1 Including the displacement current

1.1 Maxwell equations

Ampére's law with the displacement current $\partial\boldsymbol{E}/\partial t$ included reads

$$\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{J}. \tag{1}$$

Here, we must obey the constraint equation for $\boldsymbol{E}$,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\mathrm{e}}. \tag{2}$$

We also have $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, which is readily obeyed by expressing $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ in terms of the magnetic vector potential $\boldsymbol{A}$ and solving for the uncurled induction equation

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \boldsymbol{\nabla}\psi. \tag{3}$$

1.2 Charge density and Ohm's law

We compute ρ_{e} by solving the continuity equation for the charge density. It is obtained by taking the divergence of Eq. (1), i.e.,

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}, \tag{4}$$

which requires solving Ohm's law, i.e.,

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right), \tag{5}$$

so

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = \sigma\left[\boldsymbol{\nabla} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{B})\right], \tag{6}$$

where the derivatives in

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{B}) = \epsilon_{ijk}\left(u_{j,i}B_k + u_j B_{k,i}\right) \tag{7}$$

can be expressed in terms of $u_{j,i}$ and $B_{k,i}$, which are readily available.

1.3 Coulomb gauge

One way to satisfy Eq. (2) is to adopt the Coulomb gauge, $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. By taking the divergence of Eq. (3) and using the Coulomb gauge, we have $\nabla^2\psi = -\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}}$, so we have to solve a Poisson equation for the electrostatic (or scalar) potential ψ,

$$\nabla^2\psi = \rho_{\mathrm{e}}, \tag{8}$$

which is analogous to the Poisson equation for the gravitational potential.

1.4 Weyl gauge

When using the Weyl gauge (also known as the temporal gauge), we can control the longitudinal modes of $\boldsymbol{E}$, by defining

$$\Gamma = \boldsymbol{\nabla} \cdot \boldsymbol{A} \tag{9}$$

and replacing $\boldsymbol{\nabla} \times \boldsymbol{B}$ in Eq. (1) by $-\nabla^2\boldsymbol{A} + \boldsymbol{\nabla}\Gamma$, so therefore Eq. (1) becomes

$$\frac{\partial \boldsymbol{E}}{\partial t} = -\nabla^2\boldsymbol{A} + \boldsymbol{\nabla}\Gamma - \boldsymbol{J}. \tag{10}$$

The longitudinal modes of $\boldsymbol{E}$ are constrained by taking the divergence of Eq. (3), so we obtain

$$\frac{\partial \Gamma}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{E}. \tag{11}$$

Following standard procedures employed in numerical relativity, and following a suggestion from Tanmay Vachaspati, we can replace $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ by the algebraic mean of $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ and $\rho_{\rm e}$, i.e., we solve

$$\frac{\partial \Gamma}{\partial t} = -(1-w)\boldsymbol{\nabla} \cdot \boldsymbol{E} - w\rho_{\rm e}. \tag{12}$$

2 Finite axion density

When the axion density ϕ is finite, Eq. (1) is replaced by

$$\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{J} - \frac{\alpha}{f}\left(\dot{\phi}\boldsymbol{B} + \boldsymbol{\nabla}\phi \times \boldsymbol{E}\right). \tag{13}$$

Now Eq. (13) becomes

$$\frac{\partial \boldsymbol{E}}{\partial t} = -\nabla^2 \boldsymbol{A} + \boldsymbol{\nabla}\Gamma - \boldsymbol{J} - \frac{\alpha}{f}\left(\dot{\phi}\boldsymbol{B} + \boldsymbol{\nabla}\phi \times \boldsymbol{E}\right). \tag{14}$$

The longitudinal modes of $\boldsymbol{E}$ are constrained by

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\rm e} - \frac{\alpha}{f}\boldsymbol{B} \cdot \boldsymbol{\nabla}\phi. \tag{15}$$

Instead of Eq. (11), we now solve

$$\frac{\partial \Gamma}{\partial t} = -(1-w)\boldsymbol{\nabla} \cdot \boldsymbol{E} - w\rho_{\rm e}^{\rm tot}, \tag{16}$$

where $\rho_{\rm e}^{\rm tot} \equiv \rho_{\rm e} - (\alpha/f)\, \boldsymbol{B} \cdot \boldsymbol{\nabla}\phi$. In the conducting case, we obtain $\rho_{\rm e}^{\rm tot}$ by solving the continuity equation for the charge density. It is obtained by taking the divergence of Eq. (13), i.e.,

$$\frac{\partial \rho_{\rm e}^{\rm tot}}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}^{\rm tot}, \tag{17}$$

where

$$\boldsymbol{J}^{\rm tot} = \boldsymbol{J} + \frac{\alpha}{f}\left(\dot{\phi}\boldsymbol{B} + \boldsymbol{\nabla}\phi \times \boldsymbol{E}\right), \tag{18}$$

In summary, when $\sigma = 0$ and $\alpha \neq 0$, we have to solve three dynamical equations: Eq. (3), (14), and Eq. (16). When $\sigma = 0$ and $\alpha = 0$, the equations are linear and we just need to solve the two dynamical equations Eqs. (3) and (10), as was done in BS21 and BHS21. When $\sigma \neq 0$, we also have to solve Eq. (17).