Numerical Optimization HW04

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1 Multivariate Optimization

Multivariate Optimization for smooth function - The method of steppest descent - Newton's method - Quasi-Newton's method : SR1, BFGS

In this notebook, we will optimize three functions

$f(x, y) = (x + 2y - 6)^2 + (2x + y - 6)^2$
$f(x, y) = 50(y - x^2)^2 + (1 - x)^2$
$f(x, y) = (1.5 - x + xy)^2 + (2.25 - x + xy^2)^2 + (2.625 - x + xy^3)^2$

1.0.1 Iteration formular

$x_{k+1} := x_k + \alpha_k p_k$ where α_k is step length and p_k is search direction

Method of Steppest Descent $p_k = -\nabla f(x_k)$

Newton's Method $p_k = -(\nabla^2 f(x_k))^{-1} \nabla f(x_k)$

Quasi Newton's Method $p_k = -\mathbf{B}_k^{-1} \nabla f(x_k)$ where $\mathbf{B}_k \approx (\nabla^2 f(x_k))$

---

2 Implementation

[1]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.tri as tri
import math as m
import time as t
from tabulate import tabulate
```

2.0.1 Define Function

[2]:
```
def f_1(p):
    x, y = p[0][0], p[1][0]
    return (x+2*y-6)**2+(2*x+y-6)**2
def f_2(p):
    x, y = p[0][0], p[1][0]
    return 50*(y-x**2)**2+(1-x)**2
def f_3(p):
    x, y = p[0][0], p[1][0]
    return (1.5-x+x*y)*2+(2.25-x+x*(y**2))**2+(2.625-x+x*(y**3))**2
```

2.0.2 Define Plotting Function

[3]:
```
def plotContour(f, rangeX=(0, 0), rangeY=(0, 0), grid=(0, 0)):
    x = np.arange(*rangeX, 0.01)
    y = np.arange(*rangeY, 0.01)
    X, Y = np.meshgrid(x, y)
    Z = f(([X], [Y]))
    plt.figure(figsize=grid)
    cp = plt.contour(X, Y, Z, levels=np.linspace(
        Z.reshape(-1, 1).min(), Z.reshape(-1, 1).max(), 120), linewidths=0.1) ␣
↪# plt.colorbar(cp)


def plotLine(x1, x2, color="black"):
    plt.plot([x1[0], x2[0]], [x1[1], x2[1]], color, linewidth=.8)


def plotPoint(x, color="black"):
    plt.plot([x[0]], [x[1]], color, marker='o', markersize=4)


def plotText(x, str=None):
    plt.plot([x[0]], [x[1]], "black", marker='o', markersize=4)
    if str != None:
        plt.text(x[0]+.03, x[1]+.03, str, fontsize=20)
    else:
        plt.text(x[0]+.03, x[1]+.03, "(%.3f, %.3f)" % x, fontsize=20)


def resultsInTable(x, f):
    print("\n%s results at (%.2f, %.2f) " % (f.__name__, x.T[0][0], x.T[0][1]))
    if f == f_1:
        print("minimum value of f1 is 0.0 at (2,2)")
    elif f == f_2:
        print("minimum value of f2 is 0.0 at (1,1)")
```

```
    elif f == f_3:
        print("minimum value of f3 is 6.16298e-32 at (3,0.5)")
    plotText(x, "x0")
    sd = steepestDescent(x.copy(), f, color="purple")
    ne = newton(x.copy(), f, color="green")
    sr1 = quasiNewtonSR1(x.copy(), f, color="red")
    bfgs = quasiNewtonBFGS(x.copy(), f, color="blue")
    print(tabulate([["S.D", *sd], ["Newton", *ne], ["SR1", *sr1], ["BFGS",␣
↪*bfgs]],headers=["f(x)", "x", "iter", "time"], tablefmt="github",␣
↪floatfmt=(".5f",".5f")))
    return
def printVal(xk, f, iter, start):
    return "%.8f" % f(xk), "(%.8f, %.8f)" % (xk.T[0][0], xk.T[0][1]), iter, "%4.
↪2f"%((t.time()-start)*1000)
```

2.0.3 Gradient and Hessian Implementation

[4]:
```
def grad(p, f):
    dx = np.array([[[-h], [0]], [[h], [0]]])
    dy = np.array([[[0], [-h]], [[0], [h]]])
    g1 = (f(p+dx[1])-f(p+dx[0]))/(2*h)
    g2 = (f(p+dy[1])-f(p+dy[0]))/(2*h)
    g = np.array([[g1], [g2]])
    return g


def hessian(p, f):
    dx = np.array([[[-h], [0]], [[h], [0]]])
    dy = np.array([[[0], [-h]], [[0], [h]]])
    H11 = (f(p+2*dx[1])-2*f(p)+f(p+2*dx[0]))/(4*h**2)
    H12 = (f(p+dx[1]+dy[1])-f(p+dx[1]+dy[0]) -
           f(p+dx[0]+dy[1])+f(p+dx[0]+dy[0]))/(4*h**2)
    H21 = H12
    H22 = (f(p+2*dx[1])-2*f(p)+f(p+2*dx[0]))/(4*h**2)
    H = np.array([[H11, H12], [H21, H22]])
    return H
```

2.0.4 Backtracking Method for step length

[5]:
```
def stepLength(x0, p, f): # Use backtrakingMethod
    a0 = 3
    r = 0.9
    c = 0.6
    a = a0
    while True:
        if f(x0+a*p) > a*c*np.dot(grad(x0, f).T, p)+f(x0):
            a *= r
```

```
        else:
            break
    return a
```

2.0.5 Method of Steepest descent

[6]:
```
def steepestDescent(x0, f, color="black"):
    iter, start = 0, t.time()
    xk = x0
    while True:
        pk = -grad(xk, f)
        pk /= np.linalg.norm(pk)
        sk = stepLength(xk, pk, f)*pk
        plotLine(xk, xk+sk, color)
        plotPoint((xk.T[0][0], xk.T[0][1]), color)

        xk += sk

        iter += 1
        if np.linalg.norm(sk) <= eps:
            plt.plot(xk.T[0][0], xk.T[0][1], color, marker='o', markersize=6)
            break
    return printVal(xk, f, iter, start)
```

2.0.6 Newton's Method

[7]:
```
def newton(x0, f, color="black"):
    iter, start = 0, t.time()
    xk = x0
    while True:
        dfx = grad(xk, f)
        Hk = hessian(xk, f)
        pk = -np.matmul(np.linalg.inv(Hk), dfx)
        pk /= np.linalg.norm(pk)
        sk = stepLength(xk, pk, f)*pk
        plotLine(xk, xk+sk, color)
        plotPoint((xk.T[0][0], xk.T[0][1]), color)
        xk += sk
        iter += 1

        if np.linalg.norm(sk) <= eps:
            plt.plot(xk.T[0][0], xk.T[0][1], color, marker='o', markersize=6)
            break
    return printVal(xk, f, iter, start)
```

2.0.7 Quasi-Newton's Method with rank 1 : SR1 Method

[8]:
```
def quasiNewtonSR1(x0, f, color="black"):
    iter, start = 0, t.time()
    I = np.array([[1, 0], [0, 1]])
    sk, yk, Hk = np.array([[1, 0]]).T, np.array([[1, 0]]).T, I
    pk = np.array([[1, 0]]).T
    xk = x0
    while True:
        dfx = grad(xk, f)
        pk = -np.matmul(Hk, dfx)
        pk /= np.linalg.norm(pk)
        sk = stepLength(xk, pk, f)*pk
        plotLine(xk, xk+sk, color)
        plotPoint((xk.T[0][0], xk.T[0][1]), color)
        xk += sk
        yk = grad(xk, f)-dfx
        Hk = Hk+(sk-np.matmul(Hk, yk))*(sk-np.matmul(Hk, yk)).T / (np.
↪dot((sk-np.matmul(Hk, yk)).T, yk))
        iter += 1
        if np.linalg.norm(sk) <= eps:
            plt.plot(xk.T[0][0], xk.T[0][1], color, marker='o', markersize=6)
            break
    return printVal(xk, f, iter, start)
```

2.0.8 Quasi-Newton's Method with rank 2 : BFGS

[9]:
```
def quasiNewtonBFGS(x, f, color="black"):
    iter, start = 0, t.time()
    I = np.array([[1, 0], [0, 1]])
    sk, yk, Hk = np.array([[1, 0]]).T, np.array([[1, 0]]).T, I
    pk = np.array([[1, 0]]).T
    xk = x
    while True:
        dfx = grad(xk, f)
        pk = -np.matmul(Hk, dfx)
        pk /= np.linalg.norm(pk)

        sk = stepLength(xk, pk, f)*pk
        plotLine(xk, xk+sk, color)
        plotPoint((xk.T[0][0], xk.T[0][1]), color)
        xk += sk
        yk = grad(xk, f)-dfx

        rk = 1/(np.dot(yk.T, sk))
        Hk = (I-rk*sk*yk.T)*Hk*(I-rk*yk*sk.T)+rk*sk*sk.T
```

```
        iter += 1
        if np.linalg.norm(sk) <= eps:
            plt.plot(xk.T[0][0], xk.T[0][1], color, marker='o', markersize=6)
        break
    return printVal(xk, f, iter, start)
```

---

3 Results

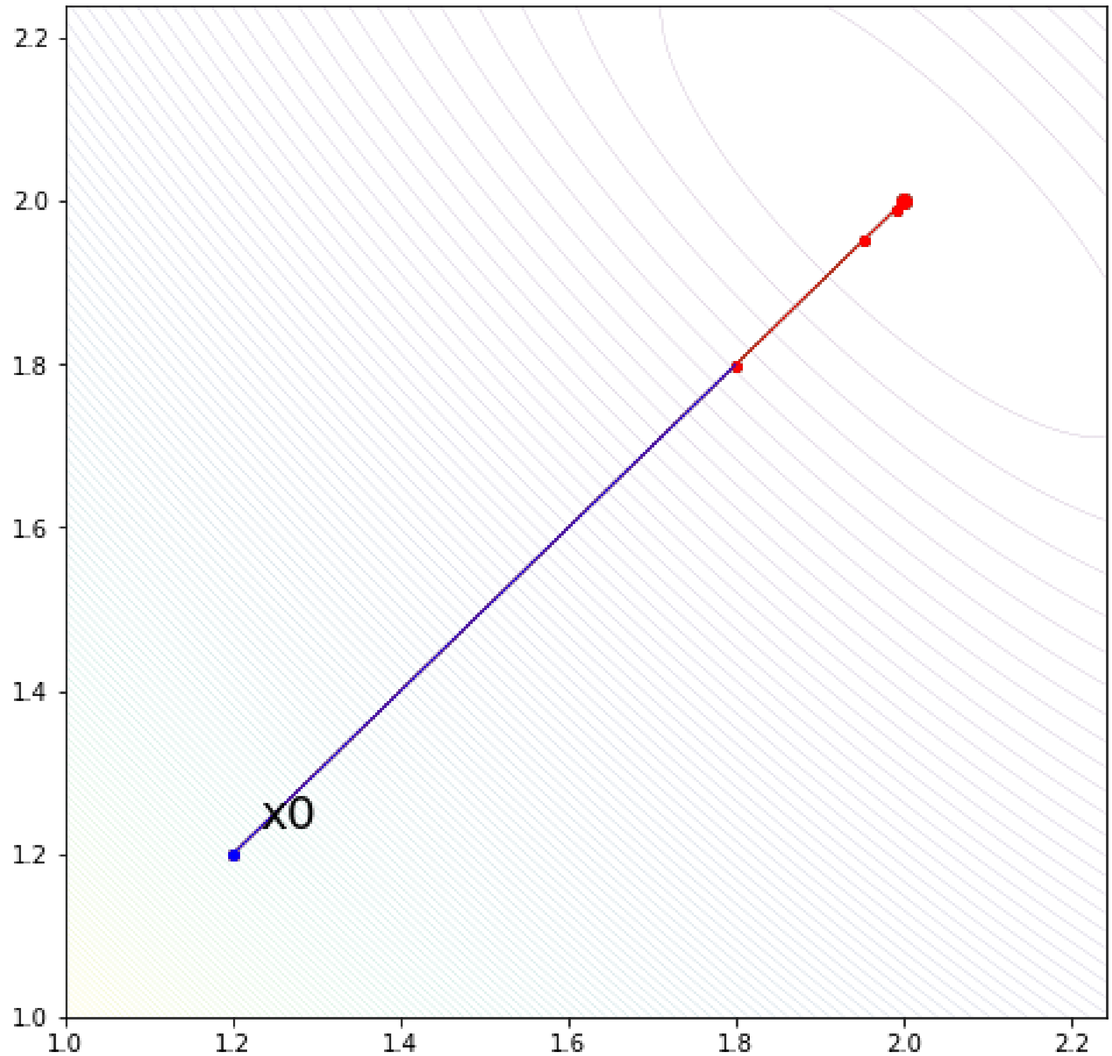
3.0.1 Optimize result at point $x_0 = (1.2, 1.2)$

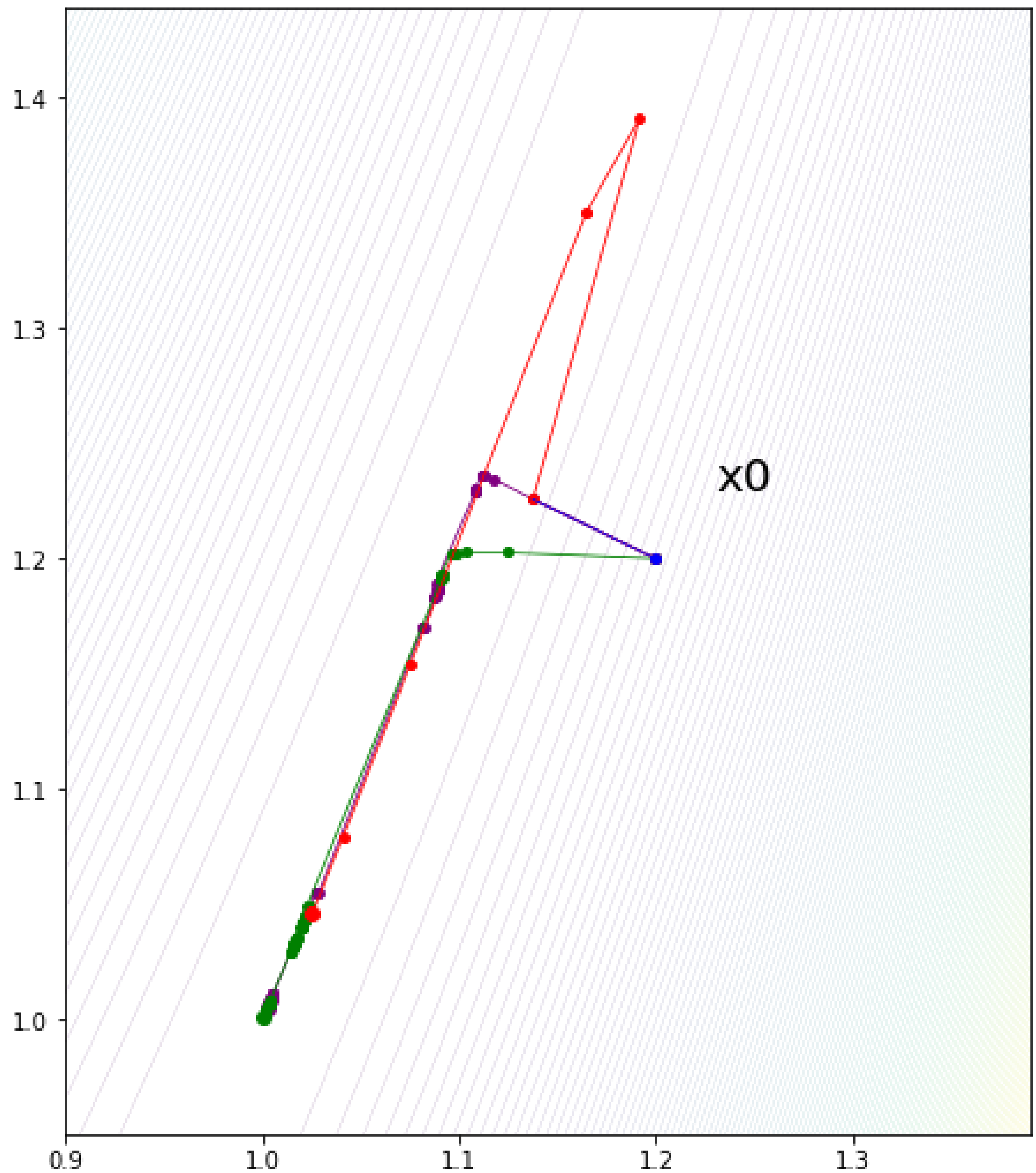
[10]:

```
h = 1e-7
eps = 1e-5
x0 = np.array([[1.2, 1.2]]).T
plotContour(f_1, rangeX=(1, 2.25), rangeY=(1, 2.25), grid=(8, 8))
resultsInTable(x0.copy(), f_1)
plotContour(f_2, rangeX=(0.9, 1.4), rangeY=(0.95, 1.45), grid=(7.5, 9))
resultsInTable(x0.copy(), f_2)
plotContour(f_3, rangeX=(-0.5, 5.0), rangeY=(-1, 1.3), grid=(16, 8))
resultsInTable(x0.copy(), f_3)
```

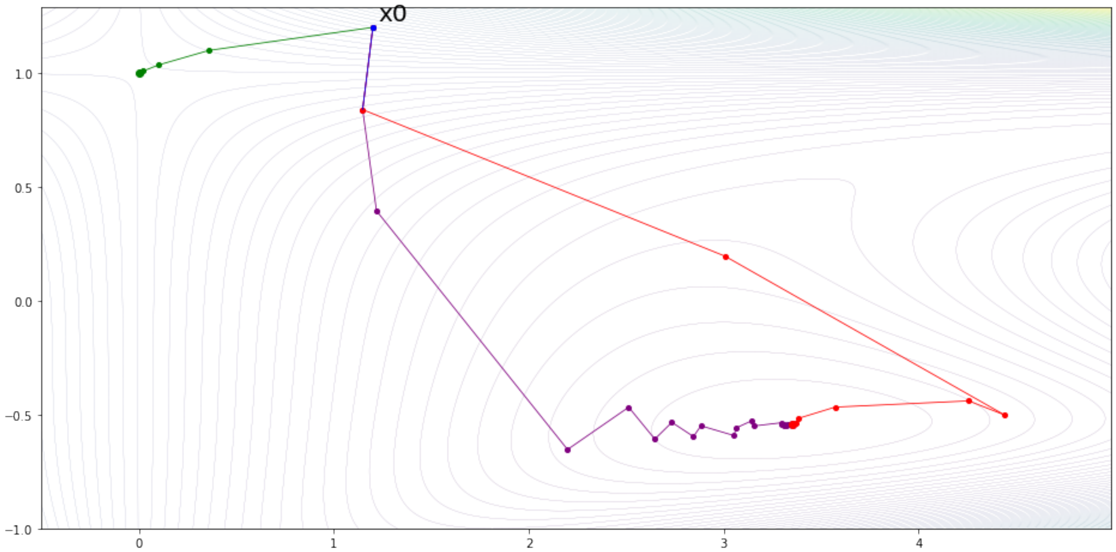
```
f_1 results at (1.20, 1.20)
minimum value of f1 is 0.0 at (2,2)
|        |    f(x) | x                        |    iter |    time |
|--------|---------|--------------------------|---------|---------|
| S.D    | 0.00000 | (1.99999769, 1.99999769) |       9 |  37.4   |
| Newton | 0.00000 | (1.99999769, 1.99999769) |       9 |  37.23  |
| SR1    | 0.00000 | (1.99999769, 1.99999769) |       9 |  30.57  |
| BFGS   | 0.72632 | (1.79912352, 1.79912352) |       1 |   1.41  |

f_2 results at (1.20, 1.20)
minimum value of f2 is 0.0 at (1,1)
|        |    f(x) | x                        |    iter |    time |
|--------|---------|--------------------------|---------|---------|
| S.D    | 0.00001 | (1.00257874, 1.00518998) |     141 | 752.91  |
| Newton | 0.00000 | (1.00065820, 1.00132666) |      83 | 417.3   |
| SR1    | 0.00223 | (1.02552932, 1.04609425) |       7 |  35.49  |
| BFGS   | 0.25134 | (1.13755014, 1.22584132) |       1 |   3.11  |

f_3 results at (1.20, 1.20)
minimum value of f3 is 6.16298e-32 at (3,0.5)
|        |     f(x) | x                          |    iter |    time |
|--------|----------|----------------------------|---------|---------|
| S.D    | -5.73452 | (3.34993582, -0.54359493)  |      52 | 274.54  |
| Newton | 14.95312 | (0.00000068, 1.00000028)   |      10 |  44.74  |
| SR1    | -5.73452 | (3.35002111, -0.54359041)  |      14 |  51.86  |
```

```
| BFGS    | 10.92526 | (1.14759193, 0.83905493)  |       1 |    2.07 |
```





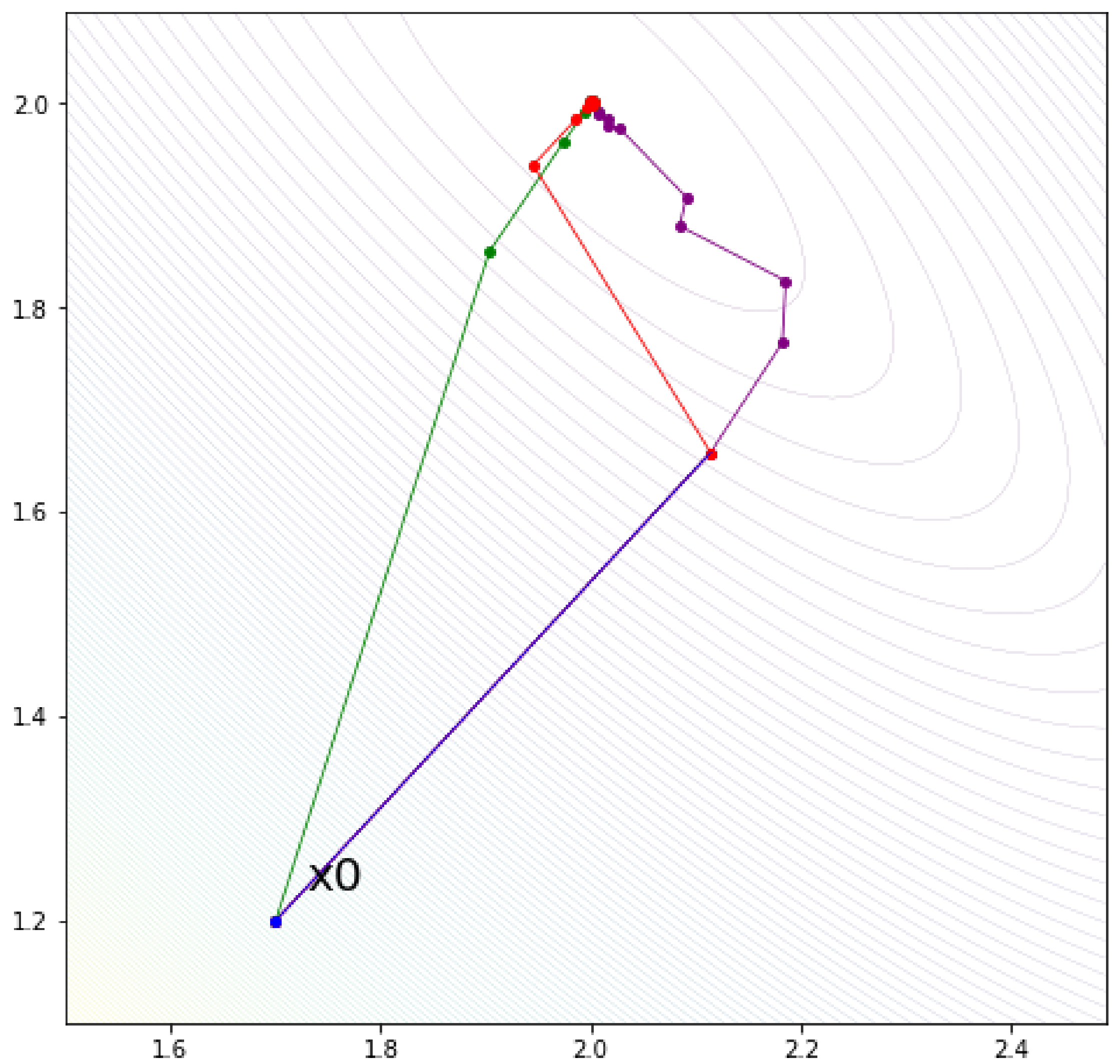
x0
1.4
1.3
1.2
1.1
1.0
0.9
1.0
1.1
1.2
1.3



3.0.2 $f(x, y) = (x + 2y - 6)^2 + (2x + y - 6)^2$ with initial point $x_0 = (1.7, 1.2)$

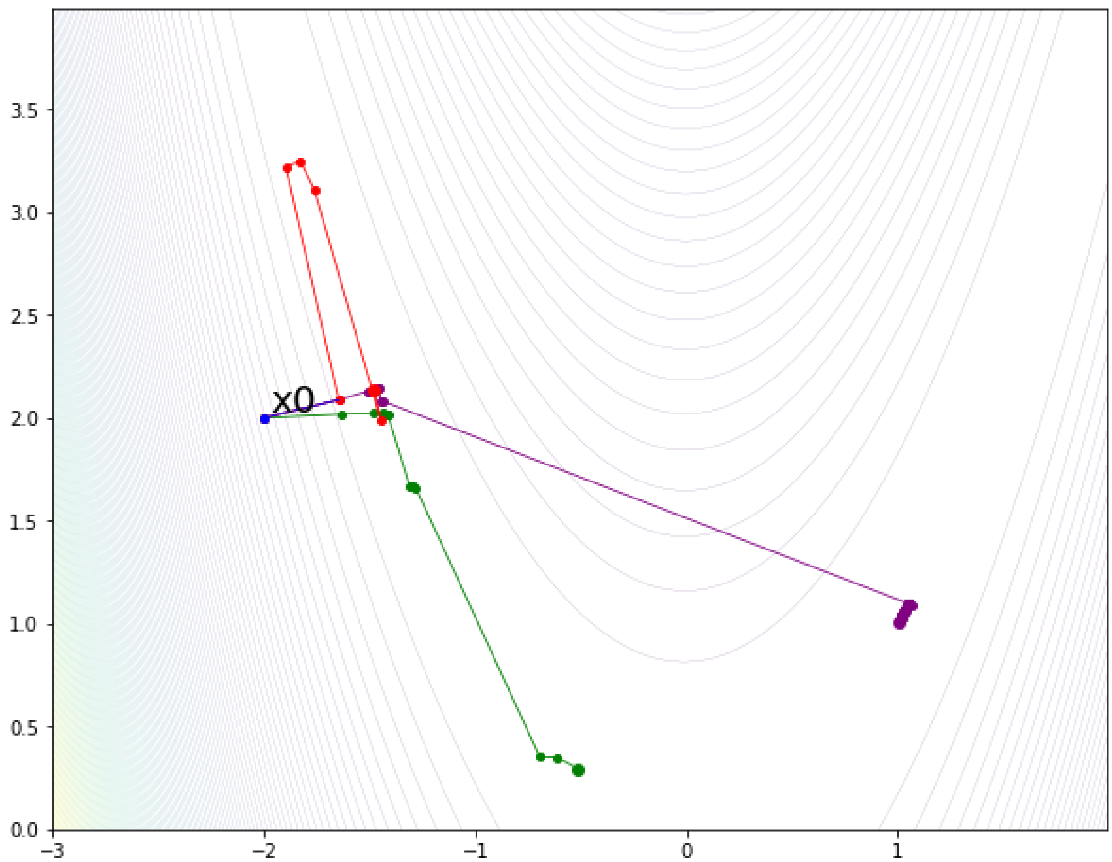
[11]:
```
x0 = np.array([[1.7, 1.2]]).T
plotContour(f_1, rangeX=(1.5, 2.5), rangeY=(1.1, 2.1), grid=(8, 8))
resultsInTable(x0.copy(), f_1)
```

```
f_1 results at (1.70, 1.20)
minimum value of f1 is 0.0 at (2,2)
|        |    f(x) | x                        |    iter |    time |
|--------|---------|--------------------------|---------|---------|
| S.D    | 0.00000 | (2.00002634, 1.99997339) |      23 |   86.45 |
| Newton | 0.00000 | (1.99999892, 1.99999842) |       9 |   30.16 |
| SR1    | 0.00000 | (1.99999762, 1.99999738) |       9 |   30.12 |
| BFGS   | 0.33703 | (2.11417451, 1.65823563) |       1 |    1.47 |
```



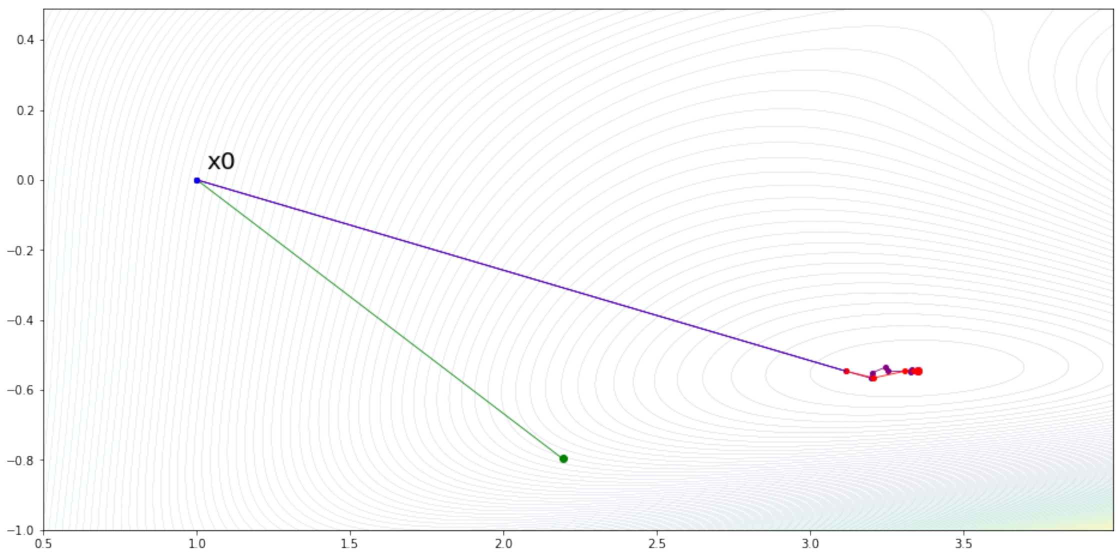
[12]:
```
x0 = np.array([[-2.0, 2.0]]).T
plotContour(f_2, rangeX=(-3, 2), rangeY=(0, 4), grid=(10, 8))
resultsInTable(x0.copy(), f_2)
```

```
f_2 results at (-2.00, 2.00)
minimum value of f2 is 0.0 at (1,1)
|        |     f(x) | x                          |   iter |   time |
|--------|----------|----------------------------|--------|--------|
| S.D    |  0.00001 | (1.00263885, 1.00531048)   |    175 | 785.67 |
| Newton |  2.33778 | (-0.51228736, 0.29430263)  |     12 | 181.97 |
| SR1    |  6.23975 | (-1.47807752, 2.14024307)  |      7 |  28.5  |
| BFGS   | 26.31410 | (-1.64600549, 2.08783983)  |      1 |   2    |
```



[13]:
```
x0 = np.array([[1.0, 0.0]]).T
plotContour(f_3, rangeX=(0.5, 4), rangeY=(-1, 0.5), grid=(16, 8))
resultsInTable(x0.copy(), f_3)
```

```
f_3 results at (1.00, 0.00)
minimum value of f3 is 6.16298e-32 at (3,0.5)
|        |      f(x) | x                          |    iter |    time |
|--------|-----------|----------------------------|---------|---------|
| S.D    | -5.73452 | (3.35000389, -0.54358946) |      25 |   98.77 |
| Newton | -2.32565 | (2.19349702, -0.79654014) |       2 |   16.6  |
| SR1    | -5.73452 | (3.35001809, -0.54359066) |      10 |   34.8  |
| BFGS   | -5.63581 | (3.11762251, -0.54648323) |       1 |    1.05 |
```


x0
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5

[ ]: