Algorithmic Game Theory, 2023 Spring
Homework 2

Zhengyang Liu

Instructions:

1. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
2. Please type your solutions if possible in LaTeX or Word whatever is suitable.
3. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.

Problem 1. (2pt) Pick your favourite result in our class and state your reason.

Problem 2. (4pt per question) Given a set function $v : 2^M \to \mathbb{R}_{\geq 0}$, and for any subsets $S, T \subseteq M$ we have that

$$v(S) + v(T) \geq v(S \cap T) + v(S \cup T).$$

For simplicity, we abuse notations $S \cup \{j\}$ as $S + j$ and $S \setminus \{j\}$ as $S - j$, for any $j \in M$.

1. Prove that for any $S \subseteq T \subseteq M$ and $j \in M$, we have
$$\frac{v(T+j)}{v(T)} \leq \frac{v(S+j)}{v(S)}.$$
2. Prove that for any $T \subseteq M$ we have
$$v(T) \geq \sum_{k \in T} \left[v(T) - v(T-k) \right].$$

Problem 3.

1. (3pt) Show that EFX always exists for identical valuations.
2. (2pt) Show that EFX always exists when $n = 2$. (Hint: you can use the first result.)

Problem 4. (5pt) Given a valuation function $v : 2^{[m]} \to \mathbb{R}$ which is *normalized* (i.e., $v(\emptyset) = 0$) and *monotone* (i.e., $v(S) \leq v(T)$ once $S \subseteq T \subseteq [m]$). v is *additive* iff $v(S) = \sum_{i \in S} v(i)$. v is *submodular* iff $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$ for any $S, T \subseteq [m]$. And v is *XOS* iff there exist t additive valuations $\{a_1, \ldots, a_t\}$ such that $v(S) = \max_{r \in [t]} a_r(S)$ for every $S \subseteq [m]$.

Prove that any monotone and normalized submodular function can be written as an XOS function.