Representation Theory of $GL_2(\mathbb{F}_q)$

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7. Tensor product of representations give a representation

Size of $GL_2(\mathbb{F}_q)$

$$GL_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q,\ ad - bc \neq 0 \right\}$$

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- $(q^2 - q)$ choices for (c, d)

$$\left| GL_2(\mathbb{F}_q) \right| \;=\; (q^2 - 1)(q^2 - q).$$

Characteristic polynomial

Characteristic polynomial of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q)$:

$$\lambda^2 - (a+d)\lambda + (ad - bc)$$

Roots: (λ_1, λ_2)

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Conjugacy Classes

- Type A: $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
- Type B: $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$
- Type C: $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$
- Type D: $\begin{pmatrix} a & db \\ b & a \end{pmatrix}$

Conjugacy Classes

- Type A: $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \longrightarrow (q-1)$ classes
- Type B: $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \longrightarrow \frac{(q-1)(q-2)}{2}$ classes
- Type C: $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \longrightarrow (q-1)$ classes
- Type D: $\begin{pmatrix} a & db \\ b & a \end{pmatrix} \longrightarrow \frac{q(q-1)}{2}$ classes

Conjugacy Classes

Partial Result:

$$\sum_{\text{matrix type}} (\#\text{ of conjugacy classes}) = q^2 - 1$$

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Number of irreducible representations: $q^2 - 1$

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For our purposes:

$$|\text{Size of conjugacy class of } M| = \frac{|G|}{|\text{Matrices commuting with } M|}$$

Conjugacy Classes

Matrix Type	Element	No. of classes	Size of class
A	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$q-1$	1
B	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	$\frac{(q-1)(q-2)}{2}$	$q(q+1)$
C	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$(q-1)$	$q(q+1)$
D	$\begin{pmatrix} a & db \\ b & a \end{pmatrix}$	$\frac{q^2-q}{2}$	$q(q-1)$

Character

Given the character table:

	A	B	C	D
χ_φ	x_A	x_B	x_C	x_D

We have:

$$\begin{aligned} \left\|\chi_\varphi\right\|^2 &= \langle \chi_\varphi, \chi_\varphi \rangle \\ &= \frac{1}{|G|} \sum (x_{\text{matrix type}})^2 \cdot (\text{Number of classes}) \cdot (\text{Size}) \end{aligned}$$

Induced Representation

Given G, a finite group, $H \subseteq G$, and $\pi : H \to GL(\mathbb{C})$, a representation:

Definition

$$\mathrm{Ind}_H^G(\pi) \;=\; \{f : G \to \mathbb{C} \mid \pi(h)f(g) = f(gh^{-1}) \; \forall h^{-1} \in H\}.$$

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Consider $\pi = triv : G \to GL(\mathbb{C})$.

Definition

$$\begin{aligned} \mathrm{Ind}_H^G(triv) \; &= \; \{f : G \to \mathbb{C} \mid f(g) = f(gh^{-1}) \; \forall h^{-1} \in H\} \\ &= \; \{f : G \to \mathbb{C} \mid f \text{ constant on } gH\} \\ &= \{f : G/H \to \mathbb{C}\} \end{aligned}$$

Borel subgroup

Consider family of matrices

$$T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \subseteq GL_2(\mathbb{F}_q)$$

We want $\mathrm{Ind}_T^G(triv) = \{f : G/T \to \mathbb{C}\}$.

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Problem: G/T is too large

“Inflating”

Consider family of matrices (“Borel subgroup”):

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$$\{gB\} \;\longleftrightarrow\; \mathbb{P}^1 \text{ in } \mathbb{F}_q^2$$

Permutation Representation

$$\text{Vector space } M \text{ over } \mathbb{P}^1(\mathbb{F}_q) \;=\; span\left\{ \begin{pmatrix} 1 \\ r \end{pmatrix} : r \in \mathbb{F}_q \cup \{\infty\} \right\}$$

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$$\text{perm. rep} \;=\; \text{triv. subrep.} \oplus \varphi$$

Character table

	A	B	C	D
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χ_{φ}	q	1	0	-1

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$$\begin{aligned} \left\|\chi_{\varphi}\right\|^2 &= \frac{1}{|G|}\sum (\text{char. value})^2 \cdot (\#\text{ of classes}) \cdot (\text{size of class}) \\ &= 1 \end{aligned}$$

Other irreducible representations

1. Type A matrices

Other irreducible representations

1. Type A matrices
2. Type B, C matrices: Principal series representations
3. Type D matrices: Cuspidal representations

References

- https://arxiv.org/pdf/0712.4051
- https://math.ntnu.edu.tw/ li/note/REPGL2K.pdf