6.853: Topics in Algorithmic Game Theory

Lecture 9

Fall 2011

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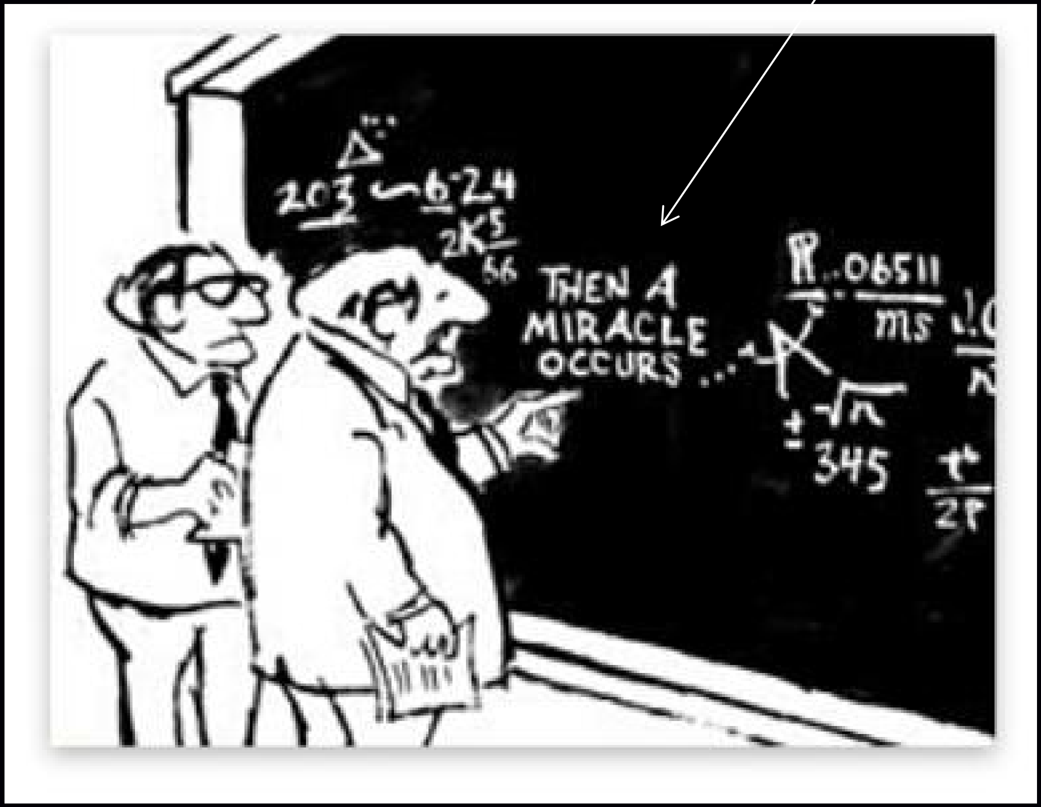
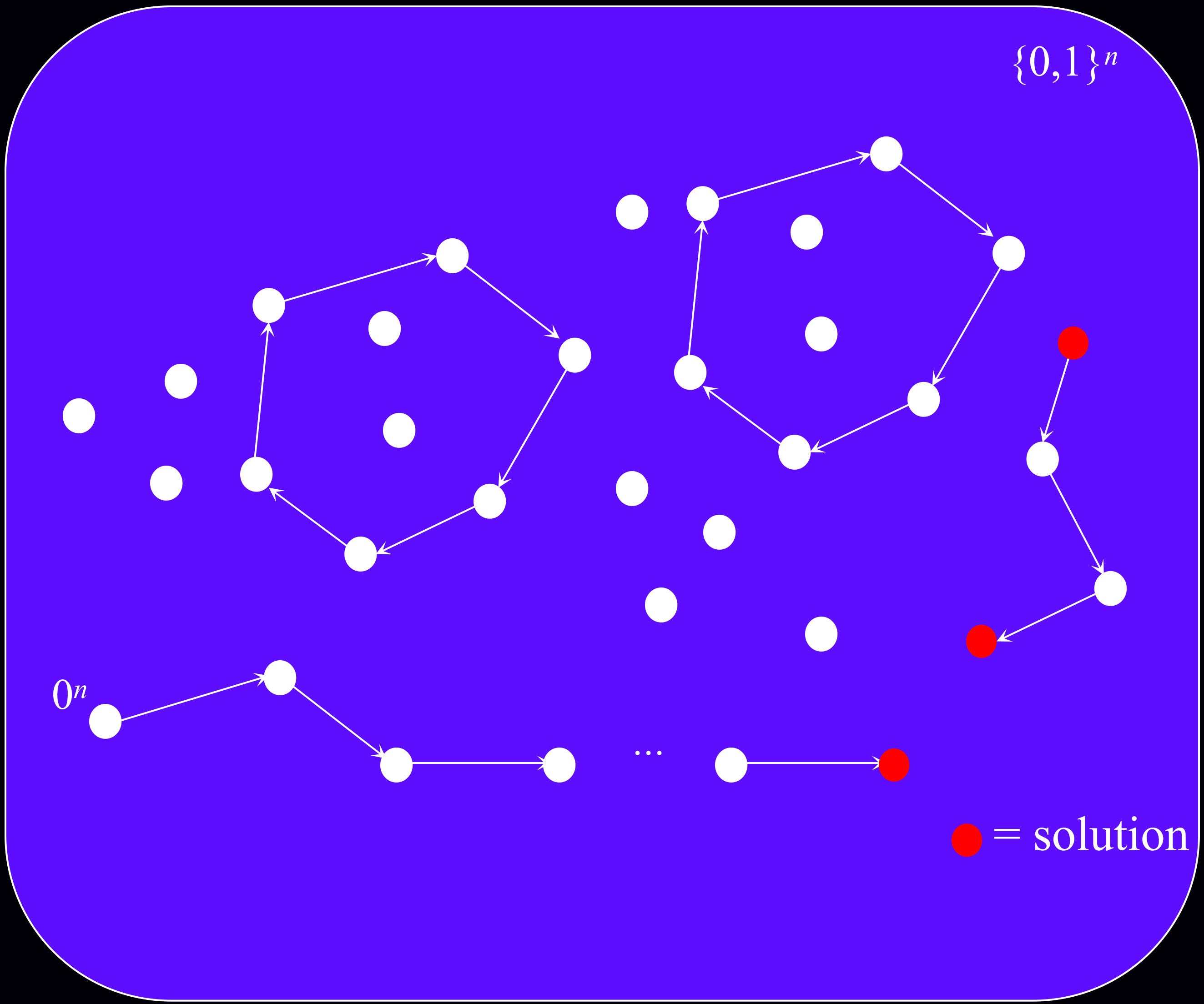
Last Time…

Non-constructive step in the proof of Sperner?



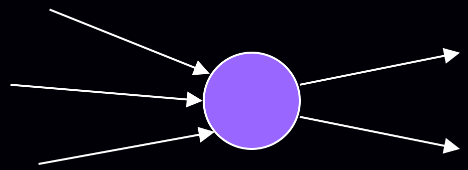
Remember this figure?



The Non-Constructive Step

an easy parity lemma:

a directed graph with an unbalanced node (a node with indegree ≠ outdegree) must have another.



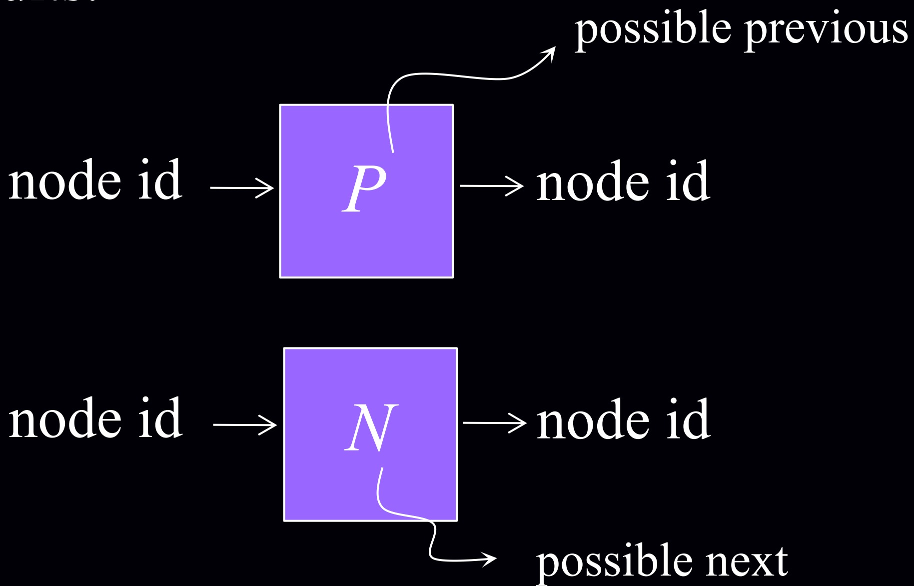
but, why is this non-constructive?

given a directed graph and an unbalanced node, isn't it trivial to find another unbalanced node?

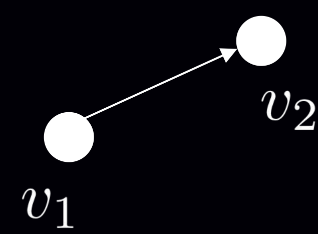
the graph can be exponentially large, but has succinct description…

The PPAD Class [Papadimitriou '94]

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:



$$P(v_2) = v_1 \wedge N(v_1) = v_2$$



END OF THE LINE: Given P and N: If 0^n is an unbalanced node, find another unbalanced node. Otherwise say "yes".

PPAD = *{ Search problems in FNP reducible to END OF THE LINE}*

Inclusions

(i) $\mathrm{PPAD} \subseteq \mathrm{FNP}$

(ii) $\mathrm{SPERNER} \in \mathrm{PPAD}$

PROOF: Sufficient to define appropriate circuits P and N as follows:

- Every simplex in the SPERNER instance is identified with an element of $\{0,1]\}^n$. for some n=n(d, m) that depends on d, the dimension of the SPERNER instance, and m, the discretization accuracy in every dimension.

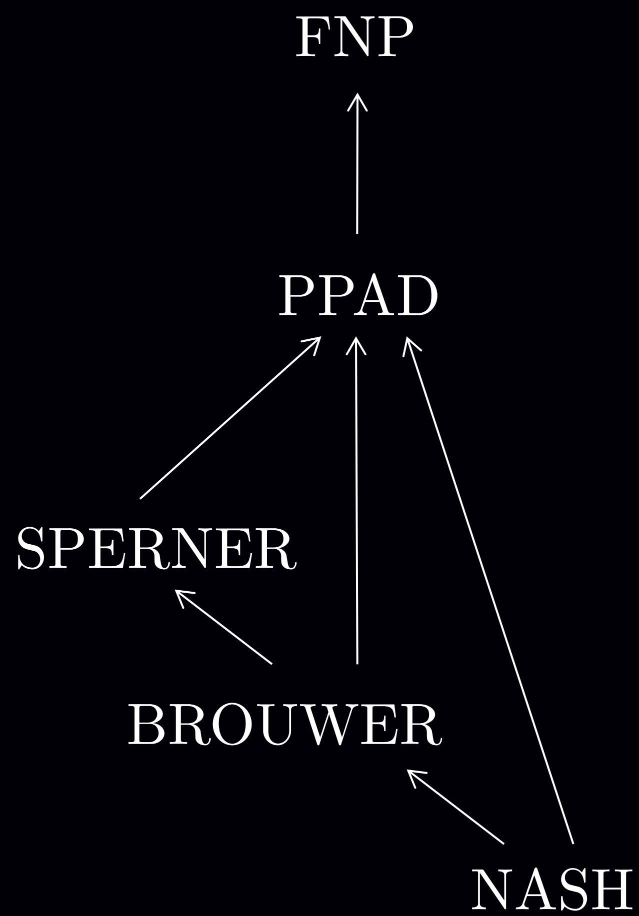
- Starting Simplex $\rightsquigarrow$ 0^n

- Define: $P(0^n) = 0^n$; make $N(0^n)$ output the simplex S sharing the colorful facet with the starting simplex; also set $P(S)$=0^n (this makes sure that 0^n is a source vertex pointing to vertex S)

- Now, if a simplex S is neither colorful nor panchromatic, then set $P(S)$=S and $N(S)$=0^n (this makes sure that S is an isolated vertex)

important here that the directions are efficiently computable locally, and consistent

- if a simplex S has a colorful facet f shared with another simplex S', then if the sign of f in S is $(-1)^{\lfloor \frac{d-1}{2} \rfloor}$ then set $N(S)$=S'; otherwise set $P(S)$=S'.


FNP
PPAD
SPERNER
BROUWER
NASH

Other arguments of existence, and resulting complexity classes

"If a graph has a node of odd degree, then it must have another."

PPA

"Every directed acyclic graph must have a sink."

PLS

"If a function maps n elements to n-1 elements, then there is a collision."

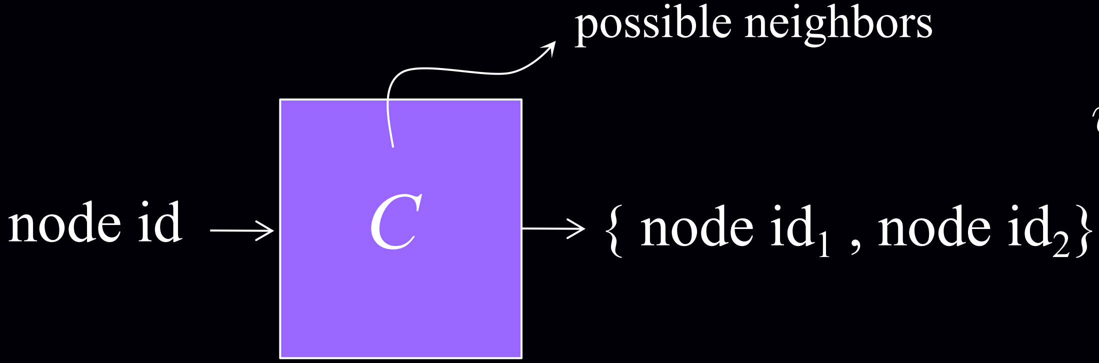
PPP

Formally?

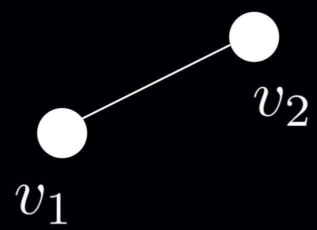
The Class PPA [Papadimitriou '94]

"If a graph has a node of odd degree, then it must have another."

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



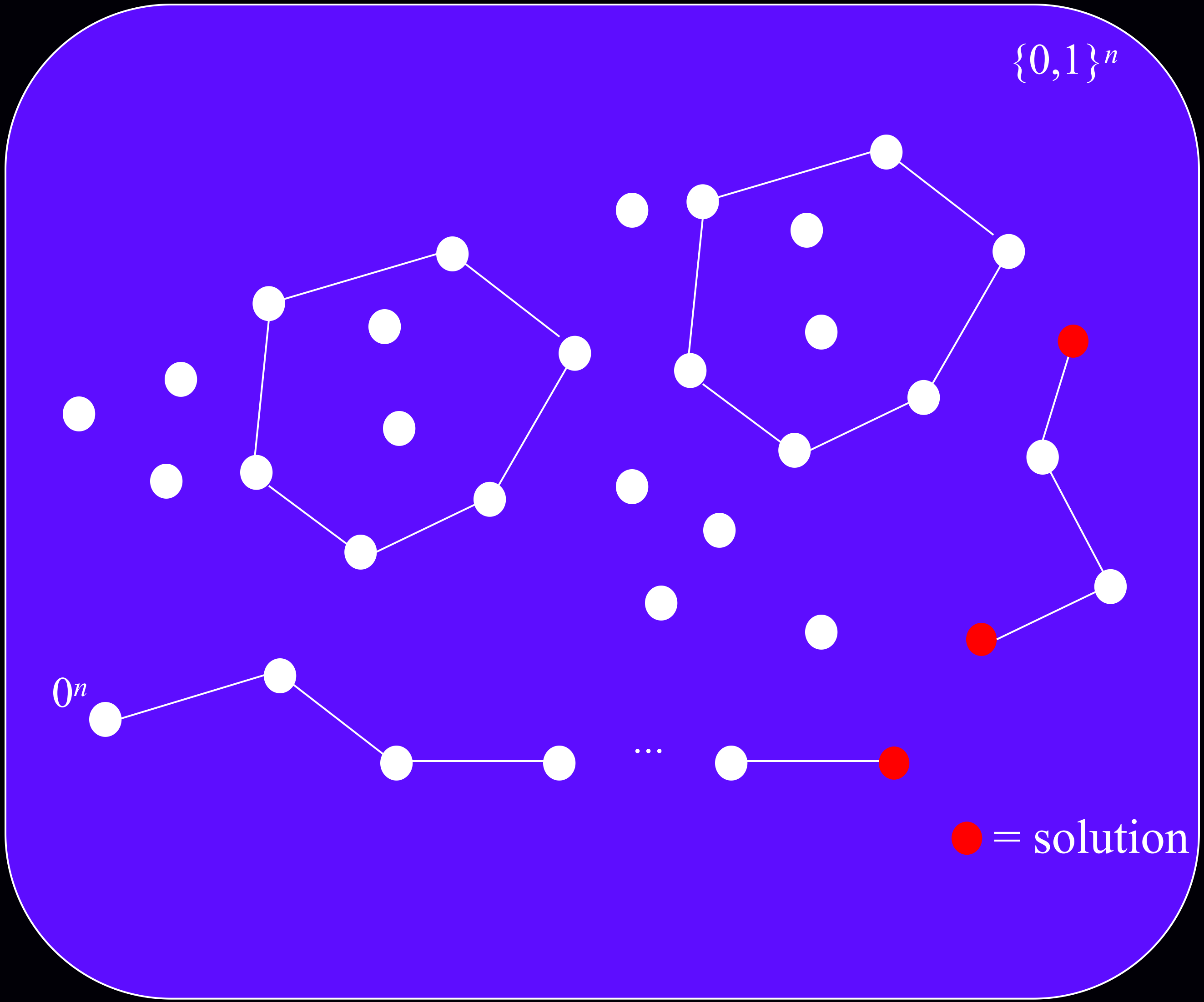
$$v_1 \in C(v_2) \quad \wedge \quad v_2 \in C(v_1)$$



ODD DEGREE NODE: Given C: If 0^n has odd degree, find another node with odd degree. Otherwise say "yes".

PPA = $\{$ *Search problems in FNP reducible to* ODD DEGREE NODE $\}$

The Undirected Graph



The Class PLS [JPY '89]

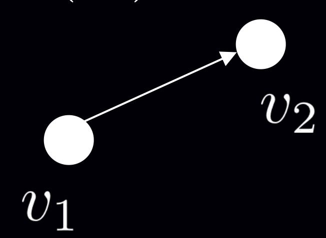
"Every DAG has a sink."

Suppose that a DAG with vertex set $\{0,1\}^n$ is defined by two circuits:

node id $\longrightarrow$ $\boxed{C}$ $\longrightarrow$ {node id$_1$, …, node id$_k$}

node id $\longrightarrow$ $\boxed{F}$ $\longrightarrow$ $\mathbb{R}$

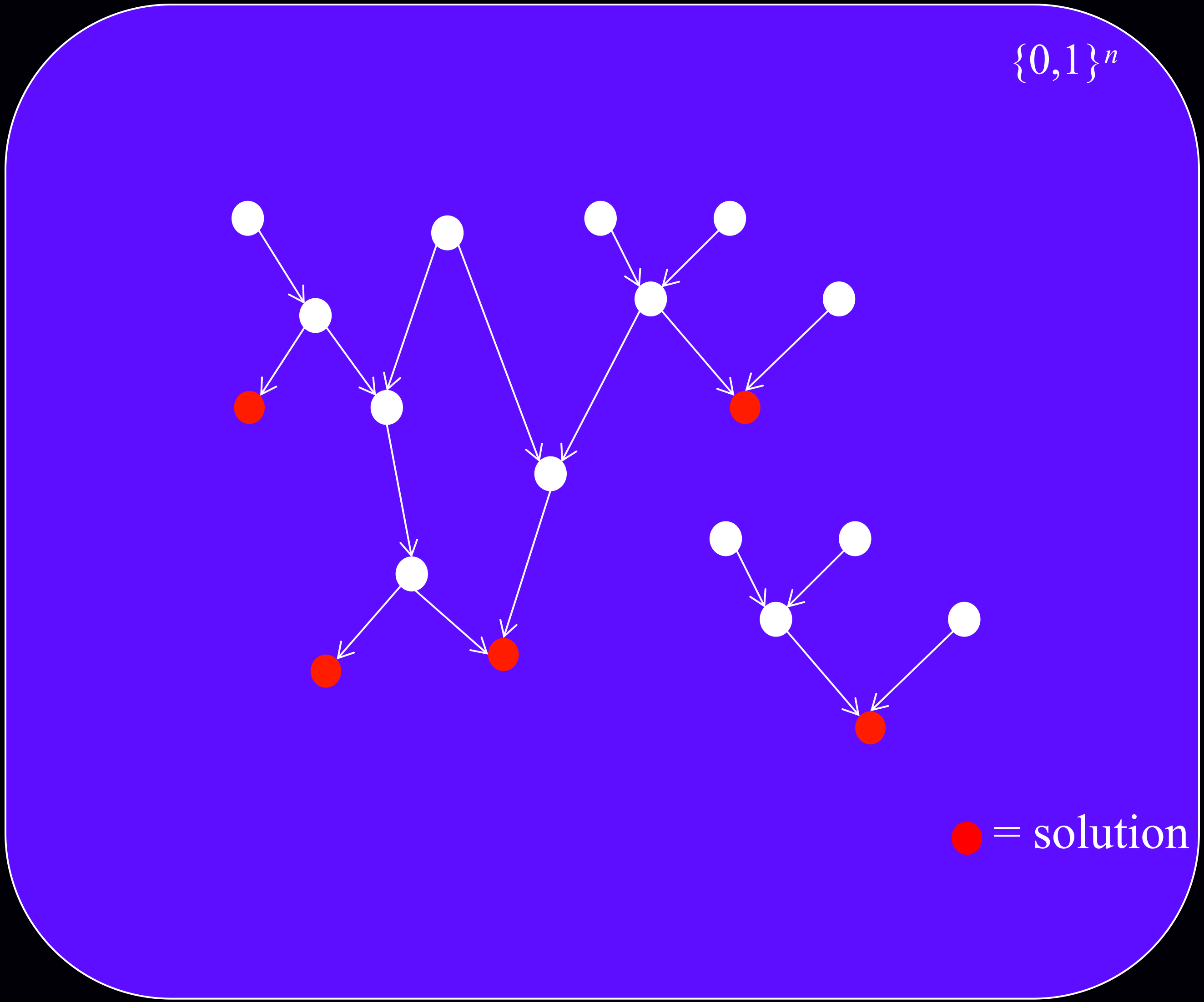
$$v_2 \in C(v_1) \quad \wedge \quad F(v_2) > F_{(}v_1)$$



FIND SINK: Given *C*, *F*: Find x s.t. F(x) ≥ F(y), for all $y \in C(x)$.

PLS = { *Search problems in FNP reducible to* FIND SINK}

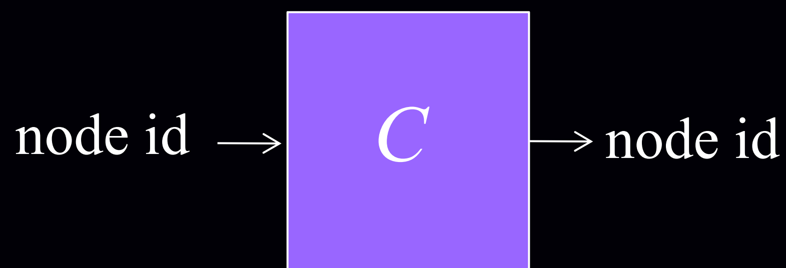
The DAG



The Class PPP [Papadimitriou '94]

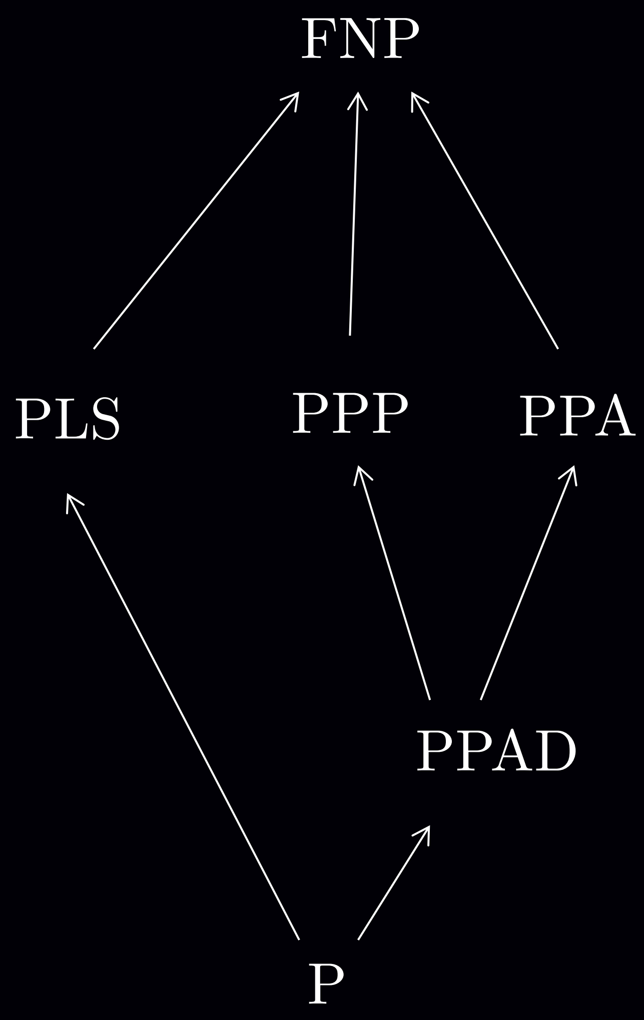
"If a function maps n elements to n-1 elements, then there is a collision."

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



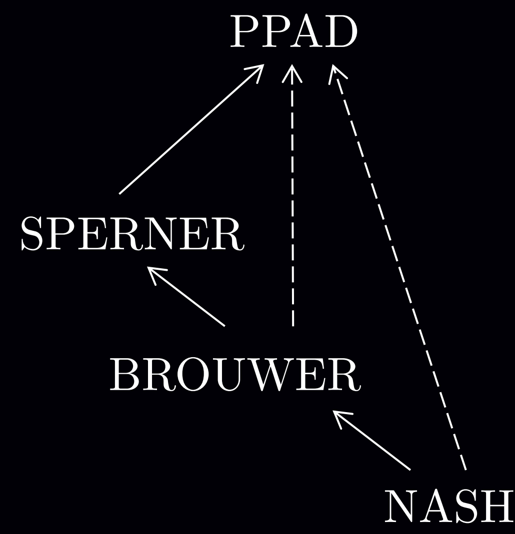
COLLISION: Given C: Find x s.t. $C(x) = 0^n$; or find $x \neq y$ s.t. $C(x)=C(y)$.

PPP = { *Search problems in FNP reducible to* COLLISION }

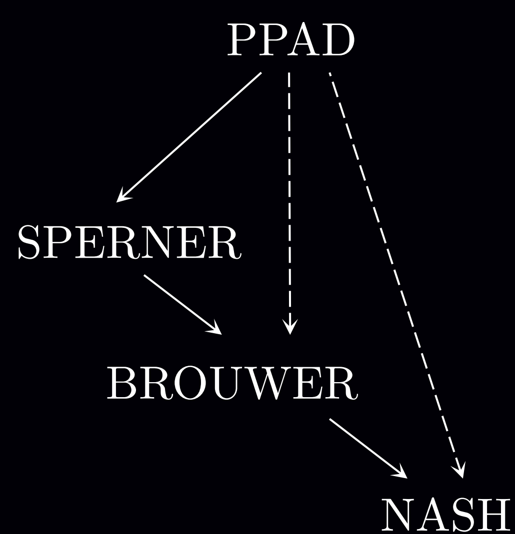

FNP
PLS
PPP
PPA
PPAD
P

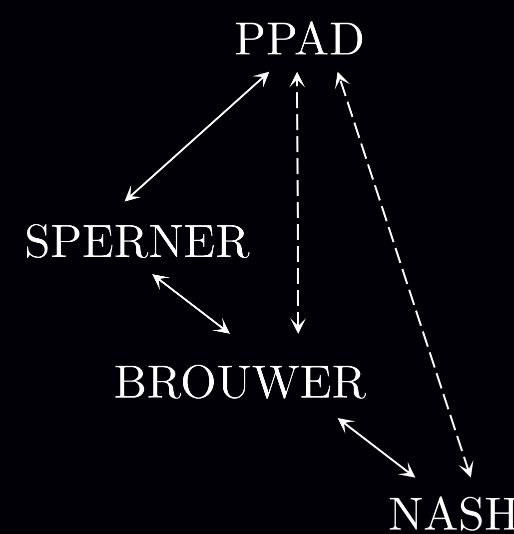
Hardness Results

Inclusions we have already established:



Our next goal:

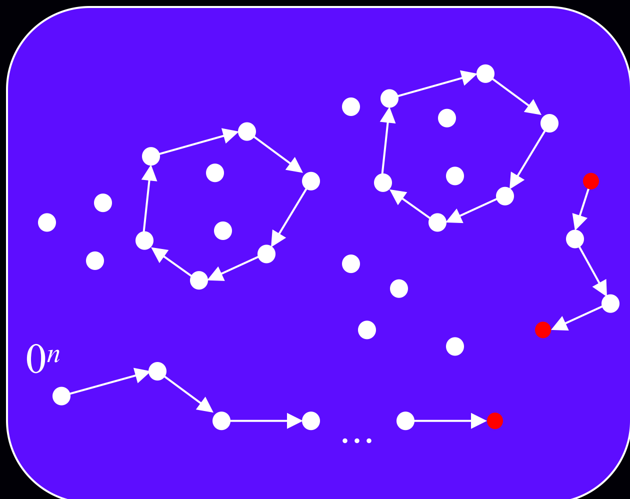




The PLAN

DGP = Daskalakis, Goldberg, Papadimitriou

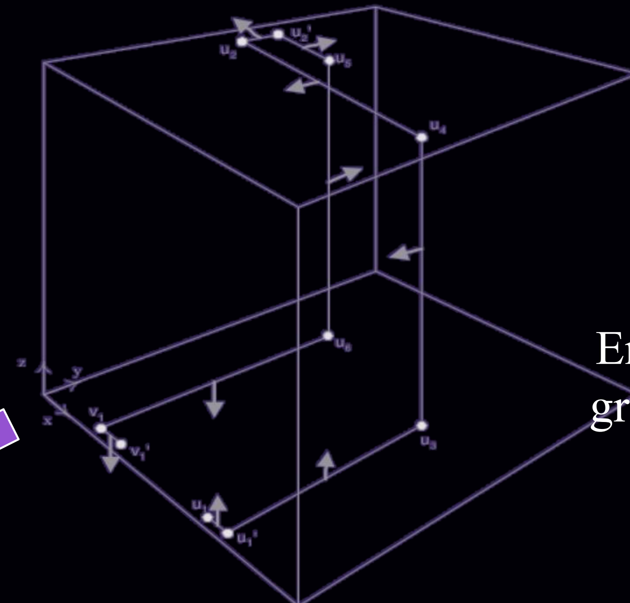
CD = Chen, Deng



Generic PPAD

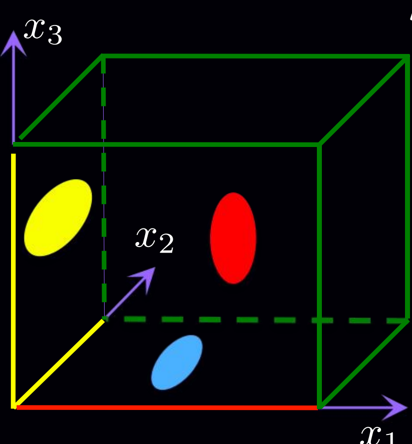
[Pap '94]

[DGP '05]



Embed PPAD graph in $[0,1]^3$

[DGP '05]



3D-SPERNER

[DGP '05]

p.w. linear BROUWER

[DGP '05]

multi-player NASH

[DGP '05]

4-player NASH

[DP '05]

[CD'05]

3-player NASH

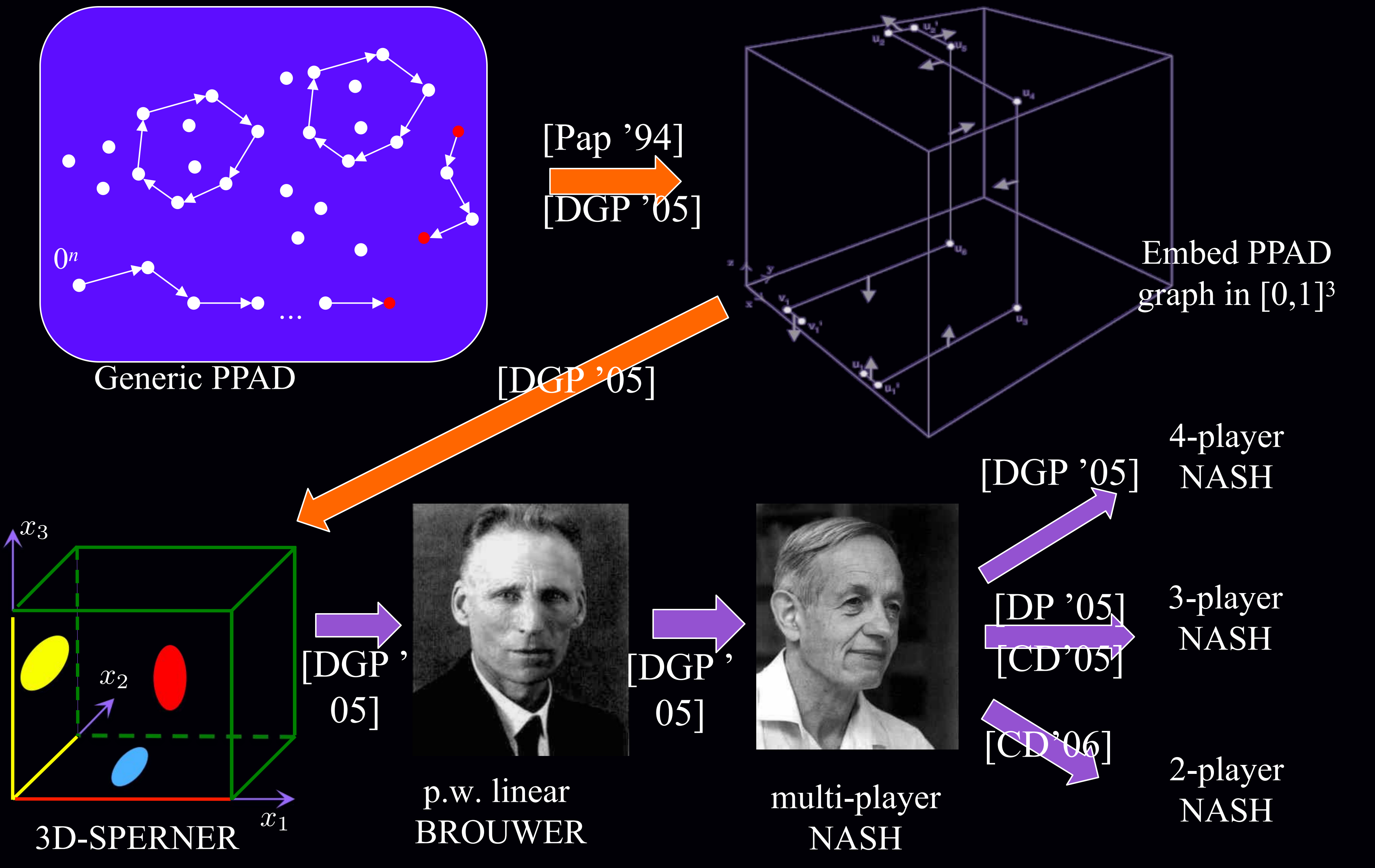
[CD'06]

2-player NASH

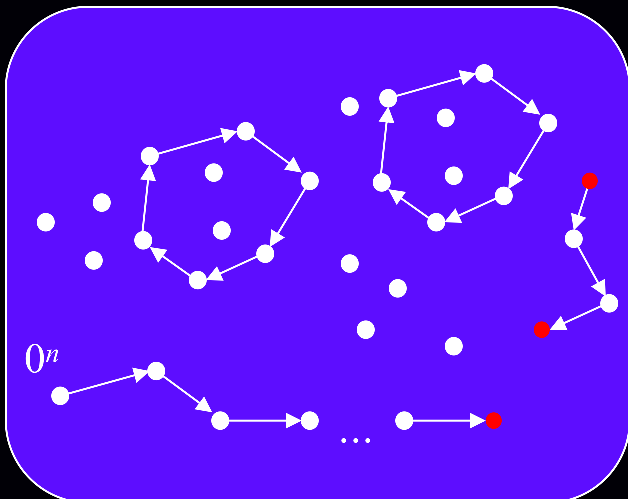
This Lecture

DGP = Daskalakis, Goldberg, Papadimitriou

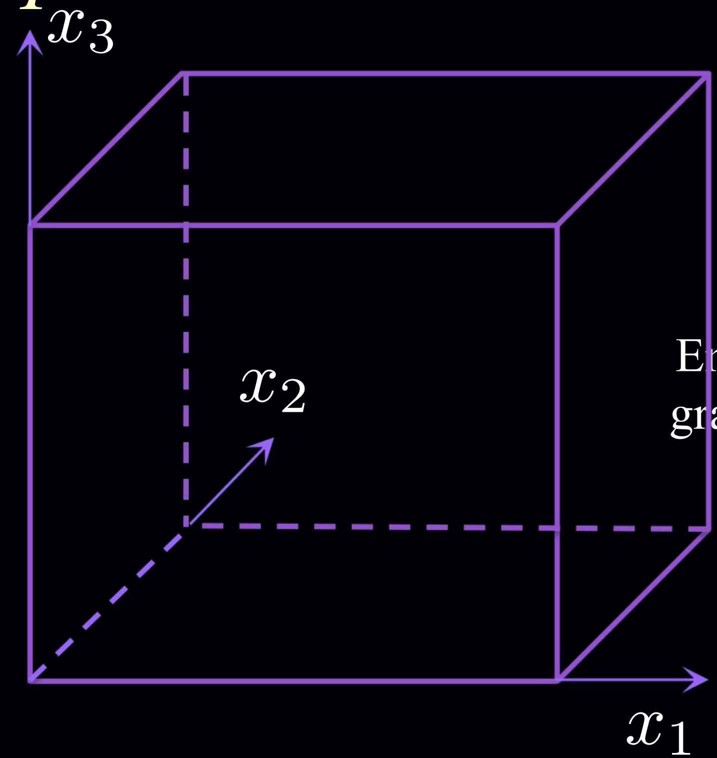
CD = Chen, Deng



First Step



Generic PPAD

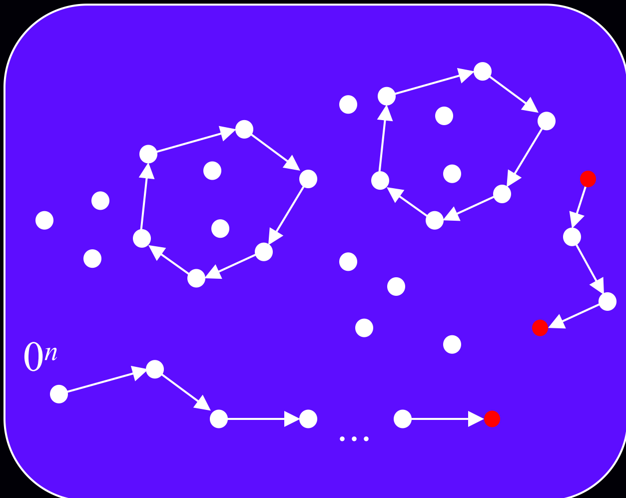


Embed PPAD graph in $[0,1]^3$

n $\longrightarrow$ $m = n + 4$

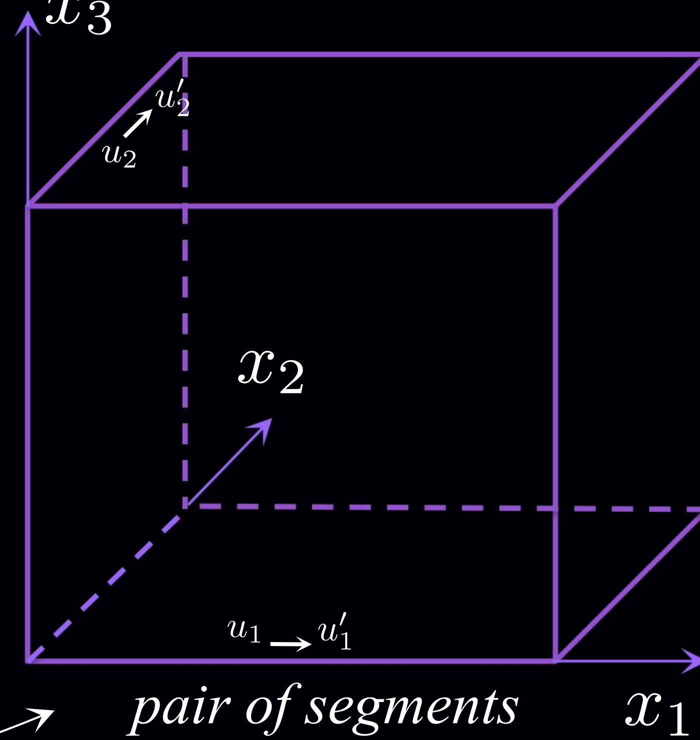
*our goal is to identify a piecewise linear, single dimensional subset of the cube, corresponding to the PPAD graph; we call this subset **L***

Non-Isolated Nodes map to pairs of segments



Generic PPAD

Non-Isolated Node



pair of segments

$$u \in \{0,1\}^n$$

$$u_1 = (8\langle u \rangle + 2, 3, 3) \qquad u_1' = (8\langle u \rangle + 6, 3, 3)$$

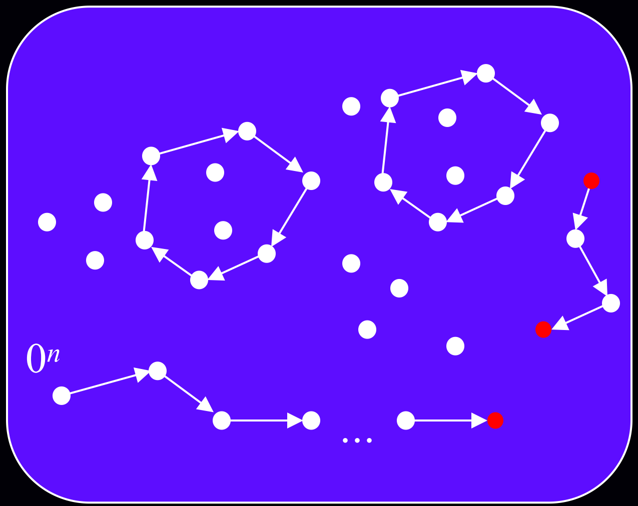
main segment

$$u_2 = (3, 8\langle u \rangle + 6, 2^m - 3)$$

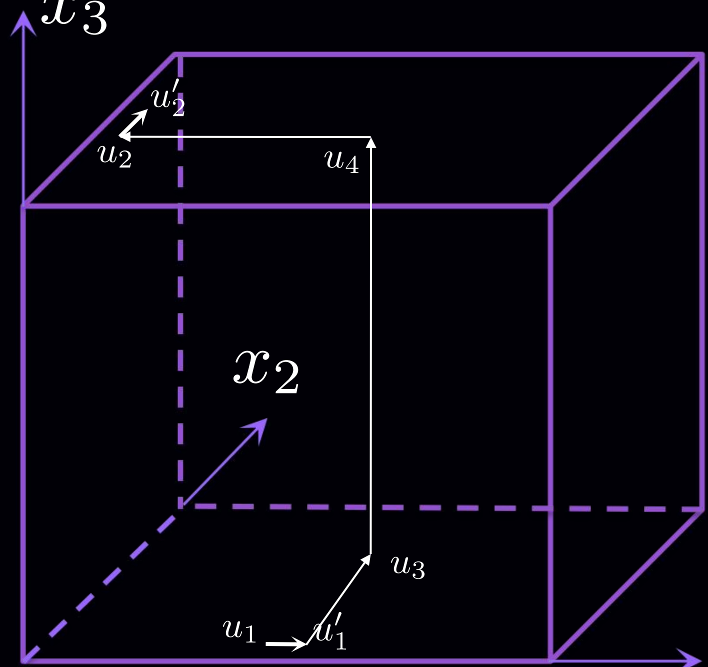
auxiliary segment

$$u_2' = (3, 8\langle u \rangle + 10, 2^m - 3)$$

Non-Isolated Nodes map to pairs of segments



Generic PPAD



Non-Isolated Node

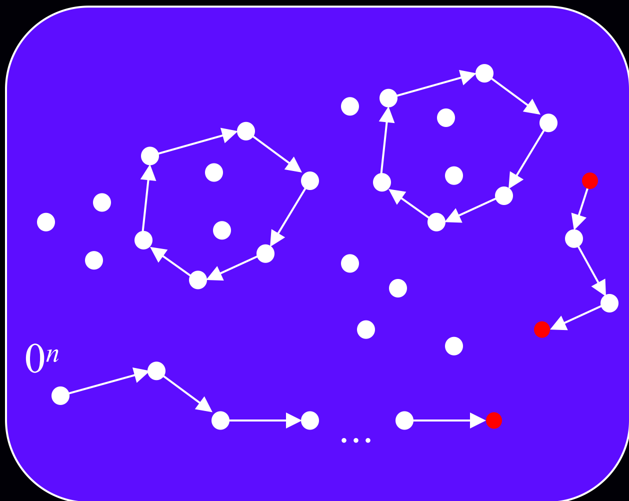
pair of segments

also, add an orthonormal path connecting the end of main segment and beginning of auxiliary segment

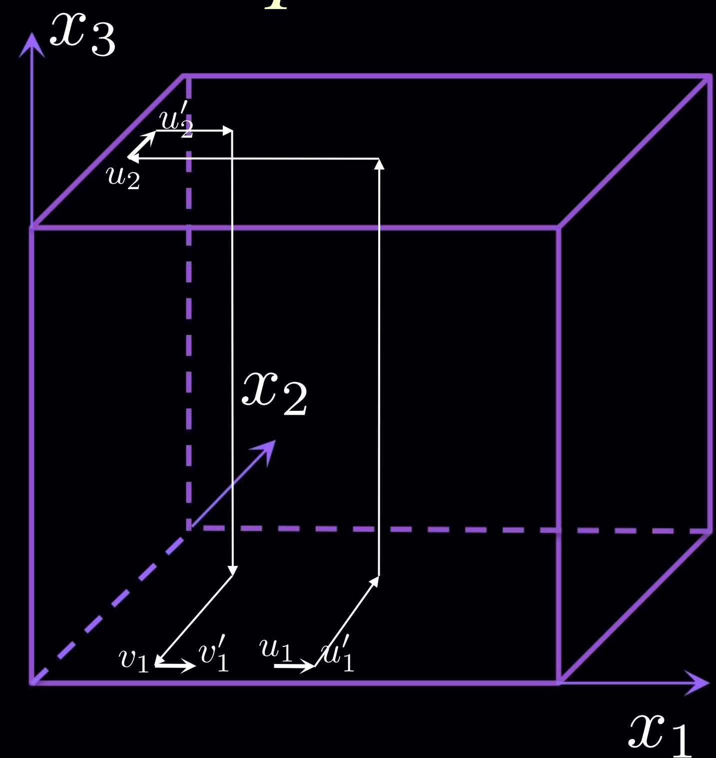
breakpoints used:

$$u_3 = (8\langle u\rangle + 6, 8\langle u\rangle + 6, 3)$$
$$u_4 = (8\langle u\rangle + 6, 8\langle u\rangle + 6, 2^m - 3)$$

Edges map to orthonormal paths



Generic PPAD



Edge between u and v

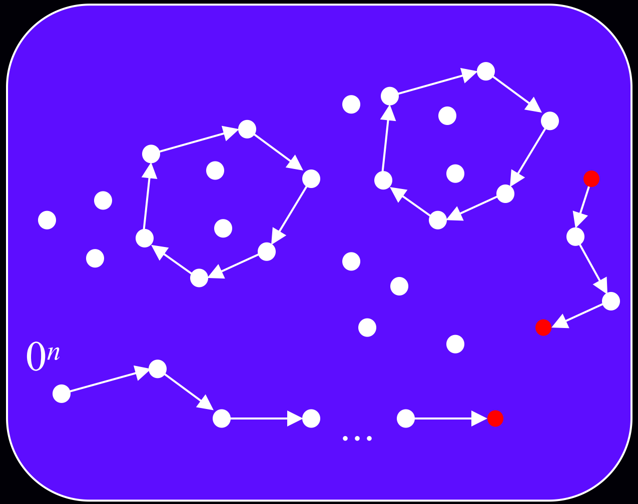
orthonormal path connecting the end of the auxiliary segment of u with beginning of main segment of v

breakpoints used:

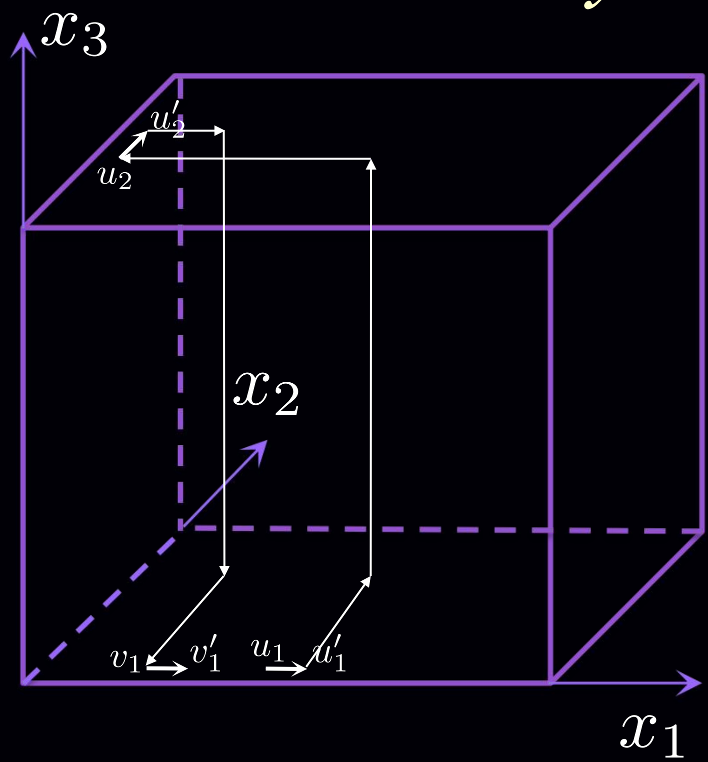
$$(8\langle v\rangle + 2, 8\langle u\rangle + 10, 2^m - 3)$$

$$(8\langle v\rangle + 2, 8\langle u\rangle + 10, 3)$$

Exceptionally 0^n is closer to the boundary…



Generic PPAD



Modifications of main segment and first breakpoint for 0^n:

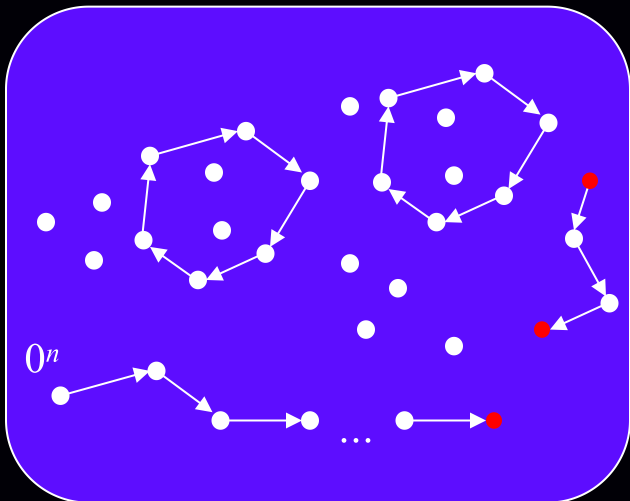
$$0_1 = (2, 2, 2)$$

$$0_1' = (6, 2, 2)$$

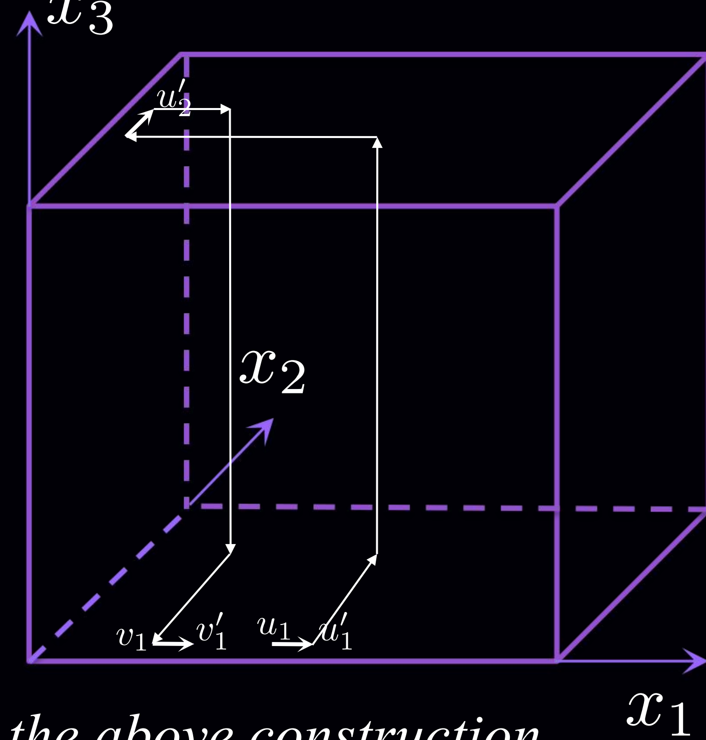
$$0_3 = (6, 6, 2)$$

This is not necessary for the embedding of the PPAD graph to the cube, but will be crucial later in the definition of the Sperner instance…

Finishing the Embedding



Generic PPAD



Call L the orthonormal line defined by the above construction.

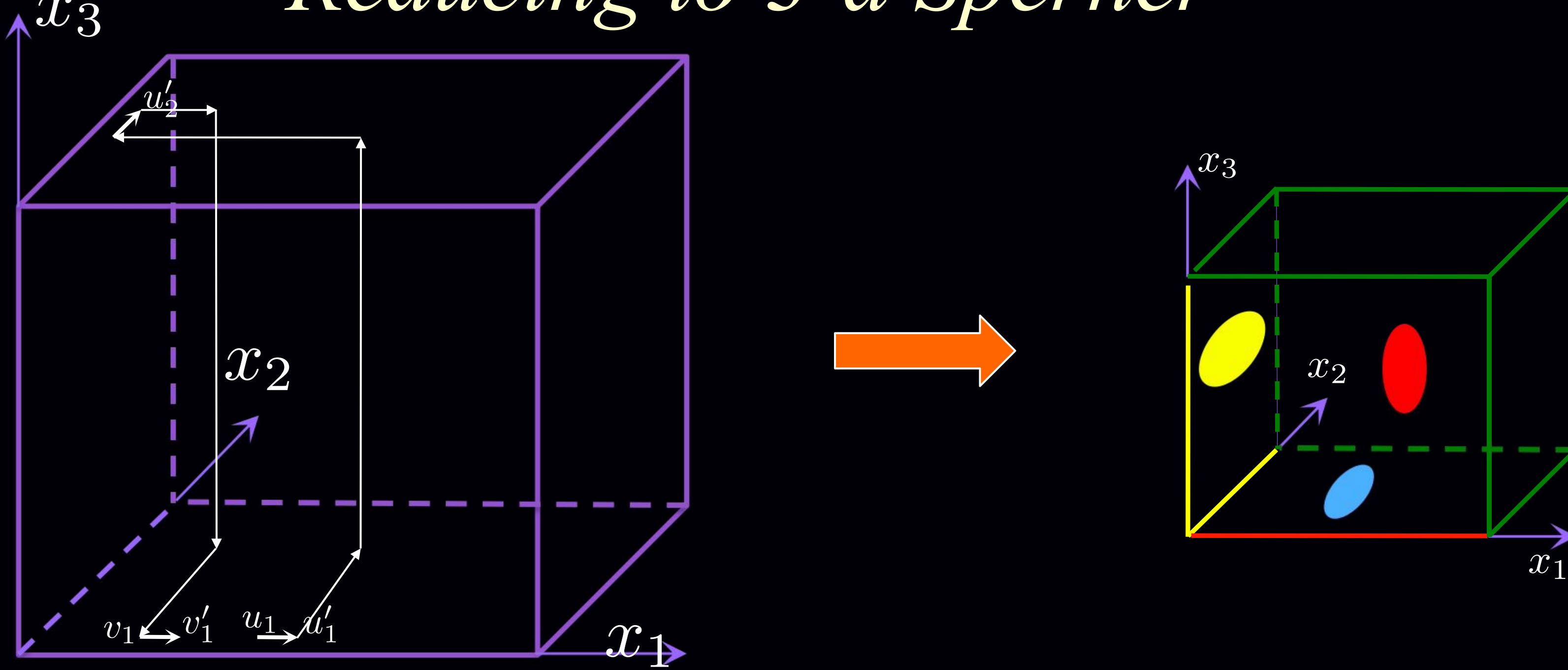
Claim 1: Two points p, p' of L are closer than $3{\cdot}2^{-m}$ in Euclidean distance only if they are connected by a part of L that has length $8{\cdot}2^{-m}$ or less.

Claim 2: Given the circuits P, N of the END OF THE LINE instance, and a point x in the cube, we can decide in polynomial time if x belongs to L.

Claim 3: u is a sink in PPAD graph $\Leftrightarrow$ L is disconnected at u'_2

u is a source in PPAD graph $\Leftrightarrow$ L is disconnected at u_1

Reducing to 3-d Sperner

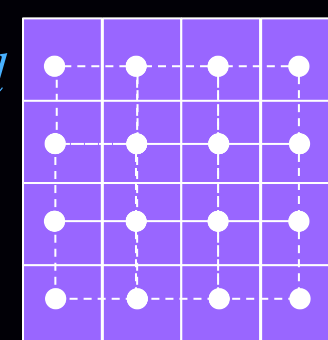


For convenience we reduce to dual-SPERNER

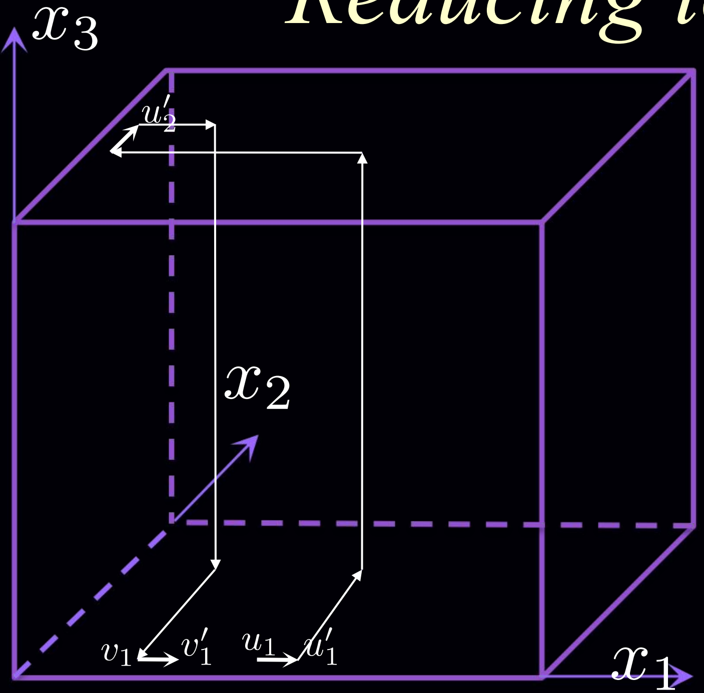
Differences between dual-SPERNER and SPERNER:

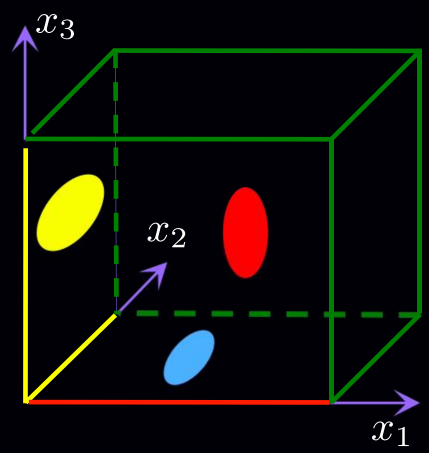
a) Instead of coloring vertices of the subdivision (the points of the cube whose coordinates are integer multiples of 2^{-m}), color the centers of the cubelets; i.e. work with simplicization of the dual graph. For convenience define:

$$K_{ijk} \;:\; \text{center of cubelet whose least significant corner has coordinates } (i,j,k)\cdot 2^{-m}$$



Reducing to 3-d Sperner





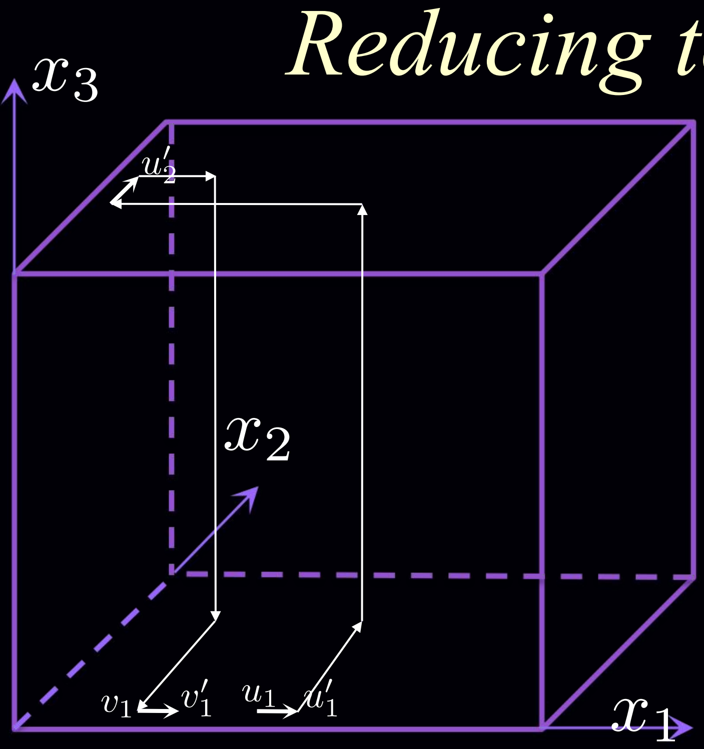
For convenience we reduce to dual-SPERNER

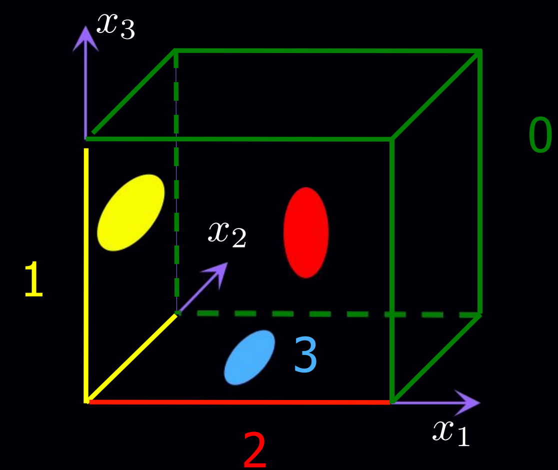
Differences between dual-SPERNER and SPERNER:

b) Solution to dual-SPERNER: a vertex of the subdivision such that all colors are present among the centers of the cubelets using this vertex as a corner. Such vertex is called panchromatic.

Lemma: If the canonical simplicization of the dual graph has a panchromatic simplex, then this simplex contains a vertex of the subdivision that is panchromatic.

Reducing to 3-d Sperner





For convenience we reduce to dual-SPERNER

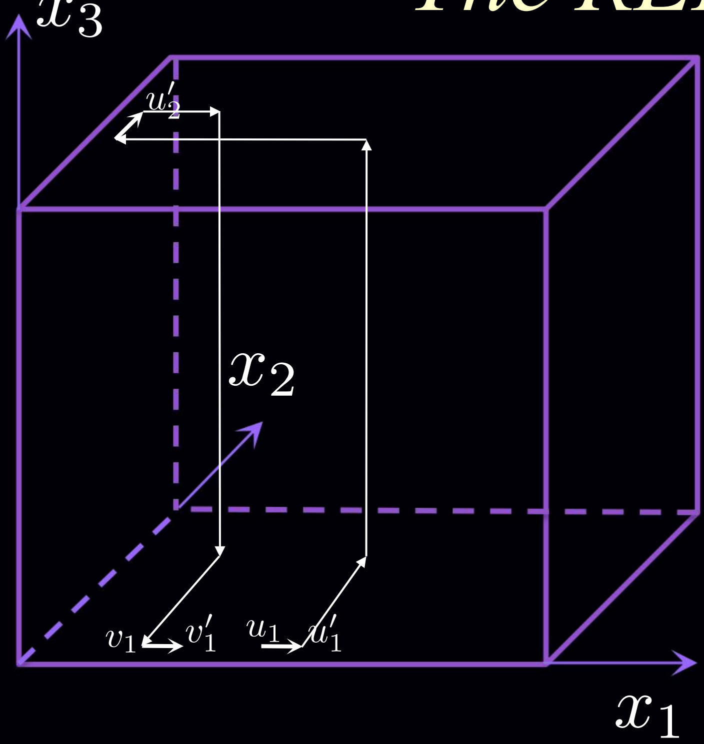
Differences between dual-SPERNER and SPERNER:

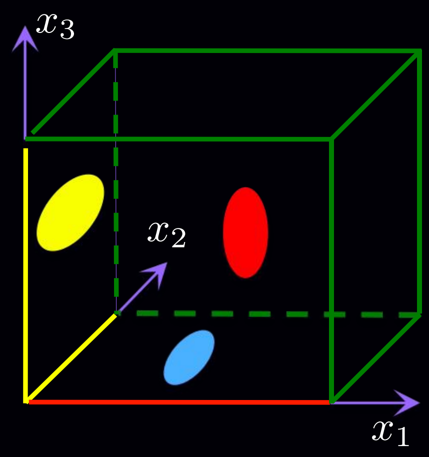
c) Canonical boundary coloring is (for convenience) slightly different than before, as per the following coloring algorithm (see also figure):

$$\left.\begin{array}{ll} K_{ijk} \leftarrow 0, & \text{if any of } i,\ j,\ k \text{ is } 2^m-1 \\ K_{ijk} \leftarrow 1, & \text{if } i=0\text{, unless already colored} \\ K_{ijk} \leftarrow 2, & \text{if } j=0\text{, unless already colored} \\ K_{ijk} \leftarrow 3, & \text{if } k=0\text{, unless already colored} \end{array}\right\}$$

Lemma: Modified boundary coloring still guarantees existence of panchromatic simplex.

The REDUCTION



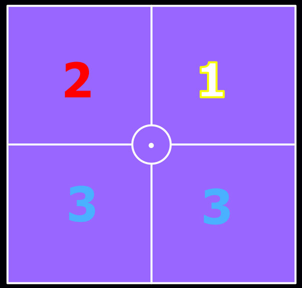


dual-SPERNER

Coloring INSIDE: *All cubelets get color **0**, unless they touch line L.*

*The cubelets surrounding line L at any given point are colored with colors **1**, 2, 3 in a way that "protects" the line from touching color 0.*

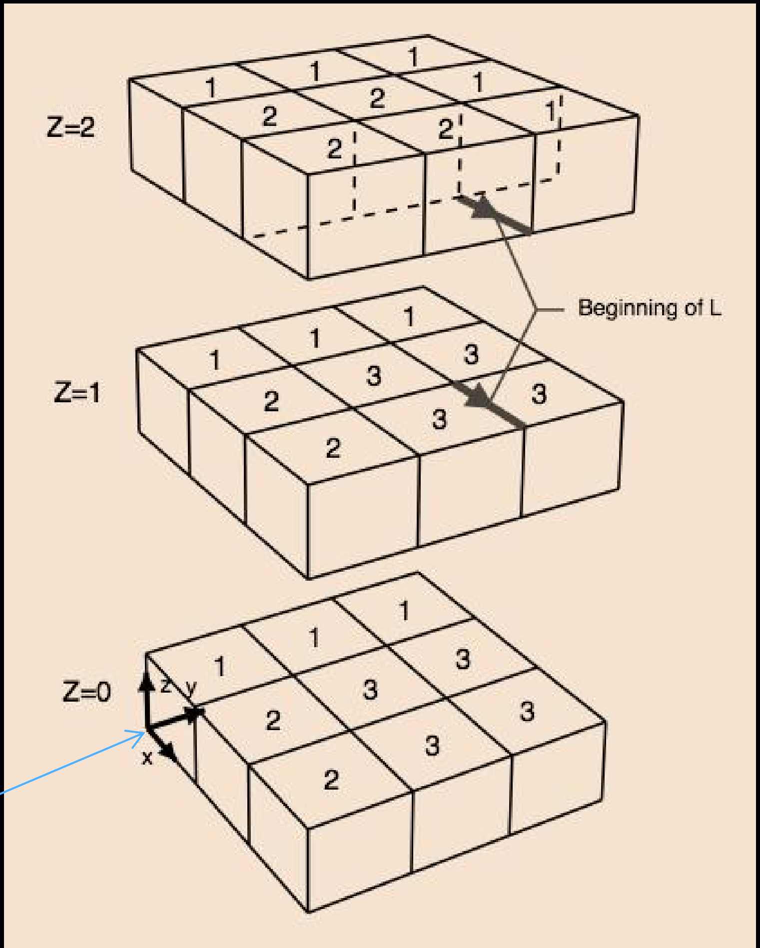
Coloring around L



colors 1, 2, 3 are placed in a clockwise arrangement for an observer who is walking on L

two out of four cubelets are colored 3, one is colored 1 and the other is colored 2

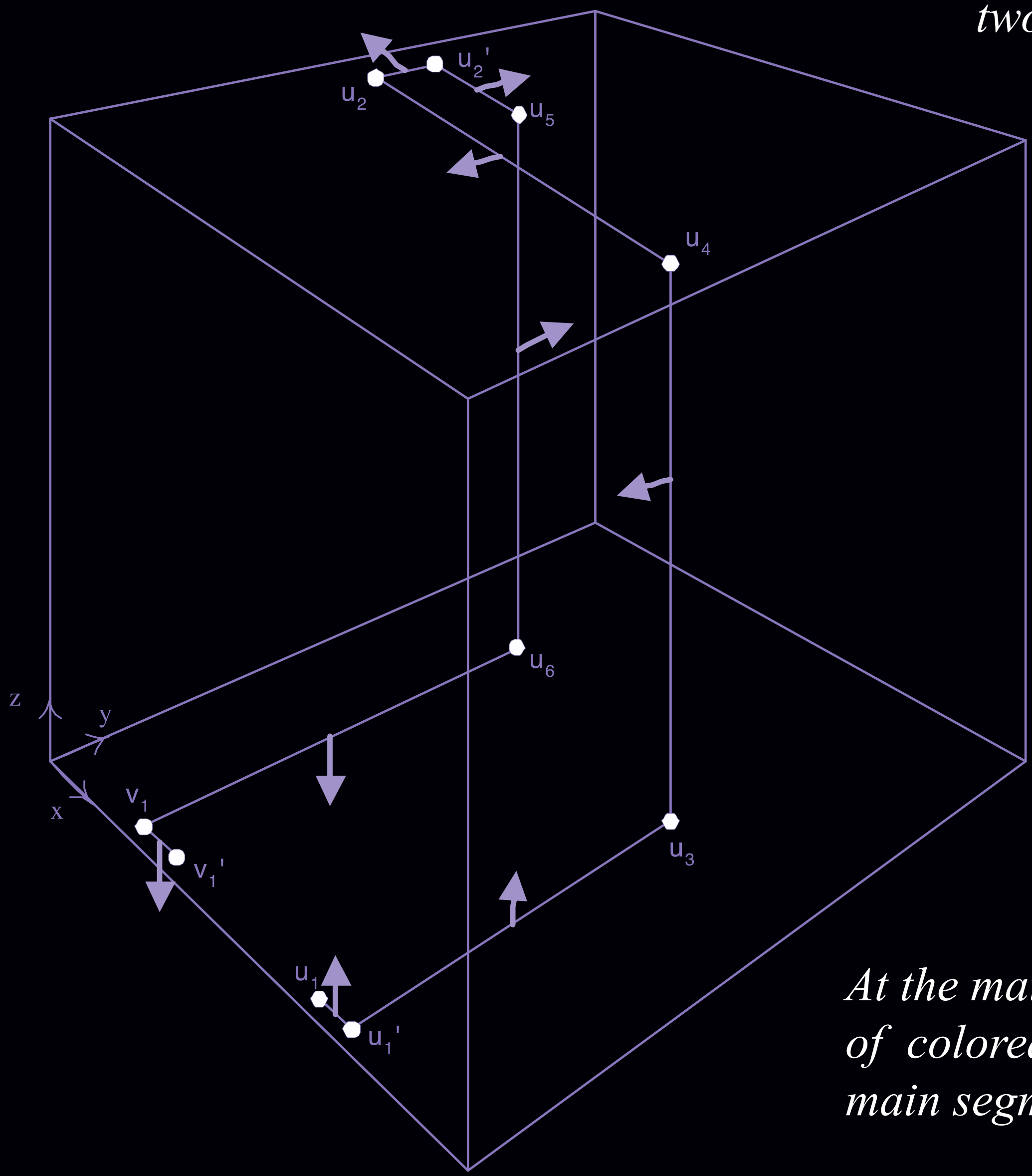
The Beginning of L at 0^n



$(0, 0, 0)$

notice that given the coloring of the cubelets around the beginning of L (on the left), there is no point of the subdivision in the proximity of these cubelets surrounded by all four colors…

Coloring at the Turns..



Out of the four cubelets around L which two are colored with color 3 ?

- in the figure on the left, the arrow points to the direction in which the two cubelets colored 3 lie;

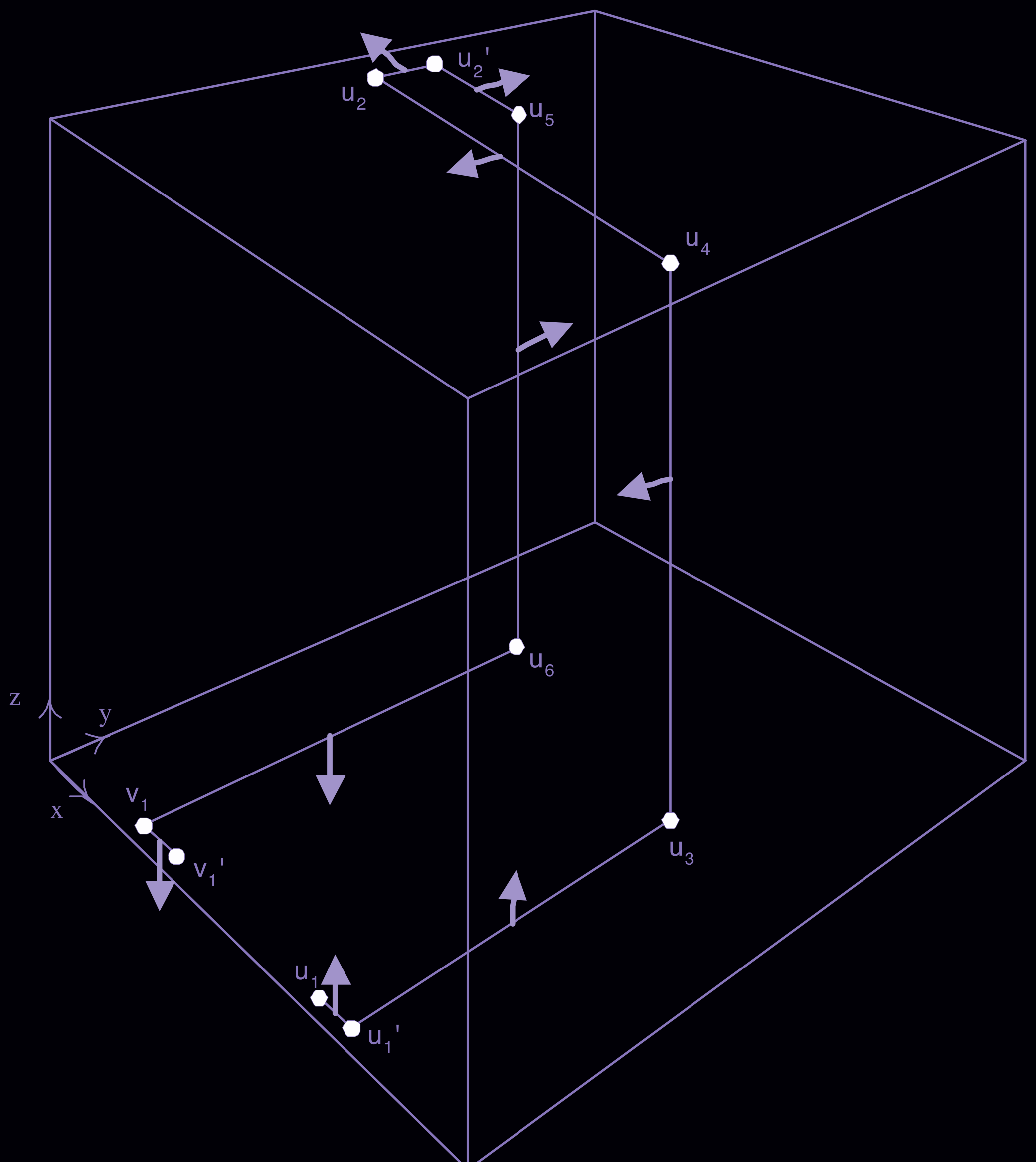
- observe also the way the turns of L affect the location of these cubelets with respect to L; our choice makes sure that no panchromatic vertices arise at the turns.

IMPORTANT directionality issue:

The picture on the left shows the evolution of the location of the pair of colored 3 cubelets along the subset of L corresponding to an edge (u, v) of the PPAD graph…

At the main segment corresponding to u the pair of colored 3 cubelets lies above L, while at the main segment corresponding to v they lie below L.

Coloring at the Turns..



the flip in the directions makes it impossible to efficiently decide locally where the colored 3 cubelets should lie!

Claim1: This is W.L.O.G.

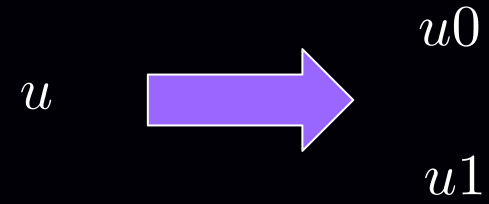
to resolve this we assume that all edges (u,v) of the PPAD graph join an odd u (as a binary number) with an even v (as a binary number) or vice versa

for even u's we place the pair of 3-colored cubelets below the main segment of u, while for odd u's we place it above the main segment

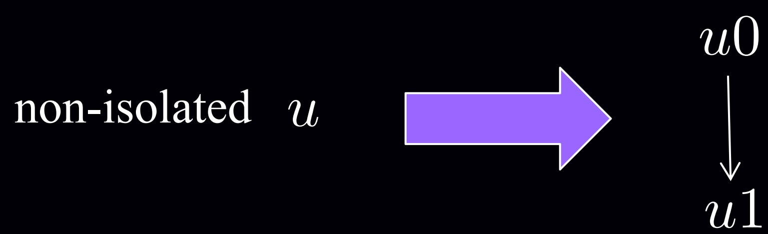
convention agrees with coloring around main segment of 0^n

Proof of Claim of Previous Slide

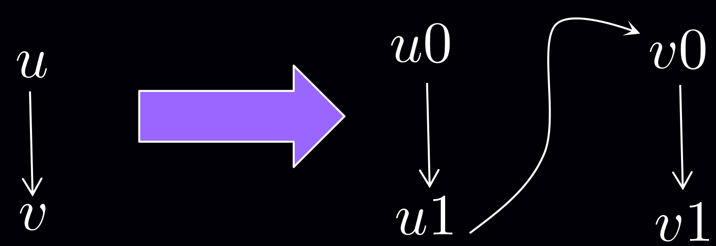
- Duplicate the vertices of the PPAD graph



- If node u is non-isolated include an edge from the 0 to the 1 copy



- Edges connect the 1-copy of a node to the 0-copy of its out-neighbor



Finishing the Reduction

Claim 1: *A point in the cube is panchromatic in the constructed coloring iff it is:*

- *an endpoint u_2' of a sink vertex u of the PPAD graph, or*
- *an endpoint u_1 of a source vertex $u \neq 0^n$ of the PPAD graph.*

Claim 2: *Given the description P, N of the PPAD graph, there is a polynomial-size circuit computing the coloring of every cubelet* K_{ijk}.