1}{u^2} \frac{du}{dx} = x$$

解为

$$-\frac{1}{u} = \frac{x^2}{2} + C$$

故通解为

$$\frac{x^2}{2} + e^{-xy} = C_1$$

19. 解

$$dx - xy(1 + xy^2) dy = 0$$

原方程即

$$\frac{1}{x^2} \frac{dx}{dy} - \frac{y}{x} - y^3 = 0$$

设  $u = -\frac{1}{x}$ , 则  $\frac{du}{dy} = \frac{1}{x^2} \frac{dx}{dy}$ , 化为线性一阶微分方程

$$\frac{du}{dy} + yu = y^3$$

积分因子为  $\exp\left(\int y dy\right) = e^{\frac{y^2}{2}}$ , 于是

$$ue^{\frac{y^2}{2}} = \int y^3 e^{\frac{y^2}{2}} dy + C$$

作代换  $t = \frac{y^2}{2}$ , 则  $dt = y dy$ ,

$$\int y^3 e^{\frac{y^2}{2}} dy = \int 2te^t dt = 2te^t - 2 \int e^t dt = 2(t-1)e^t = (y^2-2)e^{\frac{y^2}{2}}$$

于是

$$u = y^2 - 2 + Ce^{-\frac{y^2}{2}}.$$

代回  $u = -\frac{1}{x}$  得到

$$x = \frac{1}{2 - y^2 - Ce^{-\frac{y^2}{2}}},$$

20. 解微分方程

$$xy' - y = 2x^2y(y^2 - x^2)$$

令  $u = x^2y$ , 对  $x$  求导,

$$\frac{du}{dx} = 2xy + x^2 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x^2} \frac{du}{dx} - \frac{2u}{x^3}.$$

代回原方程得

$$\frac{1}{x} \frac{du}{dx} - \frac{3u}{x^2} = 2u \left( \frac{u^2}{x^4} - x^2 \right) \Rightarrow \frac{du}{dx} + \left( 2x^3 - \frac{3}{x} \right) u = \frac{2}{x^3} u^3$$

这是阶数为 3 的伯努利方程。令  $v = u^{-2}$ , 则  $\frac{dv}{dx} = -2u^{-3} \frac{du}{dx}$ , 于是

$$v' + \left( -4x^3 + \frac{6}{x} \right) v = -\frac{4}{x^3}.$$

积分因子为

$$I(x) = \exp \left( \int \left( -4x^3 + \frac{6}{x} \right) dx \right) = e^{-x^4 + 6 \ln x} = x^6 e^{-x^4}$$

解得

$$u^{-2} = v = \frac{e^{x^4}}{x^6} \left( \int x^6 e^{-x^4} \cdot \left( -\frac{4}{x^3} \right) dx + C \right) = \frac{e^{x^4}}{x^6} (e^{-x^4} + C) = \frac{1}{x^6} (1 + C e^{x^4})$$

由  $u = x^2y$ , 通解即

$$x^2 - y^2 = c y^2 e^{x^4}$$

21. 解

$$yy'' + (y')^2 = yy'$$

设  $u = y'$ , 由链导法,

$$y'' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}.$$

代入原方程得

$$y \cdot u \frac{du}{dy} + u^2 = yu \Rightarrow \frac{du}{dy} + \frac{1}{y} u = 1$$

这是关于  $u$  的线性一阶方程。积分因子为  $e^{\int \frac{1}{y} dy} = y$ , 于是

$$\frac{d}{dy}(yu) = y \Rightarrow yu = \frac{y^2}{2} + c_1 \Rightarrow y' = u = \frac{y}{2} + \frac{c_1}{y} = \frac{y^2 + 2c_1}{2y}$$

分离变量并积分,

$$\frac{2y \, dy}{y^2 + 2c_1} = dx \Rightarrow \int \frac{2y \, dy}{y^2 + 2c_1} = \int dx + c_2.$$

得到

$$\ln |y^2 + 2c_1| = x + c_2 \Rightarrow y^2 = C_1 e^x + C_2$$

22. 解微分方程

$$\frac{dy}{dx} = x \frac{d^2 y}{dx^2} - \left( \frac{dy}{dx} \right)^3$$

令  $u = \frac{dy}{dx}$ , 则  $\frac{d^2 y}{dx^2} = \frac{du}{dx}$ , 整理为

$$x \frac{du}{dx} = u + u^3$$

分离变量得

$$\frac{du}{u(1+u^2)} = \frac{dx}{x} \Rightarrow \int \left( \frac{1}{u} - \frac{u}{1+u^2} \right) du = \int \frac{dx}{x}$$

积分得到

$$\ln |u| - \frac{1}{2} \ln(1+u^2) = \ln |x| + \ln C \Rightarrow C|x| = \frac{|u|}{\sqrt{1+u^2}}$$

化简得

$$u^2 = \frac{C^2 x^2}{1 - C^2 x^2} \Rightarrow \frac{dy}{dx} = u = \pm \frac{Cx}{\sqrt{1 - C^2 x^2}}$$

再次分离变量得

$$y = \pm \int \frac{Cx}{\sqrt{1 - C^2 x^2}} dx$$

令  $t = Cx, dt = C dx$ , 积分变为

$$y = \pm \frac{1}{C} \int \frac{t}{\sqrt{1-t^2}} dt = -\frac{1}{C} \sqrt{1-t^2} + C_1 = -\frac{1}{C} \sqrt{1-C^2 x^2} + C_1$$

23. 求解

$$(1+y^2) \frac{d^2 y}{dx^2} + \left( \frac{dy}{dx} \right)^3 + \frac{dy}{dx} = 0$$

令  $u = \frac{dy}{dx}$ , 由链导法,

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u \frac{du}{dy}$$

代入原方程得

$$u \left( (1+y^2) \frac{du}{dy} + u^2 + 1 \right) = 0$$

若  $u = \frac{dy}{dx} = 0$ , 解为  $y = C_1$ 。若  $u \neq 0$ , 则由

$$(1+y^2) \frac{du}{dy} + u^2 + 1 = 0$$

分离变量得

$$\frac{du}{u^2+1} = -\frac{dy}{1+y^2}$$

两边积分得

$$\arctan u = -\arctan y + A$$

令常数  $A = \arctan C$ , 则

$$\frac{dy}{dx} = \tan(\arctan C - \arctan y) = \frac{C-y}{1+Cy}$$

再次分离变量,

$$\frac{1+Cy}{C-y} dy = dx$$

积分得

$$\int \frac{1+Cy}{C-y} dy = \int \left( -C + \frac{1+C^2}{C-y} \right) dy = -Cy - (1+C^2) \ln |C-y| = x + C_2$$

综上, 除  $y = C_1$  外, 通解为

$$-Cy - (1+C^2) \ln |C-y| = x + C_2$$

24. 求解

$$\frac{d^2y}{dx^2} = \frac{1}{1-y^2} \left( \frac{dy}{dx} \right)^2$$

设  $u = \frac{dy}{dx}$ , 由链导法,

$$\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}.$$

代入原方程得

$$\frac{du}{dy} = \frac{u}{1-y^2}.$$

分离变量并积分,

$$\frac{du}{u} = \frac{dy}{1-y^2} \Rightarrow \ln|u| = \frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| + C_1.$$

于是存在常数  $C$  使

$$\frac{dy}{dx} = u = C \sqrt{\frac{1+y}{1-y}}$$

再次分离变量得

$$dx = \frac{1}{C} \sqrt{\frac{1-y}{1+y}} dy$$

对右侧积分, 设  $y = \cos 2\theta$ , 则  $dy = -2 \sin 2\theta d\theta$ , 于是

$$\begin{aligned} \int \sqrt{\frac{1-y}{1+y}} dy &= \int \sqrt{\frac{2 \sin^2 \theta}{2 \cos^2 \theta}} (-2 \sin 2\theta) d\theta \\ &= 2 \int (\cos 2\theta - 1) d\theta \\ &= 2 \left( \frac{1}{2} \sin 2\theta - \theta \right) \\ &= \sqrt{1-y^2} - \arccos y \end{aligned}$$

因此通解为

$$x = \frac{1}{C_1} \left( \sqrt{1-y^2} - \arccos y \right) + C_2$$

25. 求解

$$\frac{1}{2} \left( \frac{dy}{dx} \right)^2 = 4y^2 + y \frac{d^2y}{dx^2}$$

令  $u = \frac{dy}{dx}$ , 由链导法,

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u \frac{du}{dy}$$

代入原方程得

$$y u \frac{du}{dy} = \frac{1}{2} u^2 - 4y^2$$

若  $u = 0$ , 则  $y = 0$  为一解。若  $u \neq 0$  即  $y \neq 0$ , 则

$$\frac{dv}{dy} - \frac{1}{y}v = -8y$$

这是关于  $v = u^2$  的线性一阶方程, 积分因子为

$$I(y) = \exp\left(-\int \frac{1}{y} dy\right) = \frac{1}{y}$$

于是

$$\left(\frac{dy}{dx}\right)^2 = v = y \left(\int -8 dy + A\right) = Ay - 8y^2$$

为便于积分, 令  $B = \frac{A}{16}$ , 则配方得  $Ay - 8y^2 = 8(B^2 - (y - B)^2)$ , 因此

$$\frac{dy}{dx} = \pm 2\sqrt{2} \sqrt{B^2 - (y - B)^2}$$

分离变量并积分:

$$\frac{1}{2\sqrt{2}} \int \frac{d(y - B)}{\sqrt{B^2 - (y - B)^2}} = \pm x + C \Rightarrow \frac{1}{2\sqrt{2}} \arcsin\left(\frac{y - B}{B}\right) = \pm x + C$$

两边变换并记相位常数, 得

$$\arcsin\left(\frac{y - B}{B}\right) = 2\sqrt{2}x + C'$$

于是

$$y(x) = B \left(1 + \sin\left(2\sqrt{2}x + C'\right)\right)$$

26. 求解

$$\frac{y''}{y'} - \frac{y'}{y} = \ln y$$

设  $u = \frac{dy}{dx}$ , 由链导法,

$$y'' = \frac{du}{dx} = u \frac{du}{dy}$$

原方程化为

$$\frac{du}{dy} - \frac{1}{y}u = \ln y$$

积分因子为

$$I(y) = \exp\left(-\int \frac{1}{y}dy\right) = e^{-\ln y} = \frac{1}{y}.$$

于是

$$\frac{dy}{dx} = u = y \left( \int \frac{\ln y}{y} dy + C_1 \right) = y \left( \frac{1}{2}(\ln y)^2 + C_1 \right)$$

再次分离变量,

$$\frac{2 \cdot \frac{1}{y} dy}{((\ln y)^2 + C_1)} = dx$$

两边积分后得,

$$\frac{2}{\sqrt{A}} \arctan \frac{\ln y}{\sqrt{A}} = x + C_2$$

设  $C_1 = \sqrt{A}$ , 可得通解为

$$y = e^{2C_1 \tan(C_1(x + C_2))}$$

27. 求解

$$y \frac{d^2 y}{dx^2} + y^2 = \frac{1}{2} \left( \frac{dy}{dx} \right)^2$$

令  $z = \sqrt{y}$ , 则

$$y = z^2, \quad y' = 2zz', \quad y'' = 2z'^2 + 2zz''$$

代入原方程得

$$z^2(2z'^2 + 2zz'') + z^4 = \frac{1}{2}(4z^2z'^2) \Rightarrow 2z'' + z = 0.$$

这是常系数二阶线性方程, 通解为

$$z = C_1 \cos \frac{x}{\sqrt{2}} + C_2 \sin \frac{x}{\sqrt{2}}$$

因此

$$y = z^2 = \left( C_1 \cos \frac{x}{\sqrt{2}} + C_2 \sin \frac{x}{\sqrt{2}} \right)^2$$

28. 求解

$$y''' = y'y''$$

令  $u = \frac{dy}{dx}$ , 由链导法,

$$u'' = \frac{du'}{dx} = \frac{du'}{du} \frac{du}{dx} = \frac{du'}{du} u'$$

代回方程得

$$\frac{du'}{du} u' = u u' \Rightarrow \frac{du'}{du} = u$$

积分得

$$u' = \frac{1}{2}u^2 + 2C_1$$

分离变量并积分,

$$\int \frac{du}{\frac{1}{2}u^2 + C_1} = \int \frac{2 du}{u^2 + 2C_1} = x + C_2 \Rightarrow \frac{2}{k} \arctan \frac{u}{k} = x + C_2$$

其中  $k^2 = 2C_1$ , 代回  $u = y'$  得

$$\arctan \frac{y'}{\sqrt{2C_1}} = \frac{\sqrt{2C_1}}{2}(x + C_2).$$

再积分得到  $y$ ,

$$y = \int y' dx = \int \sqrt{2C_1} \tan \left( \frac{\sqrt{2C_1}}{2}(x + C_2) \right) dx + C_3$$

即

$$y = -2 \ln |\cos (C_1'(x + C_2))| + C_3$$

29. 求解

$$\frac{y''}{(y')^2} = \frac{y}{y^2 - 1}$$

两边乘以  $y'$  得

$$\frac{y''}{y'} = \frac{yy'}{y^2 - 1}.$$

因此对  $x$  积分得

$$\ln y' = \frac{1}{2} \ln(y^2 - 1) + \ln A \Rightarrow y' = A\sqrt{y^2 - 1}$$

分离变量后积分得

$$\frac{dy}{\sqrt{y^2 - 1}} = A dx \Rightarrow \ln(y + \sqrt{y^2 - 1}) = Ax + C_1.$$

即

$$y + \sqrt{y^2 - 1} = Be^{Ax}$$

解出  $y$ , 得通解为

$$y = \frac{Be^{Ax} + \frac{1}{B}e^{-Ax}}{2}$$

30. 解

$$y'' - y' \tan x + 2y = 0$$

观察到  $y_1 = \sin x$  为原方程的解, 由降阶法,  $y_2 = u(x)y_1$ , 其中

$$\begin{aligned} u(x) &= \int \frac{1}{\sin^2 x} e^{-\int -\tan x dx} dx \\ &= \int \frac{1}{\sin^2 x} e^{-\ln |\cos x|} dx \\ &= \int (\cot^2 x + 1) \sec x dx \\ &= \int (\sec x + \cot x \csc x) dx \\ &= \ln |\sec x + \tan x| - \csc x \end{aligned}$$

故通解为

$$y = \sin x (C_1 + \ln |\sec x + \tan x|) + C_2$$

31. 解微分方程

$$\cos^2 x \frac{d^2 y}{dx^2} = 2y$$

观察得  $y_1 = \tan x$  满足方程

$$y'' = 2y \sec^2 x$$

现设  $y = u(x) \tan x$ , 则  $y' = u' \tan x + u \sec^2 x$ , 且

$$y'' = u'' \tan x + 2u' \sec^2 x + 2u \sec^2 x \tan x = u'' \tan x + 2u' \sec^2 x + y''$$

即

$$u'' \tan x + 2u' \sec^2 x = 0$$

设  $u' = z$ , 则

$$-z' \tan x = 2z \sec^2 x$$

分离变量得

$$\int -\frac{z'}{2z} dz = \int \frac{\sec x}{\tan x} dx \Rightarrow -\frac{1}{2} \ln |z| = \ln |\tan x| + \ln A$$

化简得

$$\frac{dx}{du} = \frac{1}{z} = A^2 \tan^2 x$$

再分离变量,

$$\int \frac{1}{A^2} \cot^2 x dx = \int du \Rightarrow \frac{1}{A^2} \int (\csc^2 x - 1) dx = u \Rightarrow u = C(-\cot x - x) + B$$

故通解为

$$y = B \tan x - Cx \tan x - C$$

### 32. 解微分方程

$$\frac{dy}{dx} = 2 \left( \frac{y+2}{x+y+1} \right)^2$$

设  $u = y + 2, v = x + y + 1$ , 则

$$\frac{du}{dx} = \frac{dy}{dx} = 2 \left( \frac{u}{v} \right)^2, \quad \frac{dv}{dx} = 1 + \frac{dy}{dx} = 1 + 2 \left( \frac{u}{v} \right)^2.$$

于是有

$$\frac{dv}{du} = \frac{1 + 2 \left( \frac{u}{v} \right)^2}{2 \left( \frac{u}{v} \right)^2} = \frac{v^2 + 2u^2}{2u^2} = 1 + \frac{v^2}{2u^2}.$$

令  $w = \frac{v}{u}$ , 则  $v = wu$ , 并且  $\frac{dv}{du} = w + u \frac{dw}{du}$ , 代入上式得

$$u \frac{dw}{du} = \frac{w^2}{2} - w + 1$$

分离变量再积分得

$$\frac{du}{u} = \frac{dw}{\frac{1}{2}((w-1)^2 + 1)} \Rightarrow \ln |u| = 2 \arctan(w-1) + C$$

把  $u, w$  代回原变量, 因此得到

$$\ln |y+2| = 2 \arctan \left( \frac{x-1}{y+2} \right) + C$$

33. 解

$$f(x) + f'(-x) = x^2 + \alpha, \quad \alpha \in \mathbb{R}$$

由

$$f(x) + f'(-x) = x^2 + \alpha \quad (1)$$

令  $x \rightarrow -x$ , 得  $f(-x) + f'(x) = x^2 + \alpha$ , 两边求导,

$$-f'(-x) + f''(x) = 2x \quad (2)$$

由 (1) + (2),

$$f''(x) + f(x) = x^2 + 2x + \alpha$$

齐次解为

$$f_c = C_1 \sin x + C_2 \cos x$$

且发现特解为

$$f_p = x^2 + 2x + \alpha - 2$$

于是通解为

$$f(x) = C_1 \sin x + C_2 \cos x + x^2 + 2x + \alpha - 2$$

尚未结束, 与原方程比较系数得

$$f'(-x) + f(x) = (C_1 + C_2)(\cos x + \sin x) + \sqrt{x^2 + \alpha} \Rightarrow C_2 = -C_1$$

故通解为

$$f(x) = C_1(\sin x - \cos x) + x^2 + 2x + \alpha - 2$$

34. 解

$$y = xy' + \sqrt{(y')^2 + 1}$$

此方程属于克莱罗方程 (Clairaut's equation)。对  $x$  求导得

$$y' = y' + xy'' + \frac{1}{x} \frac{x \cdot 2y'y''}{\sqrt{(y')^2 + 1}}$$

整理得

$$y'' \left( x + \frac{y'}{\sqrt{(y')^2 + 1}} \right) = 0$$

若  $y'' = 0$ , 则  $y' = C \Rightarrow y = Cx + \sqrt{C^2 + 1}$ 。对于

$$\left( x + \frac{y'}{\sqrt{(y')^2 + 1}} \right) = 0$$

化简得

$$y' = \pm \frac{x}{\sqrt{1 - x^2}}$$

照常分离系数后积分得

$$\int dy = \pm \int \frac{x}{\sqrt{1 - x^2}} dx \Rightarrow y = \pm \sqrt{1 - x^2} + A$$

将  $y = \pm \sqrt{1 - x^2} + A$  代入原方程解得  $A = 0$ , 故解为

$$y = \pm \sqrt{1 - x^2} \Rightarrow x^2 + y^2 = 1$$

35. 求解

$$\sqrt{\tan x} \frac{dy}{dx} = x$$

分离系数得,

$$\frac{1}{y} dy = \sqrt{\cot x} dx \Rightarrow \ln y = \int \sqrt{\cot x} dx$$

设  $\cot x = u^2$ , 则  $-\csc^2 x dx = 2u du \Rightarrow dx = -\frac{2u du}{1 + u^4}$ ,

$$I = \int \sqrt{\cot x} dx = - \int \frac{2u du}{1 + u^4} = - \int \frac{2 du}{u^2 + \frac{1}{u^2}} = - \int \frac{1 + \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du - \int \frac{1 - \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du = -(I_1 + I_2)$$

其中设  $u - \frac{1}{u} = t$ ,  $\left(1 + \frac{1}{u^2}\right) du = dt$ , 则  $u^2 + \frac{1}{u^2} = t^2 + 2$ ,

$$I_1 = \int \frac{1 + \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du = \int \frac{dt}{t^2 + 2} = \frac{1}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}} \arctan \left( \frac{u^2 - 1}{u\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \arctan \left( \frac{\cot x - 1}{\sqrt{2} \cot x} \right)$$

且设  $u + \frac{1}{u} = \phi$ ,  $\left(1 - \frac{1}{u^2}\right) du = d\phi$ , 则  $u^2 + \frac{1}{u^2} = \phi^2 - 2$ ,

$$\begin{aligned} I_2 &= \int \frac{1 - \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du I_2 = \int \frac{d\phi}{\phi^2 - 2} = \frac{1}{2\sqrt{2}} \int \left( \frac{1}{\phi - \sqrt{2}} - \frac{1}{\phi + \sqrt{2}} \right) d\phi = \frac{1}{2\sqrt{2}} \ln \left| \frac{\phi - \sqrt{2}}{\phi + \sqrt{2}} \right| \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{u^2 - u\sqrt{2} + 1}{u^2 + u\sqrt{2} + 1} \right| = \frac{1}{2\sqrt{2}} \ln \left| \frac{\cot x - \sqrt{2} \cot x + 1}{\cot x + \sqrt{2} \cot x + 1} \right| \end{aligned}$$

于是  $\ln y = I$  给出

$$y = C e^{-\frac{1}{2\sqrt{2}} \arctan \left( \frac{\cot x - 1}{\sqrt{2} \cot x} \right) \left( \frac{\cot x + \sqrt{2} \cot x + 1}{\cot x - \sqrt{2} \cot x + 1} \right)^{\frac{1}{2\sqrt{2}}}}$$

36. 解

$$y'' - y' - 2y = 10 \sin x, \quad y(0) = 0, \quad y'(0) = 1.$$

对应齐次方程的特征方程为

$$r^2 - r - 2 = (r - 2)(r + 1) = 0 \Rightarrow r = 2, -1$$

因此齐次通解为

$$y_c(x) = C_1 e^{2x} + C_2 e^{-x}.$$

取特解

$$y_p(x) = A \cos x + B \sin x$$

代入方程

$$y_p'' - y_p' - 2y_p = (B - 3A) \cos x + (-A - 3B) \sin x = 0$$

比较系数解得  $A = -1, B = -3$ , 因此特解为

$$y_p(x) = -\cos x - 3 \sin x$$

通解为

$$y(x) = y_c + y_p = C_1 e^{2x} + C_2 e^{-x} - \cos x - 3 \sin x.$$

使用初值条件  $y(0) = 0, y'(0) = 1$ ,

$$C_1 + C_2 - 1 = 0 \Rightarrow C_1 + C_2 = 1.$$

$$y'(x) = 2C_1 e^{2x} - C_2 e^{-x} + \sin x - 3 \cos x \implies y'(0) = 2C_1 - C_2 - 3 = 1.$$

解得  $C_1 = 1, C_2 = 0$ , 因此初值问题的解为

$$y(x) = e^{2x} - \cos x - 3 \sin x$$

37. 设初值条件为  $y(0) = 2, y'(0) = 0$ , 求解方程

$$y'' + 4y = 16x \cos 2x$$

对应的齐次方程为  $y'' + 4y = 0$ , 其通解为

$$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$$

非齐次项为  $16x \cos 2x$ , 原方程的特解包含了  $A_0 \cos 2x + B_0 \sin 2x$ , 将与  $y_c(x)$  重叠。因此特解需取

$$y_p(x) = x((A_0 + A_1 x) \cos 2x + (B_0 + B_1 x) \sin 2x).$$

将  $y_p$  代入原方程, 比较系数可得  $A_0 = 1, A_1 = 0, B_0 = 0, B_1 = 2$ , 因此特解为

$$y_p(x) = x \cos 2x + 2x^2 \sin 2x.$$

所以通解为

$$y(x) = y_c + y_p = (C_1 + x) \cos 2x + (C_2 + 2x^2) \sin 2x$$

由  $y(0) = 2$  得  $C_1 = 2$ , 求导得

$$y'(x) = (2x - 2C_1) \sin 2x + (4x^2 + 2C_2 + 1) \cos 2x$$

代入  $x = 0, y'(0) = 0$ , 得到  $C_2 = -\frac{1}{2}$ , 因此满足初值条件的解为

$$y(x) = (x + 2) \cos 2x + \left(2x^2 - \frac{1}{2}\right) \sin 2x$$

38. 设连续函数  $f(x)$  满足

$$f(x) = \frac{1}{4}x^2 + \frac{1}{2} \cos x - \int_0^x (x-t)f(t) dt,$$

求  $f(x)$ 。

即解微分方程

$$y''(x) + y(x) = \frac{1 - \cos x}{2}, \quad y(0) = \frac{1}{2}, \quad y'(0) = 0$$

通解为

$$y = y_h + y_p = C_1 \cos x + C_2 \sin x + \frac{1}{2} - \frac{1}{4}x \sin x$$

代入初值

$$y(0) = C_1 + \frac{1}{2} = \frac{1}{2} \Rightarrow C_1 = 0; \quad y'(0) = C_2 + 0 = 0 \Rightarrow C_2 = 0$$

因此

$$f(x) = \frac{1}{2} - \frac{1}{4}x \sin x$$

39. 考虑二阶线性齐次方程

$$y'' + p(x)y' + q(x)y = 0,$$

其中  $p(x), q(x)$  在某区间  $I$  上连续。

(a) 证明: 若存在常数  $r$  使得对所有  $x \in I$  有

$$r^2 + rp(x) + q(x) = 0,$$

则  $y(x) = e^{rx}$  为该方程的一个解。

直接代入检验, 对于  $y = e^{rx}$ ,

$$y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

代入方程得

$$y'' + p(x)y' + q(x)y = (r^2 + rp(x) + q(x))e^{rx} = 0,$$

因此  $y = e^{rx}$  确为方程的解。

(b) 求出当

$$p(x) = -2 \left( 1 + \frac{1}{x} \right), \quad q(x) = 1 + \frac{2}{x}$$

时方程的通解。

先求常数根  $r$ . 将形式  $r^2 + rp(x) + q(x) = 0$  代入:

$$r^2 + r \left( -2 \left( 1 + \frac{1}{x} \right) \right) + 1 + \frac{2}{x} = 0 \Rightarrow (r^2 - 2r + 1) + \frac{2}{x}(1 - r) = 0.$$

该等式对所有  $x$  成立, 比较系数得

$$r^2 - 2r + 1 = 0, \quad 1 - r = 0.$$

由第二式得  $r = 1$ , 并满足第一式, 于是其中一解为

$$y_1(x) = e^x$$

由降阶法公式, 第二个线性独立解为

$$y_2 = y_1 \int \frac{1}{y_1^2} \exp \left( \int p(x) dx \right) dx = e^x \int e^{-2x} \exp \left( 2 \int \left( 1 + \frac{1}{x} \right) dx \right) = e^x \int x^2 dx = \frac{1}{3} x^3 e^x$$

因此通解为

$$y(x) = (C_1 + C_2 x^3) e^x$$

40. 以换元

$$x = \int e^{-\frac{t^2}{2}} dt,$$

解微分方程

$$y''(t) + ty'(t) + e^{-t^2} y(t) = 0$$

令  $z(x) = y(t)$ , 由链导法,

$$y' = \frac{dy}{dt} = \frac{dz}{dx} \frac{dx}{dt} = e^{-\frac{t^2}{2}} \frac{dz}{dx}$$

且

$$\begin{aligned} y'' &= \frac{d}{dt} \left( e^{-\frac{t^2}{2}} \frac{dz}{dx} \right) = -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-\frac{t^2}{2}} \frac{d}{dt} \left( \frac{dz}{dx} \right) \\ &= -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-\frac{t^2}{2}} \frac{d}{dx} \left( \frac{dz}{dx} \right) \frac{dx}{dt} \\ &= -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-\frac{t^2}{2}} \frac{d^2 z}{dx^2} e^{-\frac{t^2}{2}} \\ &= -te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-t^2} \frac{d^2 z}{dx^2}. \end{aligned}$$

把  $y', y''$  代入原方程得

$$\left(-te^{-\frac{t^2}{2}} \frac{dz}{dx} + e^{-t^2} \frac{d^2z}{dx^2}\right) + t \left(e^{-\frac{t^2}{2}} \frac{dz}{dx}\right) + e^{-t^2} z = e^{-t^2} \frac{d^2z}{dx^2} + e^{-t^2} z = 0 \Rightarrow \frac{d^2z}{dx^2} + z = 0$$

解为

$$z(x) = C_1 \cos x + C_2 \sin x \Rightarrow y(t) = C_1 \cos \left(\int e^{-\frac{t^2}{2}} dt\right) + C_2 \sin \left(\int e^{-\frac{t^2}{2}} dt\right)$$

#### 41. 解方程

$$ty''(t) + (t^2 - 1)y'(t) + t^3y(t) = 0$$

欲求合适的  $x = v(t)$ , 令  $z(x) = y(t)$ , 则

$$y' = \frac{dy}{dt} = \frac{dz}{dx} \frac{dx}{dt} = \frac{dz}{dx} \frac{dv}{dt},$$

且

$$\begin{aligned} y'' &= \frac{d}{dt} \left( \frac{dv}{dt} \frac{dz}{dx} \right) = \frac{d^2v}{dt^2} \frac{dz}{dx} + \frac{dv}{dt} \frac{d}{dt} \left( \frac{dz}{dx} \right) \\ &= \frac{d^2v}{dt^2} \frac{dz}{dx} + \frac{dv}{dt} \frac{d}{dx} \left( \frac{dz}{dx} \right) \frac{dx}{dt} \\ &= \frac{d^2v}{dt^2} \frac{dz}{dx} + \left( \frac{dv}{dt} \right)^2 \frac{d^2z}{dx^2} \end{aligned}$$

代入原方程, 得到

$$t \left( \frac{dv}{dt} \right)^2 \frac{d^2z}{dx^2} + \left[ t \frac{d^2v}{dt^2} + (t^2 - 1) \frac{dv}{dt} \right] \frac{dz}{dx} + t^3z = 0$$

为使方程为常系数方程, 不妨考虑变换

$$t \left( \frac{dv}{dt} \right)^2 = t^3 \Rightarrow \frac{dv}{dt} = t \Rightarrow v = \frac{t^2}{2}$$

此时原方程可化为

$$\frac{d^2z}{dx^2} + \frac{dz}{dx} + z = 0$$

其通解为

$$z(x) = e^{-\frac{x}{2}} \left( C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right)$$

代回  $x = \frac{t^2}{2}$ , 即

$$y(t) = e^{-\frac{t^2}{4}} \left( C_1 \cos \frac{\sqrt{3}t^2}{4} + C_2 \sin \frac{\sqrt{3}t^2}{4} \right)$$

42. 设  $y_1, y_2$  为方程

$$y'' + p(x)y' + q(x)y = 0, \quad x > 0 \quad (1)$$

的两个解, 其中  $p(x), q(x)$  在  $x > 0$  上连续。已知

$$y_1 = \frac{\sin x}{\sqrt{x}}$$

且  $y_1, y_2$  的朗斯基行列式 (Wronskian) 为

$$W(x) = \frac{1}{x}$$

(a) 证明阿贝尔恒等式 (Abel's Identity):

$$W(x) = C \exp \left( - \int p(x) dx \right)$$

由朗斯基行列式的定义,

$$W(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2$$

于是

$$\begin{aligned} W'(x) &= (y_1 y_2' - y_1' y_2)' = (y_1' y_2' + y_1 y_2'') - (y_1'' y_2 + y_1' y_2') \\ &= y_1 y_2'' - y_1'' y_2 = y_1(-p y_2' - q y_2) - (-p y_1' - q y_1) y_2 \\ &= -p y_1 y_2' + p y_1' y_2 = -p(y_1 y_2' - y_1' y_2) = -p(x)W(x) \end{aligned}$$

因此

$$W'(x) + p(x)W(x) = 0$$

由此可解得

$$W(x) = C \exp \left( - \int p(x) dx \right)$$

即得证。

(b) 求方程 (1) 中的函数  $p(x), q(x)$ 。

由阿贝尔恒等式,

$$p(x) = -\frac{W'(x)}{W(x)} = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$$

由于  $y_1$  是原方程的解,

$$y_1'' + py_1' + qy_1 = 0 \Rightarrow q(x) = -\frac{y_1'' + py_1'}{y_1} = 1 - \frac{1}{4x^2}$$

其中

$$y_1' = \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{\frac{3}{2}}}, \quad y_1'' = \frac{3 \sin x}{4x^{\frac{5}{2}}} - \frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x^{\frac{3}{2}}}$$

因此原微分方程为

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0, \quad x > 0$$

(c) 求方程 (1) 的通解。

利用朗斯基行列式求  $y_2$ 。已知

$$W(x) = y_1y_2' - y_1'y_2 = \frac{1}{x} \Rightarrow y_2' - \frac{y_1'}{y_1}y_2 = \frac{1}{xy_1}$$

即为一阶线性方程

$$y_2' + \left(\frac{1}{2x} - \cot x\right)y_2 = \frac{1}{x \sin x}.$$

其积分因子为

$$I(x) = \exp\left(\int \left(\frac{1}{2x} - \cot x\right) dx\right) = \exp(\ln \sqrt{x} - \ln |\sin x|) = \pm \frac{\sqrt{x}}{\sin x}$$

取  $I(x) = \frac{\sqrt{x}}{\sin x}$ , 由此

$$\begin{aligned} y_2(x) &= \frac{\sin x}{\sqrt{x}} \left( \int \frac{\sqrt{x}}{\sin x} \cdot \frac{1}{x \sin x} dx + C \right) \\ &= \frac{\sin x}{\sqrt{x}} \left( \int \frac{1}{\sin^2 x} dx + C \right) \\ &= \frac{\sin x}{\sqrt{x}} (-\cot x + C) \\ &= -\frac{\cos x}{\sqrt{x}} + Cy_1 \end{aligned}$$

因此通解为

$$y(x) = C_1 \frac{\sin x}{\sqrt{x}} + C_2 \frac{\cos x}{\sqrt{x}}.$$

43. (a) 证明法国数学家刘维尔 (Liouville) 的结论: 若黎卡提 (Riccati) 方程

$$y' = p(x)y^2 + q(x)y + r(x)$$

已知一个特解  $\bar{y}(x)$ , 令  $y = z + \bar{y}$ , 则用此代换可把方程化为一个阶数  $n = 2$  的伯努利 (Bernoulli) 方程, 从而可通过解一个一阶线性方程得到一般解。

设  $y = z + \bar{y}$ , 代入原方程得

$$z' + \bar{y}' = p(z + \bar{y})^2 + q(z + \bar{y}) + r = pz^2 + 2pz\bar{y} + qz + (p\bar{y}^2 + q\bar{y} + r)$$

由于  $\bar{y}$  是原方程的一个解, 故  $\bar{y}' = p\bar{y}^2 + q\bar{y} + r$ , 于是

$$z' = pz^2 + 2pz\bar{y} + qz \Rightarrow z' - (2p\bar{y} + q)z = pz^2$$

这正是阶数  $n = 2$  的伯努利方程。由通解公式,

$$z^{1-n} = e^{-\int(1-n)(2p\bar{y}+q)dx} \left\{ \int (1-n)p(x)e^{\int(1-n)(2p\bar{y}+q)dx} dx + C \right\}.$$

代入  $n = 2$  即得

$$z^{-1} = e^{-\int(2p\bar{y}+q)dx} \left\{ -\int p(x)e^{\int(2p\bar{y}+q)dx} dx + C \right\}.$$

由此得到  $z$ , 再由  $y = z + \bar{y}$  得原方程的通解, 证毕。

(b) 求下列黎卡提方程的通解:

i.  $y'e^{-x} + y^2 - 2ye^x = 1 - e^{2x}$ , 已知特解  $\bar{y}(x) = e^x$ 。

原方程可写为

$$y' = -e^x y^2 + 2e^{2x} y + e^x(1 - 2e^{2x})$$

因此

$$p(x) = -e^x, \quad q(x) = 2e^{2x}, \quad r(x) = e^x(1 - 2e^{2x}), \quad \bar{y} = e^x$$

且

$$\int (2p\bar{y} + q) dx = \int (2(-e^x)e^x + 2e^{2x}) dx = 0$$

于是上式积分指数因子为常数, 代入公式得

$$z^{-1} = -\int (-e^x) dx + C = e^x + C \Rightarrow z = \frac{1}{C + e^x}$$

代回  $y = z + \bar{y}$ , 得到通解

$$y = e^x + \frac{1}{C + e^x}$$

ii.  $x^2(y' + y^2) = 2$ , 已知特解  $\bar{y}(x) = -\frac{1}{x}$ 。

将方程写成标准形,

$$y' = -y^2 + \frac{2}{x^2}$$

令  $y = z + \bar{y} = z - \frac{1}{x}$ , 代入并化简可得关于  $z$  的方程

$$z' - \frac{2}{x}z = -z^2,$$

是阶数  $n = 2$  的伯努利方程, 由公式计算得

$$z^{-1} = e^{\int \frac{2}{x} dx} \left( \int e^{-\int \frac{2}{x} dx} dx + C \right) = \frac{1}{x^2} \left( \int x^2 dx + C \right) = \frac{1}{x^2} \left( \frac{x^3}{3} + C \right)$$

即

$$z = \frac{3x^2}{x^3 + 3C}$$

代回  $y = z - \frac{1}{x}$ , 化为

$$y = \frac{3x^2}{x^3 + 3C} - \frac{1}{x} = \frac{2x^3 - 3C}{x(3x^3 + 3C)}.$$

令常数  $C_1 = 3C$ ,

$$xy = \frac{2x^3 - C_1}{3x^3 + C_1}$$

44. 求解

$$x^3 - \frac{dy}{dx} = \frac{y}{x}(y - 2)$$

发现此为一黎卡提方程, 观察得  $y_1 = x^2$  是一特解。设  $y = x^2 + \frac{1}{u}$ , 则  $\frac{dy}{dx} = 2x - \frac{1}{u^2} \frac{du}{dx}$ , 原方程给出

$$x^3 - \left( 2x - \frac{1}{u^2} \frac{du}{dx} \right) = x(x^2 - 2) + \frac{2(x^2 - 1)}{xu} + \frac{1}{xu^2}$$

再化简得一阶线性方程

$$\frac{du}{dx} - \left( 2x - \frac{2}{x} \right) u = \frac{1}{x}$$

积分因子为

$$I(x) = \exp \left( \int - \left( 2x - \frac{2}{x} \right) dx \right) = \exp (-x^2 + 2 \ln |x|) = x^2 e^{-x^2}$$

所以

$$u = \frac{e^{-x^2}}{x^2} \left( \int x e^{-x^2} + C_1 \right) = \frac{C_1 e^{x^2} - \frac{1}{2}}{x^2}$$

回代  $y = x^2 + \frac{1}{u}$  得

$$y = x^2 + \frac{x^2}{C_1 e^{x^2} - \frac{1}{2}} = x^2 \frac{C_1 e^{x^2} + \frac{1}{2}}{C_1 e^{x^2} - \frac{1}{2}}.$$

令  $C_2 = 2C_1$ , 则等价地写成

$$y = x^2 \frac{C_2 e^{x^2} + 1}{C_2 e^{x^2} - 1}$$

45. 考虑柯西-欧拉方程 (Cauchy-Euler Equation)

$$ax^2 y'' + bxy' + cy = 0, \quad x > 0,$$

一般解法为作变换  $x = e^t$  或  $t = \ln x$ , 则原方程变为常系数方程

$$az''(t) + (b-a)z'(t) + cz(t) = 0,$$

解得  $z(t)$  后代回  $y(x) = z(\ln x)$ 。据此, 求解

(a)  $x^2 y'' - 3xy' + 4y = 0$

方程为  $x^2 y'' - 3xy' + 4y = 0$ , 所以  $a = 1, b = -3, c = 4$ , 对应  $z$  方程为

$$z'' - 4z' + 4z = 0$$

特征方程为

$$r^2 - 4r + 4 = (r - 2)^2 = 0 \Rightarrow r = 2$$

因此解得

$$z(t) = C_1 e^{2t} + C_2 t e^{2t} \Rightarrow y(x) = z(\ln x) = C_1 x^2 + C_2 x^2 \ln x$$

(b)  $x^2 y'' - xy' - 35y = 0$

方程为  $x^2 y'' - xy' - 35y = 0$ , 所以  $a = 1, b = -1, c = -35$ 。对应  $z$  方程

$$z'' + (b-a)z' + cz = z'' + (-1-1)z' - 35z = z'' - 2z' - 35z = 0$$

其中特征方程为

$$r^2 - 2r - 35 = (r - 7)(r + 5) = 0 \Rightarrow r_1 = 7, r_2 = -5$$

因此通解为

$$z(t) = C_1 e^{7t} + C_2 e^{-5t} \Rightarrow y(x) = z(\ln x) = C_1 x^7 + C_2 x^{-5}$$

46. 解非齐次欧拉方程

$$x^3 y''' + x^2 y'' - x y' = 24x \ln x, \quad x > 0$$

令  $x = e^t \Rightarrow t = \ln x$ , 由链导法,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \\ \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left( \frac{dy}{dt} \frac{dt}{dx} \right) = \frac{d}{dx} \left( \frac{1}{x} \cdot \frac{dy}{dt} \right) = -\frac{dy}{x^2 dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \frac{1}{x} = \frac{1}{x^2} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \\ \frac{d^3 y}{dx^3} &= \frac{d}{dx} \left( \frac{1}{x^2} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \right) = -\frac{2}{x^3} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + \frac{1}{x^2} \left( \frac{d^3 y}{dt^3} \cdot \frac{1}{x} - \frac{d^2 y}{dt^2} \cdot \frac{1}{x} \right) \\ &= \frac{1}{x^3} \frac{d^3 y}{dt^3} - \frac{3}{x^3} \frac{d^2 y}{dt^2} + \frac{2}{x^3} \frac{dy}{dt} \end{aligned}$$

代入方程得到

$$\frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y = 24e^{tt}$$

齐次方程  $y''' - 3y'' + 3y' - y = 0$  的特征方程

$$r^3 - 3r^2 + 3r - 1 = 0 \Rightarrow r = 1$$

故齐次解为

$$y_c = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

设特解为

$$y_p = t^3 (At + B) e^t = (At^4 + Bt^3) e^t$$

代入方程解得

$$24At + 6B = 24t \Rightarrow A = 1, B = 0$$

因此通解为

$$y = t^4 e^t + C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

代回  $t = \ln x$ , 原方程的解为

$$y = x(\ln x)^4 + C_1 x + C_2 x \ln x + C_3 x (\ln x)^2$$

## 47. 以参数变换法解

$$y'' + 6y' + 9y = \frac{2e^{-4x}}{x^2 + 1}.$$

对应齐次方程通解为

$$y_c(x) = C_1 e^{-3x} + C_2 x e^{-3x}$$

设特解为

$$y_p(x) = u_1(x)e^{-3x} + u_2(x)xe^{-3x}$$

其中  $u_1, u_2$  满足

$$\begin{bmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2e^{-4x}}{x^2+1} \end{bmatrix}$$

朗斯基行列式为

$$W(y_1, y_2) = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{vmatrix} = e^{-6x}$$

由克兰姆法则,

$$u'_1 = \frac{-xe^{-3x} \cdot \frac{2e^{-4x}}{x^2+1}}{e^{-6x}} = -\frac{2x}{x^2+1} \Rightarrow u_1 = -\ln(1+x^2),$$

$$u'_2 = \frac{e^{-3x} \cdot \frac{2e^{-4x}}{x^2+1}}{e^{-6x}} = \frac{2}{x^2+1} \Rightarrow u_2 = 2 \tan^{-1} x.$$

因此特解为

$$y_p(x) = e^{-3x} (-\ln(1+x^2) + 2x \tan^{-1} x)$$

通解为

$$y(x) = e^{-3x} (C_1 + C_2 x + 2x \tan^{-1} x - \ln(1+x^2))$$

## 48. 求解

$$y'' - 2y' + y = 4e^x x^3 \ln x, \quad x > 0$$

对应齐次方程的通解为

$$y_c(x) = C_1 e^x + C_2 x e^x,$$

设特解为

$$y_p(x) = u_1(x)e^x + u_2(x)xe^x,$$

其中  $u_1, u_2$  满足

$$\begin{bmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4e^x x^3 \ln x \end{bmatrix}.$$

两边同时除以  $e^x$  得

$$\begin{bmatrix} 1 & x \\ 1 & x+1 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4x^3 \ln x \end{bmatrix}.$$

朗斯基行列式为  $(x+1) - x = 1$ , 由克兰姆法则,

$$u'_1 = -x \cdot 4x^{-2} \ln x = -4x^{-2} \ln x, \quad u'_2 = 4x^{-3} \ln x$$

于是

$$\begin{aligned} u_1 &= -4 \int x^{-2} \ln x \, dx = \frac{4 \ln x}{x} - 4 \int \frac{1}{x^2} \, dx = \frac{4(\ln x + 1)}{x} \\ u_2 &= 4 \int x^{-3} \ln x \, dx = -\frac{2 \ln x}{x^2} + 2 \int \frac{1}{x^3} \, dx = \frac{-2 \ln x - 1}{x^2} \end{aligned}$$

因此特解为

$$y_p(x) = e^x u_1 + x e^x u_2 = e^x \left( \frac{4(\ln x + 1)}{x} + x \cdot \frac{-2 \ln x - 1}{x^2} \right) = e^x \frac{2 \ln x + 3}{x}$$

所以方程通解为

$$y(x) = e^x \left( C_1 + C_2 x + \frac{2 \ln x + 3}{x} \right)$$

#### 49. 求解方程

$$y'' + 4y' + 4y = 15e^{-2x} \ln x + 25 \cos x, \quad x > 0$$

对应齐次方程的通解为

$$y_c(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

第一项非齐次  $f_1(x) = 15e^{-2x} \ln x$ : 设特解为

$$y_{p1} = u_1(x) e^{-2x} + u_2(x) x e^{-2x}$$

其中  $u_1, u_2$  满足

$$\begin{bmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 15e^{-2x} \ln x \end{bmatrix}$$

两边除以  $e^{-2x}$  得

$$\begin{bmatrix} 1 & x \\ -2 & 1-2x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \ln x \end{bmatrix}$$

解得

$$u_1' = -15x \ln x \Rightarrow u_1 = -15 \int x \ln x dx = -\frac{15}{2}x^2 \ln x + 15 \int \frac{x}{2} dx = \frac{15}{4}x^2(-2 \ln x + 1)$$

$$u_2' = 15 \ln x \Rightarrow u_2 = 15 \int \ln x dx = 15x(\ln x - 1)$$

因此

$$y_{p1}(x) = e^{-2x} \left[ \frac{15}{4}x^2(-2 \ln x + 1) + 15x(\ln x - 1) \right] = \frac{15}{4}x^2(2 \ln x - 3)e^{-2x}$$

第二项非齐次  $f_2(x) = 25 \cos x$ : 设特解

$$y_{p2}(x) = A \cos x + B \sin x$$

代入方程得

$$(3A + 4B) \cos x + (-4A + 3B) \sin x = 25 \cos x$$

比较系数解得  $A = 3, B = 4$ , 所以

$$y_{p2}(x) = 3 \cos x + 4 \sin x$$

综上, 方程通解为

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{15}{4}x^2(2 \ln x - 3)e^{-2x} + 3 \cos x + 4 \sin x$$

50. 解

$$y''' - y'' + y' - y = e^{-x} \sin x$$

齐次方程

$$y''' - y'' + y' - y = 0$$

的特征方程为

$$(r-1)(r^2+1)=0 \Rightarrow r=1, \pm i$$

所以齐次解为

$$y_c(x) = C_1 e^x + C_2 \cos x + C_3 \sin x$$

对于非齐次项  $e^{-x} \sin x$ , 尝试特解

$$y_p(x) = e^{-x}(A \cos x + B \sin x)$$

则

$$y_p' = -e^{-x}((A+B) \sin x + (A-B) \cos x)$$

$$y_p'' = 2e^{-x}(A \sin x + B \cos x)$$

$$y_p''' = 2e^{-x}((B-A) \sin x + (A+B) \cos x)$$

代入原方程解得  $A = \frac{1}{5}, B = 0$ , 于是

$$y_p(x) = -\frac{1}{5}e^{-x} \cos x$$

因此通解为

$$y(x) = C_1 e^x + C_2 \cos x + C_3 \sin x - \frac{1}{5}e^{-x} \cos x$$

51. 求解微分方程

$$y''' + 3y'' + 3y' + y = \frac{2e^{-x}}{1+x^2}$$

齐次解为

$$y_c(x) = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$$

使用参数变换法, 设

$$y_p(x) = u_1(x)e^{-x} + u_2(x)xe^{-x} + u_3(x)x^2e^{-x}$$

其中  $u_1, u_2, u_3$  满足

$$\begin{cases} e^{-x}u_1' + (xe^{-x})u_2' + (x^2e^{-x})u_3' = 0 \\ (e^{-x})'u_1' + (xe^{-x})'u_2' + (x^2e^{-x})'u_3' = 0 \\ (e^{-x})''u_1' + (xe^{-x})''u_2' + (x^2e^{-x})''u_3' = \frac{2e^{-x}}{1+x^2} \end{cases}$$

化简得

$$\begin{cases} u_1' + xu_2' + x^2u_3' = 0 \\ -u_1' + (1-x)u_2' + (2x-x^2)u_3' = 0 \\ u_1' + (-2+x)u_2' + (2-4x+x^2)u_3' = \frac{2}{1+x^2} \end{cases}$$

其中增广矩阵为

$$\left[ \begin{array}{ccc|c} 1 & x & x^2 & 0 \\ -1 & 1-x & 2x-x^2 & 0 \\ 1 & -2+x & 2-4x+x^2 & \frac{2}{1+x^2} \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & x & x^2 & 0 \\ 0 & 1 & 2x & 0 \\ 0 & 0 & 1 & \frac{1}{1+x^2} \end{array} \right]$$

逐步解得

$$u_3' = \frac{1}{1+x^2} \Rightarrow u_3 = \tan^{-1} x$$

$$u_2' = -2xu_3' = -\frac{2x}{1+x^2} \Rightarrow u_2 = -\ln(1+x^2)$$

$$u_1' = -xu_2' - x^2u_3' = \frac{x^2}{1+x^2} \Rightarrow u_1 = x - 2\tan^{-1} x$$

于是特解为

$$y_p = e^{-x} [(2x - 2\tan^{-1} x) + (-\ln(1+x^2))x + (\tan^{-1} x)x^2]$$

整理得通解为

$$y(x) = e^{-x} [C_1 + C_2x + C_3x^2 + (x^2 - 2)\tan^{-1} x + x - x\ln(1+x^2)]$$

## 52. 解方程

$$y''' + y' = \sec x$$

齐次方程  $y''' + y' = 0$  的特征方程为

$$r^3 + r = 0 \Rightarrow r = 0, \pm i$$

所以齐次解为

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

使用参数变易法, 设特解为

$$y_p = u_1y_1 + u_2y_2 + u_3y_3$$

其中  $u_1, u_2, u_3$  满足

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sec x \end{bmatrix}$$

其中增广矩阵为

$$\left[ \begin{array}{ccc|c} 1 & \cos x & \sin x & 0 \\ 0 & -\sin x & \cos x & 0 \\ 0 & -\cos x & -\sin x & \sec x \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 1 & \cos x & \sin x & 0 \\ 0 & -\sin x & \cos x & 0 \\ 1 & 0 & 0 & \sec x \end{array} \right]$$

于是

$$\begin{cases} u'_1 = \sec x \\ u'_2 = \frac{\cos x}{-\sin x} u'_3 = -1 \\ u'_3 = \frac{\sin x}{\cos x} \end{cases} \Rightarrow \begin{cases} u_1 = |\sec x + \tan x| \\ u_2 = -x \\ u_3 = \ln |\cos x| \end{cases}$$

因此通解为

$$y = |\sec x + \tan x| - x \cos x + \ln |\cos x| \sin x + C_1 + C_2 \cos x + C_3 \sin x$$

### 53. 解联立微分方程式

$$\begin{cases} x'_1 = x_1 + 2x_2 \\ x'_2 = 2x_1 - 2x_2 \end{cases}$$

$$x'_1 = x_1 + 2x_2 \quad (1)$$

$$x'_2 = 2x_1 - 2x_2 \quad (2)$$

对 (1) 求导, 且由 (2),

$$x''_1 = x'_1 + 2x'_2 = x'_1 + 2(2x_1 - 2x_2) = x'_1 + 4x_1 - 4x_2 = x'_1 + 4x_1 - 2(x'_1 - x_1) = -x'_1 + 6x_1$$

得关于  $x_1$  的二阶常微分方程, 因此  $x_1$  的通解为

$$x_1(t) = C_1 e^{-3t} + C_2 e^{2t}$$

由 (1),

$$x_2 = \frac{1}{2}(x_1' - x_1) = -2C_1e^{-3t} + \frac{C_2}{2}e^{2t}$$

最终将系统的通解写成向量形式:

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{2t}$$

54. 求解

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$$

设系数矩阵为  $\mathbf{A}$ , 首先求  $\mathbf{A}$  的特征值, 通过解

$$\det(\mathbf{A} - \lambda \mathbf{I}_3) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda + 1)^2(\lambda - 2)$$

因此特征值为

$$\lambda_1 = -1 \ (m_1 = 2), \quad \lambda_2 = 2$$

接着求特征向量。对  $\lambda_1 = -1$ , 解

$$(\mathbf{A} - \lambda_1 \mathbf{I}_3) \mathbf{v} = \mathbf{0} \implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

得到方程

$$v_1 + v_2 + v_3 = 0$$

有两个自由变量, 设  $v_2 = r, v_3 = s$ , 则  $v_1 = -r - s$ , 因此特征空间由两个线性无关向量张成:

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

对应解为

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

对  $\lambda_2 = 2$ , 解

$$(\mathbf{A} - \lambda_2 \mathbf{I}_3) \mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

化简矩阵得到行阶梯形:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{2}R_1 + R_2} \begin{pmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xrightarrow[R_3 \rightarrow \frac{2}{3}R_3]{R_3 \rightarrow R_2 + R_3} \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

秩为 2, 设  $v_3 = r$ , 则  $v_1 = v_2 = r$ , 得到特征向量

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

对应解为

$$\mathbf{x}_3(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

因此系统的一般解为

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

55. 求解

$$\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$$

设系数矩阵为  $\mathbf{A}$ 。首先求特征值, 通过解

$$\det(\mathbf{A} - \lambda \mathbf{I}_3) = \begin{vmatrix} -3 - \lambda & 0 & 2 \\ 1 & -1 - \lambda & 0 \\ -2 & -1 & -\lambda \end{vmatrix} = -(\lambda + 2)(\lambda^2 + 2\lambda + 3)$$

得特征值

$$\lambda_{1,2} = -1 \pm \sqrt{2}i, \quad \lambda_3 = -2$$

接着求对应特征向量。对  $\lambda_1 = -1 + \sqrt{2}i$ , 解

$$(\mathbf{A} - \lambda_1 \mathbf{I}_3) \mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -2 - \sqrt{2}i & 0 & 2 \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1 - \sqrt{2}i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

化为行阶梯形矩阵

$$\begin{pmatrix} -2 - \sqrt{2}i & 0 & 2 \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1 - \sqrt{2}i \end{pmatrix} \xrightarrow{R_1 \rightarrow (-2 + \sqrt{2}i)R_1/6} \begin{pmatrix} 1 & 0 & \frac{1}{3}(-2 + \sqrt{2}i) \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1 - \sqrt{2}i \end{pmatrix}$$
$$\xrightarrow[\begin{smallmatrix} R_1 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + 2R_1 \end{smallmatrix}]{\begin{smallmatrix} R_2 \rightarrow R_2 / (\sqrt{2}i) \\ R_3 \rightarrow -R_3 \end{smallmatrix}} \begin{pmatrix} 1 & 0 & \frac{1}{3}(-2 + \sqrt{2}i) \\ 0 & -\sqrt{2}i & \frac{1}{3}(2 - \sqrt{2}i) \\ 0 & -1 & -\frac{1}{3}(1 + \sqrt{2}i) \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & \frac{1}{3}(-2 + \sqrt{2}i) \\ 0 & 1 & \frac{1}{3}(1 + \sqrt{2}i) \\ 0 & 0 & 0 \end{pmatrix}$$

令  $v_3 = r$  为自由变量, 则

$$\mathbf{v} = r \begin{pmatrix} -\frac{1}{3}(-2 + \sqrt{2}i) \\ -\frac{1}{3}(1 + \sqrt{2}i) \\ 1 \end{pmatrix}.$$

取  $r = -3$  并利用欧拉公式分离实部和虚部

$$\begin{aligned} e^{(-1+\sqrt{2}i)t} \begin{pmatrix} -2 + \sqrt{2}i \\ 1 + \sqrt{2}i \\ -3 \end{pmatrix} &= e^{-t} (\cos(\sqrt{2}t) + i \sin(\sqrt{2}t)) \begin{pmatrix} -2 + \sqrt{2}i \\ 1 + \sqrt{2}i \\ -3 \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) + i(\sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t)) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) + i(\sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t)) \\ -3 \cos(\sqrt{2}t) - 3i \sin(\sqrt{2}t) \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ -3 \cos(\sqrt{2}t) \end{pmatrix} + i e^{-t} \begin{pmatrix} \sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3 \sin(\sqrt{2}t) \end{pmatrix} \end{aligned}$$

得到两个线性无关解

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ -3 \cos(\sqrt{2}t) \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} \sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3 \sin(\sqrt{2}t) \end{pmatrix}$$

对  $\lambda_3 = -2$ , 解

$$(\mathbf{A} - \lambda_3 \mathbf{I}_3) \mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

化简矩阵得到

$$\begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 + R_1} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

令  $v_3 = s$ , 得到特征向量

$$\mathbf{v}_3 = s \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

对应解为

$$\mathbf{x}_3(t) = e^{-2t} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

因此系统的一般解为

$$\begin{aligned} \mathbf{x}(t) = & c_1 e^{-t} \begin{pmatrix} -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \\ -3 \cos(\sqrt{2}t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sqrt{2} \cos(\sqrt{2}t) - 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3 \sin(\sqrt{2}t) \end{pmatrix} \\ & + c_3 e^{-2t} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

## 56. 判断矩阵

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

是否缺陷, 并求系统  $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$  的通解。

首先求  $\mathbf{A}$  的特征值:

$$\det(\mathbf{A} - \lambda \mathbf{I}_3) = \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & 0 & 4 - \lambda \end{vmatrix} = (1 - \lambda)^2(4 - \lambda)$$

因此特征值为  $\lambda_1 = 4, \lambda_2 = 1$  ( $m_2 = 2$ )。对于  $\lambda_1 = 4$ , 解

$$(\mathbf{A} - 4\mathbf{I}_3)\mathbf{v} = \mathbf{0} \implies \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

得到特征向量及其对应解

$$\mathbf{v} = r \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad r \neq 0, \quad \mathbf{x}_1(t) = e^{4t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

对于  $\lambda_2 = 1$ , 解

$$(\mathbf{A} - \mathbf{I}_3)\mathbf{v} = \mathbf{0} \implies \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

得到特征向量及其对应解

$$\mathbf{v} = s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s \neq 0, \quad \mathbf{x}_2(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

总共有两个线性无关特征向量, 因此矩阵  $\mathbf{A}$  是缺陷的。为求第三个线性无关解, 设

$$\mathbf{x}_3(t) = (\mathbf{v}t + \mathbf{w})e^t$$

其中  $\mathbf{v}$  和  $\mathbf{w}$  满足

$$(\mathbf{A} - \mathbf{I}_3)\mathbf{v} = \mathbf{0}, \quad (\mathbf{A} - \mathbf{I}_3)\mathbf{w} = \mathbf{v} \implies (\mathbf{A} - \mathbf{I}_3)^2\mathbf{w} = \mathbf{0}$$

解得

$$(\mathbf{A} - \mathbf{I}_3)^2 \mathbf{w} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies w_3 = 0$$

自由选择  $w_1, w_2$ , 取

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = (\mathbf{A} - \mathbf{I}_3)\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

因此得到第三个解

$$\mathbf{x}_3(t) = (\mathbf{v}t + \mathbf{w})e^t = e^t \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

最终通解为

$$\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

### 57. 求解非齐次系统

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{F}(t),$$

其中

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix}, \quad \mathbf{F}(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

首先求  $\mathbf{A}$  的特征值和特征向量:

$$\det(\mathbf{A} - \lambda \mathbf{I}_2) = \begin{vmatrix} -\lambda & 1 \\ -9 & 6 - \lambda \end{vmatrix} = -\lambda(6 - \lambda) + 9 = (\lambda - 3)^2$$

因此特征值为  $\lambda = 3$  ( $m = 2$ ), 对  $\lambda = 3$ , 解

$$(\mathbf{A} - 3\mathbf{I}_2)\mathbf{v} = \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得到特征向量

$$\mathbf{v} = r \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad r \neq 0$$

对应解为

$$\mathbf{x}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

矩阵  $\mathbf{A}$  是缺陷的, 第二个线性无关解取形如

$$\mathbf{x}_2(t) = (\mathbf{v}t + \mathbf{w})e^{3t}$$

其中  $\mathbf{v}$  和  $\mathbf{w}$  满足

$$(\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{v} = \mathbf{0}, \quad (\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{w} = \mathbf{v} \implies (\mathbf{A} - \lambda \mathbf{I}_2)^2 \mathbf{w} = \mathbf{0}$$

解得

$$\begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

取

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \mathbf{v} = (\mathbf{A} - 3\mathbf{I}_2)\mathbf{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

于是第二个解为

$$\mathbf{x}_2(t) = (\mathbf{v}t + \mathbf{w})e^{3t} = \begin{pmatrix} t \\ 3t + 1 \end{pmatrix} e^{3t}$$

因此齐次系统通解为

$$\mathbf{x}_c(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} t \\ 3t + 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t + 1)e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

记基础矩阵为  $\mathbf{X}$ , 寻找特解

$$\mathbf{x}_p(t) = \mathbf{X}\mathbf{u} \implies \mathbf{X}\mathbf{u}' = \mathbf{F} \implies \mathbf{u}' = \mathbf{X}^{-1}\mathbf{F}$$

计算得到

$$\begin{aligned} \mathbf{X}^{-1}\mathbf{F} &= e^{-3t} \begin{pmatrix} 3t + 1 & -t \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} = e^{-3t} \begin{pmatrix} -t^2 \\ t \end{pmatrix} \\ \mathbf{u} &= \int \mathbf{X}^{-1}\mathbf{F} dt = \begin{pmatrix} \int -t^2 e^{-3t} dt \\ \int t e^{-3t} dt \end{pmatrix} = e^{-3t} \begin{pmatrix} \frac{1}{3}t^2 + \frac{2}{9}t + \frac{2}{27} \\ -\frac{1}{3}t - \frac{1}{9} \end{pmatrix} \end{aligned}$$

于是得到特解

$$\mathbf{x}_p(t) = \mathbf{X}\mathbf{u} = \begin{pmatrix} 1 & t \\ 3 & 3t + 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3}t^2 + \frac{2}{9}t + \frac{2}{27} \\ -\frac{1}{3}t - \frac{1}{9} \end{pmatrix} = \begin{pmatrix} \frac{1}{9}t + \frac{2}{27} \\ \frac{1}{9} \end{pmatrix}.$$

最终非齐次系统通解为

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_c(t) = \begin{pmatrix} \frac{1}{9}t + \frac{2}{27} \\ \frac{1}{9} \end{pmatrix} + c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} t \\ 3t + 1 \end{pmatrix}.$$

# 参考出处



- Arts of Problem Solving (AoPS): Contest Collections
- Brilliant (Kudos to the unmonetized version in the past)
- International Mathematics Competition (IMC) for University Students
- Missouri Collegiate Mathematics Competition
- American Mathematics Competition 10
- University of Waterloo CEMC - Euclid, Fermat, Cayley, Hypatia, Galois
- Lehigh University High School Math Contest
- Joint Entrance Examination (Advanced)
- The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics
- LetsSolveMathProblems - Weekly Math Challenges
- Maths 505 - Differential Equations
- Michael Penn
- Prime Newtons
- TRML
- MadasMaths
- 日本留学試験-理系数学
- 中国复旦大学往年试题
- 雪隆森中学华罗庚杯数学比赛
- 厦门大学马来西亚分校-陈景润杯中学数学竞赛
- 微积分福音
- 线代启示录
- 福气老师
- 指考历届试题
- 统测历届试题与解答
- 印度人的作业
- 北京高考在线
- 朱式幸福教甄
- 普通型高级中学数学科能力竞赛 (决赛)
- 2014-2024 全国中学生数学竞赛联赛试题及答案汇总
- 如此醉的图书馆
- 菁优网
- 08 高考文科试题分类圆锥曲线
- 08 高考文科试题分类数列
- 福伦-隆中高数笔记
- 曾龙文师-高二数理培训队笔记