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The Orlicz Brunn–Minkowski inequality ☆

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ABSTRACT

The Orlicz Brunn–Minkowski theory originated with the work of Lutwak, Yang, and Zhang in 2010. In this paper, we first introduce the Orlicz addition of convex bodies containing the origin in their interiors, and then extend the L_p Brunn–Minkowski inequality to the Orlicz Brunn–Minkowski inequality. Furthermore, we extend the L_p Minkowski mixed volume inequality to the Orlicz mixed volume inequality by using the Orlicz Brunn–Minkowski inequality.

1. Introduction

The classical Brunn–Minkowski inequality was inspired by questions around the isoperimetric problem. Many other consequences in convex geometry make it a cornerstone of the Brunn–Minkowski theory, which provides a beautiful and powerful apparatus

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for conquering all sorts of geometrical problems involving metric quantities such as volume, surface area, and mean width.

The classical Brunn–Minkowski inequality (see [12]) states that for convex bodies K, L in Euclidean n-space $\mathbb{R}^n$, the volume of the bodies and of their Minkowski sum $K + L = \{x + y : x \in K \text{ and } y \in L\}$ are related by

$$V(K + L)^{\frac{1}{n}} \geqslant V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}, \tag{1.1}$$

with equality if and only if K and L are homothetic.

In his survey, Gardner [12] summarized the history of the Brunn–Minkowski inequality and some applications in other fields such as: probability and statistics, information theory, physics, elliptic partial differential equations, combinatorics, interacting gases, shapes of crystals and algebraic geometry.

In the early 1960s, Firey [11] defined for each $p \geqslant 1$, what have become known as Minkowski–Firey L_p-additions (or simply L_p-additions) of convex bodies. For the L_p-additions, Firey [11] also established the L_p Brunn–Minkowski inequality (an inequality that is also known as the Brunn–Minkowski–Firey inequality, see [33]). If $p > 1$, and $K, L \subset \mathbb{R}^n$ are convex bodies containing the origin in their interiors, then

$$V(K +_p L)^{\frac{p}{n}} \geqslant V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}, \tag{1.2}$$

with equality if and only if K and L are dilates.

The mixed volume $V_1(K, L)$ of convex bodies K, L is defined by

$$V_1(K, L) := \frac{1}{n} \lim_{\epsilon \to 0^+} \frac{V(K + \epsilon L) - V(K)}{\epsilon} = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS_K(u), \tag{1.3}$$

where $S_K(\cdot)$ is the surface area measure of K.

The Minkowski mixed volume inequality for convex bodies K, L states that

$$V_1(K, L) \geqslant V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}, \tag{1.4}$$

with equality if and only if K and L are homothetic.

For $p > 1$, the L_p mixed volume of convex bodies K, L containing the origin in their interiors is defined by Lutwak [33] as

$$V_p(K, L) := \frac{p}{n} \lim_{\epsilon \to 0^+} \frac{V(K +_p \epsilon \cdot L) - V(K)}{\epsilon}.$$

Lutwak [33] showed that the L_p mixed volume has the following integral representation:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L(u)}{h_K(u)} \right)^p h_K(u) dS_K(u). \tag{1.5}$$

Lutwak's L_p Minkowski mixed volume inequality [33] states

$$V_p(K,L) \geqslant V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \tag{1.6}$$

with equality if and only if K and L are dilates.

In the mid 1990s, it was shown in [33] and [34] that a study of the volume of Minkowski–Firey L_p-additions leads to an L_p Brunn–Minkowski theory. This theory has expanded rapidly (see, e.g., [1–9,16–18,20–22,26–30,32–43,46–49,51–53,56]).

The Orlicz Brunn–Minkowski theory originated with the work of Lutwak, Yang, and Zhang in 2010. Precisely, Lutwak, Yang, and Zhang [44,45] introduced Orlicz projection bodies and Orlicz centroid bodies, and they successively established the fundamental affine inequalities for these bodies. Haberl, Lutwak, Yang, and Zhang [19] dealt with the even Orlicz Minkowski problem. For related work, see also [20,21,30,31]. Ludwig and Reitzner [31] introduced what soon came to be seen as Orlicz affine surface area and Ludwig [30] established its fundamental affine inequalities. For the development of the Orlicz Brunn–Minkowski theory, see [23,25,54,57].

It seems natural, now, to define the Orlicz addition and to give the Orlicz Brunn–Minkowski inequality. We consider the Orlicz addition, which is an extension of L_p-addition.

Let $\mathcal{C}$ be the class of convex, strictly increasing functions $\phi : [0,\infty) \to [0,+\infty)$ satisfying $\phi(0) = 0$. It is not hard to conclude from [50, pp. 23–24] that $\phi \in \mathcal{C}$ is continuous on $[0,+\infty)$, and the left derivative ϕ'_l and right derivative ϕ'_r exist. Furthermore, ϕ'_l is left-continuous on $(0,+\infty)$, ϕ'_r is right-continuous on $[0,+\infty)$, and ϕ'_l and ϕ'_r are positive on $(0,+\infty)$.

Definition 1. Let $\phi \in \mathcal{C}$, and let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. We define the *Orlicz sum* $K +_\phi L$ by

$$h_{K+_\phi L}(u) = \inf\left\{\tau > 0 : \phi\left(\frac{h_K(u)}{\tau}\right) + \phi\left(\frac{h_L(u)}{\tau}\right) \leqslant 1\right\}.$$

If $\phi(t) = t^p$, $p \geqslant 1$, then $K +_\phi L = K +_p L$.

Theorem 1 is what we are calling the *Orlicz Brunn–Minkowski inequality*.

Theorem 1. *Let $\phi \in \mathcal{C}$, and let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. Then, we have*

$$\phi\left(\frac{V(K)^{\frac{1}{n}}}{V(K +_\phi L)^{\frac{1}{n}}}\right) + \phi\left(\frac{V(L)^{\frac{1}{n}}}{V(K +_\phi L)^{\frac{1}{n}}}\right) \leqslant 1. \tag{1.7}$$

Equality holds if K and L are dilates. When ϕ is strictly convex, equality holds if and only if K and L are dilates.

Next we give the definition of Orlicz combination.

Definition 2. Let $\phi \in \mathcal{C}$, and let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. Suppose $\alpha > 0$ and $\beta \geqslant 0$. We define the *Orlicz combination* $M_\phi(\alpha, \beta; K, L)$ (or the Orlicz mean of convex bodies) by

$$h_{M_\phi(\alpha,\beta;K,L)}(u) = \inf\left\{\tau > 0 : \alpha\phi\left(\frac{h_K(u)}{\tau}\right) + \beta\phi\left(\frac{h_L(u)}{\tau}\right) \leqslant 1\right\}. \tag{1.8}$$

Since the function $z \to \alpha\phi(\frac{h_K(u)}{z}) + \beta\phi(\frac{h_L(u)}{z})$ is strictly decreasing, we have

$$h_{M_\phi(\alpha,\beta;K,L)}(u) = \tau_u, \quad \text{if and only if} \quad \alpha\phi\left(\frac{h_K(u)}{\tau_u}\right) + \beta\phi\left(\frac{h_L(u)}{\tau_u}\right) = 1. \tag{1.9}$$

It is obvious that $M_\phi(1, 1; K, L) = K +_\phi L$.

In Section 2, we will show that $h_{M_\phi(\alpha,\beta;K,L)}(\cdot)$ is indeed a support function of a convex body which contains the origin in its interior. When $\phi(t) = t^p$ $(p \geqslant 1)$, the convex body $M_\phi(\alpha, \beta; K, L)$ is precisely the Firey combination (see [11,33]) $\alpha \cdot K +_p \beta \cdot L$. However, for general $\phi \in \mathcal{C}$, the “ $\cdot$ ” could not be defined, which means we cannot write $\alpha \cdot K +_\phi \beta \cdot L$ instead of $M_\phi(\alpha, \beta; K, L)$.

Definition 3. Let $\phi \in \mathcal{C}$ satisfy $\phi(1) = 1$, and let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. The *Orlicz mixed volume* is defined by

$$V_\phi(K, L) = \frac{\phi_l'(1)}{n} \lim_{\epsilon \to 0^+} \frac{V(M_\phi(1, \epsilon; K, L)) - V(K)}{\epsilon}.$$

The following theorem shows that the limit in Definition 3 exists and has an integral representation, which is an extension of (1.5).

Theorem 2. *Let $\phi \in \mathcal{C}$ satisfy $\phi(1) = 1$, and let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. Then, we have*

$$\begin{aligned} V_\phi(K, L) &= \frac{\phi_l'(1)}{n} \lim_{\epsilon \to 0^+} \frac{V(M_\phi(1, \epsilon; K, L)) - V(K)}{\epsilon} \\ &= \frac{1}{n} \int\limits_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) dS_K(u). \end{aligned} \tag{1.10}$$

The following is the *Orlicz mixed volume inequality*.

Theorem 3. *Let $\phi \in \mathcal{C}$ satisfy $\phi(1) = 1$, and let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. Then,*

$$V_\phi(K, L) \geqslant V(K)\phi\left(\frac{V(L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\right). \tag{1.11}$$

Equality holds if K and L are dilates. When ϕ is strictly convex, equality holds if and only if K and L are dilates.

If $\phi(t) = t^p$ $(p \geqslant 1)$, then the corresponding results of Theorems 1–3 in the L_p Brunn–Minkowski theory are obtained.

This paper is organized as follows. Section 2 contains the basic definition and notations, and shows that the Orlicz combination of convex bodies is also a convex body. Section 3 lists the elementary properties of Orlicz combination. In Section 4, we prove a general case of Theorem 1 using Steiner symmetrization, which is one of the methods to prove the original Brunn–Minkowski inequality (1.1) (see e.g. [10, Chapter 5, Section 5] or [55, pp. 310–314]). However, for the Orlicz case, our proof is quite different. Section 5 gives the proof of Theorem 2 and Theorem 3.

When we were about to submit our paper, we were informed that Gardner, Hug, and Weil [14] had also obtained an Orlicz Brunn–Minkowski inequality and posted their results on the arXiv.org a couple of days before. Please note that we use a completely different approach technique of Steiner symmetrization, although our results coincide with theirs.

2. Preliminaries

For quick later reference we collect some notations and basic facts about convex bodies. Good general references for the theory of convex bodies are the books of Gardner [13], Gruber [15], Leichtweiss [24], and Schneider [50].

Let S^{n-1} denote the unit sphere, B^n the unit n-ball, ω_n the volume of B^n, and o the origin in the Euclidean n-dimensional space $\mathbb{R}^n$. Denote by $\mathcal{K}^n$ the class of convex bodies (compact, convex sets with non-empty interiors) in $\mathbb{R}^n$, and let $\mathcal{K}_o^n$ be the class of members of $\mathcal{K}^n$ containing the origin in their interiors.

By $\operatorname{int} A$ and ∂A we denote, respectively, the interior and boundary of $A \subset \mathbb{R}^n$. The sets $\operatorname{relint} A$ and $\operatorname{relbd} A$ are the relative interior and relative boundary, that is, the interior and boundary of A relative to its affine hull.

We say a sequence $\{\phi_i\} \subset \mathcal{C}$ is such that $\phi_i \to \phi \in \mathcal{C}$, provided

$$\max_{t \in I} \left|\phi_i(t) - \phi(t)\right| \to 0,$$

for every compact interval $I \subset [0, \infty)$.

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact convex set $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \tag{2.1}$$

and it uniquely determines the compact convex set.

Obviously, for a pair of compact convex sets $K, L \subset \mathbb{R}^n$, we have

$$h_K \leqslant h_L \quad \text{if and only if} \quad K \subseteq L.$$

A function is a support function of a compact convex set if and only if it is positively homogeneous of degree one and subadditive.

Let $K \in \mathcal{K}^n$ and $x \in \partial K$. Denote by $\nu(x)$ an *outer normal vector* of K at x. Obviously,

$$h_K\big(\nu(x)\big) = x \cdot \nu(x).$$

Then, the hyperplane $\{y \in \mathbb{R}^n \mid y \cdot \nu(x) = h_K(\nu(x))\}$ is a *support hyperplane* of K at x.

We shall use δ to denote the *Hausdorff metric* on $\mathcal{K}^n$. If $K, L \in \mathcal{K}^n$, the Hausdorff distance $\delta(K, L)$ is defined by

$$\delta(K, L) = \min\big\{\alpha : K \subseteq L + \alpha B^n \text{ and } L \subseteq K + \alpha B^n\big\},$$

or equivalently,

$$\delta(K, L) = \max_{u \in S^{n-1}} \big|h_K(u) - h_L(u)\big|.$$

A class of convex bodies $\{K_i\}$ is said to converge to a convex body K if

$$\delta(K_i, K) \to 0, \quad \text{as } i \to \infty.$$

Let $K \in \mathcal{K}^n$. The *surface area measure* $S_K(\cdot)$ of K is a measure on S^{n-1} defined by

$$S_K(\omega) = \int\limits_{x \in \partial K,\, \nu(x) \in \omega} d\mathcal{H}^{n-1}(x), \quad \omega \subset S^{n-1},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. The surface area measure has the following property:

$$K_i \to K \quad \Rightarrow \quad S_{K_i} \to S_K \quad \text{weakly}. \tag{2.2}$$

Let $\phi \in \mathcal{C}$, $K, L \in \mathcal{K}_o^n$, $\alpha > 0$ and $\beta \geqslant 0$. The definition of Orlicz Minkowski addition and Orlicz combination are given in Section 1. In the following, we check that the Orlicz Minkowski combination $M_\phi(\alpha, \beta; K, L)$ is indeed a convex body containing the origin in its interior. Set $M = M_\phi(\alpha, \beta; K, L)$; in fact, we need to show that the function $h_M(\cdot)$ is homogeneous of degree one and subadditive, and that h_M is positive.

First, for $\gamma > 0$ we have

$$h_M(\gamma u) = \inf\left\{\tau > 0 : \alpha\phi\left(\frac{h_K(\gamma u)}{\tau}\right) + \beta\phi\left(\frac{h_L(\gamma u)}{\tau}\right) \leqslant 1\right\}$$

$$
\begin{aligned}
&= \gamma \inf\left\{ \frac{\tau}{\gamma} > 0 : \alpha\phi\left(\frac{h_K(u)}{\tau/\gamma}\right) + \beta\phi\left(\frac{h_L(u)}{\tau/\gamma}\right) \leqslant 1 \right\} \\
&= \gamma h_M(u).
\end{aligned}
$$

Next, we show that $h_M(\cdot)$ is subadditive. Set $h_M(u) = \tau_u$ and $h_M(v) = \tau_v$; then we have $\alpha\phi(\frac{h_K(u)}{\tau_u}) + \beta\phi(\frac{h_L(u)}{\tau_u}) = 1$ and $\alpha\phi(\frac{h_K(v)}{\tau_v}) + \beta\phi(\frac{h_L(v)}{\tau_v}) = 1$. Furthermore,

$$
\begin{aligned}
1 &= \frac{\tau_u}{\tau_u + \tau_v}\alpha\phi\left(\frac{h_K(u)}{\tau_u}\right) + \frac{\tau_v}{\tau_u + \tau_v}\alpha\phi\left(\frac{h_K(v)}{\tau_v}\right) \\
&\quad + \frac{\tau_u}{\tau_u + \tau_v}\beta\phi\left(\frac{h_L(u)}{\tau_u}\right) + \frac{\tau_v}{\tau_u + \tau_v}\beta\phi\left(\frac{h_L(v)}{\tau_v}\right) \\
&\geqslant \alpha\phi\left(\frac{h_K(u) + h_K(v)}{\tau_u + \tau_v}\right) + \beta\phi\left(\frac{h_L(u) + h_L(v)}{\tau_u + \tau_v}\right) \\
&\geqslant \alpha\phi\left(\frac{h_K(u+v)}{\tau_u + \tau_v}\right) + \beta\phi\left(\frac{h_L(u+v)}{\tau_u + \tau_v}\right),
\end{aligned}
$$

which implies that $h_M(u+v) \leqslant h_M(u) + h_M(v)$.

Finally, since

$$
\alpha\phi\left(\frac{h_K(u)}{h_K(u)/\phi^{-1}(\frac{1}{\alpha})}\right) + \beta\phi\left(\frac{h_L(u)}{h_K(u)/\phi^{-1}(\frac{1}{\alpha})}\right) \geqslant 1,
$$

from (1.8), we have $h_M(u) \geqslant h_K(u)/\phi^{-1}(\frac{1}{\alpha}) > 0$. Thus, $M_\phi(\alpha, \beta; K, L)$ contains o in its interior.

3. Properties of Orlicz combination

Suppose $\phi \in \mathcal{C}$, $a, b, \alpha > 0$, and $\beta \geqslant 0$. Since the function $z \mapsto \alpha\phi(\frac{a}{z}) + \beta\phi(\frac{b}{z})$ is strictly decreasing, we define a positive function $C_\phi(\alpha, \beta; a, b)$ by

$$
z = C_\phi(\alpha, \beta; a, b), \quad \text{if and only if} \quad \alpha\phi\left(\frac{a}{z}\right) + \beta\phi\left(\frac{b}{z}\right) = 1. \tag{3.1}
$$

The functions $C_\phi(\alpha, \beta; a, b)$ have some properties listed in the following lemma.

Lemma 3.1. *Suppose $\phi \in \mathcal{C}$ and $a, b > 0$. Let $\alpha > 0$, $\beta \geqslant 0$.*

(i) *If $d > 0$, then $C_\phi(\alpha, \beta; ad, bd) = dC_\phi(\alpha, \beta; a, b)$.*
(ii) *Suppose $\phi_1, \phi_2 \in \mathcal{C}$. If $\phi_2 \geqslant \phi_1$, then $C_{\phi_2}(\alpha, \beta; a, b) \geqslant C_{\phi_1}(\alpha, \beta; a, b)$.*
(iii) *Suppose $a_i, b_i > 0$ are such that $a_i \to a$ and $b_i \to b$. Then, $C_\phi(\alpha, \beta; a_i, b_i) \to C_\phi(\alpha, \beta; a, b)$.*
(iv) *Suppose $\{\phi_i\} \subset \mathcal{C}$ are such that $\phi_i \to \phi$. Then, $C_{\phi_i}(\alpha, \beta; a, b) \to C_\phi(\alpha, \beta; a, b)$.*

(v) *Suppose* $\alpha_i > 0$, $\beta_i \geqslant 0$ *are such that* $\alpha_i \to \alpha$ *and* $\beta_i \to \beta$. *Then*, $C_\phi(\alpha_i, \beta_i; a, b) \to C_\phi(\alpha, \beta; a, b)$.

Proof. (i) Suppose $d > 0$. By (3.1), we have

$$1 = \alpha\phi\left(\frac{ad}{C_\phi(\alpha,\beta;ad,bd)}\right) + \beta\phi\left(\frac{bd}{C_\phi(\alpha,\beta;ad,bd)}\right)$$
$$= \alpha\phi\left(\frac{a}{C_\phi(\alpha,\beta;ad,bd)/d}\right) + \beta\phi\left(\frac{b}{C_\phi(\alpha,\beta;ad,bd)/d}\right),$$

and

$$1 = \alpha\phi\left(\frac{a}{C_\phi(\alpha,\beta;a,b)}\right) + \beta\phi\left(\frac{b}{C_\phi(\alpha,\beta;a,b)}\right).$$

Thus, we have $C_\phi(\alpha,\beta;ad,bd) = dC_\phi(\alpha,\beta;a,b)$.

(ii) Set $C_{\phi_i}(\alpha,\beta;a,b) = z_i$, $i = 1,2$. Since $\phi_2 \geqslant \phi_1$, we have

$$1 = \alpha\phi_2\left(\frac{a}{z_2}\right) + \beta\phi_2\left(\frac{b}{z_2}\right) \geqslant \alpha\phi_1\left(\frac{a}{z_2}\right) + \beta\phi_1\left(\frac{b}{z_2}\right),$$

which implies $z_2 \geqslant z_1$.

(iii) Set $z_i = C_\phi(\alpha,\beta;a_i,b_i)$, $i = 1,2,\ldots$, and $z_0 = C_\phi(\alpha,\beta;a,b)$. We will prove (iii) by showing that every subsequence of $\{z_i\}$ has a subsequence converging to $C_\phi(\alpha,\beta;a,b)$. From

$$1 = \alpha\phi\left(\frac{a_i}{z_i}\right) + \beta\phi\left(\frac{b_i}{z_i}\right) < (\alpha+\beta)\phi\left(\frac{a_i+b_i}{z_i}\right),$$

we have $z_i < (a_i+b_i)/\phi^{-1}(\frac{1}{\alpha+\beta})$, and since $a_i \to a$, $b_i \to b$, there is a constant $R > 0$ such that $z_i \leqslant R$, $i = 1,2,\ldots$. Let $\{z_i\}$ denote a subsequence of $\{z_i\}$. Then $\{z_i\}$ has a convergent subsequence, also denoted by $\{z_i\}$, and we suppose that $z_i \to z_0'$. Since ϕ is continuous, we have $z_0' > 0$ and

$$\alpha\phi\left(\frac{a}{z_0'}\right) + \beta\phi\left(\frac{b}{z_0'}\right) = \lim_{i\to\infty}\left[\alpha\phi\left(\frac{a_i}{z_i}\right) + \beta\phi\left(\frac{b_i}{z_i}\right)\right] = 1,$$

which implies $z_0' = z_0$.

(iv) Set $\tau_i = C_{\phi_i}(\alpha,\beta;a,b)$, $i = 1,2,\ldots$, and $\tau_0 = C_\phi(\alpha,\beta;a,b)$. We claim that

$$\lim_{i\to\infty}\phi_i^{-1}(x) = \phi^{-1}(x), \tag{3.2}$$

for all $x > 0$. Let $\eta > 0$ be arbitrary. Since $\phi \in \mathcal{C}$, we conclude that ϕ^{-1} is concave on $[0,\infty)$. Hence ϕ^{-1} is continuous on $(0,\infty)$. Then, there exists a $\delta \in (0,x)$, such that

$$\phi^{-1}(x-\delta) > \phi^{-1}(x) - \eta, \tag{3.3}$$
$$\phi^{-1}(x+\delta) < \phi^{-1}(x) + \eta. \tag{3.4}$$

Since $\phi_i \to \phi$ uniformly on $[\phi^{-1}(x-\delta), \phi^{-1}(x+\delta)]$, there exists an $N > 0$, such that

$$\phi_i\big(\phi^{-1}(x-\delta)\big) < \phi\big(\phi^{-1}(x-\delta)\big) + \delta = x, \tag{3.5}$$
$$\phi_i\big(\phi^{-1}(x+\delta)\big) > \phi\big(\phi^{-1}(x+\delta)\big) - \delta = x, \tag{3.6}$$

for all $i > N$. Then, by (3.3), (3.4), (3.5), and (3.6), we have

$$\phi^{-1}(x) - \eta < \phi_i^{-1}(x) < \phi^{-1}(x) + \eta,$$

for all $i > N$. Since $\eta > 0$ is arbitrary, we complete the proof of our claim.

From

$$1 = \alpha\phi_i\left(\frac{a}{\tau_i}\right) + \beta\phi_i\left(\frac{b}{\tau_i}\right) < (\alpha+\beta)\phi_i\left(\frac{a+b}{\tau_i}\right),$$

we have $\tau_i < (a+b)/\phi_i^{-1}(\frac{1}{\alpha+\beta})$. By (3.2), there is a constant $r > 0$, such that $\phi_i^{-1}(\frac{1}{\alpha+\beta}) > r$, $i = 1, 2, \ldots$. Thus, $\{\tau_i\}$ is bounded. Then, each subsequence of $\{\tau_i\}$ has a convergent subsequence, also denoted by $\{\tau_i\}$, and we suppose it converges to τ_0'. Since

$$1 = \alpha\phi_i\left(\frac{a}{\tau_i}\right) + \beta\phi_i\left(\frac{b}{\tau_i}\right) \geqslant \alpha\phi_i\left(\frac{a}{\tau_i}\right),$$

then,

$$\tau_i \geqslant \frac{a}{\phi_i^{-1}(1/\alpha)}.$$

Thus, by (3.2), we have $\tau_0' \geqslant \frac{a}{\phi^{-1}(1/\alpha)} > 0$.

By the continuity of $\phi_i(\cdot)$, and $\phi_i \to \phi$, we have

$$\alpha\phi\left(\frac{a}{\tau_0'}\right) + \beta\phi\left(\frac{b}{\tau_0'}\right) = \lim_{i\to\infty}\left[\alpha\phi_i\left(\frac{a}{\tau_i}\right) + \beta\phi_i\left(\frac{b}{\tau_i}\right)\right] = 1,$$

which implies $\tau_0 = \tau_0'$.

(v) Set $\mu_i = C_\phi(\alpha_i, \beta_i; a, b)$, $i = 1, 2, \ldots$, and $\mu_0 = C_\phi(\alpha, \beta; a, b)$. From

$$1 = \alpha_i\phi\left(\frac{a}{\mu_i}\right) + \beta_i\phi\left(\frac{b}{\mu_i}\right) > (\alpha_i+\beta_i)\phi\left(\frac{a+b}{\mu_i}\right),$$

we obtain $\mu_i < (a+b)/\phi^{-1}(\frac{1}{\alpha_i+\beta_i})$. Noticing that ϕ^{-1} is continuous, we have that $\{\mu_i\}$ is bounded. Hence, each subsequence of $\{\mu_i\}$ has a convergent subsequence, denoted also by $\{\mu_i\}$, converging to some μ_0'. By the continuity of ϕ, we have $\mu_0' > 0$ and

$$\alpha\phi\left(\frac{a}{\mu_0'}\right)+\beta\phi\left(\frac{b}{\mu_0'}\right)=\lim_{i\to\infty}\left[\alpha_i\phi\left(\frac{a}{\mu_i}\right)+\beta_i\phi\left(\frac{b}{\mu_i}\right)\right]=1,$$

which implies $\mu_0=\mu_0'$. $\square$

Notice that $h_{M_\phi(\alpha,\beta;K,L)}(u)=C_\phi(\alpha,\beta;h_K(u),h_L(u))$, and that the convergence of convex bodies is equivalent to the pointwise convergence of the corresponding support functions on S^{n-1} (see e.g. [50, pp. 53–54]). Therefore, we obtain the following properties of Orlicz combination.

Lemma 3.2. *Suppose $\phi\in\mathcal{C}$ and $K,L\in\mathcal{K}_o^n$. Let $\alpha>0$, $\beta\geqslant 0$.*

(i) *If $d>0$, then $M_\phi(\alpha,\beta;dK,dL)=dM_\phi(\alpha,\beta;K,L)$.*
(ii) *Suppose $\phi_1,\phi_2\in\mathcal{C}$. If $\phi_2\geqslant\phi_1$, then $M_{\phi_2}(\alpha,\beta;K,L)\supseteq M_{\phi_1}(\alpha,\beta;K,L)$.*
(iii) *Suppose $K_i,L_i\in\mathcal{K}_o^n$ are such that $K_i\to K$ and $L_i\to L$. Then, $M_\phi(\alpha,\beta;K_i,L_i)\to M_\phi(\alpha,\beta;K,L)$.*
(iv) *Suppose $\{\phi_i\}\subset\mathcal{C}$ are such that $\phi_i\to\phi$. Then, $M_{\phi_i}(\alpha,\beta;K,L)\to M_\phi(\alpha,\beta;K,L)$.*
(v) *Suppose $\alpha_i>0$, $\beta_i\geqslant 0$ are such that $\alpha_i\to\alpha$ and $\beta_i\to\beta$. Then, $M_\phi(\alpha_i,\beta_i;K,L)\to M_\phi(\alpha,\beta;K,L)$.*

Properties (ii) and (iv) are not used in this paper, but Properties (i), (iii) and (v) are basic for our proofs.

4. The Orlicz Brunn–Minkowski inequality

Let $K\subset\mathbb{R}^n$ be a convex body. For $u\in S^{n-1}$, denote by K_u the image of the orthogonal projection of K onto $u^\perp$. We write $\bar{\ell}_u(K;y'):K_u\to\mathbb{R}$ and $\underline{\ell}_u(K;y'):K_u\to\mathbb{R}$ for the *overgraph* and *undergraph functions* of K in the direction u; i.e.

$$K=\left\{y'+tu:-\underline{\ell}_u\left(K;y'\right)\leqslant t\leqslant\bar{\ell}_u\left(K;y'\right)\text{ for }y'\in K_u\right\}. \tag{4.1}$$

Thus the *Steiner symmetral* S_uK of $K\in\mathcal{K}^n$ in the direction u can be defined as the body whose orthogonal projection onto $u^\perp$ is identical to that of K and whose overgraph and undergraph functions are given by

$$\bar{\ell}_u\left(S_uK;y'\right)=\underline{\ell}_u\left(S_uK;y'\right)=\frac{1}{2}\left[\bar{\ell}_u\left(K;y'\right)+\underline{\ell}_u\left(K;y'\right)\right]. \tag{4.2}$$

In this paper, we use the following notations: when $u\in S^{n-1}$ is fixed, the point $x=(x',s)$ always means $x'+su$, where $x'\in u^\perp$ and $s\in\mathbb{R}$. We will usually write $h_K(x',s)$ rather than $h_K((x',s))$.

Suppose $K\in\mathcal{K}^n$ and $x_1',x_2'\in u^\perp$. By (4.1), for $(a',s)\in K$, we have

$$\left(a',s\right)\cdot\left(x_1',1\right)=a'\cdot x_1'+s\leqslant a'\cdot x_1'+\bar{\ell}_u\left(K;a'\right),$$

then,

$$h_K(x_1',1)=\max_{(a',s)\in K}\{(x_1',1)\cdot(a',s)\}\leqslant\max_{a'\in K_u}\{x_1'\cdot a'+\bar{\ell}_u(K;a')\}.$$

On the other hand, noticing that $(a',\bar{\ell}_u(K;a'))\in K$ for arbitrary $a'\in K_u$, we have

$$h_K(x_1',1)=\max_{(a',s)\in K}\{(x_1',1)\cdot(a',s)\}\geqslant\max_{a'\in K_u}\{x_1'\cdot a'+\bar{\ell}_u(K;a')\}.$$

Thus, we get that

$$h_K(x_1',1)=\max_{a'\in K_u}\{x_1'\cdot a'+\bar{\ell}_u(K;a')\}. \tag{4.3}$$

In a similar way, we get that

$$h_K(x_2',-1)=\max_{a'\in K_u}\{x_2'\cdot a'+\underline{\ell}_u(K;a')\}. \tag{4.4}$$

The following lemma will be used in the proof of our theorem.

Lemma 4.1. *(See [45, Lemma 1.2].) Suppose $K\in\mathcal{K}_o^n$ and $u\in S^{n-1}$. For $y'\in\operatorname{relint}K_u$, the overgraph and undergraph functions of K in direction u are given by*

$$\bar{\ell}_u(K;y')=\min_{x'\in u^\perp}\{h_K(x',1)-x'\cdot y'\}$$

and

$$\underline{\ell}_u(K;y')=\min_{x'\in u^\perp}\{h_K(x',-1)-x'\cdot y'\}.$$

We refer to [45] for a proof. See [3] for an application in the proof of the L_p Busemann–Petty centroid inequality.

In addition to Lemma 4.1, note the following elementary fact: given a convex body K and a direction $u\in S^{n-1}$, for each $y'\in\operatorname{relint}K_u$, every outer normal vector at the upper boundary point $(y',\bar{\ell}_u(K;y'))$ can be written as $(x_1',1)$, while every outer normal vector at the lower boundary point $(y',-\underline{\ell}_u(K;y'))$ can be written as $(x_2',-1)$, where $x_1',x_2'\in u^\perp$.

The following lemma will be used in the proofs of our theorems.

Lemma 4.2. *Suppose $K\in\mathcal{K}^n$. Let $u\in S^{n-1}$ and $x_1',x_2'\in u^\perp$. Then,*

$$h_K(x_1',1)+h_K(x_2',-1)\geqslant 2h_{S_uK}\left(\frac{x_1'+x_2'}{2},1\right), \tag{4.5}$$

and

$$h_K(x_1',1)+h_K(x_2',-1)\geqslant 2h_{S_uK}\left(\frac{x_1'+x_2'}{2},-1\right). \tag{4.6}$$

Proof. For arbitrary $a_0'\in K_u$, noticing that $(a_0',\bar{\ell}_u(S_uK;a_0'))\in K$, we have

$$h_K(x_1',1)=\max_{(a',s)\in K}\{(x_1',1)\cdot(a',s)\}\geqslant x_1'\cdot a_0'+\bar{\ell}_u(K;a_0'). \tag{4.7}$$

In a similar way, we have

$$h_K(x_2',-1)=\max_{(a',s)\in K}\{(x_2',-1)\cdot(a',s)\}\geqslant x_2'\cdot a_0'+\underline{\ell}_u(K;a_0'). \tag{4.8}$$

Then,

$$h_K(x_1',1)+h_K(x_2',-1)\geqslant(x_1'+x_2')\cdot a_0'+[\bar{\ell}_u(K;a_0')+\underline{\ell}_u(K;a_0')], \tag{4.9}$$

for all $a_0'\in K_u$.

By (4.2), (4.3), and (4.4), we have

$$h_{S_uK}\left(\frac{x_1'+x_2'}{2},1\right)=\max_{a'\in K_u}\left\{\frac{x_1'+x_2'}{2}\cdot a'+\frac{\bar{\ell}_u(K;a')+\underline{\ell}_u(K;a')}{2}\right\}, \tag{4.10}$$

and

$$h_{S_uK}\left(\frac{x_1'+x_2'}{2},-1\right)=\max_{a'\in K_u}\left\{\frac{x_1'+x_2'}{2}\cdot a'+\frac{\bar{\ell}_u(K;a')+\underline{\ell}_u(K;a')}{2}\right\}. \tag{4.11}$$

Since (4.9) holds for all $a_0'\in K_u$, Eqs. (4.10) and (4.11) imply that (4.5) and (4.6) hold. $\square$

Lemma 4.3. *Let $\phi\in\mathcal{C}$, $\alpha>0$, $\beta\geqslant 0$, and $u\in S^{n-1}$. If $K,L\in\mathcal{K}_o^n$, then*

$$M_\phi(\alpha,\beta;S_uK,S_uL)\subseteq S_u(M_\phi(\alpha,\beta;K,L)).$$

Proof. Set $M=M_\phi(\alpha,\beta;K,L)$ and $M_S=M_\phi(\alpha,\beta;S_uK,S_uL)$.

By Lemma 4.1, for arbitrary $y'\in\operatorname{relint}M_u$, there are points $x_1',x_2'\in u^\perp$ such that

$$\bar{\ell}_u(M;y')=h_M(x_1',1)-x_1'\cdot y'$$

and

$$\underline{\ell}_u(M;y')=h_M(x_2',-1)-x_2'\cdot y'.$$

Suppose $z_1 = h_M(x_1', 1)$ and $z_2 = h_M(x_2', -1)$. Then,

$$\alpha\phi\left(\frac{h_K(x_1',1)}{z_1}\right) + \beta\phi\left(\frac{h_L(x_1',1)}{z_1}\right) = 1, \tag{4.12}$$

and

$$\alpha\phi\left(\frac{h_K(x_2',-1)}{z_2}\right) + \beta\phi\left(\frac{h_L(x_2',-1)}{z_2}\right) = 1. \tag{4.13}$$

By adding (4.12) multiplied with z_1 and (4.13) multiplied with z_2, using the convexity of ϕ, and by Lemma 4.2, we get

$$\begin{aligned}
z_1 + z_2 &= z_1\alpha\phi\left(\frac{h_K(x_1',1)}{z_1}\right) + z_2\alpha\phi\left(\frac{h_K(x_2',-1)}{z_2}\right) \\
&\quad + z_1\beta\phi\left(\frac{h_L(x_1',1)}{z_1}\right) + z_2\beta\phi\left(\frac{h_L(x_2',-1)}{z_2}\right) \\
&\geqslant (z_1+z_2)\left[\alpha\phi\left(\frac{h_K(x_1',1)+h_K(x_2',-1)}{z_1+z_2}\right)\right. \\
&\quad \left. + \beta\phi\left(\frac{h_L(x_1',1)+h_L(x_2',-1)}{z_1+z_2}\right)\right] \qquad (4.14) \\
&\geqslant (z_1+z_2)\left[\alpha\phi\left(\frac{2h_{S_uK}(\frac{x_1'+x_2'}{2},1)}{z_1+z_2}\right) + \beta\phi\left(\frac{2h_{S_uL}(\frac{x_1'+x_2'}{2},1)}{z_1+z_2}\right)\right]. \qquad (4.15)
\end{aligned}$$

Therefore, we obtain

$$\alpha\phi\left(\frac{h_{S_uK}(\frac{x_1'+x_2'}{2},1)}{(z_1+z_2)/2}\right) + \beta\phi\left(\frac{h_{S_uL}(\frac{x_1'+x_2'}{2},1)}{(z_1+z_2)/2}\right) \leqslant 1, \tag{4.16}$$

which implies that

$$\frac{z_1+z_2}{2} \geqslant h_{M_S}\left(\frac{x_1'+x_2'}{2}, 1\right). \tag{4.17}$$

Now (4.17) and Lemma 4.1 show that

$$\begin{aligned}
\bar{\ell}_u(S_uM; y') &= \frac{1}{2}\bar{\ell}_u(M;y') + \frac{1}{2}\underline{\ell}_u(M;y') \\
&= \frac{1}{2}(z_1 - x_1'\cdot y') + \frac{1}{2}(z_2 - x_2'\cdot y') \\
&\geqslant h_{M_S}\left(\frac{x_1'+x_2'}{2}, 1\right) - \frac{x_1'+x_2'}{2}\cdot y' \\
&\geqslant \min_{x'\in u^\perp}\{h_{M_S}(x',1) - x'\cdot y'\} \\
&= \bar{\ell}_u(M_S; y').
\end{aligned}$$

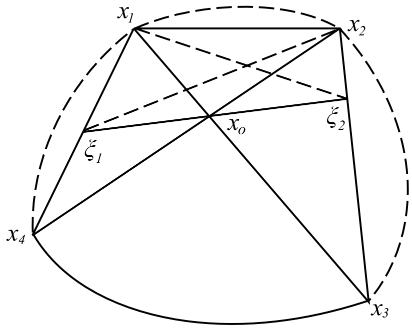

Fig. 1. Method to find the chords.

In the same way, we obtain

$$\underline{\ell}_u\big(S_u M; y'\big) \geqslant \underline{\ell}_u\big(M_S; y'\big).$$

Since $y' \in \operatorname{relint} M_u$ is arbitrary, this completes the proof of the lemma. $\square$

We say a chord $[x, y]$ of a convex body K is an *interior chord* of K if $(x, y) \subset \operatorname{int} K$, where (x, y) denotes the relative interior of $[x, y]$. We say a chord is a *boundary chord* of a convex body if it is contained in the boundary of this convex body.

It can be concluded from [50, Theorem 1.1.8] that a chord of a convex body K is an interior chord if and only if there is an interior point of K that lies in this chord. Therefore, a chord of a convex body is either an interior chord or a boundary chord.

In order to get the equality condition of (1.7), we need the following elementary observation.

Lemma 4.4. *Suppose $K \in \mathcal{K}^n$. If $x_1, x_2 \in \partial K$ are two distinct boundary points of K, then x_1 and x_2 can be connected by k interior chords with $k \leqslant 3$.*

Proof. Since K is a convex body, we can suppose x_o is an interior point of K. Suppose $n \geqslant 2$, since when $n = 1$ it is obvious.

Next, we describe how to find the k interior chords (see Fig. 1).

(i) If $[x_1, x_2]$ is an interior chord of K, then $[x_1, x_2]$ is the chord which we are searching for.
(ii) If we suppose $[x_1, x_2]$ is not an interior chord of K, then $[x_1, x_2] \subset \partial K$. There exists a unique point $x_3 \in \partial K$, such that $x_o \in (x_1, x_3)$. If $[x_3, x_2]$ is an interior chord of K, then $[x_1, x_3], [x_3, x_2]$ are the chords which we are searching for.
(iii) If we suppose $[x_1, x_2]$ and $[x_3, x_2]$ are not interior chords of K, then $[x_3, x_2] \subset \partial K$. There exists a unique point $x_4 \in \partial K$, such that $x_o \in (x_2, x_4)$. If $[x_4, x_1]$ is an interior chord of K, then $[x_1, x_4], [x_4, x_2]$ are the chords which we are searching for.

(iv) We suppose $[x_1, x_2]$, $[x_3, x_2]$ and $[x_4, x_1]$ are not interior chords of K. By our construction, the points x_o, x_1, x_2, x_3, x_4 lie in a 2-dimensional plane. Let ξ_1 be the midpoint of the chord $[x_1, x_4]$. Then $[\xi_1, x_2]$ is an interior chord of K because $(\xi_1, x_2) \cap (x_1, x_3) \neq \varnothing$. There exists a unique point $\xi_2 \in (x_2, x_3)$ such that $x_o \in (\xi_1, \xi_2)$. So, $\xi_2 \in \partial K$. It is clear that the chords $[\xi_1, \xi_2]$ and $[x_1, \xi_2]$ are interior chords of K. Then, $[x_1, \xi_2], [\xi_2, \xi_1], [\xi_1, x_2]$ are the chords which we are searching for. $\square$

Suppose $\phi \in \mathcal{C}$ is strictly convex; the following lemma gives the necessary equality condition in the inequality of Lemma 4.3.

Lemma 4.5. *Suppose $\phi \in \mathcal{C}$ is strictly convex. Let $K, L \in \mathcal{K}_o^n$, and $\alpha, \beta > 0$. If*

$$M_\phi(\alpha, \beta; S_u K, S_u L) = S_u\big(M_\phi(\alpha, \beta; K, L)\big) \tag{4.18}$$

for all $u \in S^{n-1}$, then K and L are dilates.

Proof. Set $M = M_\phi(\alpha, \beta; K, L)$. Suppose $[\xi_1, \xi_2]$ is an arbitrary interior chord of the convex body M. Let $u = (\xi_1 - \xi_2)/\|\xi_1 - \xi_2\| \in S^{n-1}$, where $\|\cdot\|$ denotes the Euclidean norm. Then, $[\xi_1, \xi_2]$ is parallel to u, ξ_1 is the upper boundary point, and ξ_2 is the lower boundary point. Thus, there exists $y' \in \operatorname{relint} M_u$, such that $\xi_1 = (y', \bar{\ell}_u M(M; y'))$, and $\xi_2 = (y', -\underline{\ell}_u(M; y'))$.

Since $y' \in \operatorname{relint} M_u$, each outer normal vector of M at ξ_1 can be written as $(x_1', 1)$, and each outer normal vector at ξ_2 can be written as $(x_2', -1)$, where $x_1', x_2' \in u^\perp$. Then, we have

$$h_M\big(x_1', 1\big) = \big(x_1', 1\big) \cdot \big(y', \bar{\ell}_u\big(M; y'\big)\big),$$

hence,

$$\bar{\ell}_u\big(M; y'\big) = h_M\big(x_1', 1\big) - x_1' \cdot y'.$$

Similarly, we have

$$\underline{\ell}_u\big(M; y'\big) = h_M\big(x_2', -1\big) - x_2' \cdot y'.$$

By the same argument as that for Lemma 4.3, we can establish inequalities (4.14), (4.15), and (4.17). If (4.18) holds for all $u \in S^{n-1}$, then (4.14), (4.15), and (4.17) are all equalities. Since ϕ is strictly convex, (4.14) is an equality if and only if

$$\frac{h_K(x_1', 1)}{z_1} = \frac{h_K(x_2', -1)}{z_2}, \quad \text{and} \quad \frac{h_L(x_1', 1)}{z_1} = \frac{h_L(x_2', -1)}{z_2},$$

and then there is a positive constant c_0 such that

$$c_0 = \frac{h_K(x_1', 1)}{h_L(x_1', 1)} = \frac{h_K(x_2', -1)}{h_L(x_2', -1)}. \tag{4.19}$$

For every direction $v \in S^{n-1}$, there is a point $\xi_3 \in \partial M$, such that v is an outer normal vector at ξ_3. If $\xi_3 \neq \xi_1$, by Lemma 4.4, there are $k \leqslant 3$ interior chords of M, such that they connect ξ_1 to ξ_3. Clearly, for each interior chord of M, there is a similar equality as (4.19). Then, we obtain that

$$c_0 = \frac{h_K(x_1', 1)}{h_L(x_1', 1)} = \frac{h_K(v)}{h_L(v)}.$$

If $\xi_3 = \xi_1$, then v is a normal vector of M at ξ_1. Since (4.19) holds for each normal vector of M at ξ_1, we have

$$c_0 = \frac{h_K(x_2', -1)}{h_L(x_2', -1)} = \frac{h_K(v)}{h_L(v)}.$$

Therefore K and L are dilates because v is arbitrary. $\square$

From Lemma 4.3 and Lemma 4.5 we get the following theorem, which is indeed an original version of Orlicz Brunn–Minkowski inequality.

Theorem 4.6. *Suppose $\phi \in \mathcal{C}$, $K, L \in \mathcal{K}_o^n$, and $\alpha, \beta > 0$. Let $V(K) = a^n\omega_n$, and $V(L) = b^n\omega_n$, then*

$$V\big(M_\phi(\alpha, \beta; K, L)\big) \geqslant C_\phi(\alpha, \beta; a, b)^n\omega_n. \tag{4.20}$$

Equality holds if K and L are dilates. When ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. There is a sequence of directions $\{u_i\}$, such that the sequence $\{K_i\}$ converges to aB^n and $\{L_i\}$ converges to bB^n, where the sequences $\{K_i\}$ and $\{L_i\}$ are defined by

$$K_i = S_{u_i} \cdots S_{u_1} K, \quad \text{and} \quad L_i = S_{u_i} \cdots S_{u_1} L.$$

Since the Steiner symmetrization preserves the volume, by Lemma 4.3 we have

$$V\big(M_\phi(\alpha, \beta; K, L)\big) \geqslant V\big(M_\phi\big(\alpha, \beta; aB^n, bB^n\big)\big).$$

From the definition of Orlicz combination of convex bodies, we get that $M_\phi(\alpha, \beta; aB^n, bB^n)$ is an n-ball with radius $C_\phi(\alpha, \beta; a, b)$. This implies (4.20).

If K and L are dilates, there exists a convex body $A \in \mathcal{K}_o^n$ whose volume is ω_n, such that A, K, and L are dilates. That is, $K = aA$, and $L = bA$. By (1.9), we have

$$\alpha\phi\left(\frac{ah_A(u)}{\tau_u}\right)+\beta\phi\left(\frac{bh_A(u)}{\tau_u}\right)=1, \quad \text{for all } u\in S^{n-1},$$

where $\tau_u = h_{M_\phi(\alpha,\beta;K,L)}(u)$. This implies that

$$h_{M_\phi(\alpha,\beta;K,L)}(u)=C_\phi(\alpha,\beta;a,b)h_A(u), \quad \text{for all } u\in S^{n-1}.$$

Therefore, $V(M_\phi(\alpha,\beta;K,L))=C_\phi(\alpha,\beta;a,b)^n\omega_n$.

Suppose ϕ is strictly convex. If equality holds in (4.20), then

$$M_\phi(\alpha,\beta;S_uK,S_uL)=S_u\big(M_\phi(\alpha,\beta;K,L)\big),$$

for all $u\in S^{n-1}$. By Lemma 4.5, we conclude that K and L are dilates. □

The following theorem is the general version of Theorem 1.

Theorem 4.7. *Suppose $\phi\in\mathcal{C}$ and $K,L\in\mathcal{K}_o^n$. If $\alpha,\beta>0$, then*

$$\alpha\phi\left(\frac{V(K)^{\frac{1}{n}}}{V(M_\phi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right)+\beta\phi\left(\frac{V(L)^{\frac{1}{n}}}{V(M_\phi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right)\leqslant 1. \tag{4.21}$$

Equality holds if K and L are dilates. When ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. Let $V(K)=a^n\omega_n$ and $V(L)=b^n\omega_n$ with $a,b>0$. By (4.20), we have

$$V\big(M_\phi(\alpha,\beta;K,L)\big)^{\frac{1}{n}}\geqslant C_\phi(\alpha,\beta;a,b)\omega_n^{\frac{1}{n}}.$$

Since $V(K)^{\frac{1}{n}}=a\omega_n^{\frac{1}{n}}$, we get

$$\frac{V(K)^{\frac{1}{n}}}{V(M_\phi(\alpha,\beta;K,L))^{\frac{1}{n}}}\leqslant\frac{a}{C_\phi(\alpha,\beta;a,b)}.$$

Therefore,

$$\phi\left(\frac{V(K)^{\frac{1}{n}}}{V(M_\phi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right)\leqslant\phi\left(\frac{a}{C_\phi(\alpha,\beta;a,b)}\right).$$

In the same way, we also get

$$\phi\left(\frac{V(L)^{\frac{1}{n}}}{V(M_\phi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right)\leqslant\phi\left(\frac{b}{C_\phi(\alpha,\beta;a,b)}\right).$$

Hence,

$$\begin{aligned}1 &= \alpha\phi\left(\frac{a}{C_\phi(\alpha,\beta;a,b)}\right)+\beta\phi\left(\frac{b}{C_\phi(\alpha,\beta;a,b)}\right)\\ &\geqslant \alpha\phi\left(\frac{V(K)^{\frac{1}{n}}}{V(M_\phi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right)+\beta\phi\left(\frac{V(L)^{\frac{1}{n}}}{V(M_\phi(\alpha,\beta;K,L))^{\frac{1}{n}}}\right).\end{aligned}$$

The equality condition can be obtained as in Theorem 4.6. $\square$

Taking $\alpha=\beta=1$ in Theorem 4.7, we obtain Theorem 1.

5. The Orlicz mixed volumes

In this section, we study the Orlicz mixed volume, which is defined by Definition 3. Since $\phi\in\mathcal{C}$, the left derivative ϕ_l' and right derivative ϕ_r' exist, ϕ_l' is left-continuous and ϕ_r' is right-continuous on $[0,+\infty)$. Furthermore, ϕ_l' and ϕ_r' are positive on $(0,+\infty)$.

Lemma 5.1. *Let $\phi\in\mathcal{C}$ satisfy $\phi(1)=1$, let $a,b>0$, and $\beta\geqslant 0$. Then, $C_\phi(1,\beta;a,b)$ is differentiable at $\beta=0$, and*

$$C_\phi'(1,0;a,b)=\lim_{\beta\to 0^+}\frac{C_\phi(1,\beta;a,b)-C_\phi(1,0;a,b)}{\beta}=\frac{a}{\phi_l'(1)}\phi\left(\frac{b}{a}\right).$$

Proof. Set $z_\beta=C_\phi(1,\beta;a,b)$, $y_\beta=\phi(\frac{a}{z_\beta})$, for all $\beta\geqslant 0$. Obviously, $z_0=a$ and $y_0=1$. It follows by Lemma 3.1 (v) that $z_\beta\to a^+$ and $y_\beta\to 1^-$ as $\beta\to 0^+$.

Since ϕ_l' and ϕ_r' are positive on $(0,+\infty)$, we have

$$\left(\phi^{-1}\right)_l'(t)=\frac{1}{\phi_l'(\phi^{-1}(t))}\quad\text{and}\quad\left(\phi^{-1}\right)_r'(t)=\frac{1}{\phi_r'(\phi^{-1}(t))},\quad t\in(0,+\infty).$$

Since $1-y_\beta=\beta\phi(\frac{b}{z_\beta})$, we have

$$\lim_{\beta\to 0^+}\frac{1-\frac{a}{z_\beta}}{\beta}=\lim_{\beta\to 0^+}\frac{1-y_\beta}{\beta}\lim_{\beta\to 0^+}\frac{1-\frac{a}{z_\beta}}{1-y_\beta}=\phi\left(\frac{b}{a}\right)\lim_{y_\beta\to 1^-}\frac{1-\frac{a}{z_\beta}}{1-y_\beta}=\phi\left(\frac{b}{a}\right)\frac{1}{\phi_l'(1)}.$$

Hence, we get

$$\begin{aligned}C_\phi'(1,0;a,b)&=\lim_{\beta\to 0^+}\frac{z_\beta-z_0}{\beta}\\ &=\lim_{\beta\to 0^+}z_\beta\cdot\lim_{\beta\to 0^+}\frac{1-\frac{a}{z_\beta}}{\beta}\\ &=\frac{a}{\phi_l'(1)}\phi\left(\frac{b}{a}\right).\end{aligned}\qquad\square$$

The following lemma shows that $h_{M_\phi(1,\epsilon;K,L)}(u)$ is uniformly differentiable at $\epsilon = 0$. This fact plays a key role in the proof of Theorem 2.

Lemma 5.2. *Let $\phi \in \mathcal{C}$ satisfy $\phi(1) = 1$, and let $K, L \in \mathcal{K}_o^n$. Then the convergence in*

$$\lim_{\epsilon \to 0^+} \frac{h_{M_\phi(1,\epsilon;K,L)}(u) - h_K(u)}{\epsilon} = \frac{h_K(u)}{\phi_l'(1)} \phi\left(\frac{h_L(u)}{h_K(u)}\right) \tag{5.1}$$

is uniform on S^{n-1}.

Proof. Set $K_\epsilon = M_\phi(1, \epsilon; K, L)$ for all $\epsilon \geqslant 0$. From Lemma 3.2 (v), $K_\epsilon \to K$. Since $h_{K_\epsilon}(u) = C_\phi(1, \epsilon, h_K(u), h_L(u))$ for each $u \in S^{n-1}$, by Lemma 5.1, we have

$$\lim_{\epsilon \to 0^+} \frac{h_{K_\epsilon}(u) - h_K(u)}{\epsilon} = \frac{h_K(u)}{\phi_l'(1)} \phi\left(\frac{h_L(u)}{h_K(u)}\right).$$

Let $g : [0, +\infty) \to [0, +\infty)$ be a concave function, and let $x > y > 0$. Then,

$$g_l'(x)(x - y) < g(x) - g(y) < g_l'(y)(x - y). \tag{5.2}$$

Let $y_\epsilon(u) = \phi(\frac{h_K(u)}{h_{K_\epsilon}(u)})$. Then, $y_\epsilon(u) \to 1^-$ as $\epsilon \to 0^+$. From $\phi \in \mathcal{C}$ we conclude that ϕ^{-1} is concave on $[0, +\infty)$. By substituting $g = \phi^{-1}$ into (5.2), and the facts that $y_0(u) = 1$ and $\phi^{-1}(y_0(u)) = 1$, we have

$$\frac{1}{\phi_l'(1)}\big(1 - y_\epsilon(u)\big) \leqslant 1 - \phi^{-1}\big(y_\epsilon(u)\big) \leqslant \frac{1}{\phi_l'(\phi^{-1}(y_\epsilon(u)))}\big(1 - y_\epsilon(u)\big).$$

Notice that

$$\frac{h_{K_\epsilon}(u) - h_K(u)}{\epsilon} = h_{K_\epsilon}(u) \frac{1 - \phi^{-1}(y_\epsilon(u))}{\epsilon},$$

and

$$1 - y_\epsilon(u) = \epsilon \phi\left(\frac{h_L(u)}{h_{K_\epsilon}(u)}\right).$$

Therefore, we have

$$\frac{h_{K_\epsilon}(u)}{\phi_l'(1)} \phi\left(\frac{h_L(u)}{h_{K_\epsilon}(u)}\right) \leqslant \frac{h_{K_\epsilon}(u) - h_K(u)}{\epsilon} \leqslant \frac{h_{K_\epsilon}(u)}{\phi_l'(h_K(u)/h_{K_\epsilon}(u))} \phi\left(\frac{h_L(u)}{h_{K_\epsilon}(u)}\right). \tag{5.3}$$

Since $h_{K_\epsilon}(u) \to h_K(u)$ (as $\epsilon \to 0^+$) uniformly on S^{n-1}, we have h_L/h_{K_ϵ} converges to h_L/h_K uniformly, and h_K/h_{K_ϵ} converges to 1 uniformly. Thus, h_L/h_{K_ϵ} and h_K/h_{K_ϵ} are uniformly bounded and they lie in a compact interval I, and $\phi(t)$ is uniformly continuous on I. So the left side of (5.3) converges to $\frac{h_K(u)}{\phi_l'(1)} \phi(\frac{h_L(u)}{h_K(u)})$ uniformly.

Notice that $\phi_l'(t)$ is left-continuous at $t = 1$, h_K/h_{K_ϵ} converges to 1 uniformly, and $h_K/h_{K_\epsilon} \leqslant 1$. For arbitrary $\eta > 0$, there exists a $\delta > 0$, such that $|\phi_l'(t) - \phi_l'(1)| < \eta$ for all $1 - \delta < t \leqslant 1$. For this δ, there exists a $\theta > 0$, such that

$$1 - \delta < \frac{h_K(u)}{h_{K_\epsilon}(u)} \leqslant 1,$$

for all $u \in S^{n-1}$ and $0 \leqslant \epsilon < \theta$. Then,

$$\left|\phi_l'\left(\frac{h_K(u)}{h_{K_\epsilon}(u)}\right) - \phi_l'(1)\right| < \eta,$$

for all $u \in S^{n-1}$ and $0 \leqslant \epsilon < \theta$. Therefore, $\phi_l'(h_K/h_{K_\epsilon})$ converges uniformly to $\phi_l'(1)$, and the right side of (5.3) converges to $\frac{h_K(u)}{\phi_l'(1)}\phi(\frac{h_L(u)}{h_K(u)})$ uniformly. Thus the convergence in (5.1) is uniform. $\square$

Applying the method in Lutwak [33] (see also [19, Lemma 1]), we get the proof of Theorem 2 by Lemma 5.2.

Proof of Theorem 2. Set $K_\epsilon = M_\phi(1, \epsilon; K, L)$, $\epsilon \geqslant 0$. By Property (v) in Lemma 3.2, we have that $K_\epsilon \to K$ as $\epsilon \to 0^+$, which implies that the surface area measure $S_{K_\epsilon} \to S_K$ weakly.

Since the measures S_{K_ϵ} are finite, converging weakly to S_K, by Lemma 5.2, we have

$$\lim_{\epsilon\to 0^+} \int\limits_{S^{n-1}} \frac{h_{K_\epsilon}(u) - h_K(u)}{\epsilon} dS_{K_\epsilon}(u) = \int\limits_{S^{n-1}} \frac{h_K(u)}{\phi_l'(1)} \phi\left(\frac{h_L(u)}{h_K(u)}\right) dS_K(u),$$

and

$$\lim_{\epsilon\to 0^+} \int\limits_{S^{n-1}} \frac{h_{K_\epsilon}(u) - h_K(u)}{\epsilon} dS_K(u) = \int\limits_{S^{n-1}} \frac{h_K(u)}{\phi_l'(1)} \phi\left(\frac{h_L(u)}{h_K(u)}\right) dS_K(u).$$

Hence, we have

$$\begin{aligned}\lim_{\epsilon\to 0^+} \frac{V(K_\epsilon) - V_1(K_\epsilon, K)}{\epsilon} &= \lim_{\epsilon\to 0^+} \frac{V_1(K, K_\epsilon) - V(K)}{\epsilon} \\ &= \frac{1}{n\phi_l'(1)} \int\limits_{S^{n-1}} h_K(u)\phi\left(\frac{h_L(u)}{h_K(u)}\right) dS_K(u). \qquad (5.4)\end{aligned}$$

Set

$$l = \frac{1}{n\phi_l'(1)} \int\limits_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) dS_K(u). \qquad (5.5)$$

From (5.4) and (1.4), we have

$$\begin{aligned} l &= \lim_{\epsilon\to 0^+} \frac{V(K_\epsilon) - V_1(K_\epsilon, K)}{\epsilon} \\ &\leqslant \liminf_{\epsilon\to 0^+} \frac{V(K_\epsilon)^{\frac{n-1}{n}}(V(K_\epsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}})}{\epsilon} \\ &= V(K)^{\frac{n-1}{n}} \liminf_{\epsilon\to 0^+} \frac{V(K_\epsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}}}{\epsilon}, \end{aligned}$$

and

$$\begin{aligned} l &= \lim_{\epsilon\to 0^+} \frac{V_1(K, K_\epsilon) - V(K)}{\epsilon} \\ &\geqslant \limsup_{\epsilon\to 0^+} \frac{V(K)^{\frac{n-1}{n}}(V(K_\epsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}})}{\epsilon} \\ &= V(K)^{\frac{n-1}{n}} \limsup_{\epsilon\to 0^+} \frac{V(K_\epsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}}}{\epsilon}. \end{aligned}$$

Thus, we obtain

$$l = V(K)^{\frac{n-1}{n}} \lim_{\epsilon\to 0^+} \frac{V(K_\epsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}}}{\epsilon}.$$

Therefore,

$$\begin{aligned} \lim_{\epsilon\to 0^+} \frac{V(K_\epsilon) - V(K)}{\epsilon} &= \lim_{\epsilon\to 0^+} \frac{(V(K_\epsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}})\sum_{i=0}^{n-1} V(K_\epsilon)^{\frac{i}{n}} V(K)^{\frac{n-1-i}{n}}}{\epsilon} \\ &= nV(K)^{\frac{n-1}{n}} \lim_{\epsilon\to 0^+} \frac{V(K_\epsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}}}{\epsilon} \\ &= nl. \end{aligned} \tag{5.6}$$

Combining with (5.6) and (5.5), we complete the proof of Theorem 2. □

Based on Theorem 2, we give two proofs of Theorem 3. The first uses the Orlicz Brunn–Minkowski inequality, while the second uses Jensen's inequality. However, the first proof only establishes the inequality, while the equality condition can be obtained in the second proof.

First proof of Theorem 3. By Theorem 2, the following limit exists:

$$V_\phi(K, L) = \frac{\phi_l'(1)}{n} \lim_{\epsilon\to 0^+} \frac{V(M_\phi(1, \epsilon; K, L)) - V(K)}{\epsilon}.$$

By the convexity of ϕ (note that $\phi(1) = 1$), we have

$$\phi_l'(1)(1-x) \geqslant 1-\phi(x). \tag{5.7}$$

By Theorem 2, (5.6), (5.7), and Theorem 4.7, we have

$$\begin{aligned}
V_\phi(K,L) &= \frac{\phi_l'(1)}{n} \lim_{\epsilon\to 0^+} \frac{V(K_\epsilon)-V(K)}{\epsilon} \\
&= \phi_l'(1)V(K)^{\frac{n-1}{n}} \lim_{\epsilon\to 0^+} \frac{V(K_\epsilon)^{\frac{1}{n}}-V(K)^{\frac{1}{n}}}{\epsilon} \\
&= \phi_l'(1)V(K) \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(1-\frac{V(K)^{\frac{1}{n}}}{V(K_\epsilon)^{\frac{1}{n}}}\right) \\
&\geqslant V(K) \liminf_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(1-\phi\left(\frac{V(K)^{\frac{1}{n}}}{V(K_\epsilon)^{\frac{1}{n}}}\right)\right) \\
&\geqslant V(K) \lim_{\epsilon\to 0^+} \phi\left(\frac{V(L)^{\frac{1}{n}}}{V(K_\epsilon)^{\frac{1}{n}}}\right) \\
&= V(K)\phi\left(\frac{V(L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\right).
\end{aligned}$$

Thus we have established inequality (1.11). □

The second proof of Theorem 3 uses the Jensen's inequality.

Second proof of Theorem 3. By Theorem 2, we have

$$V_\phi(K,L) = \frac{1}{n\phi_l'(1)} \int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)\,dS_K(u).$$

Since

$$\frac{1}{n}\int_{S^{n-1}} h_K(u)\,dS_K(u) = V(K),$$

$\frac{h_K(\cdot)S_K(\cdot)}{nV(K)}$ is a probability measure on S^{n-1}. By Jensen's inequality and (1.4), we have

$$\begin{aligned}
\frac{V_\phi(K,L)}{V(K)} &= \int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) \frac{h_K(u)\,dS_K(u)}{nV(K)} \\
&\geqslant \phi\left(\int_{S^{n-1}} \frac{h_L(u)}{h_K(u)} \frac{h_K(u)\,dS_K(u)}{nV(K)}\right)
\end{aligned}$$

$$= \phi\left(\frac{V_1(K,L)}{V(K)}\right)$$
$$\geqslant \phi\left(\frac{V(L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\right).$$

If K, L are dilates, it is easy to see that equality holds in (1.11).

Now suppose ϕ is strictly convex. If equality holds, then, by the equality condition of Jensen's inequality, there exists an $s > 0$ such that $h_L(u) = sh_K(u)$ for almost every $u \in S^{n-1}$ with respect to the measure $\frac{h_K(\cdot)S_K(\cdot)}{nV(K)}$. Then, we have

$$\frac{V_\phi(K,L)}{V(K)} = \phi(s) = \phi\left(\frac{V(L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\right).$$

Thus, $s = V(L)^{1/n}/V(K)^{1/n}$. Furthermore, the equality condition of (1.4) implies that K and L are homothetic. Then, $L = sK + t$ for some $t \in \mathbb{R}^n$. Since K has interior points, the support of the measure $\frac{h_K(\cdot)S_K(\cdot)}{nV(K)}$ cannot be contained in the great sphere of S^{n-1} orthogonal to t. Hence $t = 0$, which implies that K, L are dilates. □

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