Numerical Optimization HW05

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1 Conjugate Gradient method

1. Implement linear Conjugate Gradient method for the following function

- $f(x, y) = (x + 2y - 7)^2 + (2x + y - 5)^2$

2. Implement nonlinear Conjugate Gradient method for the following function

- $f(x, y) = 40(y - x^2)^2 + (1 - x)^2$
- $f(x, y) = (1.5 - x + xy)^2 + (2.25 - x + xy^2)^2 + (2.625 - x + xy^3)^2$

2 Linear Conjugate Gradient method

Linear conjuage gradient method use relation between two problem

- Solving linear system $\mathbf{Ax} = \mathbf{b} \Leftrightarrow$ Solving mizimize problem $\arg\min\left\{\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Ax} - \mathbf{b}^{\mathrm{T}}\mathbf{x}\right\}$

Let the observation function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Ax} - \mathbf{b}^T\mathbf{x}$, then $\nabla f(\mathbf{x}) = \mathbf{Ax} - \mathbf{b}$

$f(x)$ has optimal point $\mathbf{x}^*$ where $\nabla f(\mathbf{x}^*) = 0$

Find the quadratic form of $f(x, y) = (x + 2y - 7)^2 + (2x + y - 5)^2$

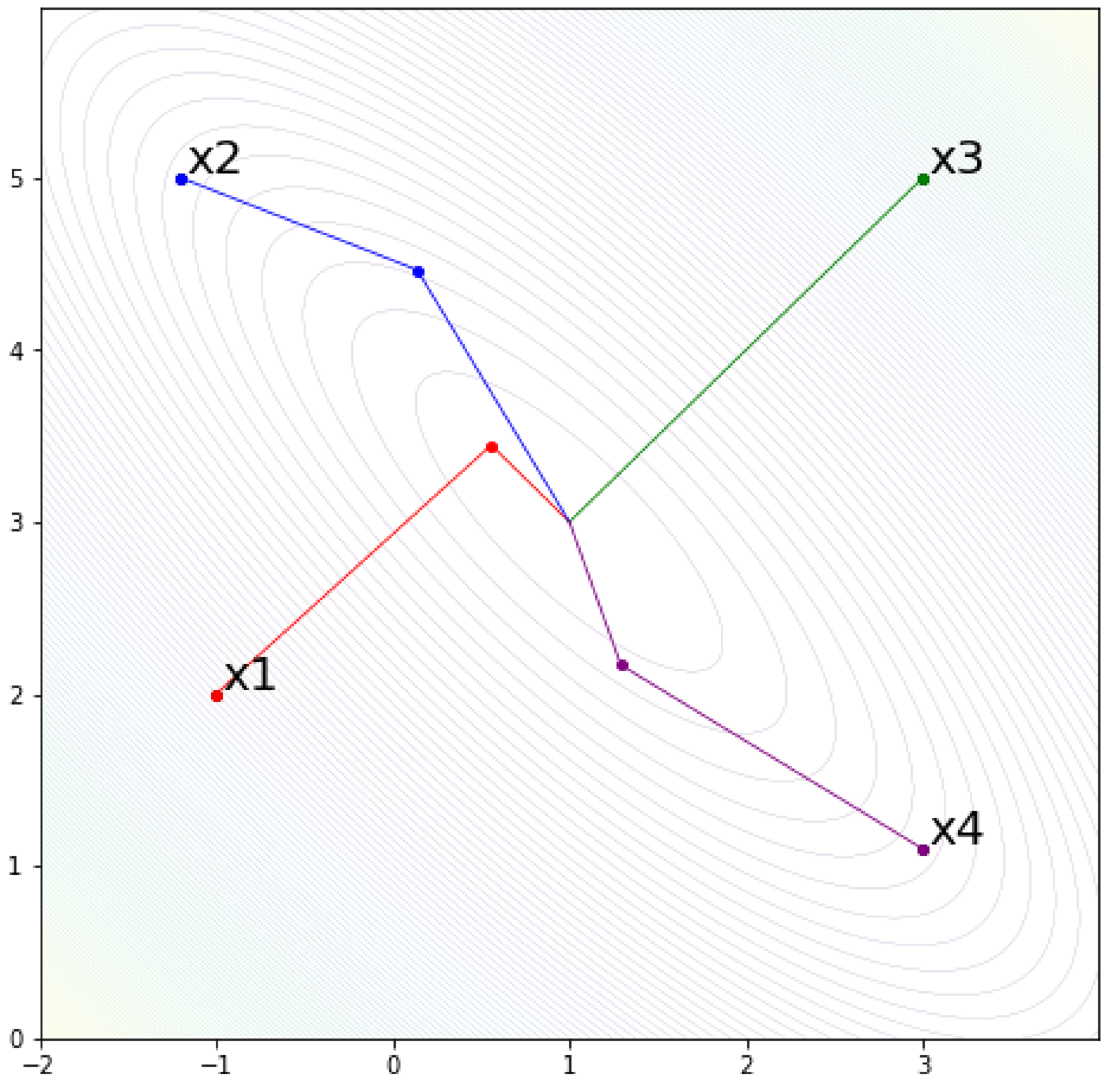
$$
\begin{aligned}
f(x, y) &= (x + 2y - 7)^2 + (2x + y - 5)^2 \\
&= 5x^2 + 8xy + 5y^2 - 34x - 38y + 74 \\
&= \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + f \\
&\therefore a = 10, b = 8, c = 10, d = -34, e = -38, f = 74 \\
&\Leftrightarrow \arg\min\left\{\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Ax} - \mathbf{b}^{\mathrm{T}}\mathbf{x} + \mathrm{f}\right\} \\
&= \arg\min\left\{\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Ax} - \mathbf{b}^{\mathrm{T}}\mathbf{x}\right\} \\
&\therefore \mathbf{A} = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 34 \\ 38 \end{bmatrix}
\end{aligned}
$$

*since linear CG method use minimization problem, we remove constant term temporary in $f(x, y)$

[6]:
```
def linearCG(x0, f=f_1, color='red'):
    iter = 0
    start = t.time()
    A = np.array([[10, 8], [8, 10]])
    b = np.array([[34, 38]]).T

    rk = A.dot(x0)-b
    pk = -rk
    bk = np.array([[0, 0]]).T
    xk = x0
    cnt = 0
    while np.linalg.norm(rk) >1e-4:
        iter += 1
        ak = rk.T.dot(rk)/np.matmul(np.matmul(pk.T, A), pk)
        plotLine(xk, xk+ak*pk, color)
        plotPoint((xk.T[0][0], xk.T[0][1]), color)
        xk = xk+ak*pk
        rk1 = rk+ak*np.matmul(A, pk)

        bk = rk1.T.dot(rk1)/rk.T.dot(rk)
        rk = rk1
        pk = -rk+bk*pk

    return printVal(xk, f, iter, start)
```

[14]:
```
def printpt(x):
    return "(%.1f, %.1f)" % (x.T[0][0], x.T[0][1])


plotContour(f_1, rangeX=(-2, 4), rangeY=(0, 6), grid=(8, 8))
x = np.array([[[-1], [2]], [[-1.2], [5]], [[3], [5]], [[3], [1.1]]])
c = ['red', 'blue', 'green', 'purple']
val = []
for i in range(4):
    plotText(x[i], "x%d" % (i+1))
    lcg = linearCG(x[i], color=c[i])
    val.append([printpt(x[i]), *lcg])
print(tabulate(val, headers=[
    "init", "f(x)", "x", "iter", "time"], tablefmt="github", floatfmt=(".5f", ".
↪5f"), stralign="center", numalign="center"))
```

```
|     init      |  f(x)   |            x             |  iter  |  time  |
|---------------|---------|--------------------------|--------|--------|
|  (-1.0, 2.0)  | 0.00000 | (1.00000000, 3.00000000) |   2    |  2.57  |
|  (-1.2, 5.0)  | 0.00000 | (1.00000000, 3.00000000) |   2    |  1.78  |
|  (3.0, 5.0)   | 0.00000 | (1.00000000, 3.00000000) |   1    |  0.87  |
|  (3.0, 1.1)   | 0.00000 | (1.00000000, 3.00000000) |   2    |  1.8   |
```



Check result with linear system $\mathbf{Ax} = \mathbf{b}$

[20]:
```
A = np.array([[10, 8], [8, 10]])
b = np.array([[34, 38]]).T
x = np.array([[1, 3]]).T

bb = np.matmul(A, x)
print("Ax=b")
print("b=(%.1f, %.1f)" % (bb.T[0][0], bb.T[0][1]))
```

```
Ax=b
b=(34.0, 38.0)
```

3 Non-linear Conjugate Gradient method

[8]:
```
h = 1e-7



def grad(p, f):
    dx = np.array([[[-h], [0]], [[h], [0]]])
    dy = np.array([[[0], [-h]], [[0], [h]]])
    g1 = (f(p+dx[1])-f(p+dx[0]))/(2*h)
    g2 = (f(p+dy[1])-f(p+dy[0]))/(2*h)
    return np.array([[g1], [g2]])
```

[9]:
```
def stepLength(x0, p, f):  # Use backtrakingMethod
    a, r, c = 3, 0.9, 0.6
    while f(x0+a*p) > a*c*np.dot(grad(x0, f).T, p)+f(x0):
        a *= r
    return a
```

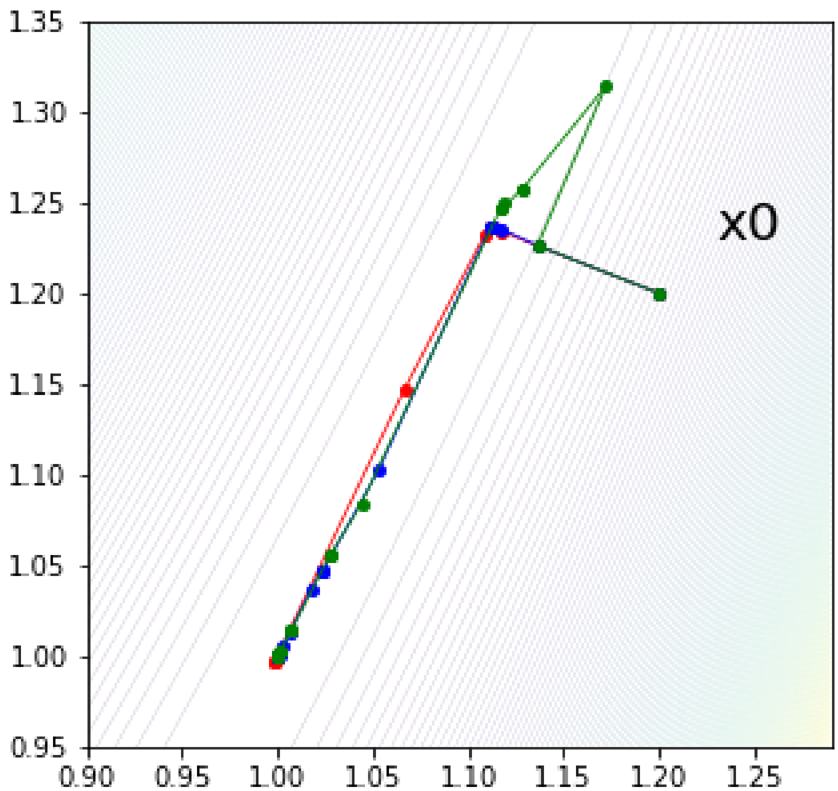
[10]:
```
def CG_FR(df1,df,pk,f): # Flectcher-Reeves
    return df1.T.dot(df1)/df.T.dot(df)
def CG_PR(df1,df,pk,f): # Polak-Ribiere
    return df1.T.dot(df1-df)/df.T.dot(df)
def CG_HS(df1,df,pk,f): # Hestenes-Stiefel
    return df1.T.dot(df1-df)/(df1-df).T.dot(pk)
```

[11]:
```
def nonLinearCG(x0, f, beta, color='black'):
    start = t.time()
    df = grad(x0, f)
    pk = -df
    bk = np.array([[0, 0]]).T
    xk = x0
    iter = 0
    while np.linalg.norm(df) > 1e-4:
        iter += 1
        ak = stepLength(xk, pk, f)
        plotLine(xk, xk+ak*pk, color)
        plotPoint((xk.T[0][0], xk.T[0][1]), color)
        xk1 = xk+ak*pk
        df1 = grad(xk1, f)
        bk = beta(df1, df, pk, f)
        pk = -df1+bk*pk
        xk,df = xk1,df1
    return printVal(xk, f, iter, start)
```

[12]:
```
x0 = np.array([[1.2, 1.2]]).T

plotContour(f_2, rangeX=(0.9, 1.3), rangeY=(0.95, 1.35), grid=(5,5))
```

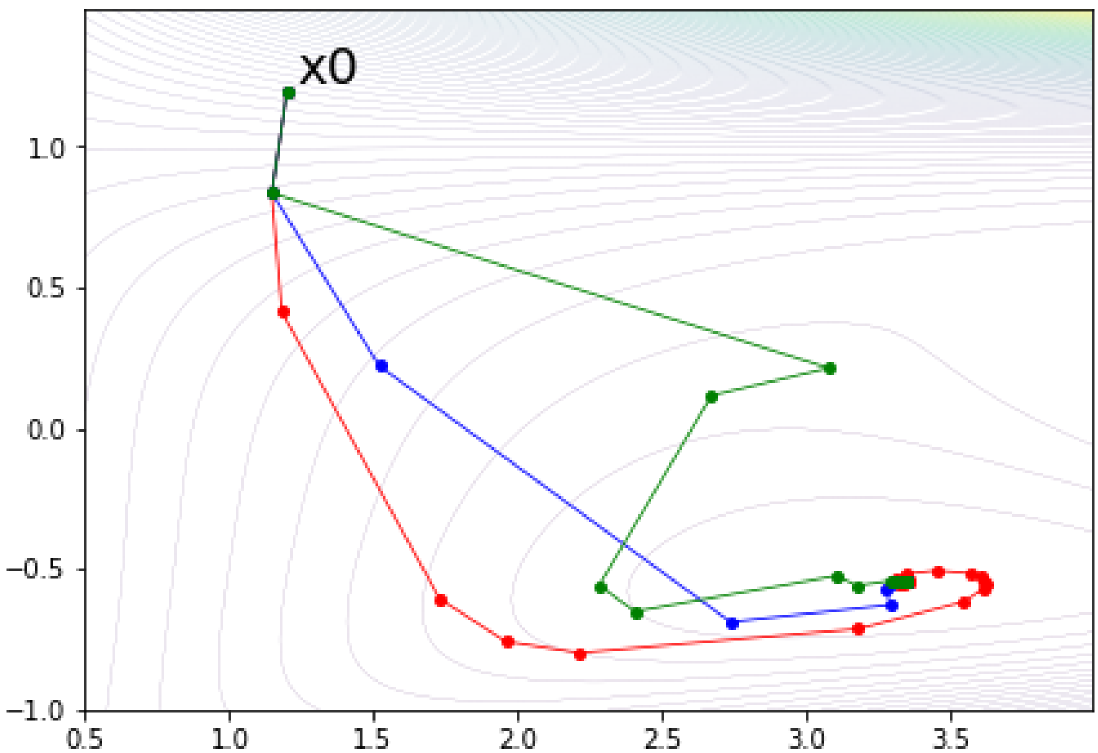
```
resultsInTable(x0.copy(), f_2)
```

```
f_2 results at (1.20, 1.20)
|       |  f(x)   |            x             |  iter  |  time  |
|-------|---------|--------------------------|--------|--------|
| CG-FR | 0.00000 | (1.00005897, 1.00011874) |   20   | 70.17  |
| CG-PR | 0.00000 | (1.00002044, 1.00004097) |   27   | 96.77  |
| CG-HS | 0.00000 | (1.00002180, 1.00004348) |   23   | 95.92  |
```



[13]:
```
plotContour(f_3, rangeX=(0.5, 4.0), rangeY=(-1, 1.5), grid=(7, 5))
resultsInTable(x0.copy(), f_3)
```

```
f_3 results at (1.20, 1.20)
|       |   f(x)   |             x              |  iter  |  time  |
|-------|----------|----------------------------|--------|--------|
| CG-FR | -5.73452 | (3.35003123, -0.54359099) |   54   | 163.14 |
| CG-PR | -5.73452 | (3.35000673, -0.54359075) |   19   | 47.71  |
| CG-HS | -5.73452 | (3.35000916, -0.54359129) |   21   | 50.27  |
```


x0
1.0
0.5
0.0
−0.5
−1.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5