

A Little Charity Guarantees Almost Envy-Freeness

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Abstract

Fair division of indivisible goods is a very well-studied problem. The goal of this problem is to distribute m goods to n agents in a “fair” manner, where every agent has a valuation for each subset of goods. We assume general valuations.

Envy-freeness is the most extensively studied notion of fairness. However, envy-free allocations do not always exist when goods are indivisible. The notion of fairness we consider here is “envy-freeness up to any good” (EFX) where no agent envies another agent after the removal of any single good from the other agent’s bundle. It is not known if such an allocation always exists.

We show there is always a partition of the set of goods into $n + 1$ subsets (X_1, \dots, X_n, P) where for $i \in [n]$, X_i is the bundle allocated to agent i and the set P is unallocated (or donated to charity) such that we have:

- envy-freeness up to any good,
- no agent values P higher than her own bundle, and
- fewer than n goods go to charity, i.e., $|P| < n$ (typically $m \gg n$).

Our proof is constructive and leads to a **pseudo-polynomial time** algorithm to find such an allocation. When agents have additive valuations and $|P|$ is large (i.e., when $|P|$ is close to n), our allocation also has a good maximin share (MMS) guarantee. Moreover, a minor variant of our algorithm also shows the existence of an allocation that is $4/7$ *groupwise* maximin share (GMMS): this is a notion of fairness stronger than MMS. This improves upon the current best bound of $1/2$ known for an approximate GMMS allocation.

This version of the paper goes beyond the preliminary version published in SODA 2020 [17] in two key points:

1. The pseudo-polynomial algorithm works when agents have general valuation functions (not just gross-substitute valuations).
2. We introduce a relaxed definition of “most envious agent”.

1 Introduction

Fair division of goods among competing agents is a fundamental problem in Economics and Computer Science. There is a set M of m goods and the goal is to *allocate* goods among n agents in a *fair* way. An allocation is a partition of M into disjoint subsets X_1, \dots, X_n where X_i is the set of goods given to agent i . When can an allocation be considered “fair”? One of the most well-studied notions of fairness is *Envy-freeness*. Every agent has a *value* associated with each subset of M and agent i *envies* agent j if i values X_j more than X_i . An allocation is *envy-free* if no agent envies another. An envy-free allocation can be regarded as a fair and

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$m \gg n$

desirable partition of M among the n agents since no agent envies another; as mentioned in [26], such a mechanism of partitioning land dates back to the Bible.

Unlike land which is divisible, goods in our setting are *indivisible* and an envy-free allocation of the given set of goods need not exist. Consider the following simple example with two agents and a single good that both agents desire: one of the agents has to receive this good and the other agent envies her. Since **envy-free allocations need not exist**, several relaxations have been considered.

Relaxations. Budish [11] introduced the notion of *EF1*: this is an allocation of goods that is “envy-free up to *one* good”. In an EF1 allocation, agent i may envy agent j , however this envy would vanish as soon as *some* good is removed from X_j . Note that no good is really removed from X_j : this is simply a way of assessing how much i values X_j more than X_i . That is, if i values X_j more than X_i , then there exists some $g \in X_j$ such that i values X_i at least as much as $X_j \setminus \{g\}$. Going back to the example of two agents and a single good, the allocation where one agent receives this good is EF1. It is known that EF1 allocations always exist; as shown by Lipton et al. [25], such an allocation can be efficiently computed.¹

Caragiannis et al. [13] introduced a notion of envy-freeness called *EFX* that is stronger than EF1. An EFX allocation is one that is “envy-free up to *any* good”. In an EFX allocation, agent i may envy agent j , however this envy would vanish as soon as *any* good is removed from X_j . Thus every EFX allocation is also EF1 but not every EF1 allocation is EFX.

| | a | b | c |
|---------|-----|-----|-----|
| Agent 1 | 1 | 1 | 2 |
| Agent 2 | 1 | 1 | 2 |

Consider the simple example given above: there are three goods a, b, c and two agents with *additive valuations*² (formally defined in Section 1.1) as described below. Both agents value c twice as much as a or b . The allocation where agent 1 gets $\{a\}$ and agent 2 gets $\{b, c\}$ is EF1 but not EFX. On the other hand, the allocation where agent 1 gets $\{a, b\}$ and agent 2 gets $\{c\}$ is EFX. Indeed, the latter allocation seems fairer than the former allocation. As said in [12]: “*Arguably, EFX is the best fairness analog of envy-freeness of indivisible items*”. While it is known that EF1 allocations always exist, the question of whether EFX allocations always exist is still an open problem (despite significant effort, according to [13]).

Plaut and Roughgarden [26] showed that EFX allocations always exist (i) when there are only two agents or (ii) when all n agents have the same valuations. Moreover, it was shown in [26] that exponentially many value queries may be needed to determine EFX allocations even in the restricted setting where there are only two agents with identical *submodular* valuation functions³. It was not known until very recently, if an EFX allocation always exists even when there are only three agents with additive valuations. A positive answer to this question was very recently given by Chaudhury et al. [15] who showed that EFX allocations always exist when there are only three agents with additive valuations. It was remarked in [26]: “*We suspect that at least for general valuations, there exist instances where no EFX allocation exists*”.

A relaxation of EFX. Very recently, Caragiannis et al. [12] introduced a more relaxed notion of EFX called *EFX-with-charity*. This is a partial allocation that is EFX, i.e., the entire set of goods need not be distributed among the agents. So some goods may be left *unallocated* and it is assumed that these unallocated goods are donated to charity. There is a very simple allocation that is EFX-with-charity where *no* good is assigned to any agent—thus all goods are

¹The algorithm in [25] was published in 2004 with a different property and EF1 was proposed in 2011.

²The value of a set $S \subseteq M$ is the sum of values of goods in S .

³These are valuation functions with decreasing marginal values.

donated to charity. Obviously, this is not an interesting allocation and one seeks allocations with better guarantees and one such allocation was shown in [12].

Let $X^* = \langle X_1^*, \dots, X_n^* \rangle$ be an optimal *Nash social welfare* allocation⁴ on the entire set of goods. It was shown in [12] that there always exists an EFX-with-charity allocation $X = \langle X_1, \dots, X_n \rangle$ where every agent receives at least half the value of her allocation in X^* . Interestingly, as shown in [12], $X_i \subseteq X_i^*$ for all i . Unfortunately, there are no upper bounds on the *number* of unallocated goods or on the *value* any agent has for the set of goods donated to charity.

We believe these are important questions to ask. The ideal allocation is one that is EFX and allocates *all* goods; so we would like a guarantee that a large number of goods have been allocated to agents. Moreover, since EFX allocations guarantee envy-freeness once any good is removed from another agent's set, it is in the same spirit that we seek an EFX (partial) allocation where nobody envies the set of unallocated goods. The allocation in [12] gives no guarantee either on the number of unallocated goods or on whether any agent values the set of unallocated goods more than her own bundle. Here we consider the notion of *EFX-with-bounded-charity*. That is, we seek EFX-with-charity allocations with bounds on the set given to charity, i.e., a bound on the size and a bound on the value of the set of goods donated to charity.


1.1 Our Results

Let $N = [n]$ be the set of agents. Every agent $i \in [n]$ has a valuation function $v_i : 2^M \rightarrow \mathbb{R}_{\geq 0}$, where M is the set of m goods.

1.1.1 General valuations

We show our main existence result for *general* valuation functions, i.e., the only assumptions we make on any valuation function v_i are that

- (i) it is *normalized*, i.e., $v_i(\emptyset) = 0$, and (ii) it is *monotone*, i.e., $S \subseteq T$ implies $v_i(S) \leq v_i(T)$.

In contrast, the EFX-with-charity allocation in [12] works only for *additive valuations*, i.e., for any $S \subseteq M$ and $i \in [n]$, we have $v_i(S) = \sum_{g \in S} v_i(\{g\})$. 

- We show there always exists an allocation⁵ $X = \langle X_1, \dots, X_n \rangle$ that satisfies the following properties:

1. X is EFX, i.e., for any two agents i, j : $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for any $g \in X_j$;
2. $v_i(X_i) \geq v_i(P)$ for all agents i , where $P = M \setminus \bigcup_{i=1}^n X_i$ is the set of unallocated goods;
3. $|P| < n$ (recall that n is the number of agents).

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Note that our result implies that among the n agents, if there is just *one* agent who has no preferences (say, for agent i , we have $v_i(S) = 0$ for all $S \subseteq M$) then a complete EFX allocation always exists for general valuations: Find an EFX allocation satisfying all the three properties mentioned above, on the set of agents $N \setminus \{i\}$. Note that the allocation among the agents in $N \setminus \{i\}$ is EFX (by condition 1) and we now allocate P (the set of unallocated goods) to agent i . Observe that agent i envies nobody as i has the same valuation for all subsets of goods and nobody envies i as nobody envies P (by condition 2). Thus we have a complete EFX allocation!

Our proof is constructive. We start with no goods being allocated to the agents and find the claimed allocation by at most nmV/Δ applications of three simple update rules, where n

⁴This is an allocation that maximizes $\prod_{i=1}^n v_i(X_i^*)$, where v_i is agent i 's valuation function.

⁵Henceforth, allocations are *partial* and we will use "complete allocation" to refer to one where all goods are allocated.

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is the number of agents, m is the number of goods, $V = \max_i v_i(M)$ is the maximum valuation of any agent, and $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ is the minimum difference between distinct valuations.

The update rules use a *minimal-envied-subset-oracle*: given $S \subseteq M$ such that there is an agent who values S more than her own bundle, find an inclusion-wise minimal subset $Z \subseteq S$ such that there is an agent who values Z more than her own bundle. This oracle can be realized by a simple algorithm that uses at most nm value queries.

- For general valuations, an EFX allocation with properties 1-3 can be computed with $\text{poly}(n, m, V, 1/\Delta)$ value queries, i.e., in pseudo-polynomial time.

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1.1.2 Identical valuations

It also follows from our proof that when all agents have the same valuation function, our allocation is *complete*. That is, $|P| = 0$. This is an alternate proof to the existence of complete EFX allocations for identical (general) valuations, originally shown in [26].

1.1.3 Additive valuations

The most well-understood class of valuation functions is the set of *additive* valuations. We consider the case when all agents have additive valuations and show that our allocation or very minor variants of our allocation can guarantee several other notions of fairness.

Ensuring high Nash social welfare. We show that modifying the starting step of our algorithm ensures that our allocation X , that satisfies properties 1-3 stated above, also has a high Nash social welfare. That is, $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ as promised in [12], where $X^* = \langle X_1^*, \dots, X_n^* \rangle$ is an optimal Nash social welfare allocation. Here we use the allocation computed in [12] as a black box in our starting step, thus our result can be regarded as an extension of the result in [12].

Number of Unallocated Goods and MMS Guarantee. Another interesting and well-studied notion of fairness is *maximin share*. Suppose agent i has to partition M into n bundles (or sets) knowing that she would receive the worst bundle with respect to her valuation. Then i will choose a partition of M that maximizes the valuation of the worst bundle (wrt her valuation). The value of this worst bundle is the maximin share of agent i . An important question here is: does there always exist an allocation of M where every agent gets a bundle worth at least her maximin share?

Formally, let N and M be the sets of n agents and m goods, respectively. We define the maximin share of an agent (say, i) as follows: (here \mathcal{X} is the set of all complete allocations)

$$MMS_i(n, M) = \max_{\langle X_1, \dots, X_n \rangle \in \mathcal{X}} \min_{j \in [n]} v_i(X_j).$$

The goal is to determine an allocation $\langle X_1, X_2, \dots, X_n \rangle$ of M such that for every i we have $v_i(X_i) \geq MMS_i(n, M)$. This question was first posed by Budish [11]. Procaccia and Wang [27] showed that such an allocation need not exist, even in the restricted setting of only three agents! Thereafter, *approximate-MMS* allocations were studied [27, 20, 22, 21] and there are polynomial time algorithms to find allocations where for all i , agent i gets a bundle of value at least $\alpha \cdot MMS_i(n, M)$; the current best guarantee for α is $3/4 - \epsilon$ by Ghodsi et al. [22] (for any $\epsilon > 0$) and this was very recently improved to $3/4$ by Garg and Taki [21].

Amanatidis et al. [1] showed that any complete EFX allocation is also a $\frac{4}{7}$ -MMS allocation. We show that our allocation promises better MMS guarantees when the number of unallocated goods is large. Let $X = \langle X_1, \dots, X_n \rangle$ be our allocation as described by properties 1-3 above and P be the set of unallocated goods. For any agent $i \in [n]$, we have:

$$v_i(X_i) \geq \frac{1}{2 - |P|/n} MMS_i(n, M).$$

Hence, larger the number of unallocated goods, better are the guarantees that we get on MMS. The extreme values are $|P| = 0$ and $|P| = n - 1$. When $|P| = 0$, we have a complete EFX allocation and when $|P| = n - 1$, we have an EFX allocation that is an *almost-MMS* allocation: $v_i(X_i) \geq (1 - 1/n) \cdot \text{MMS}_i(n, M)$ for all i .

Improved Guarantees for Groupwise MMS. Barman et al. [6] recently introduced a notion of fairness called *groupwise maximin share* (GMMS) which is stronger than MMS. An allocation is said to be GMMS if the MMS condition is satisfied for *every subgroup* of agents and the union of the sets of goods allocated to them. Formally, a complete allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ is α -GMMS if for any $N' \subseteq N$, we have $v_i(X_i) \geq \alpha \cdot \text{MMS}_i(n', \bigcup_{i \in N'} X_i)$ where $n' = |N'|$. Every GMMS allocation, i.e. $\alpha = 1$, is also a complete EFX allocation [6].

It is known [6] that GMMS strictly generalizes MMS. In particular, it was shown in [6] that GMMS allocations rule out some very unsatisfactory allocations that have MMS guarantees. For example, consider an instance with n agents with additive valuations and a set M of $n - 1$ goods and every agent has a valuation of one for each good. Since the number of goods is less than the number of agents, we have $\text{MMS}_i(n, M) = 0$ for every agent i . So any allocation has MMS guarantees. It is not hard to see that the only allocation with a GMMS guarantee is one where $n - 1$ agents get one good each and one agent is left without any goods. See Subsection 2.1 in [6] for more discussion. Naturally, it is a harder problem to approximate GMMS than MMS. While $\frac{3}{4}$ -MMS allocations always exist, the largest α for which α -GMMS allocations are known to exist is $\frac{1}{2}$ [6]. We extend the result of Amanatidis et al. [1] for MMS to show the following:

- A $\frac{4}{7}$ -GMMS allocation always exists and can be computed in pseudo-polynomial time.

In particular, we show that modifying the last step of our algorithm results in a complete allocation that is $\frac{4}{7}$ -GMMS. Very recently and independently, Amanatidis et al. [3] showed the same approximation.

1.2 Our Techniques

We now give an overview of the main ideas used to find our EFX allocation. We first recall the algorithm of Lipton et al. [25] for finding an EF1 allocation. They use the notion of an *envy-graph*: here each vertex corresponds to an agent and there is an edge (i, j) iff i envies j . The invariant maintained is that the envy-graph is a DAG: a cycle corresponds to a cycle of envy and by swapping bundles along a cycle, every agent becomes better-off and the number of envy edges does not increase. More precisely, if $i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{\ell-1} \rightarrow i_0$ is a cycle in the envy graph, then reassigning $X_{i_{j+1}}$ to agent i_j for $0 \leq j < \ell$ (indices are to be read modulo ℓ) will increase the valuation of every agent in the cycle. Also if there was an edge from s to some i_k where s is not a part of the cycle, this edge just gets directed now from s to i_{k+1} after we exchange bundles along the cycle. Thus the number of envy edges in the graph does not increase and the valuations of the agents in the cycle goes up. Thus cycles can be eliminated.

The algorithm in [25] runs in rounds and always maintains an allocation that is also EF1. At the beginning of every round, an unenvied agent s (this is a source vertex in this DAG) is identified and an unallocated good g is allocated to s . The new allocation is also EF1, as nobody will envy the bundle of s after removing the good g .

The Reallocation Operation. We now highlight a key difference between an EF1 allocation and an EFX allocation. From the algorithm of Lipton et al. [25], it is clear that given an EF1 allocation on a set M_0 of goods, one can determine an EF1 allocation on $M_0 \cup M_1$, for any $M_1 \subseteq M \setminus M_0$, by simply adding goods from M_1 one-by-one to the existing bundles and changing the owners (if necessary) in a clever way. Intuitively, we never need to cut or merge the bundles formed in any EF1 allocation. We can just append the unallocated goods appropriately to the current bundles.

The above strategy is very far from true for EFX. Consider the example illustrated below with three agents with additive valuations and four goods a, b, c , and d .

| | a | b | c | d |
|----------------|-----|-----|-----|-----|
| <i>Agent 1</i> | 0 | 1 | 1 | 2 |
| <i>Agent 2</i> | 1 | 0 | 1 | 2 |
| <i>Agent 3</i> | 1 | 1 | 0 | 2 |

An EFX allocation for the first three goods has to give exactly one of a, b, c to each of the three agents. However an EFX allocation for all the four goods has to allocate the singleton set $\{d\}$ to some agent (say, agent 1) and say, $\{a\}$ to agent 2 and $\{b, c\}$ to agent 3. Thus the allocation needs to be *cut and merged*. When there are many agents - each with her own valuation, figuring out the cut-and-merge operations is the difficult step. Here we implement our global reallocation operation as follows.

Improving Social Welfare. Suppose we have an EFX allocation $X = \langle X_1, \dots, X_n \rangle$ on some subset $M_0 \subset M$. We would now like to add a good $g \in M \setminus M_0$. However we will not be able to guarantee an EFX allocation on $M_0 \cup \{g\}$. What we will ensure is that either case (i) or case (ii) occurs:

- (i) We have an EFX allocation $X' = \langle X'_1, \dots, X'_n \rangle$ on a subset of $M_0 \cup \{g\}$ such that $v_i(X'_i) \geq v_i(X_i)$ for all i and for at least one agent j we have $v_j(X'_j) > v_j(X_j)$. Thus $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$; in other words, the *social welfare* strictly improves.
- (ii) We have an EFX allocation on $M_0 \cup \{g\}$ and the social welfare does not decrease.

Hence in each step of our algorithm, we either increase social welfare or we increase the number of allocated goods without decreasing social welfare—thus we always make progress. This is similar to the approach used by Plaut and Roughgarden [26] to guarantee the existence of $\frac{1}{2}$ -EFX⁶ when agents have subadditive valuations. We now outline how we ensure that one of cases (i) and (ii) has to happen.

For simplicity of exposition, we assume the envy-graph corresponding to our starting EFX allocation X has a single source s . Add g to s 's bundle: if nobody envies s up to any good then we are in an easy case as we have an EFX allocation on $M_0 \cup \{g\}$. In this case, we “decycle” the envy-graph (if cycles are created) and continue. Observe that swapping bundles along a cycle in the envy-graph increases social welfare.

Most Envious Agent. Assume now that there are one or more agents who envy s up to any good after g is allocated to s . To resolve this, we introduce the concept of the *most envious agent*. Let i be an agent who envies s up to any good, so $v_i(X_i) < v_i(S')$ for some $S' \subset X_s \cup \{g\}$. Let S_i be any inclusion-wise minimal subset of $X_s \cup \{g\}$ such that $v_i(X_i) < v_i(S_i)$ (break ties arbitrarily). So for any $T \subset S_i$, we have $v_i(X_i) \geq v_i(T)$.

- An agent i such that $v_i(X_i) < v_i(S_i)$ for some $S_i \subset X_s \cup \{g\}$ and no strict subset of S_i is envied by *any* agent will be called the *most envious agent* of $X_s \cup \{g\}$ (break ties arbitrarily).

Let t be the most envious agent of $X_s \cup \{g\}$. The crucial observation is that no agent envies S_t *up to any good*—otherwise it would contradict $S_t \subset X_s \cup \{g\}$ being an “inclusion-wise minimal envied set”, i.e., no proper subset of S_t is envied by any agent. Recall the assumption that s is the only source, so there is a path $s \rightarrow i_1 \rightarrow \dots \rightarrow i_{k-1} \rightarrow t$ in the envy-graph. We do

⁶An allocation $X = (X_1, \dots, X_n)$ is $\frac{1}{2}$ -EFX if for any two agents i, j : $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_j \setminus \{g\})$ for all $g \in X_j$.

a leftwise shift of bundles along this path: so s gets i_1 's bundle, and for $1 \leq r \leq k-1$: i_r gets i_{r+1} 's bundle (where $i_k = t$), and finally t gets S_t . The goods in $X_s \cup \{g\} \setminus S_t$ are thrown back into the pool of unallocated goods.

Observe that every agent in this path is strictly better-off now than in the allocation X and no other agent is worse-off. Moreover, by the definition of S_t , there is no agent envying any other agent up to any good. Thus we have a desired EFX allocation X' . When there are multiple sources, we can adapt this technique *provided* there are enough unallocated goods; in particular, the number of unallocated goods must be at least the number of sources in the envy-graph. We describe this in detail in Section 2.

We would like to contrast the above approach with other EFX algorithms [26, 12]. The $\frac{1}{2}$ -EFX algorithm by Plaut and Roughgarden [26] either merges g (the new good) with an existing bundle or allocates the singleton set $\{g\}$ to an agent. The EFX-with-charity algorithm by Caragiannis et al. [12] takes an allocation of maximum Nash social welfare as input and then permanently removes some goods from the instance. We regard the notion of “most envious agent” that shows a natural way of breaking up a bundle to preserve envy-freeness up to any good as one of the innovative contributions of our work.

Our Other Results. Regarding our result with approximate MMS guarantee, if the number of unallocated goods in our EFX allocation is large, then the number of sources also has to be large: these are unenvied agents. Moreover, no agent envies the set of unallocated goods. Suppose for now that $|P| = n - 1$. This means every agent is a source. So no agent envies the bundle of any other agent and also the set of unallocated goods. Thus for each agent i , we have:

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n+1} && \geq (1 + 1/n)^{-1} \cdot \frac{v_i(M)}{n} \\ &&& \geq (1 - 1/n) \cdot \text{MMS}_i(n, M), \end{aligned}$$

where the constraint that $v_i(M)/n \geq \text{MMS}_i(n, M)$ holds for additive valuations. We show our result for approximate-MMS allocation and our improved bound for approximate-GMMS allocation in Section 3.

1.3 Related Work

Fair division of divisible resources is a classical and well-studied subject starting from the 1940's [28]. Fair division of indivisible goods among competing agents is a young and exciting topic with recent work on EF1 and EFX allocations [13, 8, 26, 9, 12], approximate maximin share allocations [11, 10, 2, 7, 24, 22, 20], and approximation algorithms for maximizing Nash social welfare and generalizations [18, 16, 14, 4, 19, 5].

As mentioned earlier, Caragiannis et al. [13] introduced the notion of EFX and it is now known that EFX allocations always exist for three agents with additive valuations [15]. Whether EFX allocations always exist with general valuations or with a larger number of agents is an enigmatic open problem. It was shown in [13] that there always exists an EF1 allocation that is also Pareto-optimal⁷ and Barman et al. in [8] showed a pseudo-polynomial time algorithm to compute such an allocation.

Applications. Fair division of goods or resources occurs in many real-world scenarios and this is demonstrated by the popularity of the website Spliddit (<http://www.spliddit.org>) that implements mechanisms for fair division where users can log in, define what needs to be divided, and enter their valuations. This website guarantees an EF1 allocation that is also Pareto-optimal and since its launch in 2014, it has been used tens of thousands of times [13]. We refer to [23, 26] for details on the diverse applications for which Spliddit has been used: these

⁷An allocation $X = \langle X_1, \dots, X_n \rangle$ is Pareto-optimal if there is no allocation $Y = \langle Y_1, \dots, Y_n \rangle$ where $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$ and $v_i(Y_j) > v_i(X_j)$ for some j .

range from rent division and taxi fare division to credit assignment for an academic paper or group project. Another such website is Fair Outcomes, Inc. (<http://www.fairoutcomes.com>). An interesting application is also *Course Allocate* used at Wharton School that guarantees certain fairness properties to allocate courses among students [26].

1.4 Improvements with respect to the SODA version

In the conference version of the paper [17], we gave a pseudo-polynomial time algorithm to determine an EFX allocation with bounded charity when agents have “gross-substitute” valuations. In this version we show a pseudo-polynomial time algorithm to determine an EFX allocation with bounded charity even when agents have general valuations. This is realized with a relaxed definition of the “most envious agent” (see Definition 3) that helps us to implement the *minimal-envied-subset* oracle efficiently (see Subsection 2.1).

2 Existence of an EFX-Allocation with Bounded Charity

We prove our main result on EFX-with-bounded-charity allocations in this section. We will define three *update rules*. Each update rule takes a pair (X, P) consisting of an allocation X and a set P of unallocated goods (we will call P the *pool*) and returns a modified pair (X', P') .

Each application of an update rule will ensure that either (i) the *social welfare* $\phi(X) = \sum_{i \in [n]} v_i(X_i)$ of the current allocation increases or (ii) the size of the pool decreases and the social welfare does not decrease, so $|P'| < |P|$ in this case. Hence the update process will terminate. The overall structure of the algorithm is given in Algorithm 1.

Algorithm 1 Algorithm for Computing an EFX-Allocation

Postcondition: X is EFX, $|P| < n$ and $v_i(P) \leq v_i(X_i)$ for all $i \in [n]$.

- 1: $X_i \leftarrow \emptyset$ for $i \in [n]$; $P \leftarrow M$;
 - 2: **while** one of the update rules shown in Algorithm 2 is applicable **do**
Invariant: X is EFX and the envy-graph G_X is acyclic
 - 3: Let U_ℓ be an applicable update rule;
 - 4: $(X, P) \leftarrow U_\ell(X, P)$;
 - 5: Decycle the envy-graph;
 - 6: **end while**
-

In order to define our update rules, we need the concepts of *envy-graph* and the *most envious agent* for a bundle of goods. These were discussed in Section 1.2 and we formally define them below.

Definition 1. The envy-graph G_X for an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ has the set of agents as vertices and there is a directed edge from agent i to agent j if and only if $v_i(X_i) < v_i(X_j)$.

The notion of envy-graph was introduced in [25] and it is well-known that cycles can be removed from the envy-graph without destroying desirable properties (see Lemma 2). Thus we can maintain G_X as a DAG.

For each agent s , we define the *reachability component* $C(s)$ as all agents reachable from s in the envy-graph and the *sources* of the envy-graph as the vertices with indegree zero.

For ease of notation, we will use $B \setminus g$ and $B \cup g$ to denote $B \setminus \{g\}$ and $B \cup \{g\}$, respectively.

Lemma 2. Let $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{k-1} \rightarrow i_0$ be a cycle in the envy-graph. Consider the allocation X' where $X'_{i_\ell} = X_{i_{\ell+1}}$ (indices are modulo k) for $\ell \in \{0, \dots, k-1\}$ and $X'_j = X_j$ for $j \notin \{i_0, \dots, i_{k-1}\}$. If X is EFX, then X' is also EFX. Moreover, $\phi(X') > \phi(X)$.

Proof. Consider any agent i . We have $v_i(X'_i) \geq v_i(X_i)$ with strict inequality if i lies on the cycle. So $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$. Thus $\phi(X') > \phi(X)$.

Since X' is just a permutation of X , for any agent j there exists some agent j' such that $X'_j = X_{j'}$. Therefore, since X is EFX, for any good $g \in X_{j'}$ (or equivalently X'_j) we have $v_i(X'_j \setminus g) = v_i(X_{j'} \setminus g) \leq v_i(X_i) \leq v_i(X'_i)$. Thus X' is also EFX. \square

Let $S \subseteq M$. Suppose there exists an agent that considers S more valuable than her own bundle. Then we will call S an *envied* set. The following definition formalizes the notion of the “most envious agent”.

Definition 3. Let Z be an inclusion-wise minimal envied subset of S , i.e., (1) there exists an agent i such that $v_i(Z) > v_i(X_i)$ and (2) for all $j \in [n]$ we have $v_j(X_j) \geq v_j(Z')$ for all $Z' \subset Z$. The agent i (ties broken arbitrarily) will be called the *most envious agent* of S .

We are now ready to present our three update rules U_0 , U_1 , and U_2 , in Algorithm 2.

Algorithm 2 The Update Rules

- 1: **function** U_0 (allocation X , pool P)
 - Precondition:** There is a good $g \in P$ and an agent i such that allocating g to i results in an EFX allocation.
 - 2: Allocate g to i , i.e., $X'_i \leftarrow X_i \cup g$, $P' \leftarrow P \setminus g$, and $X'_j = X_j$ for $j \neq i$.
 - 3: **return** (X', P') .
 - 4: **end function**
- 5: **function** U_1 (allocation X , pool P)
 - Precondition:** There is an agent k such that $v_k(P) > v_k(X_k)$.
 - 6: Let i be the most envious agent of P ; let Z be an inclusion-wise minimal envied subset of P such that i satisfies $v_i(Z) > v_i(X_i)$.
 - 7: Set $X'_i = Z$ and $X'_j = X_j$ for $j \neq i$.
 - 8: Set $P' = X_i \cup (P \setminus Z)$.
 - 9: **return** (X', P') .
 - 10: **end function**
- 11: **function** U_2 (allocation X , pool P)
 - Precondition:** There is an $\ell \geq 1$, distinct goods $g_0, g_1, \dots, g_{\ell-1}$ in P , distinct sources $s_0, s_1, \dots, s_{\ell-1}$ of G_X and distinct agents t_1, t_2, \dots, t_ℓ such that $t_i \in C(s_i)$ and t_{i+1} is the most envious agent of $X_{s_i} \cup g_i$ for $0 \leq i \leq \ell - 1$ (indices are to be interpreted modulo ℓ).
 - 12: Let Z_i be an inclusion-wise minimal envied subset of $X_{s_i} \cup g_i$ such that $v_{t_{i+1}}(Z_i) > v_{t_{i+1}}(X_{t_{i+1}})$ for $0 \leq i \leq \ell - 1$.
 - 13: Set $P' = (P \setminus \bigcup_{i=0}^{\ell-1} \{g_i\}) \cup \bigcup_{i=0}^{\ell-1} ((X_{s_i} \cup g_i) \setminus Z_i)$.
 - 14: Let $u_0^i \rightarrow \dots \rightarrow u_{m_i}^i$ be the path of length m_i from $s_i = u_0^i$ to $t_i = u_{m_i}^i$ in $C(s_i)$ for $0 \leq i \leq \ell - 1$.
 - 15: Set $X'_{u_k^i} = X_{u_{k+1}^i}$ for all $k \in \{0, \dots, m_i - 1\}$ and all $i \in \{0, \dots, \ell - 1\}$.
 - 16: Set $X'_{t_i} = Z_{i-1}$ for all $i \in \{1, \dots, \ell\}$.
 - 17: Set $X'_j = X_j$ for all other j .
 - 18: **return** (X', P') .
 - 19: **end function**

Rule U_0 : Rule U_0 is the easiest of the update rules. It is applicable whenever adding a good from the pool to some source of G_X does not destroy the EFX-property (see Algorithm 2).

Lemma 4 (Rule U_0).

- (a) Rule U_0 returns an EFX allocation. An application of the rule does not decrease social welfare and decreases the size of the pool.
- (b) If rule U_0 is not applicable then for any source i of G_X and good $g \in P$, there will be an agent $j \neq i$ such that $v_j(X_i \cup g) > v_j(X_j)$. In particular, an inclusion-wise minimal envied subset of $X_i \cup g$ has size at most $|X_i|$.

Proof. The first part of a) follows directly from the precondition of the rule. The second part holds since the valuations are monotone and because $|P'| = |P| - 1$.

The first sentence in part b) is obvious. We come to the second sentence. Since adding g to X_i destroys the EFX-property, there must be some $g' \in X_i \cup g$ and some $j \in [n]$ such that $v_j(X_i \cup g \setminus g') > v_j(X_j)$ for some $j \in [n]$. Thus an inclusion-wise minimal envied subset of $X_i \cup g$ has size at most $|X_i|$. \square

Rule U_1 : Rule U_1 is applicable whenever there is an agent that values the pool higher than her current bundle (see Algorithm 2).

Lemma 5 (Rule U_1). *Rule U_1 increases the social welfare and returns an EFX allocation.*

Proof. Since there is an agent that values the pool higher than her own bundle, there is an inclusion-wise minimal envied subset Z of P and an agent i such that $v_i(X_i) < v_i(Z)$. Let X' be the allocation defined in Algorithm 2, line 7. Then $v_i(X'_i) > v_i(X_i)$ and $v_j(X'_j) = v_j(X_j)$ for $j \neq i$. Thus $\phi(X') > \phi(X)$. It remains to show that the allocation X' is EFX, i.e., for every pair of agents j and k and any good $g \in X'_k$, we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$.

Since X is EFX, this is obvious if neither j nor k is equal to i . If $j = i$, then $v_i(X'_i) > v_i(X_i) \geq v_i(X_k \setminus g) = v_i(X'_k \setminus g)$ for all $g \in X'_k$ (or equivalently $g \in X_k$). Finally, we consider $k = i$. Since Z is an inclusion-wise minimal envied subset of P , we have $v_j(X'_j) = v_j(X_j) \geq v_j(Z \setminus g) = v_j(X'_i \setminus g)$ for any $g \in Z$, where $v_j(X_j) \geq v_j(Z \setminus g)$ follows from the inclusion-wise minimality of Z as an envied set. \square

Rule U_2 : Rule U_2 is our most complex rule. It is applicable if for some $\ell \geq 1$, there are distinct goods $g_0, g_1, \dots, g_{\ell-1}$ in P , distinct sources $s_0, s_1, \dots, s_{\ell-1}$ of G_X and distinct agents t_1, t_2, \dots, t_ℓ (indices are to be interpreted modulo ℓ) such that for each i : (1) t_i is the most envious agent of $X_{s_{i-1}} \cup g_{i-1}$ and (2) t_i is reachable from s_i . We first show that rule U_2 is applicable if rule U_0 is not applicable and the pool contains at least n goods.

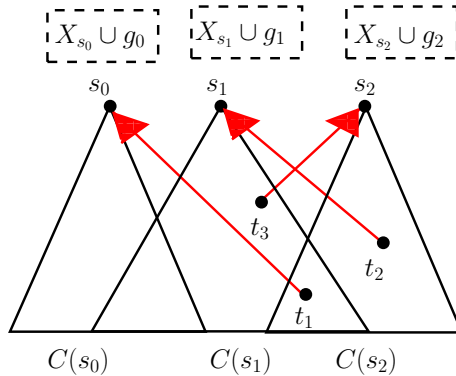


Figure 1: We have t_i as the most envious agent of $X_{s_{i-1}} \cup g_{i-1}$. Moreover, $t_i \notin C(s_0) \cup \dots \cup C(s_{i-1})$ for $i = 1, 2$ and $t_3 \in C(s_0) \cup \dots \cup C(s_2)$. Note that $j = 1$ is the largest index such that $t_3 \in C(s_j)$. The cycle is defined by s_1, s_2, g_1, g_2, t_2 and t_3 .

Lemma 6. *If $|P| \geq n$ and rule U_0 is not applicable then there is an $\ell \geq 1$, distinct goods $g_0, g_1, \dots, g_{\ell-1}$ in P , distinct sources $s_0, s_1, \dots, s_{\ell-1}$ of G_X , and distinct agents t_1, t_1, \dots, t_ℓ*

such that $t_i \in C(s_i)$ and t_i is the most envious agent of $X_{s_{i-1}} \cup g_{i-1}$ for $i \in \{0, \dots, \ell - 1\}$ (indices are modulo ℓ).

Proof. Since rule U_0 is not applicable, for every source s of G_X and every good $g \in P$, we have $v_j(S') > v_j(X_j)$ for some $S' \subset X_s \cup g$ and $j \in [n]$. Construct a sequence of triples (s_i, g_i, t_{i+1}) , $i \geq 0$ defined as follows. Let s_0 be an arbitrary source of G_X and g_0 be an arbitrary good in P . Assume we have defined s_{i-1} and g_{i-1} . Let t_i be the most envious agent of $X_{s_{i-1}} \cup g_{i-1}$. If $t_i \notin C(s_0) \cup \dots \cup C(s_{i-1})$, let s_i be such that $t_i \in C(s_i)$. Also, let g_i be a good in P distinct from g_0 to g_{i-1} . If $t_i \in C(s_0) \cup \dots \cup C(s_{i-1})$ then stop the construction of the sequence and let j be the maximum index such that $t_i \in C(s_j)$. Set $\ell = i - j$ and return $s_j, \dots, s_{i-1}, g_j, \dots, g_{i-1}$ and t_{j+1}, \dots, t_i ,⁸ see Figure 1 for an illustration.

The construction is well-defined since $|P| \geq n$ and hence we cannot run out of goods. The sources and goods are pairwise distinct by construction. The agents t_1 to t_{i-1} are also distinct by construction. Finally, agent t_i is distinct from any agent t_k for $j < k < i$ since $t_i \in C(s_j)$ and by definition $t_k \notin (C(s_0) \cup \dots \cup C(s_{k-1}))$ and so $t_k \notin C(s_j)$. \square

For each i , let $u_0^i \rightarrow u_1^i \rightarrow \dots \rightarrow u_{m_i}^i$ be the path of length m_i from $s_i = u_0^i$ to $t_i = u_{m_i}^i$ in $C(s_i)$. Rule U_2 assigns (i) $X'_{u_k^i} = X_{u_{k+1}^i}$ for all $k \in \{0, \dots, m_i - 1\}$ and all $i \in \{0, \dots, \ell - 1\}$ and (ii) $X'_{t_i} = Z_{i-1}$ for all $i \in \{1, \dots, \ell\}$, where Z_i is defined in Algorithm 2 (see line 12). For all other j , we have $X'_j = X_j$.

Lemma 7 (Rule U_2). *Rule U_2 increases social welfare and returns an EFX allocation.*

Proof. We first observe that the valuations of the agents for their bundles have either increased or remained the same (since either the agents are left with their old bundles or assigned bundles that they envied). In particular, the valuations of all the agents in $\bigcup_{i=0}^{\ell-1} \bigcup_{k=0}^{m_i} \{u_k^i\}$ are strictly larger, where the vertices u_k^i are defined above. Thus $\phi(X') > \phi(X)$.

It remains to show that the allocation X' is EFX, i.e., for every pair of agents j and k and any good $g \in X'_k$ we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$. Let $T = \{t_1, t_2, \dots, t_\ell\}$. For every agent $r \notin T$ we have $X'_r = X_r$ for some r' . Now consider two cases depending on k :

- $k \notin T$: Note that the agents' valuation for their current bundle (in X') is at least as good as their valuation for their old bundle (in X). We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(X_{k'} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$ (or equivalently, $g \in X_{k'}$).
- $k \in T$: Let $k = t_i$ for some i . We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(Z_{i-1} \setminus g)$ for any $g \in Z_{i-1}$ (by the inclusion-wise minimality of Z_{i-1} as an envied set) and $v_j(Z_{i-1} \setminus g) = v_j(X'_{t_i} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$. \square

We can now summarize. Let $V = \max_i v_i(M)$ be the maximum valuation of any agent and $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ be the minimum difference between distinct valuations. Each application of rule U_1 or rule U_2 increases the social welfare by at least Δ and hence there can be no more than nV/Δ applications of these rules. Each application of rule U_0 decreases the size of the pool by one and hence there cannot be more than m successive applications of this rule. We conclude that the number of iterations is at most nmV/Δ . Thus we have shown the following theorem.

Theorem 8. *For normalized and monotone valuations, there is always an allocation X and a pool P of unallocated goods such that*

- X is EFX,
- $v_i(X_i) \geq v_i(P)$ for all agents i , and

⁸We took j to be the maximum index such that $t_i \in C(s_j)$ so that the cycle defined by $s_j, \dots, s_{i-1}, g_j, \dots, g_{i-1}$ and t_{j+1}, \dots, t_i is simple.

- $|P|$ is less than the number of sources in the envy-graph; in particular, $|P| < n$.

Algorithm 1 determines such an allocation in at most nmV/Δ iterations.

2.1 Finding an inclusion-wise minimal envied subset

We now describe how to implement a *minimal-envied-subset* oracle efficiently, i.e., given an envied set $S \subseteq M$, we need to efficiently find an inclusion-wise minimal subset Z of S that is envied. Algorithm 3 finds such a set $Z \subseteq S$ for monotone valuations.

Algorithm 3 Algorithm for finding an inclusion-wise minimal envied subset

Precondition: There is an agent that values $S \subseteq M$ more than her own bundle.

Postcondition: $Z \subseteq S$ such that there is some agent i with $v_i(Z) > v_i(X_i)$ and for all $j \in [n]$ we have $v_j(X_j) \geq v_j(Z')$ for all $Z' \subset Z$.

```

1:  $Z = S$ ;
2: for every agent  $i$  do
3:   for every good  $g \in Z$  do
4:     if  $v_i(X_i) < v_i(Z \setminus g)$  then
5:        $Z = Z \setminus g$ ;
6:     end if
7:   end for
8: end for
9: Return  $Z$ .
```

In Algorithm 3, if the “if condition” is not invoked for any agent i and good $g \in Z$, then Z is an inclusion-wise minimal envied subset. Suppose agent i envies $Z \setminus g$ for some $g \in Z$. Then we continue our check with $Z \setminus g$ as the new Z . Since the new Z is a subset of the old Z , observe that for each agent $i \in [n]$ and each good $g \in Z$, our algorithm needs to check exactly once if $v_i(X_i) \geq v_i(Z \setminus g)$ or not. Thus our algorithm makes nm value queries. We can conclude the following theorem.

Theorem 9. *For normalized and monotone valuations, we can determine an allocation X and a pool of unallocated goods P that satisfies the 3 conditions in Theorem 8 using at most n^2m^2V/Δ value queries.*

2.2 An FPTAS to Determine an “Almost” Desired Allocation

Algorithm 1 is pseudo-polynomial, since the increase in individual valuations of the agents when we perform the update rules could be very small. Suppose we just wanted an “almost” EFX property, i.e., for every pair of agents i and j , we are happy to ensure that $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j)$ and also $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(P)$ for every i . Then we have an algorithm that runs in $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ time and finds a desired allocation.

Theorem 10. *For normalized and monotone valuations, given any $\varepsilon > 0$, using $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ value queries, we can determine an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ and a pool of unallocated goods P such that*

- for any pair of agents i and j , we have $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$,
- for any agent i , we have $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(P)$, and
- $|P| < n$.

The proof follows in a straightforward manner from the proof of Theorem 8 in Section 2. The key idea is that the “almost” EFX property is violated if and only if $(1 + \varepsilon) \cdot v_i(X_i) < v_i(X_j \setminus g)$ for some $i, j \in [n]$ or $(1 + \varepsilon) \cdot v_i(X_i) < v_i(P)$ for some $i \in [n]$. So every time we apply update rules U_1 or U_2 , there is a multiplicative improvement (by a factor of $1 + \varepsilon$) in the valuation of some agents. Since these valuations are upper-bounded by V we get a bound of $\text{poly}(n, m, \log_{(1+\varepsilon)} V)$ on the number of iterations.

2.3 New Proof of a Result from [26] for Identical Valuations

For agents with identical (general) valuations, it was shown by Plaut and Roughgarden [26] that an allocation that maximizes the minimum valuation, then maximizes the size of this bundle, then maximizes the second minimum valuation, then maximizes the size of this bundle, and so on is EFX. We now show that Algorithm 1 gives another proof that when all the agents have identical valuations, a complete allocation that is EFX always exists.

Recall that Algorithm 1 consists of applying 3 update rules: U_0, U_1, U_2 – whichever of these is applicable. Moreover, if a certain precondition is satisfied (see Algorithm 2), then rule U_2 is applicable.

We will now show that when all the agents have identical valuations and rule U_0 is not applicable, then the precondition of rule U_2 is satisfied as long as there is some unallocated good. Let $X = \langle X_1, \dots, X_n \rangle$ be the current allocation and $P = M \setminus \cup_{i=1}^n X_i$ be the set of unallocated goods in X .

Lemma 11. *Let s be any source vertex in the envy-graph G_X . If $|P| \geq 1$ and rule U_0 is not applicable then s can be chosen as the most envious agent of $X_s \cup g$ for any $g \in P$.*

Proof. Let $g \in P$ and s be any source in the envy-graph G_X . Since rule U_0 is not applicable, there is an agent t such that $v(X_t) < v(S')$ for some $S' \subset S = X_s \cup g$, where v is the common valuation function of all agents. Let $Z \subseteq S$ be an inclusion-wise minimal envied subset of S such that $v(X_t) < v(Z)$. Since s is a source in G_X , we have $v(X_s) \leq v(X_t)$. So $v(X_s) < v(Z)$; thus s can be chosen as the most envious agent of $X_s \cup g$. \square

Lemma 11 implies that while $P \neq \emptyset$, either rule U_0 or rule U_2 is applicable. Whenever we apply rule U_2 , we add any good g in P to the bundle of a source s in G_X and determine an inclusion-wise minimal envied set $Z \subseteq X_s \cup g$. Note that $v(Z) > v(X_s)$. We then throw the goods in $(X_s \cup g) \setminus Z$ back into the pool P and set $X_s = Z$.

This makes agent s strictly better-off and no agent is worse-off: thus we have made progress. So when Algorithm 1 terminates, we have an EFX allocation with $P = \emptyset$. Thus we have a complete allocation $X = \langle X_1, \dots, X_n \rangle$ that is EFX.

EFX for 2 agents. As observed in [26], an interesting consequence of the existence of EFX when all agents have identical valuations is the existence of EFX for 2 agents with distinct valuations, say v_1 and v_2 . First, an EFX allocation $\langle X_0, M \setminus X_0 \rangle$ is obtained for 2 agents – both with the same valuation function v_1 . The *more valuable* set in $\{X_0, M \setminus X_0\}$ with respect to valuation function v_2 is assigned to agent 2 while agent 1 is assigned the other bundle. It is easy to see that this is an EFX allocation.

However finding such an EFX allocation may be hard, as shown by Plaut and Roughgarden [26], and this is so even for two agents with identical submodular valuations. It was shown in [26] that an exponential number of value queries may be required to determine an EFX allocation here. It follows from Theorem 9 that an EFX allocation for two agents with general (possibly distinct) valuations can be determined using pseudo-polynomial many value queries. Moreover, even when agents have general valuations functions, we can determine a $(1 - \varepsilon)$ EFX allocation with charity (as in Theorem 10) with $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ many value queries.

3 Guarantees on Other Notions of Fairness

In this section we assume that all agents have additive valuations. We show that a minor variant of our algorithm finds an allocation with a good guarantee on Nash social welfare *and* groupwise maximin share (GMMS).

Guarantee in Terms of Nash Social Welfare. We claimed in Section 1 that for additive valuations, it can also be ensured that for each i , we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ where $X^* = \langle X_1^*, \dots, X_n^* \rangle$ is an optimal *Nash social welfare* allocation and X is the allocation in Theorem 8. This is easy to see from Algorithm 1:

- rather than initialize $X_i = \emptyset$, we will initialize X_i to the bundle corresponding to the allocation determined by the algorithm in [12].

So we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$, to begin with. As the algorithm progresses, our invariant is that $v_i(X_i)$ never decreases for any i . So if $X' = \langle X'_1, \dots, X'_n \rangle$ is the final allocation computed by our algorithm, then we have $v_i(X'_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ for all $i \in [n]$.

Lemma 12. *Given a set N of agents with additive valuations and a set M of goods, there exists an allocation $X = \langle X_1, \dots, X_n \rangle$ and a pool P of unallocated goods that satisfy all the conditions of Theorem 8 and $v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where $X^* = \langle X_1^*, \dots, X_n^* \rangle$ is an optimal Nash social welfare allocation.*

3.1 An Approximate MMS Allocation for Large $|P|$

We now show that if $|P|$ (the number of unallocated goods in our allocation) is sufficiently large, then our EFX allocation X has a very good MMS guarantee. Recall that our algorithm continues till $|P|$ is smaller than the number of sources in the envy-graph G_X and recall that sources are *unenvied* agents. In particular, if $|P| = n - 1$, then the number of sources in G_X is n ; so no agent envies another. That is, for each i , we have $v_i(X_i) \geq v_i(X_j)$ for all $j \in [n]$. Moreover, $v_i(X_i) \geq v_i(P)$. So we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n+1} &&\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot \frac{v_i(M)}{n} \\ &&&\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot MMS_i(n, M), \end{aligned}$$

where for every agent i , the inequality $MMS_i(n, M) \leq v_i(M)/n$ holds for additive valuations. We formalize the above intuition in Theorem 14. The following proposition from [20] will be useful. It states that if we exclude any set of agents and at most the same number of any goods from N and M , respectively, the maximin share of any remaining agent can only increase.

Proposition 13 ([20]). *Let N be a set of n agents with additive valuations and M be a set of m goods. If $N' \subseteq N$ and $M' \subseteq M$ are such that $|N \setminus N'| \geq |M \setminus M'|$ then for any agent $i \in N'$, we have $MMS_i(n', M') \geq MMS_i(n, M)$ where $n' = |N'|$.*

Theorem 14. *Given a set N of n agents with additive valuations and a set M of m goods, there exists an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ and set P of unallocated goods that satisfies:*

- the 3 conditions stated in Theorem 8;
- $v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where X^* is an optimal Nash social welfare allocation;
- $v_i(X_i) \geq MMS_i(n, M) / \left(2 - \frac{k}{n}\right)$ for every $i \in N$, where $k = |P|$.

Proof. Let (X, P) be the allocation guaranteed by Lemma 12. Hence the first two conditions given in the theorem statement are satisfied by (X, P) . So what we need to show now is that for any agent i , we have $v_i(X_i) \geq MMS_i(n, M) / (2 - \frac{k}{n})$.

We fix some agent i and let $N' \subseteq N$ be the set of agents consisting of all sources of G_X , agent i and all other agents j with $|X_j| \geq 2$. Let M' be the set of goods allocated to the agents in N' . Observe that every agent in $N \setminus N'$ is allocated at most one good and so $|N \setminus N'| \geq |M \setminus (M' \cup P)|$. By Proposition 13, it holds that $MMS_i(n', M' \cup P) \geq MMS_i(n, M)$ where $n' = |N'|$. Thus, it suffices to show that $v_i(X_i) \geq MMS_i(n', M' \cup P) / (2 - \frac{k}{n})$.

Consider any agent $j \in N'$ with $|X_j| \geq 2$. Because X is EFX, it holds that $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$. Since the valuations are additive, we have

$$v_i(X_i) \geq \left(1 - \frac{1}{|X_j|}\right) \cdot v_i(X_j) \geq \frac{1}{2} \cdot v_i(X_j).$$

We know the following inequalities hold:

$$v_i(X_i) \geq v_i(P), \tag{1}$$

$$v_i(X_i) \geq v_i(X_j) \text{ for all } j \text{ that were sources in } G_X, \tag{2}$$

$$2v_i(X_i) \geq v_i(X_j) \text{ for all other } j \in N'. \tag{3}$$

Recall that the number of sources is at least $|P| + 1 = k + 1$. Summing up all inequalities in (1)-(3) and using the fact that v_i is additive, we have $(2(n' - (k + 1)) + k + 2) \cdot v_i(X_i) \geq v_i(M' \cup P)$.

Hence we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M' \cup P)}{2n' - k} \\ &\geq \frac{v_i(M' \cup P)}{n'} \cdot \frac{n'}{2n' - k} \\ &\geq MMS_i(n', M' \cup P) \cdot \frac{n'}{2n' - k} \quad (\text{since } v_i \text{ is additive}) \\ &= MMS_i(n', M' \cup P) / (2 - \frac{k}{n'}) \\ &\geq MMS_i(n', M' \cup P) / (2 - \frac{k}{n}) \quad (\text{since } n' \leq n). \quad \square \end{aligned}$$

3.2 An Improved Bound for Approximate-GMMS

As mentioned in Section 1, a new notion of fairness called *groupwise maximin share* (GMMS) was recently introduced by Barman et al. [6]. We formally define a GMMS allocation below.

Definition 15. *Given a set N of n agents and a set M of m goods, an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ is α -GMMS if for every $N' \subseteq N$, we have $v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{i \in N'} X_i)$ where $n' = |N'|$.*

Observe that a GMMS allocation is also an MMS allocation. Since MMS allocations do not always exist in a given instance [27], GMMS allocations also need not always exist. Interestingly, $\frac{1}{2}$ -GMMS allocations always exist [6]. We now describe how to modify our allocation so that the resulting allocation is $\frac{4}{7}$ -GMMS.

Let $X = \langle X_1, \dots, X_n \rangle$ be the allocation and P be the pool of unallocated goods that satisfy the conditions of Lemma 12. Without loss of generality, assume that agent 1 is a source in the envy-graph G_X . Define the complete allocation $Y = \langle Y_1, \dots, Y_n \rangle$ as follows:

$$* Y_1 = X_1 \cup P \text{ and } Y_i = X_i \text{ for all } i \neq 1.$$

Theorem 16 shows that Y is our desired allocation. The proof of Theorem 16 is similar to [1, Proposition 3.4]. We also remark that one can use the proof of [1, Proposition 3.4] to show that any EFX allocation is a $4/7$ -GMMS. However note that Y is not necessarily an EFX allocation. But it has sufficiently nice properties so that we can still show that it is $4/7$ -GMMS.

Theorem 16. *Given a set N of n agents with additive valuations and a set M of m goods, there exists a complete allocation $Y = \langle Y_1, Y_2, \dots, Y_n \rangle$ of M such that*

- Y is $\frac{4}{7}$ -GMMS.
- $v_i(Y_i) \geq \frac{1}{2}v_i(X_i^*)$ for all $i \in N$ where X^* is the optimal Nash social welfare allocation.⁹

Proof. Observe that the bound on Nash social welfare holds for allocation X and thus for allocation Y (since $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$). So what we need to show now is the guarantee on GMMS. That is, we need to show that for every $\tilde{N} \subseteq N$ and all $i \in \tilde{N}$, we have $v_i(Y_i) \geq \frac{4}{7}MMS_i(\tilde{n}, \tilde{M})$ where $\tilde{n} = |\tilde{N}|$ and $\tilde{M} = \bigcup_{j \in \tilde{N}} Y_j$.

Fix some $i \in \tilde{N}$. Define N' as the subset of \tilde{N} that contains i and all agents that have been allocated at least two goods in Y , i.e., $j \in N'$ if and only if $j = i$ or $|Y_j| \geq 2$. Let $M' = \bigcup_{j \in N'} Y_j$.

Note that Y allocates all goods of $\tilde{M} \setminus M'$ to agents in $\tilde{N} \setminus N'$. Since every agent in $\tilde{N} \setminus N'$ has been allocated at most one good, we have $|\tilde{N} \setminus N'| \geq |\tilde{M} \setminus M'|$. Proposition 13 tells us that $MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$ where $n' = |N'|$. Thus it suffices to show $v_i(Y_i) \geq \frac{4}{7} \cdot MMS_i(n', M')$.

We now classify the goods in M' as good or bad. A good is *good* if it contained in a Y_j of cardinality at least three or is contained in Y_1 or Y_i . All other goods are *bad*, i.e., a bad good is contained in a bundle of cardinality two different from Y_1 and Y_i . A bundle in Y containing good goods is good.

We will next reduce the problem further. Let x be the number of bad goods in M' . Since the bad goods are contained in bundles of cardinality two, the good goods in M' come from $n' - x/2$ good bundles of Y . As long as $x > n'$, we will apply a reduction step. Each reduction step will reduce the number of bad goods in M' by two, the number of agents by one, will not decrease the MMS_i -value, and will leave the quantity $n' - x/2$ and set of good goods in M' unchanged.

Let $Z = \langle Z_1, Z_2, \dots, Z_{n'} \rangle$ be an optimal MMS partition for agent i of the set M' of goods. If there are more than n' bad goods in M' , there is a set Z_k with at least *two* bad goods, say g_1 and g_2 . We distribute the goods in $Z_k \setminus \{g_1, g_2\}$ arbitrarily among the other sets in Z . So we have a partition of the set $M' \setminus \{g_1, g_2\}$ of goods into $n' - 1$ many bundles. The value for agent i of any remaining bundles did not decrease. We set M' to $M' \setminus \{g_1, g_2\}$ and decrement n' . Note that we reduced the number of bad goods by two, the number of agents by one, did not decrease $MMS_i(n', M')$ and the set of good goods in M' still come from the $n' - x/2$ good bundles in Y . We keep repeating this reduction until M' contains at most n' bad goods. At this point, we have a set M' of goods and an integer n' with the following properties:

- (1) The number x of bad goods in M' is at most n' .
- (2) $MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$, and
- (3) The set of good goods in M' has not changed. They come from $n' - x/2$ good bundles in Y .

We will next relate the value of good and bad goods to the value of Y_i .

Lemma 17. *We have*

- a) *For any bad good g , $v_i(g) \leq v_i(Y_i)$.*
- b) *$v_i(Y_1) \leq 2v_i(Y_i)$.*
- c) *If $j \neq 1$ and Y_j is a good bundle then $v_i(Y_j) \leq 3/2 \cdot v_i(Y_i)$.*

Proof. a) Let $Y_j = \{g, g'\}$ be the bundle containing g . Since $j \neq 1$ (by definition of a bad good), we have $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus g') = v_i(Y_j \setminus g') = v_i(g)$.

⁹In private communication we are aware that Jugal Garg and Setareh Taki have obtained independently related results. For **additive** valuations, they can show that there is an EFX-allocation after donating at most $n - 1$ goods to charity. However, there is no bound on the value of the goods donated to charity. Thus they obtain a $4/7$ -GMMS-allocation after removing $n - 1$ goods from the original set of goods.

- b) Since agent 1 is a source, $v_i(X_i) \geq v_i(X_1)$. By Theorem 8, $v_i(X_i) \geq v_i(P)$. Therefore, $v_i(Y_1) = v_i(X_1 \cup P) = v_i(X_1) + v_i(P) \leq v_i(X_i) + v_i(X_i) = 2v_i(X_i) = 2v_i(Y_i)$.
- c) Let $g \in Y_j$ be such that $v_i(g)$ is minimal. Then $v_i(Y_j - g) \leq v_i(Y_i)$ and $v_i(g) \leq v_i(Y_j)/|Y_j|$. Thus

$$\begin{aligned} v_i(Y_i) &\geq \left(1 - \frac{1}{|Y_j|}\right) \cdot v_i(Y_j) \geq \left(1 - \frac{1}{3}\right) \cdot v_i(Y_j) \\ &= \frac{2}{3} \cdot v_i(Y_j). \end{aligned}$$

□

Now we are ready to show the bound on GMMS. We have x bad goods in M' . The good goods in M' come from $n' - x/2$ good bundles. Also $x \leq n'$. The total value of the good goods for agent i is at most

$$\frac{3}{2}(n' - \frac{x}{2} - 2) \cdot v_i(Y_i) + v_i(Y_i) + 2v_i(Y_i) = \frac{3}{2}(n' - \frac{x}{2}) \cdot v_i(Y_i),$$

since there are at most $n' - x/2 - |\{1, i\}|$ good bundles different from Y_1 and Y_i : each of value at most $(3/2 \cdot v_i(Y_i))$, and Y_1 has value at most $2v_i(Y_i)$. Also, the total value of the bad goods for agent i is at most $x \cdot v_i(Y_i)$, since there are x many bad goods and each bad good is worth at most $v_i(Y_i)$. Therefore,

$$\begin{aligned} v_i(M') &= v_i(\text{bad goods in } M') + v_i(\text{good goods in } M') \\ &\leq x \cdot v_i(Y_i) + \frac{3}{2}(n' - \frac{x}{2}) \cdot v_i(Y_i) \\ &= \frac{6n' + x}{4} \cdot v_i(Y_i) \leq \frac{7n'}{4} \cdot v_i(Y_i). \end{aligned}$$

□

4 Conclusions and Open Problems

We studied the existence of EFX allocations when agents have general valuations. We showed that we can ensure such an allocation always exists when we donate a small number of goods that nobody envies to charity. The major open problem here is whether complete EFX allocations always exist. Plaut and Roughgarden [26] remarked that an instance with *no* complete EFX allocation may be easier to find in the setting of general valuations. Our result on “almost-EFX” allocations for general valuations allows one to hope that complete EFX allocations always exist, at least for more structured valuations such as additive.

We also showed that we get guarantees in terms of other notions of fairness when agents have additive valuations. To the best of our knowledge, allocations with good guarantees (i.e., constant factor approximation) on Nash social welfare and MMS (as well as GMMS) were not known prior to our work. It would also be interesting to investigate whether these guarantees can be improved or if instances can be constructed where our guarantees are tight. We believe that our work is just the beginning towards determining an allocation that gives good guarantees with respect to several notions of fairness: an allocation that is *universally* fair.

Very recently, Amanatidis et al. [3] have announced an allocation that has good approximation guarantees simultaneously with respect to four notions of fairness when valuation functions are additive: in particular, their allocation is $(\phi - 1)$ -EFX and $2/(\phi + 2)$ -GMMS, where $\phi \approx 1.618$ is the golden ratio. Moreover, a fine-tuned version of their algorithm also achieves $4/7$ -GMMS which matches our result (see Theorem 16). They also show that for additive valuations, when

m is at most $n + 2$, where m is the number of goods and n is the number of agents, GMMS (and hence, EFX) allocations always exist.

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