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Sections and Projections of L_p-Zonoids and Their Polars

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Abstract Let $1 \leq k \leq n$. Sharp upper and lower bounds for the volume of k-dimensional projections (or sections) of L_p-zonoids (or their polars) with even isotropic generating measures are established. As special cases, sharp volume inequalities for k-dimensional sections and projections of ℓ_p^n-balls are recovered. The necessary conditions of equalities are new.

Keywords L_p-zonoid · Section · Projection · ℓ_p^n-ball

Mathematics Subject Classification 52A40

1 Introduction

We shall use $|\cdot|$ to denote k-dimensional volume (Lebesgue measure on the corresponding subspace), and $\|\cdot\|$ for the standard Euclidean norm on $\mathbb{R}^k$, $k = 1, \ldots, n$. For $1 \leq p \leq \infty$, let B_p^n denote the unit ball of the ℓ_p^n-space, that is,

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$$B_p^n = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{n} |x \cdot e_i|^p \right)^{\frac{1}{p}} \leq 1 \right\}, \quad 1 \leq p < \infty,$$
$$B_\infty^n = \{x \in \mathbb{R}^n : |x \cdot e_i| \leq 1, \ i = 1, \ldots, n\}, \quad p = \infty,$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$ and $x \cdot e_i$ denotes the standard inner product of x and e_i.

Estimating the volumes of sections and projections of ℓ_p^n-balls has attracted much attention. In [18], Hensley showed that if H is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ then

$$1 \leq \frac{\left|H \cap B_\infty^n\right|}{\left|B_\infty^{n-1}\right|} \leq 5,$$

and conjectured that the best upper bound is $\sqrt{2}$. Ball [2] solved this conjecture and further proved that [3] if H is a k-dimensional subspaces, then the upper bound is $(\sqrt{2})^{n-k}$, being best possible for some H if $n \geq 2(n-k)$. He also showed that if k divides n, the upper bound $(\frac{n}{k})^{\frac{k}{2}}$ is best possible. However, when $k < n/2$ and k is not a divisor of n, the best upper bound is still unknown. The lower bound for k-dimensional subspaces was obtained by Vaaler [44] who showed that it is always bigger than 1. His method was pushed further by Meyer and Pajor [37] who proved that for any k-dimensional subspace H of $\mathbb{R}^n$, $1 \leq k \leq n$, and for every $1 \leq q \leq p \leq \infty$,

$$\frac{\left|H \cap B_q^n\right|}{\left|B_q^k\right|} \leq \frac{\left|H \cap B_p^n\right|}{\left|B_p^k\right|}. \tag{1}$$

The Meyer–Pajor inequality (1) immediately yields an upper bound for sections of B_p^n for $1 \leq p \leq 2$. The case $2 \leq p \leq \infty$ was established by Barthe [6]:

$$\frac{\left|H \cap B_p^n\right|}{\left|B_p^k\right|} \leq \begin{cases} 1, & 1 \leq p \leq 2, \\ \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p}\right)}, & 2 \leq p \leq \infty. \end{cases} \tag{2}$$

Especially, if k divides n, the constant in (2) is best possible for $2 \leq p \leq \infty$.

The dual version of the Meyer–Pajor inequality for $(n-1)$-dimensional subspaces was established by Barthe and Naor [7]: Let $1 \leq q \leq p \leq \infty$ and let P_H be the orthogonal projection onto an $(n-1)$-dimensional subspace H of $\mathbb{R}^n$. Then

$$\frac{|\mathrm{P}_H B_q^n|}{|B_q^{n-1}|} \leq \frac{|\mathrm{P}_H B_p^n|}{|B_p^{n-1}|}. \tag{3}$$

For the orthogonal projection P_H onto a k-dimensional subspace H of $\mathbb{R}^n$, Barthe [6] showed that

$$\frac{\left|\mathrm{P}_H B_p^n\right|}{\left|B_p^k\right|} \geq \begin{cases} \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p}\right)}, & 1 \leq p \leq 2, \\ 1, & 2 \leq p \leq \infty. \end{cases} \tag{4}$$

Especially, if k divides n, the constant in (4) is best possible for $1 \leq p \leq 2$. For more information about sections and projections of ℓ_p^n-balls, see, e.g., [8,9,12,13,19,36,39,40,46].

The results described above motivated us to investigate the volumes of sections and projections of more general convex bodies than ℓ_p^n-balls. We will consider L_p-zonoids and their polars, which were introduced by Schneider and Weil [42], and thoroughly investigated by Lutwak et al. [32]. Note that ℓ_p^n-balls are special cases of polars of L_p-zonoids.

A Borel measure (always assumed to be nonnegative and finite) μ on S^{n-1} is *isotropic* if

$$\int_{S^{n-1}} u \otimes u \, \mathrm{d}\mu(u) = I_n, \tag{5}$$

where $u \otimes u$ is the rank-one orthogonal projection onto the space spanned by u and I_n is the identity map on $\mathbb{R}^n$. Note that it is impossible for an isotropic measure to be concentrated on a proper subspace of $\mathbb{R}^n$. The measure μ is said to be *even* if it assumes the same value on antipodal sets. A *cross measure* on S^{n-1}, as a special case of even isotropic measures, is concentrated uniformly on $\{\pm u_1, \ldots, \pm u_n\}$, where $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbb{R}^n$.

Assume that μ is an even Borel measure on S^{n-1}, whose support, $\operatorname{supp} \mu$, is not contained in a subsphere of S^{n-1}. For each $1 \leq p \leq \infty$, define the L_p-zonoid $Z_p = Z_p(\mu)$ (with generating measure μ) to be the origin-symmetric convex body in $\mathbb{R}^n$ whose support function, for $x \in \mathbb{R}^n$, is given by

$$h_{Z_p}(x) = \left(\int_{S^{n-1}} |x \cdot u|^p \mathrm{d}\mu(u)\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and for $p = \infty$, is given by

$$h_{Z_\infty}(x) = \lim_{p \to \infty} h_{Z_p}(x) = \max_{u \in \operatorname{supp} \mu} |x \cdot u|.$$

The bodies Z_1 are usually called zonoids, and are precisely the class of bodies obtained as limits of Minkowski sums of line segments. L_p-zonoids belong to the L_p Brunn–Minkowski theory, beginning largely with [26,27], and expanding rapidly thereafter (see, e.g., [1,10,11,15–17,20,23–35,38,45]).

A series of papers by Ball [4], Barthe [5], and Lutwak et al. [32] provide sharp upper and lower estimates for the volumes of L_p-zonoids Z_p and their polars Z_p^*, under the assumption that their generating measure is even and isotropic. In this paper, we will establish the sharp upper and lower bounds for the volume of k-dimensional projections (or sections) of L_p-zonoids (or their polars) with even isotropic generating measures.

Denote by ω_n the volume of the Euclidean unit ball B_2^n in $\mathbb{R}^n$. For $1 \le k \le n$, define $c_{p,k}$ by

$$c_{p,k} = \left(\frac{\Gamma\left(1+\frac{k}{2}\right)\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(\frac{k+p}{2}\right)} \right)^{\frac{k}{p}}, \ 1 \le p < \infty,$$

and define $c_{\infty,k} = \lim_{p\to\infty} c_{p,k} = 1$. For $1 \le p \le \infty$, let p^* denote the Hölder conjugate of p; i.e., $1/p + 1/p^* = 1$. If μ is a Borel measure on S^{n-1}, then the restriction of μ to a Borel set $A \subseteq S^{n-1}$ is denoted by $\mu \llcorner A$. Let $H^\perp$ denote the orthogonal complement of a subspace H of $\mathbb{R}^n$. The main results of this paper are the following. The case $k = n$ was proved by Lutwak et al. [32].

Theorem 1.1 *Let $1 \le k \le n$ and let H be a k-dimensional subspace of $\mathbb{R}^n$. If μ is an even isotropic measure on S^{n-1}, then*

$$\frac{\omega_k}{c_{p,k}} \left(\frac{\mu(S^{n-1} \setminus H^\perp)}{k} \right)^{-\frac{k}{p}} \le |H \cap Z_p^*(\mu)| \le \begin{cases} |B_p^k|, & 1 \le p \le 2, \\ \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p}\right)} |B_p^k|, & 2 \le p \le \infty, \end{cases} \tag{6}$$

and

$$\omega_k c_{p,k} \left(\frac{\mu(S^{n-1} \setminus H^\perp)}{k} \right)^{\frac{k}{p}} \ge |\mathrm{P}_H Z_p(\mu)| \ge \begin{cases} |B_{p^*}^k|, & 1 \le p \le 2, \\ \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p^*}\right)} |B_{p^*}^k|, & 2 \le p \le \infty. \end{cases} \tag{7}$$

For $1 \le p < 2$, there is equality in the right side inequalities of (6) or (7) if and only if μ is concentrated on $H \cup H^\perp$ and $\mu \llcorner (S^{n-1} \cap H)$ is a cross measure on $S^{n-1} \cap H$. For $2 < p \le \infty$, there is equality in the right side inequalities of (6) or (7) if and only if there is an orthonormal basis $\{\omega_1, \dots, \omega_k\}$ of H such that

$$\mu\left(S^{n-1} \cap (H^\perp + \sqrt{k/n}\,\omega_i)\right) = \mu\left(S^{n-1} \cap (H^\perp - \sqrt{k/n}\,\omega_i)\right) = \frac{n}{2k}, \ i = 1, \dots, k. \tag{8}$$

There is equality in the left side inequalities of (6) or (7) if and only if μ is concentrated on $H \cup H^\perp$ with $\mu(S^{n-1} \setminus H^\perp) = k$ and $H \cap Z_p^*$ is a Euclidean ball in H.

The extremal measures for the inequalities of Theorem 1.1 have intuitive geometric meanings. It is easy to see that, for $1 \le p < 2$, if $\mu \llcorner (S^{n-1} \cap H)$ is a cross measure on $S^{n-1} \cap H$ and μ is concentrated on $H \cup H^\perp$, then $H \cap Z_p^*$ is the ball B_p^k in H up to a rotation. When $2 < p \le \infty$, condition (8) tells us that the extremal measure μ is concentrated on the $2k$ intersections of unit spheres and translations of $H^\perp$. This also implies that $H \cap Z_p^*$ is a dilatation of B_p^k up to a rotation (see Sect. 3 for details). Moreover, some special convex bodies are related to the extremals of the left sides of (6) and (7). For example, when $k = n - 1$, the double cone for $p = 1$, and the cylinder for $p = \infty$.

Our proof of Theorem 1.1, contained in Sect. 3, is based on a refinement of the approach by Lutwak et al. [32], which, in turn, uses ideas of Ball [4] and Barthe [5]. More precisely, we apply the Ball–Barthe inequality and mass transportation. The approach we choose is self-contained and elementary. For more applications about this approach, see e.g., [20–22,32,34,43].

As special cases of Theorem 1.1, in Sect. 4, we will recover inequalities (2) and (4) in Theorems 4.1 and 4.2. We will show that the corresponding constants in (2) and (4) are best possible *only if* k divides n.

2 Background Materials

We list basic facts about convex bodies. As a general reference, we refer the reader to the books of Gardner [14] and Schneider [41].

Throughout this paper, a convex body K in Euclidean n-space $\mathbb{R}^n$ is a compact convex set that contains the origin in its interior. Its polar body K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

The support function of K, $h_K(\cdot) : \mathbb{R}^n \to (0, \infty)$, is defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

The Minkowski functional $\|\cdot\|_K$ of a convex body K is defined by

$$\|x\|_K = \min\{t > 0 : x \in tK\} \tag{9}$$

and related to the support function of K by

$$\|x\|_K = h_{K^*}(x). \tag{10}$$

For each $p \in (0, \infty)$, the volume of K is given by

$$|K| = \frac{1}{\Gamma\left(1 + \frac{n}{p}\right)} \int_{\mathbb{R}^n} e^{-\|x\|_K^p} dx, \tag{11}$$

where the integral is with respect to Lebesgue measure on $\mathbb{R}^n$. In particular, we have

$$\left|B_p^n\right| = \frac{\left(2\Gamma\left(1 + \frac{1}{p}\right)\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)}.$$

The polar coordinate formula for the volume of K will also be used later:

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|u\|_K^{-n} du, \tag{12}$$

where du is the spherical Lebesgue measure on S^{n-1}.

Suppose that $1 \leq k \leq n$ and H is a k-dimensional subspace of $\mathbb{R}^n$. Let μ be a Borel measure on S^{n-1}. If $\bar{\mu}$ is the Borel measure on $S^{n-1} \cap H$ defined by

$$\bar{\mu}(A) = \int_{S^{n-1}\setminus H^{\perp}} \mathbf{1}_A \left(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} \right) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u) \tag{13}$$

for Borel sets $A \subset S^{n-1} \cap H$, then for any continuous function $f : S^{n-1} \cap H \to \mathbb{R}$,

$$\int_{S^{n-1}\cap H} f(w) \mathrm{d}\bar{\mu}(w) = \int_{S^{n-1}\setminus H^{\perp}} f \left(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} \right) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u). \tag{14}$$

If μ is an isotropic measure on S^{n-1}, then for an arbitrary $y \in H$, we have

$$\begin{aligned} \int_{S^{n-1}\cap H} (y \cdot w)^2 \mathrm{d}\bar{\mu}(w) &= \int_{S^{n-1}\setminus H^{\perp}} \left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} \right)^2 \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u) \\ &= \int_{S^{n-1}} (y \cdot u)^2 \mathrm{d}\mu(u) = \|y\|^2. \end{aligned}$$

Hence, $\bar{\mu}$ is isotropic on $S^{n-1} \cap H$. Moreover, we have

$$\bar{\mu}(S^{n-1} \cap H) = k. \tag{15}$$

From the definition of Z_p and (10), it is easy to see that for $1 \leq p < \infty$,

$$H \cap Z_p^* = \left\{ y \in H : \left(\int_{S^{n-1}\setminus H^{\perp}} |y \cdot \mathrm{P}_H u|^p \mathrm{d}\mu(u) \right)^{\frac{1}{p}} \leq 1 \right\}, \tag{16}$$

and for $p = \infty$,

$$H \cap Z_\infty^* = \left\{ y \in H : \sup_{u \in \operatorname{supp} \mu \setminus H^{\perp}} |y \cdot \mathrm{P}_H u| \leq 1 \right\}. \tag{17}$$

Hence, for $1 \leq p < \infty$,

$$\|y\|_{H\cap Z_p^*} = h_{(H\cap Z_p^*)^*}(y) = \left(\int_{S^{n-1}\setminus H^{\perp}} |y \cdot \mathrm{P}_H u|^p \mathrm{d}\mu(u) \right)^{\frac{1}{p}}, \quad y \in H. \tag{18}$$

and for $p = \infty$,

$$\|y\|_{H\cap Z_\infty^*} = h_{(H\cap Z_\infty^*)^*}(y) = \sup_{u \in \operatorname{supp} \mu \setminus H^{\perp}} |y \cdot \mathrm{P}_H u|, \quad y \in H. \tag{19}$$

Let $L_p(\mu)$ denote the usual L_p space with respect to μ. For $f \in L_1(\mu)$ define $f^o \in H$ by

$$f^o = \int_{S^{n-1}\setminus H^\perp} \mathrm{P}_H u \|\mathrm{P}_H u\| f\Big(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\Big) d\mu(u) = \int_{S^{n-1}\cap H} w f(w) d\bar{\mu}(w).$$

The following lemma for $k = n$ was proved by Lutwak et al. [32, Lemma 3.1].

Lemma 2.1 *Suppose that $1 \le k \le n$. Let H be a k-dimensional subspace of $\mathbb{R}^n$ and let μ be an even Borel measure on S^{n-1}. If $f \in L_{p^*}(\mu)$, then*

$$\|f^o\|_{(H\cap Z_p^*)^*} \le \left(\int_{S^{n-1}\setminus H^\perp} \left| f\left(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)\right|^{p^*} \|\mathrm{P}_H u\|^{p^*} d\mu(u)\right)^{\frac{1}{p^*}}. \tag{20}$$

Proof For all $y \in H$, by the Hölder inequality we have

$$\begin{aligned}
|y \cdot f^o| &= \left|\int_{S^{n-1}\setminus H^\perp} (y \cdot \mathrm{P}_H u)\|\mathrm{P}_H u\| f\left(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right) \mathrm{d}\mu(u)\right| \\
&\le \left(\int_{S^{n-1}\setminus H^\perp} \|\mathrm{P}_H u\|^{p^*} \left| f\left(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)\right|^{p^*} \mathrm{d}\mu(u)\right)^{\frac{1}{p^*}} \\
&\quad \times \left(\int_{S^{n-1}\setminus H^\perp} |y \cdot \mathrm{P}_H u|^p \mathrm{d}\mu(u)\right)^{\frac{1}{p}} \\
&= \left(\int_{S^{n-1}\setminus H^\perp} \|\mathrm{P}_H u\|^{p^*} \left| f\left(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)\right|^{p^*} \mathrm{d}\mu(u)\right)^{\frac{1}{p^*}} h_{(H\cap Z_p^*)^*}(y).
\end{aligned}$$

This allows to conclude (20) because

$$\|f^o\|_{(H\cap Z_p^*)^*} = \sup\{|y \cdot f^o| : h_{(H\cap Z_p^*)^*}(y) \le 1, y \in H\}.$$

□

Lemma 2.1 directly implies the following by setting $p = 2$ and taking (14) into account.

Lemma 2.2 ([34]) *If $\bar{\mu}$ is an isotropic measure on $S^{n-1} \cap H$ and $f \in L_2(\bar{\mu})$, then*

$$\left\|\int_{S^{n-1}\cap H} \omega f(w) \mathrm{d}\bar{\mu}(\omega)\right\|^2 \le \int_{S^{n-1}\cap H} f(w)^2 \mathrm{d}\bar{\mu}(w).$$

The following continuous version (including the equality conditions) of the Ball–Barthe inequality is due to Lutwak et al. [32], extending the discrete case due to Ball and Barthe [5, Proposition 9].

Lemma 2.3 *Let $1 \le k \le n$ and let H be a k-dimensional subspace of $\mathbb{R}^n$. If $\bar{\mu}$ is an isotropic measure on $S^{n-1} \cap H$, then for each positive continuous function f on* supp $\bar{\mu}$

$$\exp\left\{\int_{S^{n-1}\cap H} \log f(w)\mathrm{d}\bar{\mu}(w)\right\} \le \det \int_{S^{n-1}\cap H} f(w) w \otimes w \mathrm{d}\bar{\mu}(w), \tag{21}$$

with equality if and only if $f(w_1)\cdots f(w_k)$ is constant for linearly independent unit vectors $w_1, \ldots, w_k \in$ supp $\bar{\mu}$.

3 Proofs of the Main Results

Theorem 3.1 *Let $1 \le k \le n$ and let H be a k-dimensional subspace of $\mathbb{R}^n$. If μ is an even isotropic measure on S^{n-1}, then*

$$|H \cap Z_p^*(\mu)| \le \begin{cases} |B_p^k|, & 1 \le p \le 2, \\ \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p}\right)} \left|B_p^k\right|, & 2 \le p \le \infty. \end{cases} \tag{22}$$

For $1 \le p < 2$, there is equality if and only if μ is concentrated on $H \cup H^\perp$ and $\mu \llcorner (S^{n-1} \cap H)$ is a cross measure on $S^{n-1} \cap H$. For $2 < p \le \infty$, there is equality if and only if there is an orthonormal basis $\{\omega_1, \ldots, \omega_k\}$ of H such that

$$\mu\big(S^{n-1} \cap (H^\perp + \sqrt{k/n}\omega_i)\big) = \mu\big(S^{n-1} \cap (H^\perp - \sqrt{k/n}\omega_i)\big) = \frac{n}{2k}, \quad i = 1, \ldots, k.$$

Proof **Case** $1 \le p < \infty$: For $u \in$ supp $\mu \backslash H^\perp$ and $t \in \mathbb{R}$, define the strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\frac{\|\mathrm{P}_H u\|^{1-\frac{2}{p}}}{\Gamma\left(1+\frac{1}{p}\right)} \int_{-\infty}^{t} e^{-\left|\|\mathrm{P}_H u\|^{1-\frac{2}{p}} s\right|^p} \mathrm{d}s = \frac{1}{\Gamma(\frac{3}{2})} \int_{-\infty}^{\phi(t)} e^{-|s|^2} \mathrm{d}s.$$

Differentiating both sides with respect to t and taking logarithms gives

$$\log\Gamma\left(\frac{3}{2}\right)+\left(1-\frac{2}{p}\right)\log\|\mathrm{P}_H u\|-\log\Gamma\left(1+\frac{1}{p}\right)-\|\mathrm{P}_H u\|^{p-2}|t|^p=-\phi(t)^2+\log\phi'(t). \tag{23}$$

Define $T : H \to H$ by

$$Ty = \int_{S^{n-1}\cap H} w\phi(y \cdot w)\mathrm{d}\bar{\mu}(w) \tag{24}$$

for each $y \in H$, where the measure $\bar{\mu}$ is defined by (13). Thus, the differential of T is given by

$$\mathrm{d}T(y) = \int_{S^{n-1}\cap H} w \otimes w\phi'(y \cdot w)\mathrm{d}\bar{\mu}(w). \tag{25}$$

Since $\phi' > 0$, the matrix $\mathrm{d}T(y)$ is positive definite for each $y \in H$. Hence, the transformation $T\colon H \to H$ is injective. Moreover, from Lemma 2.2 with $f(w) = \phi(y \cdot w)$ and (24), we have

$$\|Ty\|^2 \leq \int_{S^{n-1}\cap H} \phi(y \cdot \omega)^2 \mathrm{d}\bar{\mu}(\omega). \tag{26}$$

The Ball–Barthe inequality (21) with $f(w) = \phi'(y \cdot w)$ and (25) imply that

$$\exp\left\{\int_{S^{n-1}\cap H} \log \phi'(y \cdot \omega) d\bar{\mu}(\omega)\right\} \leq \det(\mathrm{d}T(y)). \tag{27}$$

From (11), (18), (23) with $t = y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}$, (15), (14), (26), and (27), we obtain by making the change of variable $z = Ty$ that

$$\begin{aligned}
\Gamma\left(1+\frac{k}{p}\right)|H \cap Z_p^*| &= \int_H e^{-\|y\|^p_{H\cap Z_p^*}}\mathrm{d}y \\
&= \int_H \exp\left\{-\int_{S^{n-1}\setminus H^\perp} |y \cdot \mathrm{P}_H u|^p \mathrm{d}\mu(u)\right\}\mathrm{d}y \\
&= \int_H \exp\Bigg\{-\int_{S^{n-1}\setminus H^\perp} \Bigg[\phi\left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)^2 \\
&\quad -\log\phi'\left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right) + \log\Gamma\left(\frac{3}{2}\right) \\
&\quad +\left(1-\frac{2}{p}\right)\log\|\mathrm{P}_H u\| - \log\Gamma\left(1+\frac{1}{p}\right)\Bigg]\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\Bigg\}\mathrm{d}y \\
&= \left(\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^k \exp\left\{\int_{S^{n-1}\setminus H^\perp}\left[\left(\frac{2}{p}-1\right)\log\|\mathrm{P}_H u\|\right]\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right\} \\
&\quad \times \int_H \exp\left\{\int_{S^{n-1}\cap H} -\phi(y\cdot\omega)^2\mathrm{d}\bar{\mu}(\omega)\right\} \\
&\quad \times \exp\left\{\int_{S^{n-1}\cap H}\log\phi'(y\cdot\omega)\mathrm{d}\bar{\mu}(\omega)\right\}\mathrm{d}y \\
&\leq \left(\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^k \left(\exp\int_{S^{n-1}\setminus H^\perp}(\log\|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right)^{\frac{2}{p}-1} \\
&\quad \times \int_H e^{-\|Ty\|^2}\det(\mathrm{d}T(y))\mathrm{d}y \\
&\leq \left(\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^k \left(\exp\int_{S^{n-1}\setminus H^\perp}(\log\|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right)^{\frac{2}{p}-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_H e^{-\|z\|^2} \mathrm{d}z \\
&= \left(2\Gamma\big(1+\frac{1}{p}\big)\right)^k \left(\exp \int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right)^{\frac{2}{p}-1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{|H\cap Z_p^*|}{|B_p^k|} &= \frac{\Gamma\big(1+\frac{k}{p}\big)|H\cap Z_p^*|}{\big(2\Gamma\big(1+\frac{1}{p}\big)\big)^k} \\
&\le \left(\exp \int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right)^{\frac{2}{p}-1}. \qquad (28)
\end{aligned}
$$

An application of Jensen's inequality yields

$$
\begin{aligned}
&\exp\left(-\int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right) \\
&= \left[\exp\left(\frac{1}{k}\int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|^{-2})\|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right)\right]^{\frac{k}{2}} \\
&\le \left(\frac{1}{k}\int_{S^{n-1}\setminus H^\perp} \mathrm{d}\mu(u)\right)^{\frac{k}{2}} \le \left(\frac{n}{k}\right)^{\frac{k}{2}}, \qquad (29)
\end{aligned}
$$

which together with the fact that $\|\mathrm{P}_H u\| \le 1$ gives

$$
\left(\frac{k}{n}\right)^{\frac{k}{2}} \le \exp\left(\int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right) \le 1. \qquad (30)
$$

If $1 \le p \le 2$, then from (28) and the right-hand side of inequality (30), we deduce

$$
\frac{|H\cap Z_p^*|}{|B_p^k|} \le 1.
$$

If $2 \le p < \infty$, then from (28) and the left-hand side of inequality (30), we obtain

$$
\frac{|H\cap Z_p^*|}{|B_p^k|} \le \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p}\right)}.
$$

Now we prove the equality conditions. Assume that equality in (28) holds . Since $\bar{\mu}$ is isotropic on $S^{n-1}\cap H$, the measure $\bar{\mu}$ is not concentrated on any subsphere of $S^{n-1}\cap H$. Thus there exist linearly independent $w_1,\dots,w_k \in \operatorname{supp}\bar{\mu}$. Since μ is even, so is $\bar{\mu}$. Hence, we have $\{\pm w_1,\dots,\pm w_k\} \subseteq \operatorname{supp}\bar{\mu}$. Assume that there exists a vector $v \in \operatorname{supp}\bar{\mu}$ such that $v \notin \{\pm w_1,\dots,\pm w_k\}$. Let $v = \lambda_1 w_1 + \cdots + \lambda_k w_k$

where at least one coefficient, say λ_1, is not zero. Then the equality conditions of the Ball-Barthe inequality imply that

$$\phi'(y \cdot w_1)\phi'(y \cdot w_2) \cdots \phi'(y \cdot w_k) = \phi'(y \cdot v)\phi'(y \cdot w_2) \cdots \phi'(y \cdot w_k)$$

for all $y \in H$. But $\phi' > 0$, and hence $\phi'(y \cdot w_1) = \phi'(y \cdot v)$ for all $y \in H$. If $p \neq 2$, then the function ϕ' is not constant. Differentiating both sides with respect to y shows that $\phi''(y \cdot w_1)w_1 = \phi''(y \cdot v)v$ for all $y \in H$. Since there exists $y \in H$ such that $\phi''(y \cdot w_1) \neq 0$ it follows that $v = \pm w_1$. Hence $\{\pm w_1, \ldots, \pm w_k\} = \operatorname{supp} \bar{\mu}$. Therefore, we have for $y \in H$

$$|y|^2 = \sum_{i=1}^{k} \bar{\mu}(\{\pm w_i\})(y \cdot w_i)^2.$$

Substituting $y = w_j \in S^{n-1} \cap H$, we see that necessarily $\bar{\mu}(\{\pm w_j\}) \leq 1$. From the fact that $\sum_{i=1}^{k} \bar{\mu}(\{\pm w_i\}) = k$, we get $\bar{\mu}(\{\pm w_j\}) = 1$. Thus, $w_j \cdot w_i = 0$ for $j \neq i$, and we obtain that $\bar{\mu}$ is a cross measure on $S^{n-1} \cap H$.

When $1 \leq p < 2$, by the right-hand inequality of (30), we have $\|\mathrm{P}_H u\| = 1$ for $u \in \operatorname{supp} \mu \setminus H^{\perp}$. This means that μ concentrates on $H \cup H^{\perp}$. Hence, it follows from (13) that $\operatorname{supp} \mu \cap H = \operatorname{supp} \bar{\mu}$ and thus $\mu \llcorner (S^{n-1} \cap H)$ is a cross measure on $S^{n-1} \cap H$.

Conversely, suppose that μ is concentrated on $H \cup H^{\perp}$ and $\mu \llcorner (S^{n-1} \cap H)$ is a cross measure on $S^{n-1} \cap H$. Without loss of generality, we assume that $\operatorname{supp} \mu \cap H = \{\pm\omega_1, \ldots, \pm\omega_k\}$, where $\{\omega_1, \ldots, \omega_k\}$ is an orthonormal basis. It follows from (16) that

$$\begin{aligned} H \cap Z_p^* &= \left\{ y \in H : \left(\int_{S^{n-1} \setminus H^{\perp}} |y \cdot \mathrm{P}_H u|^p \mathrm{d}\mu(u) \right)^{\frac{1}{p}} \leq 1 \right\} \\ &= \left\{ y \in H : \left(\sum_{i=1}^{k} |y \cdot \omega_i|^p \right)^{\frac{1}{p}} \leq 1 \right\}. \end{aligned}$$

This means that $H \cap Z_p^*$ is isometric to B_p^k, and hence equality in (22) holds for $1 \leq p < 2$.

When $2 < p < \infty$, by the equality conditions of Jensen's inequality, equalities in (29) imply that $\mu(S^{n-1} \cap H^{\perp}) = 0$ and $\|\mathrm{P}_H u\|$ is a constant for each $u \in \operatorname{supp} \mu$. From

$$\int_{S^{n-1} \setminus H^{\perp}} \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u) = n\|\mathrm{P}_H u\|^2 = k,$$

we get $\|\mathrm{P}_H u\| = \sqrt{k/n}$. Thus, μ is concentrated on $\{u \in S^{n-1} : \|\mathrm{P}_H u\| = \sqrt{k/n}\}$. We have proved that $\bar{\mu}$ is a cross measure on $S^{n-1} \cap H$, so set $\operatorname{supp} \bar{\mu} = \{\pm\omega_1, \ldots, \pm\omega_k\}$, where $\{\omega_1, \ldots, \omega_k\}$ is an orthonormal basis on $S^{n-1} \cap H$. Since $\bar{\mu}$ is even and isotropic, this implies that $\bar{\mu}(\{\omega_i\}) = \bar{\mu}(\{-\omega_i\}) = \frac{1}{2}$. For any $A \subset S^{n-1} \cap H$, we have

$$\int_{S^{n-1} \cap \{\|\mathrm{P}_H u\| = \sqrt{k/n}\|\}} \mathbf{1}_A \left(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} \right) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u) = \int_{S^{n-1} \cap H} \mathbf{1}_A(\omega) \mathrm{d}\bar{\mu}(\omega).$$

Letting $A = \{\pm\omega_i\}$, $i \in \{1, \ldots, k\}$, and using that μ is even, we get

$$\mu\big(S^{n-1} \cap (H^\perp + \sqrt{k/n}\omega_i)\big) = \mu\big(S^{n-1} \cap (H^\perp - \sqrt{k/n}\omega_i)\big) = \frac{n}{2k}.$$

Conversely, suppose that there is an orthonormal basis $\{\omega_1, \ldots, \omega_k\}$ of H such that $\mu\big(S^{n-1} \cap (H^\perp + \sqrt{k/n}\omega_i)\big) = \mu\big(S^{n-1} \cap (H^\perp - \sqrt{k/n}\omega_i)\big) = \frac{n}{2k}$, $i = 1, \ldots, k$. Thus,

$$\begin{aligned} H \cap Z_p^* &= \left\{ y \in H : \left(\int_{S^{n-1}} |y \cdot \mathrm{P}_H u|^p d\mu(u) \right)^{\frac{1}{p}} \le 1 \right\} \\ &= \left\{ y \in H : \left(\sum_{i=1}^k \frac{n}{k} \left| y \cdot \sqrt{\frac{k}{n}} \omega_i \right|^p \right)^{\frac{1}{p}} \le 1 \right\} \\ &= \left(\frac{n}{k} \right)^{\frac{1}{2} - \frac{1}{p}} \left\{ y \in H : \left(\sum_{i=1}^k |y \cdot \omega_i|^p \right)^{\frac{1}{p}} \le 1 \right\}. \end{aligned}$$

It follows that equality in (22) holds for $2 < p < \infty$.
Case $p = \infty$: For $u \in \operatorname{supp} \mu \setminus H^\perp$, define the strictly increasing function $\phi : \big(-\frac{1}{\|\mathrm{P}_H u\|}, \frac{1}{\|\mathrm{P}_H u\|}\big) \to \mathbb{R}$ by

$$\|\mathrm{P}_H u\| \int_{-\frac{1}{\|\mathrm{P}_H u\|}}^t \mathbf{1}_{\left[-\frac{1}{\|\mathrm{P}_H u\|}, \frac{1}{\|\mathrm{P}_H u\|}\right]}(s) \mathrm{d}s = \frac{1}{\Gamma(\frac{3}{2})} \int_{-\infty}^{\phi(t)} e^{-s^2} \mathrm{d}s.$$

Differentiating both sides with respect to t and taking logarithms gives

$$\log \Gamma\left(\frac{3}{2}\right) + \log \|\mathrm{P}_H u\| + \log \mathbf{1}_{\left[-\frac{1}{\|\mathrm{P}_H u\|}, \frac{1}{\|\mathrm{P}_H u\|}\right]}(t) = -\phi(t)^2 + \log \phi'(t). \tag{31}$$

By (17) we have

$$H \cap Z_\infty^* = \left\{ y \in H : \sup_{u \in \operatorname{supp} \mu \setminus H^\perp} \left| y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} \right| \le \frac{1}{\|\mathrm{P}_H u\|} \right\}.$$

Thus, for each $y \in H \cap Z_\infty^*$,

$$\exp\left\{\int_{S^{n-1}\setminus H^\perp} \log \mathbf{1}_{\left[-\frac{1}{\|\mathrm{P}_H u\|}, \frac{1}{\|\mathrm{P}_H u\|}\right]}\left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} = 1. \tag{32}$$

Define T : relint$(H \cap Z_\infty^*) \to H$ as in (24), where relint$(H \cap Z_\infty^*)$ is the relative interior of $H \cap Z_\infty^*$. Note that for all $y \in$ relint$(H \cap Z_\infty^*)$ and all $w \in \operatorname{supp} \bar{\mu}$, $y \cdot w$ is in the domain of ϕ. Clearly, the matrix $\mathrm{d}T(y)$ in (25) is positive definite for each $y \in$ relint$(H \cap Z_\infty^*)$ and the transformation T: relint$(H \cap Z_\infty^*) \to H$ is injective.

From (32), (31) with $t = y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}$, (15), (26), and (27), we obtain by making the change of variable $z = Ty$ and (29) that

$$\begin{aligned}
|H \cap Z_\infty^*| = &\int_{\mathrm{relint}(H\cap Z_\infty^*)} \exp\left\{\int_{S^{n-1}\setminus H^\perp} \log \mathbf{1}_{\left[-\frac{1}{\|\mathrm{P}_H u\|}, \frac{1}{\|\mathrm{P}_H u\|}\right]}\right.\\
&\times \left.\left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} \mathrm{d}y\\
=&\Gamma\left(\frac{3}{2}\right)^{-k} \exp\left\{\int_{S^{n-1}\setminus H^\perp} -(\log \|\mathrm{P}_H u\|) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\}\\
&\times \int_{\mathrm{relint}(H\cap Z_\infty^*)} \exp\left\{\int_{S^{n-1}\cap H} -\phi(y \cdot \omega)^2 \mathrm{d}\bar{\mu}(u)\right\}\\
&\times \exp\left\{\int_{S^{n-1}\cap H} \log \phi'(y \cdot w) \mathrm{d}\bar{\mu}(w)\right\} \mathrm{d}y\\
\le&\Gamma\left(\frac{3}{2}\right)^{-k} \exp\left\{\int_{S^{n-1}\setminus H^\perp} -(\log \|\mathrm{P}_H u\|)\ |\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\}\\
&\times \int_{\mathrm{relint}(H\cap Z_\infty^*)} e^{-\|Ty\|^2} \det(\mathrm{d}T(y)) \mathrm{d}y\\
\le&\Gamma\left(\frac{3}{2}\right)^{-k} \exp\left\{\int_{S^{n-1}\setminus H^\perp} -(\log \|\mathrm{P}_H u\|) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} \int_H e^{-\|z\|^2} \mathrm{d}z\\
\le&2^k \left(\frac{n}{k}\right)^{\frac{k}{2}}.
\end{aligned}$$

Therefore, we obtain the desired inequality

$$\frac{|H \cap Z_\infty^*|}{|B_\infty^k|} \le \left(\frac{n}{k}\right)^{\frac{k}{2}}.$$

The proof of the equality conditions is almost the same as in the case $2 < p < \infty$. $\square$

Theorem 3.2 *Let $1 \le k \le n$ and let H be a k-dimensional subspace of $\mathbb{R}^n$. If μ is an even isotropic measure on S^{n-1}, then*

$$|\mathrm{P}_H Z_p(\mu)| \ge \begin{cases} \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p^*}\right)} |B^k_{p^*}|, & 1 \le p^* \le 2, \\ |B^k_{p^*}|, & 2 \le p^* \le \infty. \end{cases} \tag{33}$$

For $1 \le p^* < 2$, there is equality if and only if there is an orthonormal basis $\{\omega_1, \dots, \omega_k\}$ of H such that

$$\mu\big(S^{n-1} \cap (H^\perp + \sqrt{k/n}\omega_i)\big) = \mu\big(S^{n-1} \cap (H^\perp - \sqrt{k/n}\omega_i)\big) = \frac{n}{2k}, \quad i = 1, \dots, k.$$

For $2 < p^* \le \infty$, there is equality if and only if μ is concentrated on $H \cup H^\perp$ and $\mu \llcorner (S^{n-1} \cap H)$ is a cross measure on $S^{n-1} \cap H$.

Proof **Case** $1 \le p^* < \infty$: For $u \in \operatorname{supp}\mu \setminus H^\perp$ and $t \in \mathbb{R}$, define the strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{-\infty}^{t} e^{-|s|^2} \mathrm{d}s = \frac{\|\mathrm{P}_H u\|^{1-\frac{2}{p^*}}}{\Gamma\left(1 + \frac{1}{p^*}\right)} \int_{-\infty}^{\phi(t)} e^{-\left|\|\mathrm{P}_H u\|^{1-\frac{2}{p^*}} s\right|^{p^*}} \mathrm{d}s.$$

Differentiating both sides with respect to t and taking logarithms gives

$$\begin{aligned} -t^2 = \log \Gamma\left(\frac{3}{2}\right) + \left(1 - \frac{2}{p^*}\right) \log \|\mathrm{P}_H u\| - \log \Gamma\left(1 + \frac{1}{p^*}\right) \\ - \|\mathrm{P}_H u\|^{p^*-2} |\phi(t)|^{p^*} + \log \phi'(t). \end{aligned} \tag{34}$$

Define $T : H \to H$ by

$$Ty = \int_{S^{n-1} \cap H} w \phi(y \cdot w) \mathrm{d}\bar{\mu}(w), \tag{35}$$

for each $y \in H$, where the measure $\bar{\mu}$ is defined by (13). Thus, the differential of T is given by

$$\mathrm{d}T(y) = \int_{S^{n-1} \cap H} w \otimes w \phi'(y \cdot w) \mathrm{d}\bar{\mu}(w). \tag{36}$$

Since $\phi' > 0$, the matrix $\mathrm{d}T(y)$ is positive definite for each $y \in H$. Hence, the transformation $T \colon H \to H$ is injective. Moreover, from Lemma 2.1 with $f(w) = \phi(y \cdot w)$ and (35), we have

$$\|Ty\|^{p^*}_{(H \cap Z_p^*)^*} \le \int_{S^{n-1} \setminus H^\perp} \left| \phi\left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} \right) \right|^{p^*} \|\mathrm{P}_H u\|^{p^*} \mathrm{d}\mu(u). \tag{37}$$

The Ball–Barthe inequality (21) with $f(w) = \phi'(y \cdot w)$ and (36) imply that

$$\exp\left\{\int_{S^{n-1}\cap H} \log\phi'(y\cdot\omega)\mathrm{d}\bar{\mu}(\omega)\right\} \leq \det(\mathrm{d}T(y)). \tag{38}$$

From (11), (34) with $t = y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}$, (15), (37), (38), and by making the change of variable $z = Ty$, using (11) again, we have

$$\begin{aligned}
\Gamma\Big(\frac{1}{2}\Big)^k &= \int_H e^{-\|y\|^2}\mathrm{d}y = \int_H \exp\left\{-\int_{S^{n-1}\setminus H^\perp}\left(y\cdot\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)^2\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right\}\mathrm{d}y\\
&= \int_H \exp\left\{-\int_{S^{n-1}\setminus H^\perp}\left[\|\mathrm{P}_H u\|^{p^*-2}\left|\phi\left(y\cdot\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)\right|^{p^*}\right.\right.\\
&\quad -\log\phi'\left(y\cdot\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)-\log\Gamma\Big(\frac{3}{2}\Big)\\
&\quad \left.\left.-\Big(1-\frac{2}{p^*}\Big)\log\|\mathrm{P}_H u\| + \log\Gamma\Big(1+\frac{1}{p^*}\Big)\right]\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right\}\mathrm{d}y\\
&= \left(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma\big(1+\frac{1}{p^*}\big)}\right)^k \exp\left\{\int_{S^{n-1}\setminus H^\perp}\left[\Big(1-\frac{2}{p^*}\Big)\log\|\mathrm{P}_H u\|\right]\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right\}\\
&\quad \times\int_H \exp\left\{\int_{S^{n-1}\setminus H^\perp} -\left|\phi\left(y\cdot\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)\right|^{p^*}\|\mathrm{P}_H u\|^{p^*}\mathrm{d}\mu(u)\right\}\\
&\quad \times\exp\left\{\int_{S^{n-1}\cap H}\log\phi'(y\cdot\omega)\mathrm{d}\bar{\mu}(\omega)\right\}\mathrm{d}y\\
&\leq \left(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma\big(1+\frac{1}{p^*}\big)}\right)^k\left(\exp\int_{S^{n-1}\setminus H^\perp}(\log\|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right)^{1-\frac{2}{p^*}}\\
&\quad \times\int_H e^{-\|Ty\|^{p^*}_{(H\cap Z_p^*)^*}}\det(\mathrm{d}T(y))\mathrm{d}y\\
&\leq \left(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma\big(1+\frac{1}{p^*}\big)}\right)^k\left(\exp\int_{S^{n-1}\setminus H^\perp}(\log\|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right)^{1-\frac{2}{p^*}}\\
&\quad \times\int_H e^{-\|z\|^{p^*}_{(H\cap Z_p^*)^*}}\mathrm{d}z\\
&= \left(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma\big(1+\frac{1}{p^*}\big)}\right)^k\left(\exp\int_{S^{n-1}\setminus H^\perp}(\log\|\mathrm{P}_H u\|)\|\mathrm{P}_H u\|^2\mathrm{d}\mu(u)\right)^{1-\frac{2}{p^*}}\\
&\quad \times|(H\cap Z_p^*)^*|\Gamma\Big(1+\frac{k}{p^*}\Big).
\end{aligned}$$

Note that (see e.g., [14, (0.38)])

$$\left(H \cap Z_p^*\right)^* = \mathrm{P}_H Z_p, \tag{39}$$

where the polar operation on the left is taken in H. Thus, we have

$$\begin{aligned} \frac{|\mathrm{P}_H Z_p|}{|B_{p^*}^k|} = \frac{|(H \cap Z_p^*)^*|}{|B_{p^*}^k|} &= \frac{\Gamma\left(1+\frac{k}{p^*}\right)|(H \cap Z_{p^*}^*)^*|}{\left(2\Gamma\left(1+\frac{1}{p^*}\right)\right)^k} \\ &\geq \left(\exp \int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|) \|\mathrm{P}_H u\ |^2 \mathrm{d}\mu(u)\right)^{\frac{2}{p^*}-1}. \end{aligned} \tag{40}$$

If $1 \leq p^* \leq 2$, then from (40) and the left-hand side of inequality (30), we have

$$\frac{|\mathrm{P}_H Z_p|}{|B_{p^*}^k|} \geq \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p^*}\right)}.$$

If $2 \leq p^* < \infty$, then from (40) and the right-hand side of inequality (30), we have

$$\frac{|\mathrm{P}_H Z_p|}{|B_{p^*}^k|} \geq 1.$$

Case $p^* = \infty$: For $u \in \operatorname{supp}\mu \setminus H^\perp$, define the strictly increasing function $\phi : \mathbb{R} \to \left(-\frac{1}{\|\mathrm{P}_H u\|}, \frac{1}{\|\mathrm{P}_H u\|}\right)$ by

$$\frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{-\infty}^{t} e^{-s^2} \mathrm{d}s = \|\mathrm{P}_H u\| \int_{-\infty}^{\phi(t)} \mathbf{1}_{\left[-\frac{1}{\|\mathrm{P}_H u\|}, \frac{1}{\|\mathrm{P}_H u\|}\right]}(s)\mathrm{d}s$$

Differentiating both sides with respect to t and taking logarithms yields

$$-t^2 = \log \Gamma\left(\frac{3}{2}\right) + \log \|\mathrm{P}_H u\| + \log \phi'(t). \tag{41}$$

Define $T : H \to H$ as in (35). Since $|\phi| < \frac{1}{\|\mathrm{P}_H u\|}$, Lemma 2.1, (39) and (35) show that $\|Ty\|_{\mathrm{P}_H Z_1} = \|Ty\|_{(H \cap Z_1^*)^*} \leq 1$ for all $y \in H$. But this and (9) immediately give $Ty \in \mathrm{P}_H Z_1$ for all $y \in H$. Thus,

$$T(H) \subseteq \mathrm{P}_H Z_1. \tag{42}$$

Clearly, the matrix $\mathrm{d}T(y)$ in (36) is positive definite for each $y \in H$ and the transformation $T \colon H \to \mathrm{P}_H Z_1 \subset H$ is injective.

From (11), (41) with $t = y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}$, (15), (38), and by making the change of variable $z = Ty$ and using (42), we have

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right)^k &= \int_H e^{-\|y\|^2} \mathrm{d}y = \int_H \exp\left\{-\int_{S^{n-1}\setminus H^\perp} \left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)^2 \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} \mathrm{d}y \\
&= \int_H \exp\left\{\int_{S^{n-1}\setminus H^\perp} \left(\log \Gamma\left(\frac{3}{2}\right) + \log \|\mathrm{P}_H u\|\right.\right. \\
&\qquad \left.\left. + \log \phi'\left(y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right)\right) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} \mathrm{d}y \\
&\le \Gamma\left(\frac{3}{2}\right)^k \exp\left\{\int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} \int_H \det(\mathrm{d}T(y)) \mathrm{d}y \\
&\le \Gamma\left(\frac{3}{2}\right)^k \exp\left\{\int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} \int_{\mathrm{P}_H Z_1} \mathrm{d}z \\
&= \Gamma\left(\frac{3}{2}\right)^k \exp\left\{\int_{S^{n-1}\setminus H^\perp} (\log \|\mathrm{P}_H u\|) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} |\mathrm{P}_H Z_1|.
\end{aligned}$$

Therefore, by (30) we obtain the desired inequality

$$\frac{|\mathrm{P}_H Z_1|}{|B_\infty^k|} \ge \exp\left\{\int_{S^{n-1}\setminus H^\perp} (-\log \|\mathrm{P}_H u\|) \|\mathrm{P}_H u\|^2 \mathrm{d}\mu(u)\right\} \ge 1.$$

The proofs of the equality conditions are basically the same as in the proof of Theorem 3.1 when $1 \le p^* < 2$ (or $2 < p \le \infty$), and $2 < p^* \le \infty$ (or $1 \le p < 2$), and we therefore will not repeat them here. □

Next, we prove the left-hand side inequalities of (6) and (7).

Theorem 3.3 *Let $1 \le k \le n$ and let H be a k-dimensional subspace of $\mathbb{R}^n$. If μ is an even isotropic measure on S^{n-1}, then for $1 \le p \le \infty$,*

$$|H \cap Z_p^*(\mu)| \ge \frac{\omega_k}{c_{p,k}} \left(\frac{\mu(S^{n-1} \setminus H^\perp)}{k}\right)^{-\frac{k}{p}}$$
$$\text{and} \quad |\mathrm{P}_H Z_p(\mu)| \le \omega_k c_{p,k} \left(\frac{\mu(S^{n-1} \setminus H^\perp)}{k}\right)^{\frac{k}{p}}.$$

There is equality in either inequality if and only if μ is concentrated on $H \cup H^\perp$ with $\mu(S^{n-1} \setminus H^\perp) = k$ and $H \cap Z_p^*$ is a Euclidean ball in H.

Proof From (18), Fubini's theorem and the fact that $\|\mathrm{P}_H u\| \le 1$, we have

$$\begin{aligned}
\frac{1}{k\omega_k} \int_{S^{n-1}\cap H} \|v\|_{H\cap Z_p^*}^p \mathrm{d}v &= \frac{1}{k\omega_k} \int_{S^{n-1}\cap H} \int_{S^{n-1}\setminus H^\perp} \left|v \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right|^p \|\mathrm{P}_H u\|^p \mathrm{d}\mu(u) \mathrm{d}v \\
&= \frac{c_{p,k}^{p/k}}{k} \int_{S^{n-1}\setminus H^\perp} \|\mathrm{P}_H u\|^p \mathrm{d}\mu(u)
\end{aligned}$$

$$\leq \frac{c_{p,k}^{p/k}}{k}\int_{S^{n-1}\setminus H^{\perp}} \mathrm{d}\mu(u) = \frac{\mu(S^{n-1}\setminus H^{\perp})c_{p,k}^{p/k}}{k}. \tag{43}$$

Thus, by the polar coordinate formula (12), the Hölder inequality, and (43), we obtain

$$\left(\frac{|H\cap Z_p^*|}{\omega_k}\right)^{-\frac{p}{k}} = \left(\frac{1}{k\omega_k}\int_{S^{n-1}\cap H} \|v\|_{H\cap Z_p^*}^{-k}\mathrm{d}v\right)^{-\frac{p}{k}} \leq \frac{1}{k\omega_k}\int_{S^{n-1}\cap H} \|v\|_{H\cap Z_p^*}^{p}\mathrm{d}v$$
$$\leq \frac{\mu(S^{n-1}\setminus H^{\perp})c_{p,k}^{p/k}}{k}.$$

On the other hand, by the Urysohn inequality (see, e.g., Schneider [41, p. 382]), the Hölder inequality, and (43), we have

$$\left(\frac{|\mathrm{P}_H Z_p|}{\omega_k}\right)^{\frac{1}{k}} \leq \frac{1}{k\omega_k}\int_{S^{n-1}\cap H} \|v\|_{H\cap Z_p^*}\mathrm{d}v \leq \left(\frac{1}{k\omega_k}\int_{S^{n-1}\cap H} \|v\|_{H\cap Z_p^*}^{p}\mathrm{d}v\right)^{\frac{1}{p}}$$
$$\leq c_{p,k}^{1/k}\left(\frac{\mu(S^{n-1}\setminus H^{\perp})}{k}\right)^{\frac{1}{p}}.$$

Observe that equality in (43) implies that

$$\int_{(S^{n-1}\setminus H^{\perp})\setminus(S^{n-1}\cap H)} \|\mathrm{P}_H u\|^p \mathrm{d}\mu(u) = \int_{(S^{n-1}\setminus H^{\perp})\setminus(S^{n-1}\cap H)} \mathrm{d}\mu(u).$$

This can only happen when $S^{n-1}\setminus H^{\perp} = S^{n-1}\cap H$. The isotropy of μ shows that μ cannot be concentrated on a great subsphere of S^{n-1}. Thus the measure μ is concentrated on $S^{n-1}\cap H$ and $S^{n-1}\cap H^{\perp}$. Moreover, it follows from (14) that $\mu(S^{n-1}\setminus H^{\perp}) = \bar{\mu}(S^{n-1}\cap H) = k$. The equality conditions of the Hölder inequality tell us that $H\cap Z_p^*$ is a Euclidean ball in H. □

Now, Theorem 1.1 follows from Theorems 3.1, 3.2, and 3.3.

4 Special Cases

The following theorem, as a special case of our main results, was proved by Meyer and Pajor [37], and Barthe [6]. The case $p = \infty$ is due to Ball [3]. The necessary conditions of equalities are new.

Theorem 4.1 *Let $1 \leq k \leq n$ and let H be a k-dimensional subspace of $\mathbb{R}^n$. Then*

$$\frac{|H\cap B_p^n|}{|B_p^k|} \leq \begin{cases} 1, & 1\leq p\leq 2, \\ \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p}\right)}, & 2\leq p\leq\infty. \end{cases} \tag{44}$$

For $1 \leq p < 2$, there is equality if and only if H is a coordinate subspace. For $2 < p \leq \infty$, the constant on the right hand side is best possible if and only if k divides n.

Proof **Case** $1 \le p \le 2$: Inequality (44) follows by taking supp $\mu = \{\pm e_1, \dots, \pm e_n\}$ in Theorem 3.1. Obviously, if H is a coordinate subspace of $\mathbb{R}^n$, then $H \cap B_p^n = B_p^k$.

Conversely, if equality in (44) holds for $p \ne 2$, then by Theorem 3.1, there exists a k-dimensional subspace H such that μ is concentrated on $H \cup H^\perp$ and $\mu\llcorner(S^{n-1} \cap H)$ is a cross measure on H. Since $\mu\llcorner(S^{n-1} \cap H)$ is a restriction of μ to $S^{n-1} \cap H$ and supp $\mu = \{\pm e_1, \dots, \pm e_n\}$, we have that supp $(\mu\llcorner(S^{n-1} \cap H)) \subset \{\pm e_1, \dots, \pm e_n\}$, which shows that H is a coordinate subspace of $\mathbb{R}^n$.

Case $2 \le p \le \infty$: Taking supp $\mu = \{\pm e_1, \dots, \pm e_n\}$ in Theorem 3.1, we immediately get (44).

If k divides n, let $l = n/k$. Let $v_j = e_{(j-1)l+1} + \cdots + e_{jl}$ and $w_j = v_j/|v_j|$ for $j = 1, \dots, k$. Denote by $H = \operatorname{span}\{w_1, \dots, w_k\}$. Note that $\omega_1, \dots, \omega_k$ forms an orthonormal basis of H. Moreover, for $i \in (j-1)l + 1, \dots, jl$, we have

$$\mathrm{P}_H e_i = (e_i \cdot w_j) w_j = \sqrt{\frac{k}{n}} w_j. \tag{45}$$

We claim that $H \cap B_p^n$ is isometric to $(\frac{n}{k})^{\frac{1}{2}-\frac{1}{p}} B_p^k$. In fact, by (45), we have

$$\begin{aligned} H \cap B_p^n &= \left\{ y \in H : \left(\sum_{i=1}^n |y \cdot \mathrm{P}_H e_i|^p \right)^{\frac{1}{p}} \le 1 \right\} \\ &= \left\{ y \in H : \left(\sum_{j=1}^k \frac{n}{k} \left| y \cdot \sqrt{\frac{k}{n}} \omega_j \right|^p \right)^{\frac{1}{p}} \le 1 \right\} \\ &= \left(\frac{n}{k}\right)^{\frac{1}{2}-\frac{1}{p}} \left\{ y \in H : \left(\sum_{j=1}^k |y \cdot \omega_j|^p \right)^{\frac{1}{p}} \le 1 \right\}. \end{aligned}$$

Thus, $H \cap B_p^n$ is isometric to $(\frac{n}{k})^{\frac{1}{2}-\frac{1}{p}} B_p^k$ and hence equality in (44) holds.

Conversely, if equality in (44) holds for $p \ne 2$, then by Theorem 3.1, there exists an orthonormal basis $\{\omega_1, \dots, \omega_k\}$ of H such that

$$\mu\big(S^{n-1} \cap (H^\perp + \sqrt{k/n}\,\omega_j)\big) = \mu\big(S^{n-1} \cap (H^\perp - \sqrt{k/n}\,\omega_j)\big) = \frac{n}{2k},\ j = 1, \dots, k.$$

This implies that $e_i \in S^{n-1} \cap (H^\perp \pm \sqrt{k/n}\,\omega_{j(i)})$ for a $j(i) \in \{1, \dots, k\}$. Hence, we have $\mathrm{P}_H e_i \in \{\pm\sqrt{k/n}\,\omega_{j(i)}\}$, and thus,

$$e_i \cdot \omega_{j(i)} = \pm \|\mathrm{P}_H e_i\| = \pm\sqrt{\frac{k}{n}}, \tag{46}$$

and $e_i \cdot \omega_{j'} = 0$ for $j' \ne j(i)$. In other words, we can divide $e_1, \cdots, e_n$ into k parts such that each $\omega_{j(i)}$ is a linear combination of some elements of $\{e_1, \dots, e_n\}$. Write $\omega_{j(i)} =$

$\lambda_{j_1} e_{j_1} + \cdots + \lambda_{j_m} e_{j_m}$ with $\lambda_{j_1}^2 + \cdots + \lambda_{j_m}^2 = 1$. From (46), we know that $\lambda_{j_1} = \cdots = \lambda_{j_m} = \pm\sqrt{k/n}$. Together with $\lambda_{j_1}^2 + \cdots + \lambda_{j_m}^2 = 1$, we must have $m = n/k$, which is an integer. This is the desired result. □

The same argument used in the proof of Theorem 4.1, together with Theorem 3.2, yields the following dual volume inequalities established by Barthe [6]. The necessary conditions of equalities are new.

Theorem 4.2 *Let* $1 \le k \le n$ *and let* H *be a* k*-dimensional subspace* H *of* $\mathbb{R}^n$. *Then*

$$\frac{|\mathrm{P}_H B_p^n|}{|B_p^k|} \ge \begin{cases} \left(\frac{n}{k}\right)^{k\left(\frac{1}{2}-\frac{1}{p}\right)}, & 1 \le p \le 2, \\ 1, & 2 \le p \le \infty. \end{cases}$$

For $1 \le p < 2$, *the constant on the right-hand side is best possible if and only if* k *divides* n. *For* $2 < p \le \infty$, *there is equality if and only if* H *is a coordinate subspace.*

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