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Volume inequalities of convex bodies from cosine transforms on Grassmann manifolds ☆

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The L_p cosine transform on Grassmann manifolds naturally induces finite dimensional Banach norms whose unit balls are origin-symmetric convex bodies in $\mathbb{R}^n$. Reverse isoperimetric type volume inequalities for these bodies are established, which extend results from the sphere to Grassmann manifolds.

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1. Introduction

The solution to the classical isoperimetric problem says that among all convex bodies of given surface area in the Euclidean space $\mathbb{R}^n$, only the ball has maximal volume. It is usually written as the following isoperimetric inequality,

$$S(K) \geq n\omega_n^{\frac{1}{n}} V(K)^{\frac{n-1}{n}},$$

with equality if and only if the convex body K is a ball, where $S(K)$ and $V(K)$ denote the surface area and volume of K, respectively, and ω_n is the volume of the Euclidean unit ball. A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior.

Note that the volume of a convex body can be arbitrarily small when its surface area is fixed. Thus, the isoperimetric inequality can not be reversed with a different constant factor. Establishing a reverse isoperimetric inequality that characterizes cubes, simplices, or other non-spherical convex bodies is a highly interesting problem in convex geometry.

The celebrated work of Keith Ball [2,3] is a landmark in the study of the reverse isoperimetric problem. He proved that for any symmetric convex body K in $\mathbb{R}^n$ there is a volume preserving linear transformation ψ so that the surface area of ψK is no larger than that of a cube of the same volume. If symmetry is not assumed, he proved the similar remarkable result for simplices. In Ball's works [2–4], the notion of isotropy of measures on the unit sphere (see (2.8)) and the Brascamp–Lieb inequality played critical roles. By using the method of mass transportation, Barthe [7] found a new proof of the Brascamp–Lieb inequality and established the reverse Brascamp–Lieb inequality. He then used the inequalities to show new reverse isoperimetric inequalities, and also showed the uniqueness of equality cases of his and Ball's reverse isoperimetric inequalities, see [5–8]. The remarkable work of Ball and Barthe has motivated a series of new studies, see for example, [1,10–13,18–21,24–26,31–33,41–45,56,57]. Some of Ball and Barthe's results were generalized in [42,44] from discrete to arbitrary isotropic measures on the unit sphere and from polytopes to arbitrary convex bodies in $\mathbb{R}^n$. Most recently, Schuster and Weberndorfer [60] proved important reverse isoperimetric inequalities for Wulff shapes of arbitrary isotropic measures and L_2 functions on the unit sphere which further generalize and unify the results of Ball [2,3], Barthe [7], and [44,45].

The purpose of this paper is to extend volume inequalities arising from measures on the unit sphere to volume inequalities for measures on Grassmann manifolds. We study the L_p cosine transforms on Grassmann manifolds which include the spherical cosine and sine transforms as special cases. Reverse isoperimetric inequalities are established for convex bodies that are naturally associated with cosine transforms on Grassmann manifolds, which generalize and unify the results on the unit sphere. New concepts and techniques are introduced for proving these results.

A tool in harmonic analysis that is useful for the reverse isoperimetric problem is the spherical cosine transform. The L_p cosine transform C_p on the unit sphere S^{n-1} gives a natural analytical operator in convex geometric analysis. It induces n-dimensional norms which are the norms of n-dimensional subspaces of L_p spaces. The unit balls of these subspaces constitute an important class of origin-symmetric convex bodies which are called L_p-*balls*. They are among the main objects of study in both convex geometric analysis and asymptotic functional analysis, see for example, [16,22,29,34–37,40,46,52,58,59]. For a nonnegative finite Borel measure μ on S^{n-1}, the spherical cosine transform $\mathrm{C}_p\mu$ of μ is a convex function homogeneous of degree p, $1 \le p < \infty$, in $\mathbb{R}^n$ defined by

$$(\mathrm{C}_p\mu)(x) = \int_{S^{n-1}} |x \cdot u|^p \, d\mu(u), \quad x \in \mathbb{R}^n,$$

where $x \cdot u$ denotes the inner product of vectors in $\mathbb{R}^n$. The cosine transform can also be defined appropriately when $p < 1$, and the case $p = -1$ is closely related to the spherical Radon transform. Connections of the spherical cosine transform with the Fourier transform and important applications of these integral transforms in convex geometry were presented in [9,15,17,28,30,38,53–55,64].

When μ is not concentrated on a great subsphere, the p-th root of $\mathrm{C}_p\mu$ is an n-dimensional norm, denoted by $\|\cdot\|_{Z_p^*}$ with unit ball $Z_p^* = Z_p^*(\mu)$. Thus, Z_p^* is an L_p-ball (the unit ball of an L_p-space), which is the polar of another origin-symmetric convex body $Z_p = Z_p(\mu)$. In other words, Z_p is the unit ball of the dual norm of $\|\cdot\|_{Z_p^*}$. When μ is the Lebesgue measure on the unit sphere, Z_p^* is a Euclidean ball. When μ is a cross measure, which is an even discrete measure that concentrates equally on n orthogonal antipodal pairs of directions, Z_p^* is obtained by a dilation and a rotation from B_p^n (the ℓ_p^n-ball), i.e., $B_p^n = \{x \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p \le 1\}$.

An important problem is to establish an isoperimetric characterization of the ℓ_p^n-ball among all L_p balls. This was achieved by the following result by Ball [3], Barthe [7], and [42].

Theorem 1.1. *If μ is an even isotropic measure on S^{n-1} with fixed total mass and $p \in [1,\infty]\backslash\{2\}$, then the volume $V(Z_p^*(\mu))$ is maximized and the volume $V(Z_p(\mu))$ is minimized if and only if μ is a cross measure.*

In [47], Maresch and Schuster applied the spherical sine transform successfully to isotropic measures for proving reverse isoperimetric inequalities. For a finite Borel measure μ on S^{n-1}, the sine transform of μ is defined by

$$(\mathrm{S}\mu)(x) = \int_{S^{n-1}} (|x|^2 - (x \cdot u)^2)^{\frac{1}{2}} \, d\mu(u), \quad x \in \mathbb{R}^n.$$

When μ is not concentrated on a pair of antipodal points, the sine transform $\mathrm{S}\mu$ is a norm whose unit ball $Z_{n-1,1}^*$ is an origin-symmetric convex body. As pointed out in [47], when the measure μ is the surface area measure of a convex body in $\mathbb{R}^n$, the sine transform and the convex body $Z_{n-1,1}^*$ arise naturally in geometric tomography.

When μ is isotropic, Maresch and Schuster proved reverse isoperimetric inequalities for the convex body $Z^*_{n-1,1}$ and its polar body $Z_{n-1,1}$ that are asymptotically sharp when the dimension n is large.

Denote by $G_{n,m}$ the Grassmann manifold of m-dimensional linear subspaces in $\mathbb{R}^n$, $1 \le m \le n-1$. For $\xi \in G_{n,m}$, let $\mathrm{P}_\xi : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection map with range space ξ.

For $p \in [1, \infty)$, the L_p *cosine transform* $\mathrm{C}_{m,p}\mu$ of a nonnegative finite Borel measure μ on $G_{n,m}$ is the continuous function defined by

$$(\mathrm{C}_{m,p}\mu)(x) = \int\limits_{G_{n,m}} |\mathrm{P}_\xi x|^p d\mu(\xi), \quad x \in \mathbb{R}^n.$$

Identify an even measure on the unit sphere S^{n-1} with a measure on $G_{n,1}$, or $G_{n,n-1}$. Then $2\mathrm{C}_{1,p}\mu$ is the usual spherical L_p cosine transform C_p and $2\mathrm{C}_{n-1,p}\mu$ is the spherical L_p sine transform

$$(\mathrm{S}_p\mu)(x) = \int\limits_{S^{n-1}} \left(|x|^2 - (x \cdot u)^2\right)^{\frac{p}{2}} d\mu(u). \tag{1.1}$$

If the measure μ on $G_{n,m}$ is not concentrated on a great sub-Grassmannian (the Grassmann manifold $G_{n-1,m}$ of m-dimensional subspaces in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$), the L_p cosine transform $\mathrm{C}_{m,p}$ induces an n-dimensional Banach norm, $\left(\frac{n}{m}(\mathrm{C}_{m,p}\mu)(\cdot)\right)^{1/p}$, whose unit ball, denoted by $Z^*_{m,p} = Z^*_{m,p}(\mu)$, is an origin-symmetric convex body in $\mathbb{R}^n$ given by

$$Z^*_{m,p} = \left\{ x \in \mathbb{R}^n : \left(\frac{n}{m}(\mathrm{C}_{m,p}\mu)(x)\right)^{\frac{1}{p}} \le 1 \right\}, \tag{1.2}$$

which is the polar body of the origin-symmetric convex body $Z_{m,p} = Z_{m,p}(\mu)$, whose support function is given by

$$h_{Z_{m,p}}(x) = \left(\frac{n}{m} \int\limits_{G_{n,m}} |\mathrm{P}_\xi x|^p d\mu(\xi)\right)^{\frac{1}{p}}, \quad p \in [1, \infty). \tag{1.3}$$

When $p = \infty$,

$$h_{Z_{m,\infty}}(x) = \lim_{p\to\infty} h_{Z_{m,p}}(x) = \max_{\xi \in \operatorname{supp}\mu} |\mathrm{P}_\xi x|. \tag{1.4}$$

The body $Z^*_{m,\infty}$ as the polar of $Z_{m,\infty}$ is thus also defined.

Let $1 \le m \le n-1$. A finite Borel measure μ on $G_{n,m}$ is *isotropic* if it satisfies

$$\frac{1}{|\mu|} \int\limits_{G_{n,m}} \mathrm{P}_\xi \, d\mu(\xi) = \frac{m}{n} I_n, \tag{1.5}$$

where I_n is the identity transformation of $\mathbb{R}^n$ and $|\mu|$ is the total mass of μ. Note that an isotropic measure on the Grassmann manifold $G_{n,m}$ is not concentrated on a great sub-Grassmannian.

For $q \geq 0$ and $p \geq 1$, let

$$\omega_q = \frac{\pi^{\frac{q}{2}}}{\Gamma(1+\frac{q}{2})}, \quad \alpha_n(m,p) = \frac{\left(\omega_m \Gamma(1+\frac{m}{p})\right)^{\frac{n}{m}}}{\Gamma(1+\frac{n}{p})}, \quad \gamma_n(m,p) = \left(\frac{\omega_m \omega_{n+p-2}}{\omega_n \omega_{m+p-2}}\right)^{\frac{n}{p}}.$$

For $p = \infty$, let $\alpha_n(m,\infty) = \omega_m^{\frac{n}{m}}$ and $\gamma_n(m,\infty) = 1$. Denote by p' the conjugate of p, that is, $p' = \frac{p}{p-1}$.

The main result of the paper is the following theorem.

Theorem 1.2. *Suppose $p \in [1,\infty]$. If μ is an isotropic probability measure on $G_{n,m}$, then*

$$\alpha_n(m,p) \geq V(Z^*_{m,p}(\mu)) \geq \frac{\omega_n}{\gamma_n(m,p)}, \tag{1.6}$$

$$\alpha_n(m,p') \leq V(Z_{m,p}(\mu)) \leq \omega_n \gamma_n(m,p). \tag{1.7}$$

There is equality in the right side inequalities of (1.6) and (1.7) if μ is the normalized Lebesgue measure. When n is divisible by m and $p \neq 2$, there is equality in the left side inequalities of (1.6) and (1.7) if and only if μ is a cross measure on $G_{n,m}$.

When $m = 1$, Theorem 1.2 becomes Theorem 1.1, which was proved in [42], while the discrete case was shown by Ball [3] and Barthe [7]. When $m = n-1$ and $p = 1$, the right side inequalities of (1.6) and (1.7) were shown by Maresch and Schuster [47], while they obtained slightly different constants in the left side inequalities in this case. The right side inequality of (1.6) or (1.7) is an isoperimetric inequality, which can be shown by classical results. The left side inequality of (1.6) or (1.7) is a reverse isoperimetric inequality, which requires new techniques to prove. When n is not divisible by m, it is an interesting open problem to establish the sharp inequalities and to characterize the extremal bodies.

Extending geometric concepts and results associated with the unit sphere to Grassmann manifolds not only is non-trivial, but also leads to new concepts and techniques. For the reverse isoperimetric problem associated with the L_p cosine transform on a Grassmann manifold other than the unit sphere, the ℓ_p^n-ball is no longer an extremal body. The extremal bodies are a new class of convex bodies that are extensions of the ℓ_p^n-ball on Grassmann manifolds. Defining these new convex bodies relies on defining cross measures on Grassmann manifolds. These cross measures are discrete and are in a certain sense the opposite of the Haar measure. We give a definition of cross measures on Grassmann manifolds in Section 4, and thus introduce a new class of convex bodies called $\ell_p^{n,m}$-balls.

The spherical Ball–Barthe inequality of isotropic measures on the unit sphere is a key tool for proving reverse isoperimetric inequalities on the unit sphere. A Grassmannian

Ball–Barthe inequality for isotropic measures on Grassmann manifolds is proved in Section 3. The proof is not similar to that of the spherical case. The difficulty arises from the fact that the unit sphere S^{n-1} as a hypersurface of $\mathbb{R}^n$ has a globally defined continuous normal vector field, while the Grassmann manifold $G_{n,m}$, $1 < m < n-1$, does not have a similar property. Thus a more complicated new proof is needed. Without equality conditions, the spherical Ball–Barthe inequality can be obtained by a simple approximation from the discrete case (see Barthe [8]). For Grassmann manifolds, it is not clear yet how to do such an approximation for either the Grassmannian Ball–Barthe inequality, or the reverse isoperimetric inequalities in Theorem 1.2.

The spherical Ball–Barthe inequality was used by Barthe [8] to prove a continuous version of the spherical Brascamp–Lieb inequality. The Grassmannian Ball–Barthe inequality may be used to prove a continuous version of the Brascamp–Lieb inequality on Grassmann manifolds.

We remark that the mass transportation technique of Barthe [7] is used. However, since known regularity properties of the Brenier map used by Barthe [7] are not sufficient for our proofs and properties of eigenvalues of the Hessian associated with the Brenier map are needed, we give a direct and explicit construction which is a higher dimensional version of the 1-dimensional case used in [42].

Finally, we note that reverse affine isoperimetric inequalities arising from spherical isotropic measures depend on certain volume preserving affine transformations of a convex body (or equivalently certain ellipsoids associated with a convex body). An integral geometric concept of affine surface area in $\mathbb{R}^n$ that does not depend on affine transformations of a convex body is obtained as the affine mean of shadow areas (that is, the integral of shadow area with negative n-th power) of a convex body. The associated affine isoperimetric inequality called the Petty projection inequality is stronger than the Euclidean isoperimetric inequality, see [58]. Its reverse affine isoperimetric inequality that characterizes simplices was proved in [62]. New proofs and generalizations were given in [27,63]. These results can be viewed as spherical cases. Their Grassmannian analogs are important unsolved open problems, see Lutwak [39].

2. Preliminaries

We collect some basic facts about convex bodies. Good general references for the theory of convex bodies are provided by the books of Gardner [16], Gruber [23], Schneider [58], and Thompson [61].

Denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. For $x \in \xi \in G_{n,m}$, write $|x|_\xi$ for the Euclidean norm of x in ξ. When not causing confusion, we will also suppress the subscript in the norm. Denote the Euclidean unit ball and unit sphere of $\mathbb{R}^n$ by B_2^n and S^{n-1}, respectively.

A convex body K in $\mathbb{R}^n$ is a compact convex set with nonempty interior. If K is a convex body that contains the origin in its interior, its polar body K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. It is easy to check that $(K^*)^* = K$ and for real $\lambda > 0$

$$(\lambda K)^* = \frac{1}{\lambda} K^*. \tag{2.2}$$

The support function of a convex body K, $h_K : \mathbb{R}^n \to \mathbb{R}$, is defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = \max\{x \cdot y : y \in K\}. \tag{2.3}$$

The Minkowski functional $\|\cdot\|_K$ of a convex body K that contains the origin in its interior is defined by

$$\|x\|_K = \min\{t > 0 : x \in tK\}, \quad x \in \mathbb{R}^n.$$

In this case

$$\|x\|_K = h_{K^*}(x). \tag{2.4}$$

When K is origin-symmetric, $\|\cdot\|_K$ is the norm with unit ball K. Obviously, for real $\lambda > 0$, $\|\lambda x\|_K = \lambda \|x\|_K$.

Recall that for each $p \in (0, \infty)$, the volume $V(K)$ of K has the formula,

$$V(K) = \frac{1}{\Gamma(1 + \frac{n}{p})} \int\limits_{\mathbb{R}^n} e^{-\|x\|_K^p} \, dx, \tag{2.5}$$

where the integral is with respect to Lebesgue measure on $\mathbb{R}^n$; in particular,

$$\int\limits_{\mathbb{R}^n} e^{-|x|^p} dx = \omega_n \Gamma\big(1 + \frac{n}{p}\big). \tag{2.6}$$

Let K be a convex body containing the origin in its interior. The polar coordinate volume formula of K will also be used later:

$$V(K) = \frac{1}{n} \int\limits_{S^{n-1}} h_{K^*}(u)^{-n} \, du, \tag{2.7}$$

where du is the Lebesgue measure on S^{n-1}.

A nonnegative finite Borel measure μ on S^{n-1} is said to be *isotropic* if

$$\frac{1}{|\mu|} \int\limits_{S^{n-1}} u \otimes u \, d\mu(u) = \frac{1}{n} I_n, \tag{2.8}$$

where $u \otimes u$ denotes the rank one projection defined by $u \otimes u(x) = (u \cdot x)u$ for all $x \in \mathbb{R}^n$ and I_n is the identity matrix. Note that an isotropic measure is not concentrated on a proper subspace of $\mathbb{R}^n$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. A measure on S^{n-1} is said to be *even* if it assumes the same value on antipodal sets. The two most important examples of even isotropic measures on S^{n-1} are the spherical Lebesgue measure and the *cross measure*. The *basic* cross measure on S^{n-1} is an even isotropic discrete measure concentrated equally on $\pm e_1, \dots, \pm e_n$. A cross measure on S^{n-1} is just a rotation of a basic cross measure; i.e., it is concentrated equally on $O\{\pm e_1, \dots, \pm e_n\}$, where $O \in \mathrm{O}(n)$.

For a subspace $\xi \subset \mathbb{R}^n$ of dimension m, let $\{u_1, \dots, u_m\}$ be an orthonormal basis of ξ, and let Q be the $n \times m$ matrix with column vectors $u_1, \dots, u_m$. This is equivalent to the condition

$$Q^t Q = I_m, \tag{2.9}$$

where Q^t is the transpose of Q. Then

$$\mathrm{P}_\xi = QQ^t, \tag{2.10}$$

or equivalently,

$$\mathrm{P}_\xi x = \sum_{i=1}^{m} (x \cdot u_i) u_i, \quad x \in \mathbb{R}^n. \tag{2.11}$$

The matrix Q^t gives the isometry $\xi \to \mathbb{R}^m$, $x_1 u_1 + \cdots + x_m u_m \to (x_1, \dots, x_m)$, and vanishes on $\xi^\perp$. The matrix Q gives the isometry $\mathbb{R}^m \to \xi$, $(x_1, \dots, x_m) \to x_1 u_1 + \cdots + x_m u_m$.

We observe that (1.5) is equivalent to

$$\frac{1}{|\mu|} \int\limits_{G_{n,m}} |\mathrm{P}_\xi x|^2 \, d\mu(\xi) = \frac{m}{n} |x|^2, \quad x \in \mathbb{R}^n. \tag{2.12}$$

If we let $d\nu = \frac{n}{m|\mu|} d\mu$, then $|\nu| = \frac{n}{m}$ and (1.5) becomes

$$\int\limits_{G_{n,m}} \mathrm{P}_\xi \, d\nu(\xi) = I_n, \tag{2.13}$$

or equivalently,

$$\int\limits_{G_{n,m}} |\mathrm{P}_\xi x|^2 \, d\nu(\xi) = |x|^2, \quad x \in \mathbb{R}^n. \tag{2.14}$$

Let $\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$, and $S^{n-m-1} \subset \mathbb{R}^{n-m}$, $S^{m-1} \subset \mathbb{R}^m$ be the unit spheres in the subspaces. The generalized spherical coordinates (see [22], p. 99) are for $1 \le m \le n-1$,

$$u = (u_1 \sin\varphi, u_2 \cos\varphi) \in S^{n-1},$$

where $u_1 \in S^{n-m-1}$, $u_2 \in S^{m-1}$ and $0 \le \varphi \le \frac{\pi}{2}$. The surface area elements du, du_1, and du_2 of S^{n-1}, S^{n-m-1}, and S^{m-1} satisfy

$$du = \sin^{n-m-1}\varphi \cos^{m-1}\varphi d\varphi du_1 du_2. \tag{2.15}$$

Denote by $\mathbb{M}_n$ the set of $n \times n$ matrices of real numbers. The set $\mathbb{M}_n$ is a manifold of dimension n^2 when an element of $\mathbb{M}_n$ is identified with a point in $\mathbb{R}^{n^2}$. Then the Euclidean norm $|\cdot|_2$ in $\mathbb{R}^{n^2}$ becomes a norm in $\mathbb{M}_n$. Let $\mathbb{L}_n$ be the vector space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. When an orthonormal basis of $\mathbb{R}^n$ is chosen, $\mathbb{M}_n$ and $\mathbb{L}_n$ are identified. Denote by $\mathbb{S}_n$ the subset of $\mathbb{M}_n$ that consists of symmetric matrices. For a subspace $\xi \subset \mathbb{R}^n$, denote by $\mathbb{L}_\xi$ the vector space of linear transformations from ξ to ξ.

For $A \in \mathbb{M}_n$, let $\|A\|_2$ be the spectral norm of A which is the square root of the maximal eigenvalue of $A^t A$, that is

$$\|A\|_2 = \max_{|x|\neq 0} \frac{|Ax|}{|x|}. \tag{2.16}$$

It is a basic fact that

$$\|\cdot\|_2 \le |\cdot|_2 \le \sqrt{n}\|\cdot\|_2. \tag{2.17}$$

We also need the following basic facts about matrices. For completeness, we include simple proofs.

Lemma 2.1. *Let $A \in \mathbb{S}_n$ with null space $\xi = \{x \in \mathbb{R}^n : Ax = 0\}$ and non-zero eigenvalues $\lambda_1, \ldots, \lambda_k$. If $|\lambda_1| \ge \cdots \ge |\lambda_k| > 0$, then*

$$|Ax| \le |\lambda_1||x|, \quad x \in \mathbb{R}^n, \tag{2.18}$$

$$|Ax| \ge |\lambda_k||x|, \quad x \in \xi^\perp. \tag{2.19}$$

Proof. Choose an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ so that $e_1, \ldots, e_n$ are eigenvectors of A and $e_1, \ldots, e_k$ span $\xi^\perp$. Let $x = x_1 e_1 + \cdots + x_n e_n$. Then

$$|Ax| = (\lambda_1^2 x_1^2 + \cdots + \lambda_k^2 x_k^2)^{\frac{1}{2}}.$$

This gives (2.18) and (2.19). □

We review basics regarding mixed discriminants. Let $M_1, \dots, M_m$ be positive semi-definite $n \times n$ matrices, and $s_1, \dots, s_m \geq 0$. The determinant of the linear combination $s_1 M_1 + \cdots + s_m M_m$ is a homogeneous polynomial of degree n in the s_i,

$$\det(s_1 M_1 + \cdots + s_m M_m) = \sum_{1 \leq i_1, \dots, i_n \leq m} s_{i_1} \cdots s_{i_n} D(M_{i_1}, \dots, M_{i_n}).$$

The coefficient $D(M_{i_1}, \dots, M_{i_n})$, which depends only on $M_{i_1}, \dots, M_{i_n}$, is defined to be symmetric in its arguments and is called the *mixed discriminant* of $M_{i_1}, \dots, M_{i_n}$. We will denote

$$D(\underbrace{M, \dots, M}_{n-k}, \underbrace{I_n, \dots, I_n}_{k})$$

by $D_k(M)$. Obviously, $D_0(M) = \det(M)$ while $nD_{n-1}(M) = \operatorname{tr}(M)$ is the trace of M.

Let $M_j = (a_{ik}^{(j)})$, $j = 1, \dots, n$. Then the mixed discriminant of $M_1, \dots, M_n$ is given by

$$D(M_1, \dots, M_n) = \frac{1}{n!} \sum_{(j_1, \dots, j_n)} \begin{vmatrix} a_{11}^{(j_1)} & \cdots & a_{1n}^{(j_n)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{(j_1)} & \cdots & a_{nn}^{(j_n)} \end{vmatrix}, \tag{2.20}$$

where we sum over all permutations.

Note that the mixed discriminant $D(M_1, \dots, M_n)$ is linear for each M_j with respect to matrix addition.

The following basic formula of the mixed discriminant is established by Petty [51]: Let $M_1, \dots, M_n$ be the positive semi-definite $n \times n$ matrices defined by

$$M_j = x^{(j)} \otimes x^{(j)} = \begin{pmatrix} \left(x_1^{(j)}\right)^2 & \cdots & x_1^{(j)} x_n^{(j)} \\ \vdots & \ddots & \vdots \\ x_1^{(j)} x_n^{(j)} & \cdots & \left(x_n^{(j)}\right)^2 \end{pmatrix},$$

where $(x^{(j)})^t = (x_1^{(j)}, \dots, x_n^{(j)})$, $j = 1, \dots, n$. Then

$$\begin{aligned} D(M_1, \dots, M_n) &= \frac{1}{n!} \sum_{(j_1, \dots, j_n)} \begin{vmatrix} \left(x_1^{(j_1)}\right)^2 & \cdots & x_1^{(j_n)} x_n^{(j_n)} \\ \vdots & \ddots & \vdots \\ x_1^{(j_1)} x_n^{(j_1)} & \cdots & \left(x_n^{(j_n)}\right)^2 \end{vmatrix} \\ &= \frac{1}{n!} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}^2. \end{aligned} \tag{2.21}$$

Let $r_1, \dots, r_n$ be column vectors in $\mathbb{R}^n$. Denote the determinant of $n \times n$ matrix $(r_1, \dots, r_n)$ by $|r_1, \dots, r_n|$. For an integer j, $1 \le j \le n$, let m_j, $1 \le m_j \le n$, be an integer depending on j. Define the $n \times m_j$ matrices Q^j by

$$Q^j = \big(r_1^{(j)}, \dots, r_{m_j}^{(j)}\big).$$

Thus,

$$Q^j(Q^j)^t = \sum_{i=1}^{m_j} r_i^{(j)}(r_i^{(j)})^t = \begin{pmatrix} \sum_{i=1}^{m_j}\big(r_{i1}^{(j)}\big)^2 & \cdots & \sum_{i=1}^{m_j} r_{i1}^{(j)} r_{in}^{(j)} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{m_j} r_{i1}^{(j)} r_{in}^{(j)} & \cdots & \sum_{i=1}^{m_j}\big(r_{in}^{(j)}\big)^2 \end{pmatrix}. \tag{2.22}$$

Obviously, the $n \times n$ matrix $M_j = Q^j(Q^j)^t$ is positive semi-definite.

Lemma 2.2. *Let M_j be the positive semi-definite matrix defined by $M_j = Q^j(Q^j)^t$, $j = 1, \dots, n$. Then*

$$D(M_1, \dots, M_n) = \frac{1}{n!} \sum_{\substack{k_1 \in \{1,\dots,m_1\} \\ \cdots \\ k_n \in \{1,\dots,m_n\}}} \big|r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)}\big|^2.$$

Proof. From (2.22), linearity of mixed discriminant, and (2.21), we have

$$\begin{aligned} D(M_1, \dots, M_n) &= D\Big(\sum_{i=1}^{m_1} r_i^{(1)} \otimes r_i^{(1)}, \dots, \sum_{i=1}^{m_n} r_i^{(n)} \otimes r_i^{(n)}\Big) \\ &= \sum_{\substack{k_1 \in \{1,\dots,m_1\} \\ \cdots \\ k_n \in \{1,\dots,m_n\}}} D(r_{k_1}^{(1)} \otimes r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)} \otimes r_{k_n}^{(n)}) \\ &= \frac{1}{n!} \sum_{\substack{k_1 \in \{1,\dots,m_1\} \\ \cdots \\ k_n \in \{1,\dots,m_n\}}} \big|r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)}\big|^2. \qquad \square \end{aligned}$$

Let X_j be a compact Hausdorff space and ν_j a finite Borel measure in X_j, $j = 1, \dots, n$. If $M_j : X_j \to \mathbb{M}_n$ is ν_j-integrable, $j = 1, \dots, n$, then by (2.20) it is easily seen that

$$\begin{aligned} &D\Big(\int_{X_1} M_1(v_1)\, d\nu_1(v_1), \dots, \int_{X_n} M_n(v_n)\, d\nu_n(v_n)\Big) \\ &= \int_{X_1} \cdots \int_{X_n} D(M_1(v_1), \dots, M_n(v_n))\, d\nu_1(v_1) \cdots d\nu_n(v_n). \end{aligned} \tag{2.23}$$

3. The Grassmannian Ball–Barthe inequality

The spherical Ball–Barthe inequality (see [42]) is as follows:

The spherical Ball–Barthe inequality. *If ν is an isotropic measure on S^{n-1} with total mass $|\nu| = n$, then for each positive continuous function f on S^{n-1},*

$$\det \int_{S^{n-1}} f(u) u \otimes u \, d\nu(u) \geq \exp\Big(\int_{S^{n-1}} \log f(u) \, d\nu(u) \Big), \tag{3.1}$$

with equality if and only if $f(u_1) \cdots f(u_n)$ is constant for linearly independent unit vectors $u_1, \ldots, u_n \in \operatorname{supp} \nu$.

The spherical Ball–Barthe inequality is crucial for the proof of Theorem 1.1. For a short proof of the discrete case, see Barthe [7]. In this section, we establish the Grassmannian Ball–Barthe inequality.

For $A \in \mathbb{S}_n$, denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A with corresponding eigenspaces $E_1(A), \ldots, E_n(A)$. We always assume that

$$\lambda_1(A) \geq \cdots \geq \lambda_n(A). \tag{3.2}$$

For each $1 \leq i \leq n$, we define a map $\mathcal{P}_i : \mathbb{S}_n \to \mathbb{M}_n$ by

$$\mathcal{P}_i(A) = \mathrm{P}_{E_i(A)}, \quad A \in \mathbb{S}_n. \tag{3.3}$$

Unfortunately, the map $\mathcal{P}_i$ is not continuous. The following lemma about the piecewise continuity of $\mathcal{P}_i$ is needed.

Lemma 3.1. *The map $\mathcal{P}_i : \mathbb{S}_n \to \mathbb{M}_n$ is piecewise continuous, and thus Borel measurable. Moreover, $\mathcal{P}_i$ is continuous in any Borel subset of $\mathbb{S}_n$ whose matrices keep multiplicities of their eigenvalues.*

Proof. Let i be fixed. We will show that there are Borel sets Q_j, $j = 1, \ldots, n$, so that $\mathbb{S}_n = Q_1 \cup \cdots \cup Q_n$, and $\mathcal{P}_i$ is continuous in each Q_j. Moreover, each Q_j is a finite union of Borel sets, and in each of the Borel sets eigenvalues of the matrices keep their multiplicities.

Define Q_j, $j = 1, \ldots, n$, by

$$Q_j = \{A \in \mathbb{S}_n : \lambda_i(A) \text{ is of multiplicity } j\}.$$

Then obviously, $\mathbb{S}_n = Q_1 \cup \cdots \cup Q_n$. Note that $\lambda_1(A), \ldots, \lambda_n(A)$ are continuous with respect to $A \in \mathbb{S}_n$ when they are ordered as in (3.2). Thus for any positive integers $m_1, \ldots, m_k$, $1 \leq k \leq n$, satisfying $m_1 + \cdots + m_k = n$, the set

$$\begin{aligned}\big\{A \in \mathbb{S}_n : \lambda_1(A) = \cdots = \lambda_{m_1}(A) > \lambda_{m_1+1}(A) = \cdots = \lambda_{m_1+m_2}(A) > \cdots \\ > \lambda_{m_1+\cdots+m_{k-1}+1}(A) = \cdots = \lambda_{m_1+\cdots+m_k}(A)\big\}\end{aligned}$$

is a Borel set, in which the multiplicities of eigenvalues do not change, and Q_j as a finite union of such Borel sets is a Borel set.

Fix an orthonormal basis of $\mathbb{R}^n$ and identify $\mathbb{M}_n$ with $\mathbb{L}_n$. For $1 \le j \le n$, define

$$U_j = \big\{\mathrm{P}_\xi \in \mathbb{M}_n : \xi \in G_{n,j}\big\},$$

then $U_j \cap U_l = \emptyset$ when $j \ne l$. It is easily seen that $\mathcal{P}_i(Q_j) = U_j$.

For any $A \in \mathbb{S}_n$ and $x \in \mathbb{R}^n$, $\mathcal{P}_i(A)x \in E_i(A)$ is an eigenvector of A. Thus,

$$(A - \lambda_i(A)I_n)\mathcal{P}_i(A) = 0. \tag{3.4}$$

By the continuity of eigenvalues, for a fixed $A_0 \in \mathbb{S}_n$ and any $\epsilon > 0$, there is $\delta > 0$ so that

$$|(A - \lambda_i(A)I_n) - (A_0 - \lambda_i(A_0)I_n)|_2 < \epsilon, \tag{3.5}$$

when $|A - A_0|_2 < \delta$ and $A \in \mathbb{S}_n$. By (3.4), (2.16), (2.17), and (3.5), for $|x| = 1$, we have

$$\begin{aligned}|(A_0 - \lambda_i(A_0)I_n)\mathcal{P}_i(A)x| &= |(A_0 - \lambda_i(A_0)I_n)\mathcal{P}_i(A)x - (A - \lambda_i(A)I_n)\mathcal{P}_i(A)x| \\ &\le \|(A_0 - \lambda_i(A_0)I_n) - (A - \lambda_i(A)I_n)\|_2\,|\mathcal{P}_i(A)x| \\ &\le |(A_0 - \lambda_i(A_0)I_n) - (A - \lambda_i(A)I_n)|_2 \\ &< \epsilon.\end{aligned} \tag{3.6}$$

For a fixed $A_0 \in Q_j$, let

$$\eta_0 = \min_l \big\{|\lambda_l(A_0) - \lambda_i(A_0)| : \lambda_l(A_0) \ne \lambda_i(A_0)\big\}.$$

Then $\eta_0 > 0$. For $A \in Q_j$, again by the continuity of eigenvalues, when A is close to A_0, $\lambda_l(A_0) \ne \lambda_i(A_0)$ implies $\lambda_l(A) \ne \lambda_i(A)$, and also $\lambda_l(A_0) = \lambda_i(A_0)$ implies $\lambda_l(A) = \lambda_i(A)$ because $\lambda_i(A_0)$ and $\lambda_i(A)$ have the same multiplicity j. Thus, $\lambda_l(A) \ne \lambda_i(A)$ if and only if $\lambda_l(A_0) \ne \lambda_i(A_0)$ when A is close to A_0. By this fact and the continuity of eigenvalues, there exists $0 < \delta_1 < \delta$ so that

$$\min_l \big\{|\lambda_l(A) - \lambda_i(A)| : \lambda_l(A) \ne \lambda_i(A)\big\} > \frac{\eta_0}{2}, \tag{3.7}$$

when $|A - A_0|_2 < \delta_1$ and $A \in Q_j$.

By (3.4), (3.7), and (2.19), we have

$$\begin{aligned} |(A-\lambda_i(A)I_n)x| &= |(A-\lambda_i(A)I_n)(\mathrm{P}_{E_i(A)}x+\mathrm{P}_{E_i^\perp(A)}x)| \\ &= |(A-\lambda_i(A)I_n)\mathrm{P}_{E_i^\perp(A)}x| \\ &\geq \frac{\eta_0}{2}|\mathrm{P}_{E_i^\perp(A)}x|, \end{aligned} \tag{3.8}$$

when $|A-A_0|_2<\delta_1$, $A\in Q_j$, and $x\in\mathbb{R}^n$.

By (3.8) and (3.6), for $|x|=1$, $|A-A_0|_2<\delta_1$, and $A\in Q_j$, we have

$$\begin{aligned} |\mathcal{P}_i(A)x-\mathcal{P}_i(A_0)\mathcal{P}_i(A)x| &= |\mathrm{P}_{E_i^\perp(A_0)}\mathcal{P}_i(A)x| \\ &\leq \frac{2}{\eta_0}|(A_0-\lambda_i(A_0)I_n)\mathcal{P}_i(A)x| \\ &< \frac{2\epsilon}{\eta_0}, \end{aligned}$$

and similarly,

$$|\mathcal{P}_i(A_0)x-\mathcal{P}_i(A)\mathcal{P}_i(A_0)x|<\frac{2\epsilon}{\eta_0}.$$

Therefore, when $|A-A_0|_2<\delta_1$ and $A\in Q_j$, by (2.16) and (2.17), we have

$$|\mathcal{P}_i(A)-\mathcal{P}_i(A_0)\mathcal{P}_i(A)|_2\leq\frac{2\sqrt{n}\epsilon}{\eta_0}, \tag{3.9}$$

$$|\mathcal{P}_i(A_0)-\mathcal{P}_i(A)\mathcal{P}_i(A_0)|_2\leq\frac{2\sqrt{n}\epsilon}{\eta_0}. \tag{3.10}$$

Applying transpose to the matrices in (3.10) gives

$$|\mathcal{P}_i(A_0)-\mathcal{P}_i(A_0)\mathcal{P}_i(A)|_2\leq\frac{2\sqrt{n}\epsilon}{\eta_0}, \tag{3.11}$$

when $|A-A_0|_2<\delta_1$ and $A\in Q_j$.

Finally, by (3.9) and (3.11), we conclude that

$$|\mathcal{P}_i(A_0)-\mathcal{P}_i(A)|_2\leq\frac{4\sqrt{n}\epsilon}{\eta_0},$$

for a fixed $A_0\in Q_j$ and any $A\in Q_j$ with $|A-A_0|_2<\delta_1$. This shows the continuity of $\mathcal{P}_i$ in Q_j. It follows that $\mathcal{P}_i$ is piecewise continuous and thus Borel measurable. $\square$

For $\xi\in G_{n,m}$, let $\mathrm{P}_\xi:\mathbb{R}^n\to\xi$ be the projection map, and let $I_\xi:\xi\to\mathbb{R}^n$ be the inclusion map, which is the identity map from ξ to ξ. For each $\xi\in G_{n,m}$, associate with it a positive definite linear transformation $A_\xi \in \mathbb{L}_\xi$. Then, we define a map $A : G_{n,m} \to \mathbb{S}_n$ by

$$A(\xi) = I_\xi A_\xi \mathrm{P}_\xi. \tag{3.12}$$

The matrix $A(\xi)$ can be written explicitly if an orthonormal basis of ξ is chosen. Let $\{u_1, \ldots, u_m\}$ be an orthonormal basis of ξ. Let $Q = (u_1, \ldots, u_m)$ be the $n \times m$ matrix of column vectors in $\mathbb{R}^n$. Let $S = (a_{ij})$ be the symmetric matrix of A_ξ under the basis of $\{u_1, \ldots, u_m\}$. Then

$$A(\xi) = QSQ^t. \tag{3.13}$$

This can be easily seen because for $x \in \mathbb{R}^n$,

$$A_\xi u_i = \sum_{j=1}^m a_{ji} u_j, \quad QSQ^t x = \sum_{i,j=1}^m a_{ji}(x \cdot u_i) u_j,$$

therefore, by (2.11), we have

$$\begin{aligned} I_\xi A_\xi \mathrm{P}_\xi x &= I_\xi A_\xi \Big(\sum_{i=1}^m (x \cdot u_i) u_i \Big) = I_\xi \Big(\sum_{i,j=1}^m (x \cdot u_i) a_{ji} u_j \Big) \\ &= \sum_{i,j=1}^m a_{ji}(x \cdot u_i) u_j = QSQ^t x. \end{aligned}$$

Thus, $A(\xi)$ is a positive semi-definitive matrix. Let $\lambda_1(A(\xi)), \ldots, \lambda_n(A(\xi))$ be the eigenvalues of $A(\xi)$ so that $\lambda_1(A(\xi)) \geq \cdots \geq \lambda_n(A(\xi))$. Then $\lambda_1(A(\xi)) \geq \cdots \geq \lambda_m(A(\xi)) > 0$, and $\lambda_{m+1}(A(\xi)) = \cdots = \lambda_n(A(\xi))=0$. Obviously, $\lambda_1(A(\xi)), \ldots, \lambda_m(A(\xi))$ are exactly the eigenvalues of A_ξ, denoted by $\lambda_1(A_\xi), \ldots, \lambda_m(A_\xi)$. In particular,

$$\det A_\xi = \det S = \lambda_1(A_\xi) \cdots \lambda_m(A_\xi). \tag{3.14}$$

It is also noted that the null space of $A(\xi)$ is $\xi^\perp$, that is,

$$\xi^\perp = \{x \in \mathbb{R}^n : A(\xi)x = 0\}, \tag{3.15}$$

and ξ is the direct sum of the eigenspaces of nonzero eigenvalues of $A(\xi)$. We denote by $E_1(A_\xi), \ldots, E_m(A_\xi)$, the eigenspaces corresponding to the eigenvalues $\lambda_1(A_\xi), \ldots, \lambda_m(A_\xi)$. Let $m_i(A_\xi)$ be the dimension of $E_i(A_\xi)$, which is also the multiplicity of $\lambda_i(A_\xi)$, $k = 1, \ldots, m$.

The following lemma gives decomposition formulas for the projection map P_ξ and the matrix $A(\xi)$, which are needed to establish the Grassmannian Ball–Barthe inequality.

Lemma 3.2. *For $\xi \in G_{n,m}$, if A_ξ is a positive definite linear transformation in ξ and $A(\xi)$ is the positive semi-definite matrix $\mathrm{I}_\xi A_\xi \mathrm{P}_\xi$, then*

$$\mathrm{P}_\xi = \frac{1}{m_1(A_\xi)}\mathrm{P}_{E_1(A_\xi)} + \cdots + \frac{1}{m_m(A_\xi)}\mathrm{P}_{E_m(A_\xi)}, \tag{3.16}$$

and

$$A(\xi) = \frac{\lambda_1(A_\xi)}{m_1(A_\xi)}\mathrm{P}_{E_1(A_\xi)} + \cdots + \frac{\lambda_m(A_\xi)}{m_m(A_\xi)}\mathrm{P}_{E_m(A_\xi)}. \tag{3.17}$$

Proof. For an orthonormal basis $\{u_1, ..., u_m\}$ of ξ, let Q be the $n \times m$ matrix with column vectors $u_1, ..., u_m$, then

$$\mathrm{P}_\xi = QQ^t. \tag{3.18}$$

Choose $u_1, ..., u_m$ as eigenvectors of the positive definite linear transformation A_ξ corresponding to eigenvalues $\lambda_1(A_\xi), \dots, \lambda_m(A_\xi)$. Then the matrix of A_ξ under the basis $\{u_1, ..., u_m\}$ is a diagonal matrix $\mathrm{diag}\{\lambda_1(A_\xi), \dots, \lambda_m(A_\xi)\}$. By (3.13),

$$A(\xi) = Q\,\mathrm{diag}\{\lambda_1, \dots, \lambda_m\}\,Q^t. \tag{3.19}$$

Assume that A_ξ has k distinct eigenvalues,

$$\lambda_1 = \cdots = \lambda_{l_1} > \lambda_{l_1+1} = \cdots = \lambda_{l_2} > \cdots > \lambda_{l_{k-1}+1} = \cdots = \lambda_{l_k} = \lambda_m, \tag{3.20}$$

where $l_i - l_{i-1} = m_{l_i}$, $i = 1, \dots, k$, that is, $\lambda_{l_1}, \dots, \lambda_{l_k}$ are distinct eigenvalues with multiplicities $m_1, \dots, m_k$.

Let E_{l_i} be the eigenspace corresponding to λ_{l_i}, which is spanned by $u_{l_{i-1}+1}, \dots, u_{l_i}$, $i = 1, \dots, k$. Then

$$E_1 = \cdots = E_{l_1},\ E_{l_1+1} = \cdots = E_{l_2},\ \dots,\ E_{l_{k-1}+1} = \cdots = E_{l_k} = E_m. \tag{3.21}$$

Let $Q_i = (u_{l_{i-1}+1}, \dots, u_{l_i})$, $i = 1, \dots, k$. Then $Q = (Q_1, \dots, Q_k)$. Thus, by (2.10),

$$QQ^t = Q_1Q_1^t + \cdots + Q_kQ_k^t = \mathrm{P}_{E_{l_1}} + \cdots + \mathrm{P}_{E_{l_k}}. \tag{3.22}$$

By (3.18), (3.21), and (3.22), we get (3.16). We also have

$$\begin{aligned} Q\,\mathrm{diag}\{\lambda_1, \dots, \lambda_m\}\,Q^t &= \lambda_{l_1}Q_1Q_1^t + \cdots + \lambda_{l_k}Q_kQ_k^t \\ &= \lambda_{l_1}\mathrm{P}_{E_{l_1}} + \cdots + \lambda_{l_k}\mathrm{P}_{E_{l_k}}. \end{aligned} \tag{3.23}$$

By (3.19), (3.20), (3.21), and (3.23), we get (3.17). □

We shall require the following elementary inequality.

Lemma 3.3. *Suppose that Y is a measure space with measure ν. For $i = 1, \dots, l$, let $a_i : Y \to (0, \infty)$ be positive functions and let $f_i : Y \to [0, \infty)$ be nonnegative functions. If $\int_Y \sum_{i=1}^l f_i(v) d\nu(v) = 1$ and $\sum_{i=1}^l a_i(v) f_i(v)$ is integrable, then*

$$\int\limits_Y \Big(\sum_{i=1}^l a_i(v) f_i(v) \Big) d\nu(v) \geq \exp\Big[\int\limits_Y \log \Big(\prod_{i=1}^l a_i(v)^{f_i(v)} \Big) d\nu(v) \Big],$$

with equality if and only if $a_i(v)$ is constant whenever $f_i(v) > 0$, $i = 1, \dots, l$, for ν-a.e. $v \in Y$.

Proof. Let

$$E = \Big\{ v \in Y : \sum_{i=1}^l f_i(v) \neq 0 \Big\}.$$

Then we have

$$\int\limits_Y \sum_{i=1}^l f_i(v) d\nu(v) = \int\limits_E \sum_{i=1}^l f_i(v) d\nu(v) = 1.$$

By Jensen's inequality and the geometric–arithmetic inequality, it follows that

$$\begin{aligned}
\int\limits_Y \sum_{i=1}^l a_i(v) f_i(v) d\nu(v) &= \int\limits_E \frac{\sum_{i=1}^l a_i(v) f_i(v)}{\sum_{i=1}^l f_i(v)} \sum_{i=1}^l f_i(v) d\nu(v) \\
&\geq \exp\Big[\int\limits_E \log \Big(\frac{\sum_{i=1}^l a_i(v) f_i(v)}{\sum_{i=1}^l f_i(v)} \Big) \sum_{i=1}^l f_i(v) d\nu(v) \Big] \\
&\geq \exp\Big[\int\limits_E \log \Big(\prod_{i=1}^l a_i(v)^{\frac{f_i(v)}{\sum_{i=1}^l f_i(v)}} \Big) \sum_{i=1}^l f_i(v) d\nu(v) \Big] \\
&= \exp\Big[\int\limits_E \log \Big(\prod_{i=1}^l a_i(v)^{f_i(v)} \Big) d\nu(v) \Big] \\
&= \exp\Big[\int\limits_Y \log \Big(\prod_{i=1}^l a_i(v)^{f_i(v)} \Big) d\nu(v) \Big].
\end{aligned}$$

Equalities in the inequalities hold if and only if $\frac{\sum_{i=1}^l a_i(v) f_i(v)}{\sum_{i=1}^l f_i(v)}$ is constant for ν-a.e. $v \in E$, and $a_i(v) = c$ (constant) for those i so that $f_i(v) > 0$ for ν-a.e. $v \in E$. This means that $a_i(v) = c$ whenever $f_i(v) > 0$, $i = 1, \dots, l$, for ν-a.e. $v \in Y$. $\square$

The following is the Grassmannian Ball–Barthe inequality on Grassmann manifolds. For the case of unit sphere, it becomes the spherical Ball–Barthe inequality (3.1).

Lemma 3.4 *(Grassmannian Ball–Barthe inequality). Let ν be a finite Borel measure on $G_{n,m}$. Associate with each $\xi \in G_{n,m}$ a positive definite $A_\xi \in \mathbb{L}_\xi$. Suppose that the map $A : G_{n,m} \to \mathbb{S}_n$ defined by $A(\xi) = I_\xi A_\xi \mathrm{P}_\xi$ is continuous and ν satisfies*

$$\int_{G_{n,m}} \mathrm{P}_\xi \, d\nu(\xi) = I_n.$$

Then

$$\det \int_{G_{n,m}} A(\xi)\, d\nu(\xi) \geq \exp\Big(\int_{G_{n,m}} \log(\det A_\xi)\, d\nu(\xi)\Big), \tag{3.24}$$

with equality if and only if $\lambda_{k_1}(A(\xi_1)) \cdots \lambda_{k_n}(A(\xi_n))$ is constant for $\xi_j \in \operatorname{supp}\nu$ whenever there exist n linearly independent eigenvectors belonging to positive eigenvalues $\lambda_{k_1}(A(\xi_1)), \ldots, \lambda_{k_n}(A(\xi_n))$ of $A(\xi_1), \ldots, A(\xi_n)$, $k_j = 1, \ldots, m$, and $j = 1, ..., n$.

Proof. For simplicity, write $\mathcal{P}_i(\xi) = \mathcal{P}_i(A(\xi))$, $m_i(\xi) = m_i(A(\xi))$, and $\lambda_i(\xi) = \lambda_i(A(\xi))$, where $\xi \in G_{n,m}$. Let

$$L = \big\{(k_1, ..., k_n) : k_i \in \{1, ..., m\},\ i \in \{1, 2, ..., n\}\big\},$$
$$L_i = \big\{(k_1, ..., \hat{k}_i, ..., k_n) : k_j \in \{1, ..., m\},\ j \in \{1, ..., \hat{\imath}, \ldots, n\}\big\},$$

where $i \in \{1, ..., n\}$, and $\hat{k}_i$ and $\hat{\imath}$ mean that the i-th entry k_i and i are missing.

Since $m_i(\xi) = \dim E_i(A(\xi))$, the function m_i is piecewise continuous on $G_{n,m}$. The map $\mathcal{P}_i(\xi)$ is obviously bounded on $G_{n,m}$. By these facts, the continuity of $A(\xi)$, and Lemma 3.1, $\mathcal{P}_i(\xi)$ and $m_i(\xi)$ are ν-integrable. Therefore, for each $(k_1, \ldots, k_n)$, the discriminant $D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)}\Big)$ is integrable with respect to $\nu \otimes \cdots \otimes \nu$.

By the isotropy of ν, (3.16), and (2.23), we have

$$\begin{aligned} 1 &= \det \int_{G_{n,m}} \mathrm{P}_\xi \, d\nu(\xi) \\ &= D\Big(\int_{G_{n,m}} \mathrm{P}_\xi \, d\nu(\xi), \ldots, \int_{G_{n,m}} \mathrm{P}_\xi \, d\nu(\xi)\Big) \\ &= \int_{G_{n,m}} \cdots \int_{G_{n,m}} \sum_L D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)}\Big)\, d\nu(\xi_1) \cdots d\nu(\xi_n), \end{aligned} \tag{3.25}$$

and

$$
\begin{aligned}
&D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, I_n, \ldots, I_n\Big) \\
&\quad = D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \int_{G_{n,m}} \mathrm{P}_\xi \, d\nu(\xi), \ldots, \int_{G_{n,m}} \mathrm{P}_\xi \, d\nu(\xi)\Big) \\
&\quad = \int_{G_{n,m}} \cdots \int_{G_{n,m}} \sum_{L_1} D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \frac{\mathcal{P}_{k_2}(\xi_2)}{m_{k_2}(\xi_2)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)}\Big) \, d\nu(\xi_2) \cdots d\nu(\xi_n).
\end{aligned}
\tag{3.26}
$$

Since $A(\xi)$ is continuous, the eigenvalue $\lambda_i(\xi)$ is continuous. Therefore, by (3.17), (2.23), and Lemma 3.3, we have

$$
\begin{aligned}
&\det \int_{G_{n,m}} A(\xi) d\nu(\xi) \\
&= D\Big(\int_{G_{n,m}} A(\xi_1) d\nu(\xi_1), .., \int_{G_{n,m}} A(\xi_n) d\nu(\xi_n)\Big) \\
&= \int_{G_{n,m}} \cdots \int_{G_{n,m}} D\Big(\sum_{k=1}^{m} \frac{\lambda_k(\xi_1)}{m_k(\xi_1)} \mathcal{P}_k(\xi_1), \ldots, \sum_{k=1}^{m} \frac{\lambda_k(\xi_n)}{m_k(\xi_n)} \mathcal{P}_k(\xi_n)\Big) \, d\nu(\xi_1) \cdots d\nu(\xi_n) \\
&= \int_{G_{n,m}} \cdots \int_{G_{n,m}} \sum_{L} \lambda_{k_1}(\xi_1) \cdots \lambda_{k_n}(\xi_n) D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)}\Big) \, d\nu(\xi_1) \cdots d\nu(\xi_n) \\
&\geq \exp\Big\{\int_{G_{n,m}} \cdots \int_{G_{n,m}} \log\Big(\prod_{L} (\lambda_{k_1} \cdots \lambda_{k_n})^{D(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)})}\Big) \, d\nu(\xi_1) \cdots d\nu(\xi_n)\Big\} \\
&= \exp\Big\{\int_{G_{n,m}} \cdots \int_{G_{n,m}} \log\Big(\prod_{i=1}^{n} \prod_{k_i=1}^{m} \lambda_{k_i}(\xi_i)^{\sum_{L_i} D(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)})} \, d\nu(\xi_1) \cdots d\nu(\xi_n)\Big)\Big\} \\
&= \exp\Big\{\int_{G_{n,m}} \cdots \int_{G_{n,m}} \sum_{i=1}^{n} \Big(\sum_{k_i=1}^{m} \Big(\log \lambda_{k_i}(\xi_i) \sum_{L_i} D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \frac{\mathcal{P}_{k_2}(\xi_2)}{m_{k_2}(\xi_2)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)}\Big)\Big)\Big) \\
&\qquad d\nu(\xi_1) \cdots d\nu(\xi_n)\Big\} \\
&= \exp\Big\{\int_{G_{n,m}} \cdots \int_{G_{n,m}} n \sum_{k_1=1}^{m} \Big(\log \lambda_{k_1}(\xi_1) \sum_{L_1} D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \frac{\mathcal{P}_{k_2}(\xi_2)}{m_{k_2}(\xi_2)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)}\Big)\Big) \\
&\qquad d\nu(\xi_1) \cdots d\nu(\xi_n)\Big\}.
\end{aligned}
\tag{3.27}
$$

The last equality used the following equation,

$$
\int_{G_{n,m}} \cdots \int_{G_{n,m}} \sum_{k_1=1}^{m} \Big(\log \lambda_{k_1}(\xi_1) \sum_{L_1} D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \frac{\mathcal{P}_{k_2}(\xi_2)}{m_{k_2}(\xi_2)}, \ldots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)}\Big)\Big) d\nu(\xi_1) \cdots d\nu(\xi_n)
$$

$$= \int\limits_{G_{n,m}} \cdots \int\limits_{G_{n,m}} \sum_{k_i=1}^{m} \Big(\log \lambda_{k_i}(\xi_i) \sum_{L_i} D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \frac{\mathcal{P}_{k_2}(\xi_2)}{m_{k_2}(\xi_2)}, \dots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)} \Big) \Big)$$
$$d\nu(\xi_1) \cdots d\nu(\xi_n).$$

Since $nD\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, I_n, ..., I_n\Big) = \operatorname{trace}(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}) = 1$, by (3.27) and (3.26), we have

$$\det \int\limits_{G_{n,m}} A(\xi)\, d\nu(\xi) \geq \exp\Big(\int\limits_{G_{n,m}} \sum_{k_1=1}^{m} \log \lambda_{k_1}(\xi_1)\, d\nu(\xi_1) \Big)$$
$$= \exp\Big(\int\limits_{G_{n,m}} \log(\det A_\xi)\, d\nu(\xi) \Big).$$

By the equality conditions in Lemma 3.3, equality holds in the last inequality if and only if $\lambda_{k_1}(\xi_1) \cdots \lambda_{k_n}(\xi_n)$ is constant when

$$D\Big(\frac{\mathcal{P}_{k_1}(\xi_1)}{m_{k_1}(\xi_1)}, \dots, \frac{\mathcal{P}_{k_n}(\xi_n)}{m_{k_n}(\xi_n)} \Big) \neq 0, \tag{3.28}$$

for almost all $(\xi_1, ..., \xi_n)$. Since $\lambda_{k_1}(\xi_1) \cdots \lambda_{k_n}(\xi_n)$ is continuous, the equality condition is equivalent to $\lambda_{k_1}(\xi_1) \cdots \lambda_{k_n}(\xi_n)$ is constant whenever (3.28) holds for $(\xi_1, ..., \xi_n) \in \operatorname{supp}(\nu \times \cdots \times \nu)$.

By Lemma 2.2, (3.28) is equivalent to there exist linearly independent unit vectors $r_{k_i}(\xi_i) \in E_{k_i}(A_{\xi_i})$, $i = 1, \dots, n$. This, together with the identity $\operatorname{supp}(\nu \times \cdots \times \nu) = \operatorname{supp} \nu \times \cdots \times \operatorname{supp} \nu$, implies that the equality in the inequality (3.24) holds if and only if $\lambda_{k_1}(\xi_1) \cdots \lambda_{k_n}(\xi_n)$ is constant whenever there exist linearly independent unit vectors $r_{k_i} \in E_{k_i}(A_{\xi_i})$, for $\xi_i \in \operatorname{supp} \nu$, $i = 1, ..., n$. □

4. Cross measures on Grassmann manifolds

An ℓ_p^n-ball is the spherical L_p cosine transform of a cross measure on the unit sphere. We extend the concept of cross measure to Grassmann manifolds. The L_p cosine transform of cross measures on Grassmann manifolds defines a new class of convex bodies which are generalizations of the ℓ_p^n-balls and are extremal bodies of our volume inequalities.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. For $1 \leq m \leq n-1$, let $[m, n]$ be the least common multiple of m and n. For $n \leq j \leq [m, n]$, define e_j by

$$e_j = e_i, \quad \text{if } j = i \bmod(n) \text{ and } i = 1, 2, \dots, n.$$

Let

$$\tilde{\xi}_k = \operatorname{span}\{e_{(k-1)m+1}, e_{(k-1)m+2}, \dots, e_{km}\}, \quad 1 \leq k \leq l, \quad l = \frac{[m, n]}{m}, \tag{4.1}$$

be the m-dimensional subspace spanned by the vectors.

A discrete measure on $G_{n,m}$ that is concentrated on the subspaces $\tilde{\xi}_1, \ldots, \tilde{\xi}_l$ with equal mass on each subspace is an isotropic measure on $G_{n,m}$. In fact, for $1 \le k \le l$ and $x \in \mathbb{R}^n$, we have

$$|\mathrm{P}_{\tilde{\xi}_k} x|^2 = \sum_{i=1}^{m} |x \cdot e_{(k-1)m+i}|^2. \tag{4.2}$$

Thus,

$$\sum_{k=1}^{l} |\mathrm{P}_{\tilde{\xi}_k} x|^2 = \sum_{k=1}^{l} \sum_{i=1}^{m} |x \cdot e_{(k-1)m+i}|^2 = \frac{[m,n]}{n} \sum_{i=1}^{n} |x \cdot e_i|^2 = \frac{[m,n]}{n} |x|^2.$$

This gives

$$\frac{1}{l} \sum_{k=1}^{l} |\mathrm{P}_{\tilde{\xi}_k} x|^2 = \frac{m}{n} |x|^2,$$

which can be written as

$$\frac{1}{l} \sum_{k=1}^{l} \mathrm{P}_{\tilde{\xi}_k} = \frac{m}{n} I_n. \tag{4.3}$$

Therefore, by (1.5), a discrete measure on $G_{n,m}$ that is concentrated equally on $\tilde{\xi}_k$, $1 \le k \le l$, is isotropic. Moreover, a discrete measure on $G_{n,m}$ that is concentrated equally on $O\tilde{\xi}_k$, $1 \le k \le l$, where $O \in \mathrm{O}(n)$ is a rotation in $\mathbb{R}^n$, is isotropic. Such a discrete measure on $G_{n,m}$ is called a *cross measure* on the Grassmann manifold $G_{n,m}$.

If $m = 1$, then $\frac{[m,n]}{m} = n$. A cross measure on $G_{n,1}$ is concentrated equally on n orthogonal 1-dimensional subspaces of $\mathbb{R}^n$. It is the same as a cross measure on S^{n-1}.

If $\frac{n}{m}$ is an integer, then $[m,n] = n$, and thus $\frac{n}{m} = \frac{[m,n]}{m} = l$. The l orthogonal m-dimensional subspaces $\tilde{\xi}_k$ are the following,

$$\tilde{\xi}_1 = \mathrm{span}\{e_1, \ldots, e_m\}, \ \tilde{\xi}_2 = \mathrm{span}\{e_{m+1}, \ldots, e_{2m}\}, \ \ldots, \ \tilde{\xi}_l = \mathrm{span}\{e_{n-m+1}, \ldots, e_n\}.$$

In this case, by (4.3), a cross measure μ satisfies $\mathrm{supp}\,\mu = \{\tilde{\xi}_1, \ldots, \tilde{\xi}_l\}$ with

$$\sum_{k=1}^{l} \mathrm{P}_{\tilde{\xi}_k} = I_n. \tag{4.4}$$

Using the subspaces from (4.1), we define the convex body

$$B_p^{n,m} = \Big\{ x \in \mathbb{R}^n : \frac{1}{l} \sum_{k=1}^{l} |\mathrm{P}_{\tilde{\xi}_k} x|^p \le \frac{m}{n} \Big\}. \tag{4.5}$$

This is the convex body $Z_{m,p}^*(\mu)$ when μ is a cross measure on the Grassmannian $G_{n,m}$. The case $m = 1$ is the ℓ_p^n-ball, that is, $B_p^{n,1} = B_p^n$. We call the convex body $B_p^{n,m}$ the $\ell_p^{n,m}$-ball.

Let a be a positive real number. Denote by $\lfloor a \rfloor$ the largest integer such that $\lfloor a \rfloor \leq a$.

Lemma 4.1. *Let $l = \lfloor \frac{n}{m} \rfloor$. If μ is an isotropic probability measure on $G_{n,m}$, then there are at least l elements $\xi_1, \ldots, \xi_l \in \operatorname{supp} \mu$, and $\operatorname{supp} \mu$ contains exactly l elements if and only if $l = \frac{n}{m}$ and μ is a cross measure.*

Proof. Assume that $\{\xi_1, \ldots, \xi_k\} = \operatorname{supp} \mu$. Since μ is isotropic, $\operatorname{span}\{\xi_1, \ldots, \xi_k\} = \mathbb{R}^n$. This implies that $km \geq n$, and thus, $k \geq \frac{n}{m} \geq l$.

Suppose $k = l$. Then $l = \frac{n}{m}$. Let $\delta_i = \mu(\{\xi_i\})$. Since μ is an isotropic probability measure on $G_{n,m}$, we have for all $x \in \mathbb{R}^n$

$$\sum_{i=1}^{l} \delta_i |\mathrm{P}_{\xi_i} x|^2 = \frac{m}{n} |x|^2. \tag{4.6}$$

Let $u \in \xi_j$ be a unit vector. (4.6) gives

$$\sum_{i=1}^{l} \delta_i |\mathrm{P}_{\xi_i} u|^2 = \frac{m}{n}. \tag{4.7}$$

This implies that $\delta_j \leq \frac{m}{n}$ because $|\mathrm{P}_{\xi_j} u| = 1$. Since μ is a probability measure, $\sum_{j=1}^{l} \delta_j = 1$, and hence, $\delta_j = \frac{m}{n}$. From this and (4.7) we see that $|\mathrm{P}_{\xi_i} u| = 0$ for $i \neq j$. Therefore, $\xi_1, \ldots, \xi_l$ are mutually orthogonal, and thus μ is in fact a cross measure on $G_{n,m}$. $\square$

5. Volume of convex bodies associated with cross measures

For $p \in [1, \infty]$, let $Z_{m,p}^* = Z_{m,p}^*(\mu)$ and $Z_{m,p} = Z_{m,p}(\mu)$ be the convex bodies defined in (1.2)–(1.4). Recall that the support function of $Z_{m,p}$ is given by

$$h_{Z_{m,p}}(x) = \Big(\frac{n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi x|^p \, d\mu(\xi) \Big)^{\frac{1}{p}}, \quad x \in \mathbb{R}^n, \tag{5.1}$$

and for $p = \infty$,

$$h_{Z_{m,\infty}}(x) = \max_{\xi \in \operatorname{supp} \mu} |\mathrm{P}_\xi x|, \quad x \in \mathbb{R}^n. \tag{5.2}$$

Observe that a measure μ on $G_{n,m}$ is isotropic if and only if $Z_{m,2}$ is a ball. Our normalization implies that a probability measure μ on $G_{n,m}$ is isotropic if and only if $Z_{m,2}$ is the unit ball B_2^n.

Suppose $\frac{n}{m} = l$ is an integer. If μ is a cross measure on $G_{n,m}$, the volumes of $Z_{m,p}^*$ and $Z_{m,p}$ can be explicitly calculated.

Lemma 5.1. *Suppose that $\frac{n}{m} = l$ is an integer and $p \in [1, \infty]$. If μ is a probability cross measure on $G_{n,m}$, then we have $V(Z^*_{m,p}) = \alpha_n(m, p)$.*

Proof. Note that when μ is a cross measure on $G_{n,m}$ with $|\mu| = 1$, the mass of μ at a point is $\frac{m}{n}$. Let $p \in [1, \infty)$. From (2.5), (2.4), (5.1), and (4.2), the volume of $Z^*_{m,p}$ is given by

$$\begin{aligned}
V\big(Z^*_{m,p}\big) &= \frac{1}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} e^{-\|x\|^p_{Z^*_{m,p}}} dx \\
&= \frac{1}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \exp\Bigg(-\frac{n}{m} \sum_{i=1}^{l} \frac{m}{n} |\mathrm{P}_{\tilde{\xi}_i} x|^p \Bigg) dx \\
&= \frac{1}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \exp\Bigg(-\sum_{i=1}^{l} \Big(\sum_{j=1}^{m} |x \cdot e_{(i-1)m+j}|^2 \Big)^{\frac{p}{2}} \Bigg) dx \\
&= \frac{1}{\Gamma(1+\frac{n}{p})} \Bigg(\int_{\tilde{\xi}_1} e^{-({x_1}^2+\cdots+x_m^2)^{\frac{p}{2}}} dx_1 \cdots dx_m \Bigg)^l \\
&= \frac{1}{\Gamma(1+\frac{n}{p})} \Bigg(\int_{\mathbb{R}^m} e^{-|z|^p} dz \Bigg)^{\frac{n}{m}} \\
&= \frac{\big(\omega_m \Gamma(1+\frac{m}{p})\big)^{\frac{n}{m}}}{\Gamma(1+\frac{n}{p})} \\
&= \alpha_n(m, p).
\end{aligned}$$

When $p = \infty$, by (5.2), (2.1), and (2.4), we have

$$Z^*_{m,\infty} = \{x \in \mathbb{R}^n : \max_{\xi \in \operatorname{supp} \mu} |\mathrm{P}_\xi x| \le 1\}.$$

Thus, when μ is the cross measure on $G_{n,m}$ with $|\mu| = 1$, we get for each $x \in Z^*_{m,\infty}$

$$\exp\Bigg(\sum_{i=1}^{l} \log \mathbf{1}_{\tilde{\xi}_i \cap B_2^n} (\mathrm{P}_{\tilde{\xi}_i} x) \Bigg) = 1.$$

Then we have

$$\begin{aligned}
V(Z^*_{m,\infty}) &= \int_{\mathbb{R}^n} \exp\Bigg(\sum_{i=1}^{l} \log \mathbf{1}_{\tilde{\xi}_i \cap B_2^n} (\mathrm{P}_{\tilde{\xi}_i} x) \Bigg) dx \\
&= \int_{\mathbb{R}^n} \exp\Bigg(\sum_{i=1}^{l} \log \mathbf{1}_{\tilde{\xi}_i \cap B_2^n} (x_{(i-1)m+1}, \cdots, x_{im}) \Bigg) dx
\end{aligned}$$

$$= \Big(\int\limits_{\mathbb{R}^n \cap \tilde{\xi}_1} \mathbf{1}_{\tilde{\xi}_1 \cap B_2^n}(x_1, \cdots, x_m)\, dx_1 \cdots dx_m \Big)^l$$

$$= \omega_m^{\frac{n}{m}}. \qquad \square$$

6. A dual definition for the norm $\|\cdot\|_{Z_{m,p}}$

The definition of the support function of convex body $Z_{m,p}$ is given by the L_p cosine transform, which will be used to show the isoperimetric inequality in the volume inequalities for $Z_{m,p}$. To prove the reverse isoperimetric inequality in the volume inequalities for $Z_{m,p}$, we give a definition for the norm $\|\cdot\|_{Z_{m,p}}$. The classical duality between the ℓ_p^n-ball and the $\ell_{p'}^n$-ball is generalized to $\ell_p^{n,m}$-balls.

Let μ be a finite Borel measure on $G_{n,m}$, and let $f : G_{n,m} \to \mathbb{R}^n$ be a continuous map so that for each $\xi \in G_{n,m}$, $f(\xi) \in \xi$. Write $f(\xi)$ as f_ξ. Define

$$\|f : \mu\|_p = \Big(\frac{n}{m} \int\limits_{G_{n,m}} |f_\xi|^p d\mu(\xi) \Big)^{\frac{1}{p}}, \quad p \in [1, \infty),$$

and

$$\|f : \mu\|_\infty = \sup_\xi |f_\xi|.$$

Define $\tilde{f} \in \mathbb{R}^n$ by

$$\tilde{f} = \frac{n}{m} \int\limits_{G_{n,m}} f_\xi \, d\mu(\xi) = \frac{n}{m} \int\limits_{G_{n,m}} \mathrm{P}_\xi f_\xi \, d\mu(\xi). \tag{6.1}$$

Obviously, for $\lambda > 0$, we have

$$\widetilde{\lambda f_\xi} = \lambda \tilde{f}. \tag{6.2}$$

For $1 \le p < \infty$, define M_p as the closure,

$$M_p = \mathrm{cl}\{\tilde{f} \in \mathbb{R}^n : \|f : \mu\|_{p'} \le 1\}, \tag{6.3}$$

while for $p = \infty$, define M_∞ as the closure,

$$M_\infty = \mathrm{cl}\{\tilde{f} \in \mathbb{R}^n : \|f : \mu\|_1 \le 1\}. \tag{6.4}$$

It is easily shown that M_p is a convex body in $\mathbb{R}^n$ for all $p \in [1, \infty]$ and that M_p converges to M_∞ as $p \to \infty$ under the Hausdorff metric. We will show that $M_p = Z_{m,p}$ and give a dual definition for the norm $\|\cdot\|_{Z_{m,p}}$.

Lemma 6.1. *Suppose that $p \in [1, \infty]$ and μ is a finite Borel measure on $G_{n,m}$. Then*

$$M_p = Z_{m,p}. \tag{6.5}$$

Proof. Since $Z_{m,p}$ and M_p converge to $Z_{m,\infty}$ and M_∞, respectively, as $p \to \infty$, we only need to show the case $1 \le p < \infty$. For $u \in S^{n-1}$, from (2.3), (6.3), (6.1), the Hölder inequality, and the definition (5.1), we have

$$\begin{aligned}
h_{M_p}(u) &= \sup_{\|f:\mu\|_{p'} \le 1} u \cdot \tilde{f} \\
&= \sup_{\|f:\mu\|_{p'} \le 1} \frac{n}{m} \int_{G_{n,m}} u \cdot \mathrm{P}_\xi f_\xi \, d\mu(\xi) \\
&= \sup_{\|f:\mu\|_{p'} \le 1} \frac{n}{m} \int_{G_{n,m}} (\mathrm{P}_\xi u) \cdot f_\xi \, d\mu(\xi) \\
&\le \sup_{\|f:\mu\|_{p'} \le 1} \frac{n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi u| |f_\xi| \, d\mu(\xi) \\
&\le \sup_{\|f:\mu\|_{p'} \le 1} \|f:\mu\|_{p'} \Big(\frac{n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi u|^p \, d\mu(\xi) \Big)^{\frac{1}{p}} \\
&= \Big(\frac{n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi u|^p d\mu(\xi) \Big)^{\frac{1}{p}} \\
&= h_{Z_{m,p}}(u).
\end{aligned}$$

Therefore, $M_p \subseteq Z_{m,p}$.

On the other hand, for $u \in S^{n-1}$ and $1 \le p < \infty$, let $c = h_{Z_{m,p}}(u)$, and let $f_1(\xi) = c^{-p/p'} |\mathrm{P}_\xi u|^{p/p'-1} \mathrm{P}_\xi u$. Then

$$\Big(\frac{n}{m} \int_{G_{n,m}} |f_1(\xi)|^{p'} d\mu(\xi) \Big)^{\frac{1}{p'}} = \Big(\frac{c^{-p} n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi u|^p d\mu(\xi) \Big)^{\frac{1}{p'}} = 1,$$

and thus $\tilde{f}_1 \in M_p$. Moreover,

$$\begin{aligned}
h_{M_p}(u) &\ge u \cdot \tilde{f}_1 \\
&= u \cdot \frac{n}{m} \int_{G_{n,m}} (\mathrm{P}_\xi f_1(\xi)) d\mu(\xi) \\
&= \frac{n}{m} \int_{G_{n,m}} u \cdot \big(c^{-p/p'} |\mathrm{P}_\xi u|^{p/p'-1} \mathrm{P}_\xi u \big) d\mu(\xi)
\end{aligned}$$

$$= \frac{c^{-p/p'} n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi u|^{p/p'-1} |\mathrm{P}_\xi u|^2 d\mu(\xi)$$
$$= \frac{c^{-p/p'} n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi u|^{p/p'+1} d\mu(\xi)$$
$$= h_{Z_{m,p}}(u).$$

Therefore, $M_p \supseteq Z_{m,p}$. □

The following lemma gives a dual definition for the norm $\|\cdot\|_{Z_{m,p}}$.

Lemma 6.2. *Suppose that $p \in [1, \infty]$ and μ is a finite Borel measure on $G_{n,m}$. Then, for $y \in \mathbb{R}^n$,*

$$\|y\|_{Z_{m,p}} = \inf \Big\{ \Big(\frac{n}{m} \int_{G_{n,m}} |f_\xi|^{p'} d\mu(\xi) \Big)^{\frac{1}{p'}} : y = \frac{n}{m} \int_{G_{n,m}} f_\xi \, d\mu(\xi) \Big\}. \tag{6.6}$$

Proof. By the definition of Minkowski functional and Lemma 6.1,

$$\begin{aligned}
\|y\|_{Z_{m,p}} &= \inf \Big\{ t > 0 : \frac{y}{t} \in Z_{m,p} \Big\} \\
&= \inf \Big\{ t > 0 : \frac{y}{t} = \frac{n}{m} \int_{G_{n,m}} f_\xi \, d\mu(\xi), \ \|f : \mu\|_{p'} \le 1 \Big\} \\
&= \inf \Big\{ t > 0 : y = \frac{n}{m} \int_{G_{n,m}} t f_\xi \, d\mu(\xi), \ \Big(\frac{n}{m} \int_{G_{n,m}} |t f_\xi|^{p'} d\mu(\xi) \Big)^{\frac{1}{p'}} \le t \Big\} \\
&\ge \inf \Big\{ \Big(\frac{n}{m} \int_{G_{n,m}} |f_\xi|^{p'} d\mu(\xi) \Big)^{\frac{1}{p'}} : y = \frac{n}{m} \int_{G_{n,m}} f_\xi \, d\mu(\xi) \Big\}.
\end{aligned}$$

Let f_i be a sequence such that

$$t_i = \Big(\frac{n}{m} \int_{G_{n,m}} |f_i(\xi)|^{p'} d\mu(\xi) \Big)^{\frac{1}{p'}}, \quad y = \frac{n}{m} \int_{G_{n,m}} f_i(\xi) d\mu(\xi),$$

and t_i tends to the last infimum above. Then

$$\Big(\frac{n}{m} \int_{G_{n,m}} \Big| \frac{f_i(\xi)}{t_i} \Big|^{p'} d\mu(\xi) \Big)^{\frac{1}{p'}} = 1, \quad \frac{y}{t_i} = \frac{n}{m} \int_{G_{n,m}} \frac{f_i(\xi)}{t_i} \, d\mu(\xi).$$

It follows that $\frac{y}{t_i} \in M_p = Z_{m,p}$, and hence $t_i \ge \|y\|_{Z_{m,p}}$. Therefore, the desired result is proved. □

The following corollary extends the well-known duality of ℓ_p^n-balls,

$$(B_p^n)^* = B_{p'}^n, \quad p \in [1, \infty].$$

Corollary 6.3. *Let $p \in [1, \infty]$. If μ is an isotropic probability measure on $G_{n,m}$, then we have*

$$Z_{m,p}^* \subseteq Z_{m,p'}. \tag{6.7}$$

The equality of the inclusion holds if μ is a cross measure on $G_{n,m}$ and n is divisible by m.

Proof. Since μ is the isotropic probability measure on $G_{n,m}$, we have for $x \in \mathbb{R}^n$

$$\frac{n}{m} \int\limits_{G_{n,m}} \mathrm{P}_\xi x \, d\mu(\xi) = x. \tag{6.8}$$

By Lemma 6.2, we obtain that

$$h_{Z_{m,p}^*}(x) = \|x\|_{Z_{m,p}} \le \Big(\frac{n}{m} \int\limits_{G_{n,m}} |\mathrm{P}_\xi x|^{p'} d\mu(\xi) \Big)^{\frac{1}{p'}} = h_{Z_{m,p'}}(x),$$

which gives the desired inclusion.

By the continuity of $Z_{m,p}$ with respect to p, it is enough to show the equality condition for $p \in (1, \infty)$. Let $n/m = l$. Notice that when μ is a cross measure on $G_{n,m}$, from (4.4), we have for $x \in \mathbb{R}^n$

$$\sum_{i=1}^{l} \mathrm{P}_{\tilde{\xi}_i} x = x.$$

Define $y \in \mathbb{R}^n$ by

$$y = \sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{(p'-2)} \mathrm{P}_{\tilde{\xi}_i} x, \quad x \in \mathbb{R}^n.$$

Since $\tilde{\xi}_i$, $i = 1, \ldots, l$, are orthogonal subspaces which are spanned by different coordinate vectors, we have

$$\mathrm{P}_{\tilde{\xi}_i} y = |\mathrm{P}_{\tilde{\xi}_i} x|^{(p'-2)} \mathrm{P}_{\tilde{\xi}_i} x.$$

Then for such $x, y \in \mathbb{R}^n$

$$x \cdot y = \sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{(p'-2)} x \cdot \mathrm{P}_{\tilde{\xi}_i} x = \sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{p'} \tag{6.9}$$

and from $(p'-1)p = p'$

$$h_{Z_{m,p}}(y) = \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} y|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{p'}\right)^{\frac{1}{p}}. \tag{6.10}$$

For $x, y \in \mathbb{R}^n$, since $\frac{x}{h_{Z^*_{m,p}}(x)}$ and $\frac{y}{h_{Z_{m,p}}(y)}$ lie on the boundary of $Z_{m,p}$ and $Z^*_{m,p}$, respectively, the definition of polar body (2.1) gives that

$$\frac{x}{h_{Z^*_{m,p}}(x)} \cdot \frac{y}{h_{Z_{m,p}}(y)} \leq 1.$$

Thus, from (6.9), (5.1), and (6.10), we obtain for such $x, y \in \mathbb{R}^n$

$$\begin{aligned} h_{Z^*_{m,p}}(x) h_{Z_{m,p}}(y) &\geq x \cdot y \\ &= \sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{p'} \\ &= \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{p'}\right)^{\frac{1}{p'}} \left(\sum_{i=1}^{l} |\mathrm{P}_{\tilde{\xi}_i} x|^{p'}\right)^{\frac{1}{p}} \\ &= h_{Z_{m,p'}}(x) h_{Z_{m,p}}(y), \end{aligned}$$

which gives $h_{Z^*_{m,p}}(x) \geq h_{Z_{m,p'}}(x)$ for all $x \in \mathbb{R}^n$. This proves that $Z^*_{m,p} = Z_{m,p'}$ for $p \in (1, \infty)$ when μ is a cross measure on $G_{n,m}$ and n is divisible by m. $\square$

From Lemma 5.1 and Corollary 6.3, we obtain the following corollary.

Corollary 6.4. *If μ is a probability cross measure on $G_{n,m}$ and n is divisible by m, then for $p \in [1, \infty]$,*

$$V(Z_{m,p}) = \alpha_n(m, p'). \tag{6.11}$$

7. Volume inequalities for $Z_{m,p}$ and $Z^*_{m,p}$

The following lemma is needed to establish the volume inequalities for $Z_{m,p}$ and $Z^*_{m,p}$. Denote by $B_2^m(r)$ the Euclidean open ball in $\mathbb{R}^m$ centered at the origin and with radius r. We say that a function f is *rotationally invariant* in $\mathbb{R}^m$ if for $O \in \mathrm{O}(m)$ and $x \in \mathbb{R}^m$,

$$f(x) = f(Ox).$$

Lemma 7.1. *For $a, b \in (0, \infty]$, let $f : B_2^m(a) \to (0, \infty)$, $g : B_2^m(b) \to (0, \infty)$ be continuous positive probability density functions. If f, g are rotationally invariant in $\mathbb{R}^m$, then there exists a rotationally invariant and strictly convex function ψ of class C^2 on $B_2^m(a)$ such that $\nabla\psi : B_2^m(a) \to B_2^m(b)$ and for $x \in B_2^m(a)$*

$$f(x) = g(\nabla\psi(x)) \det(\nabla^2\psi(x)). \tag{7.1}$$

Proof. Let $0 \le t < a$. Define a strictly increasing continuous function $\phi_1 : [0, a) \to [0, b)$ by

$$\int_{B_2^m(t)} f(x)dx = \int_{B_2^m(\phi_1(t))} g(x)dx, \tag{7.2}$$

where we let $B_2^m(0) = \{0\}$. Obviously, $\phi_1(0) = 0$. Since f, g are positive, continuous and rotationally invariant, there exist continuous functions $f_1 : [0, a) \to (0, \infty)$, $g_1 : [0, b) \to (0, \infty)$ such that $f(x) = f_1(|x|)$, $g(x) = g_1(|x|)$. Thus, by polar coordinates, we have

$$\int_0^t f_1(r)r^{m-1}dr = \int_0^{\phi_1(t)} g_1(r)r^{m-1}dr, \tag{7.3}$$

which implies that

$$f_1(t)t^{m-1} = g_1(\phi_1(t))\phi_1^{m-1}(t)\phi_1'(t). \tag{7.4}$$

Equation (7.3) and the mean value theorem show that there exist r_1 and r_2, $0 < r_1 < t$, $0 < r_2 < \phi_1(t)$, so that

$$\left(\frac{\phi_1(t)}{t}\right)^m = \frac{f_1(r_1)}{g_1(r_2)},$$

and

$$\left(\frac{\phi_1(t)}{t}\right)^m \to \frac{f_1(0)}{g_1(0)}, \quad t \to 0. \tag{7.5}$$

Since $f_1 > 0$ is continuous on $[0, a)$, $g_1 > 0$ is continuous on $[0, b)$ and $\phi_1 : [0, a) \to [0, b)$ is strictly increasing and continuous, it follows that $\phi_1'(t) > 0$ is continuous on $[0, a)$. Let $\psi_1 : [0, a^2) \to [0, \infty)$ be defined by

$$\phi_1(t) = 2\psi_1'(t^2)t, \quad \psi_1(0) = 0, \tag{7.6}$$

that is,

$$\psi_1(t^2) = \int_0^t \phi_1(\tau)\, d\tau.$$

Then

$$\phi_1'(t) = 4\psi_1''(t^2)t^2 + 2\psi_1'(t^2). \tag{7.7}$$

Equation (7.4) becomes

$$f_1(t)t^{m-1} = g_1\big(2\psi_1'(t^2)t\big)\big(2\psi_1'(t^2)t\big)^{m-1}\big(4\psi_1''(t^2)t^2 + 2\psi_1'(t^2)\big),$$

that is,

$$f_1(t) = g_1\big(2\psi_1'(t^2)t\big)\big(2\psi_1'(t^2)\big)^{m-1}\big(4\psi_1''(t^2)t^2 + 2\psi_1'(t^2)\big). \tag{7.8}$$

Define $\psi(x) = \psi_1(|x|^2)$. Then we have

$$\nabla\psi(x) = 2\psi_1'(|x|^2)x, \tag{7.9}$$

where $\nabla\psi : B_2^m(a) \to B_2^m(b)$, and

$$\nabla^2\psi(x) = 4\psi_1''(|x|^2)x \otimes x + 2\psi_1'(|x|^2)I_m.$$

Thus

$$\det(\nabla^2\psi(x)) = \big(2\psi_1'(|x|^2)\big)^{m-1}\big(4\psi_1''(|x|^2)|x|^2 + 2\psi_1'(|x|^2)\big).$$

This together with (7.8) and (7.9) yields for $x \in B_2^m(a)$

$$f(x) = g(\nabla\psi(x))\det(\nabla^2\psi(x)).$$

Since $f_1 > 0, g_1 > 0$, it follows from (7.8) that $\det(\nabla^2\psi(x)) > 0$, and thus $\psi \in C^2$ is convex on $B_2^m(a)$. This completes the proof. $\square$

The map $\nabla\psi$ in Lemma 7.1 is a special case of the Brenier map, see [14,48], and [49]. This simple construction gives the regularity of the Brenier map for the rotationally invariant case, which will be needed for the proof of the next theorem and later for uniqueness arguments. See [50] for results and references on the regularity of the Brenier map of general domains. We have not found an immediate reference to Lemma 7.1, and thus include a proof.

Theorem 7.2. *Suppose $p \in [1,\infty)$ and $q \in [1,\infty]$. If μ is an isotropic probability measure on $G_{n,m}$, then*

$$V(Z^*_{m,p})/\alpha_n(m,p) \le V(Z_{m,q})/\alpha_n(m,q'). \tag{7.10}$$

Proof. First, assume $q \in (1,\infty]$. Define probability densities in $\mathbb{R}^m$, $f(s), g(s) : \mathbb{R}^m \to (0,\infty)$ by

$$f(s) = \frac{1}{\omega_m\Gamma(1+\frac{m}{p})}e^{-|s|^p},$$

$$g(s) = \frac{1}{\omega_m\Gamma(1+\frac{m}{q'})}e^{-|s|^{q'}}.$$

By Lemma 7.1, there exists a convex function ψ in $\mathbb{R}^m$ such that for $s \in \mathbb{R}^m$

$$e^{-|s|^p} = c_{m,p,q} e^{-|\nabla\psi(s)|^{q'}} \det(\nabla^2\psi(s)), \tag{7.11}$$

where $c_{m,p,q} = \Gamma(1+\frac{m}{p})/\Gamma(1+\frac{m}{q'})$.

For $\xi \in G_{n,m}$, choose an orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_m\}$ of ξ. Then for $s \in \xi$, $s = s_1\varepsilon_1 + \cdots + s_m\varepsilon_m$, $(s_1, \dots, s_m) \in \mathbb{R}^m$. Identify $s \in \xi$ with $(s_1, \dots, s_m) \in \mathbb{R}^m$ when no confusion is caused.

Let $\Psi_\xi : \xi \to \xi$ be the map defined by

$$\Psi_\xi(s) = \frac{\partial\psi}{\partial s_1}(s)\varepsilon_1 + \cdots + \frac{\partial\psi}{\partial s_m}(s)\varepsilon_m, \tag{7.12}$$

which is the gradient of ψ in ξ. To see that Ψ_ξ does not depend on the choice of orthonormal basis of ξ, let

$$\varepsilon_i = \sum_{j=1}^m \alpha_{ij}u_j, \quad s = \sum_{i=1}^m s_i\varepsilon_i = \sum_{j=1}^m t_j u_j,$$

where $\{u_1, \dots, u_m\}$ is another orthonormal basis and (α_{ij}) is an orthogonal matrix. Then

$$\begin{aligned} t_j &= \sum_{i=1}^m s_i\alpha_{ij}, \\ \frac{\partial\psi}{\partial s_i} &= \sum_{j=1}^m \frac{\partial\psi}{\partial t_j}\frac{\partial t_j}{\partial s_i} = \sum_{j=1}^m \frac{\partial\psi}{\partial t_j}\alpha_{ij}, \\ \sum_{i=1}^m \frac{\partial\psi}{\partial s_i}\varepsilon_i &= \sum_{i=1}^m\sum_{j=1}^m \frac{\partial\psi}{\partial t_j}\alpha_{ij}\varepsilon_i = \sum_{j=1}^m \frac{\partial\psi}{\partial t_j}u_j. \end{aligned}$$

Thus, (7.12) is well-defined.

The differential of Ψ_ξ at s is a linear transformation in ξ, defined by

$$\begin{aligned} d\Psi_\xi(s) :\ & \xi \to \xi, \\ d\Psi_\xi(s)(\varepsilon_i) &= \sum_{j=1}^m \frac{\partial^2\psi(s)}{\partial s_i\partial s_j}\varepsilon_j. \end{aligned} \tag{7.13}$$

Since ψ is strictly convex, $d\Psi_\xi(s)$ is a positive definite symmetric transformation.

View $\varepsilon_1, \dots, \varepsilon_m$ as column vectors in $\mathbb{R}^n$ and let $Q = (\varepsilon_1, \dots, \varepsilon_m)$, then Q is an $n \times m$ matrix. By (7.13),

$$I_\xi d\Psi_\xi(s)\mathrm{P}_\xi x = \sum_{j=1}^m \frac{\partial^2\psi(s)}{\partial s_i\partial s_j}(x\cdot\varepsilon_i)\varepsilon_j, \quad x \in \mathbb{R}^n.$$

Thus,

$$I_\xi d\Psi_\xi(s)\mathrm{P}_\xi = Q\nabla^2\psi(s)Q^t. \tag{7.14}$$

Let

$$\begin{aligned} A_\xi(x) &= d\Psi_\xi(\mathrm{P}_\xi x), \\ A(\xi, x) &= I_\xi A_\xi(x)\mathrm{P}_\xi = Q\nabla^2\psi(QQ^t x)Q^t. \end{aligned} \tag{7.15}$$

Since ψ is C^2, (7.15) shows that $A(\xi, x)$ is continuous with respect to (ξ, x).

Equation (7.11) gives for all $s \in \xi$

$$|s|^p = |\Psi_\xi(s)|^{q'} - \log c_{m,p,q} - \log\det(d\Psi_\xi(s)). \tag{7.16}$$

Define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \frac{n}{m}\int_{G_{n,m}} I_\xi \Psi_\xi\big(\mathrm{P}_\xi x\big)\, d\mu(\xi), \quad x \in \mathbb{R}^n. \tag{7.17}$$

By (7.17) and (7.15), the differential of T is given by

$$dT(x) = \frac{n}{m}\int_{G_{n,m}} I_\xi d\Psi_\xi\big(\mathrm{P}_\xi x\big)\mathrm{P}_\xi\, d\mu(\xi) = \frac{n}{m}\int_{G_{n,m}} A(\xi, x)\, d\mu(\xi), \tag{7.18}$$

for $x \in \mathbb{R}^n$. Note that if μ is an isotropic probability measure on $G_{n,m}$, then

$$\frac{n}{m}\int_{G_{n,m}} \mathrm{P}_\xi\, d\mu(\xi) = I_n.$$

Since $A(\xi, x)$ is positive semi-definite and continuous with respect to ξ, the product of the positive eigenvalues of $A(\xi, x)$, $\det A_\xi(x)$, has a positive minimum over $G_{n,m}$. Therefore, the matrix $dT(x)$ is positive definite since from the Grassmannian Ball–Barthe inequality (3.24),

$$\det\Big(\frac{n}{m}\int_{G_{n,m}} A(\xi, x)\, d\mu(\xi)\Big) \geq \exp\Big(\frac{n}{m}\int_{G_{n,m}} \log\big(\det A_\xi(x)\big)\, d\mu(\xi)\Big) > 0.$$

In particular, for all $y \neq 0$ in $\mathbb{R}^n$,

$$y \cdot dT(x)y > 0,$$

so T is injective.

Moreover, (7.17) and Lemma 6.2 show that

$$\|Tx\|_{Z_{m,q}}^{q'} \le \frac{n}{m} \int_{G_{n,m}} \left|\Psi_\xi\big(\mathrm{P}_\xi x\big)\right|^{q'} d\mu(\xi), \tag{7.19}$$

for $x \in \mathbb{R}^n$.

By (2.5), (2.4), (5.1), (7.16), the Grassmannian Ball–Barthe inequality (3.24), (7.18), (7.19), the change of variables $y = Tx$, and (2.5) again, we have

$$\begin{aligned}
&\Gamma(1+\frac{n}{p})V(Z^*_{m,p}) \\
&\quad = \int_{\mathbb{R}^n} e^{-\|x\|^p_{Z^*_{m,p}}}\, dx \\
&\quad = \int_{\mathbb{R}^n} \exp\Big(-\frac{n}{m}\int_{G_{n,m}} |\mathrm{P}_\xi x|^p\, d\mu(\xi)\Big)\, dx \\
&\quad = \int_{\mathbb{R}^n} \exp\Big(-\frac{n}{m}\int_{G_{n,m}} \big(|\Psi_\xi(\mathrm{P}_\xi x)|^{q'} - \log c_{m,p,q} - \log\det(A_\xi(x))\big)\, d\mu(\xi)\Big)\, dx \\
&\quad = (c_{m,p,q})^{\frac{n}{m}} \int_{\mathbb{R}^n} \exp\Big(-\frac{n}{m}\int_{G_{n,m}} |\Psi_\xi(\mathrm{P}_\xi x)|^{q'}\, d\mu(\xi)\Big) \\
&\qquad \times \exp\Big(\frac{n}{m}\int_{G_{n,m}} \log\det(A_\xi(x))\, d\mu(\xi)\Big)\, dx \\
&\quad \le (c_{m,p,q})^{\frac{n}{m}} \int_{\mathbb{R}^n} \exp\Big(-\frac{n}{m}\int_{G_{n,m}} |\Psi_\xi(\mathrm{P}_\xi x)|^{q'}\, d\mu(\xi)\Big)\det(dT(x))\, dx \\
&\quad \le (c_{m,p,q})^{\frac{n}{m}} \int_{\mathbb{R}^n} \exp\big(-\|Tx\|^{q'}_{Z_{m,q}}\big)\det(dT(x))\, dx \\
&\quad \le (c_{m,p,q})^{\frac{n}{m}} \int_{\mathbb{R}^n} e^{-\|y\|^{q'}_{Z_{m,q}}}\, dy \\
&\quad = (c_{m,p,q})^{\frac{n}{m}}\Gamma(1+\frac{n}{q'})V(Z_{m,q}).
\end{aligned}$$

This is the desired inequality (7.10).

For $q = 1$, define probability densities

$$f(s) = \frac{1}{\omega_m\Gamma(1+\frac{m}{p})} e^{-|s|^p},$$

and

$$g(s) = \frac{1}{\omega_m} \mathbf{1}_{B_2^m(1)}(s),$$

where $B_2^m(1)$ is the unit open ball in ξ.

Lemma 7.1 gives that there exists a convex function ψ in $\mathbb{R}^m$ such that for $s \in \mathbb{R}^m$, $\nabla\psi : \mathbb{R}^m \to B_2^m(1)$ and

$$|s|^p = -\log c_{m,p} - \log \det(\nabla^2 \psi(s)), \tag{7.20}$$

where $c_{m,p} = \Gamma(1 + \frac{m}{p})$. For this ψ, let $\Psi_\xi : \xi \to \xi$ be the map defined by (7.12).

Define the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by using (7.17). The differential dT is given by (7.18). For all $y \neq 0$ in $\mathbb{R}^n$, $y \cdot dT(x)y > 0$, so T is injective.

Since $|\Psi_\xi| < 1$, Lemma 6.2 and (7.17) show that $\|Tx\|_{Z_{m,1}} \le 1$, for all $x \in \mathbb{R}^n$. This means that $Tx \in Z_{m,1}$, for all $x \in \mathbb{R}^n$. Hence we have

$$T(\mathbb{R}^n) \subseteq Z_{m,1}. \tag{7.21}$$

By (2.5), (2.4), (5.1), (7.20), the Grassmannian Ball–Barthe inequality (3.24), (7.18), the change of variables $y = Tx$, and (7.21), we have

$$\begin{aligned}
\Gamma\big(1 + \frac{n}{p}\big) V(Z^*_{m,p}) &= \int_{\mathbb{R}^n} e^{-\|x\|^p_{Z^*_{m,p}}}\, dx \\
&= \int_{\mathbb{R}^n} \exp\Big(-\frac{n}{m} \int_{G_{n,m}} |\mathrm{P}_\xi x|^p d\mu(\xi)\Big)\, dx \\
&= \int_{\mathbb{R}^n} \exp\Big(\frac{n}{m} \int_{G_{n,m}} \big[\log c_{m,p} + \log\det(A_\xi(x))\big]\, d\mu(\xi)\Big)\, dx \\
&= (c_{m,p})^{\frac{n}{m}} \int_{\mathbb{R}^n} \exp\Big(\frac{n}{m} \int_{G_{n,m}} \log\det(A_\xi(x))\, d\mu(\xi)\Big)\, dx \\
&\le (c_{m,p})^{\frac{n}{m}} \int_{\mathbb{R}^n} \det(dT(x))\, dx \\
&\le (c_{m,p})^{\frac{n}{m}} \int_{Z_{m,1}} dy \\
&= (c_{m,p})^{\frac{n}{m}} V(Z_{m,1}).
\end{aligned}$$

Therefore, (7.10) holds when $q = 1$. □

The case $p \neq \infty$ in the following theorem follows from Theorem 7.2 immediately by setting $q = 2$.

Theorem 7.3. *Suppose $p \in [1, \infty]$. If μ is an isotropic probability measure on $G_{n,m}$, then*

$$\alpha_n(m,p) \geq V(Z^*_{m,p}). \tag{7.22}$$

Proof. We only need to show the case $p = \infty$. Define probability densities

$$f(s) = \frac{1}{\omega_m} \mathbf{1}_{B_2^m(1)}(s),$$

and

$$g(s) = \frac{1}{\omega_m \Gamma(1 + \frac{m}{2})} e^{-|s|^2}.$$

By Lemma 7.1, there exists a convex function ψ such that $\nabla\psi : B_2^m(1) \to \mathbb{R}^m$ and

$$\Gamma(1 + m/2) = e^{-|\nabla\psi(s)|^2} \det(\nabla^2\psi(s)), \quad s \in B_2^m(1). \tag{7.23}$$

By (5.2) and (2.1), we have

$$\operatorname{int} Z^*_{m,\infty} = \{x \in \mathbb{R}^n : \max_{\xi \in \operatorname{supp} \mu} |\mathrm{P}_\xi x| < 1\}. \tag{7.24}$$

By (7.24), we have for each $x \in \operatorname{int} Z^*_{m,\infty}$

$$\exp\left\{\frac{n}{m} \int\limits_{\operatorname{supp} \mu} \log \mathbf{1}_{B_2^m(1)}(\mathrm{P}_\xi x)\, d\mu(\xi)\right\} = 1. \tag{7.25}$$

For $\xi \in G_{n,m}$, choose an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_m\}$ of ξ. Then we have for $s \in \xi$, $s = s_1\varepsilon_1 + \cdots + s_m\varepsilon_m$. Identify s with $(s_1, \ldots, s_m) \in \mathbb{R}^m$.

For $\xi \in G_{n,m}$, define $\Psi_\xi : B_2^m(1) \to \xi$ by (7.12), where we consider $B_2^m(1)$ as the open unit ball in ξ. Define the map $T : \operatorname{int} Z^*_{m,\infty} \to \mathbb{R}^n$ by (7.17). The differential dT is given by (7.18). Since $dT(x)$ is positive definite for each $x \in \operatorname{int} Z^*_{m,\infty}$, the map T is injective.

Equation (7.24) shows that, for $x \in \operatorname{int} Z^*_{m,\infty}$ and $\xi \in \operatorname{supp} \mu$, $\mathrm{P}_\xi x$ is in the domain of A_ξ. It follows from (7.17) and Lemma 6.2 that

$$\|Tx\|^2_{Z_{m,2}} \leq \frac{n}{m} \int\limits_{G_{n,m}} |\Psi_\xi(\mathrm{P}_\xi x)|^2 d\mu(\xi). \tag{7.26}$$

By (7.25), (7.23), (7.18), the Grassmannian Ball–Barthe inequality (3.24), (7.26), the change of variables $y = Tx$, (2.5), and the identity $Z_{m,2} = B_2^n$, we have

$$V(Z^*_{m,\infty})$$
$$= \int\limits_{\operatorname{int} Z^*_{m,\infty}} \exp\left(\frac{n}{m} \int\limits_{\operatorname{supp} \mu} \log \mathbf{1}_{B_2^m(1)}(\mathrm{P}_\xi x)\, d\mu(\xi)\right) dx$$

$$
\begin{aligned}
&= \int\limits_{\operatorname{int} Z^*_{m,\infty}} \exp\Big(\frac{n}{m}\int\limits_{\operatorname{supp}\mu} \log\Big[\Gamma\big(1+\frac{m}{2}\big)^{-1} e^{-|\Psi_\xi(\mathrm{P}_\xi x)|^2}\det(A_\xi(x))\Big]\, d\mu(\xi)\Big)\, dx \\
&= \Gamma\big(1+\frac{m}{2}\big)^{-\frac{n}{m}} \int\limits_{\operatorname{int} Z^*_{m,\infty}} \exp\Big(\frac{n}{m}\int\limits_{\operatorname{supp}\mu} -|\Psi_\xi(\mathrm{P}_\xi x)|^2\, d\mu(\xi)\Big) \\
&\quad \times \exp\Big(\frac{n}{m}\int\limits_{\operatorname{supp}\mu} \log\det(A_\xi(x))\, d\mu(\xi)\Big)\, dx \\
&\le \Gamma\big(1+\frac{m}{2}\big)^{-\frac{n}{m}} \int\limits_{\operatorname{int} Z^*_{m,\infty}} \exp\Big(\frac{n}{m}\int\limits_{\operatorname{supp}\mu} -|\Psi_\xi(\mathrm{P}_\xi x)|^2 d\mu(\xi)\Big)\det(dT(x))\, dx \\
&\le \Gamma\big(1+\frac{m}{2}\big)^{-\frac{n}{m}} \int\limits_{\operatorname{int} Z^*_{m,\infty}} e^{-\|Tx\|^2_{Z_{m,2}}}\det(dT(x))\, dx \\
&\le \Gamma\big(1+\frac{m}{2}\big)^{-\frac{n}{m}} \int\limits_{\mathbb{R}^n} e^{-\|y\|^2_{Z_{m,2}}}\, dy \\
&= \Gamma\big(1+\frac{m}{2}\big)^{-\frac{n}{m}}\Gamma(1+\frac{n}{2})V(Z_{m,2}) \\
&= \Gamma\big(1+\frac{m}{2}\big)^{-\frac{n}{m}}\Gamma(1+\frac{n}{2})V(B_2^n) \\
&= \alpha_n(m,\infty). \qquad \square
\end{aligned}
$$

The next theorem follows immediately from Theorem 7.2.

Theorem 7.4. *Suppose $p \in [1,\infty]$. If μ is an isotropic probability measure on $G_{n,m}$, then*

$$\alpha_n(m,p') \le V(Z_{m,p}). \tag{7.27}$$

Note that $\alpha_n(m,2) = \omega_n$ and $Z_{m,2} = B_2^n$. Setting $(p,q) = (p,2)$ and $(p,q) = (2,q)$ in (7.10) gives

$$V(Z^*_{m,p})/\alpha_n(m,p) \le 1 \le V(Z_{m,q})/\alpha_n(m,q'). \tag{7.28}$$

Lemma 7.5. *For any $\xi \in G_{n,m}$, there is the formula,*

$$\int\limits_{S^{n-1}} |\mathrm{P}_\xi u|^p\, du = \frac{m\omega_m\omega_{n+p-2}}{\omega_{m+p-2}}.$$

Proof. By the rotation invariance of the integral, we can assume that $\xi = \mathbb{R}^m$. The general spherical coordinates and the formula (2.15) give that

$$\int\limits_{S^{n-1}} |\mathrm{P}_\xi u|^p\, du = \int\limits_{S^{m-1}}\int\limits_{S^{n-m-1}}\int\limits_0^{\frac{\pi}{2}} (\sin\varphi)^{n-m-1}(\cos\varphi)^{m+p-1}\, d\varphi du_1 du_2$$

$$
\begin{aligned}
&= m\omega_m(n-m)\omega_{n-m}\frac{1}{2}B\Big(\frac{n-m}{2},\frac{m+p}{2}\Big) \\
&= \frac{m\omega_m\omega_{n+p-2}}{\omega_{m+p-2}}. \qquad \square
\end{aligned}
$$

In order to finish the proof of the inequalities in Theorem 1.2, we only need:

Theorem 7.6. *Suppose $p \in [1,\infty]$. If μ is an isotropic probability measure on $G_{n,m}$, then*

$$
\frac{\omega_n}{\gamma_n(m,p)} \le V(Z_{m,p}^*) \quad \text{and} \quad V(Z_{m,p}) \le \omega_n\gamma_n(m,p)
$$

where $\gamma_n(m,p) = \left(\frac{\omega_m\omega_{n+p-2}}{\omega_n\omega_{m+p-2}}\right)^{\frac{n}{p}}$. There is equality in either of the inequalities if μ is the normalized Lebesgue measure.

Proof. By the polar coordinate formula (2.7), the Hölder inequality, (5.1), Fubini's theorem, and Lemma 7.5, we have

$$
\begin{aligned}
\left(\frac{V(Z_{m,p}^*)}{\omega_n}\right)^{-\frac{1}{n}} &= \left(\frac{1}{n\omega_n}\int\limits_{S^{n-1}} h_{Z_{m,p}}(u)^{-n}du\right)^{-\frac{1}{n}} \\
&\le \left(\frac{1}{n\omega_n}\int\limits_{S^{n-1}} h_{Z_{m,p}}(u)^{p}du\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{n\omega_n}\int\limits_{S^{n-1}}\int\limits_{G_{n,m}} \frac{n}{m}|\mathrm{P}_\xi u|^p\, d\mu(\xi)du\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{m\omega_n}\int\limits_{G_{n,m}}\int\limits_{S^{n-1}} |\mathrm{P}_\xi u|^p\, dud\mu(\xi)\right)^{\frac{1}{p}} \\
&= \left(\frac{\omega_m\omega_{n+p-2}}{\omega_n\omega_{m+p-2}}\right)^{\frac{1}{p}},
\end{aligned}
$$

with equality if and only if $Z_{m,p}$ is a ball. When μ is the normalized Lebesgue measure, $Z_{m,p}$ is a ball.

To establish the second inequality, recall the classical Urysohn inequality (see, e.g., Schneider [58], p. 318). By the Urysohn inequality, the Hölder inequality, (5.1), Fubini's theorem, and Lemma 7.5, we obtain

$$
\begin{aligned}
\left(\frac{V(Z_{m,p})}{\omega_n}\right)^{\frac{1}{n}} &\le \frac{1}{n\omega_n}\int\limits_{S^{n-1}} h_{Z_{m,p}}(u)du \\
&\le \left(\frac{1}{n\omega_n}\int\limits_{S^{n-1}} h_{Z_{m,p}}(u)^{p}du\right)^{\frac{1}{p}}
\end{aligned}
$$

$$= \left(\frac{1}{m\omega_n} \int\limits_{S^{n-1}} \int\limits_{G_{n,m}} |\mathrm{P}_\xi u|^p \, d\mu(\xi) du \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{m\omega_n} \int\limits_{G_{n,m}} \int\limits_{S^{n-1}} |\mathrm{P}_\xi u|^p \, du d\mu(\xi) \right)^{\frac{1}{p}}$$

$$= \left(\frac{\omega_m \omega_{n+p-2}}{\omega_n \omega_{m+p-2}} \right)^{\frac{1}{p}},$$

with equality if and only if $Z_{m,p}$ is a ball. □

8. Equality conditions for the volume inequalities

In this section, we will focus on the case $\frac{n}{m} = l$, an integer, and obtain the necessary condition of equality for the left inequalities in (1.6) and (1.7).

Let $f(s)$ and $g(s)$ be the rotational invariant functions in the proof of Theorem 7.2 when $1 \le p < \infty$ and Theorem 7.3 when $p = \infty$. By Lemma 7.1, there exists a convex function $\psi \in C^2$ in $B_2^m(a)$ such that

$$f(s) = g(\nabla\psi(s)) \det(\nabla^2\psi(s)). \tag{8.1}$$

By the proof of Lemma 7.1, we can define $\psi_1 : [0, a^2) \to \mathbb{R}$ by $\psi(s) = \psi_1(s \cdot s)$ such that

$$\nabla\psi(s) = 2\psi_1'(s \cdot s)s \tag{8.2}$$

and

$$\nabla^2\psi(s) = 4\psi_1''(s \cdot s)s \otimes s + 2\psi_1'(s \cdot s)I_m. \tag{8.3}$$

Thus, the eigenvectors of $\det(\nabla^2\psi(s))$ are the vector cs, $c \neq 0$, and all vectors in $\mathbb{R}^m$ which are orthogonal to s. Their corresponding eigenvalues are

$$\lambda(s) := 4\psi_1''(s \cdot s)|s|^2 + 2\psi_1'(s \cdot s), \tag{8.4}$$

and $2\psi_1'(s \cdot s)$. Since ψ is of class C^2, we get that the eigenvalue functions $a_k(s) > 0$, $k = 1, \dots, m$, are continuous in $\mathbb{R}^m$, and $a_1(s) = \lambda(s)$, $a_2(s) = \cdots = a_m(s) = 2\psi_1'(s \cdot s)$.

Lemma 8.1. *The eigenvalue functions $a_k(s)$, $k = 1, \dots, m$, of $\nabla^2\psi$, are continuous. If $p \neq q'$, then for any fixed $u \in S^{n-1}$ and $k \in \{1, \dots, m\}$, $a_k(tu)$, as a function of t, is not constant in any non-empty open sub-interval of $(0, a)$.*

Proof. Continuity has already been explained above. Only the second statement needs to be proved. First, we consider the case when f, g are defined by

$$f(s) = \frac{1}{\omega_m \Gamma(1+\frac{m}{p})} e^{-|s|^p}, \quad g(s) = \frac{1}{\omega_m \Gamma(1+\frac{m}{q'})} e^{-|s|^{q'}}.$$

Function $\psi_1'(t)$ is obviously not constant in a non-empty open interval. Otherwise, by (8.1), (8.2), and (8.3), $f(s) = c^m g(cs)$ for certain constant c when $|s|$ is in a non-empty open interval. This is impossible when $p \neq q'$. Thus, the statement is true for $a_k(s)$, $k = 2, \ldots, m$. Then it is sufficient to prove that $\lambda(tu)$ defined in (8.4), as a function of t, is not constant in a non-empty open interval.

We argue by contradiction. Assume that $\lambda(tu)$, as a function of $t > 0$, is constant in a non-empty open interval. Then from (7.7) we have

$$\phi_1'(t) = c_0 > 0 \quad \text{(constant)},$$

in a non-empty open interval. It follows that

$$\phi_1(t) = c_0 t + c_1,$$

in a non-empty open interval. Therefore, by (7.4),

$$e^{-t^p} = c_2 e^{-(c_0 t + c_1)^{q'}} \left(\frac{c_0 t + c_1}{t}\right)^{m-1},$$

in a non-empty open interval, where $c_2 = c_0 \Gamma(1+\frac{m}{p})/\Gamma(1+\frac{m}{q'})$. By analytic continuation, this equation is true for all $t > 0$. It is impossible when $p \neq q'$.

Next, consider the case when f, g are defined by

$$f(s) = \frac{1}{\omega_m} \mathbf{1}_{B_2^m(a)}(s), \quad g(s) = \frac{1}{\omega_m \Gamma(1+\frac{m}{2})} e^{-|s|_\xi^2}.$$

Similar to the arguments above, if $\psi_1'(t)$ is constant in a non-empty open sub-interval of $(0, a)$, then

$$\frac{1}{\omega_m} = \frac{c^m}{\pi^{\frac{m}{2}}} e^{-(ct)^2},$$

for certain constant c when t is in a non-empty open sub-interval of $(0, a)$. This is impossible. If $\phi_1'(t)$ is constant in a non-empty open sub-interval of $(0, a)$, then

$$\frac{1}{\omega_m} = c_2 e^{-(c_0 t + c_1)^2} \left(\frac{c_0 t + c_1}{t}\right)^{m-1},$$

for some constants $c_0 > 0, c_1, c_2$ in a non-empty open sub-interval. This is again impossible. $\square$

Theorem 8.2. *Suppose that $\frac{n}{m} = l$ is an integer. For $p \in [1, \infty]$ and $p \neq 2$, the equality on the left of (1.6) or (1.7) holds if and only if μ is a cross measure on $G_{n,m}$.*

Proof. It is sufficient to consider the inequalities (7.10) and (7.22) and assume equality holds in (7.10) when $1 \leq p < \infty$ or in (7.22) when $p = \infty$. The equality conditions of the Grassmannian Ball–Barthe inequality (Lemma 3.4) for $G_{n,m}$ show that this implies that for fixed $x \in \mathbb{R}^n$ when $p \in [1, \infty)$, or $x \in \operatorname{int} Z^*_{m,\infty}$ when $p = 1$, the product $a_{k_1}(\xi_1) \cdots a_{k_n}(\xi_n)$ as a function of $\xi_1, ..., \xi_n \in G_{n,m}$ is constant if there are linearly independent n vectors

$$w_i \in E_{k_i}(\mathrm{P}_{\xi_i} x), \quad \xi_1, ..., \xi_n \in \operatorname{supp} \mu,$$

where $a_{k_i}(\xi_i)$ are the eigenvalues of $A_{\xi_i}(x)$ with eigenspaces $E_{k_i}(\mathrm{P}_{\xi_i} x)$.

Since μ is an isotropic measure on $G_{n,m}$, by Lemma 4.1, there exist l' elements $\xi_1, ..., \xi_{l'} \in \operatorname{supp} \mu$ such that

$$\mathbb{R}^n = \xi_1 + \cdots + \xi_{l'}, \quad \text{and} \quad l' \geq l. \tag{8.5}$$

We will show that $l' = l$. Suppose $l' > l$.

Let

$$\Xi = \xi_1^\perp \cup \cdots \cup \xi_{l'}^\perp.$$

From Lemma 8.1 we know that for each $x \in \mathbb{R}^n$, $a_1(t\mathrm{P}_{\xi_i} x) = \lambda(t\mathrm{P}_{\xi_i} x)$ (defined by (8.4)), as a function of t, cannot be of multiplicity n in any sub-interval of $(0, a)$. Otherwise, $\phi'' = 0$. Then $a_1(tu)$ is constant, where $u = \mathrm{P}_{\xi_i} x / |\mathrm{P}_{\xi_i} x|$. Therefore, there is $x \notin \Xi$, such that

$$a_1(\mathrm{P}_{\xi_i} x) \neq a_2(\mathrm{P}_{\xi_i} x) = \cdots = a_m(\mathrm{P}_{\xi_i} x) = 2\psi_1'(s \cdot s),$$

for $i = 1, ..., l'$. By the continuity of eigenvalues, there is an open set containing x, say $O_x \subset \mathbb{R}^n \backslash \Xi$, so that

$$a_1(\mathrm{P}_{\xi_i} y) \neq a_2(\mathrm{P}_{\xi_i} y) = \cdots = a_m(\mathrm{P}_{\xi_i} y) \tag{8.6}$$

holds for $i = 1, ..., l'$, and $y \in O_x$.

For this fixed x, by (8.5), there are n eigenspaces $E_{k_1}(\mathrm{P}_{\xi_{l_1}} x), ..., E_{k_n}(\mathrm{P}_{\xi_{l_n}} x)$, such that there exist linearly independent n eigenvectors

$$w_j \in E_{k_j}(\mathrm{P}_{\xi_{l_j}} x),$$

where $k_j \in \{1, ..., m\}$, $l_j \in \{1, ..., l'\}$, and $j = 1, ..., n$.

There are two cases: either $\{\xi_1, ..., \xi_{l'}\} \neq \{\xi_{l_1}, ..., \xi_{l_n}\}$ or $\{\xi_1, ..., \xi_{l'}\} = \{\xi_{l_1}, ..., \xi_{l_n}\}$. For the first case, there is $\xi' \in \{\xi_1, ..., \xi_{l'}\} \backslash \{\xi_{l_1}, ..., \xi_{l_n}\}$ and an eigenvector w' of $A_{\xi'}(x)$. For the second case, by $l' > l = \frac{n}{m}$, there is $\xi' \in \{\xi_{l_1}, ..., \xi_{l_n}\}$, so that the set

$$\xi' \cap \{w_1, ..., w_n\}$$

has no more than $m-1$ w_j's. Then, we can find an eigenvector w' of $A_{\xi'}(x)$, such that

$$w' \notin \operatorname{span}\big(\xi' \cap \{w_1, ..., w_n\}\big).$$

Since $w_1, \ldots, w_n$ are linearly independent, we always have

$$w' = c_1 w_1 + \cdots + c_n w_n.$$

If the w_j's with non-zero coefficients belong to only one subspace, say ξ_{l_k}, then $\xi_{l_k} \neq \xi'$, otherwise, $w' \in \xi_{l_k} = \operatorname{span}\big(\xi' \cap \{w_1, ..., w_n\}\big)$, a contradiction. Therefore, we can assume that $c_1 \neq 0$, $\xi' \neq \xi_{l_1}$, and hence $w', w_2, \ldots, w_n$ are linearly independent.

Let $a_{k'}(\mathrm{P}_{\xi'} x)$ be the eigenvalue corresponding to w'. By the equality conditions of Lemma 3.4,

$$a_{k_1}(\mathrm{P}_{\xi_{l_1}} x) a_{k_2}(\mathrm{P}_{\xi_{l_2}} x) \cdots a_{k_n}(\mathrm{P}_{\xi_{l_n}} x) = a_{k'}(\mathrm{P}_{\xi'} x) a_{k_2}(\mathrm{P}_{\xi_{l_2}} x) \cdots a_{k_n}(\mathrm{P}_{\xi_{l_n}} x).$$

It follows that

$$a_{k_1}(\mathrm{P}_{\xi_{l_1}} x) = a_{k'}(\mathrm{P}_{\xi'} x).$$

Recall that (8.6) holds for all $y \in O_x$. This means that the multiplicity of $a_h(\mathrm{P}_{\xi_j} y)$ will not change for $h \in \{1, ..., m\}$, $j \in \{1, ..., l'\}$, and $y \in O_x$. Since $A_{\xi_{l_j}}(x)$ is continuous of x, and Lemma 3.1 says that $\mathcal{P}_i(A)$ is continuous in the set that the eigenvalues of A keep multiplicities. Thus

$$\mathcal{P}_{k_j}(A(\xi_{l_j}, y)), \quad \mathcal{P}_{k'}(A(\xi', y))$$

are continuous for $y \in O_x$, for all $j \in \{1, ..., n\}$. Then, there is an open set $O'_x \subset O_x$ containing x, such that for all $y \in O'_x$, we have

$$\mathcal{P}_{k_1}(A(\xi_{l_1}, y)) w_1, \mathcal{P}_{k_2}(A(\xi_{l_2}, y)) w_2, ..., \mathcal{P}_{k_n}(A(\xi_{l_n}, y)) w_n$$

are linearly independent, and

$$\mathcal{P}_{k'}(A(\xi', y)) w', \mathcal{P}_{k_2}(A(\xi_{l_2}, y)) w_2, ..., \mathcal{P}_{k_n}(A(\xi_{l_n}, y)) w_n$$

are linearly independent, too. Since $0 \neq \mathcal{P}_{k'}(\mathrm{P}_{\xi_{l_i}} y) w_i \in E_{k'}(\mathrm{P}_{\xi_{l_i}} y)$ and $0 \neq \mathcal{P}_{k_i}(\mathrm{P}_{\xi_{l_i}} y) w_i \in E_{k_i}(\mathrm{P}_{\xi_{l_i}} y)$, by the equality conditions of Lemma 3.4, we conclude that

$$a_{k_1}(\mathrm{P}_{\xi_{l_1}} y) a_{k_2}(\mathrm{P}_{\xi_{l_2}} y) \cdots a_{k_n}(\mathrm{P}_{\xi_{l_n}} y) = a_{k'}(\mathrm{P}_{\xi'} y) a_{k_2}(\mathrm{P}_{\xi_{l_2}} y) \cdots a_{k_n}(\mathrm{P}_{\xi_{l_n}} y),$$

and hence

$$a_{k_1}(\mathrm{P}_{\xi_{l_1}} y) = a_{k'}(\mathrm{P}_{\xi'} y) \tag{8.7}$$

for all $y \in O'_x$.

Since $\xi_{l_1} \neq \xi'$ are two m-dimensional linear subspaces, there is $v \in \xi_{l_1}^{\perp}$ but $v \notin \xi'^{\perp}$. Consider $x + tv$ for $t \in I$, where I is a small open interval containing 0 so that $x + tv \in O'_x$ for all $t \in I$. We see that

$$\mathrm{P}_{\xi_{l_1}}(x + tv) = \mathrm{P}_{\xi_{l_1}} x.$$

Denote

$$u(t) = \mathrm{P}_{\xi'}(x + tv),$$

for $t \in I$. By the cosine formula,

$$|u(t)|^2 = |u(0)|^2 + t^2 |\mathrm{P}_{\xi'} v|^2 + 2t |u(0)| |\mathrm{P}_{\xi_{l'}} v| \big(u(0) \cdot \mathrm{P}_{\xi'} v\big).$$

The condition $v \notin \xi'^{\perp}$ implies $|\mathrm{P}_{\xi'} v| \neq 0$. Then $|u(t)|$ is continuous for $t \in I$. Moreover, either for all $t > 0$ or for all $t < 0$, there is

$$|u(t)| > |u(0)|.$$

Without loss of generality, we assume

$$|u(t)| > |u(0)| \quad \text{for} \quad t \in I, \text{ and } t > 0. \tag{8.8}$$

By (8.7), we get

$$a_{k'}(u(t)) = a_{k_1}(\mathrm{P}_{\xi_{l_1}}(x + tv)) = a_{k_1}(\mathrm{P}_{\xi_{l_1}} x) = a_{k'}(u(0)), \tag{8.9}$$

for all $t \in I$.

By the rotation invariance of eigenvalues, the eigenvalue $a_{k'}(\mathrm{P}_{\xi'} y)$ depends only on $|\mathrm{P}_{\xi'} y|$. Then for any $v_0 \in S^{n-1}$, (8.9) implies that

$$a_{k'}(|u(t)| v_0) = a_{k'}(|u(0)| v_0), \tag{8.10}$$

for all $t \in I$. But (8.8) says $|u(t)| > |u(0)|$ when $t > 0$. By this, the continuity of $|u(t)|$, and (8.10), we conclude that $a_{k'}(r v_0)$ is constant when r is in some open sub-interval of $(0, +\infty)$. By Lemma 8.1, this is a contradiction, and hence $\operatorname{supp} \nu$ has exactly l elements. $\square$

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