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A Grassmannian Loomis–Whitney inequality and its dual

Ai-Jun Li, Dongmeng Xi and Qingzhong Huang

Abstract

Based on reverse isoperimetric inequalities on Grassmann manifolds, a Grassmanian Loomis–Whitney inequality and its dual are established, which provides a lower bound for the volume of an origin-symmetric convex body in terms of its lower dimensional sections.

1. *Introduction*

Throughout this paper, we shall use $|\cdot|$ to denote k-dimensional volume (Lebesgue measure on the corresponding subspace) in the Euclidean space $\mathbb{R}^k$ for $k = 1, \ldots, n$. A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. A star body in $\mathbb{R}^n$ is a compact star-shaped set with respect to the origin whose radial function is positive and continuous. The unit ball of the ℓ_p^n space, B_p^n (the ℓ_p^n-ball), is defined as $B_p^n = \{x \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p \leqslant 1\}$ for $0 < p < \infty$, and $B_\infty^n = \{x \in \mathbb{R}^n : |x_i| \leqslant 1,\ i = 1, \ldots, n\}$ for $p = \infty$. The volume of B_2^n is exclusively denoted by $\omega_n = \pi^{n/2}/\Gamma(1 + \frac{n}{2})$. The *weighted* ℓ_p^n-ball, $B_{p,\alpha}^n$, is defined as

$$B_{p,\alpha}^n = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^n |x_i|^p \alpha_i^p \right)^{\frac{1}{p}} \leqslant 1 \right\}, \quad 0 < p < \infty,$$

and

$$B_{\infty,\alpha}^n = \{x \in \mathbb{R}^n : |x_i|\alpha_i \leqslant 1, \quad i = 1, \ldots, n\}, \quad p = \infty,$$

with real coefficients $(\alpha_i)_{i=1}^n > 0$.

The classical Loomis–Whitney inequality provides a sharp upper bound for the volume of a convex body in terms of its coordinate projections; while its dual inequality gives a sharp lower bound in terms of its coordinate sections. To be specific, let K be a convex body in $\mathbb{R}^n$ and let $\{e_1, \ldots, e_n\}$ be the standard orthogonal basis of $\mathbb{R}^n$. Then

$$\frac{n!}{n^n} \prod_{i=1}^n |K \cap e_i^\perp| \leqslant |K|^{n-1} \leqslant \prod_{i=1}^n |\mathrm{P}_{e_i^\perp} K|, \tag{1.1}$$

with equality in the right(left)-hand inequality if and only if K is a weighted ℓ_∞^n-ball up to translations (a weighted ℓ_1^n-ball), where $\mathrm{P}_{e_i^\perp} K$ is the orthogonal projection of K onto the hyperplane $e_i^\perp$ perpendicular to e_i and $K \cap e_i^\perp$ is the intersection of K with the hyperplane $e_i^\perp$. The right-hand inequality in (1.1) was proved by Loomis and Whitney [25] in 1949, while the left-hand inequality in (1.1) was established by Meyer [35] in 1988. These two inequalities are fundamental in convex geometry and have been widely used in many mathematical areas

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(see, for example, [11, 17–19, 37]). In recent years, the study of various extensions of the Loomis–Whitney inequality and its dual has received considerable attention (see, for example, [1, 2, 5, 8, 10, 12–14, 20–22, 24]).

Both inequalities can be generalized in the following setting: let $(u_i)_{i=1}^m$ be unit vectors in $\mathbb{R}^n$ and let $(c_i)_{i=1}^m$ be positive real numbers such that John's condition

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n \tag{1.2}$$

is satisfied, where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the space spanned by u_i and I_n is the identity map on $\mathbb{R}^n$. Then, for every convex body K in $\mathbb{R}^n$ whose centroid is at the origin,

$$\frac{n!}{n^n}\prod_{i=1}^{m}|K\cap u_i^{\perp}|^{c_i} \leqslant |K|^{n-1} \leqslant \prod_{i=1}^{m}|\mathrm{P}_{u_i^{\perp}}K|^{c_i}. \tag{1.3}$$

The equality conditions are exactly the same as the ones in the Loomis–Whitney and Meyer inequality, respectively. The right-hand inequality in (1.3), established by Ball [5], is closely related to the geometric Brascamp–Lieb inequality, due to an important observation by Ball [3, 4] that John's condition can be combined with the Brascamp–Lieb inequality such that the constant in the Brascamp–Lieb inequality is 1. The left-hand inequality in (1.3) was recently proved by the authors [22] by using a direct mass transport approach of Barthe [6], Lutwak, Yang, and Zhang [30].

Note that the extremals of the above inequalities are the (weighted) ℓ_1^n-ball and ℓ_∞^n-ball. A characterization of all the ℓ_p^n-balls is provided by the following L_p version of the Loomis–Whitney inequality (for $1 \leqslant p \leqslant \infty$) and its dual (for $0 < p \leqslant \infty$) in the setting of the L_p Brunn–Minkowski theory and its dual theory. The L_p Brunn–Minkowski theory has its origins in the early 1960s when Firey [15] introduced his concept of L_p combinations of convex bodies. However, these L_p Minkowski–Firey combinations were first systematically investigated by Lutwak [27, 28] which lead to an embryonic L_p Brunn–Minkowski theory. The dual Brunn–Minkowski theory initiated by Lutwak [26] in the 1970s as the dual theory to the classical Brunn–Minkowski theory is based on a conceptual duality in convex geometry. Both theories have expanded rapidly thereafter; for further details, as well as detailed bibliography on these topics we refer the reader to [38, Chapter 9] and the references therein.

A Borel measure ν on the unit sphere S^{n-1} is said to be *isotropic* if

$$\int_{S^{n-1}} u\otimes u d\nu(u) = I_n. \tag{1.4}$$

The measure ν is called even if it assumes the same value on antipodal sets. Note that condition (1.4) reduces to (1.2) if the isotropic measure ν is of the form $(1/2)\sum_{i=1}^m (c_i\delta_{u_i} + c_i\delta_{-u_i})$ on S^{n-1} (δ_x stands for the Dirac mass at x). An isotropic measure of the form $(1/2)\sum_{i=1}^n(\delta_{u_i} + \delta_{-u_i})$ with an orthonormal basis $(u_i)_1^n$ of $\mathbb{R}^n$ is called a *cross measure*.

The following L_p Loomis–Whitney inequality and its dual for even isotropic measures were established by the authors [21, 22] and Lv [33] (the case $p=\infty$, which is always understood as the limiting process $p \to \infty$).

THEOREM 1.1. *Suppose that ν is an even isotropic measure on S^{n-1}. If K is a convex body in $\mathbb{R}^n$ with the origin in its interior, then for $1 \leqslant p \leqslant \infty$,*

$$|K|^{\frac{n-p}{p}} \leqslant \exp\left\{\int_{S^{n-1}} \log h_{\Pi_p K}(u) d\nu(u)\right\}. \tag{1.5}$$

For $p \neq 2$, there is equality if and only if ν is a cross measure on S^{n-1} and K is a weighted $\ell^n_{p^*}$-ball formed by μ (up to translations when $p=1$). If K is a star body in $\mathbb{R}^n$, then for $0 < p \leqslant \infty$,

$$|K| \leqslant |B^n_p| \exp\left\{\int_{S^{n-1}} \log \|u\|_{\Gamma^*_p K} d\nu(u)\right\}. \tag{1.6}$$

For $p \neq 2$, there is equality if and only if ν is a cross measure on S^{n-1} and K is a weighted ℓ^n_p-ball formed by μ.

The number p^* is the Hölder conjugate of p; that is, $1/p + 1/p^* = 1$. Here are two important notions involved (within the L_p Brunn–Minkowski theory and its dual theory). One is the *L_p projection body* $\Pi_p K$ of a convex body K in $\mathbb{R}^n$ with the origin in its interior, whose support function is defined, for $u \in S^{n-1}$, by

$$h_{\Pi_p K}(u) = \left(\frac{1}{|B^n_{p^*}|^{\frac{p}{n}}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v)\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and

$$h_{\Pi_\infty K}(u) = \lim_{p\to\infty} h_{\Pi_p K}(u) = \frac{1}{|B^n_1|^{\frac{1}{n}}} \max_{v \in \operatorname{supp} S_K} \{|u \cdot v|/h_K(v)\}, \quad p = \infty, \tag{1.7}$$

where $dS_p(K, \cdot) = h_K^{1-p}(\cdot) dS_K(\cdot)$ is the L_p surface area measure and $dS_K(\cdot)$ is the usual surface area measure of K. Under a different normalization $\Pi_p K$ was introduced by Lutwak, Yang, and Zhang [**29**] for $p > 1$. The case $p = 1$ is the classical projection body ΠK introduced by Minkowski (see [**9**, p. 50]), which is defined by

$$h_{\Pi K}(u) = |\mathrm{P}_{u^\perp} K| = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| dS_K(v).$$

In this case, (1.5) can be rewritten as

$$|K|^{n-1} \leqslant \exp\left\{\int_{S^{n-1}} \log |\mathrm{P}_{u^\perp} K| d\nu(u)\right\}, \tag{1.8}$$

with equality if and only if K is a weighted ℓ^n_∞-ball formed by the cross measure ν (up to translations). By taking $\nu = \frac{1}{2}\sum_{i=1}^m (c_i \delta_{u_i} + c_i \delta_{-u_i})$ on S^{n-1}, inequality (1.8) is actually the right-hand inequality in (1.3).

The second important notion in Theorem 1.1 is the *polar L_p centroid body*, $\Gamma^*_p K$, of a star body K in $\mathbb{R}^n$, whose Minkowski functional is given, for $x \in \mathbb{R}^n$, by

$$\|x\|_{\Gamma^*_p K} = \left(\frac{n+p}{|K|} \int_K |x \cdot y|^p dy\right)^{\frac{1}{p}} = \left(\frac{1}{|K|} \int_{S^{n-1}} |x \cdot v|^p \rho_K^{n+p}(v) dv\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

and

$$\|x\|_{\Gamma^*_\infty K} = \lim_{p\to\infty} \|x\|_{\Gamma^*_p K} = \max_{v \in S^{n-1}} \{|x \cdot v| \rho_K(v)\}, \quad p = \infty, \tag{1.9}$$

where ρ_K is the radial function of K. Under a different normalization, $\Gamma^*_p K$ was introduced by Lutwak and Zhang [**32**] for $1 \leqslant p \leqslant \infty$. When $p = 1$, the body ΓK is the classical centroid body, which was first defined and investigated by Blaschke (see [**38**, p. 567]). In this case, by the sharp estimates of $\|\cdot\|_{\Gamma^*_p K}$ due to Fradelizi [**16**, Theorem 3], inequality (1.6) immediately yields the following dual Loomis–Whitney inequality for even isotropic measures [**22**]: for every convex body K in $\mathbb{R}^n$ whose centroid is at the origin,

$$|K|^{n-1} \geqslant \frac{n!}{n^n} \exp\left\{\int_{S^{n-1}} \log |K \cap u^\perp| d\nu(u)\right\}, \tag{1.10}$$

with equality if and only if K is a weighted ℓ_1^n-ball formed by the cross measure ν. By taking $\nu = \frac{1}{2}\sum_{i=1}^{m}(c_i\delta_{u_i} + c_i\delta_{-u_i})$ on S^{n-1}, inequality (1.10) becomes the left-hand inequality in (1.3).

In [**23**] a new Grassmannian Ball–Barthe inequality was established by using the mass transportation technique (actually, the Brenier map), as well as reverse isoperimetric type volume inequalities on Grassmann manifolds that are naturally associated with L_p cosine transforms on Grassmann manifolds, which generalize and unify the cosine and sine transforms on unit spheres (see, for example, [**30**, **34**]). For the reverse isoperimetric problem associated with the L_p cosine transform on Grassmann manifolds other than unit spheres, ℓ_p^n-balls are no longer extremal bodies. The extremal bodies are a new class of convex bodies, $\ell_{m,p}^n$-balls, that are extensions of ℓ_p^n-balls on Grassmann manifolds. Note that ℓ_p^n-balls are the spherical L_p cosine transform of a cross measure on unit spheres, while $\ell_{m,p}^n$-balls are the L_p cosine transform of a cross measure on Grassmann manifolds (see Section 3 for details).

In this paper, we shall characterize all $\ell_{m,p}^n$-balls by a Grassmannian Loomis–Whitney inequality and its dual. To do this, we first introduce two new functions which extend the definitions of the support function of L_p projection bodies and the gauge function of polar L_p centroid bodies from unit spheres to Grassmann manifolds.

Denote by $G_{n,m}$ the Grassmann manifold of m-dimensional linear subspaces in $\mathbb{R}^n$, $1 \leqslant m \leqslant n-1$. In particular, when $m=1$, $G_{n,1}$ can be identified as the hemisphere of S^{n-1}. Let $\mathrm{P}_\xi : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection map onto ξ for $\xi \in G_{n,m}$. Define the function $h_{K,m,p} : G_{n,m} \to (0,\infty)$, for a convex body K in $\mathbb{R}^n$ with the origin in its interior and $\xi \in G_{n,m}$, by

$$h_{K,m,p}(\xi) = \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}} |\mathrm{P}_\xi v|^p dS_p(K,v)\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and

$$h_{K,m,\infty}(\xi) = \lim_{p\to\infty} h_{K,m,p}(\xi) = \frac{1}{\gamma_n(m,1)^{\frac{1}{n}}} \max_{v\in \operatorname{supp} S_K} \{|\mathrm{P}_\xi v|/h_K(v)\}, \quad p=\infty,$$

where

$$\gamma_n(m,p) = \frac{\left(\omega_m\Gamma(1+\frac{m}{p})\right)^{\frac{n}{m}}}{\Gamma(1+\frac{n}{p})} \quad \text{and} \quad \gamma_n(m,\infty) = \omega_m^{\frac{n}{m}}. \tag{1.11}$$

In particular, when $m=1$, $\gamma_n(1,p) = |B_p^n|$ and the function $h_{K,1,p}$ reduces to $h_{\Pi_p K}$; that is,

$$h_{K,1,p}(u) = h_{\Pi_p K}(u), \quad u \in G_{n,1}. \tag{1.12}$$

Define another function $g_{K,m,p} : G_{n,m} \to (0,\infty)$, for a star body K in $\mathbb{R}^n$ and $\xi \in G_{n,m}$, by

$$g_{K,m,p}(\xi) = \left(\frac{n+p}{m|K|}\int_K |\mathrm{P}_\xi y|^p dy\right)^{\frac{1}{p}} = \left(\frac{1}{m|K|}\int_{S^{n-1}} |\mathrm{P}_\xi v|^p \rho_K^{n+p}(v)dv\right)^{\frac{1}{p}}, \quad 0<p<\infty,$$

and

$$g_{K,m,\infty}(\xi) = \lim_{p\to\infty} g_{K,m,p}(\xi) = \max_{v\in S^{n-1}} \{|\mathrm{P}_\xi v|\rho_K(v)\}, \quad p=\infty.$$

In particular, when $m=1$, it reduces to the function $\|\cdot\|_{\Gamma_p^* K}$; that is,

$$g_{K,1,p}(u) = \|u\|_{\Gamma_p^* K}, \quad u \in G_{n,1}. \tag{1.13}$$

The concept of isotropic measures on Grassman manifolds introduced in [**23**] is needed. A finite Borel measure μ on $G_{n,m}$ is called *isotropic* if

$$\int_{G_{n,m}} \mathrm{P}_\xi \, d\mu(\xi) = I_n. \tag{1.14}$$

Note that an isotropic measure on the Grassmann manifold $G_{n,m}$ is not concentrated on a great sub-Grassmannian. Moreover, identity (1.14) reduces to the sphere case (1.4) when $m = 1$.

The main result of the paper is the following theorem.

THEOREM 1.2. *Let μ be an isotropic measure on $G_{n,m}$. If K is a convex body in $\mathbb{R}^n$ with the origin in its interior, then for $1 \leqslant p \leqslant \infty$,*

$$|K|^{\frac{n-p}{p}} \leqslant \left(\exp \int_{G_{n,m}} \log h_{K,m,p}(\xi) d\mu(\xi)\right)^m.$$

When n is divisible by m and $p \neq 2$, there is equality if and only if μ is a cross measure on $G_{n,m}$ and K is a weighted ℓ^n_{m,p^*}-ball formed by μ (up to translations when $p = 1$). If K is a star body in $\mathbb{R}^n$, then for $0 < p \leqslant \infty$,

$$|K| \leqslant \gamma_n(m,p)\left(\exp \int_{G_{n,m}} \log g_{K,m,p}(\xi) d\mu(\xi)\right)^m. \tag{1.15}$$

When n is divisible by m and $p \neq 2$, there is equality if and only if μ is a cross measure on $G_{n,m}$ and K is a weighted $\ell^n_{m,p}$-ball formed by μ.

When $m = 1$, $\ell^n_{1,p}$-balls are exactly ℓ^n_p-balls, and by (1.12) and (1.13), Theorem 1.2 reduces to Theorem 1.1.

Moreover, from (1.15) we obtain the following volume inequality in terms of lower dimensional sections of K.

THEOREM 1.3. *Suppose that $p \geqslant 1$ and K is an origin-symmetric convex body in $\mathbb{R}^n$. If μ is an isotropic measure on $G_{n,m}$, then*

$$|K|^{\frac{n-m}{m}} \geqslant \frac{\Gamma(1+\frac{n}{p})[mB(m,n-m+1)]^{\frac{n}{p}+\frac{n}{m}}}{(\Gamma(1+\frac{m}{p}))^{\frac{n}{m}}[(n+p)B(m+p,n-m+1)]^{\frac{n}{p}}} \exp\left\{\int_{G_{n,m}} \log |K \cap \xi^{\perp}| d\mu(\xi)\right\}. \tag{1.16}$$

When $m, p = 1$, there is equality if and only if μ is a cross measure on $G_{n,1}$ and K is a weighted ℓ^n_1-ball formed by μ.

The case $m = 1$ of the above theorem was established in [**22**, Lemma 4.3] for convex bodies with the centroids at the origin. If in addition $p = 1$, then inequality (1.16) reduces to (1.10).

The rest of this paper is organized as follows: In Section 2, the basic notations and preliminaries are provided. Section 3 contains some definitions on Grassmann manifolds; for example, cross measures, $\ell^n_{m,p}$-balls, bodies $Z_{m,p,\alpha}$ and $Z^*_{m,p,\alpha}$. Some auxiliary results are presented in Section 4. To prove Theorem 1.2, two crucial inequalities are obtained in Sections 5 and 6, which generalize the Grassmannian reverse isoperimetric inequalities from [**23**]. Some properties of the functions $h_{K,m,p}$ and $g_{K,m,p}$ are studied in Section 7. The Grassmannian Loomis–Whitney inequality and its dual are established in Sections 8 and 9, respectively.

2. *Preliminaries*

For quick later reference, we recall some background materials from the L_p Brunn–Minkowski theory and its dual theory. General references are provided by the books of Gardner [**17**] and Schneider [**38**].

We denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. A convex body K in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Its support function, $h_K : \mathbb{R}^n \to \mathbb{R}$, is defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

Denote by $\mathcal{K}_o^n$ the space of convex bodies containing the origin in its interior in $\mathbb{R}^n$ endowed with the Hausdorff metric (that is, $\sup_{u\in S^{n-1}} |h_K(u) - h_L(u)|$, for $K, L \in \mathcal{K}_o^n$). If $K \in \mathcal{K}_o^n$, its polar body K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } y \in K\}.$$

It is easy to check that $(K^*)^* = K$ and for $A \in \mathrm{GL}(n)$,

$$(AK)^* = A^{-t}K, \tag{2.2}$$

where A^{-t} is the inverse and transpose of A.

A compact set $K \subset \mathbb{R}^n$ is a star-shaped set (with respect to the origin) if the intersection of every straight line through the origin with K is a line segment. Let $K \subset \mathbb{R}^n$ be a compact star-shaped set (with respect to the origin), the radial function $\rho_K : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ is defined by

$$\rho_K(x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}.$$

If ρ_K is positive and continuous, then we call K a star body. Let $\mathcal{S}_o^n$ be the space of star bodies in $\mathbb{R}^n$ endowed with the radial metric (that is, $\sup_{u\in S^{n-1}} |\rho_K(u) - \rho_L(u)|$, for $K, L \in \mathcal{S}_o^n$). Two star bodies K and L are said to be dilates (of each other) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

The Minkowski functional $\|\cdot\|_K$ of $K \in \mathcal{S}_o^n$ is defined by

$$\|x\|_K = \min\{t > 0 : x \in tK\}.$$

Obviously, for real $\lambda > 0$, $\|\lambda x\|_K = \lambda \|x\|_K$, and for $K, L \in \mathcal{S}_o^n$,

$$K \subseteq L \quad \Longleftrightarrow \quad \|\cdot\|_L \leqslant \|\cdot\|_K. \tag{2.3}$$

It is easy to see that for $K \in \mathcal{K}_o^n$,

$$\rho_K^{-1}(\cdot) = \|\cdot\|_K = h_{K^*}(\cdot), \tag{2.4}$$

and for each $u \in S^{n-1}$,

$$h_K(u) = \max_{v\in S^{n-1}} \{(u \cdot v)\rho_K(v)\}. \tag{2.5}$$

Recall that for each $0 < p < \infty$, the volume of K can be computed by

$$|K| = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} e^{-\|x\|_K^p}\, dx, \tag{2.6}$$

where the integral is with respect to Lebesgue measure on $\mathbb{R}^n$. In particular,

$$\int_{\mathbb{R}^n} e^{-|x|^p} dx = \omega_n \Gamma\left(1 + \frac{n}{p}\right). \tag{2.7}$$

Elements of the L_p Brunn–Minkowski theory. For $p \geqslant 1$, $K, L \in \mathcal{K}_o^n$, and $\varepsilon > 0$, the L_p Minkowski-Firey combination $K +_p \varepsilon \cdot L$ is the convex body whose support function is given by

$$h^p_{K+_p\varepsilon\cdot L}(\cdot) = h^p_K(\cdot) + \varepsilon h^p_L(\cdot).$$

The L_p mixed volume $V_p(K, L)$ of $K, L \in \mathcal{K}_o^n$ was defined in [27] by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon\to 0^+} \frac{|K +_p \varepsilon \cdot L| - |K|}{\varepsilon}.$$

In particular, $V_p(K,K) = |K|$. It was shown in [27] that there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} so that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(v)dS_p(K,v), \tag{2.8}$$

where $dS_p(K,\cdot) = h_K^{1-p}(\cdot)dS_K(\cdot)$ is the L_p surface area measure of K. The measure $S_K(\cdot)$ is the classical surface area measure of K; that is, for a Borel set $\omega \subset S^{n-1}$, $S_K(\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω. It is easy to verify that

$$dS_p(cK,\cdot) = c^{n-p}dS_p(K,\cdot), \quad c > 0. \tag{2.9}$$

The L_p Minkowski inequality [27] states that for $K, L \in \mathcal{K}_o^n$,

$$V_p(K,L)^n \geqslant |K|^{n-p}|L|^p, \tag{2.10}$$

with equality if and only if K and L are dilates when $p > 1$ and if and only if K and L are homothetic (that is, they coincide up to translations and dilatations) when $p = 1$. When $p = \infty$, the following L_∞ Minkowski inequality was established in [31]:

$$\max_{v\in \operatorname{supp} S_K}\{h_L(v)/h_K(v)\} \geqslant |K|^{-1/n}|L|^{1/n}, \tag{2.11}$$

with equality if and only if K and L are dilates.

Elements of the dual Brunn–Minkowski theory. For $p > 0$ and $\varepsilon > 0$, the L_p harmonic radial combination $K \widetilde{+}_p \, \varepsilon \cdot L$ of $K, L \in \mathcal{S}_o^n$ is the star body whose radial function is given by

$$\rho^{-p}_{K\widetilde{+}_p\,\varepsilon\cdot L}(\cdot) = \rho_K^{-p}(\cdot) + \varepsilon\rho_L^{-p}(\cdot).$$

The dual mixed volume $\widetilde{V}_{-p}(K,L)$ of $K, L \in \mathcal{S}_o^n$ was defined in [28] by

$$-\frac{n}{p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon\to 0^+}\frac{|K\widetilde{+}_p\,\varepsilon\cdot L| - |K|}{\varepsilon}.$$

In particular, $\widetilde{V}_{-p}(K,K) = |K|$. The polar coordinate formula for volume yields the following integral representation

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(v)\rho_L^{-p}(v)dv, \tag{2.12}$$

where the integration is with respect to the spherical Lebesgue measure.

A basic inequality for dual mixed volumes is the dual Minkowski inequality [28], which states that, for $K, L \in \mathcal{S}_o^n$,

$$\widetilde{V}_{-p}(K,L)^n \geqslant |K|^{n+p}|L|^{-p}, \quad p > 0, \tag{2.13}$$

and, for $p = \infty$ (see [32]),

$$\max_{v\in S^{n-1}}\{\rho_K(v)/\rho_L(v)\} \geqslant |K|^{\frac{1}{n}}|L|^{-\frac{1}{n}}. \tag{2.14}$$

Equality holds in each of the inequalities if and only if K and L are dilates.

3. *Some notions on Grassmann manifolds*

For a subspace $\xi \subset \mathbb{R}^n$ of dimension m, let $\{\varepsilon_1,\ldots,\varepsilon_m\}$ be an orthonormal basis of ξ, and let Q be the $n\times m$ matrix with column vectors $\varepsilon_1,\ldots,\varepsilon_m$. This is equivalent to the condition

$$Q^tQ = I_m,$$

where Q^t is the transpose of Q. Then

$$\mathrm{P}_\xi = QQ^t. \tag{3.1}$$

The matrix Q^t induces the isometry $\xi \to \mathbb{R}^m$, $x_1\varepsilon_1 + \cdots + x_m\varepsilon_m \to (x_1, \dots, x_m)$, and vanishes on $\xi^\perp$. The matrix Q induces the isometry $\mathbb{R}^m \to \xi$, $(x_1, \dots, x_m) \to x_1\varepsilon_1 + \cdots + x_m\varepsilon_m$. Clearly, identity (3.1) is equivalent to

$$\mathrm{P}_\xi x = \sum_{i=1}^{m} (x \cdot \varepsilon_i)\varepsilon_i, \quad x \in \mathbb{R}^n$$

or

$$|\mathrm{P}_\xi x| = \left(\sum_{i=1}^{m} |x \cdot \varepsilon_i|^2 \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n. \tag{3.2}$$

For $\xi_1, \xi_2 \in G_{n,m}$, a metric distance between ξ_1, ξ_2 can be defined by $\|\mathrm{P}_{\xi_1} - \mathrm{P}_{\xi_2}\|_2$, where $\|A\|_2$ is the spectral norm of the $n \times n$ matrix A defined by

$$\|A\|_2 = \max_{|x| \neq 0} \frac{|Ax|}{|x|}.$$

When we write $O\xi$ for $O \in \mathrm{O}(n)$ and $\xi \in G_{n,m}$, it means that

$$O\xi = O(\mathrm{span}\{\varepsilon_1, \dots, \varepsilon_m\}) = \mathrm{span}\{O\varepsilon_1, \dots, O\varepsilon_m\},$$

where $\{\varepsilon_1, \dots, \varepsilon_m\}$ is an orthonormal basis of ξ.

Observe that identity (1.14) is equivalent to

$$\int_{G_{n,m}} |\mathrm{P}_\xi x|^2 \, d\mu(\xi) = |x|^2, \quad x \in \mathbb{R}^n. \tag{3.3}$$

Taking $x = e_1, \dots, e_n$ in the above identity gives

$$\begin{aligned} n &= \sum_{j=1}^{n} \int_{G_{n,m}} \sum_{i=1}^{m} |e_j \cdot \varepsilon_i|^2 \, d\mu(\xi) = \sum_{i=1}^{m} \int_{G_{n,m}} \sum_{j=1}^{n} |e_j \cdot \varepsilon_i|^2 \, d\mu(\xi) \\ &= m\mu(G_{n,m}), \end{aligned}$$

that is

$$\mu(G_{n,m}) = \frac{n}{m}. \tag{3.4}$$

Let $\{e_1, \dots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. If $n/m = l$ is an integer, then define l orthogonal m-dimensional subspaces ξ_k by

$$\xi_1 = \mathrm{span}\{e_1, \dots, e_m\}, \xi_2 = \mathrm{span}\{e_{m+1}, \dots, e_{2m}\}, \dots, \xi_l = \mathrm{span}\{e_{n-m+1}, \dots, e_n\}.$$

It is easy to see that

$$\sum_{i=1}^{l} \mathrm{P}_{\xi_k} = I_n.$$

Comparing with the definition of (1.14), we see that the above measure concentrated equally on $\xi_1, \dots, \xi_l$, called the *basic cross measure* on $G_{n,m}$, is isotropic. A rotation of the basic cross measure that is concentrated on $O\{\xi_1, \dots, \xi_l\}$ with $O \in \mathrm{O}(n)$ is still isotropic, and is called a *cross measure* on $G_{n,m}$ (see [**23**]). In particular, $n/m = n$ if $m = 1$. In this case, it becomes a measure on $G_{n,1}$ that is concentrated equally on n orthogonal 1-dimensional subspaces of $\mathbb{R}^n$, which is exactly a cross measure on S^{n-1} (see, for example, [**30**]).

If $n/m = l$ is an integer, then the $\ell^n_{m,p}$-ball, $B^n_{m,p}$, was defined in [23] by

$$B^n_{m,p} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{l} |\mathrm{P}_{\xi_i} x|^p \leqslant 1 \right\}, \quad 0 < p < \infty, \tag{3.5}$$

and

$$B^n_{m,\infty} = \{x \in \mathbb{R}^n : |\mathrm{P}_{\xi_i} x| \leqslant 1 \text{ for all } i = 1, \dots, l\}, \quad p = \infty. \tag{3.6}$$

If $m = 1$, the $\ell^n_{m,p}$-ball reduces to the classical ℓ^n_p-ball, that is, $B^n_{1,p} = B^n_p$. The following result was established in [23]:

$$(B^n_{m,p})^* = B^n_{m,p^*}, \quad 1 \leqslant p \leqslant \infty. \tag{3.7}$$

This extends the well-known duality of ℓ^n_p-balls; that is, $(B^n_p)^* = B^n_{p^*}$ for $1 \leqslant p \leqslant \infty$.

Suppose that $n/m = l$ is an integer. Let μ be a cross measure on $G_{n,m}$ such that

$$\operatorname{supp} \mu = \{\tilde{\xi}_1, \dots, \tilde{\xi}_l\} = O\{\xi_1, \dots, \xi_l\}.$$

The *weighted* $\ell^n_{m,p}$-ball, $B^n_{m,p,\alpha} := B^n_{m,p,\alpha}(\mu)$, formed by μ is defined by

$$B^n_{m,p,\alpha} = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^p \alpha^p(\tilde{\xi}_i) \right)^{\frac{1}{p}} \leqslant 1 \right\}, \quad 0 < p < \infty, \tag{3.8}$$

and

$$B^n_{m,\infty,\alpha} = \left\{ x \in \mathbb{R}^n : |\mathrm{P}_{\tilde{\xi}_i} x| \alpha(\tilde{\xi}_i) \leqslant 1 \text{ for all } i = 1, \dots, l \right\}, \quad p = \infty, \tag{3.9}$$

where $(\alpha(\tilde{\xi}_i))_{i=1}^l > 0$ are the weights of μ. Thus, by (3.2), (3.5), and (3.6), we have

$$\begin{aligned}
B^n_{m,p,\alpha} &= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} \left(\sum_{j=1}^{m} |x \cdot O e_{(i-1)m+j}|^2 \right)^{\frac{p}{2}} \alpha^p(\tilde{\xi}_i) \right)^{\frac{1}{p}} \leqslant 1 \right\} \\
&= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} \left(\sum_{j=1}^{m} |A O^t x \cdot e_{(i-1)m+j}|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leqslant 1 \right\} \\
&= \left\{ O^{-t} A^{-1} x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} \left(\sum_{j=1}^{m} |x \cdot e_{(i-1)m+j}|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leqslant 1 \right\} \\
&= O A^{-1} B^n_{m,p},
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
B^n_{m,\infty,\alpha} &= \left\{ x \in \mathbb{R}^n : \left(\sum_{j=1}^{m} |x \cdot O e_{(i-1)m+j}|^2 \right)^{\frac{1}{2}} \alpha(\tilde{\xi}_i) \leqslant 1 \text{ for all } i = 1, \dots, l \right\} \\
&= \left\{ x \in \mathbb{R}^n : \left(\sum_{j=1}^{m} |A O^t x \cdot e_{(i-1)m+j}|^2 \right)^{\frac{1}{2}} \leqslant 1 \text{ for all } i = 1, \dots, l \right\} \\
&= O A^{-1} B^n_{m,\infty},
\end{aligned} \tag{3.11}$$

where $A = \mathrm{diag}\{\underbrace{\alpha(\tilde{\xi}_1), \ldots, \alpha(\tilde{\xi}_1)}_{m}, \ldots, \underbrace{\alpha(\tilde{\xi}_l), \ldots, \alpha(\tilde{\xi}_l)}_{m}\}$ is a diagonal matrix. Thus, we immediately get

$$|B^n_{m,p,\alpha}| = |OA^{-1}B^n_{m,p}| = |B^n_{m,p}| \left(\prod_{i=1}^{l} \alpha(\tilde{\xi}_i) \right)^{-m}, \quad 0 < p \leqslant \infty.$$

Moreover, by (3.10), (3.11), (2.2), and (3.7), we have, for $1 \leqslant p \leqslant \infty$,

$$(B^n_{m,p,\alpha})^* = (OA^{-1}B^n_{m,p})^* = OA^t B^n_{m,p^*} = B^n_{m,p^*,\alpha^{-1}}, \tag{3.12}$$

and thus,

$$|(B^n_{m,p,\alpha})^*| = |B^n_{m,p^*}| \left(\prod_{i=1}^{l} \alpha(\tilde{\xi}_i) \right)^{m}.$$

Let $\alpha : G_{n,m} \to (0, \infty)$ be a continuous positive function and let μ be an isotropic measure on $G_{n,m}$. Then we define the origin-symmetric convex body $Z_{m,p,\alpha} = Z_{m,p,\alpha}(\mu)$ in $\mathbb{R}^n$ by its support function which is given, for $x \in \mathbb{R}^n$, by

$$h_{Z_{m,p,\alpha}}(x) = \left(\int_{G_{n,m}} |\mathrm{P}_\xi x|^p \alpha^p(\xi) d\mu(\xi) \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \tag{3.13}$$

and

$$h_{Z_{m,\infty,\alpha}}(x) = \lim_{p\to\infty} h_{Z_{m,p,\alpha}}(x) = \max_{\xi \in \operatorname{supp}\mu} \{|\mathrm{P}_\xi x| \alpha(\xi)\}, \quad p = \infty. \tag{3.14}$$

We define the origin-symmetric star body $Z^*_{m,p,\alpha} = Z^*_{m,p,\alpha}(\mu)$ in $\mathbb{R}^n$ by its Minkowski functional which is given, for $x \in \mathbb{R}^n$, by

$$\|x\|_{Z^*_{m,p,\alpha}} = \left(\int_{G_{n,m}} |\mathrm{P}_\xi x|^p \alpha^p(\xi) d\mu(\xi) \right)^{\frac{1}{p}}, \quad 0 < p < \infty, \tag{3.15}$$

and

$$\|x\|_{Z^*_{m,\infty,\alpha}} = \lim_{p\to\infty} \|x\|_{Z^*_{m,p,\alpha}} = \max_{\xi \in \operatorname{supp}\mu} \{|\mathrm{P}_\xi x| \alpha(\xi)\}, \quad p = \infty. \tag{3.16}$$

When $m = 1$, the bodies $Z_{1,p,\alpha}$ and $Z^*_{1,p,\alpha}$ were defined in [**21**, **22**], respectively. Without the assumption of isotropicity, the bodies are L_p zonoids and their polars introduced by Schneider and Weil [**39**].

In particular, if $n/m = l$ is an integer and μ is a cross measure with $\operatorname{supp}\mu = \{\tilde{\xi}_1, \ldots, \tilde{\xi}_l\} = O\{\xi_1, \ldots, \xi_l\}$, then by (2.4), (3.8), (3.9), (2.4) again, and (3.12), we have, for $x \in \mathbb{R}^n$,

$$h_{Z_{m,p,\alpha}}(x) = \|x\|_{Z^*_{m,p,\alpha}} = \|x\|_{B^n_{m,p,\alpha}} = h_{(B^n_{m,p,\alpha})^*}(x) = h_{B^n_{m,p^*,\alpha^{-1}}}(x), \quad 1 \leqslant p \leqslant \infty, \tag{3.17}$$

and

$$\|x\|_{Z^*_{m,p,\alpha}} = \|x\|_{B^n_{m,p,\alpha}}, \quad 0 < p \leqslant \infty. \tag{3.18}$$

4. *Auxiliary results*

Let μ be a finite Borel measure on $G_{n,m}$, and let $f : G_{n,m} \to \mathbb{R}^n$ be a continuous map so that for each $\xi \in G_{n,m}$, $f(\xi) \in \xi$. Write $f(\xi)$ as f_ξ. Define

$$\|f : \mu\|_p = \left(\int_{G_{n,m}} |f_\xi|^p d\mu(\xi) \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and

$$\|f : \mu\|_\infty = \sup_{\xi \in \operatorname{supp} \mu} |f_\xi|.$$

For $\alpha : G_{n,m} \to (0, \infty)$ a continuous positive function, define $\widetilde{f} \in \mathbb{R}^n$ by

$$\widetilde{f} = \int_{G_{n,m}} f_\xi \alpha(\xi) d\mu(\xi) = \int_{G_{n,m}} \mathrm{P}_\xi f_\xi \alpha(\xi) d\mu(\xi). \tag{4.1}$$

Obviously, for $\lambda > 0$, we have

$$\widetilde{\lambda f} = \lambda \widetilde{f}.$$

Modifying Lemma 6.1 in [**23**] slightly gives the following lemma.

Lemma 4.1. *Suppose $1 \leqslant p \leqslant \infty$, μ is a finite Borel measure on $G_{n,m}$, and α is a continuous positive function on $G_{n,m}$. If $\|f : \mu\|_{p^*} < \infty$, then*

$$\|\widetilde{f}\|_{Z_{m,p,\alpha}} \leqslant \left(\int_{G_{n,m}} |f_\xi|^{p^*} d\mu(\xi) \right)^{\frac{1}{p^*}}. \tag{4.2}$$

Proof. For $1 \leqslant p < \infty$, define M_p as the closure

$$M_p = \mathrm{cl}\, \{\widetilde{f} \in \mathbb{R}^n : \|f : \mu\|_{p^*} \leqslant 1\}, \tag{4.3}$$

while for $p = \infty$, define M_∞ as the closure

$$M_\infty = \mathrm{cl}\, \{\widetilde{f} \in \mathbb{R}^n : \|f : \mu\|_1 \leqslant 1\}.$$

It is easily shown that M_p is a convex body in $\mathbb{R}^n$ for all $1 \leqslant p \leqslant \infty$ and that M_p converges to M_∞ as $p \to \infty$ under the Hausdorff metric. Since $Z_{m,p,\alpha}$ and M_p converge to $Z_{m,\infty,\alpha}$ and M_∞, respectively, as $p \to \infty$, we only need to show the case $1 \leqslant p < \infty$.

By (2.1), (4.3), (4.1), the Hölder inequality, and the definition (3.15), we have, for $u \in S^{n-1}$,

$$\begin{aligned}
h_{M_p}(u) &= \sup_{\|f:\mu\|_{p^*} \leqslant 1} u \cdot \widetilde{f} \\
&= \sup_{\|f:\mu\|_{p^*} \leqslant 1} \int_{G_{n,m}} u \cdot \mathrm{P}_\xi f_\xi \alpha(\xi)\, d\mu(\xi) \\
&= \sup_{\|f:\mu\|_{p^*} \leqslant 1} \int_{G_{n,m}} (\mathrm{P}_\xi u) \cdot f_\xi \alpha(\xi)\, d\mu(\xi) \\
&\leqslant \sup_{\|f:\mu\|_{p^*} \leqslant 1} \int_{G_{n,m}} |\mathrm{P}_\xi u| |f_\xi| \alpha(\xi)\, d\mu(\xi) \\
&\leqslant \sup_{\|f:\mu\|_{p^*} \leqslant 1} \|f : \mu\|_{p^*} \left(\int_{G_{n,m}} |\mathrm{P}_\xi u|^p \alpha^p(\xi)\, d\mu(\xi) \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \left(\int_{G_{n,m}} |\mathrm{P}_\xi u|^p \alpha^p(\xi) d\mu(\xi) \right)^{\frac{1}{p}}$$
$$= h_{Z_{m,p,\alpha}}(u).$$

Therefore, $M_p \subseteq Z_{m,p,\alpha}$. This, together with (2.3), yields the desired inequality (4.2). □

In particular, if μ is isotropic on $G_{n,m}$, $\alpha(\cdot) \equiv 1$ and $p = 2$, then $Z_{m,2,1} = B_2^n$ and inequality (4.2) becomes

$$\left| \int_{G_{n,m}} f_\xi d\mu(\xi) \right| \leqslant \left(\int_{G_{n,m}} |f_\xi|^2 d\mu(\xi) \right)^{\frac{1}{2}}. \tag{4.4}$$

The following Grassmannian Ball–Barthe inequality, extending the spherical Ball–Barthe inequality due to Lutwak, Yang, and Zhang [**30**], was established in [**23**]. For $\xi \in G_{n,m}$, let $I_\xi : \xi \to \mathbb{R}^n$ be the inclusion map, that is the identity map from ξ to ξ.

Lemma 4.2 (Grassmannian Ball–Barthe inequality). *Let μ be a finite Borel measure on $G_{n,m}$. Associate with each $\xi \in G_{n,m}$ a positive definite linear transformation $A_\xi : \xi \to \xi$. Suppose that $A(\xi) = I_\xi A_\xi \mathrm{P}_\xi$ is continuous and μ satisfies*

$$\int_{G_{n,m}} \mathrm{P}_\xi \, d\mu(\xi) = I_n.$$

Then

$$\det \int_{G_{n,m}} A(\xi) \, d\mu(\xi) \geqslant \exp\left(\int_{G_{n,m}} \log(\det A_\xi) \, d\mu(\xi) \right), \tag{4.5}$$

with equality if and only if $\lambda_{k_1}(A(\xi_1)) \cdots \lambda_{k_n}(A(\xi_n))$ is constant for $\xi_j \in \operatorname{supp} \mu$ whenever there exist n linearly independent eigenvectors belonging to positive eigenvalues $\lambda_{k_1}(A(\xi_1)), \ldots,$ $\lambda_{k_n}(A(\xi_n))$ of $A(\xi_1), \ldots, A(\xi_n)$, $k_j = 1, \ldots, m$, and $j = 1, \ldots, n$.

Denote by $B_2^m(r)$ the Euclidean open ball in $\mathbb{R}^m$ centered at the origin and with radius r. We say that a function f is rotationally invariant in $\mathbb{R}^m$ if for $O \in \mathrm{O}(m)$ and $x \in \mathbb{R}^m$,

$$f(x) = f(Ox).$$

The following lemma (see [**23**, Lemma 7.1]), providing the regularity of the Brenier map for the rotationally invariant case, is needed to establish the volume inequalities for $Z_{m,p,\alpha}$ and $Z^*_{m,p,\alpha}$.

Lemma 4.3. *For $a, b \in (0, \infty]$, let $f : B_2^m(a) \to (0, \infty)$, $g : B_2^m(b) \to (0, \infty)$ be continuous positive probability density functions. If f, g are rotationally invariant in $\mathbb{R}^m$, then there exists a rotationally invariant and strictly convex function ψ of class C^2 on $B_2^m(a)$ such that $\nabla\psi : B_2^m(a) \to B_2^m(b)$ and for $x \in B_2^m(a)$*

$$f(x) = g(\nabla\psi(x)) \det(\nabla^2\psi(x)).$$

5. *Volume inequalities for $Z_{m,p,\alpha}$*

Establishing Theorem 1.2 relies on sharp volume estimates for the bodies $Z_{m,p,\alpha}$ (Theorem 5.1) and $Z^*_{m,p,\alpha}$ (Theorem 6.1). The case $\alpha(\cdot) \equiv 1$ of Theorems 5.1 and 6.1 was established in [**23**], and in addition $m = 1$ is the well-known L_p volume ratio inequality due to Ball [**4**], Barthe

[**6**, **7**], and Lutwak, Yang, and Zhang [**30**]. The proofs of Theorems 5.1 and 6.1 are based on a refinement of the approach by Lutwak, Yang, and Zhang [**30**], which uses the Grassmannian Ball–Barthe inequality (Lemma 4.2) and the Brenier map (Lemma 4.3).

THEOREM 5.1. *Suppose $1 \leqslant p \leqslant \infty$ and α is a continuous positive function on $G_{n,m}$. If μ is an isotropic measure on $G_{n,m}$, then*

$$|Z_{m,p,\alpha}| \geqslant \gamma_n(m,p^*)\left(\exp\int_{G_{n,m}} \log\alpha(\xi)d\mu(\xi)\right)^m, \tag{5.1}$$

where $\gamma_n(m,p^*)$ is given by (1.11). When n is divisible by m and $p \neq 2$, there is equality if and only if μ is a cross measure on $G_{n,m}$.

Proof. *Case* $1 < p \leqslant \infty$: Define probability densities $f, g : \mathbb{R}^m \to (0,\infty)$ by

$$f(s) = \frac{1}{\omega_m\Gamma(1+\frac{m}{2})}e^{-|s|^2},$$

$$g(s) = \frac{1}{\omega_m\Gamma(1+\frac{m}{p^*})}e^{-|s|^{p^*}}.$$

By Lemma 4.3, there exists a convex function ψ on $\mathbb{R}^m$ such that for $s \in \mathbb{R}^m$

$$e^{-|s|^2} = c_{m,p^*}e^{-|\nabla\psi(s)|^{p^*}}\det(\nabla^2\psi(s)), \tag{5.2}$$

where $c_{m,p^*} = \Gamma(1+\frac{m}{2})/\Gamma(1+\frac{m}{p^*})$.

For $\xi \in G_{n,m}$, choose an orthonormal basis $\{\varepsilon_1,\dots,\varepsilon_m\}$ of ξ. Then for $s \in \xi$, $s = s_1\varepsilon_1 + \cdots + s_m\varepsilon_m$, $(s_1,\dots,s_m) \in \mathbb{R}^m$. Identify $s \in \xi$ with $(s_1,\dots,s_m) \in \mathbb{R}^m$ when no confusion is possible.

Let $\Psi_\xi : \xi \to \xi$ be the map defined by

$$\Psi_\xi(s) = \frac{\partial\psi}{\partial s_1}(s)\varepsilon_1 + \cdots + \frac{\partial\psi}{\partial s_m}(s)\varepsilon_m, \tag{5.3}$$

that is, Ψ_ξ is the gradient of ψ in ξ. It was shown in [**23**, p. 524] that Ψ_ξ does not depend on the choice of orthonormal basis of ξ, and thus (5.3) is well defined. The differential of Ψ_ξ at s, $d\Psi_\xi(s) :\ \xi \to \xi$, is given by

$$d\Psi_\xi(s)(\varepsilon_i) = \sum_{j=1}^m \frac{\partial^2\psi(s)}{\partial s_i\partial s_j}\varepsilon_j. \tag{5.4}$$

Since ψ is strictly convex, $d\Psi_\xi(s)$ is a positive definite symmetric transformation. Let

$$A_\xi(x) = d\Psi_\xi(\mathrm{P}_\xi x).$$

Since ψ is C^2, $I_\xi A_\xi(x)\mathrm{P}_\xi$ is continuous with respect to ξ and x, respectively.

Equation (5.2) gives for all $s \in \xi$

$$|s|^2 = -\log c_{m,p^*} + |\Psi_\xi(s)|^{p^*} - \log\det(d\Psi_\xi(s)). \tag{5.5}$$

Define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \int_{G_{n,m}} I_\xi\Psi_\xi(\mathrm{P}_\xi x)\alpha(\xi)d\mu(\xi), \quad x \in \mathbb{R}^n. \tag{5.6}$$

The differential of T is given by

$$dT(x) = \int_{G_{n,m}} I_\xi d\Psi_\xi(\mathrm{P}_\xi x)\mathrm{P}_\xi\alpha(\xi)d\mu(\xi) = \int_{G_{n,m}} I_\xi A_\xi(x)\mathrm{P}_\xi\alpha(\xi)d\mu(\xi), \tag{5.7}$$

for $x \in \mathbb{R}^n$.

Since $I_\xi A_\xi(x)\mathrm{P}_\xi\alpha(\xi)$ is positive semi-definite and continuous with respect to ξ, the product of its positive eigenvalues, $\det(A_\xi(x)\alpha(\xi))$, has a positive minimum over $G_{n,m}$. Therefore, the matrix $dT(x)$ is positive definite since from the Grassmannian Ball–Barthe inequality (4.5),

$$\det\left(\int_{G_{n,m}} I_\xi A_\xi(x)\mathrm{P}_\xi\alpha(\xi)\,d\mu(\xi)\right) \geqslant \exp\left(\int_{G_{n,m}} \log\left(\det(A_\xi(x)\alpha(\xi))\right) d\mu(\xi)\right) > 0.$$

In particular, for all $y \neq 0$ in $\mathbb{R}^n$, $y \cdot dT(x)y > 0$, so T is injective.

Moreover, it follows from (5.6) and Lemma 4.1 with $f_\xi = I_\xi \Psi_\xi(\mathrm{P}_\xi x)$ that, for $x \in \mathbb{R}^n$,

$$\|Tx\|^{p^*}_{Z_{m,p,\alpha}} \leqslant \int_{G_{n,m}} |\Psi_\xi(\mathrm{P}_\xi x)|^{p^*}\,d\mu(\xi). \tag{5.8}$$

By (2.7), (3.3), (5.5), (3.4), the Grassmannian Ball–Barthe inequality (4.5), (5.7), (5.8), the change of variables $y = Tx$, and (2.6), we have

$$\begin{aligned}
\pi^{\frac{n}{2}} &= \int_{\mathbb{R}^n} e^{-|x|^2}\,dx \\
&= \int_{\mathbb{R}^n} \exp\left\{-\int_{G_{n,m}} |\mathrm{P}_\xi x|^2 d\mu(\xi)\right\} dx \\
&= \int_{\mathbb{R}^n} \exp\left(-\int_{G_{n,m}} \Big(-\log c_{m,p^*} + |\Psi_\xi(\mathrm{P}_\xi x)|^{p^*} - \log\det(A_\xi(x))\Big) d\mu(\xi)\right) dx \\
&= (c_{m,p^*})^{\frac{n}{m}} \int_{\mathbb{R}^n} \exp\left(-\int_{G_{n,m}} |\Psi_\xi(\mathrm{P}_\xi x)|^{p^*}\,d\mu(\xi)\right) \exp\left(\int_{G_{n,m}} \log\alpha(\xi)^{-m} d\mu(\xi)\right) \\
&\quad \times \exp\left(\int_{G_{n,m}} \log\left(\det(\alpha(\xi)A_\xi(x))\right) d\mu(\xi)\right) dx \\
&\leqslant (c_{m,p^*})^{\frac{n}{m}} \int_{\mathbb{R}^n} \exp\left(-\|Tx\|^{p^*}_{Z_{m,p,\alpha}}\right) \exp\left(-m\int_{G_{n,m}} \log\alpha(\xi)d\mu(\xi)\right) \det(dT(x))\,dx \\
&\leqslant (c_{m,p^*})^{\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi)d\mu(\xi)\right)^{-m} \int_{\mathbb{R}^n} e^{-\|y\|^{p^*}_{Z_{m,p,\alpha}}}\,dy \\
&= (c_{m,p^*})^{\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi)d\mu(\xi)\right)^{-m} \Gamma\left(1+\frac{n}{p^*}\right)|Z_{m,p,\alpha}|,
\end{aligned}$$

which is the desired inequality (5.1).

Case $p = 1$: Define probability densities $f, g : \mathbb{R}^m \to (0, \infty)$ by

$$f(s) = \frac{1}{\omega_m \Gamma(1+\frac{m}{2})} e^{-|s|^2},$$

and

$$g(s) = \frac{1}{\omega_m} \mathbf{1}_{B_2^m(1)}(s),$$

where $B_2^m(1)$ is the unit open ball in ξ.

By Lemma 4.3, there exists a convex function ψ on $\mathbb{R}^m$ such that for $s \in \mathbb{R}^m$, $\nabla\psi : \mathbb{R}^m \to B_2^m(1)$ and

$$-|s|^2 = \log\Gamma\Big(1+\frac{m}{2}\Big) + \log\det(\nabla^2\psi(s)). \tag{5.9}$$

For this ψ, let $\Psi_\xi : \xi \to \xi$ be the map defined by (5.3).

Define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ as in (5.6). The differential $dT(x)$ given in (5.7) is positive definite for $x \in \mathbb{R}^n$, so T is injective. Since $|\Psi_\xi| < 1$, it follows from Lemma 4.1 and (5.6) that $\|Tx\|_{Z_{m,1,\alpha}} \leqslant 1$ for all $x \in \mathbb{R}^n$, which means that $Tx \in Z_{m,1,\alpha}$ for all $x \in \mathbb{R}^n$. Hence, we have

$$T(\mathbb{R}^n) \subseteq Z_{m,1,\alpha}. \tag{5.10}$$

Let $A_\xi(x) = d\Psi_\xi(\mathrm{P}_\xi x)$. By (2.7), (3.3), (5.9), (3.4), the Grassmannian Ball–Barthe inequality (4.5), (5.7), the change of variables $y = Tx$, and (5.10), we have

$$\begin{aligned}
\pi^{\frac{n}{2}} &= \int_{\mathbb{R}^n} e^{-|x|^2}dx \\
&= \int_{\mathbb{R}^n} \exp\left\{-\int_{G_{n,m}} |\mathrm{P}_\xi x|^2 d\mu(\xi)\right\}dx \\
&= \int_{\mathbb{R}^n} \exp\left(\int_{G_{n,m}} \left[\log\Gamma\left(1+\frac{m}{2}\right) + \log\det(A_\xi(x))\right] d\mu(\xi)\right) dx \\
&= \left(\Gamma\Big(1+\frac{m}{2}\Big)\right)^{\frac{n}{m}} \exp\left(\int_{G_{n,m}} \log\alpha(\xi)^{-m} d\mu(\xi)\right) \\
&\quad \times \int_{\mathbb{R}^n} \exp\left(\int_{G_{n,m}} \log\left(\det(\alpha(\xi)A_\xi(x))\right) d\mu(\xi)\right) dx \\
&\leqslant \left(\Gamma\Big(1+\frac{m}{2}\Big)\right)^{\frac{n}{m}} \exp\left(-m\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right) \int_{\mathbb{R}^n} \det(dT(x))dx \\
&\leqslant \left(\Gamma\Big(1+\frac{m}{2}\Big)\right)^{\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m} \int_{Z_{m,1,\alpha}} dy \\
&= \left(\Gamma\Big(1+\frac{m}{2}\Big)\right)^{\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m} |Z_{m,1,\alpha}|.
\end{aligned}$$

Therefore, (5.1) holds when $p = 1$.

For the equality conditions of inequality (5.1), it is sufficient to consider the equality conditions of the Grassmannian Ball–Barthe inequality (Lemma 4.2), which show that for fixed $x \in \mathbb{R}^n$ the product

$$\alpha(\xi_1)\lambda_{k_1}(\xi_1)\cdots\alpha(\xi_n)\lambda_{k_n}(\xi_n)$$

as a function of $\xi_1, \dots, \xi_n \in G_{n,m}$ is constant if and only if there are n linearly independent eigenvectors belonging to eigenvalues $\lambda_{k_i}(\xi_i)$ of $A_{\xi_i}(x)$, $k_i = 1, \dots m$, and $i = 1, \dots, n$. The rest of the proof is exactly the same as that of [**23**, Theorem 8.2]. That is, when n is divisible by m and $p \neq 2$, equality in (5.1) holds if and only if μ is a cross measure on $G_{n,m}$. ☐

6. *Volume inequalities for* $Z^*_{m,p,\alpha}$

THEOREM 6.1. *Suppose* $0 < p \leqslant \infty$ *and* α *is a continuous positive function on* $G_{n,m}$. *If* μ *is an isotropic measure on* $G_{n,m}$, *then*

$$|Z^*_{m,p,\alpha}| \leqslant \gamma_n(m,p)\left(\exp\int_{G_{n,m}} \log\alpha(\xi)d\mu(\xi)\right)^{-m}, \tag{6.1}$$

where $\gamma_n(m,p)$ *is given by* (1.11). *When* n *is divisible by* m *and* $p \neq 2$, *there is equality if and only if* μ *is a cross measure on* $G_{n,m}$.

Proof. Case $0<p<\infty$: For $\xi \in G_{n,m}$, define probability densities $f, g : \mathbb{R}^m \to (0,\infty)$ by

$$f(s) = \frac{\alpha(\xi)^m}{\omega_m\Gamma(1+\frac{m}{p})}e^{-\alpha^p(\xi)|s|^p},$$

$$g(s) = \frac{1}{\omega_m\Gamma(1+\frac{m}{2})}e^{-|s|^2}.$$

By Lemma 4.3, there exists a convex function ψ on $\mathbb{R}^m$ such that for $s \in \mathbb{R}^m$

$$c_{m,p}\alpha(\xi)^m e^{-\alpha^p(\xi)|s|^p} = e^{-|\nabla\psi(s)|^2}\det(\nabla^2\psi(s)),$$

where $c_{m,p} = \Gamma(1+\frac{m}{2})/\Gamma(1+\frac{m}{p})$, and thus,

$$\alpha^p(\xi)|s|^p = \log c_{m,p} + m\log\alpha(\xi) + |\Psi_\xi(s)|^2 - \log\det(d\Psi_\xi(s)), \tag{6.2}$$

where $\Psi_\xi(s)$ and $d\Psi_\xi(s)$ are given by (5.3) and (5.4), respectively.

Define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \int_{G_{n,m}} I_\xi\Psi_\xi(\mathrm{P}_\xi x)d\mu(\xi), \quad x \in \mathbb{R}^n.$$

The differential of T is given by

$$dT(x) = \int_{G_{n,m}} I_\xi d\Psi_\xi(\mathrm{P}_\xi x)\mathrm{P}_\xi d\mu(\xi) = \int_{G_{n,m}} I_\xi A_\xi(x)\mathrm{P}_\xi d\mu(\xi), \tag{6.3}$$

for $x \in \mathbb{R}^n$, where $A_\xi(x) = d\Psi_\xi(\mathrm{P}_\xi x)$. The matrix $dT(x)$ is positive definite and thus T is injective.

By (2.6), (3.15), (6.2), (3.4), the Grassmannian Ball–Barthe inequality (4.5), (6.3), inequality (4.4) with $f_\xi = I_\xi\Psi_\xi(\mathrm{P}_\xi x)$, the change of variables $y = Tx$, and (2.7), we have

$$\begin{aligned}
\Gamma\left(1+\frac{n}{p}\right)|Z^*_{m,p,\alpha}| &= \int_{\mathbb{R}^n} e^{-\|x\|^p_{Z^*_{m,p,\alpha}}}dx \\
&= \int_{\mathbb{R}^n}\exp\left(-\int_{G_{n,m}}|\mathrm{P}_\xi x|^p\alpha^p(\xi)d\mu(\xi)\right)dx \\
&= \int_{\mathbb{R}^n}\exp\left(-\int_{G_{n,m}}\big(\log c_{m,p} + m\log\alpha(\xi) + |\Psi_\xi(\mathrm{P}_\xi x)|^2 - \log\det(A_\xi(x))\big)\,d\mu(\xi)\right)dx \\
&= (c_{m,p})^{-\frac{n}{m}}\exp\left(-m\int_{G_{n,m}}\log\alpha(\xi)d\mu(\xi)\right)\int_{\mathbb{R}^n}\exp\left(-\int_{G_{n,m}}|\Psi_\xi(\mathrm{P}_\xi x)|^2\,d\mu(\xi)\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \exp\left(\int_{G_{n,m}} \log\det(A_\xi(x))\, d\mu(\xi)\right) dx \\
&\leqslant (c_{m,p})^{-\frac{n}{m}} \exp\left(-m\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right) \\
&\quad\times \int_{\mathbb{R}^n} \exp\left(-\int_{G_{n,m}} |\Psi_\xi(\mathrm{P}_\xi x)|^2\, d\mu(\xi)\right) \det(dT(x))\, dx \\
&\leqslant (c_{m,p})^{-\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m} \int_{\mathbb{R}^n} \exp\left(-|Tx|^2\right) \det(dT(x))\, dx \\
&\leqslant (c_{m,p})^{-\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m} \int_{\mathbb{R}^n} e^{-|y|^2} dy \\
&= (c_{m,p})^{-\frac{n}{m}} \pi^{\frac{n}{2}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m},
\end{aligned}
$$

which is the desired inequality (6.1).

Case $p = \infty$: For $\xi \in G_{n,m}$, define probability densities $f, g : \mathbb{R}^m \to (0,\infty)$ by

$$f(s) = \frac{\alpha^m(\xi)}{\omega_m} \mathbf{1}_{B_2^m(1)}(\alpha(\xi)s)$$

and

$$g(s) = \frac{1}{\omega_m \Gamma(1+\frac{m}{2})} e^{-|s|^2}.$$

By Lemma 4.3, there exists a convex function ψ on $\mathbb{R}^m$ such that for $s \in \mathbb{R}^m$

$$\Gamma\left(1+\frac{m}{2}\right)\alpha^m(\xi)\mathbf{1}_{B_2^m(1)}(\alpha(\xi)s) = e^{-|\nabla\psi(s)|^2} \det(\nabla^2\psi(s)). \tag{6.4}$$

By (3.16), we have

$$\operatorname{int} Z^*_{m,\infty,\alpha} = \left\{x \in \mathbb{R}^n : \max_{\xi\in\operatorname{supp}\mu} |\mathrm{P}_\xi x|\alpha(\xi) < 1\right\}.$$

Thus, for each $x \in \operatorname{int} Z^*_{m,\infty,\alpha}$,

$$\exp\left\{\int_{\operatorname{supp}\mu} \log \mathbf{1}_{B_2^m(1)}(\alpha(\xi)\mathrm{P}_\xi x)\, d\mu(\xi)\right\} = 1. \tag{6.5}$$

For $\xi \in G_{n,m}$, define $\Psi_\xi : \xi \to \xi$ by (5.3). Define the map $T : \operatorname{int} Z^*_{m,\infty,\alpha} \to \mathbb{R}^n$ by

$$Tx = \int_{G_{n,m}} I_\xi \Psi_\xi(\mathrm{P}_\xi x) d\mu(\xi), \quad x \in \mathbb{R}^n.$$

The differential dT is given by

$$dT(x) = \int_{G_{n,m}} I_\xi d\Psi_\xi(\mathrm{P}_\xi x)\mathrm{P}_\xi d\mu(\xi) = \int_{G_{n,m}} I_\xi A_\xi(x)\mathrm{P}_\xi d\mu(\xi), \tag{6.6}$$

for $x \in \mathbb{R}^n$, where $A_\xi(x) = d\Psi_\xi(\mathrm{P}_\xi x)$. Since $dT(x)$ is positive definite for each $x \in \operatorname{int} Z^*_{m,\infty}$, the map T is injective.

By (6.5), (6.4), (3.4), the Grassmannian Ball–Barthe inequality (4.5), (6.6), inequality (4.4) with $f_\xi = I_\xi \Psi_\xi(\mathrm{P}_\xi x)$, the change of variables $y = Tx$, and (2.7), we have

$$
\begin{aligned}
&|Z^*_{m,\infty,\alpha}| \\
&= \int_{\operatorname{int} Z^*_{m,\infty,\alpha}} \exp\left(\int_{\operatorname{supp}\mu} \log \mathbf{1}_{B_2^m(1)}(\alpha(\xi)\mathrm{P}_\xi x)\, d\mu(\xi)\right) dx \\
&= \int_{\operatorname{int} Z^*_{m,\infty,\alpha}} \exp\left(\int_{\operatorname{supp}\mu} \log\left[\Gamma\Big(1+\frac{m}{2}\Big)^{-1} \alpha^{-m}(\xi) e^{-|\Psi_\xi(\mathrm{P}_\xi x)|^2} \det(A_\xi(x))\right] d\mu(\xi)\right) dx \\
&= \Gamma\Big(1+\frac{m}{2}\Big)^{-\frac{n}{m}} \int_{\operatorname{int} Z^*_{m,\infty,\alpha}} \exp\left(\int_{\operatorname{supp}\mu} \log \alpha^{-m}(\xi) d\mu(\xi)\right) \exp\left(-\int_{\operatorname{supp}\mu} |\Psi_\xi(\mathrm{P}_\xi x)|^2 d\mu(\xi)\right) \\
&\quad \times \exp\left(\int_{\operatorname{supp}\mu} \log\det(A_\xi(x))\, d\mu(\xi)\right) dx \\
&\leqslant \Gamma\Big(1+\frac{m}{2}\Big)^{-\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m} \\
&\quad \times \int_{\operatorname{int} Z^*_{m,\infty,\alpha}} \exp\left(-\int_{\operatorname{supp}\mu} |\Psi_\xi(\mathrm{P}_\xi x)|^2 d\mu(\xi)\right) \det(dT(x))\, dx \\
&\leqslant \Gamma\Big(1+\frac{m}{2}\Big)^{-\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m} \int_{\operatorname{int} Z^*_{m,\infty,\alpha}} \exp\left(-|Tx|^2\right) \det(dT(x))\, dx \\
&\leqslant \Gamma\Big(1+\frac{m}{2}\Big)^{-\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m} \int_{\mathbb{R}^n} e^{-|y|^2}\, dy \\
&= \omega_m^{\frac{n}{m}} \left(\exp\int_{G_{n,m}} \log\alpha(\xi) d\mu(\xi)\right)^{-m}.
\end{aligned}
$$

The proof of the equality conditions is exactly the same as that of Theorem 8.2 in [**23**]. □

7. *Properties of* $h_{K,m,p}$ *and* $g_{K,m,p}$

Recall that the function $h_{K,m,p} : G_{n,m} \to (0,\infty)$ is defined, for $K \in \mathcal{K}^n_o$ and $\xi \in G_{n,m}$, by

$$
h_{K,m,p}(\xi) = \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}} \int_{S^{n-1}} |\mathrm{P}_\xi v|^p dS_p(K,v)\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \tag{7.1}
$$

and

$$
h_{K,m,\infty}(\xi) = \lim_{p\to\infty} h_{K,m,p}(\xi) = \frac{1}{\gamma_n(m,1)^{\frac{1}{n}}} \max_{v\in\operatorname{supp} S_K} \{|\mathrm{P}_\xi v|/h_K(v)\}, \quad p=\infty. \tag{7.2}
$$

The function $g_{K,m,p} : G_{n,m} \to (0,\infty)$ is defined, for $K \in \mathcal{S}^n_o$ and $\xi \in G_{n,m}$, by

$$
g_{K,m,p}(\xi) = \left(\frac{n+p}{m|K|}\int_K |\mathrm{P}_\xi y|^p dy\right)^{\frac{1}{p}} = \left(\frac{1}{m|K|}\int_{S^{n-1}} |\mathrm{P}_\xi v|^p \rho_K^{n+p}(v) dv\right)^{\frac{1}{p}}, \quad 0<p<\infty, \tag{7.3}
$$

and

$$g_{K,m,\infty}(\xi) = \lim_{p\to\infty} g_{K,m,p}(\xi) = \max_{v\in S^{n-1}}\{|\mathrm{P}_\xi v|\rho_K(v)\}, \quad p=\infty. \tag{7.4}$$

These two functions have the following properties.

LEMMA 7.1. *The functions $h_{K,m,p}$ and $g_{K,m,p}$ are continuous on $G_{n,m}$.*

Proof. For $1 \leqslant p < \infty$, let $r_K = \min_{v\in S^{n-1}} h_K(v)$ for $K \in \mathcal{K}_o^n$. Note that $dS_p(K,\cdot) = h_K^{1-p}(\cdot)dS_K(\cdot)$. It is clear that $|\mathrm{P}_\xi v|^p h_K^{1-p}(v) \leqslant r_K^{1-p}$ for each $\xi\in G_{n,m}, v\in S^{n-1}$. The dominated convergence theorem and (7.1) yield that the function $h_{K,m,p}$ is continuous on $G_{n,m}$.

For the case $p=\infty$, let $\xi_1,\xi_2\in G_{n,m}$ and $v_1,v_2 \in \operatorname{supp} S_K$ be such that

$$|\mathrm{P}_{\xi_i} v_i|/h_K(v_i) = \max_{v\in \operatorname{supp} S_K}\{|\mathrm{P}_{\xi_i} v|/h_K(v)\}.$$

Then

$$\begin{aligned}\frac{|\mathrm{P}_{\xi_1}v_2| - |\mathrm{P}_{\xi_2}v_2|}{R_K} &\leqslant \frac{|\mathrm{P}_{\xi_1}v_2| - |\mathrm{P}_{\xi_2}v_2|}{h_K(v_2)} \leqslant \frac{|\mathrm{P}_{\xi_1}v_1|}{h_K(v_1)} - \frac{|\mathrm{P}_{\xi_2}v_2|}{h_K(v_2)}\\ &\leqslant \frac{|\mathrm{P}_{\xi_1}v_1| - |\mathrm{P}_{\xi_2}v_1|}{h_K(v_1)} \leqslant \frac{|\mathrm{P}_{\xi_1}v_1| - |\mathrm{P}_{\xi_2}v_1|}{r_K},\end{aligned}$$

where $R_K = \max_{v\in S^{n-1}} h_K(v)$. Hence, $h_{K,m,\infty}$ is continuous on $G_{n,m}$.

The continuity of $g_{K,m,p}$ follows from a similar argument. □

LEMMA 7.2. *Suppose $1\leqslant p\leqslant\infty$, $K\in\mathcal{K}_o^n$ and $O\in \mathrm{O}(n)$. Then for $\xi\in G_{n,m}$*

$$h_{OK,m,p}(\xi) = h_{K,m,p}(O^{-1}\xi),$$

and for $c>0$,

$$h_{cK,m,p}(\xi) = c^{\frac{n-p}{p}} h_{K,m,p}(\xi). \tag{7.5}$$

Proof. For $\xi\in G_{n,m}$, choose an orthonormal basis $\{\varepsilon_1,\dots,\varepsilon_m\}$ of ξ. It follows from (7.1) and [**31**, Proposition 1.2] that, for $1\leqslant p<\infty$,

$$\begin{aligned}h_{OK,m,p}(\xi) &= \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}} |\mathrm{P}_\xi v|^p dS_p(OK,v)\right)^{\frac1p}\\ &= \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}} |\mathrm{P}_\xi (O^{-t}v)|^p dS_p(K,v)\right)^{\frac1p}\\ &= \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}} \left(\sum_{i=1}^m |\varepsilon_i\cdot O^{-t}v|^2\right)^{\frac{p}{2}} dS_p(K,v)\right)^{\frac1p}\\ &= \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}} \left(\sum_{i=1}^m |O^{-1}\varepsilon_i\cdot v|^2\right)^{\frac{p}{2}} dS_p(K,v)\right)^{\frac1p}\\ &= \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}} |\mathrm{P}_{O^{-1}\xi} v|^p dS_p(K,v)\right)^{\frac1p}\\ &= h_{K,m,p}(O^{-1}\xi).\end{aligned}$$

Equality (7.5) follows from (7.1) and (2.9).

The case $p = \infty$ follows from a similar argument. □

LEMMA 7.3. *Suppose $0 < p \leqslant \infty$, $K \in \mathcal{S}_o^n$ and $O \in \mathrm{O}(n)$. Then for $\xi \in G_{n,m}$*

$$g_{OK,m,p}(\xi) = g_{K,m,p}(O^t\xi),$$

and for $c > 0$,

$$g_{cK,m,p}(\xi) = c g_{K,m,p}(\xi). \tag{7.6}$$

Proof. For $\xi \in G_{n,m}$, choose an orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_m\}$ of ξ. It follows from (7.3) that, for $0 < p < \infty$,

$$\begin{aligned}
g_{OK,m,p}(\xi) &= \left(\frac{n+p}{m|OK|}\int_{OK} |\mathrm{P}_\xi y|^p dy\right)^{\frac{1}{p}} \\
&= \left(\frac{n+p}{m|K|}\int_K \left(\sum_{i=1}^m |\varepsilon_i \cdot Oy|^2\right)^{\frac{p}{2}} dy\right)^{\frac{1}{p}} \\
&= \left(\frac{n+p}{m|K|}\int_K \left(\sum_{i=1}^m |O^t\varepsilon_i \cdot y|^2\right)^{\frac{p}{2}} dy\right)^{\frac{1}{p}} \\
&= \left(\frac{n+p}{m|K|}\int_K |\mathrm{P}_{O^t\xi} y|^p dy\right)^{\frac{1}{p}} \\
&= g_{K,m,p}(O^t\xi).
\end{aligned}$$

It is clear that (7.6) holds.

The case $p = \infty$ follows from a similar argument. □

The following theorem gives an upper bound for $g_{K,m,p}$, which is critical for Theorem 1.3.

THEOREM 7.4. *Suppose $p \geqslant 1$ and K is an origin-symmetric convex body in $\mathbb{R}^n$. Then for $\xi \in G_{n,m}$*

$$g_{K,m,p}(\xi) \leqslant \frac{|K|^{\frac{1}{m}}[(n+p)B(m+p, n-m+1)]^{\frac{1}{p}}}{(\omega_m|K \cap \xi^\perp|)^{\frac{1}{m}}[mB(m, n-m+1)]^{\frac{1}{p}+\frac{1}{m}}}. \tag{7.7}$$

When $m = 1$, there is equality if and only if K is a double cone in the direction ξ.

Proof. By Fubini's theorem, we have

$$\int_K |\mathrm{P}_\xi y|^p dy = \int_\xi |x|^p \varphi_\xi(x) dx,$$

where $\varphi_\xi(x) = |K \cap (x + \xi^\perp)|$. By the Brunn–Minkowski inequality, the function $(\varphi_\xi(x))^{\frac{1}{n-m}}$ is concave. Since $\varphi_\xi(0) = \max_{x\in\xi} \varphi_\xi(x)$, we see that the following function of r,

$$f_u(r) = 1 - \left(\frac{\varphi_\xi(ru)}{\varphi_\xi(0)}\right)^{\frac{1}{n-m}}, \quad u \in S^{m-1} \subset \xi,$$

is convex and $f_u(0) = 0$, and thus, $f_u(r)/r$ is increasing.

Let a be defined by

$$\int_\xi h(a|x|)dx = \int_\xi h\left(1 - \left(\frac{\varphi_\xi(x)}{\varphi_\xi(0)}\right)^{\frac{1}{n-m}}\right)dx,$$

where $h(t) = [(1-t)_+]^{n-m}$, and $t_+ = \max\{t, 0\}$. Let

$$G(t) = \int_t^{+\infty} r^{m-1} \int_{S^{m-1}} h(ar) - h(f_u(r))dudr,$$

for every $t > 0$. By the definition of a, we have $G(0) = G(+\infty) = 0$. Since $f_u(r)/r$ is increasing and h is decreasing, it follows that G is first increasing and then decreasing. Therefore, $G(t) \geqslant 0$ for every $t > 0$.

By using polar coordinates, we obtain

$$\begin{aligned}
\int_\xi |x|^p \frac{\varphi_\xi(x)}{\varphi_\xi(0)} dx &= \int_0^{+\infty} r^{m+p-1} \int_{S^{m-1}} h(f_u(r))dudr \\
&= p\int_0^{+\infty} t^{p-1} \int_t^{+\infty} r^{m-1} \int_{S^{m-1}} h(f_u(r))dudrdt \\
&\leqslant p\int_0^{+\infty} t^{p-1} \int_t^{+\infty} r^{m-1} \int_{S^{m-1}} h(ar)dudrdt \\
&= \int_0^{+\infty} r^{m+p-1} \int_{S^{m-1}} h(ar)dudr \\
&= m\omega_m \int_0^{\frac{1}{a}} r^{m+p-1}(1-ar)^{n-m}dr \\
&= m\omega_m a^{-m-p} B(m+p, n-m+1).
\end{aligned}$$

On the other hand, by the definition of a and using polar coordinates again, we have

$$\begin{aligned}
\frac{|K|}{|K\cap\xi^\perp|} &= \int_\xi \frac{\varphi_\xi(x)}{\varphi_\xi(0)} dx = \int_\xi h(a|x|)dx \\
&= m\omega_m \int_0^{+\infty} r^{m-1}[(1-ar)_+]^{n-m}dr \\
&= m\omega_m a^{-m} B(m, n-m+1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_K |\mathrm{P}_\xi y|^p dy &= \int_\xi |x|^p \varphi_\xi(x)dx \\
&\leqslant \varphi_\xi(0) m\omega_m a^{-m-p} B(m+p, n-m+1) \\
&= |K\cap\xi^\perp| m\omega_m \left(\frac{|K|}{|K\cap\xi^\perp| m\omega_m B(m, n-m+1)}\right)^{\frac{m+p}{m}} B(m+p, n-m+1) \\
&= \frac{|K|^{\frac{m+p}{m}} B(m+p, n-m+1)}{(m\omega_m|K\cap\xi^\perp|)^{\frac{p}{m}} B(m, n-m+1)^{\frac{m+p}{m}}},
\end{aligned}$$

which gives inequality (7.7).

Equality in the above inequality holds if and only if $G(t) = 0$ for every $t > 0$, which implies $\int_{S^{m-1}} h(at) - h(f_u(t))du = 0$ for every $t > 0$. Thus, we have

$$m\omega_m(1-at)_+^{n-m} = \int_{S^{m-1}} h(at)du = \int_{S^{m-1}} h(f_u(t))du$$
$$= \int_{S^{m-1}} \frac{\varphi_\xi(tu)}{\varphi_\xi(0)}du = \frac{1}{|K\cap\xi^\perp|}\int_{S^{m-1}} |K\cap(tu+\xi^\perp)|du.$$

That is,

$$\left(\frac{1}{n\omega_n}\int_{S^{m-1}} |K\cap(tu+\xi^\perp)|du\right)^{\frac{1}{n-m}} = |K\cap\xi^\perp|^{\frac{1}{n-m}}(1-at)_+,$$

for every $t > 0$. In particular, when $m = 1$, the section $|K\cap(tu+\xi^\perp)|^{\frac{1}{n-1}}$ is an affine function of t; that is, K is a double cone in the direction ξ. $\square$

The case $m = 1$ of Theorem 7.4 was established by Fradelizi [**16**, Theorem 3, 4] for convex bodies with their centroids at the origin and by Milman and Pajor [**36**, Corollary 2.7] for origin-symmetric convex bodies.

8. *The Grassmannian Loomis–Whitney inequality*

THEOREM 8.1. *Suppose $1 \leqslant p \leqslant \infty$ and $K \in \mathcal{K}_o^n$. If μ is an isotropic measure on $G_{n,m}$, then*

$$|K|^{\frac{n-p}{p}} \leqslant \left(\exp\int_{G_{n,m}} \log h_{K,m,p}(\xi)d\mu(\xi)\right)^m. \tag{8.1}$$

When n is divisible by m and $p \neq 2$, there is equality if and only if μ is a cross measure on $G_{n,m}$ and K is a weighted ℓ^n_{m,p^*}-ball formed by μ (up to translations when $p = 1$).

Proof. *Case* $1 \leqslant p < \infty$: Since $h_{K,m,p}$ is continuous on $G_{n,m}$, let

$$\alpha(\xi) = h^{-1}_{K,m,p}(\xi) \tag{8.2}$$

for $\xi \in G_{n,m}$. From (2.10), (2.8), the definitions of $Z_{m,p,\alpha}$ (3.13) and $h_{K,m,p}$ (7.1), Fubini's theorem, and (3.4), we have

$$|K|^{n-p} \leqslant |Z_{m,p,\alpha}|^{-p}V_p(K, Z_{m,p,\alpha})^n = |Z_{m,p,\alpha}|^{-p}\left(\frac{1}{n}\int_{S^{n-1}} h^p_{Z_{m,p,\alpha}}(v)dS_p(K,v)\right)^n$$
$$= |Z_{m,p,\alpha}|^{-p}\left(\frac{1}{n}\int_{S^{n-1}}\int_{G_{n,m}} |\mathrm{P}_\xi v|^p\alpha^p(\xi)d\mu(\xi)dS_p(K,v)\right)^n$$
$$= |Z_{m,p,\alpha}|^{-p}\left(\frac{1}{n}\int_{G_{n,m}}\left(\int_{S^{n-1}} |\mathrm{P}_\xi v|^p dS_p(K,v)\right)\alpha^p(\xi)d\mu(\xi)\right)^n$$
$$= |Z_{m,p,\alpha}|^{-p}\left(\frac{m}{n}\int_{G_{n,m}} \gamma_n(m,p^*)^{\frac{p}{n}} h^p_{K,m,p}(\xi)\alpha^p(\xi)d\mu(\xi)\right)^n$$
$$= |Z_{m,p,\alpha}|^{-p}\gamma_n(m,p^*)^p.$$

Combining this with Theorem 5.1 and (8.2) yields

$$
\begin{aligned}
|K|^{n-p} \leqslant |Z_{m,p,\alpha}|^{-p}\gamma_n(m,p^*)^p &\leqslant \left[\gamma_n(m,p^*)\left(\exp\int_{G_{n,m}}\log\alpha(\xi)d\mu(\xi)\right)^m\right]^{-p}\gamma_n(m,p^*)^p \\
&= \left(\exp\int_{G_{n,m}}\log\alpha(\xi)d\mu(\xi)\right)^{-pm} \\
&= \left(\exp\int_{G_{n,m}}\log h_{K,m,p}(\xi)d\mu(\xi)\right)^{pm}, \qquad (8.3)
\end{aligned}
$$

which is the desired inequality.

By the L_p Minkowski inequality (2.10), equality in the first inequality of (8.3) holds if and only if K and $Z_{m,p,\alpha}$ are dilates when $p>1$ (K and $Z_{m,p,\alpha}$ are homothetic when $p=1$). Theorem 5.1 implies that equality in the second inequality of (8.3) holds if and only if μ is a cross measure on $G_{n,m}$ when $1\leqslant p\neq 2$ and n/m is an integer. Thus, by (3.17), $Z_{m,p,\alpha}$ is a weighted ℓ^n_{m,p^*}-ball, $B^n_{m,p^*,\alpha^{-1}}$, formed by μ. Hence, K is a dilation of a weighted ℓ^n_{m,p^*}-ball formed by the cross measure μ, which is still a weighted ℓ^n_{m,p^*}-ball formed by μ when $1<p\neq 2$ (K is the weighted $\ell^n_{m,\infty}$-ball formed by μ up to translations when $p=1$).

Conversely, when $1<p\neq 2$ and $n/m=l$ is an integer, we will show that equalities in (8.3) hold if K is a weighted ℓ^n_{m,p^*}-ball formed by μ; that is, there are positive numbers $(\alpha_i)_{i=1}^l$ such that

$$
K=\left\{x\in\mathbb{R}^n:\left(\sum_{i=1}^l|\mathrm{P}_{\tilde{\xi}_i}x|^{p^*}\alpha_i^{p^*}\right)^{\frac{1}{p^*}}\leqslant 1\right\}, \qquad (8.4)
$$

where μ with $\operatorname{supp}\mu=\{\tilde{\xi}_1,\dots,\tilde{\xi}_l\}=O\{\xi_1,\dots,\xi_l\}$ is a cross measure on $G_{n,m}$, $O\in\mathrm{O}(n)$. By (8.3), it is sufficient to verify that K and $Z_{m,p,\alpha}$ are dilates. From (3.10) and (3.11), we have

$$
K=OA^{-1}B^n_{m,p^*},
$$

where $A=\operatorname{diag}\{\underbrace{\alpha_1,\dots,\alpha_1}_{m},\dots,\underbrace{\alpha_l,\dots,\alpha_l}_{m}\}$ is a diagonal matrix. Thus, by (8.2), (7.1), (2.9), [**31**, Proposition 1.2], and (7.1) again, we get

$$
\begin{aligned}
&\alpha(\tilde{\xi}_i)\\
&= h^{-1}_{K,m,p}(\tilde{\xi}_i) = h^{-1}_{(OA^{-1}B^n_{m,p^*}),m,p}(\tilde{\xi}_i)\\
&= \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}}|\mathrm{P}_{\tilde{\xi}_i}v|^p dS_p(|A^{-1}|^{1/n}O(A^{-1}/|A^{-1}|^{1/n})B^n_{m,p^*},v)\right)^{-\frac{1}{p}}\\
&= |A^{-1}|^{\frac{p-n}{np}}\left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}}|\mathrm{P}_{\tilde{\xi}_i}(OA^{-1}/|A^{-1}|^{1/n})^{-t}v|^p dS_p(B^n_{m,p^*},v)\right)^{-\frac{1}{p}}\\
&= |A^{-1}|^{\frac{p-n}{np}}\left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}}\int_{S^{n-1}}\left(\sum_{j=1}^m|(OA^{-1}/|A^{-1}|^{1/n})^{-1}Oe_{(i-1)m+j}\cdot v|^2\right)^{\frac{p}{2}}dS_p(B^n_{m,p^*},v)\right)^{-\frac{1}{p}}
\end{aligned}
$$

$$= |A^{-1}|^{-\frac{1}{p}} \left(\frac{1}{m\gamma_n(m,p^*)^{\frac{p}{n}}} \int_{S^{n-1}} \left(\sum_{j=1}^{m} |Ae_{(i-1)m+j} \cdot v|^2 \right)^{\frac{p}{2}} dS_p(B^n_{m,p^*}, v) \right)^{-\frac{1}{p}}$$

$$= |A|^{\frac{1}{p}} h^{-1}_{B^n_{m,p^*},m,p}(\xi_i)\alpha_i^{-1},$$

for every $i = 1, \ldots, l$. Note that $h^{-1}_{B^n_{m,p^*},m,p}(\xi_i)$ is a constant for all $i = 1, \ldots, l$. Thus, there exists a constant $c > 0$ such that $\alpha(\tilde{\xi}_i) = c\alpha_i^{-1}$ for every $i = 1, \ldots, l$. Now, it follows from (3.17) and (8.4) that

$$Z_{m,p,\alpha} = B^n_{m,p^*,\alpha^{-1}} = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{p^*} \alpha(\tilde{\xi}_i)^{-p^*} \right)^{\frac{1}{p^*}} \leqslant 1 \right\}$$

$$= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{p^*} c^{-p^*} \alpha_i^{p^*} \right)^{\frac{1}{p^*}} \leqslant 1 \right\} = cK.$$

That is, K and $Z_{m,p,\alpha}$ are dilates when $1 < p \neq 2$. When $p = 1$, the proof is similar, together with the easy observation that $h_{K+v_0,m,1} = h_{K,m,1}$ for every $v_0 \in \mathbb{R}^n$.

Case $p = \infty$: Since $h_{K,m,\infty}$ is continuous on $G_{n,m}$, let

$$\alpha(\xi) = h^{-1}_{K,m,\infty}(\xi) \tag{8.5}$$

for $\xi \in G_{n,m}$. From (2.11), the definitions of $Z_{m,\infty,\alpha}$ (3.14), and $h_{K,m,\infty}$ (7.2), we have

$$|K| \geqslant |Z_{m,\infty,\alpha}| \left[\max_{v \in \operatorname{supp} S_K} \{ h_{Z_{m,\infty,\alpha}}(v)/h_K(v) \} \right]^{-n}$$

$$= |Z_{m,\infty,\alpha}| \left[\max_{v \in \operatorname{supp} S_K} \left\{ \max_{\xi \in \operatorname{supp} \mu} (|\mathrm{P}_\xi v| \alpha(\xi))/h_K(v) \right\} \right]^{-n}$$

$$= |Z_{m,\infty,\alpha}| \left[\max_{\xi \in \operatorname{supp} \mu} \left\{ \max_{v \in \operatorname{supp} S_K} (|\mathrm{P}_\xi v|/h_K(v)) \alpha(\xi) \right\} \right]^{-n}$$

$$= |Z_{m,\infty,\alpha}| \gamma_n(m,1)^{-1} \left[\max_{\xi \in \operatorname{supp} \mu} \{ h_{K,m,\infty}(\xi)\alpha(\xi) \} \right]^{-n}$$

$$= |Z_{m,\infty,\alpha}| \gamma_n(m,1)^{-1}.$$

Combining this with Theorem 5.1 and (8.5) yields

$$|K| \geqslant |Z_{m,\infty,\alpha}| \gamma_n(m,1)^{-1} \geqslant \left(\exp \int_{G_{n,m}} \log \alpha(\xi) d\mu(\xi) \right)^m$$

$$= \left(\exp \int_{G_{n,m}} \log h_{K,m,\infty}(\xi) d\mu(\xi) \right)^{-m}, \tag{8.6}$$

which is the desired inequality.

By the L_∞ Minkowski inequality (2.11), equality in the first inequality of (8.6) holds if and only if K and $Z_{m,\infty,\alpha}$ are dilates. Theorem 5.1 implies that equality in the second inequality of (8.6) holds if and only if μ is a cross measure on $G_{n,m}$ when n/m is an integer. Thus by (3.17),

$Z_{m,\infty,\alpha}$ is a weighted $\ell^n_{m,1}$-ball, $B^n_{m,1,\alpha^{-1}}$, formed by μ. Hence, K is a dilation of a weighted $\ell^n_{m,1}$-ball formed by the cross measure μ, which is still a weighted $\ell^n_{m,1}$-ball formed by μ.

Conversely, when $n/m = l$ is an integer, let

$$K = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x| \alpha_i \leqslant 1 \right\}, \tag{8.7}$$

where $(\alpha_i)_{i=1}^l > 0$ are the weights of a cross measure μ on $G_{n,m}$ with $\operatorname{supp}\mu = \{\tilde{\xi}_1, \dots, \tilde{\xi}_l\} = O\{\xi_1, \dots, \xi_l\}$. To show that equalities in (8.6) hold, it is sufficient to verify that K and $Z_{m,\infty,\alpha}$ are dilates. From (3.10), we have

$$K = OA^{-1} B^n_{m,1},$$

where $A = \operatorname{diag}\{\underbrace{\alpha_1, \dots, \alpha_1}_{m}, \dots, \underbrace{\alpha_l, \dots, \alpha_l}_{m}\}$ is a diagonal matrix. Then by (8.5), (7.5), (7.2), [**31**, Proposition 1.2]($p = \infty$), and (7.2) again, we get

$$\begin{aligned}
&\alpha(\tilde{\xi}_i) = h^{-1}_{K,m,\infty}(\tilde{\xi}_i) = h^{-1}_{(OA^{-1}B^n_{m,1}),m,\infty}(\tilde{\xi}_i) \\
&= h^{-1}_{(|A^{-1}|^{1/n} O(A^{-1}/|A^{-1}|^{1/n}) B^n_{m,1}),m,\infty}(\tilde{\xi}_i) \\
&= |A^{-1}|^{1/n} \left(\frac{1}{\gamma_n(m,1)^{\frac{1}{n}}} \max_{v \in \operatorname{supp} S_{O(A^{-1}/|A^{-1}|^{1/n}) B^n_{m,1}}} \left\{ |\mathrm{P}_{\tilde{\xi}_i} v| / h_{B^n_{m,1}}\left(\left(OA^{-1}/|A^{-1}|^{1/n}\right)^t v\right) \right\} \right)^{-1} \\
&= |A^{-1}|^{1/n} \left(\frac{1}{\gamma_n(m,1)^{\frac{1}{n}}} \max_{v \in \operatorname{supp} S_{B^n_{m,1}}} \left\{ |\mathrm{P}_{\tilde{\xi}_i} \left(OA^{-1}/|A^{-1}|^{1/n}\right)^{-t} v| / h_{B^n_{m,1}}(v) \right\} \right)^{-1} \\
&= |A^{-1}|^{1/n} \\
&\times \left(\frac{1}{\gamma_n(m,1)^{\frac{1}{n}}} \max_{v \in \operatorname{supp} S_{B^n_{m,1}}} \left\{ \left(\sum_{j=1}^{m} \left| \left(OA^{-1}/|A^{-1}|^{1/n}\right)^{-1} O e_{(i-1)m+j} \cdot v \right|^2 \right)^{\frac{1}{2}} / h_{B^n_{m,1}}(v) \right\} \right)^{-1} \\
&= \left(\frac{1}{\gamma_n(m,1)^{\frac{1}{n}}} \max_{v \in \operatorname{supp} S_{B^n_{m,1}}} \left\{ \left(\sum_{j=1}^{m} |A e_{(i-1)m+j} \cdot v|^2 \right)^{\frac{1}{2}} / h_{B^n_{m,1}}(v) \right\} \right)^{-1} \\
&= h^{-1}_{B^n_{m,1},m,\infty}(\xi_i) \alpha_i^{-1},
\end{aligned}$$

for every $i = 1, \dots, l$. Note that $h^{-1}_{B^n_{m,1},m,p}(\xi_i)$ is a constant for all $i = 1, \dots, l$. Thus, there exists a constant $c > 0$ such that $\alpha(\tilde{\xi}_i) = c\alpha_i^{-1}$ for every $i = 1, \dots, l$. Now, it follows from (3.17) and (8.7) that

$$\begin{aligned}
Z_{m,\infty,\alpha} = B^n_{m,1,\alpha^{-1}} &= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x| \alpha(\tilde{\xi}_i)^{-1} \leqslant 1 \right\} \\
&= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x| c^{-1} \alpha_i \leqslant 1 \right\} = cK.
\end{aligned}$$

That is, K and $Z_{m,\infty,\alpha}$ are dilates. ∎

When $m=1$, inequality (8.1) reduces to inequality (1.5). Note that the case $p=\infty$ of inequality (1.5) was established by Lv [**33**], which, together with (2.4) and the fact (by (1.7), (2.5) and (2.4)) that

$$|B_1^n|^{\frac{1}{n}}\Pi_\infty K = K^*$$

for each origin-symmetric convex body K, yields the following inequality due to Lv [**33**]: if K is an origin-symmetric convex body in $\mathbb{R}^n$ and ν is an even isotropic measure on S^{n-1}, then

$$|K| \geqslant \frac{2^n}{n!}\exp\left\{\int_{S^{n-1}}\log\rho_K(u)d\nu(u)\right\},$$

with equality if and only if ν is a cross measure on S^{n-1} and K is a weighted ℓ_1^n-ball formed by ν.

9. *The Grassmannian dual Loomis–Whitney inequality*

THEOREM 9.1. *Suppose $0<p\leqslant\infty$ and $K\in\mathcal{S}_o^n$. If μ is an even isotropic measure on $G_{n,m}$, then*

$$|K| \leqslant \gamma_n(m,p)\left(\exp\int_{G_{n,m}}\log g_{K,m,p}(\xi)d\mu(\xi)\right)^m. \tag{9.1}$$

When n is divisible by m and $p\neq 2$, there is equality if and only if μ is a cross measure on $G_{n,m}$ and K is a weighted $\ell_{m,p}^n$-ball formed by μ.

Proof. *Case* $0<p<\infty$: Since $g_{K,m,p}$ is continuous on $G_{n,m}$, let

$$\alpha(\xi) = |K|^{-\frac{1}{p}}g_{K,m,p}^{-1}(\xi) \tag{9.2}$$

for $\xi\in G_{n,m}$. From (2.13), (2.12), (2.4), the definition of $Z_{m,p,\alpha}^*$ (3.15), Fubini's theorem, (3.4), and (7.3), we have

$$\begin{aligned}
|K|^{n+p} &\leqslant |Z_{m,p,\alpha}^*|^p\widetilde{V}_{-p}(K,Z_{m,p,\alpha}^*)^n = |Z_{m,p,\alpha}^*|^p\left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n+p}(v)\rho_{Z_{m,p,\alpha}^*}^{-p}(v)dv\right)^n\\
&= |Z_{m,p,\alpha}^*|^p\left(\frac{1}{n}\int_{S^{n-1}}\left(\int_{G_{n,m}}|\mathrm{P}_\xi v|^p\alpha^p(\xi)d\mu(\xi)\right)\rho_K^{n+p}(v)dv\right)^n\\
&= |Z_{m,p,\alpha}^*|^p\left(\frac{1}{n}\int_{G_{n,m}}\left(\int_{S^{n-1}}|\mathrm{P}_\xi v|^p\rho_K^{n+p}(v)dv\right)\alpha^p(\xi)d\mu(\xi)\right)^n\\
&= |Z_{m,p,\alpha}^*|^p\left(\frac{m}{n}\int_{G_{n,m}}|K|g_{K,m,p}^p(\xi)\alpha^p(\xi)d\mu(\xi)\right)^n\\
&= |Z_{m,p,\alpha}^*|^p.
\end{aligned}$$

Combining this with Theorem 6.1, (9.2), and (3.4) yields

$$|K|^{\frac{n+p}{p}} \leqslant |Z_{m,p,\alpha}^*| \leqslant \gamma_n(m,p)\left(\exp\int_{G_{n,m}}\log\alpha(\xi)d\mu(\xi)\right)^{-m}$$

$$
\begin{aligned}
&= \gamma_n(m,p)\left(\exp\int_{G_{n,m}} \log(|K|^{\frac{1}{p}} g_{K,m,p}(\xi))d\mu(\xi)\right)^m \\
&= \gamma_n(m,p)|K|^{\frac{n}{p}}\left(\exp\int_{G_{n,m}} \log g_{K,m,p}(\xi)d\mu(\xi)\right)^m, \qquad (9.3)
\end{aligned}
$$

which is the desired inequality.

By (2.13), equality in the first inequality of (9.3) holds if and only if K and $Z^*_{m,p,\alpha}$ are dilates. Theorem 6.1 shows that equality in the second inequality of (9.3) holds if and only if μ is a cross measure on $G_{n,m}$ when $0 < p \neq 2$ and n/m is an integer. Thus, by (3.18), $Z^*_{m,p,\alpha}$ is a weighted $\ell^n_{m,p}$-ball, $B^n_{m,p,\alpha}$, formed by μ. Therefore, equality in (9.3) holds if and only if K is a dilation of a weighted $\ell^n_{m,p}$-ball formed by the cross measure μ, which is still a weighted $\ell^n_{m,p}$-ball formed by μ.

Conversely, when $0 < p \neq 2$ and $n/m = l$ is an integer, we will show that equalities in (9.3) hold if K is a weighted $\ell^n_{m,p}$-ball formed by the cross measure μ; that is, there are positive numbers $(\alpha_i)_{i=1}^l$ such that

$$
K = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^p \alpha_i^p \right)^{\frac{1}{p}} \leqslant 1 \right\},
$$

where μ with $\operatorname{supp}\mu = \{\tilde{\xi}_1, \ldots, \tilde{\xi}_l\} = O\{\xi_1, \ldots, \xi_l\}$ is a cross measure on $G_{n,m}$ with weights $(\alpha_i)_{i=1}^l$, $O \in \mathrm{O}(n)$. By (9.3), it is sufficient to verify that K and $Z^*_{m,p,\alpha}$ are dilates. From (3.10), we have

$$
K = OA^{-1}B^n_{m,p},
$$

where $A = \operatorname{diag}\{\underbrace{\alpha_1, \ldots, \alpha_1}_{m}, \ldots, \underbrace{\alpha_l, \ldots, \alpha_l}_{m}\}$ is a diagonal matrix. Thus, by (9.2) and (7.3), we get

$$
\begin{aligned}
\alpha(\tilde{\xi}_i)^{-p} &= |K| g^p_{K,m,p}(\tilde{\xi}_i) \\
&= \frac{n+p}{m}\int_K |\mathrm{P}_{\tilde{\xi}_i} y|^p dy = \frac{n+p}{m}\int_{OA^{-1}B^n_{m,p}} |\mathrm{P}_{\tilde{\xi}_i} y|^p dy \\
&= \frac{n+p}{m}\int_{OA^{-1}B^n_{m,p}} \left(\sum_{j=1}^m |Oe_{(i-1)m+j}\cdot y|^2\right)^{\frac{p}{2}} dy \\
&= \frac{n+p}{m}|\det A|^{-1}\int_{B^n_{m,p}} \left(\sum_{j=1}^m |(OA^{-1})^t Oe_{(i-1)m+j}\cdot y|^2\right)^{\frac{p}{2}} dy \\
&= \frac{n+p}{m}|\det A|^{-1}\alpha_i^{-p}\int_{B^n_{m,p}} \left(\sum_{j=1}^m |e_{(i-1)m+j}\cdot y|^2\right)^{\frac{p}{2}} dy \\
&= \frac{n+p}{m}|\det A|^{-1}\alpha_i^{-p}\int_{B^n_{m,p}} |\mathrm{P}_{\xi_i} y|^p dy,
\end{aligned}
$$

for every $i = 1, \ldots, l$. Note that $\int_{B^n_{m,p}} |\mathrm{P}_{\xi_i} y|^p dy$ is a constant for all $i = 1, \ldots, l$. Thus, there exists a constant $c > 0$ such that $\alpha(\tilde{\xi}_i) = c\alpha_i$ for every $i = 1, \ldots, l$. Recall that

$$\begin{aligned} Z^*_{m,p,\alpha} &= \left\{ x \in \mathbb{R}^n : \left(\int_{G_{n,m}} |\mathrm{P}_\xi x|^p \alpha^p(\xi) d\mu(\xi) \right)^{\frac{1}{p}} \leqslant 1 \right\} \\ &= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^p \alpha^p(\tilde{\xi}_i) \right)^{\frac{1}{p}} \leqslant 1 \right\} \\ &= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^p c^p \alpha_i^p \right)^{\frac{1}{p}} \leqslant 1 \right\} = c^{-1} K. \end{aligned}$$

That is, K and $Z^*_{m,p,\alpha}$ are dilates when $p \neq 2$.

Case $p = \infty$: Since $g_{K,m,\infty}$ is continuous on $G_{n,m}$, let

$$\alpha(\xi) = g^{-1}_{K,m,\infty}(\xi) \tag{9.4}$$

for $\xi \in G_{n,m}$. From (2.14), (2.4), the definitions of $Z^*_{m,\infty,\alpha}$ (3.16), and $g_{K,m,\infty}$ (7.4), we have

$$\begin{aligned} |K| &\leqslant |Z^*_{m,\infty,\alpha}| \left[\max_{v \in S^{n-1}} \{ \rho_K(v) / \rho_{Z^*_{m,\infty,\alpha}}(v) \} \right]^n \\ &= |Z^*_{m,\infty,\alpha}| \left[\max_{v \in S^{n-1}} \left\{ \max_{\xi \in \operatorname{supp} \mu} (\rho_K(v) |\mathrm{P}_\xi v| \alpha(\xi)) \right\} \right]^n \\ &= |Z^*_{m,\infty,\alpha}| \left[\max_{\xi \in \operatorname{supp} \mu} \left\{ \max_{v \in S^{n-1}} (|\mathrm{P}_\xi v| \rho_K(v)) \alpha(\xi) \right\} \right]^n \\ &= |Z^*_{m,\infty,\alpha}| \left[\max_{\xi \in \operatorname{supp} \mu} \{ g_{K,m,\infty}(\xi) \alpha(\xi) \} \right]^n \\ &= |Z^*_{m,\infty,\alpha}|. \end{aligned}$$

Combining this with Theorem 6.1 and (9.4) yields

$$\begin{aligned} |K| \leqslant |Z^*_{m,\infty,\alpha}| &\leqslant \gamma_n(m,\infty) \left(\exp \int_{G_{n,m}} \log \alpha(\xi) d\mu(\xi) \right)^{-m} \\ &= \gamma_n(m,\infty) \left(\exp \int_{G_{n,m}} \log g_{K,m,\infty}(\xi) d\mu(\xi) \right)^{m}, \end{aligned} \tag{9.5}$$

which is the desired inequality.

By (2.14), equality in the first inequality of (9.5) holds if and only if K and $Z^*_{m,\infty,\alpha}$ are dilates. Theorem 6.1 implies that equality in the second inequality of (9.5) holds if and only if μ is a cross measure on $G_{n,m}$ when n/m is an integer. Thus, by (3.18), $Z^*_{m,\infty,\alpha}$ is a weighted $\ell^n_{m,\infty}$-ball, $B^n_{m,\infty,\alpha}$, formed by μ. Hence, K is a dilation of a weighted $\ell^n_{m,\infty}$-ball formed by the cross measure μ, which is still a weighted $\ell^n_{m,\infty}$-ball formed by μ.

Conversely, when $n/m = l$ is an integer, let

$$K = \left\{ x \in \mathbb{R}^n : |\mathrm{P}_{\tilde{\xi}_i} x| \alpha_i \leqslant 1 \ \text{ for all } i = 1, \ldots, l \right\},$$

where $(\alpha_i)_{i=1}^l > 0$ are the weights of a cross measure μ on $G_{n,m}$ with $\operatorname{supp}\mu = \{\tilde{\xi}_1, \dots, \tilde{\xi}_l\} = O\{\xi_1, \dots, \xi_l\}$. To show that equalities in (9.5) hold, it is sufficient to verify that K and $Z^*_{m,\infty,\alpha}$ are dilates. From (3.11), we have

$$K = OA^{-1}B^n_{m,\infty},$$

where $A = \operatorname{diag}\{\underbrace{\alpha_1, \dots, \alpha_1}_{m}, \dots, \underbrace{\alpha_l, \dots, \alpha_l}_{m}\}$ is a diagonal matrix. Thus, by (9.4) and (7.4), we get

$$\begin{aligned}\alpha(\tilde{\xi}_i)^{-1} &= g_{K,m,\infty}(\tilde{\xi}_i) = g_{(OA^{-1}B^n_{m,\infty}),m,\infty}(\tilde{\xi}_i) \\ &= \max_{v\in S^{n-1}} \{|\mathrm{P}_{\tilde{\xi}_i} v|\rho_{B^n_{m,\infty}}((OA^{-1})^{-1}v) \\ &= \max_{v\in S^{n-1}} \left\{ \left(\sum_{j=1}^m |(OA^{-1})^t O e_{(i-1)m+j} \cdot v|^2 \right)^{\frac{1}{2}} \rho_{B^n_{m,\infty}}(v) \right\} \\ &= \alpha_i^{-1} g_{B^n_{m,\infty},m,\infty}(\xi_i),\end{aligned}$$

for every $i = 1, \dots, l$. Note that $g_{B^n_{m,\infty},m,\infty}(\xi_i)$ is a constant for all $i = 1, \dots, l$. Hence, there exists a constant $c > 0$ such that $\alpha(\tilde{\xi}_i) = c\alpha_i$ for every $i = 1, \dots, l$. Thus,

$$\begin{aligned}Z^*_{m,\infty,\alpha} &= \left\{x \in \mathbb{R}^n : |\mathrm{P}_{\tilde{\xi}_i} x|\alpha(\tilde{\xi}_i) \leqslant 1, \text{ for all } i = 1, \dots, l\right\} \\ &= \left\{x \in \mathbb{R}^n : |\mathrm{P}_{\tilde{\xi}_i} x| c\alpha_i \leqslant 1, \text{ for all } i = 1, \dots, l\right\} = c^{-1}K.\end{aligned}$$

That is, K and $Z^*_{m,\infty,\alpha}$ are dilates. □

When $m = 1$, inequality (9.1) reduces to inequality (1.6). Note that the case $p = \infty$ of inequality (1.6) was established by Lv [**33**], which, together with (2.4) and the fact (by (1.9) and (2.5)) that

$$\Gamma_\infty K = K$$

for each origin-symmetric convex body K, yields the following inequality due to Lv [**33**]: if K is an origin-symmetric convex body in $\mathbb{R}^n$ and ν is an even isotropic measure on S^{n-1}, then

$$|K| \leqslant 2^n \exp\left\{\int_{S^{n-1}} \log h_K(u) d\nu(u)\right\},$$

with equality if and only if ν is a cross measure on S^{n-1} and K is a weighted ℓ^n_∞-ball formed by ν.

Proof of Theorem 1.3. It is easy to see that the desired inequality (1.16) follows from (9.1), (7.7), and (3.4). So, by the equality condition of (9.1), it follows that when $m = 1$ and $1 \leqslant p \neq 2$, μ is a cross measure on $G_{n,1}$ and K is a generalized ℓ^n_p-ball formed by μ. The equality conditions of (7.7) yield that when $m = 1$, K is a double cone in the direction of the support of μ. Obviously, only the generalized ℓ^n_1-ball formed by the cross measure μ is satisfied. □

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Ai-Jun Li
School of Mathematics and Information Science
Henan Polytechnic University
Gaoxin District, Shiji Road 2001
Jiaozuo 454000
China

liaijun72@163.com

Qingzhong Huang
College of Mathematics, Physics and Information Engineering
Jiaxing University
No. 56, South Yuexiu RD
Jiaxing 314001
China

and

Department of Mathematics and Statistics
Memorial University of Newfoundland
230 Elizabeth Ave
St. John's Newfoundland
Canada A1C 5S7

hqz376560571@163.com

Dongmeng Xi
Department of Mathematics
Shanghai University
Shanghai 200444
China

dongmeng.xi@live.com