ORLICZ MOMENT REARRANGEMENT INEQUALITY

DONGMENG XI

Department of Mathematics, Shanghai University, Shanghai, China

YIMING ZHAO

Department of Mathematics, Massachusetts Institute of Technology, MA USA

ABSTRACT. A new rearrangement inequality is established by using n-times Steiner symmetrization at a step. The necessity of applying n-times Steiner symmetrization at a step, as opposed to the usual Steiner symmetrization, is demonstrated with an example. This inequality is an Orlicz extension of Lutwak, Yang & Zhang's moment-entropy inequality. Moreover, it has the L_p Blaschke-Santaló inequality as well as the classical Blaschke-Santaló inequality as its special cases.

1. INTRODUCTION

Let K be an origin-symmetric convex body (compact convex set with non-empty interior) in $\mathbb{R}^n$. Define its polar body K^* as $\{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}$. A quantity of immense interest is the *Mahler volume*

$$V(K)V(K^*). \tag{1.1}$$

It is not hard to see that the Malher volume (1.1) is invariant under linear transformations. The sharp lower bound for (1.1), except for the dimension 2, remains a mystery in convex geometry and is conjectured to be attained at cubes, cross-polytopes and their linear images in $\mathbb{R}^3$ and even more bodies in $\mathbb{R}^n$ for $n > 3$. This is known as the *Mahler conjecture*. See, e.g. page 564 in Schneider [37]. The upper bound, however, is well-known and is characterized by the celebrated Blaschke-Santaló inequality which states that the upper bound is attained precisely when K is the linear image of a ball, i.e., ellipsoid.

E-mail addresses: `dongmeng.xi@live.com, yimingzh@mit.edu`.

2000 *Mathematics Subject Classification.* 52A20, 52A40.

Key words and phrases. rearrangement inequality, Blaschke-Santaló inequality.

The Blaschke-Santaló inequality is equivalent to the affine isoperimetric inequality that characterizes the relationship between affine surface area and volume.

The following geometric inequality (see Lutwak, Yang & Zhang [32]) can be viewed as a generalization of the Blaschke-Santaló inequality. It states that for convex bodies K, L with fixed volumes and $\lambda(t) = |t|^p$ where $p \geq 1$,

$$\int_K \int_L \lambda(x \cdot y) dx dy \tag{1.2}$$

has a lower bound and is reached precisely when K and L are (up to a set of measure 0) dilates of *polar reciprocal* origin-symmetric ellipsoids, i.e. there exist $\phi \in \mathrm{GL}(n)$ and $c_1, c_2 > 0$ such that $K = c_1\phi(B)$ and $L = c_2\phi^{-t}(B)$. Here B is the unit ball in $\mathbb{R}^n$. To see that (1.2) is a generalization of the Blaschke-Santaló inequality, one simply has to note that by letting $p \to \infty$ and $L = K^*$ when K is origin-symmetric, one recovers the classical Blaschke-Santaló inequality. Inequality (1.2) was obtained as a corollary of the following L_p *moment entropy inequality*: for probability densities f and g, and $\lambda(t) = |t|^p$ where $p \geq 1$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \lambda(x \cdot y) f(x) g(y) dx dy, \tag{1.3}$$

has a lower bound if both the q-th Rényi entropy powers of f and g are fixed for some $q > \frac{n}{n+p}$. Here, the q-th Rényi entropy power of, say f, is given by

$$\left(\int_{\mathbb{R}^n} f^q(x) dx\right)^{\frac{1}{1-q}} \text{ for } q \neq 1 \text{ and } \exp\left\{-\int_{\mathbb{R}^n} f(x) \log f(x) dx\right\} \text{ for } q = 1.$$

The lower bounds are achieved precisely when f and g are almost everywhere proportional to the characteristic functions of the dilates of polar reciprocal origin-symmetric ellipsoids.

The Orlicz-Brunn-Minkowski theory stems from the two papers by Lutwak, Yang & Zhang [33, 34]. See also Gardner, Hug & Weil [12] and Xi, Jin & Leng [43]. Problems in the Orlicz-Brunn-Minkowski theory are often non-trivial extensions of their counterparts in the L_p theory. The fact that one loses homogeneity when replacing $|t|^p$ by a generic convex function often makes the proofs fundamentally different and challenging.

The main purpose of the present paper is an attempt to generalize both (1.2) and (1.3) to the Orlicz setting, along with their equality conditions, for any even convex function λ with some mild restrictions on the functions f and g. It should be noted that the approach adopted in [32] relies strongly on the fact that $\lambda(t) = |t|^p$ is homogeneous—something that is *critically* missing in the Orlicz theory.

Both (1.2) and (1.3) are isoperimetric in nature since their extremal cases are characterized by round objects—in this case, polar-reciprocal ellipsoids. There is a long history of establishing functional isoperimetric inequalities using their geometric counterparts. To do that, one needs to find a way to link functional objects with geometric ones.

One way to obtain such a link is to use solutions to Minkowski-type problems. In general, a Minkowski-type problem asks for the necessary and sufficient condition(s) on a given measure so that it is a certain geometric measure generated by a particular convex body. In using the solution to the classical Minkowski problem (the problem of prescribing Gauss curvature), Zhang [45] used a generalized Petty projection inequality and established the affine Sobolev inequality, which is stronger than the classical Sobolev inequality. Both the classical Minkowski problem and the Petty projection inequality are critical elements of the Brunn-Minkowski theory of convex bodies. The L_p Brunn-Minkowski theory, introduced by Lutwak [28, 29], is a highly non-trivial extension of the Brunn-Minkowski theory and has over the course of the last two decades become the center in the field of convex geometry, see, for example, [3–5, 8, 11, 14, 19–22, 22, 24–26, 38, 41, 42, 47–49]. The L_p Minkowski problem and the L_p Petty projection inequality (see [30]) are the counterparts of the Minkowski problem and the Petty projection inequality in the L_p theory. Using these two ingredients, Lutwak, Yang & Zhang [31] were able to establish a family of functional affine isoperimetric inequalities—known as *the L_p affine Sobolev inequality*, each of which is more powerful than the classical L_p Sobolev inequality. These inequalities have since then been extended to more general cases and inspired many functional isoperimetric inequalities, see, for example, [16–18, 39]. Despite the fact that many geometric isoperimetric inequalities have been established in the Orlicz setting (see, e.g., [2, 33, 34, 44, 46]), most of them are yet to be utilized to establish their functional counterparts. This is partially due to the fact that the Orlicz Minkowski problem (see [13]) does not yet have a complete solution. More importantly, because of the loss of homogeneity as previously mentioned, quantities in the Orlicz Brunn-Minkowski theory are often defined as solutions to optimization problems (Luxemberg norm), see, for example, the definitions of Orlicz projection and centroid body in [33, 34]. This feature makes it particularly difficult to "translate" geometric inequalities in the Orlicz Brunn-Minkowski theory.

It should be noted that many geometric isoperimetric inequalities can be established using symmetrization techniques. In particular, the classical isoperimetric inequality can be proved by showing that the surface area of a convex body is non-increasing under Steiner symmetrization. It makes sense to apply symmetrization techniques directly to functions. The resulting inequalities are known as *rearrangement inequalities*. An overview of classical rearrangement inequalities and an introduction on symmetrization techniques can be found in Burchard [6]. Perhaps the most well-known rearrangement inequality is the

Pólya-Szegö principle, from which both the classical geometric isoperimetric inequality and the Sobolev inequality (with optimal constants) can be derived. The Pólya-Szegö principle and its various extensions are still attracting much attentions, see, for example, [1, 7, 9, 10, 15, 23, 35, 40].

We shall establish the following rearrangement inequality.

Theorem 1.1. *Let f, g be two non-negative, quasi-concave, and integrable functions on $\mathbb{R}^n$. Let λ be an even convex function on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}^n}\int_{R^n}\lambda(x\cdot y)f(x)g(y)dxdy \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\lambda(x\cdot y)f^*(x)g^*(y)dxdy,$$

Moreover, if λ is strictly convex, then equality holds if and only if the closures of $\{x \in \mathbb{R}^n : f(x) > t\}$ and $\{y \in \mathbb{R}^n : g(y) > s\}$ are dilates of a common pair of polar reciprocal origin-symmetric ellipsoids, for almost all $t, s > 0$.

Here f^* and g^* are the *symmetric decreasing rearrangement* of f and g respectively.

For a convex body K, write B_K as the ball who has the same volume as K. As part of the proof, we shall establish the following isoperimetric inequality.

Theorem 1.2. *Let $K, L \subset \mathbb{R}^n$ be convex bodies and λ be an even convex function defined on $\mathbb{R}$. Then,*

$$\int_K\int_L \lambda(x\cdot y)dxdy \geq \int_{B_K}\int_{B_L}\lambda(x\cdot y)dxdy, \tag{1.4}$$

Moreover, if λ is strictly convex, equality holds in (4.16) if and only if K and L are dilates of a pair of polar reciprocal origin-symmetric ellipsoids.

This is an extension of the inequality (1.2) shown in [32], as choosing $\lambda(t) = |t|^p$ recovers it. See Theorem 4.4.

The main difficulty in establishing (1.4) is that the quantity

$$\int_K\int_L \lambda(x\cdot y)dxdy \tag{1.5}$$

could increase under one single application of Steiner symmetrization (see the example in the Appendix). To deal with it, we apply n-times Steiner symmetrization (with respect to an orthonormal basis) at a time to K and L *simultaneously* (although with opposite orders) and show that the quantity (1.5) is non-increasing under n-times Steiner symmetrizations. See Lemma 4.1. Properties of Steiner symmetrization, particularly applying it n-times at a step, are included in Section 3.

A quantity related to (1.5) is

$$\int_{\partial K}\int_{\partial L}|\sigma_K(x)\cdot\sigma_L(y)|dxdy, \tag{1.6}$$

where σ_K and σ_L are the Gauss maps of K and L, respectively. This quantity is closely related to Petty's conjecture (see Page 570 in [37]), which is one of the major problems in the area of affine isoperimetric inequality for volume of projection bodies. Lutwak [27] showed that the conjecture that the minimum of (1.6) for K and L with fixed volume is attained at a pair of polar-reciprocal ellipsoids is equivalent to Petty's conjecture. The volume of projection body shares a common feature with the central quantity (1.5) considered in the current paper: it does not necessarily decrease under one single application of Steiner symmetrization. An example of this was provided in Theorem 3 in [36]. The procedure adopted in the current paper could possibly be developed to deal with the quantity (1.6).

2. Basic notations

We will use $x^{(i)}$ to denote the i-th component of a point $x\in\mathbb{R}^n$.

Throughout the paper, by convex body, we mean a compact convex subset of $\mathbb{R}^n$ with non-empty interior. We will write $\mathcal{K}^n$ for the set of all convex bodies in $\mathbb{R}^n$. For $K\in\mathcal{K}^n$, we shall write B_K for the ball in $\mathbb{R}^n$ centered at the origin with the same volume as K.

If K contains the origin in its interior, the *polar body* of K, denoted by K^*, is the convex body given by

$$K^*=\{y\in\mathbb{R}^n : y\cdot x\le 1, \forall x\in K\}.$$

It is not hard to see that for a linear transformation ϕ, we have $(\phi K)^*=\phi^{-t}K^*$. Thus, the polar body of an origin-symmetric ellipsoid is also an origin-symmetric ellipsoid. In particular, if $E=\phi B$, then $E^*=\phi^{-t}B$. Here B is the unit ball in $\mathbb{R}^n$. Such a pair of ellipsoids are said to be *polar reciprocal* to each other.

For $u\in S^{n-1}$, denote by K_u the image of the orthogonal projection of K onto $u^\perp$. We write $\overline{\ell}_u(K;y'): K_u\to\mathbb{R}$ and $\underline{\ell}_u(K;y'): K_u\to\mathbb{R}$ for the *overgraph* and *undergraph functions* of K in the direction u; i.e.

$$K=\{y'+tu : -\underline{\ell}_u(K;y')\le t\le\overline{\ell}_u(K;y') \text{ for } y'\in K_u\}.$$

Clearly, they are concave functions if K is a convex body.

The *Steiner symmetral* S_uK of $K\in\mathcal{K}^n$ in the direction u can be defined as the body whose orthogonal projection onto $u^\perp$ is identical to that of K and whose overgraph and undergraph functions are given by

$$\bar{\ell}_u(S_uK;y') = \underline{\ell}_u(S_uK;y') = \frac{1}{2}[\bar{\ell}_u(K;y') + \underline{\ell}_u(K;y')].$$

Let f be an integrable function on $\mathbb{R}^n$. The *symmetric decreasing rearrangement* of f, denoted by f^*, is the radial symmetric and decreasing function such that for each $t \in \mathbb{R}$,

$$\mathcal{H}^n(\{x \in \mathbb{R}^n : f(x) > t\}) = \mathcal{H}^n(\{x \in \mathbb{R}^n : f^*(x) > t\}).$$

Clearly, the level sets of f^* are balls centered at the origin.

3. Steiner symmetrization and its properties

By the definition of Steiner symmetrization, we have $(S_u K)_u = K_u$ for each $u \in S^{n-1}$. Moreover, the Steiner symmetral $S_u K$ is symmetric with respect to the hyperplane $u^\perp$. Also obvious is the fact that if $K \subset L$, then

$$S_uK \subset S_uL, \tag{3.1}$$

for each $u \in S^{n-1}$.

Lemma 3.1. *Let $K \subset \mathbb{R}^n$ be a convex body and $u \in S^{n-1}$. Suppose $v \in u^\perp \cap S^{n-1}$. If K is symmetric with respect to $v^\perp$, then $S_u K$ is also symmetric with respect to $v^\perp$.*

Proof. For each $x \in \mathbb{R}^n$, write x as

$$x = tu + sv + y'',$$

where $t, s \in \mathbb{R}$ and $y'' \in u^\perp \cap v^\perp$.

Suppose $x_0 \in S_u K$ and $x_0 = t_0u + s_0v + y_0''$. Let $y_0' = s_0v + y_0'' \in u^\perp$ and $z_0' = -s_0v + y_0''$. Since K is symmetric with respect to $v^\perp$, the orthogonal image K_u is also symmetric with respect to $v^\perp$. Hence $z_0' \in K_u$. Also, since K is symmetric with respect to $v^\perp$, the point $tu+sv+y_0'' \in K$ if and only if $tu-sv+y_0'' \in K$, where $t, s \in \mathbb{R}$. Hence $\underline{l}_u(K; y_0') = \underline{l}_u(K; z_0')$ and $\bar{l}_u(K; y_0') = \bar{l}_u(K; z_0')$. Thus, we have

$$t_0u - s_0v + y_0'' \in S_u K.$$

Hence $S_u K$ is symmetric with respect to $v^\perp$. $\square$

The next corollary follows immediately from the previous lemma and that the Steiner symmetral $S_u K$ is symmetric with respect to $u^\perp$.

Corollary 3.2. *Let $K \subset \mathbb{R}^n$ be a convex body and $e_1, \cdots, e_n$ be an orthonormal basis. Define*

$$K_n = S_{e_1} S_{e_2} \cdots S_{e_n} K.$$

The convex body K_n is 1-unconditional; i.e., K_n is symmetric with respect to $e_i^\perp$ for all $i = 1, 2, \cdots, n$.

For each convex body K, write B_K for the ball centered at the origin with the same volume as K. Let $u \in S^{n-1}$. We claim that if $d_H(K, B_K)$ is less than the radius of B_K, then

$$d_H(S_u K, B_K) \le d_H(K, B_K). \tag{3.2}$$

To see this, write r_0 to be the radius of B_K. By the definition of Hausdorff distance, for any $d_H(K, B_K) < \varepsilon < r_0$, we have

$$K \subset B_K + \varepsilon B, \tag{3.3}$$

and

$$B_K \subset K + \varepsilon B,$$

or, equivalently by the fact that $\varepsilon < r_0$,

$$(r_0 - \varepsilon) B \subset K. \tag{3.4}$$

Applying Steiner symmetrization S_u to both sides of (3.3) and (3.4), and using (3.1), we have

$$S_u K \subset B_K + \varepsilon B, \tag{3.5}$$

and

$$(r_0 - \varepsilon) B \subset S_u K,$$

or, equivalently by the fact that $\varepsilon < r_0$,

$$B_K \subset S_u K + \varepsilon B. \tag{3.6}$$

Equations (3.5) and (3.6), definition of Hausdorff distance, and the fact that ε can be arbitrarily close to $d_H(K, B_K)$, immediately imply (3.2).

Lemma 3.3. *Let K be a convex body in $\mathbb{R}^n$. There exists a sequence of ordered orthonormal bases $e_1^i, \ldots, e_n^i$ such that*

$$K_i \text{ converges to } B_K \text{ in Hausdorff metric,}$$

where B_K is the ball centered at the origin with $V(B_K) = V(K)$. Here, $K_0 = K$ and $K_i = S_{e_n^i} \ldots S_{e_1^i} K_{i-1}$.

Proof. Suppose $I : e_1, \dots, e_n$. For simplicity, we shall write $S_I = S_{e_n} \cdots S_{e_1}$.

Define the set

$$\mathcal{Q} = \{S_{I_k} S_{I_{k-1}} \dots S_{I_1} K : \text{ orthonormal bases } I_1, \dots, I_k \text{ and } k > 0\}.$$

For each $Q \in \mathcal{K}_o^n$, write r_Q as the outer radius of Q, i.e., the smallest $r > 0$ such that $Q \in rB$. Set $r_0 = \inf_{Q \in \mathcal{Q}} r_Q$. Let Q_i be a sequence in $\mathcal{Q}$ such that $r_{Q_i} \to r_0$. Since the set $\mathcal{Q}$ is uniformly bounded as a result of (3.1), we can invoke Blaschke's selection theorem and assume (by possibly taking a subsequence) that Q_i converges in Hausdorff metric to a non-empty compact convex set Q_0.

By the choice of r_0, it is apparent that $Q_0 \subset r_0 B$. We claim that $Q_0 = r_0 B$. To see this, we prove by contradiction. Assume that Q_0 is strictly contained in $r_0 B$ and $Q_0 \neq r_0 B$. Therefore, there exists $x_0 \in \partial(r_0 B)$ and a neighborhood U of x_0 such that $U \cap \partial(r_0 B)$ contains non-empty interior with respect to the induced topology on $\partial(r_0 B)$ and $U \cap Q_0 = \emptyset$. Note that for any line ξ passing through $U \cap \partial(r_0 B)$ and not tangent to $r_0 B$, the length of the line segment $\xi \cap r_0 B$ is strictly larger than the length of the line segment $\xi \cap Q_0$. This suggests that for each ordered orthonormal basis $I : e_1, \dots, e_n$, the convex body $S_I K$ will not intersect $U \cap \partial(r_0 B)$ and C_i where C_i is the reflection of $U \cap \partial(r_0 B)$ with respect to $u_i^\perp$. Since $\partial(r_0 B)$ is compact, we may choose a finite number of orthonormal bases, say $I_1, \dots, I_k$, so that $U \cap \partial(r_0 B)$ together with the reflections generated by it with respect to $u^\perp$ for $u \in \cup_k I_k$ will form a finite cover of $\partial(r_0 B)$. Therefore $S_{I_k} \cdots S_{I_1} K \subset \operatorname{int} K$ and as a result, $r_Q < r_0$ for $Q = S_{I_k} \cdots S_{I_1} K \in \mathcal{Q}$, which is a contradiction to the choice of r_0.

Hence, there exists a sequence $Q_i \in \mathcal{Q}$ such that $Q_i \to r_0 B$ in Hausdorff metric. Moreover, it can be easily seen that $r_0 B = B_K$ since Steiner symmetrization preserves volume.

Towards this end, let ε_k be a sequence of sufficiently small positive numbers (less than r_0) and $\varepsilon_k \to 0$. Choose $K_1 \in \mathcal{Q}$ such that $d_H(K_1, B_K) < \varepsilon_1$. Now, applying the above argument again but this time on K_1 instead of on K allows us to conclude the existence of $I_1, I_2, \dots, I_m$ and $K_2 = S_{I_m} \cdots S_{I_1} K_1 \in \mathcal{Q}$ such that $d_H(K_2, B_{K_1}) < \varepsilon_2$. Notice that $B_{K_1} = B_K$ since Steiner symmetrization preserves volume. Hence $d_H(K_2, B_K) < \varepsilon_2$. Carrying on this process, we can find a sequence $K_i \in \mathcal{Q}$ such that $d_H(K_i, B_K) < \varepsilon_k$.

To reach the desired result, we only need to use (3.2) to conclude that the Hausdorff distance is non-increasing after applying each Steiner symmetrization. $\square$

Lemma 3.4. *Let K and L be two convex bodies in $\mathbb{R}^n$. There exists a sequence of orthonormal bases $e_1^i, \dots, e_n^i$ such that*

K_i and L_i converges to B_K and B_L in Hausdorff metric respectively,

where B_K and B_L are the balls centered at the origin with $V(B_K) = V(K)$ and $V(B_L) = V(L)$. Here, $K_0 = K$, $L_0 = L$ and

$$K_i = S_{e_n^i} \cdots S_{e_1^i} K_{i-1}, \qquad L_i = S_{e_1^i} \cdots S_{e_n^i} L_{i-1} \tag{3.7}$$

Proof. Suppose $I : e_1, \ldots, e_n$. For simplicity, we shall write $S_I = S_{e_n} \cdots S_{e_1}$ and $S_{-I} = S_{e_1} \cdots S_{e_n}$. Let ε_k be a sequence of sufficiently positive numbers such that $\varepsilon_k \to 0$.

By Lemma 3.3, there exists orthonormal bases $I_1, \ldots, I_{k_1}$ such that $d_H(\widetilde{K_1}, B_K) < \varepsilon_1$ for $\widetilde{K_1} = S_{I_{k_1}} \cdots S_{I_1} K$.

Let $\widetilde{L_1} = S_{-I_{K_1}} \cdots S_{-I_1} L$. Apply Lemma 3.3 to $\widetilde{L_1}$ and we have that there exists orthonormal bases $I_{k_1+1}, \ldots, I_{k_1+k_2}$ such that $d_H(\widetilde{L_2}, B_{\widetilde{L_1}}) = d_H(\widetilde{L_2}, B_L) < \varepsilon_2$ for $\widetilde{L_2} = S_{-I_{k_1+k_2}} \cdots S_{-I_{k_1+1}} \widetilde{L_1}$.

Set $\widetilde{K_2} = S_{I_{k_1+k_2}} \cdots S_{I_{k_1+1}} \widetilde{K_1}$. We continue in this fashion, by applying Lemma 3.3 alternatively to the sequences $\widetilde{K_i}$ and $\widetilde{L_i}$. This allows us to conclude a sequence of orthonormal bases I_i and sequences $\widetilde{K_i}$, $\widetilde{L_i}$ such that $K_i \to B_K$ and $L_i \to B_L$.

Equation (3.2) now allows us to conclude that I_i satisfies (3.7). □

The following lemma is a direct consequence of Lemma 3.4 and the fact that convergence in Hausdorff metric implies convergence of characteristic functions in L_1 norm.

Lemma 3.5. *Let K and L be two convex bodies in $\mathbb{R}^n$. There exists a sequence of ordered orthonormal bases $e_1^i, \ldots, e_n^i$ such that*

$$\lim_{i \to \infty} \|1_{K_i} - 1_{B_K}\|_1 = 0, \qquad \lim_{i \to \infty} \|1_{L_i} - 1_{B_L}\|_1 = 0, \tag{3.8}$$

where $K_0 = K$, $L_0 = L$, and

$$K_i = S_{e_n^i} \ldots S_{e_1^i} K_{i-1}, \qquad L_i = S_{e_1^i} \ldots S_{e_n^i} L_{i-1}. \tag{3.9}$$

For any fixed convex body K and $u \in S^{n-1}$, write $x \in K$ as

$$x = y' + tu,$$

where $y' \in u^\perp$ and $t \in \mathbb{R}$. Define $\phi_{K,u} : K \to S_u K$ by

$$\phi_{K,u}(x) = x - \frac{1}{2}(\bar{l}_u(K; y') - \underline{l}_u(K; y'))u,$$

where $x \in K$ and $x = y' + tu$. Intuitively, the map $\phi_{K,u}$ moves each point x in K in the direction of u so that $\phi_{K,u}(K \cap \{y' + tu : t \in \mathbb{R})$ is a line segment symmetric about the hyperplane $u^\perp$.

Note that $\phi_{K,u}$ is one-to-one and onto. Let $\psi_{K,u} : S_u K \to K$ be the inverse of $\phi_{K,u}$; i.e., for each $x \in S_u K$,

$$\psi_{K,u}(x) = x + \frac{1}{2}(\bar{l}_u(K;y') - \underline{l}_u(K;y'))u, \tag{3.10}$$

where $x = y' + tu$.

Lemma 3.6. *Let $K \subset \mathbb{R}^n$ be a convex body and $u \in S^{n-1}$. The function $\psi_{K,u}$ as defined in (3.10) is a continuous function on $S_u K$ and is differentiable except for at most countably many points. Moreover, if $x \in int\, S_u K$ is a differentiable point for $\psi_{K,u}$, then the Jacobian matrix of $\psi_{K,u}$ at x has determinant 1.*

Proof. That $\psi_{K,u}$ is continuous and almost everywhere differentiable is immediate from the fact that both $\underline{l}_u(K;\cdot)$ and $\bar{l}_u(K;\cdot)$ are concave.

That the Jacobian matrix of $\psi_{K,u}$ at each differentiable point $x \in S_u K$ comes from the fact that

$$\psi_{K,u}(x) \cdot v = x \cdot v,$$

for each $v \in u^{\perp}$. □

Let $K \subset \mathbb{R}^n$ be a convex body and $I : e_1, \ldots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. Define $K_0 = K$ and

$$K_i = S_{e_i} K_{i-1},$$

for $i = 1, \ldots, n$. Define $\Psi_{K,I} : K_n \to K_0 = K$ as

$$\Psi_{K,I} = \psi_{K_0,e_1} \circ \psi_{K_1,e_2} \circ \ldots \psi_{K_{n-1},e_n} \tag{3.11}$$

where the ψ's are as defined in (3.10). Note that by Corollary 3.2, the convex body K_n is 1-unconditional.

The map $\Psi_{K,I}$ may be expressed using the following lemma.

Lemma 3.7. *Let $K \subset \mathbb{R}^n$ be a convex body and $I : e_1, \ldots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. Define $\Psi_{K,I}$ as in (3.11). Then, for each $i = 1, \ldots n$, there exists $l_{K,i} : K_n \to \mathbb{R}$ such that $l_{K,i}$ is symmetric in its first i arguments and the i-th coordinate of $\Psi_{K,I}$ may be expressed as*

$$[\Psi_{K,I}(x_1, \ldots, x_n)]^{(i)} = x_i + l_{K,i}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = x_i + l_{K,i}(|x_1|, \ldots, |x_i|, x_{i+1}, \ldots, x_n), \tag{3.12}$$

for each $x = (x_1, \ldots, x_n) \in K_n$. Moreover, the map $\Psi_{K,I}$ is differentiable almost everywhere and its Jacobian is 1 wherever it is defined.

Proof. Note that by the definition of Steiner symmetrization, the i-th coordinate of a point $x \in K_n$ can only be changed by ψ_{K_{i-1},e_i}. Hence,

$$[\Psi_{K,I}(x_1,\ldots,x_n)]^{(i)}(x_1,\ldots,x_n) = \psi_{K_{i-1},e_i} \circ \psi_{K_i,e_{i+1}} \circ \cdots \circ \psi_{K_{n-1},e_n}(x_1,\ldots,x_n).$$

The same observation shows that the first i coordinates remain unchanged under $\psi_{K_i,e_{i+1}} \circ \cdots \circ \psi_{K_{n-1},e_n}$; that is,

$$\psi_{K_i,e_{i+1}} \circ \cdots \circ \psi_{K_{n-1},e_n}(x_1,\ldots,x_n) = (x_1,\ldots,x_i,\tilde{x}_{i+1},\ldots,\tilde{x}_n),$$

where $\tilde{x}_j = \tilde{x}_j(x_1,\ldots,x_n)$. Note that by Lemma 3.1, the convex bodies $K_i,\ldots,K_n$ are symmetric with respect to $e_1,\ldots,e_i$. This implies that $\tilde{x}_j(x_1,\ldots,x_n)$ is symmetric with respect to its first i arguments; that is,

$$\tilde{x}_j = \tilde{x}_j(x_1,\ldots,x_n) = \tilde{x}_j(|x_1|,\ldots,|x_i|,x_{i+1},\ldots,x_n). \tag{3.13}$$

By (3.10),

$$\begin{aligned} &\left[\psi_{K_{i-1},e_i} \circ \cdots \circ \psi_{K_{n-1},e_n}(x_1,\ldots,x_n)\right]^{(i)} \\ =&\left[\psi_{K_{i-1},e_i}(x_1,\ldots,x_i,\tilde{x}_{i+1},\ldots,\tilde{x}_n)\right]^{(i)} \\ =&x_i + \frac{1}{2}(\bar{l}_{e_i}(K_{i-1};(x_1,\ldots,x_{i-1},0,\tilde{x_{i+1}},\ldots,\tilde{x}_n)) - \underline{l}_{e_i}(K_{i-1};(x_1,\ldots,x_{i-1},0,\tilde{x}_{i+1},\ldots,\tilde{x}_n)))\end{aligned} \tag{3.14}$$

Note that K_{i-1} symmetric with respect to $e_1,\ldots,e_{i-1}$. Hence, both $\bar{l}_{e_i}(K_{i-1};\cdot)$ and $\underline{l}_{e_i}(K_{i-1};\cdot)$ are symmetric with respect to the first $(i-1)$ arguments. Define $l_{K,i}$ as

$$\begin{aligned} &l_{K,i}(x_1,\ldots,x_n) \\ =&\frac{1}{2}(\bar{l}_{e_i}(K_{i-1};(x_1,\ldots,x_{i-1},0,\tilde{x_{i+1}},\ldots,\tilde{x}_n)) - \underline{l}_{e_i}(K_{i-1};(x_1,\ldots,x_{i-1},0,\tilde{x}_{i+1},\ldots,\tilde{x}_n))).\end{aligned} \tag{3.15}$$

By (3.13) and the symmetry property we observed about $\bar{l}_{e_i}(K_{i-1};\cdot)$ and $\underline{l}_{e_i}(K_{i-1};\cdot)$, we conclude that $l_{K,i}$ is symmetric with respect to its first i arguments; that is,

$$l_{K,i}(x_1,\ldots,x_n) = l_{K,i}(|x_1|,\ldots,|x_i|,x_{i+1},\ldots,x_n). \tag{3.16}$$

Equations (3.14), (3.15), and (3.16) imply (3.12).

The facts that $\Psi_{K,I}$ is almost everywhere differentiable and its Jacobian is 1 wherever it is defined follow immediately from its definition, Lemma 3.6, and the fact that Steiner symmetrization is volume preserving. $\square$

4. Proof of the rearrangement inequality

Let $I : e_1, \dots, e_n$ be an ordered list of orthonormal basis in $\mathbb{R}^n$. Denote by $-I$ the reversed list; that is, $-I : e_n, \dots, e_1$. Let K, L be two convex bodies in $\mathbb{R}^n$. Define $K_0 = K$, $L_0 = L$, and

$$K_i = S_{e_i} K_{i-1}, \qquad L_i = S_{e_{n-i+1}} L_{i-1} \tag{4.1}$$

for $i = 1, \dots, n$. Consider $\Psi_{K,I}$ and $\Psi_{L,-I}$ as defined in (3.11). In particular,

$$\Psi_{L,-I} = \psi_{L_0,e_n} \circ \psi_{L_1,e_{n-1}} \circ \cdots \circ \psi_{L_{n-1},e_1}.$$

By Lemma 3.7, there exist $l_{K,i} : K_n \to \mathbb{R}$ and $l_{L,i} : L_n \to \mathbb{R}$ such that

$$\begin{aligned}
[\Psi_{K,I}(x_1, \dots, x_n)]^{(i)} &= x_i + l_{K,i}(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = x_i + l_{K,i}(|x_1|, \dots, |x_i|, x_{i+1}, \dots, x_n), \\
[\Psi_{L,I}(y_1, \dots, y_n)]^{(i)} &= y_i + l_{L,i}(y_1, \dots, y_{i-1}, y_i, \dots, y_n) = y_i + l_{L,i}(y_1, \dots, y_{i-1}, |y_i|, \dots, |y_n|).
\end{aligned} \tag{4.2}$$

For notational simplicity, write

$$d_{K,i}(x_1, \dots, x_n) = \frac{1}{2} l_{K,i}(x_1, \dots, x_n) - \frac{1}{2} l_{K,i}(-x_1, \dots, -x_n), \tag{4.3}$$

and

$$d_{L,i}(y_1, \dots, y_n) = \frac{1}{2} l_{L,i}(y_1, \dots, y_n) - \frac{1}{2} l_{L,i}(-y_1, \dots, -y_n). \tag{4.4}$$

By (4.2), $d_{K,i}$ is symmetric with respect to its first i arguments and $d_{L,i}$ is symmetric with respect to its last $(n - i + 1)$ arguments; that is

$$\begin{aligned}
d_{K,i}(x_1, \dots, x_n) &= d_{K,i}(|x_1|, \dots, |x_i|, x_{i+1}, \dots, x_n), \\
d_{L,i}(y_1, \dots, y_n) &= d_{L,i}(y_1, \dots, y_{i-1}, |y_i|, \dots, |y_n|).
\end{aligned} \tag{4.5}$$

In particular,

$$d_{K,n} = 0 = d_{L,1}. \tag{4.6}$$

Moreover, by Lemma 3.7

$$\begin{aligned}
[\Psi_{K,I}(x) - \Psi_{K,I}(-x)]^{(i)} &= 2x_i + 2d_{K,i}(x), \\
[\Psi_{L,-I}(y) - \Psi_{L,-I}(-y)]^{(i)} &= 2y_i + 2d_{L,i}(y).
\end{aligned}$$

We first show that the integral

$$\int_K \int_L \lambda(x \cdot y) dx dy$$

is non-increasing under the Steiner symmetrizations (4.1).

Lemma 4.1. *Let $K, L \subset \mathbb{R}^n$ be two convex bodies and λ be an even convex function defined on $\mathbb{R}$. Suppose $I : e_1, \ldots, e_n$ is an orthonormal basis in $\mathbb{R}^n$ and K_i and L_i are as defined in (4.1). Then*

$$\int_K \int_L \lambda(x \cdot y) dx dy \geq \int_{K_n} \int_{L_n} \lambda(x \cdot y) dx dy. \tag{4.7}$$

Proof. By Lemma 3.7, we have

$$\int_K \int_L \lambda(x \cdot y) dx dy = \int_{K_n} \int_{L_n} \lambda(\Psi_{K,I}(x) \cdot \Psi_{L,-I}(y)) dx dy. \tag{4.8}$$

Since K_n and L_n are 1-unconditional, the following four integrals are identical:

$$\int_{K_n} \int_{L_n} \lambda(\Psi_{K,I}(x) \cdot \Psi_{L,-I}(y)) dx dy, \qquad \int_{K_n} \int_{L_n} \lambda(\Psi_{K,I}(-x) \cdot \Psi_{L,-I}(y)) dx dy,$$
$$\int_{K_n} \int_{L_n} \lambda(\Psi_{K,I}(-x) \cdot \Psi_{L,-I}(-y)) dx dy, \qquad \int_{K_n} \int_{L_n} \lambda(\Psi_{K,I}(x) \cdot \Psi_{L,-I}(-y)) dx dy.$$

Since λ is convex and even, we have

$$\frac{1}{4} [\lambda(\Psi_{K,I}(x) \cdot \Psi_{L,-I}(y)) + \lambda(\Psi_{K,I}(-x) \cdot \Psi_{L,-I}(y))] \geq \frac{1}{2} \lambda \left(\frac{1}{2} (\Psi_{K,I}(x) - \Psi_{K,I}(-x)) \cdot \Psi_{L,-I}(y) \right),$$
$$\frac{1}{4} [\lambda(\Psi_{K,I}(x) \cdot \Psi_{L,-I}(-y)) + \lambda(\Psi_{K,I}(-x) \cdot \Psi_{L,-I}(-y))] \geq \frac{1}{2} \lambda \left(\frac{1}{2} (\Psi_{K,I}(x) - \Psi_{K,I}(-x)) \cdot \Psi_{L,-I}(-y) \right).$$

Adding the two inequalities together and using the fact that λ is convex and even again, we get

$$\begin{aligned}
&\int_{K_n} \int_{L_n} \lambda(\Psi_{K_I}(x) \cdot \Psi_{L,-I}(y)) dx dy \\
&\geq \int_{K_n} \int_{L_n} \lambda \left(\frac{1}{4} (\Psi_{K,I}(x) - \Psi_{K,I}(-x)) \cdot (\Psi_{L,-I}(y) - \Psi_{L,-I}(-y)) \right) dx dy \\
&= \int_{K_n} \int_{L_n} \lambda \left(\sum_{i=1}^n x_i y_i + x_i d_{L,i}(y) + y_i d_{K,i}(x) + d_{K,i}(x) d_{L,i}(y) \right) dx dy \\
&= \int_{K_n} \int_{L_n} \lambda \left(\sum_{i=1}^n f_{i,1}(x,y) + f_{i,2}(x,y) + f_{i,3}(x,y) + f_{i,4}(x,y) \right) dx dy,
\end{aligned} \tag{4.9}$$

where $f_{i,1}(x,y) = x_i y_i$, $f_{i,2}(x,y) = x_i d_{L,i}(y)$, $f_{i,3} = y_i d_{K,i}(x)$, and $f_{i,4} = d_{K,i}(x) d_{L,i}(y)$. Write $f_i = f_{i,1} + f_{i,2} + f_{i,3} + f_{i,4}$.

Let Ω be the set of all diagonal matrices whose diagonal entries are either 1 or -1. Since K_n and L_n are symmetric with respect to each $e_i^\perp$, we have

$$\int_{K_n} \int_{L_n} \lambda(f(x,y)) dx dy = \int_{K_n} \int_{L_n} \lambda \left(\sum_{i=1}^n f_i(Ax, Ay) \right) dx dy.$$

for each $A \in \Omega$. This implies that

$$\begin{aligned}\int_{K_n}\int_{L_n}\lambda\left(\sum_{i=1}^n f_i(Ax,Ay)\right)dxdy &= \int_{K_n}\int_{L_n}\frac{1}{2^n}\sum_{A\in\Omega}\lambda\left(\sum_{i=1}^n f_i(Ax,Ay)\right)dxdy,\\ &\geq \int_{K_n}\int_{L_n}\lambda\left(\sum_{i=1}^n \frac{1}{2^n}\sum_{A\in\Omega} f_i(Ax,Ay)\right)dxdy,\end{aligned} \tag{4.10}$$

where the last inequality follows from the fact λ is convex.

We claim that

$$\frac{1}{2^n}\sum_{A\in\Omega} f_i(Ax,Ay) = x_i y_i. \tag{4.11}$$

By (4.3), (4.4), and (4.5), it is obvious that

$$d_{K,i}(x_1,\dots,x_i,-x_{i+1},\dots,-x_n) = -d_{K,i}(x_1,\dots,x_i,x_{i+1},\dots,x_n),$$

and

$$d_{L,i}(-y_1,\dots,-y_{i-1},y_i,\dots,y_n) = -d_{L,i}(y_1,\dots y_{i-1},y_i,\dots,y_n).$$

This, together with (4.5), implies

$$f_{i,2}(A_0x,A_0y) + f_{i,2}(x,y) = 0$$

for $A_0 = \mathrm{diag}(-1,\dots,-1,1,\dots,1)$ where there are $(i-1)$ many (-1)'s. Hence,

$$\sum_{A\in\Omega} f_{i,2}(Ax,Ay) = 0.$$

Similarly,

$$f_{i,3}(A_1x,A_1y) + f_{i,3}(x,y) = 0, \text{ and } f_{i,4}(A_1x,A_1y) + f_{i_4}(x,y) = 0$$

for $A_1 = \mathrm{diag}(1,\dots,1,-1,\dots,-1)$ where there are i many 1's. Hence,

$$\sum_{A\in\Omega} f_{i,3}(Ax,Ay) = 0 = \sum_{A\in\Omega} f_{i,4}(Ax,Ay).$$

On the other hand, it is clear that

$$f_{i,1}(Ax,Ay) = f_{i,1}(x,y),$$

for each $A \in \Omega$. Therefore, equation (4.11) is established.

Equations (4.8), (4.9), (4.10), and (4.11) immediately imply (4.7). □

When the function λ is strictly convex, the equality condition in (4.7) is stated in the next lemma.

Lemma 4.2. *If $\lambda(\cdot) : \mathbb{R} \to \mathbb{R}$ is strictly convex, then equality holds in (4.7) if and only if there is a linear transform $T \in SL(n)$ such that $\Psi_{K,I}(x) = Tx$ and $\Psi_{L,I}(y) = T^{-t}y$.*

Proof. Suppose equality holds in (4.7). Then equality must hold in (4.10). Since λ is strictly convex, this implies that

$$\sum_{i=1}^{n} f_i(Ax, Ay) = \sum_{i=1}^{n} f_i(x, y), \qquad \text{a.e. for } x \in K_n, y \in L_n,$$

for all $A \in \Omega$. This, together with (4.11), shows that

$$\sum_{i=1}^{n} f_i(x, y) = \sum_{i=1}^{n} x_i y_i, \qquad \text{a.e. for } x \in K_n, y \in L_n.$$

Or, equivalently by the definition of f_i,

$$\sum_{i=1}^{n} \left(x_i d_{L,i}(y) + y_i d_{K,i}(x) + d_{K,i}(x) d_{L,i}(y)\right) = 0, \tag{4.12}$$

for almost all $x \in K_n$ and $y \in L_n$. By continuity of $d_{K,i}$ and $d_{L,i}$, (4.12) is valid for all $x \in K_n$ and $y \in L_n$.

On the other side, equality in (4.7) implies equality in (4.9), which implies that

$$\Psi_{K,I}(x) \cdot \Psi_{L,-I}(y) = \frac{1}{2}(\Psi_{K,I}(x) - \Psi_{K,I}(-x)) \cdot \Psi_{L,-I}(y),$$

for almost all $x \in K_n$ and $y \in L_n$. This implies that

$$\Psi_{K,I}(x) \cdot y = \frac{1}{2}(\Psi_{K,I}(x) - \Psi_{K,I}(-x)) \cdot y,$$

for almost all $x \in K_n$ and $y \in L$. Since L contains interior points, we conclude that

$$\Psi_{K,I}(x) = \frac{1}{2}\left(\Psi_{K,I}(x) - \Psi_{K,I}(-x)\right).$$

This, combined with the continuity of the map $\Psi_{K,I}$, we have

$$\Psi_{K,I}(x) = -\Psi_{K,I}(-x), \tag{4.13}$$

for all $x \in K_n$. This, in turn, implies that K is origin-symmetric. To see this, suppose $y \in K$, then there exists $x \in K_n$ such that $y = \Psi_{K,I}(x)$. Since K_n is 1-conditional, $-x \in K_n$. Hence $-y = -\Psi_{K,I}(x) = \Psi_{K,I}(-x) \in K$.

The same argument for L implies that L is also origin-symmetric. Therefore, there exists $r > 0$ such that

$$r\sqrt{n}B_n \subset \operatorname{int}(K \cap L).$$

Towards this end, for each $k = 1, 2, \dots, n$, let $x^{(k)} = (r, r, \dots, r, 0, \dots, 0) \in \mathbb{R}^n$ where r appears k times. By (4.5) and the fact that $d_{K,i}$ is odd (from (4.3)), we have

$$d_{K,i}(x^{(k)}) = 0, \tag{4.14}$$

for $i \geq k$.

We will show, by induction (on i), that there exists constants $c_{i,j}$ with $2 \leq i \leq n$ and $1 \leq j \leq i-1$ such that

$$d_{L,i}(y) = c_{i,1}y_1 + \dots + c_{i,i-1}y_{i-1}, \tag{4.15}$$

for $y \in L_n$.

Consider the case $i = 2$. Inserting $x = x^{(2)}$ in (4.12) and using (4.14), we have

$$r d_{L,1}(y) + r d_{L,2}(y) + y_1 d_{K,1}(x^{(2)}) + d_{K,1}(x^{(2)}) d_{L,1}(y) = 0.$$

This, together with (4.6), implies

$$d_{L,2}(y) = -d_{K,1}(x^{(2)})/r y_1,$$

which proves (4.15) for the case $i = 2$ by choosing $c_{2,1} = -d_{K,1}(x^{(2)})/r$.

For the inductive step, assume (4.15) is valid for $i \leq k \leq n-1$. For the case $i = k+1$, insert $x = x^{(k)}$ into (4.12). By (4.6), We have

$$r \sum_{i=2}^{k+1} d_{L,i}(y) + \sum_{i=1}^{k} y_i d_{K,i}(x^{(k)}) + \sum_{i=2}^{k} d_{K,i}(x^{(k)}) d_{L,i}(y) = 0,$$

or,

$$d_{L,k+1}(y) = -\left(r \sum_{i=2}^{k} d_{L,i}(y) + \sum_{i=1}^{k} y_i d_{K,i}(x^{(k)}) + \sum_{i=2}^{k} d_{K,i}(x^{(k)}) d_{L,i}(y) \right) \Bigg/ r.$$

This and (4.15) for the cases $i \leq k$ show that $d_{L,k+1}(y)$ is a linear combination of $y_1, \dots, y_k$, thus establishing (4.15) for the case $i = k$.

Equations (4.15) and (4.6) immediately implies the existence of an $n \times n$ matrix M_L such that

$$(d_{L,1}(y), \dots, d_{L,n}(y))^t = M_L(y_1, \dots, y_n)^t.$$

The same argument applied to K will imply the existence of an $n \times n$ matrix M_K such that

$$(d_{K,1}(x), \dots, d_{K,n}(x))^t = M_K(x_1, \dots, x_n)^t.$$

This, (4.13), the definition of $\Psi_{K,I}$ (4.2), and the definition of $d_{K,i}$ (4.3) imply that

$$\Psi_{K,I}(x) = \frac{1}{2}\left(\Psi_{K,I}(x) - \Psi_{K,I}(-x)\right) = (I + M_K)x,$$

where I is the identity matrix. Similarly,

$$\Psi_{L,I}(y) = (I + M_L)y.$$

Now, since (4.12) is valid for all $x \in K$ and and $y \in L$, we have

$$M_L^t + M_K + M_K M_L^t = 0,$$

or equivalently

$$(I + M_K)(I + M_L)^t = I.$$

This implies $\Psi_{L,I}(y) = (I + M_K)^{-t}y$. To see that $(I + M_K) \in SL(n)$, we simply use the fact that Steiner symmetrization preserves volume. This settles the "only if" part of the lemma.

To see the "if" part, assume there is $T \in SL(n)$ such that $\Psi_{K,I}(x) = Tx$ and $\Psi_{L,I}(y) = T^{-t}y$. Then $K_n = T^{-1}K$ and $L_n = T^t L$. That the equality holds in (4.7) follows trivially from a change of variable in integral. □

The following theorem shows that the integral

$$\int_K \int_L \lambda(x \cdot y) dx dy$$

is minimized at B_K and B_L.

Theorem 4.3. *Let $K, L \subset \mathbb{R}^n$ be convex bodies and λ be an even convex function defined on $\mathbb{R}$. Then,*

$$\int_K \int_L \lambda(x \cdot y) dx dy \geq \int_{B_K} \int_{B_L} \lambda(x \cdot y) dx dy, \tag{4.16}$$

Moreover, if λ is strictly convex, equality holds in (4.16) if and only if K and L are dilates of a pair of polar reciprocal origin-symmetric ellipsoids.

Proof. By Lemma 3.5, there exists a sequence of ordered orthornomal bases $e_1^i, \ldots, e_n^i$ such that (3.8) holds. Let K^i and L^i be as defined in (3.9). Repeated use of Lemma 4.1 shows that

$$\int_K \int_L \lambda(x \cdot y) dx dy \geq \int_{K^i} \int_{L^i} \lambda(x \cdot y) dx dy.$$

Let i go to ∞. By (3.8), we have

$$\int_K \int_L \lambda(x \cdot y) dx dy \geq \int_{B_K} \int_{B_L} \lambda(x \cdot y) dx dy.$$

The rest of the proof is dedicated to show the equality condition.

Suppose equality holds in (4.16). By Lemma 4.1,

$$\int_{K^{i-1}} \int_{L^{i-1}} \lambda(x \cdot y) dx dy = \int_{K^i} \int_{L^i} \lambda(x \cdot y) dx dy,$$

for each i. This and Lemma 4.2 imply that there exists $T_i \in SL(n)$ such that

$$K^{i-1} = T_i(K^i), \qquad L^{i-1} = T_i^{-t}(L^i).$$

Let $G_i = T_1 \cdots T_i$. Hence $K = G_i(K^i)$ and $L = G_i^{-t}(L^i)$. This implies that K and L are o-symmetric and there are $r_0, R_0 > 0$ such that

$$r_0 B_n \subset K, L \subset R_0 B_n.$$

Equation (3.1) implies

$$r_0 B_n \subset K^i, L^i \subset R_0 B_n$$

for all $i \geq 1$. Hence,

$$G_i x, G_i^{-t} x \in R_0 B_n$$

for each $x \in r_0 B_n$, which implies that the sequences of linear transformations G_i and G_i^{-t} are uniformly bounded. Thus, there exists a convergent subsequence, which we also denote by G_i, such that

$$G_i \to \bar{G} \in \mathrm{SL}(n), \qquad \text{and } G_i^{-t} \to \bar{G}^{-t} \in \mathrm{SL}(n).$$

By the properties of the support function,

$$\begin{aligned} |h_{G_i K^i}(u) - h_{\bar{G} B_K}(u)| &\leq |h_{G_i K^i}(u) - h_{G_i B_K}(u)| + |h_{G_i B_K}(u) - h_{\bar{G} B_K}(u)| \\ &= |h_{K^i}(G_i^t u) - h_{B_K}(G_i^t u)| + |h_{B_K}(G_i^t u) - h_{B_K}(\bar{G}^t u)| \\ &\leq \|h_{K_i} - h_{B_K}\|_\infty \cdot |G_i^t u| + |h_{B_K}(G_i^t u) - h_{B_K}(\bar{G}^t u)|. \end{aligned}$$

This, the fact that the quantity $|G_i^t u|$ is bounded, and $G_i^t u \to \bar{G}^t u$ for each $u \in S^{n-1}$, imply that

$$h_{G_i K^i}(u) \to h_{\bar{G} B_K}(u),$$

for each $u \in S^{n-1}$. Note that $G_i K^i = K$. Hence $K = \bar{G} B_K$.

Using the same argument (but this time to L), we have $L = \bar{G}^{-t} B_L$. Since both B_K and B_L are Euclidean balls, we conclude that K is an ellipsoid centered at the origin and that L is a dilation of its polar.

To see that equality holds when K is an ellipsoid centered at the origin and that L is a dilation of its polar, one simply needs to use the change of variable formula for integrals. □

The inequality (1.2) established in [32] is a special case of Theorem 4.3.

Theorem 4.4. *Let $K, L \subset \mathbb{R}^n$ be two convex bodies and $p \geq 1$. Then,*

$$\int_K \int_L |x \cdot y|^p dx dy \geq c_p |K|^{\frac{n+p}{n}} |L|^{\frac{n+p}{n}}, \tag{4.17}$$

where

$$c_p = \int_B \int_B |x \cdot y|^p dx dy.$$

When $p > 1$, equality holds in (4.17) if and only if K and L are dilates of a pair of polar reciprocal origin-symmetric ellipsoids.

Proof. Let $\lambda(t) = |t|^p$. Note that when $p \geq 1$, the function λ is even and convex. By Theorem 4.3,

$$\int_K \int_L |x \cdot y|^p dx dy \geq \int_{B_K} \int_{B_L} |x \cdot y|^p dx dy.$$

Equation (4.17) follows immediately by homogeneity.

When $p > 1$, the function λ is strictly convex and the equality condition follows directly from the equality condition of Theorem 4.3. $\square$

Using layer-cake representation, we may extend Theorem 4.3 to a more general setting.

Theorem 4.5. *Let f, g be two non-negative, quasi-concave, and integrable functions on $\mathbb{R}^n$. Let λ be an even convex function on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}^n} \int_{R^n} \lambda(x \cdot y) f(x) g(y) dx dy \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \lambda(x \cdot y) f^*(x) g^*(y) dx dy,$$

Moreover, if λ is strictly convex, then equality holds if and only if the closures of $\{x \in \mathbb{R}^n : f(x) > t\}$ and are $\{y \in \mathbb{R}^n : g(y) > s\}$ are dilates of a common pair of polar reciprocal origin-symmetric ellipsoids, for almost all $t, s > 0$.

Proof. For each $t, s > 0$, define

$$K_t = \{x \in \mathbb{R}^n : f(x) > t\}, \qquad L_s = \{y \in \mathbb{R}^n : g(y) > s\}.$$

Since f is integrable, then K_t are bounded for $t > 0$. Similarly, L_s are bounded for $s > 0$.

Since f and g are quasi-concave, both K_t and L_s are convex, and hence their boundary are of measure 0 with respect to the Lebesgue measure. By this, the layer-cake representation, Theorem 4.3, and the definition of rearrangement, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\lambda(x\cdot y)f(x)g(y)dxdy &= \int_0^\infty ds\int_0^\infty dt\int_{K_t}\int_{L_s}\lambda(x\cdot y)dxdy\\ &= \int_0^\infty ds\int_0^\infty dt\int_{\mathrm{cl}K_t}\int_{\mathrm{cl}L_s}\lambda(x\cdot y)dxdy\\ &\geq \int_0^\infty ds\int_0^\infty dt\int_{B_{K_t}}\int_{B_{L_s}}\lambda(x\cdot y)dxdy\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\lambda(x\cdot y)f^*(x)g^*(y)dxdy,\end{aligned} \tag{4.18}$$

where $\mathrm{cl}K_t$ and $\mathrm{cl}L_s$ are the closure of K_t and L_s respectively.

To see the equality condition when λ is strictly convex, assume the equality holds. Then, for almost all $t \in [0,\infty)$ and almost all $s \in [0,\infty)$,

$$\int_{\mathrm{cl}K_t}\int_{\mathrm{cl}L_s}\lambda(x\cdot y)dxdy = \int_{K_t^*}\int_{L_s^*}\lambda(x\cdot y)dxdy.$$

By the equality condition in Theorem 4.3, there is a linear tansform $T \in SL(n)$, such that for almost all $t \in [0,\infty)$ and almost all $s \in [0,\infty)$, $\mathrm{cl}K_t = TB_{K_t}$ and $\mathrm{cl}L_s = T^{-t}B_{L_s}$. This shows the "only if" part of the equality condition.

To see the "if" part of the equality condition, one only needs to use the equality condition in Theorem 4.3 to conclude that equality holds in (4.18).

□

5. Appendix

The following example in $\mathbb{R}^2$ shows precisely why we need to apply n-times Steiner symmetrization simultaneously to both K and L in Lemma 4.1.

Let m be an arbitrary integer. Consider the convex bodies

$$K = \left\{(x_1,x_2) \in \mathbb{R}^2 : -m \leq x_1 \leq m, -x_1 - \frac{1}{k} \leq x_2 \leq -x_1 + \frac{1}{k}\right\}, \tag{5.1}$$

and

$$L = \left\{(y_1,y_2) \in \mathbb{R}^2 : -m \leq y_1 \leq m, y_1 - \frac{1}{m} \leq y_2 \leq y_1 + \frac{1}{m}\right\}. \tag{5.2}$$

Note that K is the convex hull generated by $\{-m\} \times [m - \frac{1}{m}, m + \frac{1}{m}]$ and $\{m\} \times [-m - \frac{1}{m}, -m + \frac{1}{m}]$ and L is the convex hull generated by $\{-m\} \times [-m - \frac{1}{m}, -m + \frac{1}{m}]$ and $\{m\} \times [m - \frac{1}{m}, m + \frac{1}{m}]$.

Let $e = (0,1) \in S^1$. Then by the definition of Steiner symmetrization (given in Section 3), it is simple to show that $S_e K = S_e L = [-m, m] \times [-\frac{1}{m}, \frac{1}{m}]$.

Example. For sufficiently large m, the convex bodies K and L as defined above satisfy

$$\int_K \int_L |x \cdot y| dy dx < \int_{S_e K} \int_L |x \cdot y| dy dx,$$

and

$$\int_K \int_L |x \cdot y| dy dx < \int_{S_e K} \int_{S_e L} |x \cdot y| dy dx.$$

The statements above follow from direct computation. By definition of K and L,

$$\int_K \int_L |x \cdot y| dy dx = \int_{-m}^{m} \int_{-m}^{m} \int_{-x_1 - \frac{1}{m}}^{-x_1 + \frac{1}{m}} \int_{y_1 - \frac{1}{m}}^{y_1 + \frac{1}{m}} |x_1 y_1 + x_2 y_2| dy_2 dx_2 dy_1 dx_1$$

By the change of variable $u_1 = x_1/m$, $v_1 = y_1/m$, $u_2 = m(x_2 + x_1)$, and $v_2 = m(y_2 - y_1)$, we have

$$\begin{aligned}
\int_K \int_L |x \cdot y| dx dy &= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left| \frac{1}{m^2} u_2 v_2 + v_1 u_2 - u_1 v_2 \right| dv_2 du_2 dv_1 du_1 \\
&\leq \frac{4}{m^2} \int_{-1}^{1} \int_{-1}^{1} |u_2 v_2| du_2 dv_2 + \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} |v_1 u_2 - u_1 v_2| \, dv_2 du_2 dv_1 du_1 \\
&\to \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} |v_1 u_2 - u_1 v_2| \, dv_2 du_2 dv_1 du_1,
\end{aligned}$$

as $m \to \infty$.

Similarly,

$$\begin{aligned}
\int_{S_e K} \int_L |x \cdot y| dy dx &= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left| m^2 u_1 v_1 + \frac{1}{m^2} u_2 v_2 + v_1 u_2 \right| dv_2 du_2 dv_1 du_1 \\
&\geq 4m^2 \int_{-1}^{1} \int_{-1}^{1} |u_1 v_1| \, dv_1 du_1 - \frac{4}{m^2} \int_{-1}^{1} \int_{-1}^{1} |u_2 v_2| \, dv_2 du_2 \\
&\quad - 4 \int_{-1}^{1} \int_{-1}^{1} |v_1 u_2| \, du_2 dv_1 \\
&\to \infty,
\end{aligned}$$

as $m \to \infty$.

Also,

$$\begin{aligned}
\int_{S_e K} \int_{S_e L} |x \cdot y| dy dx &= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left| m^2 u_1 v_1 + \frac{1}{m^2} u_2 v_2 \right| dv_2 du_2 dv_1 du_1 \\
&\geq 4m^2 \int_{-1}^{1} \int_{-1}^{1} |u_1 v_1| \, dv_1 du_1 - \frac{4}{m^2} \int_{-1}^{1} \int_{-1}^{1} |u_2 v_2| \, dv_2 du_2 \\
&\to \infty,
\end{aligned}$$

as $m \to \infty$.

Hence, for sufficiently large m, both (5.1) and (5.2) are valid.

The above example shows that in $\mathbb{R}^2$, applying Steiner symmetrization once is not good enough to show Lemma 4.1. Similar counterexamples can be constructed in higher dimensions.

References

[1] M. Barchiesi, G. M. Capriani, N. Fusco, and G. Pisante. Stability of Pólya-Szegö inequality for log-concave functions. *J. Funct. Anal.*, 267(7):2264–2297, 2014.

[2] K. J. Böröczky. Stronger versions of the Orlicz-Petty projection inequality. *J. Differential Geom.*, 95(2):215–247, 2013.

[3] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The log-Brunn-Minkowski inequality. *Adv. Math.*, 231:1974–1997, 2012.

[4] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The logarithmic Minkowski problem. *J. Amer. Math. Soc.*, 26(3):831–852, 2013.

[5] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. Affine images of isotropic measures. *J. Differential Geom.*, 99:407–442, 2015.

[6] A. Burchard. A short course on rearrangement inequalities. 2009. Available online at http://www.math.utoronto.ca/almut/rearrange.pdf.

[7] A. Burchard and A. Ferone. On the extremals of the Pólya-Szegö inequality. *Indiana Univ. Math. J.*, 64(5):1447–1463, 2015.

[8] S. Campi and P. Gronchi. The L^p-Busemann-Petty centroid inequality. *Adv. Math.*, 167(1):128–141, 2002.

[9] A. Cianchi, L. Esposito, N. Fusco, and C. Trombetti. A quantitative Pólya-Szegö principle. *J. Reine Angew. Math.*, 614:153–189, 2008.

[10] A. Cianchi and N. Fusco. Steiner symmetric extremals in Pólya-Szegö type inequalities. *Adv. Math.*, 203(2):673–728, 2006.

[11] A. Colesanti, K. Nyström, P. Salani, J. Xiao, D. Yang, and G. Zhang. The Hadamard variational formula and the Minkowski problem for p-capacity. *Adv. Math.*, 285:1511–1588, 2015.

[12] R. J. Gardner, D. Hug, and W. Weil. The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities. *J. Differential Geom.*, 97(3):427–476, 2014.

[13] C. Haberl, E. Lutwak, D. Yang, and G. Zhang. The even Orlicz Minkowski problem. *Adv. Math.*, 224(6):2485–2510, 2010.

[14] C. Haberl and L. Parapatits. The centro-affine Hadwiger theorem. *J. Amer. Math. Soc. (JAMS)*, 27:685–705, 2014.

[15] C. Haberl, F. Schuster, and J. Xiao. An asymmetric affine Pólya-Szegö principle. *Math. Ann.*, 352(3):517–542, 2012.

[16] C. Haberl and F. E. Schuster. Asymmetric affine L_p Sobolev inequalities. *J. Funct. Anal.*, 257:641–658, 2009.

[17] C. Haberl and F. E. Schuster. General L_p affine isoperimetric inequalities. *J. Differential Geom.*, 83:1–26, 2009.

[18] J. Haddad, C. H. Jiménez, and M. Montenegro. Sharp affine Sobolev type inequalities via the L_p Busemann-Petty centroid inequality. *J. Funct. Anal.*, 271(2):454–473, 2016.

[19] M. Henk and E. Linke. Cone-volume measures of polytopes. *Adv. Math.*, 253:50–62, 2014.

[20] Y. Huang, J. Liu, and L. Xu. On the uniqueness of L_p-Minkowski problems: the constant p-curvature case in $\mathbb{R}^3$. *Adv. Math.*, 281:906–927, 2015.

[21] H. Jian, J. Lu, and X.-J. Wang. Nonuniqueness of solutions to the L_p-Minkowski problem. *Adv. Math.*, 281:845–856, 2015.

[22] J. Li and G. Leng. L_p Minkowski valuations on polytopes. *Adv. Math.*, 299:139–173, 2016.

[23] Y. Lin. Affine Orlicz Pólya-Szegö principle for log-concave functions. *J. Funct. Anal.*, 273(10):3295–3326, 2017.

[24] M. Ludwig. General affine surface areas. *Adv. Math.*, 224:2346–2360, 2010.

[25] M. Ludwig and M. Reitzner. A classification of $\mathrm{SL}(n)$ invariant valuations. *Ann. of Math. (2)*, 172:1219–1267, 2010.

[26] M. Ludwig, J. Xiao, and G. Zhang. Sharp convex Lorentz-Sobolev inequalities. *Math. Ann.*, 350(1):169–197, 2011.

[27] E. Lutwak. On a conjectured projection inequality of Petty. In *Integral geometry and tomography (Arcata, CA, 1989)*, volume 113 of *Contemp. Math.*, pages 171–182. Amer. Math. Soc., Providence, RI, 1990.

[28] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.*, 38:131–150, 1993.

[29] E. Lutwak. The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas. *Adv. Math.*, 118:244–294, 1996.

[30] E. Lutwak, D. Yang, and G. Zhang. L_p affine isoperimetric inequalities. *J. Differential Geom.*, 56:111–132, 2000.

[31] E. Lutwak, D. Yang, and G. Zhang. Sharp affine L_p Sobolev inequalities. *J. Differential Geom.*, 62:17–38, 2002.

[32] E. Lutwak, D. Yang, and G. Zhang. Moment-entropy inequalities. *Ann. Probab.*, 32(1B):757–774, 2004.

[33] E Lutwak, D. Yang, and G. Zhang. Orlicz centroid bodies. *J. Differential Geom.*, 84(2):365–387, 2010.

[34] E. Lutwak, D. Yang, and G. Zhang. Orlicz projection bodies. *Adv. Math.*, 223(1):220–242, 2010.

[35] V. H. Nguyen. New approach to the affine Pólya-Szegö principle and the stability version of the affine Sobolev inequality. *Adv. Math.*, 302:1080–1110, 2016.

[36] C. Saroglou. Volumes of projection bodies of some classes of convex bodies. *Mathematika*, 57(2):329–353, 2011.

[37] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[38] F. E. Schuster and M. Weberndorfer. Volume inequalities for asymmetric Wulff shapes. *J. Differential Geom.*, 92(2):263–283, 2012.

[39] T. Wang. The affine Sobolev-Zhang inequality on $BV(\mathbb{R}^n)$. *Adv. Math.*, 230:2457–2473, 2012.

[40] T. Wang. The affine Pólya-Szegö principle: equality cases and stability. *J. Funct. Anal.*, 265(8):1728–1748, 2013.

[41] E. Werner and D. Ye. Inequalities for mixed p-affine surface area. *Math. Ann.*, 347(3):703–737, 2010.

[42] E. M. Werner. Rényi divergence and L_p-affine surface area for convex bodies. *Adv. Math.*, 230(3):1040–1059, 2012.

[43] D. Xi, H. Jin, and G. Leng. The Orlicz Brunn-Minkowski inequality. *Adv. Math.*, 260:350–374, 2014.

[44] D. Ye. New Orlicz affine isoperimetric inequalities. *J. Math. Anal. Appl.*, 427(2):905–929, 2015.

[45] G. Zhang. The affine Sobolev inequality. *J. Differential Geom.*, 53:183–202, 1999.

[46] G. Zhu. The Orlicz centroid inequality for star bodies. *Adv. in Appl. Math.*, 48(2):432–445, 2012.

[47] G. Zhu. The logarithmic Minkowski problem for polytopes. *Adv. Math.*, 262:909–931, 2014.

[48] G. Zhu. The centro-affine Minkowski problem for polytopes. *J. Differential Geom.*, 101:159–174, 2015.

[49] G. Zhu. The L_p Minkowski problem for polytopes for $0 < p < 1$. *J. Funct. Anal.*, 269:1070–1094, 2015.