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New sine ellipsoids and related volume inequalities ☆

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Corresponding to the Legendre ellipsoid and the LYZ ellipsoid, two new sine ellipsoids are introduced in this paper. These four ellipsoids are closely related in the Pythagorean relation and duality. Several volume inequalities and the valuation properties are obtained for two new ellipsoids.

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1. Introduction

It is well acknowledged that ellipsoids play an important role in convex geometric analysis. As extremals, they often appear in (affine) isoperimetric type problems and other extremal problems (see, e.g., [1,2,20,41,42,46,51,53,54,73]). For instance, the celebrated *John ellipsoid* (or *Löwner ellipsoid*) associated with each convex body is the unique ellipsoid of maximal volume contained in the body (or minimal volume containing the body), which is extremely useful in convex geometric analysis and Banach space geometry. In particular, the isotropic characterization of the John ellipsoid was perfectly combined with the Brascamp-Lieb inequality by Ball to solve reverse isoperimetric problems that usually have simplices or, in the symmetric case, cubes and their polars, as extremals (see, e.g., [1–3,45,47,50,63]).

Among these ellipsoids there are some relationships, such as polarity or duality. In this paper, we shall search for a new relationship among ellipsoids. The duality is an important relation between convex bodies. In modern convex geometric analysis, the L_p Brunn-Minkowski theory and the dual Brunn-Minkowski theory are two fundamental ingredients, which generalize and dualize the classical Brunn-Minkowski theory. The L_p Brunn-Minkowski theory began from L_p Minkowski-Firey combinations [11] in the 1960's and came to life when Lutwak [40,41] introduced the concept of L_p *surface area measure* in the 1990's. The dual Brunn-Minkowski theory initiated by Lutwak [38] in the 1970's as the dual theory to the classical Brunn-Minkowski theory is based on a conceptual duality in convex geometric analysis. Since then, the L_p Brunn-Minkowski theory and its dual theory have expanded rapidly over the last three decades; see, e.g., [6,8,9,14,18,20,22–24,31–33,36,39,42,45–49,64–72]. For more details about both theories, we refer the reader to [57, Chapter 9] and the references therein.

Associated with each star body K in n-dimensional Euclidean space $\mathbb{R}^n$, there is a unique ellipsoid $\Gamma_2 K$ which has the same moment of inertia of K with respect to every 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the *Legendre ellipsoid*, whose support function is defined by, for $x \in \mathbb{R}^n$,

$$h^2_{\Gamma_2 K}(x) = \frac{n+2}{V(K)} \int_K |x \cdot y|^2 dy, \tag{1.1}$$

where $V(K)$ denotes the n-dimensional volume of K and $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. The Legendre ellipsoid and its polar body (the Binet ellipsoid) are well-known concepts from classical mechanics and are closely related to the long-standing unsolved maximal slicing problem; see, e.g., [28,30,51] for more information. Recall that a convex body L in $\mathbb{R}^n$ is a compact convex set with nonempty interiors. Its support function h_L is defined as $h_L(x) = \max\{x \cdot y : y \in L\}$. Let $\mathcal{K}^n_o$ denote the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. The polar body L^* of $L \in \mathcal{K}^n_o$ is defined by

$$L^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \quad \text{for all } y \in L\}. \tag{1.2}$$

A star body Q in $\mathbb{R}^n$ is a compact star-shaped set about the origin whose radial function $\rho_Q(x) = \max\{\lambda \geq 0 : \lambda x \in Q\}$ is positive and continuous. Denote the set of star bodies in $\mathbb{R}^n$ by $\mathcal{S}_o^n$. Obviously, $\mathcal{K}_o^n \subset \mathcal{S}_o^n$.

Note that the Legendre ellipsoid is an object in the dual Brunn-Minkowski theory. By this observation, Lutwak, Yang, and Zhang [43] introduced a new dual analog of the Legendre ellipsoid by using the notion of L_2 surface area measure within the L_p Brunn-Minkowski theory. This ellipsoid is now called the *LYZ ellipsoid* $\Gamma_{-2}K$ whose radial function at $x \in \mathbb{R}^n \backslash \{o\}$ is given by

$$\rho_{\Gamma_{-2}K}^{-2}(x) = \frac{1}{V(K)} \int\limits_{S^{n-1}} |x \cdot v|^2 dS_2(K, v), \tag{1.3}$$

where $S_2(K, \cdot)$ is the L_2 surface area measure of $K \in \mathcal{K}_o^n$ on the unit sphere S^{n-1}.

The L_2 surface area measure of K, also called the quadratic surface area measure, was further investigated by Lutwak, Yang, and Zhang [47] to establish a sharp reverse affine isoperimetric inequality which states that the reciprocal of the volume of $\Gamma_{-2}K$ provides a greatest lower bound for the volume of K^*. Moreover, the domain of the operator Γ_{-2} can be extended to star-shaped sets, which leads to an elegant inclusion [44]: if K is a star-shaped set in $\mathbb{R}^n$, then

$$\Gamma_{-2}K \subset \Gamma_2 K,$$

with equality if and only if K is an origin-centered ellipsoid. This inclusion is the geometric analog of one of the basic inequalities in information theory—the Cramér-Rao inequality (see, e.g., [10]).

It is well-known that the notion of valuation played a critical role in Dehn's solution of Hilbert's Third Problem. A systematic study of valuations was initiated by Hadwiger [21] who obtained his famous classification of continuous, rigid motion invariant valuations and characterization of mixed volumes. Over the past two decades the theory of valuations has become an ever more important part of convex geometric analysis (see, e.g., [7,17–19,31–37,52,58–62]). In particular, Ludwig [32] showed that the matrices corresponding to the Legendre ellipsoid and the LYZ ellipsoid are the only matrix-valued valuations on convex polytopes that are $\mathrm{GL}(n)$ covariant.

It is worth mentioning that the LYZ ellipsoid as well as the John ellipsoid belongs to a family of ellipsoids introduced by Lutwak, Yang and Zhang [46]. These ellipsoids are called the L_p John ellipsoids which provide a unified treatment for several fundamental objects in convex geometric analysis, for instance, the John ellipsoid ($p = \infty$), the LYZ ellipsoid ($p = 2$), and the Petty ellipsoid ($p = 1$). Recently, the Orlicz extension of the L_p John ellipsoids was established in [73].

The classical Pythagorean theorem states that for $x, y \in \mathbb{R}^n$

$$|x \cdot y|^2 + [x, y]^2 = |x|^2 |\mathrm{P}_x y|^2 + |x|^2 |\mathrm{P}_{x^\perp} y|^2 = |x|^2 |y|^2, \tag{1.4}$$

where $|x|$ denotes the Euclidean norm of x and $[x, y]$ denotes the 2-dimensional volume of the parallelepiped spanned by x, y. Here $\mathrm{P}_x y$, $\mathrm{P}_{x^\perp} y$ are the images of orthogonal projection of y onto the 1-dimensional subspace containing x and the 1-codimensional subspace $x^\perp$ perpendicular to x, respectively. This fundamental theorem enlightens us to introduce two new ellipsoids $\Lambda_2 K$ and $\Lambda_{-2} K$ by replacing $|x \cdot y|$ by $[x, y]$ in (1.1) and (1.3).

For each $K \in \mathcal{S}_o^n$, we define the ellipsoid $\Lambda_2 K$ whose support function at $x \in \mathbb{R}^n$ is given by

$$h^2_{\Lambda_2 K}(x) = \frac{n+2}{V(K)} \int\limits_K [x, y]^2 dy. \tag{1.5}$$

For each $K \in \mathcal{K}_o^n$, we define the ellipsoid $\Lambda_{-2} K$ whose radial function at $x \in \mathbb{R}^n \backslash \{o\}$ is given by

$$\rho^{-2}_{\Lambda_{-2} K}(x) = \frac{1}{V(K)} \int\limits_{S^{n-1}} [x, v]^2 dS_2(K, v). \tag{1.6}$$

The ellipsoids $\Lambda_2 K$ and $\Lambda_{-2} K$ are well defined because $h^2_{\Lambda_2 K}(x)$ and $\rho^{-2}_{\Lambda_{-2} K}(x)$ are positive definite quadratic forms in the variable x (see (2.19) and (2.21)). Notice that the operation $|x \cdot y|$ of $x, y \in \mathbb{R}^n$ is related to the cosine transform, but $[x, y]$ is related to the sine transform. It is reasonable to call $\Lambda_2 K$ and $\Lambda_{-2} K$ the *sine ellipsoid* of the Legendre ellipsoid and the LYZ ellipsoid, respectively. The cosine transform (a spherical variant of the Fourier transform) of a finite Borel measure on S^{n-1} turns out to yield a finite dimensional Banach norm, which gives a natural analytical operator in convex geometric analysis. Important applications of these integral transforms were presented in, e.g., [5,12,15,16,26,27,29,42,55,67]. The sine transform, appearing in different forms in geometric tomography, was applied successfully by Maresch and Schuster [50] to establish reverse isoperimetric inequalities with asymptotically optimal forms. Furthermore, both transforms were generalized and unified to the L_p cosine transform on Grassmann manifolds in [29].

These four ellipsoids, as well as their polars, have the following "Pythagorean" relations in terms of the L_2 Minkowski-Firey combination $+_2$ and the L_2 harmonic radial combination $\widetilde{+}_{-2}$:

$$\Gamma_2 K +_2 \Lambda_2 K = c_1 B^n \quad \text{and} \quad \Gamma_2^* K \widetilde{+}_{-2} \Lambda_2^* K = \tfrac{1}{c_1} B^n; \tag{1.7}$$

$$\Gamma_{-2} K \widetilde{+}_{-2} \Lambda_{-2} K = \frac{1}{c_2} B^n \quad \text{and} \quad \Gamma_{-2}^* K +_2 \Lambda_{-2}^* K = c_2 B^n. \tag{1.8}$$

Here B^n denotes the unit ball of $\mathbb{R}^n$, and $c_1^2 = (n+2)\int_K |y|^2 dy/V(K)$, $c_2^2 = S_2(K)/V(K)$. Thus, we can illustrate the relationships of these four ellipsoids in the following diagram.

$$\begin{tikzcd}
\Lambda_{-2}K \arrow[r, "\text{Pythagorean relation}"] & \Gamma_{-2}K \\
\Lambda_2 K \arrow[u, "\text{Duality}"] \arrow[r, "\text{Pythagorean relation}"] & \Gamma_2 K \arrow[u, "\text{Duality}"]
\end{tikzcd}$$

The relations between the volumes of the ellipsoids $\Gamma_2 K$, $\Gamma_{-2}K$ and the body K are demonstrated in the following inequalities:

$$V(\Gamma_2 K) \geq V(K) \tag{1.9}$$

for each $K \in \mathcal{S}_o^n$;

$$V(\Gamma_{-2} K) \leq V(K) \tag{1.10}$$

for each $K \in \mathcal{K}_o^n$. Equality holds in both inequalities if and only if K is an origin-centered ellipsoid. Inequality (1.9) goes back to Blaschke [4], John [25], Petty [53], and Milman, Pajor [51]. Both inequalities were also obtained by Lutwak, Yang, and Zhang in [43].

In this paper we shall establish the following analogs of inequalities (1.9) and (1.10) for the sine ellipsoids $\Lambda_2 K$ and $\Lambda_{-2}K$.

Theorem 1.1. *If $K \in \mathcal{S}_o^n$, then*

$$V(\Lambda_2 K) \geq (n-1)^{\frac{n}{2}} V(K). \tag{1.11}$$

If $K \in \mathcal{K}_o^n$, then

$$V(\Lambda_{-2} K) \leq (n-1)^{-\frac{n}{2}} V(K). \tag{1.12}$$

Equality holds in both inequalities if and only if K is an origin-centered ball when $n \geq 3$ and is an origin-centered ellipsoid when $n = 2$.

To prove Theorem 1.1, we introduce two new operators T_p and T_{-p} with affine natures in Sections 4 and 5 where the operators Λ_2 and Λ_{-2} are their Euclidean specializations (see (4.7) and (5.5)). Such specialization that parts of entries are replaced by Euclidean balls is an important technique in convex geometric analysis, and a good example is the relation between mixed volumes and quermassintegrals (see, e.g., [57]). Using the affine natures of T_2 and T_{-2}, two affine invariants are constructed to establish the expected sharp inequalities.

Note that when $n = 2$, inequalities (1.11) and (1.12) are affine invariant. Moreover, in plane, our new ellipsoids are just rotations of the Legendre ellipsoid and the LYZ ellipsoid by an angle $\pi/2$, respectively. See Remarks 4.6 and 5.6 below for details.

Moreover, our new ellipsoids have the following valuation properties.

Theorem 1.2. *The operator $\widetilde{\Lambda_2}$ is an L_2 Minkowski valuation over $\mathcal{S}_o^n$ and the operator $\widetilde{\Lambda_{-2}}$ is an L_{-2} radial valuation over $\mathcal{K}_o^n$, where $\widetilde{\Lambda_2}K = V(K)^{1/2}\Lambda_2 K$ and $\widetilde{\Lambda_{-2}}K = V(K)^{-1/2}\Lambda_{-2}K$.*

This paper is organized as follows: In Section 2 some background materials are provided. Section 3 contains several volume inequalities and auxiliary lemmas. The proof of Theorem 1.1 is presented in Sections 4 and 5. In Section 6, the valuation properties of two sine ellipsoids are given. Finally, two open problems will be posed in the last section.

2. Background materials

For quick later reference we collect some background materials from the L_p Brunn-Minkowski theory and its dual theory. Good general references are Gardner [13] and Schneider [57].

Throughout $\mathbb{R}^n$ denotes n-dimensional Euclidean space ($n \geq 2$). For $x \in \mathbb{R}^n$, let $|x|$ be the Euclidean norm of x. The unit ball of $\mathbb{R}^n$ is denoted by B^n, and the unit sphere by S^{n-1}. Write ω_n for the volume of B^n. A convex body K in $\mathbb{R}^n$ is a compact convex set with nonempty interiors. Its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}. \tag{2.1}$$

It is easy to see that for $x \in \mathbb{R}^n$ and $\phi \in \mathrm{GL}(n)$

$$h_{\phi K}(x) = h_K(\phi^t x), \tag{2.2}$$

where $\phi K = \{\phi x : x \in K\}$ is the image of K under ϕ and ϕ^t denotes the transpose of ϕ. In particular, for $c > 0$ and $x \in \mathbb{R}^n$,

$$h_{cK}(x) = c h_K(x), \tag{2.3}$$

where $cK = \{cx : x \in K\}$.

A set $K \subset \mathbb{R}^n$ is star-shaped about the origin if the line segment joining each point of K and the origin is completely contained in K. If K is compact and star-shaped, then the radial function $\rho_K : \mathbb{R}^n \setminus \{o\} \to [0, \infty)$ of K is defined for $x \in \mathbb{R}^n \setminus \{o\}$ by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}. \tag{2.4}$$

A star body is a compact star-shaped set about the origin whose radial function is positive and continuous. It is easy to see that for $x \in \mathbb{R}^n \setminus \{o\}$ and $\phi \in \mathrm{GL}(n)$,

$$\rho_{\phi K}(x) = \rho_K(\phi^{-1}x), \tag{2.5}$$

where ϕ^{-1} denotes the inverse of ϕ. In particular, for $c > 0$ and $x \in \mathbb{R}^n \backslash \{o\}$,

$$\rho_{cK}(x) = c\rho_K(x). \tag{2.6}$$

If $K \in \mathcal{K}_o^n$, then it follows from (2.1), (1.2), and (2.4) that

$$h_{K^*} = 1/\rho_K \quad \text{and} \quad \rho_{K^*} = 1/h_K. \tag{2.7}$$

It is easy to verify that for $\phi \in \mathrm{GL}(n)$,

$$(\phi K)^* = \phi^{-t} K^*, \tag{2.8}$$

and in particular for $c > 0$,

$$(cK)^* = \frac{1}{c} K^*. \tag{2.9}$$

Elements of the L_p Brunn-Minkowski theory. For $p \geq 1$ and $\varepsilon > 0$, the L_p Minkowski-Firey combination $K +_p \varepsilon \cdot L$ of $K, L \in \mathcal{K}_o^n$ is the convex body whose support function is given by

$$h^p_{K +_p \varepsilon \cdot L}(\cdot) = h^p_K(\cdot) + \varepsilon h^p_L(\cdot). \tag{2.10}$$

The L_p mixed volume $V_p(K, L)$ of K, L is defined by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

In particular,

$$V_p(K, K) = V(K).$$

It was shown in [40] that corresponding to each $K \in \mathcal{K}_o^n$, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1}, such that for each $L \in \mathcal{K}_o^n$,

$$V_p(K, L) = \frac{1}{n} \int\limits_{S^{n-1}} h^p_L(v) dS_p(K, v),$$

where $dS_p(K, \cdot) = h_K(\cdot)^{1-p} dS(K, \cdot)$ is the L_p surface area measure of K. The measure $S(K, \cdot)$ is the classical surface area measure of K; i.e., for a Borel set $\omega \subset S^{n-1}$, $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω. Thus, for $c > 0$,

$$S_p(cK,\cdot)=c^{n-p}S_p(K,\cdot), \tag{2.11}$$

and the L_p surface area $S_p(K)$ is

$$S_p(K)=\int\limits_{S^{n-1}} dS_p(K,v)=\int\limits_{S^{n-1}} h_K(v)^{1-p}dS(K,v). \tag{2.12}$$

The L_p Brunn-Minkowski inequality [40] states that if $K,L\in\mathcal{K}_o^n$, then for $p\geq 1$,

$$V(K+_p L)^{p/n}\geq V(K)^{p/n}+V(L)^{p/n}, \tag{2.13}$$

with equality if and only if K and L are dilates when $p>1$ and are homothetic when $p=1$.

Elements of the dual Brunn-Minkowski theory. For $p\in\mathbb{R}\backslash\{0\}$ and $\varepsilon>0$, the L_p harmonic radial combination $K\widetilde{+}_{-p}\ \varepsilon\cdot L$ of $K,L\in\mathcal{S}_o^n$ is the star body whose radial function is given by

$$\rho^{-p}_{K\widetilde{+}_{-p}\ \varepsilon\cdot L}(\cdot)=\rho_K^{-p}(\cdot)+\varepsilon\rho_L^{-p}(\cdot). \tag{2.14}$$

The dual L_p mixed volume $\widetilde{V}_{-p}(K,L)$ of $K,L\in\mathcal{S}_o^n$ was defined in [41] by

$$-\frac{n}{p}\widetilde{V}_{-p}(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K\widetilde{+}_{-p}\ \varepsilon\cdot L)-V(K)}{\varepsilon}.$$

In particular,

$$\widetilde{V}_{-p}(K,K)=V(K).$$

The polar coordinate formula for volume yields the following integral representation

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int\limits_{S^{n-1}}\rho_K^{n+p}(v)\rho_L^{-p}(v)dv,$$

where the integration is with respect to the spherical Lebesgue measure.

The dual L_p Brunn-Minkowski inequality [41] states that if $K,L\in\mathcal{S}_o^n$, then for $p>0$,

$$V(K\widetilde{+}_{-p}L)^{-p/n}\geq V(K)^{-p/n}+V(L)^{-p/n}, \tag{2.15}$$

with equality if and only if K and L are dilates.

The operators Γ_2 and Γ_{-2} have the following affine natures (see [43]): for each $\phi\in\mathrm{GL}(n)$,

$$\Gamma_2(\phi K)=\phi\Gamma_2 K, \tag{2.16}$$

and

$$\Gamma_{-2}(\phi K)=\phi\Gamma_{-2}K. \tag{2.17}$$

Let A be a positive definite $n\times n$ symmetric matrix. It is well-known that the radial function and support function of the ellipsoid $E(A)=\{y\in\mathbb{R}^n : y\cdot Ay\le 1\}$ at $x\in\mathbb{R}^n$ are given by

$$h^2_{E(A)}(x)=x\cdot A^{-1}x \quad\text{and}\quad \rho^{-2}_{E(A)}(x)=x\cdot Ax. \tag{2.18}$$

By (1.5) and (1.4), it follows that

$$\Lambda_2 K=\sqrt{\frac{n+2}{V(K)}}E(D_2(K)^{-1}), \tag{2.19}$$

with the matrix $D_2(K)$ having entries

$$\int_K (|y|^2\delta_{ij}-y_iy_j)dy, \tag{2.20}$$

where we use coordinates $y=(y_1,\ldots,y_n)$ for $\mathbb{R}^n$ and δ_{ij} for the Kronecker delta. By (1.6) and (1.4), we also have

$$\Lambda_{-2}K=\sqrt{V(K)}E(D_{-2}(K)), \tag{2.21}$$

where the matrix $D_{-2}(K)$ has entries

$$\int_{S^{n-1}}(\delta_{ij}-v_iv_j)dS_2(K,v). \tag{2.22}$$

3. Several volume inequalities and auxiliary lemmas

First, we shall prove identities (1.7) and (1.8). By (2.10), (1.1), (1.5), (1.4), and (2.3), we have, for $u\in S^{n-1}$,

$$h^2_{\Gamma_2K+_2\Lambda_2K}(u)=h^2_{\Gamma_2K}(u)+h^2_{\Lambda_2K}(u)=\frac{n+2}{V(K)}\int_K|y|^2dy=h^2_{c_1B^n}(u),$$

and by (2.14), (2.7), and (2.9),

$$\rho^{-2}_{\Gamma_2^*K\tilde{+}_{-2}\Lambda_2^*K}(u)=\rho^{-2}_{\Gamma_2^*K}(u)+\rho^{-2}_{\Lambda_2^*K}(u)=h^2_{\Gamma_2K}(u)+h^2_{\Lambda_2K}(u)=h^2_{c_1B^n}(u)=\rho^{-2}_{c_1^{-1}B^n}(u),$$

where $c_1^2 = (n+2)\int_K |y|^2 dy/V(K)$. Thus, (1.7) follows. Similarly, by (2.14), (1.3), (1.6), (1.4), (2.12), and (2.6), we have, for $u \in S^{n-1}$,

$$\rho^{-2}_{\Gamma_{-2}K\tilde{+}_{-2}\Lambda_{-2}K}(u) = \rho^{-2}_{\Gamma_{-2}K}(u) + \rho^{-2}_{\Lambda_{-2}K}(u) = \frac{S_2(K)}{V(K)} = \rho^{-2}_{c_2^{-1}B^n}(u),$$

and by (2.10), (2.7), and (2.9),

$$\begin{aligned} h^2_{\Gamma^*_{-2}K+_2\Lambda^*_{-2}K}(u) &= h^2_{\Gamma^*_{-2}K}(u) + h^2_{\Lambda^*_{-2}K}(u) = \rho^{-2}_{\Gamma_{-2}K}(u) + \rho^{-2}_{\Lambda_{-2}K}(u) = \rho^{-2}_{c_2^{-1}B^n}(u) \\ &= h^2_{c_2B^n}(u), \end{aligned}$$

where $c_2^2 = S_2(K)/V(K)$. Hence, we obtain (1.8).

A star body K is said to be in *isotropic position* if $\Gamma_2 K$ is a ball with $V(K) = 1$ (see, e.g., [51]), and a convex body K is said to be in *dual isotropic position* [43] if $\Gamma_{-2}K$ is a ball with $V(K) = 1$.

Identities (1.7) and (1.8), together with the L_2 Brunn-Minkowski inequality (2.13) ($p = 2$) and the dual L_2 Brunn-Minkowski inequality (2.15) ($p = 2$), immediately yield the following volume inequalities.

Theorem 3.1. *If $K \in \mathcal{S}^n_o$, then*

$$V(\Gamma_2 K)^{2/n} + V(\Lambda_2 K)^{2/n} \le c_1^2\omega_n^{2/n}, \tag{3.1}$$

and

$$V(\Gamma_2^* K)^{-2/n} + V(\Lambda_2^* K)^{-2/n} \le c_1^2\omega_n^{-2/n}, \tag{3.2}$$

with equalities if and only if the star body $K/V(K)^{1/n}$ is in isotropic position. If $K \in \mathcal{K}^n_o$, then

$$V(\Gamma_{-2} K)^{-2/n} + V(\Lambda_{-2} K)^{-2/n} \le c_2^2\omega_n^{-2/n}, \tag{3.3}$$

and

$$V(\Gamma_{-2}^* K)^{2/n} + V(\Lambda_{-2}^* K)^{2/n} \le c_2^2\omega_n^{2/n}, \tag{3.4}$$

with equalities if and only if the convex body $K/V(K)^{1/n}$ is in dual isotropic position.

Proof. Only the equality conditions need to be verified. The equality conditions of the L_2 Brunn-Minkowski inequality (2.13) show that equality in (3.1) holds if and only if $\Gamma_2 K$ and $\Lambda_2 K$ are dilates. But the relation (1.7) implies that the ellipsoids $\Gamma_2 K$ and $\Lambda_2 K$ must be balls. Thus, the star body $K/V(K)^{1/n}$ is in isotropic position.

The equality conditions of the dual L_2 Brunn-Minkowski inequality (2.15) show that equality in (3.3) holds if and only if $\Gamma_{-2}K$ and $\Lambda_{-2}K$ are dilates. But the relation (1.8) implies that the ellipsoids $\Gamma_{-2}K$ and $\Lambda_{-2}K$ must be balls. Thus, the convex body $K/V(K)^{1/n}$ is in dual isotropic position.

The proofs of equality conditions of (3.2) and (3.4) are similar. $\square$

Denote by $[x, y, v_3 \ldots, v_k]$ the k-dimensional volume of the parallelotope spanned by the vectors $x, y, v_3, \ldots, v_k$ whenever $k \geq 3$. The following lemma is critical for the proof of Theorem 1.1.

Lemma 3.2. *If $n \geq 3$ and $p > 0$, then for any $x, y \in \mathbb{R}^n$ satisfying $[x, y] \neq 0$,*

$$\int_{S^{n-1}} \cdots \int_{S^{n-1}} \frac{[x, y, v_3, \ldots, v_n]^p}{[x, y]^p} dv_3 \cdots dv_n \tag{3.5}$$

is a constant depending on n and p. In particular,

$$\int_{S^{n-1}} \cdots \int_{S^{n-1}} \frac{[x, y, v_3, \ldots, v_n]^2}{[x, y]^2} dv_3 \cdots dv_n = (n-2)!\omega_n^{n-2}. \tag{3.6}$$

Proof. For $x, y \in \mathbb{R}^n$ with $[x, y] \neq 0$ and $v_k \in S^{n-1}$, $k = 3, \cdots, n-1$, let

$$V_2 := \mathrm{span}\{x, y\}, \quad \text{and} \quad V_k := \mathrm{span}\{x, y, v_3, \ldots, v_k\}.$$

If $x, y, v_3, \ldots, v_n$ are linearly independent, then

$$[x, y, v_3, \ldots, v_n] = |x||\mathrm{P}_{x^\perp} y||\mathrm{P}_{V_2^\perp} v_3| \cdots |\mathrm{P}_{V_{n-1}^\perp} v_n|,$$

where $\mathrm{P}_{V_i^\perp}$ is the orthogonal projection onto $V_i^\perp$ (the orthogonal complement of V_i). Define the set Ω_k, $k = 3, \ldots, n-1$, by

$$\Omega_k = \{(x, y, v_3, \ldots, v_k) \in (S^{n-1})^k : [x, y, v_3, \ldots, v_k] \neq 0\}.$$

Note that if $v_k \in \mathrm{span}\{x, y, v_3, \ldots, v_{k-1}\}$, then we have

$$|\mathrm{P}_{V_{k-1}^\perp} v_k| = 0.$$

Hence, we obtain

$$\int_{(S^{n-1})^{n-2}} \frac{[x, y, v_3, \ldots, v_n]^p}{[x, y]^p} dv_3 \cdots dv_n$$

$$
\begin{aligned}
&= \int\limits_{\Omega_n} \frac{(|x||\mathrm{P}_{x^\perp} y||\mathrm{P}_{V_2^\perp} v_3| \cdots |\mathrm{P}_{V_{n-1}^\perp} v_n|)^p}{(|x||\mathrm{P}_{x^\perp} y|)^p} dv_3 \cdots dv_n \\
&= \int\limits_{\Omega_{n-1}} \Big(\int\limits_{S^{n-1}} |\mathrm{P}_{V_{n-1}^\perp} v_n|^p dv_n \Big) |\mathrm{P}_{V_{n-2}^\perp} v_{n-1}|^p \cdots |\mathrm{P}_{V_2^\perp} v_3|^p dv_{n-1} \cdots dv_3 \\
&\quad \cdots \\
&= \int\limits_{S^{n-1}} |\mathrm{P}_{V_{n-1}^\perp} v_n|^p dv_n \int\limits_{S^{n-1}} |\mathrm{P}_{V_{n-2}^\perp} v_{n-1}|^p dv_{n-1} \cdots \int\limits_{S^{n-1}} |\mathrm{P}_{V_2^\perp} v_3|^p dv_3. \qquad (3.7)
\end{aligned}
$$

Note that in the last step $V_i^\perp$ can be seen as a subspace of $\mathbb{R}^n$ with $\dim V_i^\perp = n - i$ for $i = 2, \ldots, n$. Thus, by the rotation invariance of the spherical Lebesgue measure, we have

$$
\int\limits_{S^{n-1}} |\mathrm{P}_{V_i^\perp} v_{i+1}|^p dv_{i+1} = \int\limits_{S^{n-1}} (v_{i+1,1}^2 + \cdots + v_{i+1,n-i}^2)^{p/2} dv_{i+1},
$$

which is a constant only depending on n and p. Here we use coordinates $v_{i+1} = (v_{i+1,1}, \cdots, v_{i+1,n-i}, \ldots, v_{i+1,n})$ for $\mathbb{R}^n$.

In particular, for $p = 2$, we further have

$$
\begin{aligned}
\int\limits_{S^{n-1}} |\mathrm{P}_{V_i^\perp} v_{i+1}|^2 dv_{i+1} &= \int\limits_{S^{n-1}} v_{i+1,1}^2 dv_{i+1} + \cdots + \int\limits_{S^{n-1}} v_{i+1,n-i}^2 dv_{i+1} \\
&= (n-i)\omega_n.
\end{aligned}
$$

Combining (3.7), we immediately get (3.6). $\square$

Applying inequality (5.133) in [57] to the determinants of real symmetric $n \times n$ matrices yields the following Aleksandrov inequality for mixed discriminants: let $Q_1, \ldots, Q_n$ be positive definite $n \times n$ matrices. Then

$$
D(Q_1, Q_2, \ldots, Q_n) \geq (\det Q_1)^{\frac{1}{n}} (\det Q_2)^{\frac{1}{n}} \cdots (\det Q_n)^{\frac{1}{n}}, \qquad (3.8)
$$

with equality if and only if $Q_i = \lambda_i Q_1$ with a real number λ_i for $i = 2, \ldots, n$.

A Borel measure μ on S^{n-1} generates a positive semi-definite $n \times n$ matrix $[\mu]$ defined by

$$
[\mu] = \int\limits_{S^{n-1}} v \otimes v d\mu(v), \qquad (3.9)
$$

where $v \otimes v$ is the rank 1 linear operator on $\mathbb{R}^n$ that takes x to $(x \cdot v)v$. It was proved in [45] that for Borel measures $\mu_1, \ldots, \mu_n$ on S^{n-1}, the mixed discriminant of $[\mu_1], \ldots, [\mu_n]$ is given by

$$D([\mu_1],\ldots,[\mu_n])=\frac{1}{n!}\int\limits_{S^{n-1}}\cdots\int\limits_{S^{n-1}}[v_1,\ldots,v_n]^2 d\mu_1(v_1)\cdots d\mu_n(v_n). \tag{3.10}$$

Combining with (3.8), we immediately obtain

$$D([\mu_1],\ldots,[\mu_n])\ge(\det[\mu_1])^{\frac{1}{n}}\cdots(\det[\mu_n])^{\frac{1}{n}}, \tag{3.11}$$

with equality if and only if $[\mu_i]=\lambda_i[\mu_1]$ with a real number λ_i for $i=2,\ldots,n$.

4. Inequalities for the ellipsoid $\Lambda_2 K$

The classical Blaschke-Santaló inequality is one of the essential affine isoperimetric inequalities in convex geometric analysis (see, e.g., [13,57]), which states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$V(K)V(K^*)\le\omega_n^2, \tag{4.1}$$

with equality if and only if K is an origin-centered ellipsoid. (A non origin-symmetric version also holds true, for which one needs to choose the Santaló point to define polarity; see, e.g., [57, p. 548].) In [49], the Blaschke-Santaló inequality (4.1) was extended to the L_p setting by Lutwak and Zhang such that the inequality (4.1) is a special case ($p=\infty$). In particular, they showed that if $K\in\mathcal{S}_o^n$, then

$$V(K)V(\Gamma_2^* K)\le\omega_n^2, \tag{4.2}$$

with equality if and only if K is an origin-centered ellipsoid. In this section, we shall establish an analog of inequality (4.2) for the sine ellipsoid $\Lambda_2^* K$.

Theorem 4.1. *If $K\in\mathcal{S}_o^n$, then*

$$V(K)V(\Lambda_2^* K)\le\frac{\omega_n^2}{(n-1)^{\frac{n}{2}}}, \tag{4.3}$$

with equality if and only if K is an origin-centered ball when $n\ge 3$ and is an origin-centered ellipsoid when $n=2$.

For $p\ge 1$ and $K_2,\ldots,K_n\in\mathcal{S}_o^n$, we define the convex body $\mathrm{T}_p(K_2,\ldots,K_n)$ whose support function at $x\in\mathbb{R}^n$ is given by

$$h^p_{\mathrm{T}_p(K_2,\ldots,K_n)}(x)=\frac{(n+p)^{n-1}}{V(K_2)\cdots V(K_n)}\int\limits_{K_2}\cdots\int\limits_{K_n}[x,x_2,\ldots,x_n]^p dx_2\cdots dx_n. \tag{4.4}$$

By the polar coordinate, we further have

$$
\begin{aligned}
&h^p_{\mathrm{T}_p(K_2,\ldots,K_n)}(x)\\
&= \frac{(n+p)^{n-1}}{V(K_2)\cdots V(K_n)} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} \int\limits_0^{\rho_{K_2}(v_2)} \cdots \int\limits_0^{\rho_{K_n}(v_n)} [x, r_2 v_2, \ldots, r_n v_n]^p \\
&\qquad\qquad \times r_2^{n-1} \cdots r_n^{n-1} dr_2 \cdots dr_n dv_2 \cdots dv_n \\
&= \frac{1}{V(K_2)\cdots V(K_n)} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} [x, v_2, \ldots, v_n]^p \rho^{n+p}_{K_2}(v_2) \cdots \rho^{n+p}_{K_n}(v_n) dv_2 \cdots dv_n. \qquad (4.5)
\end{aligned}
$$

Note that $\mathrm{T}_2(K_2, \ldots, K_n)$ is an ellipsoid since $h^2_{\mathrm{T}_2(K_2,\ldots,K_n)}(x)$ is a positive definite quadratic form in the variable x.

The affine nature of the operator T_p is as follows.

Theorem 4.2. *If $p \geq 1$ and $K_2, \ldots, K_n \in \mathcal{S}^n_o$, then for $\phi \in \mathrm{GL}(n)$,*

$$
\mathrm{T}_p(\phi K_2, \ldots, \phi K_n) = |\det \phi| \phi^{-t} \mathrm{T}_p(K_2, \ldots, K_n). \qquad (4.6)
$$

Proof. By (4.4), (2.2), and (2.3), we have, for $\phi \in \mathrm{GL}(n)$ and $x \in \mathbb{R}^n$,

$$
\begin{aligned}
&h^p_{\mathrm{T}_p(\phi K_2,\ldots,\phi K_n)}(x)\\
&= \frac{(n+p)^{n-1}}{V(\phi K_2)\cdots V(\phi K_n)} \int\limits_{\phi K_2} \cdots \int\limits_{\phi K_n} [x, x_2, \ldots, x_n]^p dx_2 \cdots dx_n \\
&= \frac{(n+p)^{n-1}}{|\det \phi|^{n-1} V(K_2)\cdots V(K_n)} \int\limits_{K_2} \cdots \int\limits_{K_n} [x, \phi x_2, \ldots, \phi x_n]^p |\det \phi|^{n-1} dx_2 \cdots dx_n \\
&= \frac{(n+p)^{n-1}}{V(K_2)\cdots V(K_n)} \int\limits_{K_2} \cdots \int\limits_{K_n} |\det \phi|^p [\phi^{-1} x, x_2, \ldots, x_n]^p dx_2 \cdots dx_n \\
&= |\det \phi|^p h^p_{\mathrm{T}_p(K_2,\ldots,K_n)}(\phi^{-1} x) = h^p_{|\det \phi| \phi^{-t} \mathrm{T}_p(K_2,\ldots,K_n)}(x),
\end{aligned}
$$

which gives the desired result. □

For $p \geq 1$ and $K \in \mathcal{S}^n_o$, define the convex body $\Lambda_p K$ whose support function at $x \in \mathbb{R}^n$ is given by

$$
h^p_{\Lambda_p K}(x) = \frac{n+p}{V(K)} \int\limits_K [x, y]^p dy.
$$

When $p = 2$, the body $\Lambda_2 K$ is exactly the ellipsoid defined in (1.5).

The following theorem shows the connection between T_p and Λ_p.

Theorem 4.3. *If $p \geq 1$ and $K \in \mathcal{S}_o^n$, then there exists a constant $c_{n,p} > 0$ depending on n, p such that*

$$\mathrm{T}_p(K, B^n, \ldots, B^n) = c_{n,p}\Lambda_p K.$$

In particular,

$$\mathrm{T}_2(K, B^n, \ldots, B^n) = ((n-2)!)^{\frac{1}{2}}\Lambda_2 K. \tag{4.7}$$

Proof. For $n \geq 3$, taking $K_2 = K$ and $K_3 = \cdots = K_n = B^n$ in (4.5) and by Lemma 3.2, we have, for $x \in \mathbb{R}^n \backslash \{o\}$,

$$\begin{aligned}
&h^p_{\mathrm{T}_p(K,B^n,\ldots,B^n)}(x)\\
=&\frac{1}{V(K)\omega_n^{n-2}}\int\limits_{S^{n-1}}\int\limits_{S^{n-1}}\cdots\int\limits_{S^{n-1}}[x, v_2, v_3, \ldots, v_n]^p\rho_K^{n+p}(v_2)dv_2dv_3\cdots dv_n\\
=&\frac{1}{V(K)\omega_n^{n-2}}\int\limits_{\{v_2\in S^{n-1}:[x,v_2]\neq 0\}}\Big(\int\limits_{S^{n-1}}\cdots\int\limits_{S^{n-1}}\frac{[x, v_2, v_3, \ldots, v_n]^p}{[x, v_2]^p}dv_3\cdots dv_n\Big)\\
&\times [x, v_2]^p\rho_K^{n+p}(v_2)dv_2\\
=&c^p_{n,p}\frac{1}{V(K)}\int\limits_{S^{n-1}}[x, v_2]^p\rho_K^{n+p}(v_2)dv_2\\
=&c^p_{n,p}\frac{n+p}{V(K)}\int\limits_K[x, y]^p dy\\
=&c^p_{n,p}h^p_{\Lambda_p K}(x) = h^p_{c_{n,p}\Lambda_p K}(x),
\end{aligned}$$

where the constant

$$c_{n,p} = \Big(\frac{1}{\omega_n^{n-2}}\int\limits_{S^{n-1}}\cdots\int\limits_{S^{n-1}}\frac{[x, v_2, v_3, \ldots, v_n]^p}{[x, v_2]^p}dv_3\cdots dv_n\Big)^{1/p}.$$

In particular, $c_{n,2} = ((n-2)!)^{\frac{1}{2}}$ by (3.6), thus (4.7) follows.

For $n = 2$, the definitions of $\Lambda_p K$ and $\mathrm{T}_p(K)$ are coincident. So the assertion still follows. $\square$

Theorem 4.4. *If $K_2, \ldots, K_n \in \mathcal{S}_o^n$, then*

$$V(\Gamma_2 K_2)\cdots V(\Gamma_2 K_n)V(\mathrm{T}_2^*(K_2, \ldots, K_n)) \leq \frac{\omega_n^n}{((n-1)!)^{\frac{n}{2}}}, \tag{4.8}$$

with equality if and only if there exists a transform $\phi \in \mathrm{GL}(n)$ such that all star bodies $\phi K_i/V(\phi K_i)^{1/n}$ are in isotropic position for $i = 2, \ldots, n$.

Proof. It follows from (4.6), (2.16), and (2.8) that the left-hand side of inequality (4.8) is $\mathrm{GL}(n)$ invariant. Thus, we may assume that

$$\mathrm{T}_2(K_2, \dots, K_n) = B^n. \tag{4.9}$$

Then, by (4.5), for $v \in S^{n-1}$,

$$1 = \frac{1}{V(K_2)\cdots V(K_n)} \int\limits_{(S^{n-1})^{n-1}} [v, v_2, \dots, v_n]^2 \rho_{K_2}^{n+2}(v_2)\cdots \rho_{K_n}^{n+2}(v_n) dv_2 \cdots dv_n.$$

Integrating both sides with respect to the spherical Lebesgue measure gives

$$n\omega_n = \frac{1}{V(K_2)\cdots V(K_n)} \int\limits_{(S^{n-1})^{n}} [v, v_2, \dots, v_n]^2 \rho_{K_2}^{n+2}(v_2)\cdots \rho_{K_n}^{n+2}(v_n) dv dv_2 \cdots dv_n. \tag{4.10}$$

By (3.9) and (3.10), equation (4.10) means that

$$D([B^n], [K_2], \dots, [K_n]) = \frac{1}{(n-1)!}, \tag{4.11}$$

where $D([B^n], [K_2], \dots, [K_n])$ is the mixed discriminant of the positive definite $n \times n$ matrices $[B^n], [K_2], \dots, [K_n]$ defined by

$$[B^n] = \frac{1}{\omega_n} \int\limits_{S^{n-1}} v \otimes v dv = I_n, \tag{4.12}$$

and

$$[K_i] = \frac{1}{V(K_i)} \int\limits_{S^{n-1}} v \otimes v \rho_{K_i}^{n+2}(v) dv = \frac{n+2}{V(K_i)} \int\limits_{K_i} y \otimes y dy, \quad i = 2, \cdots, n. \tag{4.13}$$

By (1.1) and (2.18), we have

$$\Gamma_2 K_i = \{x \in \mathbb{R}^n : x \cdot [K_i]^{-1} x \le 1\},$$

and therefore,

$$V(\Gamma_2 K_i) = (\det[K_i])^{\frac{1}{2}} \omega_n, \quad i = 2, \dots, n. \tag{4.14}$$

Moreover, from (4.11), (4.12), (3.11) with $[\mu_1] = I_n$, $[\mu_i] = [K_i]$ for $i = 2, \dots, n$, (4.14), and (4.9), we obtain

$$
\begin{aligned}
\frac{1}{(n-1)!} &= D(I_n, [K_2], \ldots, [K_n]) \\
&\geq (\det[K_2])^{\frac{1}{n}} \cdots (\det[K_n])^{\frac{1}{n}} \\
&= (V(\Gamma_2 K_2)\omega_n^{-1})^{\frac{2}{n}} \cdots (V(\Gamma_2 K_n)\omega_n^{-1})^{\frac{2}{n}} \\
&= V(\Gamma_2 K_2)^{\frac{2}{n}} \cdots V(\Gamma_2 K_n)^{\frac{2}{n}} \omega_n^{\frac{2(1-n)}{n}} \\
&= V(\Gamma_2 K_2)^{\frac{2}{n}} \cdots V(\Gamma_2 K_n)^{\frac{2}{n}} \omega_n^{-2} V(\mathrm{T}_2^*(K_2, \ldots, K_n))^{\frac{2}{n}},
\end{aligned}
$$

which is the desired inequality (4.8).

By inequality (3.11), equality in (4.8) holds if and only if $[K_i] = \lambda_i I_n$ with a real number λ_i for $i = 2, \ldots, n$. Thus, it follows from (4.13) and (1.1) that $\Gamma_2 K_i$ are all balls; i.e., star bodies $K_i/V(K_i)^{1/n}$ are in isotropic position for $i = 2, \ldots, n$. $\square$

Combining with (4.8) and (1.9), we immediately obtain

Corollary 4.5. *If $K_2, \ldots, K_n \in \mathcal{S}_o^n$, then*

$$
V(K_2) \cdots V(K_n) V(\mathrm{T}_2^*(K_2, \ldots, K_n)) \leq \frac{\omega_n^n}{((n-1)!)^{\frac{n}{2}}}, \tag{4.15}
$$

with equality if and only if K_i are origin-centered ellipsoids that are dilates for $i = 2, \ldots, n$.

Proof. We only need to verify the equality conditions of (4.15). By the equality conditions of (4.8) and (1.9), equality in (4.15) holds if and only if there exists a transform $\phi \in \mathrm{GL}(n)$ such that all star bodies $\phi K_i/V(\phi K_i)^{1/n}$, $i = 2, \ldots, n$, are all origin-centered ellipsoids and in isotropic position. It is well-known that for a star body K with $V(K) = 1$ there exists a unique transform $\phi \in \mathrm{SL}(n)$ (if we ignore orthogonal transformations) such that ϕK is in isotropic position. This further implies that among all origin-centered ellipsoids only the ball $B^n/V(B^n)^{1/n}$ is in isotropic position. Thus, the desired equality conditions follow. $\square$

When $n \geq 3$, Theorem 4.1 now follows from Corollary 4.5, (4.7), and (2.9) by taking $K_2 = K$ and $K_3 = \cdots = K_n = B^n$. When $n = 2$, Theorem 4.1 follows from Corollary 4.5 with $K_2 = K$ and (4.7).

Finally, Theorem 4.1, together with the equality conditions of inequality (4.1) and the fact that $\Lambda_2 K$ is an origin-centered ellipsoid, immediately yields inequality (1.11) and its equality conditions.

Remark 4.6. When $n = 2$, the ellipsoids $\mathrm{T}_2(K)$ and $\Lambda_2 K$ coincide according to their definitions. Thus, it follows from (4.6) that for $\phi \in \mathrm{GL}(n)$,

$$
\Lambda_2(\phi K) = |\det \phi| \phi^{-t} \Lambda_2 K. \tag{4.16}
$$

So inequalities (1.11) and (4.3) are affine invariant when $n = 2$. In fact, for any $K \in \mathcal{S}_o^2$, the sine ellipsoid $\Lambda_2 K$ is just a rotation of the Legendre ellipsoid $\Gamma_2 K$ by an angle $\pi/2$. To see this, denote by $\psi_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the rotation by an angle $\pi/2$ in $\mathbb{R}^2$. Then we have $x \perp \psi_{\pi/2} x$ for any $x \in \mathbb{R}^2$. Thus, it follows from (1.5), (1.1), and (2.2) that

$$\begin{aligned} h^2_{\Lambda_2 K}(x) &= \frac{4}{V(K)} \int\limits_K [x,y]^2 dy = \frac{4}{V(K)} \int\limits_K |x|^2 |\mathrm{P}_{x^\perp} y|^2 dy \\ &= \frac{4}{V(K)} \int\limits_K |\psi_{\pi/2} x|^2 |\mathrm{P}_{\psi_{\pi/2} x} y|^2 dy = \frac{4}{V(K)} \int\limits_K |\psi_{\pi/2} x \cdot y|^2 dy \\ &= h^2_{\Gamma_2 K}(\psi_{\pi/2} x) = h^2_{\psi^t_{\pi/2} \Gamma_2 K}(x), \end{aligned} \tag{4.17}$$

which means

$$\Lambda_2 K = \psi^t_{\pi/2} \Gamma_2 K.$$

This also implies the affine property (4.16) by using (2.16) and the fact $\Lambda_2 K$ is origin-symmetric:

$$\Lambda_2(\phi K) = \psi^t_{\pi/2} \Gamma_2(\phi K) = \psi^t_{\pi/2} \phi \psi_{\pi/2} (\psi^t_{\pi/2} \Gamma_2 K) = |\det \phi| \phi^{-t} \Lambda_2 K.$$

5. Inequalities for the ellipsoid $\Lambda_{-2} K$

Recall that the projection body ΠK of a convex body K in $\mathbb{R}^n$ is the origin-symmetric convex body whose support function at $x \in \mathbb{R}^n$ is given by

$$h_{\Pi K}(x) = \frac{1}{V(K)} \int\limits_{S^{n-1}} |x \cdot v| dS(K, v).$$

Associated with the projection body there is an important affine invariant

$$V(\Pi K)/V(K)^{n-1}.$$

Its least upper bound and greatest lower bound were conjectured by Schneider [56] and Petty [54], named Schneider's projection problem and Petty's conjecture, respectively. As far as we know, both conjectures remain open. Petty [54] conjectured that the greatest lower bound of the ratio is attained if and only if K is an ellipsoid. Petty's conjecture, once established, would imply a number of important isoperimetric inequalities, including the classical isoperimetric inequality and Petty's projection inequality.

Since the operator Γ^*_{-2} is actually the L_2 projection body operator, the L_2 analog of Petty's conjecture can be formulated as follows: if $K \in \mathcal{K}^n_o$, then

$$V(K)V(\Gamma^*_{-2}K) \geq \omega_n^2, \tag{5.1}$$

with equality if and only if K is an origin-centered ellipsoid. This inequality follows directly from (4.1), the fact that $\Gamma_{-2}K$ is an origin-centered ellipsoid and inequality (1.10). In this section, we shall establish an analog of inequality (5.1) for the sine ellipsoid $\Lambda^*_{-2}K$.

Theorem 5.1. *If $K \in \mathcal{K}_o^n$, then*

$$V(K)V(\Lambda^*_{-2}K) \geq (n-1)^{\frac{n}{2}}\omega_n^2, \tag{5.2}$$

with equality if and only if K is an origin-centered ball when $n \geq 3$ and is an origin-centered ellipsoid when $n = 2$.

If $K \in \mathcal{K}_o^n$, then for $p > 0$, one can define the L_p surface area measure $dS_p(K, \cdot)$ of K (see, e.g. [46]) by

$$dS_p(K, \cdot) = h_K(\cdot)^{1-p} dS(K, \cdot).$$

Thus, for $p > 0$ and $K_2, \ldots, K_n \in \mathcal{K}_o^n$, we define the star body $\mathrm{T}_{-p}(K_2, \ldots, K_n)$ whose radial function at $x \in \mathbb{R}^n \backslash \{o\}$ is given by

$$\begin{aligned}&\rho^{-p}_{\mathrm{T}_{-p}(K_2,\ldots,K_n)}(x)\\ =&\frac{1}{V(K_2)\cdots V(K_n)}\int\limits_{S^{n-1}}\cdots\int\limits_{S^{n-1}}[x, v_2, \ldots, v_n]^p dS_p(K_2, v_2)\cdots dS_p(K_n, v_n).\end{aligned} \tag{5.3}$$

Note that $\mathrm{T}_{-2}(K_2, \ldots, K_n)$ is an ellipsoid because $\rho^{-2}_{\mathrm{T}_{-2}(K_2,\ldots,K_n)}(x)$ is a positive definite quadratic form in the variable x.

The affine nature of the operator T_{-p} is given below.

Theorem 5.2. *If $p > 0$ and $K_2, \ldots, K_n \in \mathcal{K}_o^n$, then for $\phi \in \mathrm{GL}(n)$,*

$$\mathrm{T}_{-p}(\phi K_2, \ldots, \phi K_n) = |\det \phi|\phi^{-t}\mathrm{T}_{-p}(K_2, \ldots, K_n). \tag{5.4}$$

Proof. For any $\phi \in \mathrm{SL}(n)$, it follows from definition (5.3), [46, Proposition 1.2], and (2.5) that, for $x \in \mathbb{R}^n \backslash \{o\}$,

$$\begin{aligned}&\rho^{-p}_{\mathrm{T}_{-p}(\phi K_2,\ldots,\phi K_n)}(x)\\ =&\frac{1}{V(K_2)\cdots V(K_n)}\int\limits_{S^{n-1}}\cdots\int\limits_{S^{n-1}}[x, v_2, \ldots, v_n]^p dS_p(\phi K_2, v_2)\cdots dS_p(\phi K_n, v_n)\\ =&\frac{1}{V(K_2)\cdots V(K_n)}\int\limits_{S^{n-1}}\cdots\int\limits_{S^{n-1}}|\phi^{-t}v_2|^p\cdots|\phi^{-t}v_n|^p\end{aligned}$$

$$\times \left[x, \frac{\phi^{-t}v_2}{|\phi^{-t}v_2|}, \ldots, \frac{\phi^{-t}v_n}{|\phi^{-t}v_n|}\right]^p dS_p(K_2, v_2) \cdots dS_p(K_n, v_n)$$
$$= \frac{1}{V(K_2)\cdots V(K_n)} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} [x, \phi^{-t}v_2, \ldots, \phi^{-t}v_n]^p dS_p(K_2, v_2) \cdots dS_p(K_n, v_n)$$
$$= \frac{1}{V(K_2)\cdots V(K_n)} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} [\phi^t x, v_2, \ldots, v_n]^p dS_p(K_2, v_2) \cdots dS_p(K_n, v_n)$$
$$= \rho^{-p}_{\mathrm{T}_{-p}(K_2,\ldots,K_n)}(\phi^t x) = \rho^{-p}_{\phi^{-t}\mathrm{T}_{-p}(K_2,\ldots,K_n)}(x).$$

Hence, $\mathrm{T}_{-p}(\phi K_2, \ldots, \phi K_n) = \phi^{-t}\mathrm{T}_{-p}(K_2, \ldots, K_n)$ for any $\phi \in \mathrm{SL}(n)$.

For $c > 0$, it follows from (5.3), (2.11), and (2.6) that

$$\mathrm{T}_{-p}(cK_2, \ldots, cK_n) = c^{n-1}\mathrm{T}_{-p}(K_2, \ldots, K_n).$$

Consequently, we obtain (5.4). □

For $p > 0$ and $K \in \mathcal{K}^n_o$, define the star body $\Lambda_{-p}K$ whose radial function at $x \in \mathbb{R}^n \backslash \{o\}$ is given by

$$\rho^{-p}_{\Lambda_{-p}K}(x) = \frac{1}{V(K)} \int\limits_{S^{n-1}} [x, v]^p dS_p(K, v).$$

When $p = 2$, the body $\Lambda_{-2}K$ is exactly the ellipsoid defined in (1.6).

We have the following connection between T_{-p} and Λ_{-p}.

Theorem 5.3. *If $p > 0$ and $K \in \mathcal{K}^n_o$, then there exists a constant $c_{n,p} > 0$ such that*

$$\mathrm{T}_{-p}(K, B^n, \ldots, B^n) = c^{-1}_{n,p}\Lambda_{-p}K.$$

In particular,

$$\mathrm{T}_{-2}(K, B^n, \ldots, B^n) = ((n-2)!)^{-\frac{1}{2}}\Lambda_{-2}K. \tag{5.5}$$

Proof. For $n \geq 3$, taking $K_2 = K$ and $K_3 = \cdots = K_n = B^n$ in (5.3) and by Lemma 3.2, we have, for $x \in \mathbb{R}^n \backslash \{o\}$,

$$\rho^{-p}_{\mathrm{T}_{-p}(K,B^n,\ldots,B^n)}(x)$$
$$= \frac{1}{V(K)\omega_n^{n-2}} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} [x, v_2, v_3, \ldots, v_n]^p dS_p(K, v_2) dv_3 \cdots dv_n$$
$$= \frac{1}{V(K)\omega_n^{n-2}} \int\limits_{\{v_2 \in S^{n-1}:[x,v_2]\neq 0\}} \Bigg(\int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} \frac{[x, v_2, v_3, \ldots, v_n]^p}{[x, v_2]^p} dv_3 \cdots dv_n \Bigg)$$

$$
\begin{aligned}
&\times [x,v_2]^p dS_p(K,v_2)\\
=&c_{n,p}^p \frac{1}{V(K)} \int\limits_{S^{n-1}} [x,v_2]^p dS_p(K,v_2)\\
=&c_{n,p}^p \frac{1}{V(K)} \int\limits_{S^{n-1}} [x,v_2]^p dS_p(K,v_2)\\
=&c_{n,p}^p \rho_{\Lambda_{-p}K}^{-p}(x) = \rho_{c_{n,p}^{-1}\Lambda_{-p}K}^{-p}(x),
\end{aligned}
$$

where the constant

$$c_{n,p} = \Big(\frac{1}{\omega_n^{n-2}} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} \frac{[x,v_2,v_3,\dots,v_n]^p}{[x,v_2]^p} dv_3 \cdots dv_n\Big)^{1/p}.$$

In particular, (5.5) follows since $c_{n,2} = ((n-2)!)^{\frac{1}{2}}$ by (3.6).

For $n=2$, the definitions of $\Lambda_{-p}K$ and $\mathrm{T}_{-p}(K)$ are coincident. So the assertion still follows. $\square$

Theorem 5.4. *If $K_2,\dots,K_n \in \mathcal{K}_o^n$, then*

$$V(\Gamma_{-2}K_2)\cdots V(\Gamma_{-2}K_n)V(\mathrm{T}_{-2}^*(K_2,\dots,K_n)) \ge ((n-1)!)^{\frac{n}{2}}\omega_n^n, \tag{5.6}$$

with equality if and only if there exists a transform $\phi \in \mathrm{GL}(n)$ such that all convex bodies $\phi K_i/V(\phi K_i)^{1/n}$ are in dual isotropic position for $i=2,\dots,n$.

Proof. It follows from (5.4), (2.17), and (2.8) that the left-hand side of inequality (5.6) is $\mathrm{GL}(n)$ invariant. Since $\mathrm{T}_{-2}(K_2,\dots,K_n)$ is an ellipsoid, we may assume that

$$\mathrm{T}_{-2}(K_2,\dots,K_n) = B^n. \tag{5.7}$$

Thus, by (5.3), we have, for $v \in S^{n-1}$,

$$1 = \frac{1}{V(K_2)\cdots V(K_n)} \int\limits_{(S^{n-1})^{n-1}} [v,v_2,\dots,v_n]^2 dS_2(K_2,v_2)\cdots dS_2(K_n,v_n).$$

Integrating both sides with respect to the spherical Lebesgue measure gives

$$n\omega_n = \frac{1}{V(K_2)\cdots V(K_n)} \int\limits_{(S^{n-1})^{n}} [v,v_2,\dots,v_n]^2 dv dS_2(K_2,v_2)\cdots dS_2(K_n,v_n). \tag{5.8}$$

By (3.9) and (3.10), equation (5.8) means that

$$D([B^n],[K_2],\dots,[K_n]) = \frac{1}{(n-1)!}, \tag{5.9}$$

where $D([B^n],[K_2],\dots,[K_n])$ is the mixed discriminant of the positive definite $n\times n$ matrices $[B^n],[K_2],\dots,[K_n]$ defined by

$$[B^n]=\frac{1}{\omega_n}\int_{S^{n-1}} v\otimes v dv = I_n, \tag{5.10}$$

and

$$[K_i]=\frac{1}{V(K_i)}\int_{S^{n-1}} v\otimes v dS_2(K_i,v), \quad i=2,\cdots,n. \tag{5.11}$$

By (1.3) and (2.18), we have

$$\Gamma_{-2}K_i=\{x\in\mathbb{R}^n : x\cdot[K_i]x\le 1\},$$

and therefore,

$$V(\Gamma_{-2}K_i)=(\det[K_i])^{-\frac{1}{2}}\omega_n, \quad i=2,\dots,n. \tag{5.12}$$

Moreover, from (5.9), (5.10), (3.11) with $[\mu_1]=I_n$, $[\mu_i]=[K_i]$ for $i=2,\dots,n$, (5.12), and (5.7), we obtain

$$\begin{aligned}
\frac{1}{(n-1)!} &= D(I_n,[K_2],\dots,[K_n])\\
&\ge (\det[K_2])^{\frac{1}{n}}\cdots(\det[K_n])^{\frac{1}{n}}\\
&= (V(\Gamma_{-2}K_2)^{-1}\omega_n)^{\frac{2}{n}}\cdots(V(\Gamma_{-2}K_n)^{-1}\omega_n)^{\frac{2}{n}}\\
&= V(\Gamma_{-2}K_2)^{-\frac{2}{n}}\cdots V(\Gamma_{-2}K_n)^{-\frac{2}{n}}\omega_n^{\frac{2(n-1)}{n}}\\
&= V(\Gamma_{-2}K_2)^{-\frac{2}{n}}\cdots V(\Gamma_{-2}K_n)^{-\frac{2}{n}}\omega_n^2 V(\mathrm{T}^*_{-2}(K_2,\dots,K_n))^{-\frac{2}{n}},
\end{aligned}$$

which is the desired inequality (5.6).

By inequality (3.11), equality in (5.6) holds if and only if $[K_i]=\lambda_i I_n$ with a real number λ_i for $i=2,\dots,n$. Thus, it follows from (5.11) and (1.3) that $\Gamma_{-2}K_i$ are all balls; i.e., convex bodies $K_i/V(K_i)^{1/n}$ are in dual isotropic position for $i=2,\dots,n$. □

Combining with (5.6) and (1.10), we immediately obtain

Corollary 5.5. *If $K_2,\dots,K_n\in\mathcal{K}^n_o$, then*

$$V(K_2)\cdots V(K_n)V(\mathrm{T}^*_{-2}(K_2,\dots,K_n))\ge((n-1)!)^{\frac{n}{2}}\omega_n^n, \tag{5.13}$$

with equality if and only if K_i are origin-centered ellipsoids that are dilates for $i=2,\dots,n$.

Proof. We only need to verify the equality conditions of (5.13). By the equality conditions of (5.6) and (1.10), equality in (5.13) holds if and only if there exists a transform $\phi \in \mathrm{GL}(n)$ such that $\phi K_i / V(\phi K_i)^{1/n}$, $i = 2, \dots, n$, are all origin-centered ellipsoids and in dual isotropic position. In [46], Lutwak, Yang, and Zhang proved that for a convex body K with $V(K) = 1$ there exists a unique transform $\phi \in \mathrm{SL}(n)$ (if we ignore orthogonal transformations) such that ϕK is in dual isotropic position. This further implies that among all origin-centered ellipsoids only the ball $B^n / V(B^n)^{1/n}$ is in dual isotropic position. Thus, the desired equality conditions follow. □

When $n \geq 3$, Theorem 5.1 now follows from Corollary 5.5, (5.5), and (2.9) by taking $K_2 = K$ and $K_3 = \cdots = K_n = B^n$. When $n = 2$, Theorem 5.1 follows from Corollary 5.5 with $K_2 = K$ and (5.5).

Finally, Theorem 5.1, together with the equality conditions of inequality (4.1) and the fact that $\Lambda_{-2}K$ is an origin-centered ellipsoid, immediately yields inequality (1.12) and its equality conditions.

Remark 5.6. When $n = 2$, the ellipsoids $\mathrm{T}_{-2}(K)$ and $\Lambda_{-2}K$ coincide according to their definitions. Thus, it follows from (5.4) that, for $\phi \in \mathrm{GL}(n)$ and $K \in \mathcal{K}_o^2$,

$$\Lambda_{-2}(\phi K) = |\det \phi| \phi^{-t} \Lambda_{-2}K. \tag{5.14}$$

So inequalities (1.12) and (5.2) are affine invariant when $n = 2$. The same argument in Remark 4.6 shows that for any $K \in \mathcal{K}_o^2$ the sine ellipsoid $\Lambda_{-2}K$ is just a rotation of the LYZ ellipsoid $\Gamma_{-2}K$ by an angle $\pi/2$; i.e.,

$$\Lambda_{-2}K = \psi_{\pi/2}^t \Gamma_{-2}K.$$

Moreover, the above identity also implies (5.14) by using (2.17) and the fact $\Lambda_{-2}K$ is origin-symmetric.

6. Valuation properties of two sine ellipsoids

An operator $Z : \mathcal{S}_o^n \to \mathcal{K}_o^n$ is called an L_p Minkowski valuation [33] if

$$Z(K \cup L) +_p Z(K \cap L) = ZK +_p ZL,$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{S}_o^n$. An operator $Z : \mathcal{K}_o^n \to \mathcal{S}_o^n$ is called an L_{-p} radial valuation [18] if

$$Z(K \cup L) \widetilde{+}_{-p} Z(K \cap L) = ZK \widetilde{+}_{-p} ZL,$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{K}_o^n$.

For $p \geq 1$ and $K \in \mathcal{S}_o^n$, we define the convex body $\widetilde{\Lambda_p}K$ whose support function at $x \in \mathbb{R}^n$ is given by

$$h^p_{\widetilde{\Lambda_p}K}(x) = h^p_{V(K)^{1/p}\Lambda_p K}(x) = (n+p)\int_K [x,y]^p dy.$$

For $p > 0$ and $K \in \mathcal{K}_o^n$, we define the star body $\widetilde{\Lambda_{-p}}K$ whose radial function at $x \in \mathbb{R}^n \backslash \{o\}$ is given by

$$\rho^{-p}_{\widetilde{\Lambda_{-p}}K}(x) = \rho^{-p}_{V(K)^{-1/p}\Lambda_{-p}K}(x) = \int_{S^{n-1}} [x,v]^p dS_p(K,v).$$

Theorem 6.1. *For $p \geq 1$, the operator $\widetilde{\Lambda_p} : \mathcal{S}_o^n \to \mathcal{K}_o^n$ is an L_p Minkowski valuation.*

Proof. For any $K, L \in \mathcal{S}_o^n$ and $u \in S^{n-1}$, by the polar coordinate, we have

$$\begin{aligned} h^p_{\widetilde{\Lambda_p}K}(u) &= (n+p)\int_K [u,y]^p dy = (n+p)\int_{S^{n-1}} \int_0^{\rho_K(v)} [u,rv]^p r^{n-1} dr dv \\ &= \int_{S^{n-1}} [u,v]^p \rho_K^{n+p}(v) dv = \int_{S_1 \cup S_2 \cup S_3} [u,v]^p \rho_K^{n+p}(v) dv \\ &= \int_{S_1} [u,v]^p \rho_K^{n+p}(v) dv + \int_{S_2} [u,v]^p \rho_K^{n+p}(v) dv + \int_{S_3} [u,v]^p \rho_K^{n+p}(v) dv, \end{aligned}$$

where

$$S_1 = \{v \in S^{n-1} : \rho_K(v) > \rho_L(v)\}, \quad S_2 = \{v \in S^{n-1} : \rho_K(v) < \rho_L(v)\},$$

and

$$S_3 = \{v \in S^{n-1} : \rho_K(v) = \rho_L(v)\}.$$

Then, we have

$$\begin{aligned} \int_{S_1} [u,v]^p \rho_{K\cup L}^{n+p}(v) dv &= \int_{S_1} [u,v]^p \rho_K^{n+p}(v) dv, \\ \int_{S_2} [u,v]^p \rho_{K\cup L}^{n+p}(v) dv &= \int_{S_2} [u,v]^p \rho_L^{n+p}(v) dv, \\ \int_{S_3} [u,v]^p \rho_{K\cup L}^{n+p}(v) dv &= \int_{S_3} [u,v]^p \rho_K^{n+p}(v) dv, \end{aligned}$$

and

$$\int\limits_{S_1}[u,v]^p\rho_{K\cap L}^{n+p}(v)dv=\int\limits_{S_1}[u,v]^p\rho_L^{n+p}(v)dv,$$
$$\int\limits_{S_2}[u,v]^p\rho_{K\cap L}^{n+p}(v)dv=\int\limits_{S_2}[u,v]^p\rho_K^{n+p}(v)dv,$$
$$\int\limits_{S_3}[u,v]^p\rho_{K\cap L}^{n+p}(v)dv=\int\limits_{S_3}[u,v]^p\rho_L^{n+p}(v)dv.$$

Summing up both sides of the integrals above gives

$$\int\limits_{S^{n-1}}[u,v]^p\rho_{K\cup L}^{n+p}(v)dv+\int\limits_{S^{n-1}}[u,v]^p\rho_{K\cap L}^{n+p}(v)dv$$
$$=\int\limits_{S^{n-1}}[u,v]^p\rho_K^{n+p}(v)dv+\int\limits_{S^{n-1}}[u,v]^p\rho_L^{n+p}(v).$$

Since this holds for any $u\in S^{n-1}$, it follows that

$$\widetilde{\Lambda_p}(K\cup L)+_p\widetilde{\Lambda_p}(K\cap L)=\widetilde{\Lambda_p}K+_p\widetilde{\Lambda_p}L,$$

which is the desired valuation. $\square$

Theorem 6.2. *For $p>0$, the operator $\widetilde{\Lambda_{-p}}:\mathcal{K}_o^n\to\mathcal{S}_o^n$ is an L_{-p} radial valuation.*

Proof. We shall make use of the fact that if $K,L,K\cup L\in\mathcal{K}_o^n$, then $h_{K\cup L}=\max\{h_K,h_L\}$ and $h_{K\cap L}=\min\{h_K,h_L\}$.

First, we assume that K and L are both strictly convex; i.e., the boundary of K and L contains no segment. For $u\in S^{n-1}$,

$$\rho_{\widetilde{\Lambda_{-p}}K}^{-p}(u)=\int\limits_{S^{n-1}}[u,v]^pdS_p(K,v)=\int\limits_{S_1\cup S_2\cup S_3}[u,v]^ph_K^{1-p}(v)dS(K,v)$$
$$=\int\limits_{S_1}[u,v]^ph_K^{1-p}(v)dS(K,v)+\int\limits_{S_2}[u,v]^ph_K^{1-p}(v)dS(K,v)+\int\limits_{S_3}[u,v]^ph_K^{1-p}(v)dS(K,v),$$

where

$$S_1=\{v\in S^{n-1}:h_K(v)>h_L(v)\},\quad S_2=\{v\in S^{n-1}:h_K(v)<h_L(v)\},$$

and

$$S_3 = \{v \in S^{n-1} : h_K(v) = h_L(v)\}.$$

Then, we have

$$\int_{S_1} [u,v]^p h_{K\cup L}^{1-p}(v) dS(K\cup L, v) = \int_{S_1} [u,v]^p h_K^{1-p}(v) dS(K, v),$$

$$\int_{S_2} [u,v]^p h_{K\cup L}^{1-p}(v) dS(K\cup L, v) = \int_{S_2} [u,v]^p h_L^{1-p}(v) dS(L, v),$$

$$\int_{S_3} [u,v]^p h_{K\cup L}^{1-p}(v) dS(K\cup L, v) = \int_{S_3} [u,v]^p h_K^{1-p}(v) dS(K, v),$$

and

$$\int_{S_1} [u,v]^p h_{K\cap L}^{1-p}(v) dS(K\cap L, v) = \int_{S_1} [u,v]^p h_L^{1-p}(v) dS(L, v),$$

$$\int_{S_2} [u,v]^p h_{K\cap L}^{1-p}(v) dS(K\cap L, v) = \int_{S_2} [u,v]^p h_K^{1-p}(v) dS(K, v),$$

$$\int_{S_3} [u,v]^p h_{K\cap L}^{1-p}(v) dS(K\cap L, v) = \int_{S_3} [u,v]^p h_L^{1-p}(v) dS(L, v).$$

Summing up both sides of the integrals above gives

$$\begin{aligned}\int_{S^{n-1}} [u,v]^p h_{K\cup L}^{1-p}(v) dS(K\cup L, v) + \int_{S^{n-1}} [u,v]^p h_{K\cap L}^{1-p}(v) dS(K\cap L, v)\\ = \int_{S^{n-1}} [u,v]^p h_K^{1-p}(v) dS(K, v) + \int_{S^{n-1}} [u,v]^p h_L^{1-p}(v) dS(L, v).\end{aligned}$$

Since this holds for any $u \in S^{n-1}$, it follows that

$$\widetilde{\Lambda_{-p}}(K\cup L) \widetilde{+}_{-p} \widetilde{\Lambda_{-p}}(K\cap L) = \widetilde{\Lambda_{-p}} K \widetilde{+}_{-p} \widetilde{\Lambda_{-p}} L.$$

For the general case, we shall use Weil's Approximation Lemma (see, e.g., [48]): if $K, L, K\cup L \in \mathcal{K}_o^n$, then K and L can be approximated by sequences of $K_i, L_i \in \mathcal{K}_o^n$ that are both strictly convex and smooth and such that $K_i \cup L_i \in \mathcal{K}_o^n$. Together with the weak continuity of $S_p(K, \cdot)$ (see, e.g., [41]), the desired result follows. $\square$

Theorems 6.1 and 6.2 immediately yield Theorem 1.2.

7. Open problems

Recall that a positive definite $n \times n$ symmetric matrix A generates an ellipsoid $E(A)$ in $\mathbb{R}^n$ defined by

$$E(A) = \{y \in \mathbb{R}^n : y \cdot Ay \leq 1\}.$$

Together with (1.1) and (2.18), the Legendre ellipsoid for $K \in \mathcal{S}_o^n$ can be defined by

$$\Gamma_2 K = \sqrt{\frac{n+2}{V(K)}} E(M_2(K)^{-1}),$$

where $M_2(K)$ is the moment matrix of K with entries

$$\int_K y_i y_j dy.$$

Together with (1.3) and (2.18), the LYZ ellipsoid for $K \in \mathcal{K}_o^n$ can be defined by

$$\Gamma_{-2} K = \sqrt{V(K)} E(M_{-2}(K)),$$

where the matrix $M_{-2}(K)$ has entries

$$\int_{S^{n-1}} v_i v_j dS_2(K, v).$$

In [32], Ludwig showed that only $M_2(K)$ and $M_{-2}(K)$ are Borel measurable, $\mathrm{GL}(n)$ covariant matrix valued valuations on the space of n-dimensional convex polytopes.

As in the proofs of Theorems 6.1 and 6.2, the corresponding matrices of our new sine ellipsoids defined in (2.20) and (2.22) are actually matrix valuations. Moreover, it is easy to verify that for any $O \in \mathrm{O}(n)$ and $K \in \mathcal{K}_o^n$,

$$D_2(OK) = OD_2(K)O^t \quad \text{and} \quad D_{-2}(OK) = OD_{-2}(K)O^t.$$

In particular, when $n = 2$, it follows from (4.17) that

$$\int_K [x, y]^2 dy = \int_K |\psi_{\pi/2} x \cdot y|^2 dy = \int_K |x \cdot \psi_{\pi/2}^t y|^2 dy,$$

which gives

$$D_2(K) = \psi_{\pi/2}^t M_2(K) \psi_{\pi/2}.$$

Note that the operator $\psi^t_{\pi/2}M_2\psi_{\pi/2}$ for convex polytopes in $\mathbb{R}^2$ was already characterized by Ludwig [32]. So we may ask the following question.

Question 1. *How to characterize the matrix-valued valuations of the new ellipsoids Λ_2K and $\Lambda_{-2}K$ for $n \geq 3$?*

The Cramér-Rao inequality (see, e.g., [10]) is a fundamental inequality in information theory, which states that, for a random vector $x \in \mathbb{R}^n$ with the probability density f,

$$v \cdot Cv \geq v \cdot F^{-1}v$$

holds for all $v \in \mathbb{R}^n$, where C is the covariance matrix with entries

$$\int_{\mathbb{R}^n} x_i x_j f(x)dx,$$

and F is the Fisher information matrices with entries

$$\int_{\mathbb{R}^n} \frac{\partial \log f}{\partial x_i}\frac{\partial \log f}{\partial x_j} f(x)dx.$$

Equality holds if and only if the distribution f is Gaussian. The Cramér-Rao inequality gives a lower bound on the variance of any unbiased estimator, which is very helpful in extracting useful information from noisy signals in information theory.

Lutwak, Yang, and Zhang [44] observed that there exists in fact a "dictionary" connecting the subject of information theory and the L_2 Brunn-Minkowski theory. In this dictionary a probability distribution corresponds to a convex body and the entropy power of the distribution to the volume of the body. Thus, the ellipsoid $E(C^{-1})$ corresponds to the Legendre ellipsoid Γ_2K and the ellipsoid $E(F)$ corresponds to the ellipsoid $\Gamma_{-2}K$. The Cramér-Rao inequality can be read as

$$E(F) \subset E(C^{-1}). \tag{7.1}$$

Corresponding to (7.1), Lutwak, Yang, and Zhang extended the domain of Γ_2 to star-shaped sets and established the remarkable geometric inclusion: if K is a star-shaped set in $\mathbb{R}^n$, then

$$\Gamma_{-2}K \subset \Gamma_2K, \tag{7.2}$$

with equality if and only if K is an origin-centered ellipsoid.

Remark 4.6 and Remark 5.6 show that, for $n = 2$,

$$\Lambda_2K = \psi^t_{\pi/2}\Gamma_2K \quad \text{and } \Lambda_{-2}K = \psi^t_{\pi/2}\Gamma_{-2}K.$$

Thus, it follows from (7.2) that when $n = 2$,

$$\Lambda_{-2}K \subset \Lambda_2 K,$$

with equality if and only if K is an origin-centered ellipsoid. So, the following question is of significant interest.

Question 2. *Is there a Cramér-Rao inclusion for the new ellipsoids $\Lambda_2 K$ and $\Lambda_{-2}K$ for $n \geq 3$?*

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