The Fault-Tolerant Hamiltonian Properties of $D_{n,2}$

Theorem 1. *For any $n \geq 3$ and $k = 2$, $D_{n,k}$ is $(nk-3)$-fault-tolerant Hamiltonian-connected and $(nk-2)$-fault-tolerant Hamiltonian.*

***Proof*:** In the following, for any faulty element set F in $D_{n,2}$, where $F \subset V(D_{n,2}) \cup E(D_{n,2})$, we aim to proof that:

(1) For any two distinct fault-free nodes x and y in $D_{n,2}$, there exists a (x, y)-Ham-path in $D_{n,2} - F$, where $|F| \leq 2n - 3$;
(2) There exists a Ham-cycle in $D_{n,2} - F$, where $|F| \leq 2n - 2$.

It is clear that $D_{n,2}$ contains $f_2 = n * w_1 + 1 = n(n+1)+1$ disjoint copies of $D_{n,1}$. For any $i \in \langle f_2 - 1\rangle$, we define $F_i = F \cap (V(D^i_{n,1}) \cup E(D^i_{n,1}))$, $D^{i,f}_{n,1} = D^i_{n,1} - F_i$, and $D^f_{n,2} = D_{n,2} - F$. By considering each copy of $D_{n,1}$ in $D^f_{n,2}$ as a single node and preserving the 2-dimensional fault-free edges between distinct copies of $D_{n,1}$, the $D^f_{n,2}$ can be regarded as a complete graph K_{f_2} with a faulty element set $\mathcal{F}$, where $|\mathcal{F}| \leq |F|$. Let $D^c = (V(D^c), E(D^c))$ be a graph isomorphic to $K_{f_2} - \mathcal{F}$. Since $f_2 - 1 > w_1 > |\mathcal{F}|$ and $|F|$ may be large than w_1, we define $V(D^c) = \{0, 1, \cdots, \alpha\}$, where $\alpha = f_2 - 1$ for $|F| < n + 1$, otherwise $\alpha \in \{f_2 - 2, f_2 - 1\}$. Therefore, each node $i \in V(D^c)$ corresponds to a copy $D^{i,f}_{n,1}$, and each edge $(i, j) \in E(D^c)$ corresponds to a unique fault-free 2-dimensional edge connecting two distinct copies $D^{i,f}_{n,1}$ and $D^{j,f}_{n,1}$. Since $(f_2 - 1) - 3 > |\mathcal{F}|$, it is clear that D^c is Hamiltonian-connected. Furthermore, we define $|F_\delta| = max\{|F_0|, |F_1|, \cdots, |F_\alpha|\}$ with $\delta \in \langle\alpha\rangle$. Next, we combine the 2-dimensional edges corresponding to Ham-paths in D^c with the appropriate fault-free Ham-paths in the distinct copies of $D_{n,1}$ to prove the theorem.

We have concluded that $D_{n,1}$ is $(n-3)$-fault-tolerant Hamiltonian-connected and $(n-2)$-fault-tolerant Hamiltonian for any $n \geq 3$. Next, we give two statements, which will be useful for the subsequent analysis:

Statement 1: For any $i \in \langle\alpha\rangle$, $D^{i,f}_{n,1}$ is Hamiltonian-connected when $|F_i| \leq n - 3$.

Statement 2: For any $i \in \langle\alpha\rangle$, $D^{i,f}_{n,1}$ is Hamiltonian when $|F_i| \leq n - 2$.

***Proof of* (1).** Based on the size of the faulty element set $|F_\delta|$, we prove this conclusion from the following three cases.

Case 1. $|F_\delta| \leq n - 3$.

In this case, $\alpha = f_2 - 1$ and each $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $m \in \langle f_2 - 1\rangle$. Based on whether nodes x and y are in the same copy of $D_{n,1}$, we divide the following two subcases.

Case 1.1. $x, y \in V(D^{i,f}_{n,1})$, where $i \in \langle f_2 - 1\rangle$.

According to Statement 1, there exists a (x, y)-Ham-path HP_1 in $D^{i,f}_{n,1}$. Select one node z from HP_1 such that z is adjacent one of the end-nodes of HP_1 and both z and this end-node have at least one fault-free external neighbor. Without loss of generality, we assume that z is adjacent to y in HP_1. Let $(y', y), (z, z') \in E(D^f_{n,2})$ be two external edges, where $y' \in V(D^{l,f}_{n,1})$, $z' \in V(D^{j,f}_{n,1})$, $j, l \in \langle f_2 - 1\rangle \setminus \{i\}$, and $j \neq l$. Let HP_c be a (j, l)-Ham-path in $D^c - \{i\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1. Then, a (z', y')-Ham-path HP_2 in $D^f_{n,2} - D^{i,f}_{n,1}$ can be obtained by union E_1 and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1 \rangle \setminus \{i\}$. Consequently, a (x, y)-Ham-path in $D^f_{n,2}$ can be expressed as: $HP_1(x, z) + (z, z') + HP_2 + (y', y)$.

Case 1.2. $x \in V(D^{i,f}_{n,1})$ and $y \in V(D^{j,f}_{n,1})$, where $i, j \in \langle f_2 - 1 \rangle$ and $i \neq j$.

Let HP_c be an (i, j)-Ham-path in D^c, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1. Then, a (x, y)-Ham-path in $D^f_{n,2}$ can be obtained by union E_1 and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1 \rangle$.

Case 2. $|F_\delta| = n - 2$.

In this case, $\alpha = f_2 - 1$ and $D_{n,2} - D^\delta_{n,1}$ has at most $2n - 3 - (n - 2) = n - 1$ faulty elements. Therefore, each node in $D^f_{n,2}$ has at least one fault-free external neighbor. Since $k = 2$ and $n \geq 3$, there exists at most one copy $D^{\beta,f}_{n,1}$ is Hamiltonian and the other copy $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $\beta \in \langle f_2 - 1 \rangle \setminus \{\delta\}$ and $m \in \langle f_2 - 1 \rangle \setminus \{\delta, \beta\}$. According to Statement 2, there exists a Ham-cycle HC_1 in $D^{\delta,f}_{n,1}$ and a Ham-cycle HC_2 in $D^{\beta,f}_{n,1}$. Based on whether nodes x and y are in the same copy of $D_{n,1}$, we divide the following two subcases.

Case 2.1. Nodes x and y are in the same copy of $D_{n,1}$.

Case 2.1.1. Each $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $m \in \langle f_2 - 1 \rangle \setminus \{\delta\}$.

Case 2.1.1.1. $x, y \in V(D^{\delta,f}_{n,1})$.

According to whether x and y are adjacent in HC_1, we have the following two cases.

Case 2.1.1.1.1. $(x, y) \in E(HC_1)$.

In this case, there exists a (x, y)-Ham-path $HP_1 = HC_1 - (x, y)$ in $D^{\delta,f}_{n,1}$. Select one node z adjacent to y in HP_1, and then a (x, y)-Ham-path in $D^f_{n,2}$ can be constructed using a method similar to Case 1.1.

Case 2.1.1.1.2. $(x, y) \notin E(HC_1)$.

Without loss of generality, we let $HC_1 = \langle (x, u), \cdots, (v, y), (y, w), \cdots, (z, x) \rangle$. Then, we let $(w, a), (b, u) \in E(D^f_{n,2})$ be two external edges, where $a \in V(D^{i,f}_{n,1})$, $b \in V(D^{j,f}_{n,1})$, $i, j \in \langle f_2 - 1 \rangle \setminus \{\delta\}$, and $i \neq j$. Let HP_c be an (i, j)-Ham-path in $D^c - \{\delta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1. Then, a (a, b)-Ham-path HP in $D^f_{n,2} - D^{\delta,f}_{n,1}$ can be obtained by union E_1 and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1 \rangle \setminus \{\delta\}$. Thus, a (x, y)-Ham-path in $D^f_{n,2}$ can be expressed as: $HC^{-1}_1(x, w) + (w, a) + HP + (b, u) + HC_1(u, y)$.

Case 2.1.1.2. $x, y \in V(D^{i,f}_{n,1})$, where $i \in \langle f_2 - 1 \rangle \setminus \{\delta\}$.

According to Statement 1, there exists a (x, y)-Ham-path HP_1 in $D^{i,f}_{n,1}$. Select one node z from HP_1 such that z is adjacent one of the end-nodes of HP_1, and both z and this end-node have one external neighbor in $D^f_{n,2} - D^{\delta,f}_{n,1}$. Without loss of generality, we assume that z is adjacent to x in HP_1, and let $(x, u), (v, z) \in E(D^f_{n,2})$ be two external edges, where $u \in V(D^{j,f}_{n,1})$, $v \in V(D^{l,f}_{n,1})$, $j, l \in \langle f_2 - 1 \rangle \setminus \{i, \delta\}$, and $j \neq l$. Let HP_c be a (j, l)-Ham-path in $D^c - \{i\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes a and b in $D^{\delta,f}_{n,1}$, satisfying $(a, b) \in E(HC_1)$. Then, a (u, v)-Ham-path HP_2 in $D^f_{n,2} - D^{i,f}_{n,1}$ can be obtained by union E_1, $HC_1 - (a, b)$, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1 \rangle \setminus \{i, \delta\}$. Thus, a (x, y)-Ham-path in $D^f_{n,2}$ can be expressed as: $(x, u) + HP_2 + (v, z) + HP_1(z, y)$.

Case 2.1.2. There exists one copy $D^{\beta,f}_{n,1}$ is Hamiltonian and the other copy $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $\beta \in \langle f_2 - 1 \rangle \setminus \{\delta\}$ and $m \in \langle f_2 - 1 \rangle \setminus \{\delta, \beta\}$.

In this case, $D_{n,2}-D_{n,1}^{\delta}-D_{n,1}^{\beta}$ has at most $2n-3-(n-2)-(n-2)=1$ faulty elements. Therefore, each node in $D_{n,2}^{f}$ has at least $n-1$ fault-free external neighbors.

Case 2.1.2.1. $x,y \in V(D_{n,1}^{m,f})$, where $m \in \{\delta,\beta\}$.

Without loss of generality, we assume that $x,y \in V(D_{n,1}^{\delta,f})$. According to whether x and y are adjacent in HC_1, we have the following two cases.

Case 2.1.2.1.1. $(x,y) \in E(HC_1)$.

In this case, there exists a (x,y)-Ham-path $HP_1 = HC_1-(x,y)$ in $D_{n,1}^{\delta,f}$. Select one node z adjacent to y in HP_1. Let $(y',y),(z,z') \in E(D_{n,2}^{f})$ be two external edges, where $y' \in V(D_{n,1}^{l,f})$, $z' \in V(D_{n,1}^{j,f})$, $j,l \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$, and $j \neq l$. Let HP_c be a (j,l)-Ham-path in $D^c-\{\delta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes c and d in $D_{n,1}^{\beta,f}$, satisfying $(c,d) \in E(HC_2)$. Then, a (z',y')-Ham-path HP_2 in $D_{n,2}^{f}-D_{n,1}^{\delta,f}$ can be obtained by union E_1, $HC_2-(c,d)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$. Consequently, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1(x,z)+(z,z')+HP_2+(y',y)$.

Case 2.1.2.1.2. $(x,y) \notin E(HC_1)$.

Without loss of generality, we let $HC_1 = \langle (x,u),\cdots,(v,y),(y,w),\cdots,(z,x)\rangle$. Then, we let $(w,a),(b,u) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{i,f})$, $b \in V(D_{n,1}^{j,f})$, $i,j \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$, and $i \neq j$. Let HP_c be an (i,j)-Ham-path in $D^c-\{\delta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes c and d in $D_{n,1}^{\beta,f}$, satisfying $(c,d) \in E(HC_2)$. Then, a (a,b)-Ham-path HP in $D_{n,2}^{f}-D_{n,1}^{\delta,f}$ can be obtained by union E_1, $HC_2-(c,d)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HC_1^{-1}(x,w)+(w,a)+HP+(b,u)+HC_1(u,y)$.

Case 2.1.2.2. $x,y \in V(D_{n,1}^{i,f})$, where $i \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$.

According to Statement 1, there exists a (x,y)-Ham-path HP_1 in $D_{n,1}^{i,f}$. Select one node z adjacent to x in HP_1, and let $(x,u),(v,z) \in E(D_{n,2}^{f})$ be two external edges, where $u \in V(D_{n,1}^{j,f})$, $v \in V(D_{n,1}^{l,f})$, $j,l \in \langle f_2-1\rangle \setminus \{i,\delta,\beta\}$, and $j \neq l$. Let HP_c be a (j,l)-Ham-path in $D^c-\{i\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes a and b in $D_{n,1}^{\delta,f}$ and intersects with nodes c and d in $D_{n,1}^{\beta,f}$, satisfying $(a,b) \in E(HC_1)$ and $(c,d) \in E(HC_2)$. Then, a (u,v)-Ham-path HP_2 in $D_{n,2}^{f}-D_{n,1}^{i,f}$ can be obtained by union E_1, $HC_1-(a,b)$, $HC_2-(c,d)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle f_2-1\rangle \setminus \{i,\delta,\beta\}$. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $(x,u)+HP_2+(v,z)+HP_1(z,y)$.

Case 2.2. Nodes x and y are in the distinct copies of $D_{n,1}$.

Case 2.2.1. Each $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $m \in \langle f_2-1\rangle \setminus \{\delta\}$.

Case 2.2.1.1. $x \in V(D_{n,1}^{\delta,f})$ and $y \in V(D_{n,1}^{i,f})$, where $i \in \langle f_2-1\rangle \setminus \{\delta\}$.

Select one node z from HC_1 such that $(x,z) \in E(HC_1)$ and z has one external neighbor in $D_{n,2}^{f}-D_{n,1}^{i,f}$. Then, we obtain a (x,z)-Ham-path $HP_1 = HC_1-(x,z)$ in $D_{n,1}^{\delta,f}$. Additionally, let $(z,w) \in E(D_{n,2}^{f})$ be an external edge, where $w \in V(D_{n,1}^{j,f})$ and $j \in \langle f_2-1\rangle \setminus \{i,\delta\}$. Let HP_c be a (j,i)-Ham-path in $D^c-\{\delta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1. Then, a (w,y)-Ham-path HP_2 in $D_{n,2}^{f}-D_{n,1}^{\delta,f}$ can be obtained by union E_1 and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle f_2-1\rangle \setminus \{\delta\}$. Consequently, a

(x, y)-Ham-path in $D^f_{n,2}$ can be expressed as: $HP_1 + (z, w) + HP_2$.

Case 2.2.1.2. $x \in V(D^{i,f}_{n,1})$ and $y \in V(D^{j,f}_{n,1})$, where $i, j \in \langle f_2 - 1\rangle \setminus \{\delta\}$ and $i \neq j$.

Let HP_c be an (i, j)-Ham-path in D^c, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes a and b in $D^{\delta,f}_{n,1}$, satisfying $(a, b) \in E(HC_1)$. Then, a (x, y)-Ham-path in $D^f_{n,2}$ can be obtained by union E_1, $HC_1 - (a, b)$, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1\rangle \setminus \{\delta\}$.

Case 2.2.2. There exists one copy $D^{\beta,f}_{n,1}$ is Hamiltonian and the other copy $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $\beta \in \langle f_2 - 1\rangle \setminus \{\delta\}$ and $m \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$.

In this case, $D_{n,2} - D^{\delta}_{n,1} - D^{\beta}_{n,1}$ has at most $2n - 3 - (n - 2) - (n - 2) = 1$ faulty elements. Therefore, each node in $D^f_{n,2}$ has at least $n - 1$ fault-free external neighbors.

Case 2.2.2.1. $x \in V(D^{i,f}_{n,1})$ and $y \in V(D^{j,f}_{n,1})$, where $i \in \{\delta, \beta\}$ and $j \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$.

Without loss of generality, we let $x \in V(D^{\delta,f}_{n,1})$. Select one node z adjacent to x in HC_1, and let $(z, w) \in E(D^f_{n,2})$ be an external edge, where $w \in V(D^{l,f}_{n,1})$ and $l \in \langle f_2 - 1\rangle \setminus \{j, \delta, \beta\}$. Then, we obtain a (x, z)-Ham-path $HP_1 = HC_1 - (x, z)$ in $D^{\delta,f}_{n,1}$. Let HP_c be a (l, j)-Ham-path in $D^c - \{\delta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes c and d in $D^{\beta,f}_{n,1}$, satisfying $(c, d) \in E(HC_2)$. Then, a (w, y)-Ham-path HP_2 in $D^f_{n,2} - D^{\delta,f}_{n,1}$ can be obtained by union E_1, $HC_2 - (c, d)$, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$. Consequently, a (x, y)-Ham-path in $D^f_{n,2}$ can be expressed as: $HP_1 + (z, w) + HP_2$.

Case 2.2.2.2. $x \in V(D^{\delta,f}_{n,1})$ and $y \in V(D^{\beta,f}_{n,1})$.

Let z and u be two nodes such that $(x, z) \in E(HC_1)$ and $(y, u) \in E(HC_2)$. Then, we obtain a (x, z)-Ham-path $HP_1 = HC_1 - (x, z)$ in $D^{\delta,f}_{n,1}$ and a (u, y)-Ham-path $HP_2 = HC_2 - (u, y)$ in $D^{\beta,f}_{n,1}$. Additionally, let $(z, w), (u, v) \in E(D^f_{n,2})$ be two external edges, where $w \in V(D^{i,f}_{n,1})$, $v \in V(D^{j,f}_{n,1})$, $i, j \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$, and $i \neq j$. Let HP_c be an (i, j)-Ham-path in $D^c - \{\delta, \beta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1. Then, a (w, v)-Ham-path HP_3 in $D^f_{n,2} - D^{\delta,f}_{n,1} - D^{\beta,f}_{n,1}$ can be obtained by union E_1 and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$. Consequently, a (x, y)-Ham-path in $D^f_{n,2}$ can be expressed as: $HP_1 + (z, w) + HP_3 + (v, u) + HP_2$.

Case 2.2.2.3. $x \in V(D^{i,f}_{n,1})$ and $y \in V(D^{j,f}_{n,1})$, where $i, j \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$ and $i \neq j$.

Let HP_c be an (i, j)-Ham-path in D^c, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes a and b in $D^{\delta,f}_{n,1}$ and intersects with nodes c and d in $D^{\beta,f}_{n,1}$, satisfying $(a, b) \in E(HC_1)$ and $(c, d) \in E(HC_2)$. Then, a (x, y)-Ham-path in $D^f_{n,2}$ can be obtained by union E_1, $HC_1 - (a, b)$, $HC_2 - (c, d)$, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$.

Case 3. $n - 1 \leq |F_\delta| \leq 2n - 3$.

In this case, $\alpha \in \{f_2 - 2, f_2 - 1\}$ and $D_{n,2} - D^{\delta}_{n,1}$ has at most $2n - 3 - (n - 1) = n - 2$ faulty elements. Therefore, each node in $D^f_{n,2}$ has at least two fault-free external neighbors. Since $k = 2$ and $n \geq 3$, there is at most one copy $D^{\beta,f}_{n,1}$ is Hamiltonian and the other copy $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $\beta \in \langle \alpha\rangle \setminus \{\delta\}$ and $m \in \langle \alpha\rangle \setminus \{\delta, \beta\}$. According to Statement 2, there exists a Ham-cycle HC in $D^{\beta,f}_{n,1}$. Based on whether nodes x and y are in the same copy of $D_{n,1}$, we divide the following two subcases.

Case 3.1. Nodes x and y are in the same copy of $D_{n,1}$.

Case 3.1.1. Each $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $m \in \langle \alpha \rangle \setminus \{\delta\}$.

Case 3.1.1.1. $x, y \in V(D_{n,1}^{\delta,f})$.

For each node w in $D_{n,1}^{\delta,f} - \{x, y\}$, let $(u, w), (v, w) \in E(D_{n,2}^{f})$ be two external edges. We represent these edges as a set E_1 and refer to the copies of $D_{n,1}$ containing u and v as *bounded subgraphs* for convenience. Let $(x, a), (b, y) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{i,f})$, $b \in V(D_{n,1}^{j,f})$, $i, j \in \langle \alpha \rangle \setminus \{\delta\}$, and $i \neq j$. Let HP_c be an (i, j)-Ham-path in $D^c - \{\delta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. Then, we denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Then, a (a, b)-Ham-path HP in $D_{n,2}^{f} - \{x, y\}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle \alpha \rangle \setminus \{\delta\}$. Consequently, a (x, y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $(x, a) + HP + (b, y)$.

Case 3.1.1.2. $x, y \in V(D_{n,1}^{i,f})$, where $i \in \langle \alpha \rangle \setminus \{\delta\}$.

According to Statement 1, there exists a (x, y)-Ham-path HP_1 in $D_{n,1}^{i,f}$. In this case, there may be a node w' in $D_{n,1}^{\delta,f}$ with only two fault-free external neighbors u' and v', where u' or v' is in $D_{n,1}^{i,f}$. Then, we have the following two cases.

Case 3.1.1.2.1. There exists a node w' in $D_{n,1}^{\delta,f}$ with only two fault-free external neighbors u' and v', where u' or v' is in $D_{n,1}^{i,f}$.

Without loss of generality, we let $u' \in V(D_{n,1}^{i,f})$ and $v' \in V(D_{n,1}^{j,f})$, where $j \in \langle \alpha \rangle \setminus \{\delta, i\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{w'\}$. Select one node a adjacent to u' in HP_1, so that a is located in the longer sub-path from u' to the end-nodes. Without loss of generality, we let a in the sub-path $HP_1(u', y)$. Then, let $(a, b) \in E(D_{n,2}^{f})$ be an external edge, where $b \in V(D_{n,1}^{l,f})$ and $l \in \langle \alpha \rangle \setminus \{\delta, i, j\}$. For each node w in $D_{n,1}^{\delta,f'}$, let $(u, w), (v, w) \in E(D_{n,2}^{f})$ be two external edges, where $u, v \notin V(D_{n,1}^{m,f})$ and $m \in \{i, j, l, \delta\}$. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (j, l)-Ham-path in $D^c - \{i, \delta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. Then, we denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Then, a (v', b)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{i,f} - \{w'\}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle \alpha \rangle \setminus \{i, \delta\}$. Thus, a (x, y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1(x, u') + (u', w') + (w', v') + HP_2 + (b, a) + HP_1(a, y)$.

Case 3.1.1.2.2. There is no node w' in $D_{n,1}^{\delta,f}$ has only two fault-free external neighbors u' and v', where u' or v' is in $D_{n,1}^{i,f}$.

Select one node z adjacent to y in HP_1, and let $(a, y), (z, b) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{j,f})$, $b \in V(D_{n,1}^{l,f})$, $j, l \in \langle \alpha \rangle \setminus \{i, \delta\}$, and $j \neq l$. In this case, there may be a node w' in $D_{n,1}^{\delta,f}$ with only two fault-free external neighbors u' and v', where u' or v' equal to a or b. Then, we have the following three cases.

Case 3.1.1.2.2.1. $u' = a$ and $v' = b$.

In this case, there must exists one node w'' such that $(w', w'') \in E(D_{n,1}^{\delta,f})$. Select one edge $(w'', u'') \in E(D_{n,2}^{f})$, where $u'' \in V(D_{n,1}^{r,f})$ and $r \in \langle \alpha \rangle \setminus \{\delta, i, j, l\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{w', w''\}$ and $D_{n,1}^{j,f'} = D_{n,1}^{j,f} - \{a\}$. According to Statement 2, there exists a

Ham-cycle HC in $D_{n,1}^{j,f'}$. For each node w in $D_{n,1}^{\delta,f'}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (l,r)-Ham-path in $D^c - \{\delta, i\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs, where E_2 intersects with nodes c and d in $D_{n,1}^{j,f'}$, satisfying $(c,d) \in E(HC)$. Thus, a (b,u'')-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{i,f} - \{a, w', w''\}$ can be obtained by union E_1, E_2, $HC - (c,d)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta, i\}$. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1(x,z) + (z,b) + HP_2 + (u'',w'') + (w'',w') + (w',a) + (a,y)$.

Case 3.1.1.2.2.2. One of u' and v' equal to a or b.

Without loss of generality, we let $u' = a$ and $v' \in V(D_{n,1}^{r,f})$, where $r \in \langle\alpha\rangle \setminus \{i,j,l,\delta\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{w'\}$ and $D_{n,1}^{j,f'} = D_{n,1}^{j,f} - \{a\}$. Next, a (b,v')-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{i,f} - \{a, w'\}$ can be obtained by using a method similar to the construction of (b,u'')-Ham-path in Case 3.1.1.2.2.1. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1(x,z) + (z,b) + HP_2 + (v',w') + (w',a) + (a,y)$.

Case 3.1.1.2.2.3. There is no node w' in $D_{n,1}^{\delta,f}$ has only two fault-free external neighbors u' and v', where u' or v' equal to a or b.

For each node w in $D_{n,1}^{\delta,f}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (l,j)-Ham-path in $D^c - \{i,\delta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. Then, we denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Then, a (b,a)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{i,f}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{i,\delta\}$. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1(x,z) + (z,b) + HP_2 + (a,y)$.

Case 3.1.2. There exists one copy $D_{n,1}^{\beta,f}$ is Hamiltonian and the other copy $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $\beta \in \langle\alpha\rangle \setminus \{\delta\}$ and $m \in \langle\alpha\rangle \setminus \{\delta,\beta\}$.

In this case, there is no faulty element in $D_{n,2} - D_{n,1}^{\delta} - D_{n,1}^{\beta}$. Therefore, each node in $D_{n,2}^{f}$ has n fault-free external neighbors.

Case 3.1.2.1. $x,y \in V(D_{n,1}^{\delta,f})$.

For each node w in $D_{n,1}^{\delta,f} - \{x,y\}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges, where $u,v \notin V(D_{n,1}^{\beta,f})$. We represent these edges as a set E_1 and refer to the copies of $D_{n,1}$ containing u and v as bounded subgraphs for convenience. Let $(x,a),(b,y) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{i,f})$, $b \in V(D_{n,1}^{j,f})$, $i,j \in \langle\alpha\rangle \setminus \{\delta,\beta\}$, and $i \neq j$. Let HP_c be an (i,j)-Ham-path in $D^c - \{\delta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to this path in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs, where E_2 intersects with nodes c and d in $D_{n,1}^{\beta,f}$, satisfying $(c,d) \in E(HC)$. Then, a (a,b)-Ham-path HP in $D_{n,2}^{f} - \{x,y\}$ can be obtained by union E_1, E_2, $HC - (c,d)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta,\beta\}$. Consequently, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $(x,a) + HP + (b,y)$.

Case 3.1.2.2. $x,y \in V(D_{n,1}^{\beta,f})$.

In this case, for each node w in $D_{n,1}^{\delta,f}$, we let $(u, w), (v, w) \in E(D_{n,2}^{f})$ be two external edges, where $u, v \notin V(D_{n,1}^{\beta,f})$. Next, depending on whether x and y are adjacent in HC, we have the following two cases.

Case 3.1.2.2.1. $(x, y) \in E(HC)$.

In this case, there exists a (x, y)-Ham-path $HP_1 = HC - (x, y)$ in $D_{n,1}^{\beta,f}$. Select one node z adjacent to y in HP_1, and let $(a, y), (z, b) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{j,f})$, $b \in V(D_{n,1}^{l,f})$, $j, l \in \langle\alpha\rangle \setminus \{\delta, \beta\}$, and $j \neq l$. Then, a (b, a)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{\beta,f}$ can be obtained by using a method similar to Case 3.1.1.2.2. Thus, a (x, y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1(x, z) + (z, b) + HP_2 + (a, y)$.

Case 3.1.2.2.2. $(x, y) \notin E(HC)$.

Without loss of generality, we let $HC = \langle(x, o), \cdots, (p, y), (y, q), \cdots, (z, x)\rangle$. Then, we let $(q, a), (b, o) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{j,f})$, $b \in V(D_{n,1}^{l,f})$, $j, l \in \langle\alpha\rangle \setminus \{\delta, \beta\}$, and $j \neq l$. Then, a (a, b)-Ham-path HP in $D_{n,2}^{f} - D_{n,1}^{\beta,f}$ can be obtained by using a method similar to Case 3.1.1.2.2. Thus, a (x, y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HC^{-1}(x, q) + (q, a) + HP + (b, o) + HC(o, y)$.

Case 3.1.2.3. $x, y \in V(D_{n,1}^{i,f})$, where $i \in \langle\alpha\rangle \setminus \{\delta, \beta\}$.

According to Statement 1, there exists a (x, y)-Ham-path HP_1 in $D_{n,1}^{i,f}$. Select one node z adjacent to y in HP_1, and let $(a, y), (z, b) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{j,f})$, $b \in V(D_{n,1}^{l,f})$, $j, l \in \langle\alpha\rangle \setminus \{i, \delta, \beta\}$, and $j \neq l$. For each node w in $D_{n,1}^{\delta,f}$, let $(u, w), (v, w) \in E(D_{n,2}^{f})$ be two external edges, where $u, v \notin V(D_{n,1}^{i,f})$. Then, by adding the following condition to the method of Case 3.1.1.2.2, a (b, a)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{i,f}$ can be obtained: E_2 intersects with nodes c and d in $D_{n,1}^{\beta,f}$, satisfying $(c, d) \in E(HC)$. Thus, a (x, y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1(x, z) + (z, b) + HP_2 + (a, y)$.

Case 3.2. Nodes x and y are in the distinct copies of $D_{n,1}$.

Case 3.2.1. Each $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $m \in \langle\alpha\rangle \setminus \{\delta\}$.

Case 3.2.1.1. $x \in V(D_{n,1}^{\delta,f})$ and $y \in V(D_{n,1}^{i,f})$, where $i \in \langle\alpha\rangle \setminus \{\delta\}$.

In this case, there may be a node w' in $D_{n,1}^{\delta,f}$ with only two fault-free external neighbors u' and v', where u' or v' equal to node y. Then, we have the following two cases.

Case 3.2.1.1.1. There exists a node w' in $D_{n,1}^{\delta,f}$ with only two fault-free external neighbors u' and v', where u' or v' equal to node y.

Without loss of generality, we let $u' = y$ and $v' \in V(D_{n,1}^{j,f})$, where $j \in \langle\alpha\rangle \setminus \{\delta, i\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{x, w'\}$ and $D_{n,1}^{i,f'} = D_{n,1}^{i,f} - \{y\}$. According to Statement 2, there exists a Ham-cycle HC in $D_{n,1}^{i,f'}$. For each node w in $D_{n,1}^{\delta,f'}$, let $(u, w), (v, w) \in E(D_{n,2}^{f})$ be two external edges. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (δ, j)-Ham-path in D^c such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs, where E_2 intersects with nodes a and b in $D_{n,1}^{i,f'}$, satisfying $(a, b) \in E(HC)$. Thus, a (x, v')-Ham-path HP in $D_{n,2}^{f} - \{y, w'\}$ can be obtained by union E_1, E_2, $HC - (a, b)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta, i\}$. Thus, a (x, y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP + (v', w') + (w', y)$.

Case 3.2.1.1.2. There is no node w' in $D_{n,1}^{\delta,f}$ has only two fault-free external neighbors u' and v', where u' or v' equal to node y.

For each node w in $D_{n,1}^{\delta,f} - \{x\}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (δ, i)-Ham-path in D^c such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Thus, a (x,y)-Ham-path HP in $D_{n,2}^{f}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta\}$.

Case 3.2.1.2. $x \in V(D_{n,1}^{i,f})$ and $y \in V(D_{n,1}^{j,f})$, where $i,j \in \langle\alpha\rangle \setminus \{\delta\}$ and $i \neq j$.

In this case, there may be a node w' in $D_{n,1}^{\delta,f}$ with only two fault-free external neighbors u' and v', where u' or v' equal to nodes x or y. Then, we have the following three cases.

Case 3.2.1.2.1. $x = u'$ and $y = v'$.

In this case, there must exists one node w'' such that $(w', w'') \in E(D_{n,1}^{\delta,f})$. Select one edge $(w'', u'') \in E(D_{n,2}^{f})$, where $u'' \in V(D_{n,1}^{l,f})$ and $l \in \langle\alpha\rangle \setminus \{\delta, i, j\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{w', w''\}$ and $D_{n,1}^{i,f'} = D_{n,1}^{i,f} - \{x\}$. According to Statement 2, there exists a Ham-cycle HC in $D_{n,1}^{i,f'}$. For each node w in $D_{n,1}^{\delta,f'}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (l, j)-Ham-path in $D^c - \{\delta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs, where E_2 intersects with nodes a and b in $D_{n,1}^{i,f'}$, satisfying $(a,b) \in E(HC)$. Thus, a (u'', y)-Ham-path HP in $D_{n,2}^{f} - \{x, w', w''\}$ can be obtained by union E_1, E_2, $HC - (a,b)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta, i\}$. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $(x, w') + (w', w'') + (w'', u'') + HP$.

Case 3.2.1.2.2. One of x and y equal to u' or v'.

Without loss of generality, we let $u' = x$ and $v' \in V(D_{n,1}^{l,f})$, where $l \in \langle\alpha\rangle \setminus \{\delta, i, j\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{w'\}$ and $D_{n,1}^{i,f'} = D_{n,1}^{i,f} - \{x\}$. Next, a (v', y)-Ham-path HP in $D_{n,2}^{f} - \{x, w'\}$ can be obtained by using a method similar to the construction of (u'', y)-Ham-path in Case 3.2.1.2.1. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $(x, w') + (w', v') + HP$.

Case 3.2.1.2.3. There is no node w' in $D_{n,1}^{\delta,f}$ has only two fault-free external neighbors u' and v', where u' or v' equal to nodes x or y.

For each node w in $D_{n,1}^{\delta,f}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be an (i,j)-Ham-path in $D^c - \{\delta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta\}$.

Case 3.2.2. There exists one copy $D_{n,1}^{\beta,f}$ is Hamiltonian and the other copy $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $\beta \in \langle\alpha\rangle \setminus \{\delta\}$ and $m \in \langle\alpha\rangle \setminus \{\delta, \beta\}$.

In this case, there is no faulty element in $D_{n,2} - D_{n,1}^{\delta} - D_{n,1}^{\beta}$. Therefore, each node in $D_{n,2}^{f}$

has n fault-free external neighbors.

Case 3.2.2.1. $x \in V(D_{n,1}^{\delta,f})$ and $y \in V(D_{n,1}^{i,f})$, where $i \in \langle\alpha\rangle \setminus \{\delta,\beta\}$.

For each node w in $D_{n,1}^{\delta,f} - \{x\}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges, where $u,v \notin V(D_{n,1}^{i,f})$. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (δ,i)-Ham-path in D^c such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to this path in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs, where E_2 intersects with nodes c and d in $D_{n,1}^{\beta,f}$, satisfying $(c,d) \in E(HC)$. Thus, a (x,y)-Ham-path HP in $D_{n,2}^{f}$ can be obtained by union E_1, E_2, $HC-(c,d)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta,\beta\}$.

Case 3.2.2.2. $x \in V(D_{n,1}^{\beta,f})$ and $y \in V(D_{n,1}^{i,f})$, where $i \in \langle\alpha\rangle \setminus \{\delta,\beta\}$.

Let z be one node such that $(x,z) \in E(HC)$. Then, we obtain a (x,z)-Ham-path $HP_1 = HC - (x,z)$ in $D_{n,1}^{\beta,f}$. Let $(z,a) \in E(D_{n,2}^{f})$ be one external edge, where $a \in V(D_{n,1}^{j,f})$ and $j \in \langle\alpha\rangle \setminus \{i,\delta,\beta\}$. In this case, there may be a node w' in $D_{n,1}^{\delta,f}$ with only two fault-free external neighbors u' and v' in $D_{n,2}^{f} - D_{n,1}^{\beta,f}$, where u' or v' equal to nodes a or y. Then, we have the following three cases.

Case 3.2.2.2.1. $u' = a$ and $v' = y$.

In this case, there must exists one node w'' such that $(w',w'') \in E(D_{n,1}^{\delta,f})$. Select one edge $(w'',u'') \in E(D_{n,2}^{f})$, where $u'' \in V(D_{n,1}^{l,f})$ and $l \in \langle\alpha\rangle \setminus \{\beta,\delta,i,j\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{w',w''\}$ and $D_{n,1}^{j,f'} = D_{n,1}^{j,f} - \{a\}$. According to Statement 2, there exists a Ham-cycle HC_1 in $D_{n,1}^{j,f'}$. For each node w in $D_{n,1}^{\delta,f'}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges, where $u,v \notin V(D_{n,1}^{\beta,f})$. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (l,i)-Ham-path in $D^c - \{\delta,\beta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs, where E_2 intersects with nodes c and d in $D_{n,1}^{j,f'}$, satisfying $(c,d) \in E(HC_1)$. Thus, a (u'',y)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{\beta,f} - \{a,w',w''\}$ can be obtained by union E_1, E_2, $HC_1 - (c,d)$, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{\delta,\beta,j\}$. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1 + (z,a) + (a,w') + (w',w'') + (w'',u'') + HP_2$.

Case 3.2.2.2.2. One of a and y equal to u' or v'.

Without loss of generality, we let $u' = a$ and $v' \in V(D_{n,1}^{l,f})$, where $l \in \langle\alpha\rangle \setminus \{\delta,\beta,i,j\}$. For convenience, we let $D_{n,1}^{\delta,f'} = D_{n,1}^{\delta,f} - \{w'\}$ and $D_{n,1}^{j,f'} = D_{n,1}^{j,f} - \{a\}$. Next, a (v',y)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{\beta,f} - \{a,w'\}$ can be obtained by using a method similar to the construction of (u'',y)-Ham-path in Case 3.2.2.2.2. Thus, a (x,y)-Ham-path in $D_{n,2}^{f}$ can be expressed as: $HP_1 + (z,a) + (a,w') + (w',v') + HP_2$.

Case 3.2.2.2.3. There is no node w' in $D_{n,1}^{\delta,f}$ has only two fault-free external neighbors u' and v' in $D_{n,2}^{f} - D_{n,1}^{\beta,f}$, where u' or v' equal to nodes a or y.

For each node w in $D_{n,1}^{\delta,f}$, let $(u,w),(v,w) \in E(D_{n,2}^{f})$ be two external edges, where $u,v \notin V(D_{n,1}^{\beta,f})$. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (j,i)-Ham-path in $D^c - \{\delta,\beta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to this path in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Then, a (a, y)-Ham-path HP_2 in $D^{f}_{n,2} - D^{\beta,f}_{n,1}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle\alpha\rangle \setminus \{\delta, \beta\}$. Consequently, a (x, y)-Ham-path in $D^{f}_{n,2}$ can be expressed as: $HP_1 + (z, a) + HP_2$.

Case 3.2.2.3. $x \in V(D^{\delta,f}_{n,1})$ and $y \in V(D^{\beta,f}_{n,1})$.

Let z be one node such that $(y, z) \in E(HC)$. Then, we obtain a (z, y)-Ham-path $HP_1 = HC - (y, z)$ in $D^{\beta,f}_{n,1}$. Let $(z, a) \in E(D^{f}_{n,2})$ be an external edge, where $a \in V(D^{i,f}_{n,1})$ and $i \in \langle\alpha\rangle \setminus \{\delta, \beta\}$. In this case, there may be a node w' in $D^{\delta,f}_{n,1} - \{x\}$ with only two fault-free external neighbors u' and v' in $D^{f}_{n,2} - D^{\beta,f}_{n,1}$, where u' or v' equal to node a. Then, we have the following two cases.

Case 3.2.2.3.1. There exists a node w' in $D^{\delta,f}_{n,1} - \{x\}$ with only two fault-free external neighbors u' and v' in $D^{f}_{n,2} - D^{\beta,f}_{n,1}$, where u' or v' equal to node a.

Without loss of generality, we let $u' = a$ and $v' \in V(D^{j,f}_{n,1})$, where $j \in \langle\alpha\rangle \setminus \{\delta, \beta, i\}$. For convenience, we let $D^{\delta,f'}_{n,1} = D^{\delta,f}_{n,1} - \{x, w'\}$ and $D^{i,f'}_{n,1} = D^{i,f}_{n,1} - \{a\}$. According to Statement 2, there exists a Ham-cycle HC_1 in $D^{i,f'}_{n,1}$. For each node w in $D^{\delta,f'}_{n,1}$, let $(u, w), (v, w) \in E(D^{f}_{n,2})$ be two external edges, where $u, v \notin V(D^{\beta,f}_{n,1})$. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (δ, j)-Ham-path in $D^c - \{\beta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs, where E_2 intersects with nodes c and d in $D^{i,f'}_{n,1}$, satisfying $(c, d) \in E(HC_1)$. Thus, a (x, v')-Ham-path HP_2 in $D^{f}_{n,2} - D^{\beta,f}_{n,1} - \{a, w'\}$ can be obtained by union E_1, E_2, $HC_1 - (c, d)$, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle\alpha\rangle \setminus \{\delta, \beta, i\}$. Thus, a (x, y)-Ham-path in $D^{f}_{n,2}$ can be expressed as: $HP_2 + (v', w') + (w', a) + (a, z) + HP_1$.

Case 3.2.2.3.2. There is no node w' in $D^{\delta,f}_{n,1} - \{x\}$ has only two fault-free external neighbors u' and v' in $D^{f}_{n,2} - D^{\beta,f}_{n,1}$, where u' or v' equal to node a.

For each node w in $D^{\delta,f}_{n,1} - \{x\}$, let $(u, w), (v, w) \in E(D^{f}_{n,2})$ be two external edges, where $u, v \notin V(D^{\beta,f}_{n,1})$. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Let HP_c be a (δ, i)-Ham-path in $D^c - \{\beta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. We denote the edge-set corresponding to this path in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Then, a (x, a)-Ham-path HP_2 in $D^{f}_{n,2} - D^{\beta,f}_{n,1}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle\alpha\rangle \setminus \{\delta, \beta\}$. Consequently, a (x, y)-Ham-path in $D^{f}_{n,2}$ can be expressed as: $HP_2 + (a, z) + HP_1$.

Case 3.2.2.4. $x \in V(D^{i,f}_{n,1})$ and $y \in V(D^{j,f}_{n,1})$, where $i, j \in \langle\alpha\rangle \setminus \{\delta, \beta\}$ and $i \neq j$.

In this case, there may be a node w' in $D^{\delta,f}_{n,1}$ with only two fault-free external neighbors u' and v' in $D^{f}_{n,2} - D^{\beta,f}_{n,1}$, where u' or v' equal to nodes x or y. Then, by adding the following condition to the method of Case 3.2.1.2, a (x, y)-Ham-path in $D^{f}_{n,2}$ can be obtained: E_2 intersects with nodes c and d in $D^{\beta,f}_{n,1}$, satisfying $(c, d) \in E(HC)$.

***Proof of* (2).** According to the size of the faulty element set $|F_\delta|$, we prove this conclusion by considering the following three cases.

Case 1. $|F_\delta| \le n-3$.

In this case, $\alpha = f_2 - 1$ and each $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $m \in \langle f_2-1\rangle$. Select two fault-free distinct nodes x and y from $D^{\delta,f}_{n,1}$ and both x and y have at least one fault-free external neighbor. According to Statement 1, there exists a (y,x)-Ham-path HP_1 in $D^{\delta,f}_{n,1}$. Let $(a,x),(b,y) \in E(D^f_{n,2})$ be two external edges, where $a \in V(D^{i,f}_{n,1})$, $b \in V(D^{j,f}_{n,1})$, $i,j \in \langle f_2-1\rangle \setminus \{\delta\}$, and $i \ne j$. Let HP_c be an (i,j)-Ham-path in $D^c - \{\delta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1. Then, a (a,b)-Ham-path HP_2 in $D^f_{n,2}$ can be obtained by union E_1 and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2-1\rangle \setminus \{\delta\}$. Consequently, one of the Ham-cycles in $D^f_{n,2}$ is given by $(x,a)+HP_2+(b,y)+HP_1$.

Case 2. $|F_\delta| = n-2$.

In this case, $\alpha = f_2-1$ and $D_{n,2} - D^\delta_{n,1}$ has at most $2n-2-(n-2) = n$ faulty elements. Since $k=2$ and $n \ge 3$, there is at most one copy $D^{\beta,f}_{n,1}$ is Hamiltonian and the other copy $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $\beta \in \langle f_2-1\rangle \setminus \{\delta\}$ and $m \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$. According to Statement 2, there exists a Ham-cycle HC_1 in $D^{\delta,f}_{n,1}$ and a Ham-cycle HC_2 in $D^{\beta,f}_{n,1}$. Therefore, we give the following two subcases.

Case 2.1. Each $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $m \in \langle f_2-1\rangle \setminus \{\delta\}$.

Let x and y be two nodes such that $(x,y) \in E(HC_1)$, and both x and y have at least one fault-free external neighbor. Then, we obtain a (y,x)-Ham-path $HP_1 = HC_1 - (y,x)$ in $D^{\delta,f}_{n,1}$. Next, a Ham-cycle in $D^f_{n,2}$ can be constructed using a method similar to Case 1.

Case 2.2. There exists one copy $D^{\beta,f}_{n,1}$ is Hamiltonian and the other copy $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $\beta \in \langle f_2-1\rangle \setminus \{\delta\}$ and $m \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$.

In this case, $D_{n,2} - D^\delta_{n,1} - D^\beta_{n,1}$ has at most $2n-2-(n-2)-(n-2) = 2$ faulty elements. Therefore, each node in $D^f_{n,2}$ has at least $n-2$ fault-free external neighbors. Let x and y be two nodes such that $(x,y) \in E(HC_1)$, and let $(a,x),(b,y) \in E(D^f_{n,2})$ be two external edges, where $a \in V(D^{i,f}_{n,1})$, $b \in V(D^{j,f}_{n,1})$, $i,j \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$, and $i \ne j$. Then, we obtain a (y,x)-Ham-path $HP_1 = HC_1 - (y,x)$ in $D^{\delta,f}_{n,1}$. Let HP_c be an (i,j)-Ham-path in $D^c - \{\delta\}$, and we denote the edge-set corresponding to this path in $D_{n,2}$ as E_1, where E_1 intersects with nodes c and d in $D^{\beta,f}_{n,1}$, satisfying $(c,d) \in E(HC_2)$. Then, a (a,b)-Ham-path HP_2 in $D^f_{n,2} - D^{\delta,f}_{n,1}$ can be obtained by union E_1, $HC_2 - (c,d)$, and the appropriate Ham-path in each $D^{m,f}_{n,1}$, where $m \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$. Consequently, one of the Ham-cycles in $D^f_{n,2}$ is given by $(x,a)+HP_2+(b,y)+HP_1$.

Case 3. $|F_\delta| = n-1$.

In this case, $\alpha = f_2-1$ and $D_{n,2} - D^\delta_{n,1}$ has at most $2n-2-(n-1) = n-1$ faulty elements. Therefore, each node in $D^f_{n,2}$ has at least one fault-free external neighbors. Let f_0 be the one faulty element in F_δ and $F'_\delta = F_\delta - f_0$. Therefore, $|F'_\delta| = |F_\delta| - 1 = n-2$. We let $D^{\delta,f'}_{n,1} = D^\delta_{n,1} - F'_\delta$. Let x and y be the two adjacent nodes (or end-nodes) of f_0 in HC_1 if f_0 is a faulty node (or faulty edge). Thus, there exists a (y,x)-Ham-path $HP_1 = HC_1 - f_0$ in $D^{\delta,f'}_{n,1}$. Since $k \ge 3$ and $n = 2$, there is at most one copy $D^{\beta,f}_{n,1}$ is Hamiltonian and the other copy $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $\beta \in \langle f_2-1\rangle \setminus \{\delta\}$ and $m \in \langle f_2-1\rangle \setminus \{\delta,\beta\}$. According to Statement 2, there exists a Ham-cycle HC_1 in $D^{\delta,f'}_{n,1}$ and a Ham-cycle HC_2 in $D^{\beta,f}_{n,1}$. Therefore, we give the following two subcases.

Case 3.1. Each $D^{m,f}_{n,1}$ is Hamiltonian-connected, where $m \in \langle f_2-1\rangle \setminus \{\delta\}$.

In this case, a Ham-cycle in $D_{n,2}^{f}$ can be constructed using a method similar to Case 1.

Case 3.2. There exists one copy $D_{n,1}^{\beta,f}$ is Hamiltonian and the other copy $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $\beta \in \langle f_2 - 1\rangle \setminus \{\delta\}$ and $m \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$.

In this case, $D_{n,2} - D_{n,1}^{\delta} - D_{n,1}^{\beta}$ has at most $2n-2-(n-1)-(n-2) = 1$ faulty elements. Therefore, each node in $D_{n,2}^{f}$ has at least $n-1$ fault-free external neighbors. Let $(a, x), (b, y) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{i,f})$, $b \in V(D_{n,1}^{j,f})$, $i, j \in \langle f_2 - 1\rangle \setminus \{\delta, \beta\}$, and $i \neq j$. Next, a Ham-cycle in $D_{n,2}^{f}$ can be constructed using a method similar to Case 2.2.

Case 4. $n \leq |F_\delta| \leq 2n - 2$.

In this case, $\alpha \in \{f_2 - 2, f_2 - 1\}$ and $D_{n,2} - D_{n,1}^{\delta}$ has at most $2n - 2 - n = n - 2$ faulty elements. Therefore, each node in $D_{n,2}^{f}$ has at least two fault-free external neighbors. Since $k \geq 3$ and $n = 2$, there is at most one copy $D_{n,1}^{\beta,f}$ is Hamiltonian and the other copy $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $\beta \in \langle\alpha\rangle \setminus \{\delta\}$ and $m \in \langle\alpha\rangle \setminus \{\delta, \beta\}$. According to Statement 2, there exists a Ham-cycle HC in $D_{n,1}^{\beta,f}$. Therefore, we give the following two subcases.

Case 4.1. Each $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $m \in \langle\alpha\rangle \setminus \{\delta\}$.

For each node w in $D_{n,1}^{\delta,f}$, let $(u, w), (v, w) \in E(D_{n,2}^{f})$ be two external edges. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Select one copy $D_{n,1}^{i,f}$ from $D_{n,2}^{f}$ that contains a (y, x)-Ham-path HP_1 and let $(a, x), (b, y) \in E(D_{n,2}^{f})$ be two external edges, where $a \in V(D_{n,1}^{j,f})$, $b \in V(D_{n,1}^{l,f})$, $i, j, l \in \langle\alpha\rangle \setminus \{\delta\}$, $i \neq j \neq l$, and the three copies $D_{n,1}^{i,f}$, $D_{n,1}^{j,f}$, and $D_{n,1}^{l,f}$ are not equal to any bounded subgraph. Let HP_c be a (j, l)-Ham-path in $D^c - \{i, \delta\}$ such that the nodes corresponding to the bounded subgraphs in D^c are adjacent in HP_c. Then, we denote the edge-set corresponding to HP_c in $D_{n,2}$ as E_2, except for the edges that connect the bounded subgraphs. Then, a (a, b)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{i,f}$ can be obtained by union E_1, E_2, and the appropriate Ham-path in each $D_{n,1}^{m,f}$, where $m \in \langle\alpha\rangle \setminus \{i, \delta\}$. Thus, one of the Ham-cycles in $D_{n,2}^{f}$ is given by $(x, a) + HP_2 + (b, y) + HP_1$.

Case 4.2. There exists one copy $D_{n,1}^{\beta,f}$ is Hamiltonian and the other copy $D_{n,1}^{m,f}$ is Hamiltonian-connected, where $\beta \in \langle\alpha\rangle \setminus \{\delta\}$ and $m \in \langle\alpha\rangle \setminus \{\delta, \beta\}$.

Let x and y be two nodes such that $(x, y) \in E(HC)$. Then, we obtain a (y, x)-Ham-path $HP_1 = HC - (y, x)$ in $D_{n,1}^{\beta,f}$. For each node w in $D_{n,1}^{\delta,f}$, let $(u, w), (v, w) \in E(D_{n,2}^{f})$ be two external edges, where $u, v \notin V(D_{n,1}^{\beta,f})$. We denote all these edges as a set E_1 and the copies of $D_{n,1}$ which contains u and v as bounded subgraphs. Next, a (a, b)-Ham-path HP_2 in $D_{n,2}^{f} - D_{n,1}^{\beta,f}$ can be obtained by using a method similar to the Case 3.1.1.2.2 in Proof of (1). Thus, one of the Ham-cycles in $D_{n,2}^{f}$ is given by $(x, a) + HP_2 + (b, y) + HP_1$.

□