Bull. London Math. Soc. 46 (2014) 367–378
doi:10.1112/blms/bdt095

Affine inequalities for L_p mean zonoids

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Abstract

In this paper, the $L_p (p \geqslant 1)$ mean zonoid of a convex body K is given, and we show that it is the L_p centroid body of radial $(n+p)$th mean body of K up to a dilation. We also establish some affine inequalities of these bodies by proving that the volume of the new bodies is decreasing under Steiner symmetrization.

1. *Introduction*

The notion of zonoids is basic in the Brunn–Minkowski theory of convex bodies and appears in different contexts of the mathematical literature (see, for example, [2, 5, 9, 10]). Zonoids are defined as limits of zonotopes in the Hausdorff metric, where zonotopes are Minkowski sum of segments. A zonoid Z can also be defined as a convex body whose support function is

$$h_Z(u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \, d\mu(v) \quad \text{for all } u \in S^{n-1},$$

where μ is an even measure on the unit sphere S^{n-1}.

Let $K \subset \mathbb{R}^n$ be a convex body (compact, convex set with non-empty interior). As introduced by Zhang [11], a mean zonoid $\tilde{Z}K$ is defined by

$$h_{\tilde{Z}K}(u) = \frac{1}{V(K)^2}\int_K \int_K |u \cdot (x-y)| \, dx \, dy \quad \text{for all } u \in S^{n-1}, \tag{1.1}$$

where $V(K)$ denotes the volume of the body K.

The body $\tilde{Z}K$ is indeed a zonoid (limits of Minkowski sums of line segments). It was also shown by Zhang [11] that the volume of $V(\tilde{Z}K)$ satisfies

$$V(\tilde{Z}K) \geqslant V(\tilde{Z}B_K),$$

where B_K is the n-ball with the same volume as K. The equality holds if and only if K is an ellipsoid.

Schneider and Weil [10] introduced the notion of L_p zonoids. A finite-dimensional real normed space is isometric to a subspace of L_p if and only if the polar of its unit ball is an L_p zonoid. Specially, L_2 zonoids are ellipsoids in $\mathbb{R}^n$. For $p \geqslant 1$, an L_p zonoid can be defined by

$$h_{Z_p}(u)^p = \int_{S^{n-1}} |u \cdot v|^p \, d\mu(v) \quad \text{for all } u \in S^{n-1}, \tag{1.2}$$

where μ is a finite even Borel measure on the unit sphere S^{n-1}. We refer the reader to [4, 5] for the study on this subject.

It is natural for us to consider a class of bodies $\tilde{Z}_pK$ named L_p mean zonoids.

Received 12 December 2012; revised 24 October 2013; published online 3 February 2014.

2010 *Mathematics Subject Classification* 52A20 (primary), 52A40 (secondary).

The authors would like to acknowledge the support from the National Natural Science Foundation of China (10971128), the National Natural Science Foundation of China (11271244) and Shanghai Leading Academic Discipline Project (S30104).

Definition. Let $K \subset \mathbb{R}^n$ be a convex body, and let $p \geqslant 1$. Then, the L_p mean zonoid $\tilde{Z}_p K$ of K is defined by

$$h_{\tilde{Z}_p K}(z) = \left(\frac{1}{V(K)^2} \int_K \int_K |z \cdot (x-y)|^p \, dx \, dy \right)^{1/p} \quad \text{for all } z \in \mathbb{R}^n \backslash \{0\}. \tag{1.3}$$

The case $p = 1$ is just the $\tilde{Z}K$ defined by Zhang [**11**]. We will show that $\tilde{Z}_p K$ is an L_p zonoid in Section 2.

In their paper, Lutwak and Zhang [**7**] introduced the L_p centroid body $\Gamma_p K$, $p \geqslant 1$, with $\Gamma_1 K = \Gamma K$. It was also shown that the volume of the polar of $\Gamma_p K$ is maximized if the volume of K is given. This gives an L_p version of the Blaschke–Santaló inequality. In [**4**], it was proved that the volume of $\Gamma_p K$ is minimized if the volume of K is given. These results have found applications in asymptotic functional analysis.

Similar to the centroid body ΓK, the L_p centroid body $\Gamma_p K$ is origin dependent. In this paper, the L_p mean zonoid $\tilde{Z}_p K$, which is defined by (1.3), is a translation invariant analogue of the L_p centroid body. Further relationship between the L_p centroid body and the L_p mean zonoid can be seen in Section 2.

Throughout this paper, we assume that

$$C_{n,p} = \omega_n \left[\frac{2^{n+p} \omega_{n+p} \omega_{2n+p}}{\omega_2^2 \omega_{p-1} \omega_{n+p-1}} \right]^{n/p},$$

where $\omega_p = \pi^{p/2} / \Gamma(1 + p/2)$.

The main result of this paper is the following theorem. The result of the case $p = 1$, in our theorem, is first proved in [**11**].

Theorem 1. *Let $K \subset \mathbb{R}^n$ be a convex body, and let $p \geqslant 1$. Then, the volumes of $\tilde{Z}_p K$ and K satisfy the following inequality*:

$$V(\tilde{Z}_p K) \geqslant C_{n,p} V(K), \tag{1.4}$$

with equality if and only if K is an ellipsoid.

This paper is organized as follows. In Section 2, we observe that $\tilde{Z}_p K$ is a dilation of $\Gamma_p(R_{n+p} K)$, where $\Gamma_p(\cdot)$ is the L_p centroid operation, and $R_{n+p} K$ is the radial $(n+p)$th mean body of K. Section 3 contains the proofs of our main results. Note that the volume ratio $V(\Gamma_p L)/V(L)$ takes the minimum if and only if L is an ellipsoid (see [**1**]), while the volume ratio $V(R_{n+p} K)/V(K)$ takes the minimum if and only if K is a simplex (see [**3**, p. 522]); we found that Theorem 1 cannot be obtained from the results above in [**1**, **3**]. However, it seems that we could not get a proof using the same method as in [**11**]. Inspired by the work of Lutwak, Yang and Zhang [**6**], we prove our main theorem by utilizing the Steiner symmetrization. In Section 4, two inclusion relationships are established: one of them is between $\tilde{Z}_p K$ and $\Gamma_p \Pi^* K$, the other is between $\tilde{Z}_p K$ and $\tilde{Z}_p(\Delta K)$, where $\Pi^* K$ is the polar projection body of K, and $\Delta K = (K - K)/2$. The results of the case $p = 1$ are first obtained in [**11**].

2. *Preliminary*

2.1. *Definitions and notation*

Let $\mathbb{R}^n$ denote the Euclidean n-dimensional space. Let S^{n-1} denote the unit sphere, B^n the unit n-ball and o the origin in $\mathbb{R}^n$. Denote by $\mathcal{K}^n$ the class of convex bodies (compact, convex sets with non-empty interiors) in $\mathbb{R}^n$, let $\mathcal{K}_o^n$ be the class of members of $\mathcal{K}^n$ containing the origin in their interiors, and let $\mathcal{K}_s^n$ be the class of o-symmetric members of $\mathcal{K}^n$. If $u \in S^{n-1}$, then we denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u, by l_u the line through o parallel to u and by $l_u(x)$ the line through the point x parallel to u.

Lebesgue k-dimensional measure V_k in $\mathbb{R}^n$, $k = 1, \ldots, n$, can be identified with k-dimensional Hausdorff measure in $\mathbb{R}^n$. We also generally write V instead of V_n. Let $\omega_n = V(B^n)$, and thus $n\omega_n = V_{n-1}(S^{n-1})$. Associated with a convex body K is its support function h_K defined for all $x \in \mathbb{R}^n \backslash \{0\}$ by

$$h_K(x) := \max\{x \cdot y : y \in K\}.$$

We shall use δ to denote the Hausdorff metric on $\mathcal{K}^n$: If $K, L \in \mathcal{K}^n$, then $\delta(K, L)$ is defined by

$$\delta(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|,$$

or equivalently,

$$\delta(K, L) = \min\{\lambda : K \subseteq L + \lambda B^n \text{ and } L \subseteq K + \lambda B^n\}.$$

The projection body ΠK of a convex body K is defined in [**8**] by

$$h_{\Pi K}(u) = V_{n-1}(K|u^\perp) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \, dS_K(v),$$

for each $u \in S^{n-1}$, where $K|u^\perp$ is the orthogonal projection of K on $u^\perp$, $S_K(\cdot)$ is the surface area measure of K.

A set L is star-shaped with respect to the point x if every line passing through x crosses the boundary of L at exactly two points different from x. If L is a compact set that is star-shaped with respect to x, then its radial function $\rho_L(x, z) : \mathbb{R}^n \backslash \{x\} \to [0, \infty)$ with respect to x is defined by

$$\rho_L(x, z) = \max\{c : x + cz \in L\} \quad \text{for all } z \in \mathbb{R}^n \backslash \{x\}. \tag{2.1}$$

When x is the origin, we also denote $\rho_L(o, z)$ by $\rho_L(z)$ and refer to it simply as the radial function of L. By a star body, we mean a compact set L whose radial function is positive and continuous. Note that this implies $o \in \mathrm{int} L$.

The L_p centroid body $\Gamma_p K$ of a star body K is defined by

$$h_{\Gamma_p K}(u)^p = \frac{1}{V(K)}\int_K |u \cdot x|^p \, dx = \frac{1}{(n+p)V(K)}\int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p} \, dv, \tag{2.2}$$

for all $u \in S^{n-1}$. We denote the polar body of K by K^*. The difference body DK of K is defined by $DK = K - K$.

Let $K \in \mathcal{K}^n$, for all $r \geqslant 0$ and $u \in S^{n-1}$, define

$$E_K(r, u) = \{y \in u^\perp : |l_u(y) \cap K| \geqslant r\}$$

and

$$a_K(r, u) = V_{n-1}(E_K(r, u)).$$

In [**11**], $a_K(r, u)$ is called the restricted chord projection function of K. It is clear that $E_K(0, u) = K|u^\perp$, and $a_K(0, u) = h_{\Pi K}(u)$. When $r > \rho_{DK}(u)$, $a_K(r, u) = 0$.

The following inequality can be found in [**3**] or [**11**] that

$$V(K) \geqslant \frac{1}{n} a_K(0, u)\rho_{DK}(u), \tag{2.3}$$

with equality if and only if K is a simplex.

If $\lambda > 0$, then Zhang (see [**11**, (2.2)]) proved that

$$\int_0^\infty a_K(r,u) r^\lambda \, dr \leqslant n^{\lambda+1}\beta(\lambda+1,n)V(K)^{\lambda+1}a_K(0,u)^{-\lambda}, \tag{2.4}$$

with equality if and only if K is a simplex.

Let K be a convex body and $p > -1$. The radial pth mean body R_pK is defined in [**3**] by

$$\rho_{R_pK}(z)^p = \frac{1}{V(K)}\int_K \rho_K(x,z)^p \, dx, \tag{2.5}$$

and it has been shown that

$$\int_K \rho_K(x,u)^p \, dx = \int_0^\infty a_K(r,u) r^p \, dr = \int_0^{\rho_{DK}(u)} a_K(r,u) r^p \, dr. \tag{2.6}$$

The formula

$$V(K) = \int_0^\infty a_K(r,u) \, dr = \int_0^{\rho_{DK}(u)} a_K(r,u) \, dr \tag{2.7}$$

can be obtained from (2.6).

When $p \geqslant 0$, it has also been proved in [**3**] that R_pK is an o-symmetric convex body.

2.2. *The L_p mean zonoids*

Let K be a convex body and $p \geqslant 1$. Then the L_p mean zonoid $\tilde{Z}_pK$ of K is defined by (1.3). We can consistently define $\tilde{Z}_\infty K$ by

$$h_{\tilde{Z}_\infty K}(u) = \max_{x,y\in K} |u\cdot(x-y)| \quad \text{for all } u \in S^{n-1}.$$

In fact, $\tilde{Z}_\infty K = DK$. From Jensen's inequality, it is obvious that

$$\tilde{Z}_pK \subseteq \tilde{Z}_qK \subseteq DK \quad \text{for } 1 \leqslant p \leqslant q.$$

By (1.3), (2.1), the Fubini theorem, (2.2) and (2.5)

$$\begin{aligned} h_{\tilde{Z}_pK}(z) &= \left(\frac{1}{V(K)^2}\int_K\int_K |z\cdot(x-y)|^p \, dx \, dy\right)^{1/p} \\ &= \left(\frac{1}{V(K)^2}\int_K\int_{S^{n-1}}\int_0^{\rho_K(y,v)} |z\cdot v|^p r^{n+p-1} \, dr \, dv \, dy\right)^{1/p} \\ &= \left(\frac{1}{(n+p)V(K)^2}\int_{S^{n-1}} |z\cdot v|^p \int_K \rho_K(y,v)^{n+p} \, dy \, dv\right)^{1/p} && (2.8) \\ &= \left(\frac{1}{(n+p)V(K)}\int_{S^{n-1}} |z\cdot v|^p \rho_{R_{n+p}K}(v)^{n+p} \, dv\right)^{1/p} && (2.9) \\ &= \left(\frac{V(R_{n+p}K)}{(n+p)V(K)}\right)^{1/p} h_{\Gamma_p(R_{n+p}K)}(z). && (2.10) \end{aligned}$$

From (2.10), $h_{\tilde{Z}_pK}(z)$ is obviously a support function, and

$$\tilde{Z}_pK = \left(\frac{V(R_{n+p}K)}{(n+p)V(K)}\right)^{1/p} \Gamma_p(R_{n+p}K). \tag{2.11}$$

Since $R_{n+p}K$ is o-symmetric, $(1/(n+p)V(K))\rho_{R_{n+p}K}(v)^{n+p}$ can be seen as a density function of an even Borel measure, thus $\tilde{Z}_pK$ is an L_p zonoid from (1.2) and (2.9).

Using (2.6) and (2.8), we have the following useful formula:

$$h_{\tilde{Z}_p K}(z) = \left(\frac{1}{(n+p)V(K)^2} \int_{S^{n-1}} |z \cdot v|^p \int_0^{\rho_{DK}(u)} a_K(r,u) r^{n+p}\, dr\, dv \right)^{1/p}. \tag{2.12}$$

3. *Proof of main result*

We first give some notation and definitions about Steiner symmetrization that will be used in this section. Let K be a convex body and $u \in S^{n-1}$, denote by K_u the image of the orthogonal projection of K onto $u^\perp$. We write $\bar{\ell}_u(K; y') : K_u \to \mathbb{R}$ and $\underline{\ell}_u(K; y') : K_u \to \mathbb{R}$ for the overgraph and undergraph functions of K in the direction u; that is,

$$K = \{y' + tu : -\underline{\ell}_u(K; y') \leqslant t \leqslant \bar{\ell}_u(K; y') \text{ for } y' \in K_u\}.$$

Thus, the Steiner symmetral $S_u K$ of $K \in \mathcal{K}^n$ in direction u can be defined as the body whose orthogonal projection onto $u^\perp$ is identical to that of K and whose overgraph and undergraph functions are given by

$$\bar{\ell}_u(S_u K; y') = \underline{\ell}_u(S_u K; y') = \tfrac{1}{2}[\bar{\ell}_u(K; y') + \underline{\ell}_u(K; y')].$$

For $y' \in K_u$, define $m_{y'} = m_{y'}(u)$ by

$$m_{y'}(u) = \tfrac{1}{2}[\bar{\ell}_u(K; y') - \underline{\ell}_u(K; y')].$$

So that the midpoint of the chord $K \cap l_u(y')$ is $y' + m_{y'}(u)u$, where $l_u(y')$ is the line through y' parallel to u. The length $|K \cap l_u(y')|$ of this chord is denoted by $\sigma_{y'} = \sigma_{y'}(u)$.

Throughout this section, we denote $x = (x', s) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and we will usually write $h_K(x', s)$ rather than $h_K((x', s))$.

The following lemma will be used in the proofs of our theorems.

Lemma 3.1 [6, Lemma 1.2]. *Suppose $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$. For $y' \in \operatorname{relint} K_u$, the overgraph and undergraph functions of K in direction u are given by*

$$\bar{\ell}_u(K; y') = \min_{x' \in u^\perp} \{h_K(x', 1) - x' \cdot y'\} \tag{3.1a}$$

and

$$\underline{\ell}_u(K; y') = \min_{x' \in u^\perp} \{h_K(x', -1) - x' \cdot y'\}. \tag{3.1b}$$

We refer the reader to [6] for a proof, and [1] for an application to the proof of the L_p Busemann–Petty centroid inequality.

We next show the operation $\tilde{Z}_p : \mathcal{K}^n \to \mathcal{K}_s^n$ is continuous.

Lemma 3.2. *Suppose $p \geqslant 1$, $K_i \in \mathcal{K}^n$ and $K_i \to K \in \mathcal{K}^n$, then $\tilde{Z}_p K_i \to \tilde{Z}_p K$.*

Proof. Suppose $u_0 \in S^{n-1}$. We will show that

$$h_{\tilde{Z}_p K_i}(u_0) \to h_{\tilde{Z}_p K}(u_0).$$

Since $K_i \to K$ implies $\{K_i\}$ are uniformly bounded, there is $R > 0$, such that $K_i \subseteq RB^n$.

By (1.3) and Minkowski's inequality, we have

$$\begin{aligned}
&|h_{\tilde{Z}_p K_i}(u_0) - h_{\tilde{Z}_p K}(u_0)| \\
&= \left| \left(\frac{1}{V(K_i)^2} \int_{RB^n} \int_{RB^n} 1_{K_i}(x) 1_{K_i}(y) |u_0 \cdot (x-y)|^p \, dx \, dy \right)^{1/p} \right. \\
&\qquad \left. - \left(\frac{1}{V(K)^2} \int_{RB^n} \int_{RB^n} 1_K(x) 1_K(y) |u_0 \cdot (x-y)|^p \, dx \, dy \right)^{1/p} \right| \\
&\leqslant \left(\frac{1}{V(K_i)^2} \int_{RB^n} \int_{RB^n} |1_{K_i}(x) 1_{K_i}(y) - 1_K(x) 1_K(y)| |u_0 \cdot (x-y)|^p \, dx \, dy \right)^{1/p} \\
&\qquad + \left| \left(\left(\frac{1}{V(K_i)^2} - \frac{1}{V(K)^2} \right) \int_{RB^n} \int_{RB^n} 1_K(x) 1_K(y) |u_0 \cdot (x-y)|^p \, dx \, dy \right)^{1/p} \right|.
\end{aligned}$$

Obviously these two integrals converge to 0 when $K_i \to K$ in the Hausdorff metric, thus $h_{\tilde{Z}_p K_i}(u_0) \to h_{\tilde{Z}_p K}(u_0)$.

Since for support functions on S^{n-1} pointwise and uniform convergence are equivalent, we complete the proof. □

We show that the operator $\tilde{Z}_p : \mathcal{K}^n \to \mathcal{K}^n_s$ is $\mathrm{GL}(n)$ covariant in the following lemma.

LEMMA 3.3. *Suppose $p \geqslant 1$. For a convex body $K \in \mathcal{K}^n$, and a linear transform $\phi \in \mathrm{GL}(n)$, then*

$$\tilde{Z}_p(\phi K) = \phi(\tilde{Z}_p K).$$

Proof. By (1.3) and the substitution $x = \phi x_1,\ y = \phi y_1$, we have

$$\begin{aligned}
h_{\tilde{Z}_p(\phi K)}(z) &= \left(\frac{1}{V(\phi K)^2} \int_{\phi K} \int_{\phi K} |z \cdot (x-y)|^p \, dx \, dy \right)^{1/p} \\
&= \left(\frac{1}{V(\phi K)^2} |\phi|^2 \int_K \int_K |z \cdot (\phi x - \phi y)|^p \, dx_1 \, dy_1 \right)^{1/p} \\
&= \left(\frac{1}{V(K)^2} \int_K \int_K |\phi^t z \cdot (x_1 - y_1)|^p \, dx_1 \, dy_1 \right)^{1/p} \\
&= h_{\tilde{Z}_p K}(\phi^t z) = h_{\phi(\tilde{Z}_p K)}(z).
\end{aligned}$$

Thus, $\tilde{Z}_p(\phi K) = \phi(\tilde{Z}_p K)$. □

The following lemma plays a key role in the proof of Theorem 1.

LEMMA 3.4. *Let $K \in \mathcal{K}^n, p \geqslant 1$ and $u \in S^{n-1}$. If $z_1', z_2' \in u^\perp$, then*

$$h_{\tilde{Z}_p(S_u K)}\left(\frac{z_1' + z_2'}{2}, 1 \right) \leqslant \frac{1}{2} h_{\tilde{Z}_p K}(z_1', 1) + \frac{1}{2} h_{\tilde{Z}_p K}(z_2', -1) \tag{3.2a}$$

and

$$h_{\tilde{Z}_p(S_u K)}\left(\frac{z_1' + z_2'}{2}, -1 \right) \leqslant \frac{1}{2} h_{\tilde{Z}_p K}(z_1', 1) + \frac{1}{2} h_{\tilde{Z}_p K}(z_2', -1). \tag{3.2b}$$

Equality in (3.2a) or (3.2b) implies that all of the chords of K parallel to u have midpoints that lie in a hyperplane.

Proof. From the definition of L_p mean zonoid, we have

$$
\begin{aligned}
&h_{\tilde{Z}_pK}(z_1',1)\\
&= \left(\frac{1}{V(K)^2}\int_K\int_K |(z_1',1)\cdot(x-y)|^p\,dx\,dy\right)^{1/p}\\
&= \left(\frac{1}{V(K)^2}\int_{K_u}\int_{m_{y'}-1/2\sigma_{y'}}^{m_{y'}+1/2\sigma_{y'}}\int_{K_u}\int_{m_{x'}-1/2\sigma_{x'}}^{m_{x'}+1/2\sigma_{x'}} |(z_1',1)\cdot((x',s_1)-(y',s_2))|^p\,ds_1\,dx'\,ds_2\,dy'\right)^{1/p}\\
&= \left(\frac{1}{V(K)^2}\int_{K_u}\int_{m_{y'}-1/2\sigma_{y'}}^{m_{y'}+1/2\sigma_{y'}}\int_{K_u}\int_{m_{x'}-1/2\sigma_{x'}}^{m_{x'}+1/2\sigma_{x'}} |z_1'\cdot(x'-y')+s_1-s_2|^p\,ds_1\,dx'\,ds_2\,dy'\right)^{1/p}\\
&= \left(\frac{1}{V(K)^2}\int_{K_u}\int_{-1/2\sigma_{y'}}^{1/2\sigma_{y'}}\int_{K_u}\int_{-1/2\sigma_{x'}}^{1/2\sigma_{x'}} |z_1'\cdot(x'-y')+t_1-t_2+m_{x'}-m_{y'}|^p\,dt_1\,dx'\,dt_2\,dy'\right)^{1/p}\\
&= \left(\frac{1}{V(S_uK)^2}\int_{S_uK}\int_{S_uK} |z_1'\cdot(x'-y')+t_1-t_2+m_{x'}-m_{y'}|^p\,dt_1\,dx'\,dt_2\,dy'\right)^{1/p},
\end{aligned}
$$

by making the change of variables $t_1=-m_{x'}+s_1$, $t_2=-m_{y'}+s_2$, and

$$
\begin{aligned}
&h_{\tilde{Z}_pK}(z_2',-1)\\
&= \left(\frac{1}{V(K)^2}\int_K\int_K |(z_2',-1)\cdot(x-y)|^p\,dx\,dy\right)^{1/p}\\
&= \left(\frac{1}{V(K)^2}\int_{K_u}\int_{m_{y'}-1/2\sigma_{y'}}^{m_{y'}+1/2\sigma_{y'}}\int_{K_u}\int_{m_{x'}-1/2\sigma_{x'}}^{m_{x'}+1/2\sigma_{x'}} |(z_2',-1)\cdot((x',s_1)-(y',s_2))|^p\,ds_1\,dx'\,ds_2\,dy'\right)^{1/p}\\
&= \left(\frac{1}{V(K)^2}\int_{K_u}\int_{m_{y'}-1/2\sigma_{y'}}^{m_{y'}+1/2\sigma_{y'}}\int_{K_u}\int_{m_{x'}-1/2\sigma_{x'}}^{m_{x'}+1/2\sigma_{x'}} |z_2'\cdot(x'-y')-s_1+s_2|^p\,ds_1\,dx'\,ds_2\,dy'\right)^{1/p}\\
&= \left(\frac{1}{V(K)^2}\int_{K_u}\int_{-1/2\sigma_{y'}}^{(1/2)\sigma_{y'}}\int_{K_u}\int_{-1/2\sigma_{x'}}^{1/2\sigma_{x'}} |z_2'\cdot(x'-y')+t_1-t_2-m_{x'}+m_{y'}|^p\,dt_1\,dx'\,dt_2\,dy'\right)^{1/p}\\
&= \left(\frac{1}{V(S_uK)^2}\int_{S_uK}\int_{S_uK} |z_2'\cdot(x'-y')+t_1-t_2-m_{x'}+m_{y'}|^p\,dt_1\,dx'\,dt_2\,dy'\right)^{1/p},
\end{aligned}
$$

by making the change of variables $t_1=m_{x'}-s_1$, $t_2=m_{y'}-s_2$.

Then, by Minkowski's inequality, we have

$$
\begin{aligned}
&2h_{\tilde{Z}_p(S_uK)}\left(\frac{z_1'+z_2'}{2},1\right)\\
&\quad= 2\left(\frac{1}{V(S_uK)^2}\int_{S_uK}\int_{S_uK}\left|\left(\frac{z_1'+z_2'}{2},1\right)\cdot(x-y)\right|^p\,dx\,dy\right)^{1/p}\\
&\quad= \left(\frac{1}{V(S_uK)^2}\int_{S_uK}\int_{S_uK}|(z_1'+z_2')\cdot(x'-y')+2t_1-2t_2|^p\,dt_1\,dx'\,dt_2\,dy'\right)^{1/p}\\
&\quad\leqslant \left(\frac{1}{V(S_uK)^2}\int_{S_uK}\int_{S_uK}|z_1'\cdot(x'-y')+t_1-t_2+m_{x'}-m_{y'}|^p\,dt_1\,dx'\,dt_2\,dy'\right)^{1/p}\\
&\qquad+ \left(\frac{1}{V(S_uK)^2}\int_{S_uK}\int_{S_uK}|z_2'\cdot(x'-y')+t_1-t_2-m_{x'}+m_{y'}|^p\,dt_1\,dx'\,dt_2\,dy'\right)^{1/p}\\
&\quad= h_{\tilde{Z}_pK}(z_1',1)+h_{\tilde{Z}_pK}(z_2',-1).
\end{aligned}
$$

Thus, we established (3.2a), and (3.2b) can be obtained by the same way.

Since the inequality is obtained by Minkowski's inequality, it takes equality in (3.2a) or (3.2b) if and only if there is $\lambda \geqslant 0$, such that

$$z_1' \cdot (x'-y') + t_1 - t_2 + m_{x'} - m_{y'} = \lambda(z_2' \cdot (x'-y') + t_1 - t_2 - m_{x'} + m_{y'}),$$

for all $(x', t_1), (y', t_2) \in K$. This is equivalent to

$$(z_1' - \lambda z_2') \cdot (x'-y') + (1+\lambda)(m_{x'} - m_{y'}) = (\lambda - 1)(t_1 - t_2), \tag{3.3}$$

for all $(x', t_1), (y', t_2) \in K$.

If we fix x', y' and change t_1, t_2 in (3.3) such that $(x', t_1), (y', t_2) \in K$, then the left of (3.3) will not change; this implies that $\lambda = 1$. Thus, equality in (3.2a) or (3.2b) implies all of the chords of K parallel to u have midpoints that lie in a hyperplane. $\square$

The theorems will be proved using the following lemma.

Lemma 3.5. *Let $K \in \mathcal{K}^n, p \geqslant 1$ and $u \in S^{n-1}$, then*

$$\tilde{Z}_p(S_u K) \subseteq S_u(\tilde{Z}_p K). \tag{3.4}$$

If the inclusion is an identity, then all of the chords of K parallel to u have midpoints that lie in a hyperplane.

Proof. Suppose $y' \in \operatorname{relint}(\tilde{Z}_p K)_u$. By Lemma 3.1, there exist $z_1' = z_1'(y')$ and $z_2' = z_2'(y')$ in $u^{\perp}$ such that

$$\begin{aligned}\overline{\ell}_u(\tilde{Z}_p K; y') &= h_{\tilde{Z}_p K}(z_1', 1) - z_1' \cdot y', \\ \underline{\ell}_u(\tilde{Z}_p K; y') &= h_{\tilde{Z}_p K}(z_2', -1) - z_2' \cdot y'.\end{aligned}$$

By (3.1) and (3.2), we have

$$\begin{aligned}\overline{\ell}_u(S_u(\tilde{Z}_p K); y') &= \frac{1}{2}\overline{\ell}_u(\tilde{Z}_p K; y') + \frac{1}{2}\underline{\ell}_u(\tilde{Z}_p K; y') \\ &= \frac{1}{2}(h_{\tilde{Z}_p K}(z_1', 1) - z_1' \cdot y') + \frac{1}{2}(h_{\tilde{Z}_p K}(z_2', -1) - z_2' \cdot y') \\ &= \frac{1}{2}h_{\tilde{Z}_p K}(z_1', 1) + \frac{1}{2}h_{\tilde{Z}_p K}(z_2', -1) - \left(\frac{1}{2}z_1' + \frac{1}{2}z_2'\right) \cdot y' \\ &\geqslant h_{\tilde{Z}_p(S_u K)}\left(\frac{z_1' + z_2'}{2}, 1\right) - \left(\frac{1}{2}z_1' + \frac{1}{2}z_2'\right) \cdot y' \\ &\geqslant \min_{x' \in u^{\perp}} \{h_{\tilde{Z}_p(S_u K)}(x', 1) - x' \cdot y'\} \\ &= \overline{\ell}_u(\tilde{Z}_p(S_u K); y'),\end{aligned}$$

and

$$\begin{aligned}\underline{\ell}_u(S_u(\tilde{Z}_p K); y') &= \frac{1}{2}\overline{\ell}_u(\tilde{Z}_p K; y') + \frac{1}{2}\underline{\ell}_u(\tilde{Z}_p K; y') \\ &= \frac{1}{2}(h_{\tilde{Z}_p K}(z_1', 1) - z_1' \cdot y') + \frac{1}{2}(h_{\tilde{Z}_p K}(z_2', -1) - z_2' \cdot y')\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} h_{\tilde{Z}_p K}(z_1', 1) + \frac{1}{2} h_{\tilde{Z}_p K}(z_2', -1) - \left(\frac{1}{2} z_1' + \frac{1}{2} z_2'\right) \cdot y' \\
&\geqslant h_{\tilde{Z}_p(S_u K)}\left(\frac{z_1' + z_2'}{2}, -1\right) - \left(\frac{1}{2} z_1' + \frac{1}{2} z_2'\right) \cdot y' \\
&\geqslant \min_{x' \in u^{\perp}} \{h_{\tilde{Z}_p(S_u K)}(x', -1) - x' \cdot y'\} \\
&= \underline{\ell}_u(\tilde{Z}_p(S_u K); y').
\end{aligned}
$$

If the inclusion is an identity, then it must take equality in both (3.2a) and (3.2b); this implies all of the chords of K parallel to u have midpoints that lie in a hyperplane. $\square$

Proof of Theorem 1. Choose a sequence of directions $\{u_i\}$ such that the sequence $\{K_i\}$ defined by

$$K_{i+1} = S_{u_i} K_i, \quad K_0 = K$$

converges to B_K, where B_K is the n-ball such that $V(K) = V(B_K)$.

Since the Steiner transform keeps the volume, by Lemmas 3.2 and 3.5 we have

$$V(\tilde{Z}_p K) \geqslant V(\tilde{Z}_p B_K).$$

If K is an ellipsoid, then $V(\tilde{Z}_p K) = V(\tilde{Z}_p B_K)$ according to Lemma 3.3.

Conversely, if $V(\tilde{Z}_p K) = V(\tilde{Z}_p B_K)$, then the inclusion in (3.4) must be identity for all $u \in S^{n-1}$. This shows that all of the chords of K parallel to u have midpoints that lie in a hyperplane for all $u \in S^{n-1}$, and thus K is an ellipsoid. $\square$

Thus, we have

$$V(\tilde{Z}_p K) \geqslant V(\tilde{Z}_p B_K), \tag{3.5}$$

where B_K is the n-ball with the same volume as K, with equality if and only if K is an ellipsoid.

We claim that

$$
\begin{aligned}
\left[\frac{V(\tilde{Z}_p B_K)}{\omega_n}\right]^{1/n} &= \left(\frac{1}{V(B_K)^2} \int_{B_K} \int_{B_K} |u \cdot (x - y)|^p \, dx \, dy\right)^{1/p} \\
&= \left(\frac{(n+p)\omega_{n+p}}{n \omega_2 \omega_{p-1} \omega_n V(K)^2} \int_{B_K} \int_{B_K} |x - y|^p \, dx \, dy\right)^{1/p}.
\end{aligned}
$$

Since $h_{\tilde{Z}_p B_K}(u)$ is a constant independent of u, it is clear that $\tilde{Z}_p B_K$ is an n-ball, then we obtain the first equality. The second equality is obtained by integral

$$\int_{S^{n-1}} |u \cdot (x - y)|^p \, du = \frac{(n+p)\omega_{n+p}}{\omega_2 \omega_{p-1}} |x - y|^p.$$

By using the spherical polar coordinates, (2.1), the Fubini theorem, and (2.6),

$$
\begin{aligned}
\int_{B_K} \int_{B_K} |(x - y)|^p \, dx \, dy &= \int_{B_K} \int_{S^{n-1}} \int_0^{\rho_{B_K}(y,u)} r^{n+p-1} \, dr \, du \, dy \\
&= \frac{1}{n+p} \int_{S^{n-1}} \int_{B_K} \rho_{B_K}(y, u)^{n+p} \, dy \, du \\
&= \frac{1}{n+p} \int_{S^{n-1}} \int_0^{\rho_{DB_K}(u)} a_{B_K}(r, u) r^{n+p} \, dr \, du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\omega_{n-1} V(K)^{(2n+p)/n}}{n+p} \int_{S^{n-1}} \int_0^2 \left(1-\left(\frac{r}{2}\right)^2\right)^{(n-1)/2} r^{n+p}\, dr\, du \\
&= \frac{2^{n+p} n\omega_n \omega_{n-1}}{n+p} \beta\left(\frac{n+1}{2}, \frac{n+p+1}{2}\right) V(K)^{2+p/n} \\
&= \frac{2^{n+p} n\omega_n \omega_{2n+p}}{(n+p)\omega_2 \omega_{n+p-1}} V(K)^{2+p/n}.
\end{aligned}
$$

Combining these together, we have

$$
\left[\frac{V(\tilde{Z}_p B_K)}{\omega_n}\right]^{1/n} = \left[\frac{2^{n+p}\omega_{n+p}\omega_{2n+p}}{\omega_2^2 \omega_{p-1}\omega_{n+p-1}}\right]^{1/p} V(K)^{1/n}. \tag{3.6}
$$

From (3.5) and (3.6), we obtain (1.4).

4. *Further results*

In this section, we assume that

$$
C_K(n,p) = n^{(n+p+1)/p} \beta(n+p+1, n)^{1/p} V(K)^{(n+p-1)/p} V(\Pi^* K)^{1/p}.
$$

THEOREM 4.1. *Let $K \in \mathcal{K}^n$ and $p \geqslant 1$, then*

$$
\tilde{Z}_p K \subseteq C_K(n,p) \Gamma_p \Pi^* K,
$$

with equality if and only if K is a simplex.

Proof. Let $u \in S^{n-1}$. We will prove

$$
h_{\tilde{Z}_p K}(u) \leqslant C_K(n,p) h_{\Gamma_p \Pi^* K}(u), \tag{4.1}
$$

with equality for some u if and only if K is a simplex.

By (2.4) and (2.12), we have

$$
h_{\tilde{Z}_p K}(u) \leqslant \left(n^{n+p} \beta(n+p, n+1) V(K)^{n+p-1} \int_{S^{n-1}} |uv|^p a_K(0,v)^{-n-p}\, dv\right)^{1/p}. \tag{4.2}
$$

By (2.2) and the fact that $\rho_{\Pi^* K}(v) = a_K(0,v)^{-1}$,

$$
h_{\Gamma_p \Pi^* K}(u) = \left(\frac{1}{(n+p)V(\Pi^* K)} \int_{S^{n-1}} |uv|^p a_K(0,v)^{-n-p}\, dv\right)^{1/p}. \tag{4.3}
$$

Then, (4.2) and (4.3) imply (4.1), with equality if and only if K is a simplex. □

The following lemma is crucial in the proof of Theorem 4.3.

LEMMA 4.2. *Suppose $p \geqslant 1$, $K \in \mathcal{K}^n$. Let $\Delta K = \frac{1}{2}(K-K)$, $r \geqslant 0$ and $u \in S^{n-1}$, then*

$$
a_K(r,u) \leqslant a_{\Delta K}(r,u), \tag{4.4}
$$

with equality for all (r,u) if and only if K is symmetric.

Proof. Let $L \in \mathcal{K}^n$, denote $G_L(r,u)$ by

$$
G_L(r,u) = \{x \in L | \rho_L(x,u) + \rho_L(x,-u) \geqslant r\}.
$$

We first prove that

$$\tfrac{1}{2}G_K(r,u) - \tfrac{1}{2}G_K(r,u) \subseteq G_{\Delta K}(r,u). \tag{4.5}$$

Let $x, y \in G_K(r,u)$, then it is easy to see that

$$x + \rho_K(x,u)u - y + \rho_K(y,-u)u \in K - K,$$
$$x - \rho_K(x,-u)u - y - \rho_K(y,u)u \in K - K.$$

Then

$$\rho_{\Delta K}\left(\frac{x-y}{2}, u\right) + \rho_{\Delta K}\left(\frac{x-y}{2}, -u\right) \geqslant \frac{\rho_K(x,u) + \rho_K(y,-u)}{2} + \frac{\rho_K(x,-u) + \rho_K(y,u)}{2} \geqslant r,$$

we have $(x-y)/2 \in G_{\Delta K}(r,u)$, for arbitrary $x, y \in K$, thus we get (4.5).

Noticing that $G_K(r,u)|u^{\perp} = E_K(r,u)$, using Brunn–Minkowski inequality, we have

$$\begin{aligned} a_{\Delta K}(r,u)^{1/(n-1)} &= V[(K/2 - K/2)_r|u^{\perp}]^{1/(n-1)} \\ &\geqslant V[(G_K(r,u)/2 - G_K(r,u)/2)|u^{\perp}]^{1/(n-1)} \\ &\geqslant \tfrac{1}{2}V(G_K(r,u)|u^{\perp})^{1/(n-1)} + \tfrac{1}{2}V(-G_K(r,u)|u^{\perp})^{1/(n-1)} \\ &= a_K(r,u)^{1/(n-1)}. \end{aligned}$$

If K is symmetric, then it is clear that $a_K(r,u) = a_{\Delta K}(r,u)$ for all (r,u).

Conversely, if equality holds in (4.4) for all (r,u), then we have $V(K) = V(\Delta K)$ by (2.7). By using Brunn–Minkowski inequality, we have $V(\Delta K) \geqslant V(K)$, with equality if and only if K is a translation of $-K$. Thus, we complete the proof. □

THEOREM 4.3. *Suppose* $p \geqslant 1$. *Let* $K \in \mathcal{K}^n$ *and* $\Delta K = \frac{1}{2}(K-K)$, *then*

$$\tilde{Z}_p K \subseteq [V(\Delta K)/V(K)]^{2/p}\tilde{Z}_p(\Delta K),$$

with equality if and only if K *is symmetric.*

Proof. For each $u \in S^{n-1}$, by (2.12) and Lemma 4.2, we have

$$h_{\tilde{Z}_p K}(u) \leqslant [V(\Delta K)/V(K)]^{2/p} h_{\tilde{Z}_p(\Delta K)}(u).$$

Then

$$\tilde{Z}_p K \subseteq [V(\Delta K)/V(K)]^{2/p}\tilde{Z}_p(\Delta K),$$

with equality if and only if

$$a_K(r,u) = a_{\Delta K}(r,u),$$

for all $r \geqslant 0$ and $u \in S^{n-1}$; this is equivalent to K is symmetric by Lemma 4.2. □

Acknowledgements. The authors are grateful to the referees for their comments and valuable suggestions.

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