Torque Equation
(See section 4.9)

Our goal is to combine the state-space voltage equations with the state-space torque equations.

To achieve this, we need to do the following three things to the torque equation:
1. Address the difference in power bases.
2. Address the difference in speed (time) bases.
3. Express the electromagnetic torque in terms of i_d and i_q quantities instead of a-b-c quantities.

Let's take them in that order.

1. Power Base (see first part of Section 4.9):

Consider the electrical torque in MKS units (ntn-meters); denote it as T_e.

The electrical torque that is computed from the voltage equations will be on a per-phase base, because, as we have seen, all quantities in the voltage equations are per-unitized on a per-phase base. Let's denote this torque as $T_{e\phi u}$ (A&F call it $T_{e\phi}$ - see eqt. 4.80 in text). Therefore,

$$T_{e\phi u} = \frac{T_e}{(S_B / \omega_B)}$$

However, the swing equation is usually written on a 3-phase power base, e.g.,

In (4.79), ω_B is used in place of ω_{Re}→

$$\frac{2H}{\omega_{Re}}\dot{\omega} = T_{mu} - T_{eu} \quad \text{in pu} \qquad (4.79)$$

Here, we have that T_{eu} is in pu on a 3-phase power base, i.e.,

$$T_{eu} = \frac{T_e}{(3S_B / \omega_B)}$$

We will continue to write the swing equation (and use it) like this, because the network equations are typically given on a three-phase base. In addition, this is the convention in the literature.

But because our voltage equations have been per-unitized on a per-phase base, we need to divide the torque obtained from the voltage equations by 3 before using it in the swing equation, i.e.,

$$T_{eu} = \frac{T_{e\phi u}}{3}$$

2. <u>Speed (time) Base (see Section 4.9.1)</u>

In the voltage equations, both speed and time were per-unitized, so we also need to do this in the swing equation.

$$\frac{2H}{\omega_{\mathrm{Re}}} \frac{d\omega}{dt} = T_{mu} - T_{eu}$$

Let's substitute for speed and time according to $\omega=\omega_u\omega_B$ and $t=t_u t_B$, resulting in

$$\frac{2H}{\omega_{\mathrm{Re}}} \frac{d(\omega_u \omega_B)}{d(t_u t_B)} = T_{mu} - T_{eu} \Rightarrow \frac{2H}{\omega_{\mathrm{Re}}} \frac{d(\omega_u \omega_B)}{d(t_u / \omega_B)} = T_{mu} - T_{eu}$$

$$\Rightarrow \frac{2H}{\omega_{\mathrm{Re}}} \omega_B{}^2 \frac{d(\omega_u)}{d(t_u)} = T_{mu} - T_{eu}$$

With $\omega_{\mathrm{Re}} = \omega_B$, we have that

$$2H\omega_B \frac{d\omega_u}{dt_u} = T_{mu} - T_{eu}$$

Now define $\tau_j = 2H\omega_B$, and the swing equation becomes

$$\tau_j \frac{d\omega_u}{dt_u} = T_{mu} - T_{eu} \qquad (4.82)$$

Mechanical starting time: total time required to accelerate the unit from standstill to rated speed ω_R if rated torque (T_{au}=1.0) is applied as a step function at t=0.

<u>Aside</u>: Recall (eq. (46) of "Swing equation" notes, and pg. 450 A&F) that the *mechanical starting time* is T_4=2H, therefore we see $\tau_j = T_4\omega_B$.

3. The electromagnetic torque (see Section 4.10)

Our basic approach is to obtain an expression for the electric power in terms of the 0dq quantities and then use that to obtain an expression for the electric torque in terms of the 0dq quantities.

In what follows, assume that all quantities are in pu on a per-phase base. This treatment is similar to that done in "macheqts," p. 13.

The instantaneous 3-phase power is given in terms of a-b-c quantities as

$$p_{out} = v_a i_a + v_b i_b + v_c i_c = \underline{v}_{abc}^{T}\,\underline{i}_{abc}$$

We want it in terms of the 0dq quantities.

Recall that $\underline{v}_{odq}=\underline{P}\underline{v}_{abc}$ and $\underline{i}_{odq}=\underline{P}\underline{i}_{abc}$, implying that $\underline{v}_{abc}=\underline{P}^{-1}\underline{v}_{0dq}$ and $\underline{i}_{abc}=\underline{P}^{-1}\underline{i}_{0dq}$.

But we need $\underline{v}_{abc}^{T}$, which will be $\underline{v}_{abc}^{T}=[\,\underline{P}^{-1}\underline{v}_{0dq}\,]^{T}$.

How do we deal with the transpose of a vector product? Consider:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^{T} = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 39 \end{bmatrix}$$

From the above illustration, we may infer that

$$\underline{v}_{abc}^{T}=[\,\underline{P}^{-1}\underline{v}_{0dq}\,]^{T}=\underline{v}_{0dq}^{T}[\underline{P}^{-1}]^{T}$$

But here, we recall that $\underline{P}$ is orthogonal. Then $[\underline{P}^{-1}]^{T}=\underline{P}$.

So finally, we have that $\underline{v}_{abc}^T = \underline{v}_{0dq}^T \underline{P}$. Therefore, we can substitute into the instantaneous power expression to obtain:

$$p_{out} = \underline{v}_{abc}^T \underline{i}_{abc} = [\underline{v}_{0dq}^T \underline{P}][\underline{P}^{-1} \underline{i}_{0dq}] = \underline{v}_{0dq}^T \underline{i}_{0dq}$$

The above proves that our version of Park's transformation is power invariant, i.e., the instantaneous power is obtained from either the a-b-c quantities or the 0-d-q quantities using the same form of expression, according to:

$$p_{out} = v_a i_a + v_b i_b + v_c i_c = v_0 i_0 + v_d i_d + v_q i_q$$

<u>Aside</u>: Observe that power invariance depends on the orthogonality of $\underline{P}$. Without an orthogonal $\underline{P}$, then $[\underline{P}^{-1}]^T \neq \underline{P}$, and

$$p_{out} = \underline{v}_{abc}^T \underline{i}_{abc} = [\underline{v}_{0dq}^T (\underline{P}^{-1})^T][\underline{P}^{-1} \underline{i}_{0dq}] \neq \underline{v}_{0dq}^T \underline{i}_{0dq}$$

We will again consider only balanced conditions so that zero-sequence quantities are zero, and

$$p_{out} = v_d i_d + v_q i_q$$

Returning to the voltage equations we had before we folded in the speed voltage terms (see p. 31 of "perunitization notes), we can extract the expressions for v_d and v_q as:

$$v_d = -r i_d - L_d \dot{i}_d - kM_F \dot{i}_F - kM_D \dot{i}_D - \omega\lambda_q$$

$$v_q = -r i_q - L_q \dot{i}_q - kM_Q \dot{i}_Q - kM_G \dot{i}_G + \omega\lambda_d$$

Now substitute this into the expression for p_{out} to obtain:

$$p_{out} = -r i_d^2 - L_d \dot{i}_d i_d - kM_F \dot{i}_F i_d - kM_D \dot{i}_D i_d - \omega\lambda_q i_d +$$

$$-r i_q^2 - L_q \dot{i}_q i_q - kM_Q \dot{i}_Q i_q - kM_G \dot{i}_G i_q + \omega\lambda_d i_q$$

Gathering together

- The derivative terms
- The ω terms

- The resistive terms

we obtain:

$$p_{out} = -i_d \overbrace{(L_d \dot{i}_d + kM_F \dot{i}_F + kM_D \dot{i}_D)}^{\dot{\lambda}_d} - i_q \overbrace{(L_q \dot{i}_q - kM_Q \dot{i}_Q - kM_G \dot{i}_G)}^{\dot{\lambda}_q}$$
$$+\omega(\lambda_d i_q - \lambda_q i_d)$$
$$-r(i_d{}^2 + i_q{}^2)$$

Note the expressions in brackets of the first line are flux linkage derivatives according to the notes in "macheqts" (see eq. 4.20').

Making the substitution indicated by the brackets above the first line of the expression,

$$p_{out} = \underbrace{-[i_d \dot{\lambda}_d + i_q \dot{\lambda}_q]}_{TERM\ 1} + \underbrace{\omega(\lambda_d i_q - \lambda_q i_d)}_{TERM\ 2} - \underbrace{r(i_d{}^2 + i_q{}^2)}_{TERM\ 3} \qquad \text{eq. 4.94'}$$

Note that this is identical to eq. 4.94 in the text except for the minus sign in front of the term 1. I believe that this is an error in the text. But it does not matter, because we will not use this term anyway.

The text, on page 106, and Charles Concordia in his book on synchronous machines (see page 28 of "Synchronous Machines: Theory and Performance," 1951) indicate that the three terms may be understood to represent:

- Term 1: rate of change in the stator magnetic field energy (recognizing flux linkage derivatives as voltages, $i_d \dot{\lambda}_d$ is d-axis winding power and $i_q \dot{\lambda}_q$ is q-axis winding power, where "power" here is of course reactive).
- Term 2: Power crossing the air gap (the speed-voltage terms)
- Term 3: Stator ohmic losses due to the armature resistance

Therefore, terms 1 and 3 represent power that is entirely on the stator side. But we need the power transferred from the rotor to the stator, which corresponds to the electromagnetic torque. Therefore, we are only interested in term 2.

From any text on mechanics or electromechanics (see, for example, pg. 104 of Fitzgerald, Kingsley, and Kusko), we know that a body experiencing a force f over a distance ∂x undergoes a change in energy according to

$$\partial W = f\partial x$$

Analogously, a body experiencing a torque T over an angle $\partial\theta$ undergoes a change in energy according to

$$\partial W = T\partial\theta$$

For magnetically coupled coils for which at least one of them may experience rotation, the exerted electromagnetic torque is related to the variation in field energy with angular motion according to

$$T_{fld} = \frac{\partial W_{fld}}{\partial\theta_m}$$

But we may write this in terms of time derivatives according to:

$$T_{fld} = \frac{\partial W_{fld}}{\partial t}\frac{\partial t}{\partial\theta_m} = \frac{\partial W_{fld}/\partial t}{\partial\theta_m/\partial t}$$

Note that the numerator is the power and the denominator is the speed, therefore:

$$T_{fld} = \frac{\partial W_{fld}/\partial t}{\partial\theta_m/\partial t} = \frac{P_{fld}}{\omega_m}$$

Note that $T_{fld} = \frac{P_{fld}}{\omega_m}$ is expressed in MKS units. In per-unit, we have:

$$T_{fldu} = \frac{P_{fld}/S_B}{\omega_m/\omega_{mB}} = \frac{P_{fldu}}{\omega_e/\omega_B} = \frac{P_{fldu}}{\omega_u}$$

Here ω_u is the same as ω in our voltage equation, eq. 4.94’ above. Therefore,

$$T_{fldu} = \frac{P_{fldu}}{\omega_u} = \frac{\omega(\lambda_d i_q - \lambda_q i_d)}{\omega} = \lambda_d i_q - \lambda_q i_d$$

This is $T_{e\phi u}$, as discussed on page 1 above.

Now, from "macheqts" (see page 28), eqt. 4.20', we have:

$$\lambda_d = L_d i_d + kM_F i_F + kM_D i_D$$

$$\lambda_q = L_q i_q + kM_Q i_Q + kM_G i_G$$

Substitution of the above flux linkage relations into our torque expression yields:

$$T_{e\phi u} = \lambda_d i_q - \lambda_q i_d = (L_d i_q) i_d + (kM_F i_q) i_F + (kM_D i_q) i_D$$
$$- (L_q i_d) i_q - (kM_Q i_d) i_Q - (kM_G i_d) i_G$$

The above can be written as the product of 2 vectors, according to

$$T_{e\phi u} = \begin{bmatrix} L_d i_q & kM_F i_q & kM_D i_q & -L_q i_d & -kM_Q i_d & -kM_G i_d \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix} \quad (4.98')$$

Recall the swing equation (see p. 2 above):

$$\tau_j \frac{d\omega_u}{dt_u} = T_{mu} - T_{eu} \quad (4.82)$$

where $\tau_j = 2H\omega_B$ and the torque is given on a three-phase base.

<u>Three issues</u>:

1. As discussed before, we must divide $T_{e\phi u}$ in (4.98') by 3 to account for power base difference before using it in the above.
2. We will bring in a damping term.
3. Drop the per-unit notation, and realize that per-unit is implied throughout.

So the swing equation becomes:

$$\tau_j \frac{d\omega}{dt} = T_m - T_e = T_m - \frac{T_{e\phi}}{3} - T_d$$

Here, the damping term is T_d. Typically, it is written as a linear function of speed with the constant of proportionality D; thus, $T_d = D\omega$, and we have:

$$\tau_j \frac{d\omega}{dt} = T_m - T_e = T_m - \frac{T_{e\phi}}{3} - D\omega$$

We want a state-space equation so as to combine with our state-space "current-form" of the voltage equations (given by eq. 4.75), which is

$$\underline{v} = -(\underline{R} + \omega\underline{N})\underline{i} - \underline{L}\dot{\underline{i}} \qquad \text{(eq. 4.75)}$$

$$\dot{\underline{i}} = -\underline{L}^{-1}(\underline{R} + \omega\underline{N})\underline{i} - \underline{L}^{-1}\underline{v} \qquad \text{(eq. 4.76)}$$

where each term is defined on pg. 27 of the "per-unitization" notes. So let's divide both sides of the swing equation above by τ_j.

$$\dot{\omega} = \frac{T_m}{\tau_j} + \frac{1}{3\tau_j}[-T_{e\phi}] + \left[\frac{-D}{\tau_j}\right]\omega$$

Substituting into the last expression eqt. 4.98' for $T_{e\phi}$, we have

$$\dot{\omega} = \frac{T_m}{\tau_j} + \begin{bmatrix} \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix} + \left[\frac{-D}{\tau_j}\right]\omega \qquad \text{(4.101')}$$

Now let’s bring in ω into the state vector….

$$\dot{\omega}=\frac{T_m}{\tau_j}+\begin{bmatrix}\frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{-D}{\tau_j}\end{bmatrix}\begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \\ \omega \end{bmatrix}$$

Finally, we recall that there are two states for each machine: speed and angle, yet in the above, we only have angle. But we must be careful here, and use per-unit.

Recall that:

$$\theta=\omega_{\mathrm{Re}}t+\delta+\frac{\pi}{2} \;\Rightarrow\; \dot{\theta}=\omega=\omega_{\mathrm{Re}}+\dot{\delta}$$

Dividing through by $\omega_B=\omega_{Re}$, we obtain that

$$\omega_u=1+\dot{\delta}_u \;\Rightarrow\; \dot{\delta}_u=\omega_u-1$$

Dropping the per-unit notation, we have

$$\dot{\delta}=\omega-1 \qquad (4.102)$$

Now we have three different sets of state equations, summarized as follows:

$$\dot{\underline{i}} = -\underline{L}^{-1}(\underline{R} + \omega\underline{N})\underline{i} - \underline{L}^{-1}\underline{v} \qquad \text{(eq. 4.75)}$$

$$\dot{\omega} = \frac{T_m}{\tau_j} + \begin{bmatrix} \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{-D}{\tau_j} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \\ \omega \end{bmatrix}$$

$$\dot{\delta} = \omega - 1 \qquad (4.102)$$

And we can put all of this together into a single state equation that looks like the following:

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_Q \\ \dot{i}_G \\ \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} -\underline{L}^{-1}(\underline{R} + \omega\underline{N}) & \underline{0} \\ \begin{bmatrix} \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{-D}{\tau_j} & 0 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \\ \omega \\ \delta \end{bmatrix} + \begin{bmatrix} \left[-\underline{L}^{-1}\underline{v}\right] \\ \begin{bmatrix} \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \end{bmatrix}$$

The above relation is called the “**current state-space model**.” We will derive a “**flux-linkage state-space model**” in the next set of notes.