Algorithms on Strings

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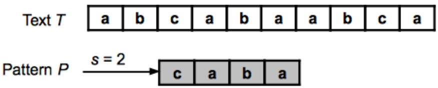
String searching

- String matching problem: find one or all occurrences of a pattern in a given text
- Applications
  - information retrieval
  - Text editors
  - computational biology (DNA sequences)
- Formal formulation
  - A text is an array $T[1..n]$ and a pattern is an array $P[1..m]$ $(m \neq n)$
  - $T[i], P[j] \in$ a finite alphabet $\sum$ (e.g., $\sum = \{0, 1\}$ or $\sum = \{a, \ldots, z\}$)
  - We say that pattern P **occurs with shift s** in T if $0 \leq s \leq n - m$ and $T[s + 1..s + m]$=$P[1..m]$



String searching algorithms

- Naive
- Boyer-Moore
- Rabin-Karp
- Knuth-Morris-Pratt (KMP)

Naive algorithm

Algorithm 1: NaiveSM(P, T)

foreach s = *0..n-m* **do**

  $i \leftarrow 1$;

  while $i \leq m$ *and* $P[i] = T[i+s]$ **do**

    $i \leftarrow i + 1$;

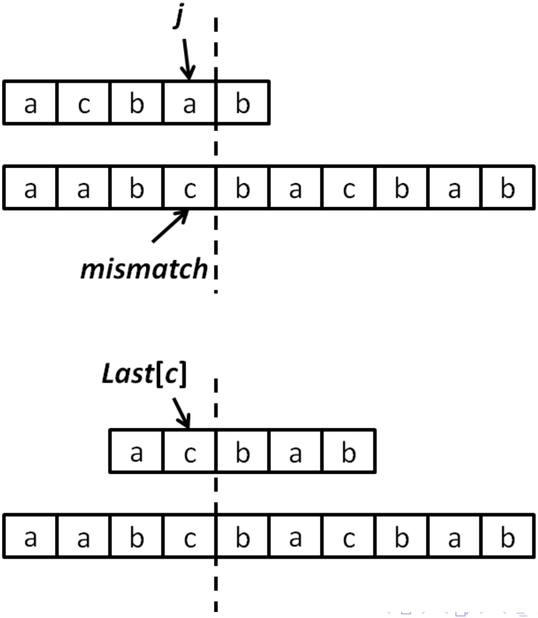
  if $i \geq m$ **then**

    Output(s);

Boyer-Moore algorithm

- Left to right shift
- Right to left scan
- Use information gained by preprocessing P in order to skip as many alignment as possible
- Bad character shift rule
  - $last[c]$: the right-most occurrence of c in P
  - When mismatch: shift P right by $max\{j - last[c], 1\}$ where j is the position of mismatch character of P

Boyer-Moore algorithm



Boyer-Moore algorithm

```
void computeLast(){
  for(int c = 0; c < 256; c++)
   last[c] = 0;
  for(int i = m; i >= 1; i--){
    if(last[P[i]] == 0)
      last[P[i]] = i;
  }
}
void BoyerMoore(){
  int s = 0;
  while(s <= n-m){
    int j = m;
    while(j > 0 && T[j+s] == P[j])  j--;
    if(j == 0){
      Output(s);
      s = s + 1;
    }else{
      int k = last[T[j+s]];
```

Rabin-Karp algorithm

- Convert the pattern $P[1..m]$ to a number:

$$p = P[1] * d^{m-1} + P[2] * d^{m-2} + \cdots + P[m] * d^0$$

  where each character $P[i]$ is viewed as a nonnegative integer $< d$, and d is the size of the alphabet

- Using Horner's rule:

$$p = P[m] + d * (P[m-1] + d * (\cdots + d * P[1]) + \dots)$$

- Convert $T[s+1..s+m]$ to the integer

$$t_s = T[s+1] * d^{m-1} + \cdots + T[s+m]$$

- **Note**: t_{s+1} can easily be computed from t_s as follows: $t_{s+1} = (t_s - T[s+1] * d^{m-1}) * d + T[s+m+1]$

Rabin-Karp algorithm

- Drawback: when m is large, then the computation of p and t_s does not take constant time
- Solution: Compute p and t_s modulo a suitable number q
    - Still problem: $p \equiv t_s (mod\ q)$ does not mean that $p = t_s$, we have to check $P[1..m]$ and $T[s+1..s+m]$ character by character to see if they are really identical
- Worst-case time is $\mathcal{O}(mn)$ where $P = a^m$ and $T = a^n$

Knuth-Morris-Pratt (KMP) algorithm

- Comparison: from left to right
- Shift: more than one position
- Preprocessing the pattern
  - Pattern $P[1..m]$
  - $\pi[q]$ is the length of the longest prefix of $P[1..q]$ which is also the **strictly** suffix of $P[1..q]$

Example

q	1	2	3	4	5	6	7	8	9	10
$P[q]$	a	b	a	b	a	b	a	b	c	a
$\pi[q]$	0	0	1	2	3	4	5	6	0	1

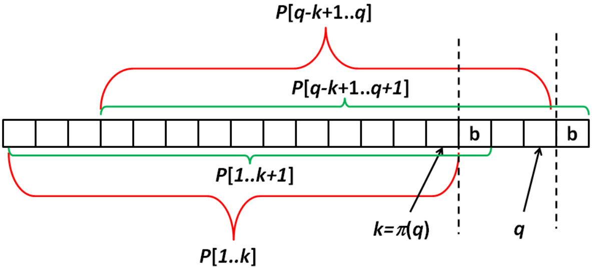
Knuth-Morris-Pratt (KMP) algorithm - preprocessing

```
void computePI(){
  pi[1] = 0;
  int k = 0;
  for(int q = 2; q <= m; q++){
    while(k > 0 && P[k+1] != P[q])
      k = pi[k];
    if(P[k+1] == P[q])
      k = k + 1;
    pi[q] = k;
  }
}
```

Knuth-Morris-Pratt (KMP) algorithm - preprocessing

Denote $k = \pi[q]$

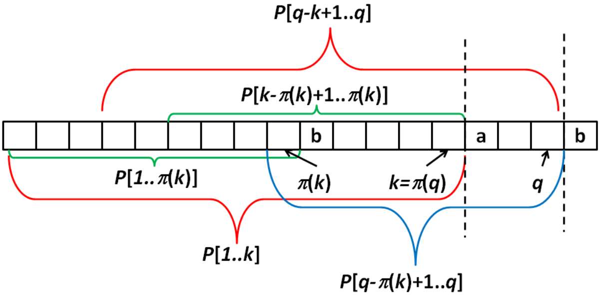
- If $P[q+1] = P[k+1]$, then $\pi[q+1] = \pi[q] + 1$



Knuth-Morris-Pratt (KMP) algorithm - preprocessing

Denote $k = \pi[q]$

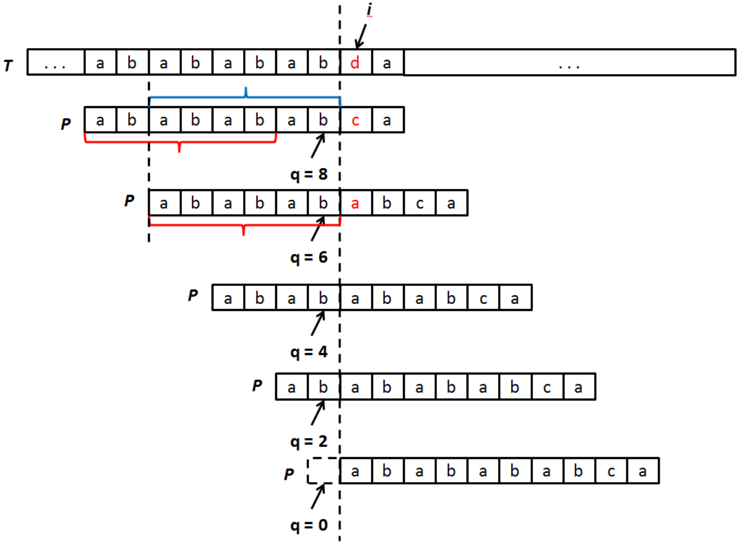
- if $P[q+1] \neq P[k+1]$ and $P[q+1] = P[\pi[k]+1] = b$:
  - $P[1..k] = P[q-k+1..q] \Rightarrow P[k-\pi[k]+1..k] = P[q-\pi[k]+1..q]$
  - Moreover, $P[k-\pi[k]+1] = P[1..\pi[k]]$, so $P[1..\pi[k]] = P[q-\pi[k]+1..q]$,
  - Hence $P[1..\pi[k]+1] = P[q-\pi[k]+1..q+1]$, this means $\pi[q+1] = \pi[k]+1$



Knuth-Morris-Pratt (KMP) algorithm

```
void kmp(){
  int q = 0;
  for(int i = 1; i <= n; i++){
    while(q > 0 && P[q+1] != T[i]){
      q = pi[q];
    }
    if(P[q+1] == T[i])
      q++;
    if(q == m){
      cout << "match at position " << i-m+1 << endl
      q = pi[q];
    }
  }
}
```

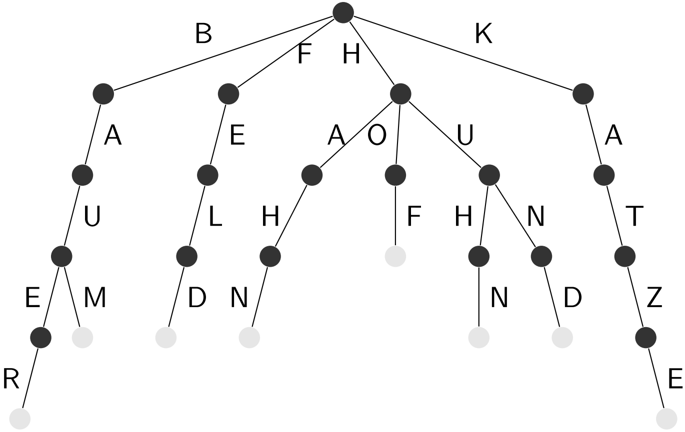
Knuth-Morris-Pratt (KMP) algorithm



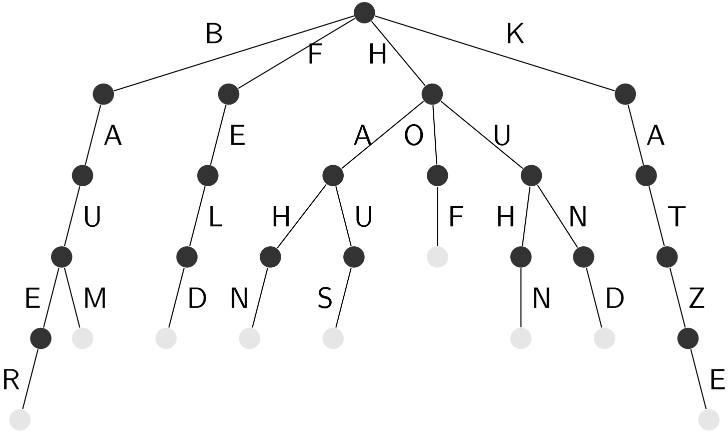
Sets of strings

- We often have sets (or maps) of strings
- Insertions and lookups usually guarantee $O(\log n)$ comparisons
- But string comparisions are actually pretty expensive...
- There are other data structures, like tries, which do this in a more clever way

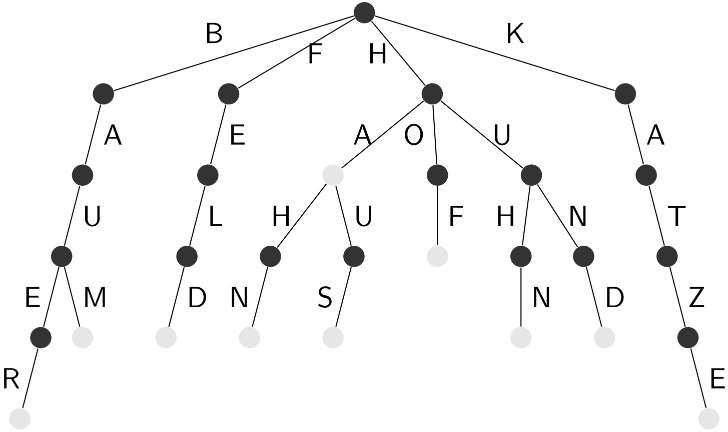
Tries



Tries



Tries



Tries

```
struct node {
    node* children[26];
    bool is_end;

    node() {
        memset(children, 0, sizeof(children));
        is_end = false;
    }
};
```

Tries

```
void insert(node* nd, char *s) {
    if (*s) {
        if (!nd->children[*s - 'a'])
            nd->children[*s - 'a'] = new node();

        insert(nd->children[*s - 'a'], s + 1);
    } else {
        nd->is_end = true;
    }
}
```

Tries

```
bool contains(node* nd, char *s) {
    if (*s) {
        if (!nd->children[*s - 'a'])
            return false;

        return contains(nd->children[*s - 'a'], s + 1);
    } else {
        return nd->is_end;
    }
}
```

Tries

```
node *trie = new node();

insert(trie, "banani");

if (contains(trie, "banani")) {
    // ...
}
```

Tries

- Time complexity?

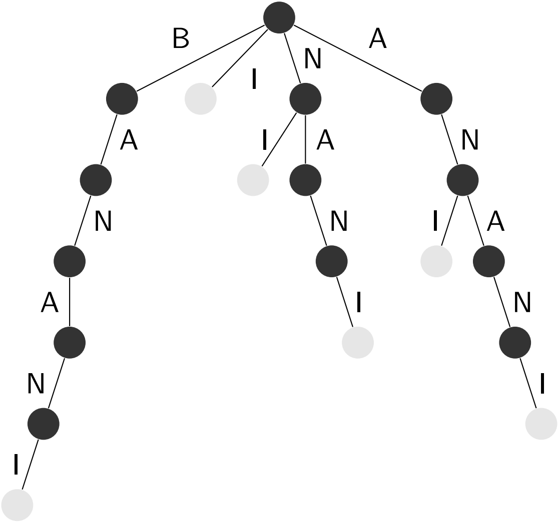
- Let k be the length of the string we're inserting/looking for
- Lookup and insertion are both $O(k)$

- Also very space efficient...

Suffix tries

- Say we're dealing with some string S of length n
- Let's insert all suffixes of S into a trie
- $S = \texttt{banani}$
  - `insert(trie, "banani");`
  - `insert(trie, "anani");`
  - `insert(trie, "nani");`
  - `insert(trie, "ani");`
  - `insert(trie, "ni");`
  - `insert(trie, "i");`

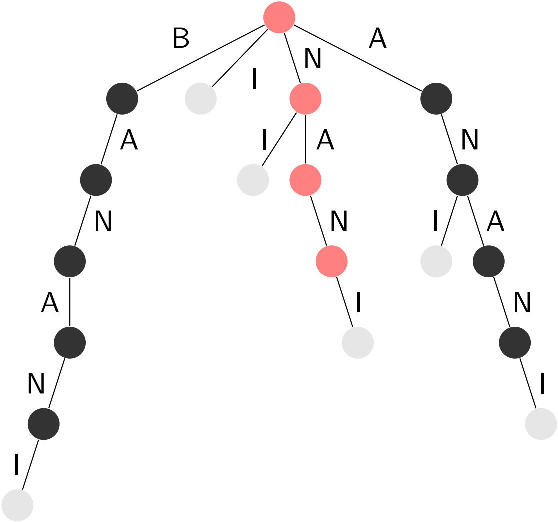
Suffix tries



Suffix tries

- There are a lot of cool things we can do with suffix tries
- Example: String matching
- If a string T is a substring in S, then (obviously) it has to start at some suffix of S
- So we can simply look for T in the suffix trie of S, ignoring whether the last node is an end node or not
- This is just $O(m)$...

Suffix tries



Suffix tries

- String matching is fast if we have the suffix trie for S

- But what is the time complexity of suffix trie construction?
- There are n suffixes, and it takes $O(n)$ to insert each of them
- So $O(n^2)$, which is pretty slow

- Can we do better?
- There can be up to n^2 nodes in the graph, so this is actually optimal...

Suffix trees

- There exists a compressed version of a suffix trie, called a suffix tree
- It can be constructed in $O(n)$, and has all the features that suffix tries have
- But the $O(n)$ construction algorithm is pretty complex, a big disadvantage for us

Suffix arrays

- A variation of the previous structures
- Can do everything the other structures can do, with a small overhead
- Can be constructed pretty quickly with relatively simple code

Suffix arrays

- Take all the suffixes of S

```
banani
anani
nani
ani
ni
i
```

- and sort them

```
anani
ani
banani
i
nani
ni
```

Suffix arrays

- We can use this array to do everything that suffix tries can do
- Like string matching

Suffix arrays

- Let's look for `nan`

```
anani
ani
banani
i
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The first letter in the string has to be `n`, so we can binary search for the range of strings starting with `n`

```
anani
ani
banani
i
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The first letter in the string has to be `n`, so we can binary search for the range of strings starting with `n`

```
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The second letter in the string has to be `a`, so we can binary search for the range of strings that have `a` as the second letter

```
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The second letter in the string has to be `a`, so we can binary search for the range of strings that have `a` as the second letter

```
nani
```

Suffix arrays

- Let's look for `nan`
- The third letter in the string has to be `n`, so we can binary search for the range of strings that have `n` as the third letter

`nani`

Suffix arrays

- Let's look for `nan`
- The third letter in the string has to be `n`, so we can binary search for the range of strings that have `n` as the third letter

`nani`

Suffix arrays

- Let's look for `nan`
- The third letter in the string has to be `n`, so we can binary search for the range of strings that have `n` as the third letter

`nani`

- If there is at least one string left, we have a match

Suffix arrays

- Time complexity?
- For each letter in T, we do two binary searches on the n suffixes to find the new range
- Time complexity is $O(m \times \log n)$
- A bit slower than doing it with a suffix trie, but still not bad

Suffix arrays

- But how do we construct a suffix array for a string?
- A simple `sort(suffixes)` is $O(n^2 \log(n))$, because comparing two suffixes is $O(n)$
- And we still have the same problem as with suffix tries, there are almost n^2 characters if we store all suffixes

Suffix arrays

- The second problem is easy to fix
- Just store the indices of the suffixes

```
anani
ani
banani
i
nani
ni
```

- becomes

```
1: anani
3: ani
0: banani
5: i
2: nani
4: ni
```

Suffix arrays

- What about the construction?
- In short, we
  - sort all suffixes by only looking at the first letter
  - sort all suffixes by only looking at the first 2 letters
  - sort all suffixes by only looking at the first 4 letters
  - sort all suffixes by only looking at the first 8 letters
  - ...
  - sort all suffixes by only looking at the first 2^i letters
  - ...
- If we use an $O(n \log n)$ sorting algorithm, this is $O(n \log^2 n)$
- We can also use an $O(n)$ sorting algorithm, since all sorted values are between 0 and n, bringing it down to $O(n \log n)$

Suffix arrays

```
struct suffix_array {
    struct entry {
        pair<int, int> nr;
        int p;

        bool operator <(const entry &other) {
            return nr < other.nr;
        }
    };

    string s;
    int n;
    vector<vector<int> > P;
    vector<entry> L;
    vi idx;

    // constructor
};
```

Suffix arrays

```
suffix_array(string _s) : s(_s), n(s.size()) {
    L = vector<entry>(n);
    P.push_back(vi(n));
    idx = vi(n);

    for (int i = 0; i < n; i++)
        P[0][i] = s[i];

    for (int stp = 1, cnt = 1; (cnt >> 1) < n; stp++, cnt <<= 1) {
        P.push_back(vi(n));
        for (int i = 0; i < n; i++) {
            L[i].p = i;
            L[i].nr = make_pair(P[stp - 1][i],
                        i + cnt < n ? P[stp - 1][i + cnt] : -1);
        }
        sort(L.begin(), L.end());
        for (int i = 0; i < n; i++) {
            if (i > 0 && L[i].nr == L[i - 1].nr)
                  P[stp][L[i].p] = P[stp][L[i - 1].p];
            else  P[stp][L[i].p] = i;
        }
    }
    for (int i = 0; i < n; i++)
        idx[P[P.size() - 1][i]] = i;
}
```

Suffix arrays

- There is also one other useful operation on suffix arrays
- Finding the longest common prefix (lcp) of two suffixes of S

```
1: anani
3: ani
0: banani
5: i
2: nani
4: ni
```

- `lcp(1,3) = 2`
- `lcp(2,1) = 0`

- This function can be implemented in $O(\log n)$ by using intermediate results from the suffix array construction

Suffix arrays

```
int lcp(int x, int y) {
    int res = 0;
    if (x == y) return n - x;
    for (int k = P.size() - 1; k >= 0 && x < n && y < n; k--) {
        if (P[k][x] == P[k][y]) {
            x += 1 << k;
            y += 1 << k;
            res += 1 << k;
        }
    }
    return res;
}
```

Practicing Problems

Longest common substring

- Given two strings S and T, find their longest common substring
- `S = banani`
- `T = kanina`
- Their longest common substring is `ani`

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