An improved fifth-order WENO scheme with symmetry-preserving smoothness indicators for hyperbolic conservation laws

Donghui Son

School of Space Research, Kyung Hee University

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Background & motivation

- The study is motivated by the need to address the limitations of existing WENO schemes, specifically WENO-NS and WENO-P, which struggle to achieve optimal convergence orders at critical points where the first and second derivatives are zero.

- The authors aim to improve the performance of these schemes by developing a new set of smoothness indicators that satisfy the symmetry-preserving property, which is not maintained by the smoothness indicators of WENO-NS and WENO-P

- Another motivation is to eliminate the reliance on user-tunable parameters, which are necessary in the previous schemes to balance contributions from substencils and to remedy symmetry issues.

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Introduction

- The WENO (Weighted Essentially Non-Oscillatory) schemes are crucial for numerically simulating time-dependent hyperbolic conservation laws, with the first scheme developed by Liu et al. achieving $(r+1)$-th-order accuracy.

- Subsequent improvements, notably the WENO-JS scheme by Jiang and Shu, enhanced convergence to (2r - 1)th-order accuracy, addressing issues of convergence loss at critical points.

- Recent developments include the WENO-M and WENO-Z schemes, which utilize different local smoothness indicators (LSIs) to improve performance, particularly near discontinuities, while also introducing computational challenges.

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- WENO-JS employs third-order ENO reconstructions and nonlinear weights calculated to ensure stability and convergence, avoiding division by zero.

- The LSIs used in WENO-JS are critical for determining the nonlinear weights, which are essential for achieving the desired accuracy in numerical simulations.

- The scheme's design allows it to achieve full-order convergence in smooth regions, although it struggles near critical points.

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- WENO-Z improves upon WENO-JS by ensuring that its nonlinear weights satisfy a sufficient condition for full-order convergence accuracy, particularly near critical points.

- This enhancement allows WENO-Z to maintain higher accuracy in simulations where WENO-JS may falter, especially in the presence of discontinuities.

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- The WENO-NS scheme introduces a user-tunable parameter to enhance the performance of the nonlinear weights, allowing for optimal-order convergence even at critical points.

- The nonlinear weights are derived from a specific function, ensuring that the scheme can adapt to various flow conditions effectively.

- WENO-P further refines the nonlinear weights to achieve optimal-order convergence at first-order critical points, addressing limitations found in previous schemes.

- The introduction of a user-tunable parameter allows for greater flexibility and adaptability in the scheme's performance across different scenarios.

- The MWENO-P scheme builds on the strengths of WENO-P, providing improved nonlinear weights that maintain optimal convergence at critical points.

- Numerical analyses confirm that MWENO-P outperforms other WENO schemes in simulating complex flows with discontinuities.

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- Numerical analyses confirm that MWENO-P outperforms other WENO schemes in simulating complex flows with discontinuities.

- The new LSIs introduced in the improved MWENO-P scheme are designed to enhance performance by incorporating symmetry-preserving properties.

- The proposed global smoothness indicator (GSI) is shown to reduce numerical dissipation near discontinuities, improving the overall accuracy of the scheme.

- The elimination of user-tunable parameters simplifies the implementation of the scheme while maintaining its robustness.

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Convergence property of IMWENO-P at critical points

- The theoretical analysis demonstrates that IMWENO-P achieves optimal convergence orders even near first-order and second-order critical points.
- Taylor series expansions of the LSIs confirm the scheme's ability to maintain accuracy in challenging numerical scenarios.

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Numerical experiments

- A series of numerical tests validate the performance of IMWENO-P, showing its superior accuracy and stability compared to other WENO schemes.

- The experiments cover various scenarios, including scalar conservation laws and Euler equations, highlighting the scheme's robustness in capturing fine structures and discontinuities.

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- The numerical tests reveal that IMWENO-P consistently outperforms WENO-JS, WENO-Z, and other schemes in terms of accuracy and resolution.

- Even in cases with critical points, IMWENO-P maintains its desired convergence orders, demonstrating its effectiveness in complex flow simulations.

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Example 1: $u_0(x) = \sin(\pi x)$, Example 2: $u_0(x) = \sin^3(\pi x)$

- The computational efficiency of IMWENO-P is assessed against other schemes, showing that it achieves lower errors with less CPU time in certain scenarios.

- The results indicate that IMWENO-P balances accuracy and computational cost effectively, making it a viable option for practical applications.

- IMWENO-P demonstrates superior performance in capturing discontinuities in the 1D Euler equations, outperforming traditional WENO schemes.

- The scheme's ability to maintain high resolution in complex flow scenarios is highlighted through various numerical tests.

Example 7: Lax shock tube

$$(\rho, u, p) = \begin{cases} (0.445, 0.698, 3.528), & \text{on the left,} \\ (0.5, 0, 0.571), & \text{on the right.} \end{cases}$$

- The performance of IMWENO-P is further validated in high-frequency wave problems, where it consistently achieves higher resolution compared to other WENO schemes.

- The results indicate that IMWENO-P effectively captures intricate flow features, showcasing its robustness in challenging numerical environments.

- In 2D simulations, IMWENO-P excels in resolving complex flow structures, outperforming WENO-JS and WENO-Z in terms of numerical dissipation.

- The scheme's ability to capture instabilities and fine details in the flow field is demonstrated through various test cases.

Conclusion

- The study concludes that IMWENO-P successfully recovers optimal convergence orders near critical points, enhancing the accuracy of numerical simulations.

- The numerical experiments confirm that IMWENO-P not only resolves fine structures effectively but also reduces dissipation near discontinuities, making it a robust choice for simulating complex flows.

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Thank you for listening!

Questions?