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ON THE DISCRETE ORLICZ MINKOWSKI PROBLEM

YUCHI WU, DONGMENG XI, AND GANGSONG LENG

ABSTRACT. In this paper, we demonstrate the existence part of the discrete Orlicz Minkowski problem, which is a non-trivial extension of the discrete L_p Minkowski problem for $0 < p < 1$.

1. INTRODUCTION

One of the cornerstones of the classical Brunn–Minkowski theory is the Minkowski problem. At the turn of the 19th into the 20th century, Minkowski proposed this problem and solved the discrete case. The Minkowski problem was completely solved by Alexandrov [1], and Fenchel and Jessen [12]. Analytic versions and algorithmic issues of this problem are still the subject of current research and highly relevant (see, e.g., Chou and Wang [10], Jerison [26], Klain [27], and references therein).

In the middle of the last century, Firey (see [38] for references) extended Minkowski addition to L_p Minkowski–Firey addition. As a part of the L_p Minkowski theory, Lutwak [30] introduced the L_p Minkowski problem. It asks for necessary and sufficient conditions on a Borel measure μ on S^{n-1} to be the L_p surface area measure of a convex body; i.e., is there a convex body K such that

$$h_K^{1-p} dS_K = d\mu \ ?$$

Here, h_K is the support function of K and S_K is the surface area measure of K. The solutions of the L_p Minkowski problem have important applications to affine isoperimetric inequalities, see, e.g., Zhang [46], Lutwak, Yang, and Zhang [33], Haberl and Schuster [18–20].

The even L_p Minkowski problem for $p > 1$ but $p \neq n$ was solved in [30]. An equivalent volume-normalized version of the L_p Minkowksi problem was proposed in [34], and the even case was also solved for $p = n$. A solution to the L_p Minkowski problem for $p > n$ was given by Chou and Wang [11], in which they also solved the problem for polytopes for all $p > 1$, while an alternate approach to this problem was presented by Hug et al., [25]. Zhu [47–49], deals with the existence for the solution to the discrete L_p Minkowski problem for $0 \leq p < 1$ and $p = -n$. Other studies with respect to the L_p Minkowski problem have also been extensively studied (see, e.g., [5, 8, 9, 22, 35, 39–41, 43, 50]). Quite recently, Huang, Lutwak, Yang, and Zhang [24]

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proposed the dual Minkowski problem and proved the existence theorem. Since [24], a number of works on the dual Minkowski problem have appeared. Zhao [44], Böröczky, Henk, and Pollehn [7], and Böröczky, Lutwak, Yang, Zhang, and Zhao [6] combined completely solved the existence part of the even dual Minkowski problem when the index $q \in (1, n)$. Zhao [45] proved both the existence and the uniqueness of the solution to the dual Minkowski problem when $q < 0$. Henk and Pollehn [21] showed a necessary condition for the even dual Minkowski problem when $q \geq n + 1$.

The Orlicz Brunn–Minkowski theory originated from the work of Lutwak, Yang, and Zhang in 2010; see [36, 37] and the 2010 work of Ludwig [28] and Ludwig and Reitzner [29]. For the development of the Orlicz Brunn–Minkowski theory, see [14,15,17,28,42]. Haberl, Lutwak, Yang, and Zhang [17] first proposed the following Orlicz Minkowski problem: Given a suitable continuous function $\varphi : (0, +\infty) \to (0, +\infty)$ and a Borel measure μ on S^{n-1}, is there a convex body K such that for some $c > 0$,

$$c\varphi(h_K)dS_K = d\mu \ ?$$

Set $\varphi(t) = t^{1-p}$ $(p \neq n)$. This problem reduces to the L_p Minkowski problem.

The even Orlicz Minkowski problem was solved by Haberl, Lutwak, Yang, and Zhang in [17] under some suitable conditions on φ. One of their results is:

Theorem 1.1 ([17]). *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is a continuous function such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t and is unbounded as $t \to \infty$ and μ is an even finite Borel measure on S^{n-1} that is not concentrated on any great subsphere of S^{n-1}. Then there exists an origin symmetric convex body $K \subset \mathbb{R}^n$ and $c > 0$ such that $c\varphi(h_K)dS_K = d\mu$.*

When $\varphi(t) = t^{1-p},\ p > 0$, we obtain the even L_p Minkowski problem for $p > 0$.

Later, the existence of the general Orlicz Minkowski problem without assuming that μ is an even measure was solved by Huang and He [23]. But besides the assumptions on φ in [17], they assume that $\varphi(s)$ tends to infinity as $s \to 0^+$. As we can see, the L_p Minkowski problem for $p > 1$ is a special case of this result. However, the L_p Minkowski problem for $0 < p < 1$ is not contained in this result.

In this paper, we aim to introduce a new version of the Orlicz Minkowski problem for polytopes, which contains the discrete L_p Minkowski problem for $0 < p < 1$.

Our main result can be formulated as follows:

Theorem 1.2. *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t and is unbounded as $t \to \infty$. If $\mu = \sum_{i=1}^N \alpha_i \delta_{u_i}$, where δ_{u_i} is Kronecker delta, $\alpha_1, \ldots, \alpha_N > 0$, and $u_1, \ldots, u_N \in S^{n-1}$ are not contained in any closed hemisphere, then there exists a polytope P which contains the origin in its interior and $c > 0$ such that*

$$\mu = c\varphi(h(P, \cdot))S(P, \cdot). \tag{1.1}$$

Letting $\varphi(s) = s^{1-p}$, $0 < p < 1$, we get Zhu's result in [47].

Corollary 1.3. *Suppose vectors $u_1, \ldots, u_N \in S^{n-1}$ are not contained in any closed hemisphere, $\alpha_1, \ldots, \alpha_N > 0$, and $\mu = \sum_{i=1}^N \alpha_i \delta_{u_i}$, where δ_{u_i} is Kronecker delta. If $0 < p < 1$, then there exists a polytope P which contains the origin in its interior* *such that*

$$\mu = h(P,\cdot)^{1-p}S(P,\cdot).$$

The work of Zhu [47] inspired us a lot. However, when it comes to the Orlicz case, the functional φ may not be homogeneous, so it is difficult to show that the map $\xi_\phi(P_r)$ has a right derivative at $r=0$, which is needed to use the Lagrange multiplier rule. Thus, we need many new steps; for details, see section 4. This paper is organized as follows: In section 2, we list some basic facts regarding convex bodies for quick reference. In section 3, we give some properties about $\Phi_P(\xi)$. In section 4, we prove the differentiability of $\xi_\phi(P_r)$. The proof of Theorem 1.2 is presented in section 5.

2. Preliminaries

In this section, we collect some terminology and notation about convex bodies. We recommend the books of Gardner [13], Gruber [16], and Schneider [38] as excellent references on convex geometry.

For $x,y\in\mathbb{R}^n$, let $[x,y]=\{(1-\lambda)x+\lambda y : 0\le\lambda\le 1\}$ and $(x,y)=\{(1-\lambda)x+\lambda y : 0<\lambda<1\}$. We also denote their inner product by $x\cdot y$ and the Euclidean norm of x by $|x|=\sqrt{x\cdot x}$. The unit sphere $\{x\in\mathbb{R}^n : |x|=1\}$ is denoted by S^{n-1}. Let V stand for n-dimensional Lebesgue measure and $|\mu|=\mu(S^{n-1})$ for a finite Borel measure μ on S^{n-1}.

A convex body is a compact convex set in $\mathbb{R}^n$ with non-empty interior. For a convex body K, the support function h_K is defined by $h_K(u)=h(K,u)=\max\{x\cdot u : x\in K\}$. We also denote $H_{u,t}=\{x\in\mathbb{R}^n : x\cdot u=t\}$ and $H^-_{u,t}=\{x\in\mathbb{R}^n : x\cdot u\le t\}$. For $u\in S^{n-1}$, the support hyperplane $F(K,u)$ in direction u is defined by

$$F(K,u)=\{x\in\mathbb{R}^n : x\cdot u=h(K,u)\};$$

the half-space $H^-(K,u)$ in direction u is defined by

$$H^-(K,u)=\{x\in\mathbb{R}^n : x\cdot u\le h(K,u)\}.$$

If the unit vectors $u_1,\dots,u_N$ $(N>n+1)$ are not contained in any closed hemisphere, we denote by $\mathcal{P}(u_1,\dots,u_N)$ a subset of polytopes which satisfies

$$P=\bigcap_{k=1}^{N} H^-(P,u_k),\quad \forall P\in\mathcal{P}(u_1,\dots,u_N).$$

It is easy to see that if $P\in\mathcal{P}(u_1,\dots,u_N)$, then P has at most N facets, and the outer unit normals of P are a subset of $\{u_1,\dots,u_N\}$. Let $\mathcal{P}_N(u_1,\dots,u_N)$ denote the subset of $\mathcal{P}(u_1,\dots,u_N)$ such that if $P\in\mathcal{P}_N(u_1,\dots,u_N)$, then P has exactly N facets.

A point z is said to be a vertex of a polytope P if it cannot be written in the form $z=(1-\lambda)x+\lambda y$ with $x,y\in P, x\neq y$, and $\lambda\in(0,1)$. The set of vertices of P is denoted by vert$P$.

For a Borel set $\omega\subset S^{n-1}$, the surface area measure $S_K(\omega)$ of the convex body K is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω; i.e.,

$$S_K(\omega)=\int_{x\in\nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of boundary points of K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Observe that for the surface area measure of cK we have

$$S_{cK} = c^{n-1} S_K, \qquad c > 0. \tag{2.1}$$

Lemma 2.1. *Suppose $\varphi : (0,\infty) \to (0,\infty)$ is differentiable, strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t. Let $\phi(0) = \lim_{t\to 0^+} \phi(t)$. Then*

$$\phi'(t) = \frac{1}{\varphi(t)} \quad \text{for } t > 0, \qquad \lim_{t\to 0^+} \frac{t}{\phi(t) - \phi(0)} = 0, \tag{2.2}$$

and ϕ is strictly concave on $[0,\infty)$.

Proof. The first equation of (2.2) is clear, and the second follows from L'Hôpital's rule. Since $\varphi : (0,\infty) \to (0,\infty)$ is differentiable, strictly increasing, we have

$$\phi'' = -\frac{\varphi'}{\varphi^2} < 0. \tag{2.3}$$

Thus ϕ is strictly concave on $(0,\infty)$. Then, for $\forall x, y \in (0,\infty)$,

$$\phi((1-\lambda)x + \lambda y) > (1-\lambda)\phi(x) + \lambda\phi(y), \quad \forall \lambda \in (0,1). \tag{2.4}$$

Letting $x \to 0^+$, we have

$$\phi(\lambda y) \geq (1-\lambda)\phi(0) + \lambda\phi(y), \quad \forall \lambda \in (0,1). \tag{2.5}$$

These two inequalities (2.4) and (2.5) imply that ϕ is concave on $[0,\infty)$. We claim that ϕ is also strictly concave on $[0,\infty)$. If not, then there exist λ', x' with $0 < \lambda' < 1, x' > 0$, such that

$$\phi(\lambda' x') = (1-\lambda')\phi(0) + \lambda'\phi(x'). \tag{2.6}$$

Then for $\lambda' < \mu < 1$, by the concavity of ϕ, we have

$$\phi(\lambda' x') = \phi(\frac{\mu - \lambda'}{\mu} \cdot 0 + \frac{\lambda'}{\mu}\mu x') \geq \frac{\mu - \lambda'}{\mu}\phi(0) + \frac{\lambda'}{\mu}\phi(\mu x'). \tag{2.7}$$

Combining with (2.6), we have

$$\phi(\mu x') \leq (1-\mu)\phi(0) + \mu\phi(x').$$

Note that $\phi(\mu x') = \phi((1-\mu)\cdot 0 + \mu x') \geq (1-\mu)\phi(0) + \mu\phi(x')$; thus

$$\phi(\mu x') = (1-\mu)\phi(0) + \mu\phi(x'). \tag{2.8}$$

From (2.6) and (2.8), it follows that

$$\phi(\mu x') = \frac{1-\mu}{1-\lambda'}\phi(\lambda' x') + \frac{\mu - \lambda'}{1-\lambda'}\phi(x'),$$

which contradicts the fact that ϕ is strictly concave on $(0,\infty)$. Therefore, ϕ is strictly concave on $[0,\infty)$. $\square$

3. An extremal problem to the Orlicz Minkowski problem

Suppose that $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not contained in any closed hemisphere, and $P \in \mathcal{P}(u_1, \ldots, u_N)$. Now we define the function $\Phi_P : P \to \mathbb{R}$ by

$$\Phi_P(\xi) = \sum_{k=1}^{N} \alpha_k \phi\left(h(P, u_k) - \xi \cdot u_k\right), \tag{3.1}$$

where ϕ is as described in Theorem 1.2 and $\phi(0) := \lim_{t \to 0^+} \phi(t)$.

In this section, we study the following extremal problem:

$$\sup\{V(Q) : \sup_{\xi \in Q} \Phi_Q(\xi) = 1 \text{ and } Q \in \mathcal{P}(u_1, \ldots, u_N)\}. \tag{3.2}$$

Next, we will prove that $\Phi_P(\xi)$ is concave on P and that there exists a unique $\xi_\phi(P) \in \operatorname{Int}(P)$ such that

$$\Phi_P(\xi_\phi(P)) = \sup_{\xi \in P} \Phi_P(\xi).$$

We want to prove that there exists a polytope with $u_1, \ldots, u_N$ as its outer unit normals and this polytope is a solution of problem (3.2). Now, we prove the concavity of $\Phi_P(\xi)$.

Lemma 3.1. *If $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not contained in any closed hemisphere, ϕ is strictly concave on $[0, \infty)$, and $P \in \mathcal{P}(u_1, \ldots, u_N)$, then $\Phi_P(\xi)$ is strictly concave on P.*

Proof. Since ϕ is strictly concave on $[0, \infty)$, for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in P$,

$$\begin{aligned}
&\lambda \Phi_P(\xi_1) + (1-\lambda)\Phi_P(\xi_2) \\
=& \sum_{k=1}^{N} \alpha_k \left[\lambda \phi\left(h(P, u_k) - \xi_1 \cdot u_k\right) + (1-\lambda)\phi\left(h(P, u_k) - \xi_2 \cdot u_k\right)\right] \\
\leq& \sum_{k=1}^{N} \alpha_k \phi\left(h(P, u_k) - (\lambda \xi_1 + (1-\lambda)\xi_2) \cdot u_k\right) \\
=& \Phi_P(\lambda \xi_1 + (1-\lambda)\xi_2),
\end{aligned}$$

with equality if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, \ldots, N$. Since $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere, $\mathbb{R}^n = \operatorname{Span}\{u_1, \ldots, u_N\}$. Thus, $\xi_1 = \xi_2$. Therefore, $\Phi_P(\xi)$ is strictly concave on P. $\square$

Next we prove the existence and uniqueness of $\xi_\phi(P)$.

Lemma 3.2. *Suppose $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not concentrated on any closed hemisphere, and $P \in \mathcal{P}(u_1, \ldots, u_N)$. If $\varphi : (0, \infty) \to (0, \infty)$ is differentiable, strictly increasing, $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t and is unbounded as $t \to \infty$, and $\phi(0) := \lim_{t \to 0^+} \phi(t)$, then there exists a unique $\xi_\phi(P) \in \operatorname{Int}(P)$ such that*

$$\Phi_P(\xi_\phi(P)) = \max_{\xi \in P} \Phi_P(\xi).$$

Proof. It follows from Lemma 2.1 and Lemma 3.1 that $\Phi_P(\xi)$ is strictly concave on P. Since P is a compact convex set, there exists a unique $\xi_\phi(P) \in P$ such that

$$\Phi_P(\xi_\phi(P)) = \max_{\xi \in P} \Phi_P(\xi).$$

We next prove that $\xi_\phi(P) \in \mathrm{Int}(P)$. Otherwise, suppose $\xi_\phi(P) \in \partial P$ with

$$h(P, u_k) - \xi_\phi(P) \cdot u_k = 0$$

for $k \in \{i_1, \dots, i_s\}$ and

$$h(P, u_k) - \xi_\phi(P) \cdot u_k > 0$$

for $k \in \{1, \dots, N\} \backslash \{i_1, \dots, i_s\}$, where $1 \le i_1 \le \cdots \le i_s \le N$ and $1 \le s \le N-1$. Choose $x_0 \in \mathrm{Int}(P)$. Let

$$u_0 = \frac{x_0 - \xi_\phi(P)}{|x_0 - \xi_\phi(P)|}$$

and

$$[h(P, u_k) - (\xi_\phi(P) + \delta u_0) \cdot u_k] - [h(P, u_k) - \xi_\phi(P) \cdot u_k] = c_k \delta, \tag{3.3}$$

where $c_k = -u_0 \cdot u_k$. Since $h(P, u_k) - \xi_\phi(P) \cdot u_k = 0$ for $k \in \{i_1, \dots, i_s\}$ and x_0 is an interior point of P, then $c_k = -u_0 \cdot u_k > 0$ for $k \in \{i_1, \dots, i_s\}$. Let

$$c_0 = \min \{h(P, u_k) - \xi_\phi(P) \cdot u_k : k \in \{1, \dots, N\} \backslash \{i_1, \dots, i_s\}\} > 0,$$

and choose $\delta > 0$ small enough so that $\xi_\phi(P) + \delta u_0 \in \mathrm{Int}(P)$ and

$$\min\{h(P, u_k) - (\xi_\phi(P) + \delta u_0) \cdot u_k : k \in \{1, \dots, N\} \backslash \{i_1, \dots, i_s\}\} > \frac{c_0}{2}.$$

Since ϕ is differentiable, strictly increasing, and concave (Lemma 2.1), for x_0, $x_0 + \Delta x \in (\frac{c_0}{2}, \infty)$, we have

$$|\phi(x_0 + \Delta x) - \phi(x_0)| < \phi'(\frac{c_0}{2}) |\Delta x|.$$

From these two inequalities, $h(P, u_k) = \xi_\phi(P) \cdot u_k$ for $k \in \{i_1, \dots, i_s\}$, $c_k > 0$ for $k \in \{i_1, \dots, i_s\}$ and equations (3.3), it follows that

$$\begin{aligned}
&\Phi_p(\xi_\phi(P) + \delta u_0) - \Phi_p(\xi_\phi(P)) \\
&= \sum_{k=1}^{N} \alpha_k [\phi(h(P, u_k) - (\xi_\phi(P) + \delta u_0) \cdot u_k) - \phi(h(P, u_k) - \xi_\phi(P) \cdot u_k)] \\
&\ge - \sum_{k \in \{1, \dots, N\} \backslash \{i_1, \dots, i_s\}} |\phi(h(P, u_k) - (\xi_\phi(P) + \delta u_0) \cdot u_k) \\
&\quad - \phi(h(P, u_k) - \xi_\phi(P) \cdot u_k)| + \sum_{k \in \{i_1, \dots, i_s\}} \alpha_k (\phi(c_k \delta) - \phi(0)) \\
&\ge - \sum_{k \in \{1, \dots, N\} \backslash \{i_1, \dots, i_s\}} \alpha_k \phi'(\frac{c_0}{2}) |c_k \delta| + \sum_{k \in \{i_1, \dots, i_s\}} \alpha_k (\phi(c_k \delta) - \phi(0)).
\end{aligned}$$

Note that $\lim_{t \to 0^+} \frac{t}{\phi(t) - \phi(0)} = 0$ (Lemma 2.1), so there exists a small enough $\delta_0 > 0$ such that $\xi_\phi(P) + \delta_0 u_0 \in \mathrm{Int}(P)$ and

$$\Phi_P(\xi_\phi(P) + \delta_0 u_0) > \Phi_P(\xi_\phi(P)),$$

which contradicts the definition of $\xi_\phi(P)$. Therefore, the conclusion follows. $\square$

Note that if $P_i \in \mathcal{P}(u_1, \dots, u_N)$ and P_i converges to a polytope P, then $P \in \mathcal{P}(u_1, \dots, u_N)$. In order to use approximation, we need the following lemma.

Lemma 3.3. *Suppose $\varphi : (0,\infty) \to (0,\infty)$ is differentiable, strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t and is unbounded as $t \to \infty$. If $\phi(0) = \lim_{t\to 0^+} \phi(t)$, $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not concentrated on any closed hemisphere, $P_i \in \mathcal{P}(u_1, \ldots, u_N)$, and P_i converges to a polytope P, then $\lim_{i\to\infty} \xi_\phi(P_i) = \xi_\phi(P)$ and*

$$\lim_{i\to\infty} \Phi_{P_i}(\xi_\phi(P_i)) = \Phi_P(\xi_\phi(P)).$$

Proof. By Lemma 3.2, $\xi_\phi(P_i)$ exists. Since $P_i \to P$ and $\xi_\phi(P_i) \in \operatorname{Int}(P_i)$, $\xi_\phi(P_i)$ is bounded. Suppose $\xi_\phi(P_i)$ does not converge to $\xi_\phi(P)$. Then there exists a subsequence P_{i_j} of P_i such that P_{i_j} converges to P, $\xi_\phi(P_{i_j}) \to \xi_0$, but $\xi_0 \neq \xi_\phi(P)$. It follows from the continuity of ϕ that $\Phi_P(\xi)$ is continuous with respect to P and ξ. Then by $\xi_0 \in P$, we have

$$\begin{aligned}\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) &= \Phi_P(\xi_0)\\ &< \Phi_P(\xi_\phi(P))\\ &= \lim_{j\to\infty} \Phi_{P_{i_j}}(\xi_\phi(P)),\end{aligned}$$

which contradicts the fact that

$$\Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) \geq \Phi_{P_{i_j}}(\xi_\phi(P)).$$

Therefore, $\lim_{i\to\infty} \xi_\phi(P_i) = \xi_\phi(P)$. Thus,

$$\lim_{i\to\infty} \Phi_{P_i}(\xi_\phi(P_i)) = \Phi_P(\xi_\phi(P)).$$ □

For convex body K in $\mathbb{R}^n$, we define

$$R(K) = \max_{x\in K} |x|.$$

The following lemma is needed to prove the boundness.

Lemma 3.4. *Suppose $\phi : [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, and $\phi(t)$ tends to infinity as $t \to \infty$. If $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not concentrated on any closed hemisphere, $P_k \in \mathcal{P}(u_1, \ldots, u_N)$, $o \in P_k$, and $R(P_k)$ is not bounded, then*

$$\sum_{i=1}^N \alpha_i \phi(h(P_k, u_i))$$

is not bounded.

Proof. By taking subsequences, we can assume that

$$\lim_{k\to\infty} R(P_k) = \infty. \tag{3.4}$$

Let

$$h_+(t) = \max\{0, t\},\ f(u) = \sum_{i=1}^N \alpha_i \phi(h_+(u_i \cdot u)),$$

where $t \in \mathbb{R},\ u \in S^{n-1}$.

Since $u_1, \ldots, u_N$ are not contained in any closed hemisphere,

$$\mathbb{R}^n = \operatorname{Span}\{u_1, \ldots, u_N\}.$$

Thus, for $u \in S^{n-1}$ there exists $i \in \{1, \dots, N\}$ such that $h_+(u_i \cdot u) > 0$. Then we have $f(u) > 0$ for all $u \in S^{n-1}$. Note that $\max\{h_+(u_i \cdot u) : i \in \{1, \dots, N\}\}$ is a continuous function on S^{n-1}. Thus, there exists a constant $a_0 > 0$ such that

$$\max\{h_+(u_i \cdot u) : i \in \{1, \dots, N\}\} > a_0, \quad \text{for all } u \in S^{n-1}. \tag{3.5}$$

Suppose that $\sum_{i=1}^N \alpha_i \phi(h(P_k, u_i))$ is bounded. Then there exists $M \in \mathbb{R}$ such that

$$\phi(h(P_k, u_i)) < M, \quad \text{for all } i \in \{1, \dots, N\}.$$

Since ϕ is continuous, strictly increasing, and $\phi(t)$ tends to infinity as $t \to \infty$, there exists a unique $t_0 \in \mathbb{R}^+$ such that $\phi(t_0) = M$. Together with $\phi(h(P_k, u_i)) < M$, we have

$$t_0 > h(P_k, u_i), \quad \forall i \in \{1, \dots, N\}. \tag{3.6}$$

Choose $v_k \in S^{n-1}$ such that $R(P_k)v_k \in P_k$. Since $o \in P_k$,

$$h(P_k, u_i) \geq h_+(R(P_k)v_k \cdot u_i) = R(P_K)h(u_i \cdot v_k).$$

Together with (3.6) and (3.5), we have

$$t_0 > R(P_k) \max_i h(u_i \cdot v_k) > R(P_k)a_0,$$

which contradicts (3.4). Thus $\sum_{i=1}^N \alpha_i \phi(h(P_k, u_i))$ is unbounded. □

4. The differentiability of $\xi_\phi(P_r)$

In fact, Lemma 3.4 guarantees that there exists a polytope P that solves (3.2). See Lemma 4.9 for details. In this section, let δ_m^k be Kronecker delta. This means that if $k = m$, then $\delta_m^k = 1$; otherwise $\delta_m^k = 0$. We want to prove that P has exactly N faces. If $P \in \mathcal{P}_\mathcal{N}(u_1, \dots, u_N)$, then the differentiability of $\xi_\phi(P_r)$ is easy. See the following two lemmas.

Lemma 4.1. *Suppose the unit vectors $u_1, \dots, u_N$ ($N \geq n+1$) are not concentrated on any closed hemisphere. Let $P \in \mathcal{P}_\mathcal{N}(u_1, \dots, u_N)$ and*

$$P_r = \bigcap_{k=1}^N \{x : x \cdot u_k \leq h(P, u_k) - r\delta_m^k\},$$

where $m \in \{1, 2, \dots, N\}$. Then there exists a number $r_0 > 0$ such that $h(P_r, u_k) = h(P, u_k) - r\delta_m^k$ for every $|r| < r_0$.

Proof. Since $P \in \mathcal{P}_\mathcal{N}(u_1, \dots, u_N)$, by Lemma 2.4.13 in [38], one can choose $r_0 > 0$, such that P_r has exactly N facets for $|r| < r_0$, which implies $h(P_r, u_k) = h(P, u_k) - r\delta_m^k$. □

Lemma 4.2. *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t. If $\phi(0) = \lim_{t \to 0^+} \phi(t)$, $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \dots, u_N$ ($N \geq n+1$) are not concentrated on any closed hemisphere, $P \in \mathcal{P}_N(u_1, \dots, u_N)$, and $|r|$ is small enough such that*

$$P_r = \bigcap_{k=1}^N \{x : x \cdot u_k \leq h(P, u_k) - r\delta_m^k\} \in \mathcal{P}_N(u_1, \dots, u_N),$$

where $m \in \{1, 2, \dots, N\}$, *then there exists a number* $r_0 > 0$ *such that* $\xi(r) = \xi_\phi(P_r)$ *is continuously differentiable with respect to* r *in* $(-r_0, r_0)$.

Proof. Let $\xi(r) = \xi_\phi(P_r)$ and

$$\begin{aligned}\Phi(r) &= \max_{\xi \in P_r} \sum_{k=1}^{N} \alpha_k \phi\left(h(P_r, u_k) - \xi \cdot u_k\right) \\ &= \sum_{k=1}^{N} \alpha_k \phi\left(h(P_r, u_k) - \xi(r) \cdot u_k\right).\end{aligned}$$

From this and the fact that $\xi(r)$ is an interior point of P_r, we have that

$$\sum_{k=1}^{N} \alpha_k \phi'\left(h(P_r, u_k) - \xi(r) \cdot u_k\right) u_{k,i} = 0, \tag{4.1}$$

for $i = 1, \dots, n$, where $u_k = (u_{k,1}, \dots, u_{k,n})^T$.

Next, we use the inverse function theorem to prove the conclusion. Let $\xi_0 = \xi(0)$ and

$$F_i(r, \xi_1, \dots, \xi_n) = \sum_{k=1}^{N} \alpha_k \phi'\left(h(P_r, u_k) - \xi \cdot u_k\right) u_{k,i},$$

where $i \in \{1, \dots, n\}$ and $\xi = (\xi_1, \dots, \xi_n)$. Since $P \in \mathcal{P}_N(u_1, \dots, u_N)$, by Lemma 4.1, we have $h(P_r, u_k) = h(P, u_k) - r\delta_m^k$. Then

$$\frac{\partial F_i}{\partial r} = -\alpha_m \phi''\left(h(P, u_m) - r - \xi \cdot u_m\right) u_{m,i}$$

and

$$\frac{\partial F_i}{\partial \xi_j} = -\sum_{k=1}^{N} \alpha_k \phi''\left(h(P_r, u_k) - \xi \cdot u_k\right) u_{k,i} u_{k,j}$$

are obviously continuous.

Let $r = 0$. Then the Jacobian matrix of $F := (F_1, \dots, F_N)$ at ξ_0 equals

$$\left(\frac{\partial F}{\partial \xi_j}\bigg|_{\xi_0} \right)_{n \times n} = -\sum_{k=1}^{N} \alpha_k \phi''\left(h(P, u_k) - \xi_0 \cdot u_k\right) u_k \cdot u_k^T,$$

where $u_k u_k^T$ is an $n \times n$ matrix.

Since $u_1, \dots, u_N$ are not contained in any closed hemisphere,

$$\mathbb{R}^n = \operatorname{Span}\{u_1, \dots, u_N\}.$$

Thus, for any $x \in \mathbb{R}^n$ with $x \neq 0$, there exists a $u_{i_m} \in \{u_1, \dots, u_N\}$ such that $u_{i_m} \cdot x \neq 0$. Together with the fact that ϕ is twice differentiable and strictly concave, we have

$$\begin{aligned}& x^T \cdot \left(-\sum_{k=1}^{N} \alpha_k \phi''\left(h(P, u_k) - \xi_0 \cdot u_k\right) u_k \cdot u_k^T \right) \cdot x \\ &= -\sum_{k=1}^{N} \alpha_k \phi''\left(h(P, u_k) - \xi_0 \cdot u_k\right) (x \cdot u_k)^2 \\ &\geq -\alpha_{i_m} \phi''\left(h(P, u_{i_m}) - \xi_0 \cdot u_k\right) (x \cdot u_{i_m})^2 > 0.\end{aligned}$$

Thus, $\left(\frac{\partial F}{\partial \xi_j}\Big|_{(0,\xi_0)}\right)$ is positive definite. From this, equation (4.1), the inverse function theorem, and the fact that F_i has continuous partial derivative for ξ and r, the conclusion follows. $\square$

Remark 4.1. For $t > 0$, by a similar method in Lemma 4.2, we have $\xi_\phi(tP)$ is continuously differentiable in a small neighborhood of t. Thus, $\xi_\phi(tP)$ is continuous for every $t > 0$. Therefore, $\Phi_{tP}(\xi_\phi(tP))$ is continuous for $t > 0$.

In order to prove that every polytope which solves (3.2) has exactly N faces, we need one-sided differentiability of $\xi_\phi(P_r)$ for $P \in \mathcal{P}(u_1, \ldots, u_N)$. First, we study the property of $h(P_r, u_k)$, for which the following three lemmas are prepared.

Lemma 4.3. *Let P be a polytope. Then for every $u \in \mathbb{S}^{n-1}$, $F(P,u)$ is the convex hull of the set* $\mathrm{vert}P \cap F(P,u)$*, where* $\mathrm{vert}P$ *denotes the vertices of P.*

Proof. If $y \in F(P,u)$, then it can be expressed by

$$y = \sum_{i=1}^{m} a_i y_i, \quad \text{where } y_i \in \mathrm{vert}P,$$

where $0 < a_i \le 1$ and $\sum a_i = 1$.

Note that $y \cdot u = h(P,u)$ and $y_i \cdot u \le h(P,u)$. We have $y_i \cdot u = h(P,u)$ for $1 \le i \le m$. Thus, the conclusion follows. $\square$

Lemma 4.4. *Let P be a polytope, $u \in \mathbb{S}^{n-1}$. Then there exists a real number $r' > 0$ such that*

$$P \cap H_{u,h(P,u)-r} \subset \mathrm{conv}\{F(P,u) \cup (P \cap H_{u,h(P,u)-r'})\}, \quad \forall r \in (0, r').$$

Proof. We can choose $r' > 0$ small enough such that

$$z \cdot u < h(P,u) - r', \quad \forall z \in \mathrm{vert}P/F(P,u).$$

Let $0 < r < r'$ and $x \in P \cap H_{u,h(P,u)-r}$. Then x has the representation

$$x = \sum_{i=1}^{p} a_i y_i + \sum_{j=1}^{q} b_j z_j, \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = 1,$$

where $0 < a_i, b_j < 1$, $p, q \in \mathbb{N}$, $y_i \in F(P,u)$, $z_j \in \mathrm{vert}P/F(P,u)$, and $z_j \cdot u < h(P,u) - r'$.

We may write

$$x = \lambda \sum_{i=1}^{p} \frac{a_i}{\lambda} y_i + (1-\lambda) \sum_{j=1}^{q} \frac{b_j}{1-\lambda} z_j, \lambda := \sum_{i=1}^{p} a_i.$$

Note that

$$\left(\sum_{i=1}^{p} \frac{a_i}{\lambda} y_i \right) \cdot u = h(P,u), \qquad \left(\sum_{j=1}^{q} \frac{b_j}{1-\lambda} z_j \right) \cdot u < h(P,u) - r'.$$

Thus, we can choose a point

$$z \in \left(\sum_{i=1}^{p} \frac{a_i}{\lambda} y_i, \sum_{j=1}^{q} \frac{b_j}{1-\lambda} z_j \right) \subset P$$

such that

$$z \cdot u = h(P,u) - r'.$$

Combining with $x \in P \cap H_{u,h(P,u)-r}$, we have $x \in \left[\sum_{i=1}^{p} \frac{a_i}{\lambda} y_i, z\right]$, which is equivalent to the assertion of the lemma. $\square$

Lemma 4.5. *Suppose the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not concentrated on any closed hemisphere. Let $P \in \mathcal{P}(u_1, \ldots, u_N)$, $r' > 0$, and*

$$P_r = \bigcap_{k=1}^{N} \left\{x : x \cdot u_k \leq h(P, u_m) - r\delta_m^k\right\}$$

such that $h(P_r, u_m) = h(P, u_m) - r$ for $0 \leq r \leq r'$, where $m \in \{1, 2, \ldots, N\}$. Then there exists a number r_m with $0 < r_m < r'$ such that

(i) $\mathrm{vert} P_r \cap F(P_r, u_m) \subset \{(1-\lambda)y + \lambda z : y \in \mathrm{vert} P \cap F(P, u_m),\ z \in \mathrm{vert} P_{r_m} \cap F(P_{r_m}, u_m)\}$, *where $r \in (0, r_m)$ and $\lambda = \frac{r}{r_m}$.*

(ii) $F(P_r, u_m) = \{(1-\lambda)y + \lambda z : y \in \mathrm{vert} P \cap F(P, u_m),\ z \in \mathrm{vert} P_{r_m} \cap F(P_{r_m}, u_m)\}$, *where $r \in (0, r_m)$ and $\lambda = \frac{r}{r_m}$.*

Proof. Since $h(P_r, u_m) = h(P, u_m) - r$ for $0 \leq r \leq r'$, then $F(P_r, u_m) = P \cap H_{u_m, h(P,u_m)-r}$ for $0 \leq r \leq r'$. Thus, by Lemma 4.4 , there exists a number r_m with $0 < r_m < r'$ such that

$$F(P_r, u_m) \subset \mathrm{conv}\{F(P, u_m) \cup (F(P_{r_m}, u_m))\}, \quad \forall r \in (0, r_m). \tag{4.2}$$

For (i), let $x \in \mathrm{vert} P_r \cap F(P_r, u_m)$. By (4.2) and Lemma 4.3, it can be expressed as

$$x = \sum_{i=1}^{p} b_i y_i + \sum_{j=1}^{q} c_j z_j,$$

where $p, q \in \mathbb{N}$, $0 < b_i, c_j < 1$, $\sum b_i = 1 - \lambda$, $\sum c_j = \lambda$, $\lambda = \frac{r}{r_m}$, and $y_i \in \mathrm{vert} P \cap F(P, u_m)$, $z_j \in \mathrm{vert} P_{r_m} \cap F(P_{r_m}, u_m)$. If $p = q = 1$, the assertion is clear. Otherwise, we can rewrite x as

$$\begin{aligned} x &= \sum_{i=1}^{p} \frac{b_i}{1-\lambda}(1-\lambda) y_i + \sum_{j=1}^{q} \frac{c_j}{\lambda} \lambda z_j \\ &= \sum_{i,j} \frac{b_i c_j}{(1-\lambda)\lambda}((1-\lambda)y_i + \lambda z_j), \end{aligned}$$

where $\sum_{i,j} \frac{b_i c_j}{(1-\lambda)\lambda} = 1$. This contradicts the fact that $x \in \mathrm{vert} P_r$, since $(1-\lambda)y_i + \lambda z_j \in F(P_r, u_m) \subset P_r$.

The assertion (ii) follows from (i), Lemma 4.3, and $((1-\lambda)y + \lambda z) \cdot u_m = h(P, u_m) - r, \forall y \in \mathrm{vert} P \cap F(P, u_m), \forall z \in \mathrm{vert} P_{r_m} \cap F(P_{r_m}, u_m)$, where $\lambda = \frac{r}{r_m}$. $\square$

Now, we prove the property of $h(P_r, u_k)$ for $P \in \mathcal{P}(u_1, \ldots, u_N)$.

Lemma 4.6. *Suppose the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not concentrated on any closed hemisphere. Let $P \in \mathcal{P}(u_1, \ldots, u_N)$ and*

$$P_r = \bigcap_{k=1}^{N} \{x : x \cdot u_k \leq h(P, u_k) - r\delta_m^k\},$$

where $m \in \{1, 2, \ldots, N\}$*. Then there exists a number* $r_0 > 0$ *such that for* $0 \le r \le r_0$*,*

$$h(P_r, u_k) = \begin{cases} h(P, u_k) - r & \text{if } k = m, \\ h(P, u_k) - a_k r & \text{if } \ k \ne m, \end{cases}$$

where a_k *is a constant with* $a_k \ge 0$*.*

Proof. We first prove $h(P_r, u_m) = h(P, u_m) - r$ for small enough r. Let $x \in \text{vert}P \cap F(P, u_m)$. Suppose that

$$h(P, u_k) = x \cdot u_k$$

for $k \in \{m, i_1, \ldots, i_s\}$ and

$$h(P, u_k) > x \cdot u_k \tag{4.3}$$

for $k \in \{1, \ldots, N\} \backslash \{m, i_1, \ldots, i_s\}$, where $1 \le i_1 \le \cdots \le i_s \le N$ and $1 \le s \le N-1$.

Note that the set $\big\{u_k : \ k \in \{m, i_1, i_2, \ldots, i_s\}\big\}$ is contained in an open hemisphere. Thus, we can choose a unit vector u_0 such that

$$u_0 \cdot u_k > 0, \quad \forall k \in \{m, i_1, i_2, \ldots, i_s\}. \tag{4.4}$$

By (4.3), there exists a number $r' > 0$ such that

$$h(P, u_k) > (x - \frac{r}{u_0 \cdot u_m} u_0) \cdot u_k \tag{4.5}$$

$\forall 0 \le r \le r'$ and $\forall k \in \{1, \ldots, N\} \backslash \{m, i_1, \ldots, i_s\}$. It follows from (4.4) and (4.5) that

$$(x - \frac{r}{u_0 \cdot u_m} u_0) \cdot u_k \le h(P, u_k) - r\delta_m^k, \quad \forall k \in \{1, \ldots, N\}.$$

Hence

$$x - \frac{r}{u_0 \cdot u_m} u_0 \in P_r \quad \text{and} \quad (x - \frac{r}{u_0 \cdot u_m} u_0) \cdot u_m = h(P, u_m) - r.$$

This implies that

$$h(P_r, u_m) = h(P, u_m) - r \quad \forall 0 \le r \le r'. \tag{4.6}$$

Now, we turn to dealing with the case $k \ne m$. If $F(P, u_k) \not\subset F(P, u_m)$, then there exists $x_k \in F(P, u_k)$ such that

$$x_k \cdot u_k = h(P, u_k) \quad \text{but} \quad x_k \cdot u_m < h(P, u_m).$$

Then there exists a number r_k such that $x_k \cdot u_m < h(P, u_m) - r$ for $r < r_k$. This implies that $x_k \in P_r$ for $r < r_k$ and

$$h(P_r, u_k) = x_k \cdot u_k = h(P, u_k). \tag{4.7}$$

If $F(P, u_k) \subset F(P, u_m)$, we claim that $F(P_r, u_k) \subset F(P_r, u_m)$ for small enough r. In fact, let $x \in \text{vert}P \setminus F(P, u_m)$. We have

$$h(P, u_k) > x \cdot u_k \quad \text{and} \quad (x - y) \cdot u_k < 0, \quad \forall y \in F(P, u_k) \subset F(P, u_m),$$
$$h(P, u_m) > x \cdot u_m \quad \text{and} \quad (x - y) \cdot u_m < 0, \quad \forall y \in F(P, u_k) \subset F(P, u_m).$$

This implies that for small enough r there exists a point $z_r \in (x, y)$ such that $z_r \cdot u_m = h(P, u_m) - r$. Therefore, $z_r \in P_r$ and $h(P_r, u_k) \ge z_r \cdot u_k > x \cdot u_k$. Thus $x \notin F(P_r, u_k)$. Then the claim is clear, since $\text{vert}P \setminus F(P, u_m)$ is a finite set.

Using the claim and Lemma 4.5 with (4.6), we can choose a $0 < r_0 < r'$ such that

$$F(P_r, u_m) = \{(1-\lambda)y + \lambda z : y \in \mathrm{vert}P \cap F(P, u_m), \\ z \in \mathrm{vert}P_{r_0} \cap F(P_{r_0}, u_m)\}, \tag{4.8}$$

and

$$F(P_r, u_k) \subset F(P_r, u_m) \quad \text{if } F(P, u_k) \subset F(P, u_m), \tag{4.9}$$

where $r \in (0, r_0)$ and $\lambda = \frac{r}{r_0}$.

For any $r \in (0, r_0)$, by (4.9) there exists $x_k \in \mathrm{vert}P_r \cap F(P_r, u_m)$ such that $h(P_r, u_k) = x_k \cdot u_k$. By the definition of support function,

$$h(P_r, u_k) = \sup\{x \cdot u_k : x \in F(P_r, u_m).$$

By (4.8), we have $h(P_r, u_k) = \{((1-\lambda)y + \lambda z) \cdot u_k : y \in \mathrm{vert}P \cap F(P, u_m),\ z \in \mathrm{vert}P_{r_0} \cap F(P_{r_0}, u_m)\}$.

Together with $F(P, u_k) \subset F(P, u_m)$ and $F(P_{r_0}, u_k) \subset F(P_{r_0}, u_m)$, it follows that

$$h(P_r, u_k) = (1-\lambda)h(P, u_k) + \lambda h(P_{r_0}, u_k),$$

which is equivalent to

$$h(P_r, u_k) = h(P, u_k) - a_k r, \tag{4.10}$$

where $a_k = \frac{h(P,u_k) - h(P_{r_0}, u_k)}{r_0} \geq 0$.

The conclusion follows from (4.6), (4.7), and (4.10). ☐

With tackle in hand, now we aim to prove that $\xi_\phi(P_r)$ has one-sided derivative at 0 for $P \in \mathcal{P}(u_1, \ldots, u_N)$.

Lemma 4.7. *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t. Assume that $\phi(0) = \lim_{t \to 0^+} \phi(t)$, $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ $(N \geq n+1)$ are not concentrated on any closed hemisphere, $P \in \mathcal{P}(u_1, \ldots, u_N)$, and $r \geq 0$ is small enough such that*

$$P_r = \bigcap_{k=1}^{N} \{x : x \cdot u_k \leq h(P, u_k) - r\delta_m^k\} \in \mathcal{P}(u_1, \ldots, u_N),$$

where $m \in \{1, 2, \ldots, N\}$. If the continuous function $\lambda : [0, \infty) \to (0, \infty)$ is continuously differentiable on $(0, \infty)$ and $\lim_{r \to 0} \lambda'(r)$ exists, then $\xi_\phi(\lambda(r)P_r)$ has right derivative at 0.

Proof. Let $F = (F_1, \ldots, F_n)$ and

$$F_i(r, \xi_1, \ldots, \xi_n) = \sum_{k=1}^{N} \alpha_k \phi' \left(h(\lambda(r)P_r, u_k) - \xi \cdot u_k \right) u_{k,i}, \tag{4.11}$$

where $i \in \{1, \ldots, n\}$ and $\xi = (\xi_1, \ldots, \xi_n)$. Since $P \in \mathcal{P}(u_1, \ldots, u_N)$, by Lemma 4.6, for small enough $r \geq 0$, we have

$$h(\lambda(r)P_r, u_k) = \begin{cases} \lambda(r)h(P, u_k) - \lambda(r)r & \text{if } k = m, \\ \lambda(r)h(P, u_k) - a_k\lambda(r)r & \text{if } k \neq m, \end{cases} \tag{4.12}$$

where a_k is a constant with $a_k \geq 0$.

By a similar method in Lemma 4.2 and the inverse function theorem, $\xi(r) := \xi_\phi(\lambda(r)P_r)$ is continuously differentiable for every $r > 0$ and

$$\begin{pmatrix} \frac{d\xi_1}{dr} \\ \frac{d\xi_2}{dr} \\ \vdots \\ \frac{d\xi_n}{dr} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial \xi_1} & \frac{\partial F_1}{\partial \xi_2} & \cdots & \frac{\partial F_1}{\partial \xi_n} \\ \frac{\partial F_2}{\partial \xi_1} & \frac{\partial F_2}{\partial \xi_2} & \cdots & \frac{\partial F_2}{\partial \xi_n} \\ & & \vdots & \\ \frac{\partial F_n}{\partial \xi_1} & \frac{\partial F_n}{\partial \xi_2} & \cdots & \frac{\partial F_n}{\partial \xi_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial r} \\ \frac{\partial F_2}{\partial r} \\ \vdots \\ \frac{\partial F_n}{\partial r} \end{pmatrix}.$$

Letting $a_m = 1$, then by (4.11) and (4.12), we have

$$\frac{\partial F_i}{\partial \xi_j} = -\sum_{k=1}^{N} \alpha_k \phi''\left(h(\lambda(r)P_r, u_k) - \xi \cdot u_k\right) u_{k,i} u_{k,j}$$

and

$$\begin{aligned} \frac{\partial F_i}{\partial r} =& \sum_{k=1}^{N} \alpha_k \phi''\left(\lambda(r)h(P, u_k) - a_k \lambda(r) r - \xi \cdot u_k\right) \\ & \cdot \left(\lambda'(r)h(P, u_k) - a_k \lambda'(r) r - a_k \lambda(r)\right) u_{k,i}. \end{aligned}$$

By a similar proof in Lemma 4.2, the matrix $(\frac{\partial F_i}{\partial \xi_j})$ is positive definite. Thus, $\lim_{r\to 0+} \xi'(r)$ exists.

It follows from the Lagrange mean value theorem that for every $r > 0$ and $1 \le i \le n$, there exists an $\varepsilon_i(r)$ with $0 < \varepsilon_i(r) < r$ such that

$$\frac{\xi_i(r) - \xi_i(0)}{r} = \xi_i'(\varepsilon_i(r)).$$

Let $r \to 0+$. Then the conclusion follows. $\square$

Now, we turn to proving that there exists a polytope with $u_1, \dots, u_N$ as its outer unit normals and that this polytope is a solution of problem (3.2). Before this, we need the following lemma.

Lemma 4.8 ([47, Lemma 3.5]). *If P is a polytope in $\mathbb{R}^n$ and $v_0 \in S^{n-1}$ with $V_{n-1}(F(P, v_0)) = 0$, then there exists a $\delta_0 > 0$ such that for $0 \le \delta < \delta_0$,*

$$V(P \cap \{x : x \cdot v_0 \ge h(P, v_0) - \delta\}) = c_n \delta^n + \cdots + c_2 \delta^2,$$

where $c_n, \dots, c_2$ are constants that depend on P and v_0.

Next, we prove the existence of a solution in (3.2).

Lemma 4.9. *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is continuously differentiable, strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)} ds$ exists for every positive t and is unbounded as $t \to \infty$. If $\phi(0) = \lim_{t\to 0^+} \phi(t)$, $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$, and the unit vectors $u_1, \dots, u_N$ $(N \ge n+1)$ are not concentrated on any closed hemisphere, then there exists a $P \in \mathcal{P}_N(u_1, \dots, u_N)$ such that $\xi_\phi(P) = o$ and*

$$V(P) = \sup\{V(Q) : \max_{\xi \in Q} \Phi_Q(\xi) = 1 \quad \textit{and} \quad Q \in \mathcal{P}(u_1, \dots, u_N)\}.$$

Proof. Note that, for $P, Q \in \mathcal{P}(u_1, \dots, u_N)$, if Q is a translate of P, then

$$\Phi_P(\xi_\phi(P)) = \Phi_Q(\xi_\phi(Q)).$$

Thus, we can choose a sequence $P_i \in \mathcal{P}(u_1, \dots, u_N)$ with $\xi_\phi(P_i) = o$ such that $V(P_i)$ converges to

$$\sup\{V(Q) : \max_{\xi \in Q} \Phi_Q(\xi) = 1 \quad \text{and} \quad Q \in \mathcal{P}(u_1, \dots, u_N)\}.$$

We claim that P_i is bounded. Otherwise, from Lemma 3.4, $\Phi_{P_i}(\xi_\phi(P_i))$ converges to $+\infty$. This contradicts $\Phi_{P_i}(\xi_\phi(P_i)) = 1$. Therefore, P_i is bounded.

From Lemma 3.3 and the Blaschke selection theorem, there exists a subsequence of P_i that converges to a polytope P such that $P \in P(u_1, \dots, u_N)$, $\xi_\phi(P) = o$, and

$$V(P) = \sup\{V(Q) : \max_{\xi \in Q} \Phi_Q(\xi) = 1 \quad \text{and} \quad Q \in \mathcal{P}(u_1, \dots, u_N)\}. \tag{4.13}$$

We next prove that $F(P, u_i)$ are facets for all $i = 1, \dots, N$. Otherwise, there exists an $i_0 \in \{1, \dots, N\}$ such that $F(P, u_{i_0})$ is not a facet of P.

Choose $\delta \geq 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, \dots, u_N)$$

and (by Lemma 4.8)

$$V(P_\delta) = V(P) - (c_n \delta^n + \cdots + c_2 \delta^2),$$

where $c_n, \dots, c_2$ are constants that depend on P and direction u_{i_0}. By Lemma 4.6, we can assume $\delta \geq 0$ is small enough so that

$$h(P_\delta, u_k) = h(P, u_k) - a_k \delta, \tag{4.14}$$

where a_k is a constant with $a_k \geq 0$ and $a_{i_0} = 1$.

From Lemma 3.3, for any $\delta_i \to 0$, it is always true that $\xi_\phi(P_{\delta_i}) \to o$. We have

$$\lim_{\delta \to 0} \xi_\phi(P_\delta) = o.$$

Let

$$\lambda(\delta) = \left(\frac{V(P_\delta)}{V(P)}\right)^{-\frac{1}{n}} = \left(1 - \frac{(c_n \delta^n + \cdots + c_2 \delta^2)}{V(P)}\right)^{-\frac{1}{n}}.$$

Then we have $V(\lambda(\delta) P_\delta) = V(P)$ and $\lambda'(0) = 0$.

Let $\xi(\delta) = \xi_\phi(\lambda(\delta) P_\delta)$ and

$$\begin{aligned} \Phi(\delta) &= \max_{\xi \in \lambda(\delta) P_\delta} \sum_{k=1}^N \alpha_k \phi\left(h(\lambda(\delta) P_\delta, u_k) - \xi \cdot u_k\right) \\ &= \sum_{k=1}^N \alpha_k \phi\left(h(\lambda(\delta) P_\delta, u_k) - \xi(\delta) \cdot u_k\right). \end{aligned} \tag{4.15}$$

From this and the fact that $\xi(\delta)$ is an interior point of $\lambda(\delta) P_\delta$, we get

$$\sum_{k=1}^N \alpha_k \phi'\left(h(P, u_k)\right) u_k = 0. \tag{4.16}$$

It follows from Lemma 4.7 that $\xi_\phi(\lambda(\delta) P_\delta)$ has right derivative at 0. Together with (4.14), (4.15), (4.16), $\lambda'(0) = 0$, and the definition of ϕ, we have the right derivative

$$\begin{aligned}\frac{d}{d\delta}\Big|_{\delta=0^+}\Phi(\delta) &= -\sum_{k=1}^{N}\alpha_k a_k \phi'\left(h(P,u_{i_0})\right) + \sum_{k=1}^{N}\alpha_k \phi'(h(P,u_k))\left(\xi_r'(0)\cdot u_k\right)\\ &= -\sum_{k=1}^{N}\alpha_k a_k \phi'\left(h(P,u_{i_0})\right) + \xi_r'(0)\cdot\sum_{k=1}^{N}\alpha_k\phi'(h(P,u_k))u_k\\ &= -\sum_{k=1}^{N}\alpha_k a_k \phi'\left(h(P,u_{i_0})\right) < 0.\end{aligned}$$

Note that $o=\xi_\phi(P)\in \operatorname{Int}P$, there exists a $\delta_0>0$ such that $P_{\delta_0}\in\mathcal{P}(u_1,\dots,u_N)$, $o\in P_{\delta_0}$, and

$$\Phi_{\delta_0 P_\delta}(\xi_\phi(\lambda_0 P_{\delta_0})) < \Phi_P(\xi_\phi(P)) = 1,$$

where $\lambda_0 = \left(\frac{V(P_{\delta_0})}{V(P)}\right)^{-\frac{1}{n}}$. Let $P_0 := \lambda_0 P_{\delta_0}$. Then $P_0\in P(u_1,\dots,u_N)$, $o\in P_0$, $V(P_0)=V(P)$, and

$$\sup_{\xi\in P_0}\Phi_{P_0}(\xi) < 1.$$

Then by Lemma 3.4 and Remark 4.1, there exists a real number $\beta>1$ such that

$$\sup_{\xi\in \beta P_0}\Phi_{\beta P_0}(\xi) = 1.$$

But $V(\beta P_0) > V(P_0) = V(P)$, which contradicts equation (4.13). Therefore, $P\in P_N(u_1,\dots,u_N)$. $\square$

5. The Orlicz Minkowski problem for polytopes

This section is devoted to the proof of our main theorem by using the Lagrange multiplier rule. In the following, we denote by $\mathbb{R}_+^N$ the set of all $x=(x_1,\dots,x_N)\in\mathbb{R}^N$ with positive components. To use the Lagrange multiplier rule, we need the following lemma.

Lemma 5.1 ([25, Lemma 3.2]). *Let $u_1,\dots,u_N\in S^{n-1}$ be pairwise distinct vectors which are not contained in any closed hemisphere. For $x\in\mathbb{R}_+^N$, let $P(x)=\bigcap_{i=1}^N H^-_{u_i,x_i}$. Then $V(P(x))$ is continuously differentiable, and $\partial_i V(P(x)) = S(P(x),\{u_i\})$ for $i=1,\dots,N$.*

Now, we turn to proving Theorem 1.2.

Proof. Let $P(x)=\bigcap_{i=1}^N H^-_{u_i,x_i}$, where $x\in\mathbb{R}_+^N$, such that

$$\max_{\xi\in P(x)}\Phi_{P(x)}(\xi) = 1.$$

Then (3.2) becomes

$$\sup\{V(P(x)) : \Phi_{P(x)}(\xi_\phi(P(x))) = 1\}.$$

From this restriction condition and the fact that $\xi_\phi(P(x))$ is an interior point of $P(x)$, we have

$$\sum_{k=1}^{N}\alpha_k\phi'\left(h(P(x),u_k) - \xi_\phi(P(x))\cdot u_k\right)u_k = o. \tag{5.1}$$

From Lemma 4.9, there exists a polytope $P \in P_N(u_1, \ldots, u_N)$ with $\xi_\phi(P) = o$ such that

$$V(P) = \sup\{V(Q) : \max_{\xi \in Q} \Phi_Q(\xi) = 1 : Q \in \mathcal{P}(u_1, \ldots, u_N)\}.$$

Let $z = (h(P, u_1), h(P, u_2), \ldots, h(P, u_N)) = (z_1, \ldots, z_N)$. Then

$$\Phi_{P(z)}(\xi_\phi(P(z))) = 1,$$

$$V(P(z)) = \sup\{V(P(x)) : \Phi_{P(x)}(\xi_\phi(P(x))) = 1\},$$

and (5.1) becomes

$$\sum_{k=1}^{N} \alpha_k \phi'(z_k)\, u_k = o. \tag{5.2}$$

Since $P \in P_N(u_1, \ldots, u_N)$, by Lemma 2.4.13 in [38] and Lemma 4.2, we can choose a small neighborhood $D(z)$ of z, such that $\forall x \in D(z)$, $h(P(x), u_i) = x_i$, and the partial differential $\partial_i \xi_\phi(P(x))$ exists, where $i \in \{1, \ldots, N\}$.

By the Lagrange multiplier rule there is some $\lambda \in \mathbb{R}$ such that

$$\nabla V(P(z)) = \lambda \nabla \left(\sum_{i=1}^{m} \alpha_i \phi(z_i - \xi_\phi(P(z)) \cdot u_k) \right),$$

where $V(P(z))$ is differentiable by Lemma 5.1, and $\phi'(z_i)$ exists since $z_i > 0$ for all $i = 1, \ldots, m$. Therefore, by (5.2)

$$\begin{aligned} S_i :=& S(P(z), u_i) \\ =& \lambda \alpha_i \phi'(z_i) - \lambda \sum_{k=1}^{N} \alpha_k \phi'(z_k) \partial_i \xi_\phi(P(z)) \cdot u_k \\ =& \lambda \alpha_i \phi'(z_i) - \lambda \partial_i \xi_\phi(P(z)) \cdot \sum_{k=1}^{N} \alpha_k \phi'(z_k) u_k \\ =& \lambda \alpha_i \phi'(z_i), \end{aligned}$$

where $i \in \{1, \ldots, N\}$.

Then, we have

$$nV(P(z)) = \sum_{i=1}^{N} S_i z_i = \lambda \sum_{i=1}^{N} \alpha_i \phi'(z_i) z_i.$$

Therefore, for $i = 1, \ldots, N$,

$$S(P(z), u_i) = S_i = \frac{1}{c} \alpha_i \phi'(z_i),$$

where $c = \frac{1}{nV(P(z))} \sum_{i=1}^{N} \alpha_i \phi'(z_i) z_i$. Indeed, from the definition of ϕ, it follows that

$$\mu = \sum_{i=1}^{N} \alpha_i \delta_{u_i} = c\varphi(h(P, \cdot)) S(P, \cdot). \qquad \square$$

Corollary 1.3 follows from this theorem and (2.1).

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Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

Email address: wuyuchi1990@126.com

Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China – and – Department of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

Email address: dongmeng.xi@live.com

Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

Email address: gleng@staff.shu.edu.cn