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ORIGINAL PAPER

On the discrete Orlicz Minkowski problem II

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Abstract
The Orlicz Minkowski problem is a generalization of the L_p Minkowski problem. For a class of appropriate functions and discrete measures that have no essential subspaces, the existence is demonstrated for the discrete Orlicz Minkowski problem. This is a non-trivial extension of the discrete L_p Minkowski problem for $p < 0$.

Keywords Convex polytope · Minkowski problem · Orlicz Minkowski problem

Mathematics Subject Classification (2000) 52A40

1 Introduction

The L_p surface area measure was introduced by Lutwak [30]. Let K be a convex body in n-dimensional Euclidean space, $\mathbb{R}^n$, that contains the origin in its interior. For $p \in \mathbb{R}$ and a Borel set ω on the unit sphere, S^{n-1}, the L_p surface area measure $S_p(K, \omega)$ of the convex body K is defined by

$$S_p(K,\omega) = \int_{x \in \nu_K^{-1}(\omega)} (x \cdot \nu_K)^{1-p} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of boundary points of K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. For $p = 1$, the L_1 surface area measure is the classic surface area measure, which

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is abbreviated as $S(K,\cdot)$ or S_K. Observe that for the surface area measure of cK, $c>0$, we have

$$S_{cK} = c^{n-1} S_K. \tag{1.1}$$

The L_p Minkowski problem, posed by Lutwak [30], is considered as one of the cornerstones of the L_p Brunn–Minkowski theory. It asks for necessary and sufficient conditions on a Borel measure μ on S^{n-1} to be the L_p surface area measure of a convex body, i.e., is there a convex body K such that

$$h_K^{1-p} dS_K = d\mu?$$

Here, h_K is the support function of K. The solutions of the L_p Minkowski problem have important applications to affine isoperimetric inequalities, see, e.g., Zhang [47], Lutwak et al. [32,33], Haberl and Schuster [18–20].

The L_1 Minkowski problem is called Minkowski problem, which is proposed Minkowski. The discrete case was solved by Minkowski himself. Minkowski problem was completely solved by Alexandrov [1], Fenchel and Jessen [12]. For analytic versions and algorithmic issues of this problem, see, e.g., Chou and Wang [10], Jerison [26], Klain [27], and the references cited there.

The even L_p Minkowski problem for $p>1$ but $p\neq n$ was solved in [30]. An equivalent volume-normalized version of the L_p Minkowksi problem was proposed in [34], and the even case was also solved for $p=n$. A systemic study of the L_p Minkowski problem can be seen in the work of Chou and Wang [11]. In particular, they solved the problem for all $p>1$, while an alternate approach to this problem was presented by Hug et al. [25]. Zhu [49–51] dealt with the existence for the solution to the discrete L_p Minkowski problem for $0\leq p<1$ and $p=-n$. Other studies with respect to the L_p Minkowski problem have also been extensively studied (see, e.g., [2–5,8,9,22,31,35,39–41,44,52]). Quite recently, Huang et al. [24] proposed the dual Minkowski problem and proved existence theorem. Since [24], a number of works on the dual Minkowski problem have appeared. Zhao [45], Böröczky, Henk and Pollehn [7] and Böröczky et al. [6] combined completely solved existence part of the even dual Minkowski problem when the index $q\in(1,n)$. Zhao [46] proved both the existence and the uniqueness of the solution to the dual Minkowski problem when $q<0$. Henk and Pollehn [21] showed a necessary condition for the even dual Minkowski problem when $q\geq n+1$.

The Orlicz Brunn–Minkowski theory originated from the work of Lutwak, Yang, and Zhang in 2010, see [36,37], and the 2010 work of Ludwig [28] and Ludwig and Reitzner [29]. For the development of the Orlicz Brunn–Minkowski theory, see [14,15,17,28,43]. Haberl et al. [17] first proposed the following Orlicz Minkowski problem: given a suitable continuous function $\varphi:(0,+\infty)\to(0,+\infty)$ and a Borel measure μ on S^{n-1}, is there a convex body K such that for some $c>0$

$$c\varphi(h_K)dS_K = d\mu?$$

Set $\varphi(t)=t^{1-p}$ $(p\neq n)$, this problem reduces to the L_p Minkowski problem.

The even Orlicz Minkowski problem was solved by Haberl, Lutwak, Yang and Zhang in [17] under some suitable conditions on φ. One of their results is.

Theorem 1.1 [17] *Suppose $\varphi:(0,\infty)\to(0,\infty)$ is a continuous function such that $\phi(t)=\int_o^t \frac{1}{\varphi(s)}ds$ exists for every positive t and is unbounded as $t\to\infty$, and μ is an even finite Borel measure on S^{n-1} that is not concentrated on any great subsphere of S^{n-1}, then there exists an origin symmetric convex body $K\subset\mathbb{R}^n$ and $c>0$ such that $c\varphi(h_K)dS_K=d\mu$.*

When $\varphi(t) = t^{1-p},\ p > 0$, we obtain the even L_p Minkowski problem for $p > 0$.

The existence part of the general Orlicz Minkowski problem which contains the L_p Minkowski problem for $p > 1$ as a special case was solved by Huang and He [23]. One version of the discrete Orlicz Minkowski problem which contains the L_p Minkowski problem for $0 < p < 1$ as a special case was solved by Wu et al. [42].

Note that the conditions in Theorem 1.1 imply that ϕ is a concave function. It is the aim of this paper to deal with the discrete Orlicz Minkowski problem when ϕ is a convex function (see Lemma 3.1).

A linear subspace X $(0 < \dim X < n)$ of $\mathbb{R}^n$ is said to be essential with respect to a Borel measure μ on S^{n-1} if $X \cap \mathrm{supp}(\mu)$ is not concentrated on any closed hemisphere of $X \cap S^{n-1}$.

Our main result can be formulated as follows:

Theorem 1.2 *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_t^\infty \frac{1}{\varphi(s)} ds$ exists for every positive t and unbounded as $t \to 0$. If $\mu = \sum_{i=1}^N \alpha_i \delta_{u_i}$ does not have essential subspace, where δ_{u_i} is* Kronecker *delta, $\alpha_1, \ldots, \alpha_N > 0$ and $u_1, \ldots, u_N \in S^{n-1}$ are not contained in any closed hemisphere, then there exists a polytope P which contains the origin in its interior and $c > 0$ such that*

$$\mu = c\varphi(h(P, \cdot))S(P, \cdot). \tag{1.2}$$

Let $\varphi(s) = s^{1-p}$, $p < 0$, we get Zhu's result in [48].

Corollary 1.3 *Suppose vectors $u_1, \ldots, u_N \in S^{n-1}$ are not contained in any closed hemisphere, $\alpha_1, \ldots, \alpha_N > 0$ and $\mu = \sum_{i=1}^N \alpha_i \delta_{u_i}$ does not have essential subspace, where δ_{u_i} is* Kronecker *delta. If $p < 0$, then there exists a polytope P which contains the origin in its interior such that*

$$\mu = h(P, \cdot)^{1-p} S(P, \cdot).$$

The work of Zhu [48] inspired us a lot. However, when it comes to the Orlicz case, the functional φ may not be homogeneous, so it is difficult to show that the map $\xi_\phi(P_r)$ has a right derivative at $r = 0$, which is needed to use calculus of variations. Thus, we need many new steps, for details, see Sect. 4. This paper is organized as follows: in Sect. 2, we list some basic facts regarding convex bodies for quick reference. In Sect. 3, we give some properties about $\Phi_P(\xi)$. In Sect. 4, we prove the differentiability of $\xi_\phi(P_r)$. The proof of Theorem 1.2 is presented in Sect. 5.

2 Preliminaries

In this section, we list some terminologies and notations about convex bodies. For more more information on convex geometry, we recommend the books of Gardner [13], Gruber [16], and Schneider [38].

For $x, y \in \mathbb{R}^n$, let $[x, y] = \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$ and $(x, y) = \{(1-\lambda)x + \lambda y : 0 < \lambda < 1\}$. We also denote their inner product by $x \cdot y$ and the Euclidean norm of x by $|x| = \sqrt{x \cdot x}$. The unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ is denoted by S^{n-1}. Let V stand for n-dimensional Lebesgue measure.

A convex body is a compact convex set in $\mathbb{R}^n$ with nonempty interior. For a convex body K, the support function h_K is defined by $h_K(u) = h(K, u) = \max\{x \cdot u : x \in K\}$. For $u \in S^{n-1}$, the support hyperplane $F(K, u)$ in direction u is defined by

$$F(K,u)=\{x\in\mathbb{R}^n : x\cdot u=h(K,u)\},$$

the half-space $H^-(K,u)$ in direction u is defined by

$$H^-(K,u)=\{x\in\mathbb{R}^n : x\cdot u\leq h(K,u)\}.$$

If the unit vectors $u_1,\ldots,u_N$ $(N>n+1)$ are not contained in any closed hemisphere, we denote by $\mathcal{P}(u_1,\ldots,u_N)$ a subset of polytopes, which satisfies

$$P=\bigcap_{k=1}^{N}H^-(P,u_k),\quad \forall P\in\mathcal{P}(u_1,\ldots,u_N).$$

It is easy to see that if $P\in\mathcal{P}(u_1,\ldots,u_N)$, then P has at most N facets, and the outer unit normals of P are a subset of $\{u_1,\ldots,u_N\}$. Let $\mathcal{P}_N(u_1,\ldots,u_N)$ denote the subset of $\mathcal{P}(u_1,\ldots,u_N)$ such that if $P\in\mathcal{P}_N(u_1,\ldots,u_N)$, then P has exactly N facets.

3 An extremal problem to the Orlicz Minkowski problem

Suppose that $\alpha_1,\ldots,\alpha_N\in\mathbb{R}^+$, the unit vectors $u_1,\ldots,u_N$ are not contained in any closed hemisphere and $P\in\mathcal{P}(u_1,\ldots,u_N)$. Now we define the function $\Phi_P: P\to\mathbb{R}$ by

$$\Phi_P(\xi)=\sum_{k=1}^{N}\alpha_k\phi\left(h(P,u_k)-\xi\cdot u_k\right),\tag{3.1}$$

where ϕ is as described in Theorem 1.2.

In this section, we study the following extremal problem

$$\sup\left\{\inf_{\xi\in\mathrm{Int}(Q)}\Phi_Q(\xi) : Q\in\mathcal{P}(u_1,\ldots,u_N)\text{ and }V(Q)=1\right\}.\tag{3.2}$$

Next, we will prove that $\Phi_P(\xi)$ is convex on $\mathrm{Int}(P)$ and that there exists a unique $\xi_\phi(P)\in\mathrm{Int}(P)$ such that

$$\Phi_P(\xi_\phi(P))=\inf_{\xi\in\mathrm{Int}(P)}\Phi_P(\xi).$$

We want to prove that there exists a polytope with $u_1,\ldots,u_N$ as its outer unit normals and this polytope is a solution of problem (3.2). Now, we prove the convexity of $\Phi_P(\xi)$.

Lemma 3.1 *Suppose $\varphi:(0,\infty)\to(0,\infty)$ is differentiable, strictly increasing and $\varphi(s)$ tends to 0 as $s\to 0^+$ such that $\phi(t)=\int_t^\infty\frac{1}{\varphi(s)}ds$ exists for every positive t. Then*

$$\phi'(t)=-\frac{1}{\varphi(t)},\quad \forall t>0,\tag{3.3}$$

and ϕ is strictly convex on $(0,\infty)$.

Proof The Eq. (3.3) is clear and the second follows from L'hopital's rule. Since $\varphi:(0,\infty)\to(0,\infty)$ is differentiable, strictly increasing, we have

$$\phi''=\frac{\varphi'}{\varphi^2}>0.\tag{3.4}$$

Thus ϕ is strictly convex on $(0,\infty)$. $\square$

Lemma 3.2 *If $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ are not contained in any closed hemisphere, ϕ is strictly convex on $(0, \infty)$, $\lim_{t\to 0^+} \phi(t) = \infty$ and $P \in \mathcal{P}(u_1, \ldots, u_N)$, then there exists a unique $\xi_\phi(P) \in \mathrm{Int}(P)$ such that*

$$\Phi_P(\xi_\phi(P)) = \inf_{\xi\in\mathrm{Int}P} \Phi_P(\xi).$$

Proof Since ϕ is strictly convex on $(0, \infty)$. Then, for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in P$,

$$\begin{aligned}
&\lambda\Phi_P(\xi_1) + (1-\lambda)\Phi_P(\xi_2)\\
&\quad= \sum_{k=1}^{N} \alpha_k \left[\lambda\phi\left(h(P, u_k) - \xi_1 \cdot u_k\right) + (1-\lambda)\phi\left(h(P, u_k) - \xi_2 \cdot u_k\right)\right]\\
&\quad\geq \sum_{k=1}^{N} \alpha_k \phi\left(h(P, u_k) - (\lambda\xi_1 + (1-\lambda)\xi_2) \cdot u_k\right)\\
&\quad= \Phi_P(\lambda\xi_1 + (1-\lambda)\xi_2),
\end{aligned}$$

with equality if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, \ldots, N$. Since $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere, $\mathbb{R}^n = \mathrm{Span}\{u_1, \ldots, u_N\}$. Thus, $\xi_1 = \xi_2$. Therefore, $\Phi_P(\xi)$ is strictly convex on $\mathrm{Int}(P)$.

From the fact that $P \in P(u_1, \ldots, u_N)$, we have, for any $x \in \partial P$, there exists a $u_{i_0} \in \{u_1, \ldots, u_N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$

Together with (3.1), we have $\Phi_P(\xi) \to \infty$ whenever $\xi \in \mathrm{Int}(P)$ and $\xi \to x$. Therefore, there exists a unique interior point $\xi_\phi(P)$ of P such that

$$\Phi_P(\xi_\phi(P)) = \inf_{\xi\in\mathrm{Int}P} \Phi_P(\xi).$$

□

Note that, if $P_i \in \mathcal{P}(u_1, \ldots, u_N)$ and P_i converges to a polytope P, then $P \in \mathcal{P}(u_1, \ldots, u_N)$. In order to use approximation, we need the following lemma.

Lemma 3.3 *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_t^\infty \frac{1}{\varphi(s)} ds$ exists for every positive t and unbounded as $t \to 0$. If $\alpha_1, \ldots, \alpha_N > 0$ and $u_1, \ldots, u_N \in S^{n-1}$ are not contained in any closed hemisphere, $P_i \in \mathcal{P}(u_1, \ldots, u_N)$ and P_i converges to a polytope P, then $\lim_{i\to\infty} \xi_\phi(P_i) = \xi_\phi(P)$ and*

$$\lim_{i\to\infty} \Phi_{P_i}(\xi_\phi(P_i)) = \Phi_P(\xi_\phi(P)).$$

Proof By Lemmas 3.1 and 3.2 , $\xi_\phi(P_i)$ exists. Since $P_i \to P$ and $\xi_\phi(P_i) \in \mathrm{Int}(P_i)$, $\xi_\phi(P_i)$ is bounded. Suppose $\xi_\phi(P_i)$ does not converge to $\xi_\phi(P)$, then there exists a subsequence P_{i_j} of P_i such that P_{i_j} converges to P, $\xi_\phi(P_{i_j}) \to \xi_0$ but $\xi_0 \neq \xi_\phi(P)$. It follows from the continuity of ϕ that $\Phi_P(\xi)$ is continuous with respect to P and ξ. Then by $\xi_0 \in P$, we have

$$\begin{aligned}
\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) &= \Phi_P(\xi_0)\\
&> \Phi_P(\xi_\phi(P))\\
&= \lim_{j\to\infty} \Phi_{P_{i_j}}(\xi_\phi(P)),
\end{aligned}$$

which contradicts the fact that

$$\Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) \leq \Phi_{P_{i_j}}(\xi_\phi(P)).$$

Therefore, $\lim_{i\to\infty} \xi_\phi(P_i) = \xi_\phi(P)$. Thus,

$$\lim_{i\to\infty} \Phi_{P_i}(\xi_\phi(P_i)) = \Phi_P(\xi_\phi(P)).$$

□

Lemma 3.4 *Suppose $\varphi : (0,\infty) \to (0,\infty)$ is continuously differentiable, strictly increasing and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_t^\infty \frac{1}{\varphi(s)} ds$ exists for every positive t and unbounded as $t \to 0$. If the unit vectors $u_1, \ldots, u_N$ are not concentrated on a closed hemisphere and $P \in P(u_1, \ldots, u_N)$, then*

$$\sum_{k=1}^N \alpha_k \phi'\left(h(P, u_k) - \xi_\phi(P) \cdot u_k\right) u_k = 0.$$

Proof Define $f : \mathrm{Int}(P) \to \mathbb{R}^n$ by

$$f(x) = \sum_{k=1}^N \alpha_k \phi(h(P, u_k) - x \cdot u_k).$$

By conditions and Lemma 3.2,

$$f(\xi_\phi(P)) = \inf_{x\in \mathrm{Int}(P)} f(x).$$

Thus,

$$\sum_{k=1}^N \alpha_k \phi'\left(h(P, u_k) - \xi_\phi(P) \cdot u_k\right) u_{k,i} = 0, \tag{3.5}$$

for $i = 1, \ldots, n$, where $u_k = (u_{k,1}, \ldots, u_{k,n})^T$. Therefore,

$$\sum_{k=1}^N \alpha_k \phi'\left(h(P, u_k) - \xi_\phi(P) \cdot u_k\right) u_k = 0. \tag{3.6}$$

□

4 The differentiability of $\xi_\phi(P_r)$

In this section, Let δ_m^k be Kronecker delta, which means if $k = m$, then $\delta_m^k = 1$, otherwise, $\delta_m^k = 0$. We want to prove that P has exactly N faces. If $P \in \mathcal{P}_{\mathcal{N}}(u_1, \ldots, u_N)$, then the differentiability of $\xi_\phi(P_r)$ is easy. See the following lemma.

Lemma 4.1 *Suppose $\varphi : (0,\infty) \to (0,\infty)$ is continuously differentiable, strictly increasing and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_t^\infty \frac{1}{\varphi(s)} ds$ exists for every positive t and unbounded as $t \to 0$. If $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere, $P \in \mathcal{P}_N(u_1, \ldots, u_N)$ and $|r|$ small enough such that*

$$P_r = \bigcap_{k=1}^N \{x : x \cdot u_k \leq h(P, u_k) - r\delta_m^k\} \in \mathcal{P}_N(u_1, \ldots, u_N),$$

where $m \in \{1, 2, \ldots, N\}$. Then there exists a number $r_0 > 0$ such that $\xi_\phi(P_r)$ is continuously differentiable with respect to r in $(-r_0, r_0)$.

Proof Let $\xi(r) = \xi_\phi(P_r)$ and

$$\begin{aligned}\Phi(r) &= \min_{\xi \in P_r} \sum_{k=1}^{N} \alpha_k \phi\left(h(P_r, u_k) - \xi \cdot u_k\right) \\ &= \sum_{k=1}^{N} \alpha_k \phi\left(h(P_r, u_k) - \xi(r) \cdot u_k\right).\end{aligned}$$

From this and the fact $\xi(r)$ is an interior point of P_r, we have

$$\sum_{k=1}^{N} \alpha_k \phi'\left(h(P_r, u_k) - \xi(r) \cdot u_k\right) u_{k,i} = 0, \tag{4.1}$$

for $i = 1, \ldots, n$, where $u_k = (u_{k,1}, \ldots, u_{k,n})^T$.

Next, we use the inverse function theorem to prove the conclusion. Let $\xi_0 = \xi(0)$ and

$$F_i(r, \xi_1, \ldots, \xi_n) = \sum_{k=1}^{N} \alpha_k \phi'\left(h(P_r, u_k) - \xi \cdot u_k\right) u_{k,i},$$

where $i \in \{1, \ldots, n\}$ and $\xi = (\xi_1, \ldots, \xi_n)$. Since $P \in \mathcal{P}_N(u_1, \ldots, u_N)$, by Lemma 2.4.13 in [38], one can choose $r_0 > 0$, such that P_r has exactly N facets for $|r| < r_0$, which implies $h(P_r, u_k) = h(P, u_k) - r\delta_m^k$. Then,

$$\begin{aligned}\frac{\partial F_i}{\partial r} &= -\alpha_m \phi''\left(h(P, u_m) - r - \xi \cdot u_m\right) u_{m,i} \text{ and} \\ \frac{\partial F_i}{\partial \xi_j} &= -\sum_{k=1}^{N} \alpha_k \phi''\left(h(P_r, u_k) - \xi \cdot u_k\right) u_{k,i} u_{k,j}\end{aligned}$$

are obviously continuous.

Let $r = 0$, then, the Jacobian matrix of $F := (F_1, \ldots, F_N)$ at ξ_0 equals

$$\left(\left.\frac{\partial F}{\partial \xi_j}\right|_{\xi_0}\right)_{n \times n} = -\sum_{k=1}^{N} \alpha_k \phi''\left(h(P, u_k) - \xi_0 \cdot u_k\right) u_k \cdot u_k^T,$$

where $u_k u_k^T$ is an $n \times n$ matrix.

Since $u_1, \ldots, u_N$ are not contained in any closed hemisphere, $\mathbb{R}^n = \text{Span}\{u_1, \ldots, u_N\}$. Thus, for any $x \in \mathbb{R}^n$ with $x \neq 0$, there exists a $u_{i_m} \in \{u_1, \ldots, u_N\}$ such that $u_{i_m} \cdot x \neq 0$. Together with the fact that ϕ is twice differentiable and strictly convex (Lemma 3.1), we have

$$\begin{aligned}& x^T \cdot \left(-\sum_{k=1}^{N} \alpha_k \phi''\left(h(P, u_k) - \xi_0 \cdot u_k\right) u_k \cdot u_k^T\right) \cdot x \\ &= -\sum_{k=1}^{N} \alpha_k \phi''\left(h(P, u_k) - \xi_0 \cdot u_k\right) (x \cdot u_k)^2 \\ &\leq -\alpha_{i_m} \phi''\left(h(P, u_{i_m}) - \xi_0 \cdot u_k\right) (x \cdot u_{i_m})^2 < 0.\end{aligned}$$

Thus, $\left(\frac{\partial F}{\partial \xi_j}\Big|_{(0,\xi_0)}\right)$ is negative definite. From this, Eq. (4.1), the inverse function theorem and the fact that F_i has continuous partial derivative for ξ and r , the conclusion follows. □

Remark 4.1 For $t > 0$, by a similar method in Lemma 4.1, we have $\xi_\phi(tP)$ is continuously differentiable in a small neighborhood of t. Thus, $\xi_\phi(tP)$ is continuous for every $t > 0$. Therefore, $\Phi_{tP}(\xi_\phi(tP))$ is continuous for $t > 0$.

In order to prove that every polytope which solves (3.2) has exactly N faces, we need one-sided differentiability of $\xi_\phi(P_r)$ for $P \in \mathcal{P}(u_1, \ldots, u_N)$. The following lemma is needed.

Lemma 4.2 [42, Lemma 4.6] *Suppose the unit vectors $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere. Let $P \in \mathcal{P}(u_1, \ldots, u_N)$ and*

$$P_r = \bigcap_{k=1}^{N} \{x : x \cdot u_k \le h(P, u_k) - r\delta_m^k\},$$

where $m \in \{1, 2, \ldots, N\}$. Then there exists a number $r_0 > 0$ such that for $0 \le r \le r_0$,

$$h(P_r, u_k) = \begin{cases} h(P, u_k) - r, & if\ k = m \\ h(P, u_k) - a_k r, & if\ k \ne m \end{cases},$$

where a_k is a constant with $a_k \ge 0$.

Now, we aim to prove that $\xi_\phi(P_r)$ has one-sided derivative at 0 for $P \in \mathcal{P}(u_1, \ldots, u_N)$.

Lemma 4.3 *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_t^\infty \frac{1}{\varphi(s)} ds$ exists for every positive t and unbounded as $t \to 0$. Assume that $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere, $P \in \mathcal{P}(u_1, \ldots, u_N)$ and $r \ge 0$ small enough such that*

$$P_r = \bigcap_{k=1}^{N} \{x : x \cdot u_k \le h(P, u_k) - r\delta_m^k\} \in \mathcal{P}(u_1, \ldots, u_N),$$

where $m \in \{1, 2, \ldots, N\}$. If the continuous function $\lambda : [0, \infty) \to (0, \infty)$ is continuously differentiable on $(0, \infty)$ and $\lim_{r \to 0} \lambda'(r)$ exists, then $\xi_\phi(\lambda(r)P_r)$ has right derivative at 0.

Proof Let $F = (F_1, \ldots, F_n)$ and

$$F_i(r, \xi_1, \ldots, \xi_n) = \sum_{k=1}^{N} \alpha_k \phi' \left(h(\lambda(r)P_r, u_k) - \xi \cdot u_k\right) u_{k,i}, \tag{4.2}$$

where $i \in \{1, \ldots, n\}$ and $\xi = (\xi_1, \ldots, \xi_n)$. Since $P \in \mathcal{P}(u_1, \ldots, u_N)$, by Lemma 4.2, for small enough $r \ge 0$, we have

$$h(\lambda(r)P_r, u_k) = \begin{cases} \lambda(r)h(P, u_k) - \lambda(r)r, & \text{if } k = m \\ \lambda(r)h(P, u_k) - a_k\lambda(r)r, & \text{if } k \ne m \end{cases}, \tag{4.3}$$

where a_k is a constant with $a_k \ge 0$.

By a similar method in Lemma 4.1 and the inverse function theorem, $\xi(r) := \xi_\phi(\lambda(r)P_r)$ is continuously differentiable for every $r > 0$ and

$$\begin{pmatrix} \frac{d\xi_1}{dr} \\ \frac{d\xi_2}{dr} \\ \vdots \\ \frac{d\xi_n}{dr} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial \xi_1} & \frac{\partial F_1}{\partial \xi_2} & \cdots & \frac{\partial F_1}{\partial \xi_n} \\ \frac{\partial F_2}{\partial \xi_1} & \frac{\partial F_2}{\partial \xi_2} & \cdots & \frac{\partial F_2}{\partial \xi_n} \\ & & \vdots & \\ \frac{\partial F_n}{\partial \xi_1} & \frac{\partial F_n}{\partial \xi_2} & \cdots & \frac{\partial F_n}{\partial \xi_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial r} \\ \frac{\partial F_2}{\partial r} \\ \vdots \\ \frac{\partial F_n}{\partial r} \end{pmatrix}.$$

Letting $a_m = 1$, then by (4.2) and (4.3), we have

$$\frac{\partial F_i}{\partial \xi_j} = -\sum_{k=1}^{N} \alpha_k \phi''\left(h(\lambda(r)P_r, u_k) - \xi \cdot u_k\right) u_{k,i} u_{k,j} \text{ and}$$

$$\frac{\partial F_i}{\partial r} = \sum_{k=1}^{N} \alpha_k \phi''\left(\lambda(r)h(P, u_k) - a_k\lambda(r)r - \xi \cdot u_k\right) \cdot \left(\lambda'(r)h(P, u_k) - a_k\lambda'(r)r - a_k\lambda(r)\right) u_{k,i},$$

By a similar proof in Lemma 4.1, the matrix $(\frac{\partial F_i}{\partial \xi_j})$ is negative definite. Thus, $\lim_{r\to 0+} \xi'(r)$ exists.

It follows from the Lagrange mean value theorem that for every $r > 0$ and $1 \le i \le n$, there exists a $\varepsilon_i(r)$ with $0 < \varepsilon_i(r) < r$ such that

$$\frac{\xi_i(r) - \xi_i(0)}{r} = \xi_i'(\varepsilon_i(r)).$$

Let $r \to 0+$, then the conclusion follows. □

5 The Orlicz Minkowski problem for polytopes

This section is devoted to the proof of our main theorem by using calculus of variations. First, we need to prove that there exists a polytope with $u_1, \dots, u_N$ as its outer unit normals and this polytope is a solution of problem (3.2). Before this, we need the following lemmas.

Lemma 5.1 [49, Lemma 3.5] *If P is a polytope in $\mathbb{R}^n$ and $v_0 \in S^{n-1}$ with $V_{n-1}(F(P, v_0)) = 0$, then there exists a $\delta_0 > 0$ such that for $0 \le \delta < \delta_0$,*

$$V(P \cap \{x : x \cdot v_0 \ge h(P, v_0) - \delta\}) = c_n\delta^n + \cdots + c_2\delta^2,$$

where $c_n, \dots, c_2$ are constants that depend on P and v_0.

Lemma 5.2 [48, Lemma 4.2] *Suppose the unit vectors $u_1, \dots, u_N$ are not concentrated on a closed hemisphere, and for any subspace, X, of $\mathbb{R}^n$ with $1 \le dimX \le n-1$, $\{u_1, \dots, u_N\} \cap X$ is concentrated on a closed hemisphere of $S^{n-1} \cap X$. If P_m is a sequence of polytopes with $P_m \in P(u_1, \dots, u_N)$ and $V(Q) = 1$, then P_m is bounded.*

Next, we prove the existence of a solution in (3.2).

Lemma 5.3 *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_t^\infty \frac{1}{\varphi(s)} ds$ exists for every positive t and unbounded as $t \to 0$. If $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \dots, u_N$ are not concentrated* *on any closed hemisphere and for any subspace X with $1 \leq dimX \leq n-1$, $\{u_1, \dots, u_N\} \cap X$ is always concentrated on a closed hemisphere of $S^{n-1} \cap X$, then there exists a $P \in \mathcal{P}_N(u_1, \dots, u_N)$ such that $\xi_\phi(P) = o$, $V(P) = 1$ and*

$$\Phi_P(o) = \sup\left\{\inf_{\xi \in \mathrm{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1\right\}.$$

Proof Note that, for $P, Q \in \mathcal{P}(u_1, \dots, u_N)$, if Q is a translate of P, then

$$\Phi_P(\xi_\phi(P)) = \Phi_Q(\xi_\phi(Q)).$$

Thus, we can choose a sequence $P_i \in \mathcal{P}(u_1, \dots, u_N)$ with $\xi_\phi(P_i) = o$ and $V(P_i) = 1$ such that $\Phi_{P_i}(o)$ converges to

$$\sup\left\{\inf_{\xi \in Int(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1\right\}.$$

By the conditions of this Lemma 5.2, P_i is bounded. From Lemma 3.3 and the Blaschke selection theorem, there exists a subsequence of P_i that converges to a polytope P such that $P \in P(u_1, \dots, u_N)$, $\xi_\phi(P) = o$, $V(P) = 1$ and

$$\Phi_P(o) = \sup\left\{\inf_{\xi \in Int(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1\right\}. \tag{5.1}$$

We next prove that $F(P, u_i)$ are facets for all $i = 1, \dots, N$. Otherwise, there exists a $i_0 \in \{1, \dots, N\}$ such that $F(P, u_{i_0})$ is not a facet of P.

Choose $\delta \geq 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, \dots, u_N)$$

and (by Lemma 5.1)

$$V(P_\delta) = V(P) - (c_n\delta^n + \cdots + c_2\delta^2),$$

where $c_n, \dots, c_2$ are constants that depend on P and direction u_{i_0}. By Lemma 4.2, we can assume $\delta \geq 0$ is small enough so that

$$h(P_\delta, u_k) = h(P, u_k) - a_k\delta, \tag{5.2}$$

where a_k is a constant with $a_k \geq 0$ and $a_{i_0} = 1$.

From Lemma 3.3, for any $\delta_i \to 0$, it is always true that $\xi_\phi(P_{\delta_i}) \to o$. We have

$$\lim_{\delta \to 0} \xi_\phi(P_\delta) = o.$$

Let

$$\lambda(\delta) = \left(\frac{V(P_\delta)}{V(P)}\right)^{-\frac{1}{n}} = \left(1 - \frac{(c_n\delta^n + \cdots + c_2\delta^2)}{V(P)}\right)^{-\frac{1}{n}}.$$

then we have $V(\lambda(\delta)P_\delta) = V(P)$ and $\lambda'(0) = 0$.

Let $\xi(\delta) = \xi_\phi(\lambda(\delta)P_\delta)$ and

$$\begin{aligned}\Phi(\delta) &= \inf_{\xi \in \mathrm{Int}(\lambda(\delta)P_\delta)} \sum_{k=1}^N \alpha_k \phi\left(h(\lambda(\delta)P_\delta, u_k) - \xi \cdot u_k\right) \\ &= \sum_{k=1}^N \alpha_k \phi\left(h(\lambda(\delta)P_\delta, u_k) - \xi(\delta) \cdot u_k\right).\end{aligned} \tag{5.3}$$

From this and the fact $\xi(\delta)$ is an interior point of $\lambda(\delta)P_\delta$, we get

$$\sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k))\, u_k = 0. \tag{5.4}$$

It follows from Lemma 4.3 that $\xi_\phi(\lambda(\delta)P_\delta)$ has right derivative at 0. Together with (5.2), (5.3), (5.4), $\lambda'(0) = 0$ and the definition of ϕ, we have the right derivative

$$\begin{aligned}
\frac{d}{d\delta}\Big|_{\delta=0^+} \Phi(\delta) &= -\sum_{k=1}^{N} \alpha_k a_k \phi'\left(h(P, u_{i_0})\right) + \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) \left(\xi_r'(0) \cdot u_k\right) \\
&= -\sum_{k=1}^{N} \alpha_k a_k \phi'\left(h(P, u_{i_0})\right) + \xi_r'(0) \cdot \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) u_k \\
&= -\sum_{k=1}^{N} \alpha_k a_k \phi'\left(h(P, u_{i_0})\right) > 0.
\end{aligned}$$

Thus, there exists a $\delta_0 > 0$ such that $P_{\delta_0} \in \mathcal{P}(u_1, \ldots, u_N)$, $o \in P_{\delta_0}$ and

$$\Phi_{\delta_0 P_{\delta_0}}(\xi_\phi(\lambda_0 P_{\delta_0})) > \Phi_P(\xi_\phi(P)),$$

where $\lambda_0 = \left(\frac{V(P_{\delta_0})}{V(P)}\right)^{-\frac{1}{n}}$. Let $P_0 = \lambda_0 P_{\delta_0}$, then $P_0 \in P(u_1, \ldots, u_N)$, $o \in P_0$, $V(P_0) = V(P) = 1$, and

$$\inf_{\xi \in \mathrm{Int}(P_0)} \Phi_{P_0}(\xi) > \Phi_P(\xi_\phi(P)),$$

which contradicts Eq. (5.1). Therefore, $P \in P_N(u_1, \ldots, u_N)$. $\square$

Now, we turn to prove Theorem 1.2.

Proof By Lemma 5.3, there exists a polytope $P \in \mathcal{P}_N(u_1, \ldots, u_N)$ such that $\xi_\phi(P) = o$, $V(P) = 1$ and

$$\Phi_P(o) = \sup\left\{\inf_{\xi \in Int(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1\right\}.$$

For $m \in \{1, \ldots, N\}$, choose $|t|$ small enough so that the polytope P_t defined by

$$P_t = \bigcap_{i=1}^{N} \{x : x \cdot u_i \le h(P, u_i) + t\delta_m^i\}$$

has exactly N facets. By [38, Lemma 7.5.3],

$$\frac{\partial V(P_t)}{dt} = S_m,$$

where S_m is the area of $F(P, u_m)$, i.e., $S_m = S(P, u_m)$.

Let $\lambda(t) = (V(P_t))^{-\frac{1}{n}}$. Then we have $V(\lambda(t)P_t) = 1$, $\lambda(t)P_t \in \mathcal{P}_N(u_1, \ldots, u_N)$ and

$$\lambda'(0) = -\frac{1}{n} S_m. \tag{5.5}$$

Let $\xi(t) = \xi_\phi(\lambda(t)P_t)$ and

$$\begin{aligned}\Phi(t) &= \inf_{\xi \in \mathrm{Int}(\lambda(t)P_t)} \sum_{k=1}^{N} \alpha_k \phi\left(h(\lambda(t)P_t, u_k) - \xi \cdot u_k\right) \\ &= \sum_{k=1}^{N} \alpha_k \phi\left(h(\lambda(t)P_t, u_k) - \xi(t) \cdot u_k\right).\end{aligned}$$

By Lemma 3.4 and $\xi_\phi(P) = o$,

$$\sum_{k=1}^{N} \alpha_k \phi'\left(h(P, u_k)\right) u_k = 0. \tag{5.6}$$

From the fact that $\Phi(0)$ is an extreme value of $\Phi(t)$, Lemma 4.1, (5.5) and (5.6), we have

$$\begin{aligned}0 =& \frac{d}{dt}\Big|_{t=0} \Phi(t) \\ =& -\sum_{k=1}^{N} \alpha_k \phi'\left(h(P, u_k)\right)\left[\lambda'(0)h(P, u_k) + \delta_m^k\right] + \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k))\left(\xi'(0) \cdot u_k\right) \\ =& -\sum_{k=1}^{N} \alpha_k \phi'\left(h(P, u_k)\right))\left[-\frac{S_m}{n} h(P, u_k) + \delta_m^k\right] + \xi'(0) \cdot \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k))u_k \\ =& \frac{S_m}{n} \sum_{k=1}^{N} \alpha_k \phi'\left(h(P, u_k)\right) h(P, u_k) - \alpha_m \phi'\left(h(P, u_m)\right).\end{aligned}$$

Together with (3.3), we have

$$\alpha_m = c\varphi(h(P, u_m))S_m = c\varphi(h(P, u_m))S(P, u_m), \tag{5.7}$$

where $c = \frac{1}{n} \sum\limits_{k=1}^{N} \alpha_k \frac{h(P, u_k)}{\varphi(h(P, u_k))}$.

The conclusion follows, since (5.7) holds for every $m \in \{1, \ldots, N\}$. □

Corollary 1.3 follows from this theorem and (1.1).

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