

ON THE ASYMPTOTIC TRANSLATION LENGTHS ON THE SPHERE COMPLEXES AND THE GENERALIZED FIBERED CONE

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ABSTRACT. In this paper, we study the asymptotic translation lengths on the sphere complexes of monodromies of a manifold fibered over the circle. Given a compact mapping torus, we define a cone in the first cohomology which we call the generalized fibered cone, and show that every primitive integral element gives a fibration over the circle. Moreover, we prove that the generalized fibered cone is a rational slice of Fried’s cone, which is defined as the dual of homological directions, an analogue of Thurston’s fibered cone.

As a consequence of our description of the generalized fibered cone, we provide each proper subcone of the generalized fibered cone with a uniform upper bound for asymptotic translation lengths of monodromies on sphere complexes of fibers in the proper subcone. Our upper bound is purely in terms of the dimension of the proper subcone. We also deduce similar estimates for asymptotic translation lengths of some mapping classes on finite graphs constructed in the works of Dowdall–Kapovich–Leininger, measured on associated free-splitting complexes and free-factor complexes.

Moreover, as an application of our result, we prove that the asymptote for the minimal asymptotic translation length of the genus g handlebody group on the disk complex is $1/g^2$, the same as the one on the curve complex.

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1. INTRODUCTION

Group actions have been proven to be fruitful in the study of groups. For instance, Thurston [Thu88] and Bers [Ber78] classified mapping classes of a closed surface according to the dynamics of their action on the Teichmüller space. Moreover, Masur–Minsky [MM99, MM00] studied the action of the mapping class group on the curve complex, and then proved the relative hyperbolicity of the mapping class group.

In [BSW21], Baik–Shin–Wu studied fibered 3-manifold groups and related dynamics on the curve complexes. Namely, for a fibered hyperbolic 3-manifold, they showed an estimation for the asymptotic translation lengths of monodromies in a fibered cone, where the asymptotic translation length is measured on the curve complex of each fiber.

In this paper, we extend this result from fibered 3-manifold groups to more general fibered manifold groups. Throughout the paper, we consider smooth connected compact manifolds and simply call them manifolds. For a manifold M and a diffeomorphism $\varphi : M \rightarrow M$, we consider the mapping torus N of φ . This gives a fibration $N \rightarrow S^1$ and a flow \mathcal{F} on N . As an analogy of Thurston’s fibered cone, we introduce a cone in the first cohomology $H^1(N)$ associated to the monodromy $\varphi : M \rightarrow M$, which we call the *generalized fibered cone* (Definition 2.2). We show that each primitive integral element in the generalized fibered cone gives a fibration over the circle with respect to the flow \mathcal{F} (Proposition 2.6).

Moreover, we prove the characterization of the generalized fibered cone as a rational slice of Fried’s cone. In [Fri82], for a flow in the manifold, Fried introduced the set of so-called homological directions as a subset of the first homology of the manifold. In the first cohomology, the dual cone of the set of homological directions is called *Fried’s cone*, and Fried showed that every primitive integral element corresponds to a fibration over the circle with respect to the given flow. We also show that the generalized fibered cone is the intersection of a rational subspace and Fried’s cone in $H^1(N)$ for the flow \mathcal{F} on N (Theorem A).

In our main theorem, we estimate the asymptotic translation lengths of monodromies coming from the generalized fibered cone, on sphere complexes of fibers (Theorem B). As an application, we also obtain the precise asymptote of minimal asymptotic translation lengths of pseudo-Anosov handlebody mapping classes on disk graphs in terms of genera (Theorem C).

1.1. Generalized fibered cone and asymptotic translation lengths.

For a compact manifold M and a diffeomorphism $\varphi : M \rightarrow M$, we consider the mapping torus N of φ and define the generalized fibered cone in $H^1(N)$ associated to the monodromy $\varphi : M \rightarrow M$ (Definition 2.2). We begin with the following characterization of the generalized fibered cone:

Theorem A (Theorem 2.4). *Let $\varphi : M \rightarrow M$ be a diffeomorphism of a compact manifold to itself. Let N be the mapping torus of φ . Then the*

generalized fibered cone in $H^1(N)$ associated to φ is the intersection of a rational subspace and Fried's cone for the suspension flow given by φ in $H^1(N)$.

Hence, one can simply regard the generalized fibered cone in the following discussion as a rational slice of Fried's cone. We are mainly interested in the asymptotic translation lengths of monodromies on sphere complexes of fibers. We define the sphere complex as follows, which is a generalization of the curve complex of a surface. A sphere in a compact manifold is called essential if it does not bound a ball or is not isotopic to a boundary component.

Definition 1.1 (Sphere complex). *For a compact manifold M and $n \geq 1$, its sphere complex $\mathcal{S}(M; n)$ is a simplicial complex whose vertices are isotopy classes of essential embedded spheres $S^n \subseteq M$, and $k+1$ number of isotopy classes S_0, \dots, S_k of spheres form a k -simplex in $\mathcal{S}(M; n)$ if and only if they can be represented by $k+1$ number of disjoint spheres.*

Each k -simplex is identified with the standard simplex in \mathbb{R}^{k+1} spanned by $(1/\sqrt{2})\vec{e}_1, \dots, (1/\sqrt{2})\vec{e}_{k+1}$ where \vec{e}_i 's are standard unit vectors. Then we endow the sphere complex $\mathcal{S}(M; n)$ with the induced path metric $d_{\mathcal{S}(M; n)}$.

We note that for a closed surface S , the sphere complex $\mathcal{S}(S; 1)$ is the usual *curve complex* of S . The 1-skeleton of the curve complex is called the *curve graph* which is quasi-isometrically embedded in the curve complex.

Remark 1.2. *Throughout the paper, most of our argument on the sphere complex $\mathcal{S}(M; n)$ does not depend on the exact value of n while interesting cases are with low codimensions. Hence, when we deal with the sphere complex, we simply use the notation $\mathcal{S}(M)$ to mean by the sphere complex $\mathcal{S}(M; n)$ for some fixed n .*

Sphere complexes have played an important role in geometric group theory and algebraic topology. For instance, the connectivity of various sphere complexes have been obtained and used to show the homological stability of automorphism groups of free groups in [Hat95].

Similar to the curve complex of a surface, a diffeomorphism $M \rightarrow M$ naturally induces the isometry $\mathcal{S}(M) \rightarrow \mathcal{S}(M)$. Furthermore, two isotopic diffeomorphisms on M induce the same isometry on $\mathcal{S}(M)$. In this regard, we come up with a question pertaining to generalizing the dynamical properties of mapping class groups on curve complexes to the sphere complexes. In particular, we consider asymptotic translation lengths on sphere complexes.

Definition 1.3 (Asymptotic translation length). *Let (\mathcal{Y}, d) be a metric space and $f : \mathcal{Y} \rightarrow \mathcal{Y}$ be an isometry. Then its asymptotic translation length $l_{\mathcal{Y}}(f)$ on \mathcal{Y} is defined as*

$$l_{\mathcal{Y}}(f) = \liminf_{n \rightarrow \infty} \frac{d(f^n(y), y)}{n}$$

for $y \in \mathcal{Y}$.

Throughout the paper, we write $0 \leq A(x) \lesssim B(x)$ if there is a constant $C > 0$ satisfying $A(x) \leq CB(x)$ for all x . In addition, when we have both $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$, we write $A(x) \asymp B(x)$. Also, since the first cohomology with \mathbb{R} -coefficients is a finite-dimensional real vector space, we take any norm $\|\cdot\|$ without specifying it. The following is our main theorem.

Theorem B. *Let $\varphi : M \rightarrow M$ be a diffeomorphism of a compact manifold to itself. Consider the generalized fibered cone of the mapping torus of φ , and let \mathcal{R} be an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone. Then we have*

$$(1.1) \quad l(\varphi_\alpha) \lesssim \|\alpha\|^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R}$ where φ_α is the corresponding monodromy and $l(\varphi_\alpha)$ is the asymptotic translation length of φ_α on the sphere complex of the fiber.

Remark 1.4. We simply denote by $l(\cdot)$ the asymptotic translation length on the sphere complex of the underlying manifold since it is of our primary interest. Similarly, in Section 6, we denote $l_{FS}(\cdot)$, $l_{FF}(\cdot)$, and $l_D(\cdot)$ for the asymptotic translation lengths of the induced isometries on the free-splitting complex, the free-factor complex, and the disk graph respectively.

We deduce similar statements for asymptotic translation lengths on free-splitting and free-factor complexes of free groups as well. In [DKL15], Dowdall–Kapovich–Leininger proved that given an expanding irreducible train track map $\psi : G \rightarrow G$ which is a homotopy equivalence on a graph G , there is an open rational cone, called the *positive cone*, in the first cohomology of a (folded) mapping torus of ψ containing the monodromy class. The positive cone is a free-by-cyclic group version of Thurston’s fibered cone in the sense that for any primitive integral cohomology class α in the positive cone, α gives an another expanding irreducible train track map $\psi_\alpha : G_\alpha \rightarrow G_\alpha$. We note that ψ_α may not be uniquely determined by α ; it requires more choices to be made to obtain ψ_α (see [DKL15, Theorem B] for the precise statement).

As in [AS11], the sphere complex of a 3-manifold with free fundamental group is related to the free-splitting complex, (which is also the simplicial completion of the Culler–Vogtmann Outer space [Vog18]). Moreover, the barycentric subdivision of the sphere complex with a marked point has something to do with the free-factor complex, as studied by Hatcher–Vogtmann [HV98]. The relation between the free-splitting complex and the free-factor complex was also studied by Kapovich–Rafi [KR14]. Observing these relations among the sphere complex, the free-splitting complex, and the free-factor complex, Theorem B has some implications on the dynamics on free-splitting complexes and free-factor complexes. We deduce analogous estimates on the asymptotic translation lengths on free-splitting complexes and free-factor complexes from Theorem B (Corollary 6.5 and Corollary 6.6). Together with ([Hir11], [HK21], [Lau74]), these can be interpreted in

terms of $\text{Out}(F_g)$ -actions of free-splitting and free-factor complexes of the free group F_g (Remark 6.7).

1.2. Minimal asymptotic translation lengths on disk graphs. Another application of our results is related to minimal asymptotic translation lengths of subgroups of mapping class groups of surfaces. Let S_g be a closed connected orientable surface of genus $g \geq 2$ and denote its mapping class group by $\text{Mod}(S_g)$.

Definition 1.5 (Minimal asymptotic translation length). *For a subgroup $H \leq \text{Mod}(S_g)$ and a metric space \mathcal{Y} on which H isometrically acts, the minimal asymptotic translation length of H on \mathcal{Y} is*

$$L_{\mathcal{Y}}(H) := \inf\{l_{\mathcal{Y}}(f) : f \in H \text{ is pseudo-Anosov}\}.$$

Minimal asymptotic translation lengths of some subgroups of mapping class groups have been studied in the settings of Teichmüller spaces (e.g. [ALM16], [FLM08], [Hir11], [Pen91]) and curve complexes (e.g. [BS20], [BSW21], [GT11], [KS19]). On Teichmüller space \mathcal{T}_g and curve complex \mathcal{C}_g of S_g , the following asymptotes are known for the whole mapping class group $\text{Mod}(S_g)$ and the Torelli group \mathcal{I}_g :

Subgroups	Teichmüller spaces	Curve complexes
$\text{Mod}(S_g)$	(Penner [Pen91]) $L_{\mathcal{T}_g}(\text{Mod}(S_g)) \asymp 1/g$	(Gadre-Tsai [GT11]) $L_{\mathcal{C}_g}(\text{Mod}(S_g)) \asymp 1/g^2$
\mathcal{I}_g	(Farb-Leininger-Margalit [FLM08]) $L_{\mathcal{T}_g}(\mathcal{I}_g) \asymp 1$	(Baik-Shin [BS20]) $L_{\mathcal{C}_g}(\mathcal{I}_g) \asymp 1/g$

We now consider the *handlebody group* $\mathcal{H}_g < \text{Mod}(S_g)$. That is, identifying S_g with the boundary ∂V_g of genus g handlebody V_g , the handlebody group \mathcal{H}_g consists of mapping classes of S_g that extends to V_g . Kin–Shin proved in [KS19] the following asymptote:

$$(1.2) \quad L_{\mathcal{C}_g}(\mathcal{H}_g) \asymp \frac{1}{g^2}.$$

On the other hand, there is a subcomplex of \mathcal{C}_g on which the handlebody group \mathcal{H}_g acts, the disk graph, which is defined analogous to the curve graph:¹

Definition 1.6 (Disk graph). *Disk graph \mathcal{D}_g of the handlebody V_g is a graph whose vertices are isotopy classes of embedded disks $(D^2, \partial D^2) \subseteq (V_g, \partial V_g)$*

¹One can indeed consider the disk complex, but for simplicity, we consider the disk graph which is the 1-skeleton of the disk complex.

such that ∂D^2 is essential and two vertices are adjacent if they are represented by two disjoint disks. We endow the disk graph with a metric so that each edge is of length 1.

It is clear from the definition that \mathcal{H}_g acts on \mathcal{D}_g by isometries. Masur–Schleimer showed that the disk graph \mathcal{D}_g is Gromov hyperbolic [MS13]. The inclusion $(D^2, \partial D^2) \subset (V_g, \partial V_g)$ induces an embedding $\mathcal{D}_g \rightarrow \mathcal{C}_g$, and the image of \mathcal{D}_g under this embedding is quasi-convex in \mathcal{C}_g as shown by Masur–Minsky [MM04]. However, the disk graph \mathcal{D}_g is distorted in the curve complex \mathcal{C}_g . Indeed, it is not quasi-isometrically embedded [MS13]. Hence it is not straightforward whether handlebody groups \mathcal{H}_g have the same asymptote for the minimal asymptotic translation lengths on disk graphs \mathcal{D}_g and on curve complexes \mathcal{C}_g . As an application of Theorem B, we answer the affirmative:

Theorem C. *We have*

$$L_{\mathcal{D}_g}(\mathcal{H}_g) \asymp \frac{1}{g^2}.$$

Note that the lower bound of the above asymptote can be deduced from (1.2).

1.3. Future directions. In [CV86], Culler–Vogtmann introduced the notion of the Outer space CV_g which is equipped with a natural action of $\text{Out}(F_g)$. Roughly speaking, CV_g is the space of marked metric graph structures on F_g of volume 1. It has a natural simplicial decomposition in terms of graphs and the vertices that can be re-interpreted as splittings of F_g as a free product or HNN extension via Bass–Serre theory [Vog18]. This allows us to identify the free-splitting complex \mathcal{FS}_g with the simplicial closure of CV_g as we noted earlier. For a general review on the geometry of Outer space, one can refer to [Vog15].

According to [HM19], a fully irreducible element in $\text{Out}(F_g)$ acts as a hyperbolic isometry on \mathcal{FS}_g which is equivalent to the sphere complex. Hence, we can say more if we could figure out a lower bound of translation lengths in Theorem B or Corollary 6.5. To be precise, let $\phi \in \text{Out}(F_g)$ be fully irreducible and let L be its quasi-axis on \mathcal{FS}_g . Then L and a geodesic connecting $x \in L$ and $\phi(x)$ pass through coarsely. Noting that the Outer space can also be defined via sphere systems in a doubled handlebody as introduced in [Hat95], the lower bound for translation lengths on \mathcal{FS}_g gives a lower bound for the number of foldings one needs to get $\phi(x)$ from x as points in CV_g , by comparing the barycentric subdivision and the dual complex of \mathcal{FS}_g . Moreover, it would be possible to make the lower bounds uniform on a positive cone in [DKL15] if we could further control various sphere complexes and monodromies from the positive cone. Indeed, if $\phi \in \text{Out}(F_g)$ further satisfies that $F_g \rtimes_\phi \mathbb{Z}$ is word-hyperbolic, monodromies from the positive cone for ϕ are fully irreducible by [DKL15].

Another important remark is that the deductions of Theorem 6.4 from Theorem B are based on the concrete geometric description of the generalized fibered cone for doubled handlebody case. However, for general manifolds other than surfaces and doubled handlebodies, we do not have a concrete description of the generalized fibered cone. Finding a concrete geometric description of the generalized fibered cone in more general setting would lead to many other applications of our approach.

Lastly, Baik–Kin–Shin–Wu conjectured in [BKS23] that if an element of the mapping class group of a surface has small asymptotic translation length on the curve complex, then the element is a normal generator of the mapping class group. When one considers the action on the Teichmüller space instead of the curve complex, such a phenomenon was obtained by Lanier–Margalit [LM22, Theorem 1.2]. One might view our present article as a beginning step toward an analogous question for $\text{Out}(F_g)$, or more generally, automorphism groups of sphere complexes.

Organization. We define the generalized fibered cone and characterize it as a rational slice of Fried’s cone in Section 2, proving Theorem A. In Section 3, we prove Theorem B. Section 4 is devoted to the application to minimal asymptotic translation lengths of handlebody groups on disk graphs (Theorem C). We provide a construction of a diffeomorphism between doubled handlebodies from a given folding sequence for a combinatorial map between graphs in Section 5. In Section 6, we discuss applications to free-splitting complexes and free-factor complexes of free groups.

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2. GENERALIZED FIBERED CONE AND SLICES OF FRIED’S CONE

Recall that a fibered cone for the surface case is a cone in the first cohomology of a fibered hyperbolic 3-manifold such that every primitive integral cohomology class in the cone corresponds to a fibration over the circle. In this section, we define a higher-dimensional analogue of the fibered cone, the *generalized fibered cone*.

Let M be a compact manifold and $\varphi : M \rightarrow M$ be a diffeomorphism. Let $\mathbb{Z}^d \cong H \leq H_1(M)$ be a free abelian subgroup invariant under φ with the connected free abelian cover \tilde{M} having Deck group H . We fix a lift

$\tilde{\varphi} : \tilde{M} \rightarrow \tilde{M}$ of $\varphi : M \rightarrow M$. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\varphi}} & \tilde{M} \\ \downarrow & & \downarrow \text{Deck group } H \\ M & \xrightarrow{\varphi} & M \end{array}$$

Let $N := M \times \mathbb{R}/(x, s) \sim (\varphi(x), s+1)$ be the mapping torus. We also set $\tilde{N}' := \tilde{M} \times \mathbb{R}$ and $\tilde{N} := \tilde{N}'/(x, s) \sim (\tilde{\varphi}(x), s+1)$ which is the mapping torus of $\tilde{\varphi}$. The associated flow $\mathcal{F} := \{\mathcal{F}_t\}_{t \in \mathbb{R}}$ on N is given by the projection of the map $(x, s) \mapsto (x, s+t)$ on \tilde{N}' at time $t \in \mathbb{R}$. We summarize the relationship among these spaces using the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\varphi}} & \tilde{M} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & M \end{array} \quad \xrightarrow{\text{mapping torus}} \quad \begin{array}{ccc} \tilde{N} & \xleftarrow[\text{Deck group } \Gamma]{} & \tilde{N}' \\ \downarrow & \swarrow & \downarrow \\ N & & \end{array}$$

$\xleftarrow[\text{Z-fold}]{} \quad \text{Deck group } \Gamma$

Then \tilde{N}' is a $\Gamma := H \oplus \mathbb{Z}$ cover of N . Here $(h, n) \in \Gamma$ applied to $(x, s) \in \tilde{N}'$ is $(\tilde{\varphi}^n h(x), s+n)$. In other words, $\Gamma = H \oplus \mathbb{Z}$ is a quotient of $H_1(N)$, hence its dual $\Gamma^* = \text{Hom}(\Gamma, \mathbb{Z})$ is a subgroup of $H^1(N)$. Throughout the paper, we also use the coordinate in Γ^* dual to Γ . Furthermore, the Γ -action is restricted to the action on \tilde{M} given by $(h, n) \cdot x = \tilde{\varphi}^n h(x)$ for $x \in \tilde{M}$. In this point of view, we sometimes identify $(h, n) = \tilde{\varphi}^n h$ when we discuss the action on \tilde{M} . By abusing notations, we use multiplication for the group operation on H when we consider elements of H as maps, and use addition when we regard H as a free abelian group.

Now fix a fundamental domain D of the cover $\tilde{M} \rightarrow M$. For a map $f : \tilde{M} \rightarrow \tilde{M}$, let us define

$$\Omega(f) := \text{CH}\{h \in H : (h \cdot D) \cap f(D) \neq \emptyset\}$$

where $\text{CH}\{\cdot\}$ denotes a convex hull of $\{\cdot\}$ in $H \otimes \mathbb{R}$. Using this notation, we define

$$\Omega := \bigcup_{t \in \mathbb{Z}} (-\Omega(\tilde{\varphi}^t) \times \{t\}) \subseteq \Gamma \otimes \mathbb{R}.$$

We then have:

Proposition 2.1. *The set Ω is contained in a pair of rational cones whose intersection with $H \otimes \mathbb{R} \times \{\pm 1\}$ is bounded.*

Proof. Let C be the convex hull of $-\Omega(\tilde{\varphi}) \cup \Omega(\tilde{\varphi}^{-1})$. Then C is a bounded polygon because D is compact. Then, because for any $n > 0$,

$$\begin{aligned}\Omega(\tilde{\varphi}^n) &= CH\{h \in H : (h \cdot D) \cap \tilde{\varphi}^n(D) \neq \emptyset\} \\ &= CH\left\{h \in H : \begin{array}{l} \exists h_1 \text{ such that } (h \cdot D) \cap \tilde{\varphi}(h_1 \cdot D) \neq \emptyset, \\ (h_1 \cdot D) \cap \tilde{\varphi}^{n-1}(D) \neq \emptyset \end{array}\right\}\end{aligned}$$

We have $\Omega(\tilde{\varphi}^n) \subset (-C) + (-C) + \cdots + (-C)$, where the addition is among n copies of $(-C)$. Here, we use conventions $-A = \{x \in H \otimes \mathbb{R} : -x \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$. Similarly, $\Omega(\tilde{\varphi}^{-n})$ is contained in the sum of n copies of C . As a consequence, Ω is contained in the set $\{(x, t) : x = 0, t = 0 \text{ or } x/t \in C\}$. \square

We denote by $\hat{\Omega} \subset \Gamma \otimes \mathbb{R}$ the asymptotic cone of Ω , i.e.,

$$\hat{\Omega} := \{x \in \Gamma \otimes \mathbb{R} : x = \lim_{i \rightarrow \infty} t_i w_i \text{ for some sequences } w_i \in \Omega \text{ and } t_i \rightarrow 0\}.$$

Now we define the *generalized fibered cone*:

Definition 2.2 (Generalized fibered cone). *In the above setting of a diffeomorphism $\varphi : M \rightarrow M$ on a compact manifold M with a choice of a fundamental domain, the generalized fibered cone is the dual cone $\mathcal{C} \subset H^1(N)$ of the asymptotic cone of Ω : for the asymptotic cone $\hat{\Omega}$ of Ω ,*

$$\mathcal{C} := \{\alpha \in \Gamma^* \otimes \mathbb{R} : \text{sign}(t)\alpha(h, t) > 0 \text{ for all non-zero } (h, t) \in \hat{\Omega}\}$$

where $\text{sign}(t) = 1$ when $t > 0$ and $\text{sign}(t) = -1$ when $t < 0$. In particular, for any $\alpha \in \mathcal{C}$, there exists $K > 0$ such that for any $(h, t) \in \Omega$ with $|t| > K$, we have $\text{sign}(t)\alpha(h, t) > 0$.

By Proposition 2.1, the generalized fibered cone always has non-empty interior. Moreover, if $\Omega \subset \Gamma \otimes \mathbb{R}$ is of finite Hausdorff distance to the union of two cones in $\Gamma \otimes \mathbb{R}$ centered at the origin, the first one with $t \geq 0$ and second one with $t \leq 0$, then \mathcal{C} is the intersection of the dual cone of the first cone and the negative of the dual cone of the second one. Although the choice of fundamental domain is involved in defining the generalized fibered cone, it is independent of the choice:

Lemma 2.3. *The generalized fibered cone does not depend on the choice of the fundamental domain D .*

Proof. This is due to the fact that the covering $\tilde{M} \rightarrow M$ is abelian. Let D and D' be fundamental domains with compact closures, and fix an H -equivariant map $f : \tilde{M} \rightarrow \tilde{M}$. If $(h \cdot D) \cap f(D) \neq \emptyset$ for $h \in H$, we then have $x_1, x_2 \in D$ such that $h(x_1) = f(x_2)$. Since D' is also a fundamental domain, there exist $h_1, h_2 \in H$ and $x'_1, x'_2 \in D'$ such that $x_1 = h_1 x'$ and $x_2 = h_2 x'_2$. Hence, we have $(hh_1)(x'_1) = (fh_2)(x'_2)$. By the equivariance of f and the fact that H is abelian, we have $(h_2^{-1}h_1h)(x'_1) = f(x'_2)$. Since both D and D' have compact closures, there are only finitely many possible h_1 and h_2 , depending only on D and D' . This implies that the set $\Omega(f)$ obtained using

D is within bounded Hausdorff distance from the one obtained using D' , and vice versa. It then follows that the generalized fibered cone is independent of the choice between D and D' . \square

2.1. Generalized fibered cone as the slice of Fried's cone. Fried showed in [Fri82] that given a flow on a compact manifold, there exists a cone in the first cohomology so that a primitive integral element gives a fibration over the circle with respect to the given flow if and only if it belongs to the cone. Applying this result to the flow \mathcal{F} on N , we obtain the cone, $\mathcal{C}_F \subset H^1(N)$, which we call Fried's cone. Our generalized fibered cone is indeed a slice of Fried's cone.

Theorem 2.4 (Theorem A). *We have*

$$\mathcal{C} = \mathcal{C}_F \cap (\Gamma^* \otimes \mathbb{R}).$$

In other words, a primitive integral element $\alpha \in \Gamma^$ gives a fibration $N \rightarrow S^1$ with respect to the flow \mathcal{F} if and only if $\alpha \in \mathcal{C}$.*

To prove this, we first recall the work of Fried [Fri82]: let X be a compact smooth manifold and $\phi = (\phi_t)_{t \in \mathbb{R}}$ be a C^1 -flow on X which is either transverse or tangent to each component of ∂X . Let \tilde{X} be a connected \mathbb{Z} -cover with Deck transformation $g : \tilde{X} \rightarrow \tilde{X}$ and a lift $\tilde{\phi}$ of the flow ϕ . We denote by $\tilde{X}/\tilde{\phi}$ the space of $\tilde{\phi}$ -flow lines in \tilde{X} . The \mathbb{Z} -cover \tilde{X} can be compactified by two points $\{\pm\infty\}$ so that $g^n x \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$ for any $x \in \tilde{X}$. Fried gave the following characterization of the cross section of ϕ in terms of the behavior of $\tilde{\phi}$:

Theorem 2.5. [Fri82, Theorem A] *In the above setting, the following are equivalent:*

- (1) *for any $x \in \tilde{X}$, $\tilde{\phi}_t(x) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$.*
- (2) *$K := \tilde{X}/\tilde{\phi}$ is a cross section of the flow ϕ in X so that we can identify $\tilde{X} = K \times \mathbb{R}$ and have $\tilde{\phi}_t(k, s) = (k, s+t)$ and $g(r(k), s) = (k, s+t(k))$ where $t(k)$ is the return time of $k \in K$ under ϕ and $r(k) = \phi_{t(k)}(k)$.*

In particular, X is fibered over the circle with fiber $K = \tilde{X}/\tilde{\phi}$.

Now recall that we have a mapping torus N of M with the monodromy φ . Using the Fried's result, we show the following proposition, which implies that $\mathcal{C} \subset \mathcal{C}_F \cap (\Gamma^* \otimes \mathbb{R})$. For $\alpha \in \Gamma^*$, we denote by $\alpha^\perp < \Gamma$ the subgroup consisting of elements whose pairing with α is 0.

Proposition 2.6. *Let $\alpha = (\cdot, n_\alpha)$ be any primitive integral class in \mathcal{C} . Then N admits another fibration over the circle respecting the flow \mathcal{F} so that the generator of the first cohomology of the circle pulls back to α , and the fiber is*

$$M_\alpha = \tilde{M}/\alpha^\perp.$$

Moreover, the monodromy φ_α has a lift $\tilde{\varphi}_\alpha$ on \tilde{M} such that $\tilde{\varphi}_\alpha^{n_\alpha} = \tilde{\varphi}$.

Here, a fibration over the circle respecting the flow \mathcal{F} refers to a fibration such that the suspension flow is conjugate to the flow \mathcal{F} .

Proof. To show the desired fibration, recall the commutative diagram of coverings:

$$\begin{array}{ccc} \tilde{N} & \xleftarrow{\mathbb{Z}\text{-fold}} & \tilde{N}' = \tilde{M} \times \mathbb{R} \\ \downarrow & \nearrow \text{Deck group } \Gamma = H \oplus \mathbb{Z} & \\ N & & \end{array}$$

and the flow \mathcal{F} can be lifted to the \mathbb{R} -translation flow on $\tilde{N}' = \tilde{M} \times \mathbb{R}$. We consider the \mathbb{Z} -cover

$$\tilde{N}'/\alpha^\perp \rightarrow N$$

and denote by $\tilde{\mathcal{F}}$ the induced flow on \tilde{N}'/α^\perp , which is the lift of \mathcal{F} as well. The Deck group is generated by $(g, m) \in H \oplus \mathbb{Z}$ such that $\alpha(g, m) = 1$. Since the set of $\tilde{\mathcal{F}}$ -flow lines in \tilde{N}'/α^\perp is identified with \tilde{M}/α^\perp , the first statement follows once we verify that Theorem 2.5(1) holds with $X = N$, $\tilde{X} = \tilde{N}'/\alpha^\perp$, $\phi = \mathcal{F}$, and $\tilde{\phi} = \tilde{\mathcal{F}}$.

We first claim that any $\tilde{\mathcal{F}}$ -flow line in \tilde{N}'/α^\perp does not accumulate. Suppose not. It means that there exist $(x, s) \in \tilde{N}' = \tilde{M} \times \mathbb{R}$ and sequences $t_i \rightarrow \pm\infty$ and $(h_i, n_i) \in \alpha^\perp < H \oplus \mathbb{Z}$ such that the sequence

$$(h_i, n_i) \cdot (x, s + t_i) = (h_i \tilde{\varphi}^{n_i}(x), s + n_i + t_i)$$

converges in \tilde{N}' . Hence, the sequence n_i is divergent and $h_i \tilde{\varphi}^{n_i}(x)$ converges in \tilde{M} . In particular, there exists $h \in H$ such that $hh_i \tilde{\varphi}^{n_i}(D) \cap D \neq \emptyset$ for all i . This implies that $hh_i \in -\Omega(\tilde{\varphi}^{n_i})$, and therefore (h_i, n_i) is contained in a bounded neighborhood of Ω . Since n_i is a divergent sequence and $\alpha(h_i, n_i) = 0$, α vanishes at some vector in the asymptotic cone of Ω . This contradicts $\alpha \in \mathcal{C}$.

Now to verify Theorem 2.5(1), let $(x, s) \in \tilde{N}' = \tilde{M} \times \mathbb{R}$. For each $t \in \mathbb{R}$, there exist $(h_t, n_t) \in \alpha^\perp$ and $k_t \in \mathbb{Z}$ such that

$$(g, m)^{k_t} (h_t, n_t) \cdot (x, s + t) = (g^{k_t} h_t \tilde{\varphi}^{k_t m + n_t}(x), s + t + k_t m + n_t)$$

is contained in a fixed compact subset. By the previous claim, as $t \rightarrow \infty$, we have either $k_t \rightarrow \infty$ or $k_t \rightarrow -\infty$. After passing to a subsequence, we may assume that the sequence

$$(g^{k_t} h_t \tilde{\varphi}^{k_t m + n_t}(x), s + t + k_t m + n_t)$$

converges in \tilde{N}' . This implies that

$$(2.1) \quad \frac{k_t m + n_t}{t} \rightarrow -1$$

as $t \rightarrow \infty$ and for some $h \in H$, we have

$$hg^{k_t} h_t \tilde{\varphi}^{k_t m + n_t}(D) \cap D \neq \emptyset$$

for all large $t > 0$. Hence $(g, m)^{k_t}(h_t, n_t) = (g^{k_t}h_t, k_tm + n_t)$ is of bounded distance from some point $(y_t, k_tm + n_t) \in -\Omega(\tilde{\varphi}^{k_tm+n_t}) \times \{k_tm + n_t\}$. Therefore,

$$\lim_{t \rightarrow \infty} \frac{k_t}{k_tm + n_t} = \lim_{t \rightarrow \infty} \frac{\alpha((g, m)^{k_t}(h_t, n_t))}{k_tm + n_t} = \lim_{t \rightarrow \infty} \alpha\left(\frac{y_t}{k_tm + n_t}, 1\right).$$

Since the vector $\left(\frac{y_t}{k_tm + n_t}, 1\right)$ converges to the vector in the asymptotic cone of Ω and $\alpha \in \mathcal{C}$, we obtain

$$\lim_{t \rightarrow \infty} \frac{k_t}{k_tm + n_t} > 0.$$

Together with (2.1), we have that $k_t \rightarrow -\infty$ as $t \rightarrow \infty$. Note that the positivity of t has not been used. Applying the same argument to the case $t \rightarrow -\infty$, we conclude that $k_t \rightarrow \mp\infty$ as $t \rightarrow \pm\infty$. Consequently, in the compactification of \tilde{N}'/α^\perp , each $\tilde{\mathcal{F}}$ -flow line is from the end $(g, m)^{-\infty}$ to the end $(g, m)^\infty$. This verifies Theorem 2.5(1), showing the first statement of the proposition.

The last assertion follows from the observation that $\alpha(0, 1) = n_\alpha$, and hence $(g, m)^{n_\alpha}(0, -1) \in \alpha^\perp$. \square

The converse might be standard to experts, but we present the proof as follows, completing the proof of Theorem 2.4:

Proposition 2.7. *Let $\alpha \in \Gamma^*$ be a primitive integral class. If N admits a circle fibration respecting the flow \mathcal{F} associated to α , then $\alpha \in \mathcal{C}$.*

Proof. Again, we consider the \mathbb{Z} -cover

$$\tilde{N}'/\alpha^\perp \rightarrow N.$$

Then from the hypothesis, the flow \mathcal{F} admits a cross section that can be lifted to \tilde{N}'/α^\perp . Recall the commutative diagram:

$$\begin{array}{ccc} \tilde{N} & \xleftarrow[\text{\mathbb{Z}-fold}]{} & \tilde{N}' = \tilde{M} \times \mathbb{R} \\ \downarrow & \nearrow \text{Deck group } \Gamma = H \oplus \mathbb{Z} & \\ N & & \end{array}$$

We also denote by $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ the lifts of the flow \mathcal{F} on \tilde{N}'/α^\perp and \tilde{N}' respectively.

Let $h_n \in H$, $n \in \mathbb{Z}$, be a sequence such that

$$(h_n \tilde{\varphi}^n \cdot D) \cap D \neq \emptyset.$$

To prove $\alpha \in \mathcal{C}$, it suffices to show that the sequence $\alpha(h_n, n)/n$ is a positive for all but finitely many n , and does not accumulate to 0. Fix an element $(g, m) \in \Gamma$ such that $\alpha(g, m) = 1$. This acts as a Deck transformation on \tilde{N}'/α^\perp .

For each $n \in \mathbb{Z}$, let $x_n \in D$ be such that $h_n \tilde{\varphi}^n(x_n) \in D$. Then for each $n \in \mathbb{Z}$, there exists $k_n \in \mathbb{Z}$ and $(g_n, t_n) \in \alpha^\perp$ such that

$$(g, m)^{k_n}(g_n, t_n)(h_n, n) \cdot \tilde{\mathcal{F}}'_{-n}(x_n, 0) \in \tilde{N}'$$

is contained in the fundamental domain for the Γ -action on \tilde{N}' containing $D \times \{0\}$. Expanding the action, this means that

$$(g^{k_n} g_n \tilde{\varphi}^{k_n m + t_n} h_n \tilde{\varphi}^n(x_n), k_n m + t_n) \in \tilde{N}'$$

belongs to the Γ -fundamental domain on \tilde{N}' . In particular, the sequence $k_n m + t_n \in \mathbb{Z}$ is bounded. Since $h_n \tilde{\varphi}^n(x_n) \in D$ for all $n \in \mathbb{Z}$, we now have that the sequence $\tilde{\varphi}^{k_n m + t_n} h_n \tilde{\varphi}^n(x_n)$ is bounded in \tilde{M} as well. Together with the boundedness of the sequence $g^{k_n} g_n \tilde{\varphi}^{k_n m + t_n} h_n \tilde{\varphi}^n(x_n) \in \tilde{M}$, it follows that the sequence $g^{k_n} g_n \in H$ is bounded. Therefore, the sequence $(g, m)^{k_n}(g_n, t_n) = (g^{k_n} g_n, k_n m + t_n) \in \Gamma$ is bounded.

Now for each $n \in \mathbb{Z}$, let $(z_n, s_n) = (g, m)^{k_n}(g_n, t_n)(h_n, n) \cdot \tilde{\mathcal{F}}'_{-n}(x_n, 0) \in \tilde{N}'$. We then have

$$(g, m)^{k_n}(g_n, t_n)(h_n, n) \cdot (x_n, 0) = \tilde{\mathcal{F}}'_n(z_n, s_n) \in \tilde{N}'.$$

If we denote by $\tilde{\alpha}'$ the 1-form on \tilde{N}' induced by α , we have

$$\alpha((g, m)^{k_n}(g_n, t_n)(h_n, n)) = \int_{(x_n, 0)}^{(z_n, s_n)} \tilde{\alpha}' + \int_{(z_n, s_n)}^{\tilde{\mathcal{F}}'_n(z_n, s_n)} \tilde{\alpha}'.$$

Since $(x_n, 0)$ and (z_n, s_n) are contained in the fixed fundamental domain of the Γ -action on \tilde{N}' , $\int_{(x_n, 0)}^{(z_n, s_n)} \tilde{\alpha}'$ is bounded. Moreover, since the return time for the flow \mathcal{F} to the fiber \tilde{M}/α^\perp is bounded from below and above by positive constants, there exist $c, c' > 1$ such that

$$\begin{aligned} c^{-1}n - c' &\leq \alpha((g, m)^{k_n}(g_n, t_n)(h_n, n)) \leq cn + c' && \text{for } n \geq 0 \\ cn - c' &\leq \alpha((g, m)^{k_n}(g_n, t_n)(h_n, n)) \leq c^{-1}n + c' && \text{for } n < 0. \end{aligned}$$

Since the sequence $(g, m)^{k_n}(g_n, t_n)$ is bounded, this finishes the proof. \square

The following lemma will be used later:

Lemma 2.8. [BSW21, Lemma 5.3] *Let \mathcal{C}_0 be a proper subcone of the generalized fibered cone. There exists $C > 0$ such that for any primitive integral element $\alpha = (\cdot, n_\alpha) \in \mathcal{C}_0$ with $n_\alpha > C$, there is some $h \in H$ which does not belong to the $Cn_\alpha^{1/d}$ -neighborhood of $\bigcup_{a \in \alpha^\perp} \Omega(a)$ in H .*

Proof. The definition of generalized fibered cone implies that Ω is contained in a Hausdorff neighborhood of the dual cone of \mathcal{C}_0 . Let α be a primitive integral element in \mathcal{C}_0 , $b = (x_1, \dots, x_d, y) \in \alpha^\perp$, and $p \in \Omega(b)$. Then Proposition 2.1 implies $d(\pm\Omega(\tilde{\varphi}^y), 0) \leq Ay + C$ for some $A, C > 0$. Furthermore, the fact that Ω is contained in a Hausdorff neighborhood of \mathcal{C}_0 implies that $x := (x_1, \dots, x_d)$ satisfies $d(x, -\Omega(\tilde{\varphi}^y)) \geq A'y - C'$ for some $A', C' > 0$. This observation implies the same statement as [BSW21, Lemma 5.1]. Together

with the fact that the covolume of the projection of α^\perp onto H is $\gtrsim |n_\alpha|$ (cf. [BSW21, Lemma 5.2]), the same proof as in [BSW21, Lemma 5.3] works and gives the desired statement. \square

3. ASYMPTOTIC TRANSLATION LENGTHS ON SPHERE COMPLEXES

Now we prove the main theorem:

Theorem B. *Let $\varphi : M \rightarrow M$ be a diffeomorphism of a compact manifold to itself. Consider the generalized fibered cone of the mapping torus of φ , and let \mathcal{R} be an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone. Then we have*

$$(1.1) \quad l(\varphi_\alpha) \lesssim \|\alpha\|^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R}$ where φ_α is the corresponding monodromy and $l(\varphi_\alpha)$ is the asymptotic translation length of φ_α on the sphere complex of the fiber.

We deduce it from the following weaker version:

Theorem 3.1. *Let $\varphi : M \rightarrow M$ be a diffeomorphism of a compact manifold to itself. Consider the generalized fibered cone of the mapping torus of φ , and let \mathcal{R} be an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone. Then there exist finitely many hyperplanes R_1, \dots, R_k , disjoint from the cohomology class corresponding to φ , such that*

$$l(\varphi_\alpha) \lesssim \|\alpha\|^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R} - \bigcup_{i=1}^k R_i$ where φ_α is the corresponding monodromy.

Proof. By Lemma 2.8, there exists $C > 0$ depending on \mathcal{R} such that for any primitive integral element $\alpha = (\cdot, n_\alpha) \in \mathcal{R}$ in coordinates of $\Gamma^* = (H \oplus \mathbb{Z})^*$ with $n_\alpha > C$, there exists $h \in H$ such that the $Cn_\alpha^{1/d}$ -neighborhood of h in H is disjoint from $\bigcup_{a \in \alpha^\perp} \Omega(a)$. In particular, h is disjoint from $\bigcup_{a \in \alpha^\perp} \Omega(a)$ and hence

$$(3.1) \quad (h \cdot D) \cap (\alpha^\perp \cdot D) = \emptyset$$

We claim that we can choose $c > 0$ small enough so that for any such α and h above, we have

$$(3.2) \quad \left(\left(h \tilde{\varphi}^{\lfloor cn_\alpha^{1/d} \rfloor} \right) \cdot D \right) \cap (\alpha^\perp \cdot D) = \emptyset.$$

See Figure 1 for the pictorial description of the claim.

Let $c > 0$ be a constant. We prove the claim by showing that we can take c small enough so that for any α and h as above, we have

$$(3.3) \quad h \cdot D \cap \left(\alpha^\perp \tilde{\varphi}^{-\lfloor cn_\alpha^{1/d} \rfloor} \right) \cdot D = \emptyset.$$

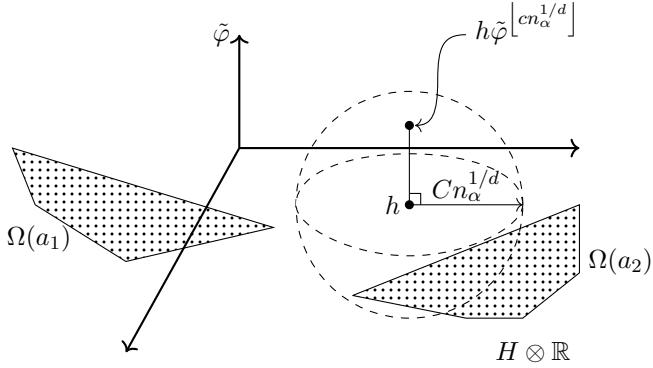


FIGURE 1. Description of $\Gamma \otimes \mathbb{R}$. The subspace $H \otimes \mathbb{R}$ is illustrated as a horizontal plane. $a_1, a_2 \in \alpha^\perp$ and the dotted regions are $\Omega(a_1)$ and $\Omega(a_2)$. The ball centered at h is of radius $Cn_\alpha^{1/d}$ in $\Gamma \otimes \mathbb{R}$. The constant c is chosen appropriately so that $h\tilde{\varphi}^{\lfloor cn_\alpha^{1/d} \rfloor}$ belongs to the ball.

Fix such α and h , and suppose that

$$(3.4) \quad h \cdot D \cap \left(a\tilde{\varphi}^{-\lfloor cn_\alpha^{1/d} \rfloor} \right) \cdot D \neq \emptyset$$

for some $a \in \alpha^\perp$. Writing $a = x\tilde{\varphi}^m$ for some $x \in H$ and $m \in \mathbb{Z}$, we have

$$(x^{-1}h \cdot D) \cap \left(\tilde{\varphi}^{m-\lfloor cn_\alpha^{1/d} \rfloor} \cdot D \right) \neq \emptyset.$$

Then there exists $y \in \tilde{\varphi}^{-\lfloor cn_\alpha^{1/d} \rfloor} \cdot D$ such that $\tilde{\varphi}^m y \in x^{-1}h \cdot D$. We can also choose $h_1 \in H$ such that $y \in h_1 \cdot D$. This implies that

$$(x^{-1}h \cdot D) \cap (\tilde{\varphi}^m h_1 \cdot D) \neq \emptyset \quad \text{and} \quad (h_1 \cdot D) \cap \left(\tilde{\varphi}^{-\lfloor cn_\alpha^{1/d} \rfloor} \cdot D \right) \neq \emptyset.$$

In particular, $h_1^{-1}x^{-1}h \in \Omega(\tilde{\varphi}^m)$ and $h_1 \in \Omega\left(\tilde{\varphi}^{-\lfloor cn_\alpha^{1/d} \rfloor}\right)$. Therefore, we have

$$h \in x\Omega(\tilde{\varphi}^m)\Omega\left(\tilde{\varphi}^{-\lfloor cn_\alpha^{1/d} \rfloor}\right).$$

Since $x\Omega(\tilde{\varphi}^m) = \Omega(a)$ and the diameter of $\Omega\left(\tilde{\varphi}^{-\lfloor cn_\alpha^{1/d} \rfloor}\right)$ is less than $c_0 cn_\alpha^{1/d}$ for some constant $c_0 > 0$ depending only on \mathcal{R} by Proposition 2.1, we now have that h is contained in the $c_0 cn_\alpha^{1/d}$ -neighborhood of $\Omega(a)$ in H . On the other hand, h was chosen so that the $Cn_\alpha^{1/d}$ -neighborhood of h is disjoint from $\bigcup_{a \in \alpha^\perp} \Omega(a)$. Consequently, if we choose $c < C/c_0$, then (3.4) cannot happen, and hence such c satisfies (3.2), proving the claim.

Now fix $c > 0$ satisfying (3.2). Since \mathcal{R} is contained in a proper subcone of the generalized fibered cone, the set $\{w \in \Omega : \alpha(w) = 0 \text{ for some } \alpha \in \mathcal{R}\}$ is finite. Denote by w_1, \dots, w_k its elements and set $R_i := \{\alpha \in \Gamma^* \otimes \mathbb{R} : \alpha(w_i) = 0\}$ for $i = 1, \dots, k$. As a result, we have finitely many hyperplanes $R_1, \dots, R_k \subset \Gamma^* \otimes \mathbb{R}$ such that for any primitive integral element $\alpha \in \mathcal{R} - \bigcup_{i=1}^k R_i$, we have $\alpha^\perp \cap \Omega = \{0\}$. Since the cohomology class corresponding to φ is the dual of $(0, 1) \in \Gamma$, the hyperplanes R_1, \dots, R_k are disjoint from this cohomology class. This implies that for any $\alpha \in \mathcal{R} - \bigcup_{i=1}^k R_i$ and non-trivial $a \in \alpha^\perp$, we have $(a \cdot D) \cap D = \emptyset$. Indeed, if $a = (x, m)$ in the coordinate of $\Gamma = H \oplus \mathbb{Z}$, then $(a \cdot D) \cap D \neq \emptyset$ implies

$$(x^{-1} \cdot D) \cap (\tilde{\varphi}^m \cdot D) \neq \emptyset.$$

Hence, $x^{-1} \in \Omega(\tilde{\varphi}^m)$, and therefore

$$a = (x, m) \in -\Omega(\tilde{\varphi}^m) \times \{m\} \subset \Omega$$

which is a contradiction.

Now let $\alpha \in \mathcal{R} - \bigcup_{i=1}^k R_i$ with $n_\alpha > C$, and $h \in H$ be the element given by Lemma 2.8. We choose embedded spheres S_1 and S_2 in D and $h \cdot D$ respectively. Since the translate of D by any non-trivial element of α^\perp is disjoint from D as observed in the previous paragraph, S_1 and S_2 are injected to the fiber $M_\alpha = \tilde{M}/\alpha^\perp$, and hence are vertices of the sphere complex $\mathcal{S}(M_\alpha)$. Abusing notations, we also denote by S_1 and S_2 their images in M_α . By (3.1), S_1 and S_2 are disjoint in M_α , and therefore they represent adjacent vertices in $\mathcal{S}(M_\alpha)$. Now it follows from Proposition 2.6 and (3.2) that

$$\begin{aligned} d_{\mathcal{S}(M_\alpha)} \left(S_2, \varphi_\alpha^{n_\alpha \cdot \lfloor cn_\alpha^{1/d} \rfloor} (S_2) \right) &\leq d_{\mathcal{S}(M_\alpha)} (S_2, S_1) + d_{\mathcal{S}(M_\alpha)} \left(S_1, \varphi_\alpha^{n_\alpha \cdot \lfloor cn_\alpha^{1/d} \rfloor} (S_2) \right) \\ &= 1 + d_{\mathcal{S}(M_\alpha)} \left(S_1, \varphi^{\lfloor cn_\alpha^{1/d} \rfloor} (S_2) \right) = 2. \end{aligned}$$

Therefore, we can estimate the asymptotic translation length of φ_α as follows:

$$l(\varphi_\alpha) \leq \limsup_{m \rightarrow \infty} \frac{d_{\mathcal{S}(M_\alpha)} \left(S_2, \varphi_\alpha^{n_\alpha \cdot \lfloor cn_\alpha^{1/d} \rfloor \cdot m} (S_2) \right)}{n_\alpha \cdot \lfloor cn_\alpha^{1/d} \rfloor \cdot m} \leq \frac{2}{n_\alpha \cdot \lfloor cn_\alpha^{1/d} \rfloor}.$$

Since n_α is comparable to $\|\alpha\|$ for $\alpha \in \mathcal{R}$, this completes the proof of the estimate. \square

Proof of Theorem B. By Theorem 3.1, we have finitely many hyperplanes R_1, \dots, R_k so that the desired inequality (1.1) holds on $\mathcal{R} - \bigcup_{i=1}^k R_i$.

Now for each R_i , choose a primitive integral element $\alpha_i \in R_i \cap \mathcal{R}$. By Proposition 2.6, this corresponds to a fibration $N \rightarrow S^1$ with the monodromy φ_{α_i} and the fiber $M_{\alpha_i} = \tilde{M}/\alpha_i^\perp$, with respect to the flow \mathcal{F} . We now apply

Theorem 3.1 to $\varphi_{\alpha_i} : M_{\alpha_i} \rightarrow M_{\alpha_i}$. Since its mapping torus is also N and the associated flow is \mathcal{F} as well, the generalized fibered cone for φ_{α_i} , which is equal to Fried's cone for \mathcal{F} in $H^1(N)$ by Theorem 2.4, is the same as the generalized fibered cone for the original diffeomorphism $\varphi : M \rightarrow M$. In particular, \mathcal{R} is still an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone for φ_{α_i} . Hence, we can apply Theorem 3.1 to $\varphi_{\alpha_i} : M_{\alpha_i} \rightarrow M_{\alpha_i}$ and get finitely many hyperplanes $R_1^{(i)}, \dots, R_{n_i}^{(i)}$ so that $\alpha_i \notin \bigcup_{j=1}^{n_i} R_j^{(i)}$ and (1.1) holds on $\mathcal{R} - \bigcup_{j=1}^{n_i} R_j^{(i)}$. Since $\alpha_i \in R_i$ while α_i is disjoint from $\bigcup_{j=1}^{n_i} R_j^{(i)}$, the intersection $R_i \cap R_j^{(i)}$ is a codimension 2 subspaces in $H^1(N)$.

By the above argument, we now have finitely many codimension 2 subspaces $R_i \cap R_j^{(i)}$, $i = 1, \dots, k$ and $j = 1, \dots, n_i$, so that (1.1) holds on $\mathcal{R} - \bigcup_{i=1}^k \bigcup_{j=1}^{n_i} R_i \cap R_j^{(i)}$. By proceeding the above argument inductively, we finally obtain finitely many one-dimensional subspaces L_1, \dots, L_ℓ in $H^1(N)$ such that (1.1) holds on $\mathcal{R} - \bigcup_{i=1}^\ell L_i$. Since there are only finitely many primitive integral elements in $\bigcup_{i=1}^\ell L_i$, this completes the proof. \square

4. MINIMAL TRANSLATION LENGTHS OF HANDLEBODY GROUPS

Recall that we denote by \mathcal{H}_g and \mathcal{D}_g the handlebody group and the disk graph of a closed connected orientable surface of genus g respectively. In this section, we prove:

Theorem C. *We have*

$$L_{\mathcal{D}_g}(\mathcal{H}_g) \asymp \frac{1}{g^2}.$$

Proof. We first deduce the lower bound from [GT11]. Recall that the inclusion $(D^2, \partial D^2) \subseteq (V_g, \partial V_g)$ induces a map from the vertices of \mathcal{D}_g to the vertices of \mathcal{C}_g . Moreover, two non-isotopic disjoint disks in V_g have non-isotopic disjoint boundary as noted in [Hen18, Lemma 2.3]. This implies that the above map on vertices indeed gives the graph embedding $\mathcal{D}_g \rightarrow \mathcal{C}_g$. In other words, the disk graph \mathcal{D}_g is a subgraph of the curve graph of ∂V_g , and therefore

$$L_{\mathcal{C}_g}(\mathcal{H}_g) \leq L_{\mathcal{D}_g}(\mathcal{H}_g)$$

for all $g \geq 2$. Since $L_{\mathcal{C}_g}(\text{Mod}(S_g)) \asymp 1/g^2$ as proved by Gadre–Tsai [GT11] and $L_{\mathcal{C}_g}(\mathcal{H}_g) \geq L_{\mathcal{C}_g}(\text{Mod}(S_g))$, the lower bound follows.

To prove the upper bound, we apply our main argument together with a similar construction as in [Hir11, Section 6]. We begin with a genus 2 handlebody V_2 and a Torelli pseudo-Anosov $\varphi \in \mathcal{H}_2$. Such an element φ indeed exists: consider a separating curve $b \subset \partial V_2$ which bounds a disk in V_2 as in Figure 2. Taking any pseudo-Anosov $\varphi_0 \in \mathcal{H}_2$, two separating curves b and $\varphi_0^n b$ fill ∂V_2 for large enough $n \in \mathbb{N}$ by [MM99], and they bound disks in V_2 . Therefore, φ can be obtained by Thurston's construction [Thu88] or Penner's construction [Pen88] applied to the pair b and $\varphi_0^n b$.

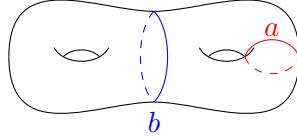
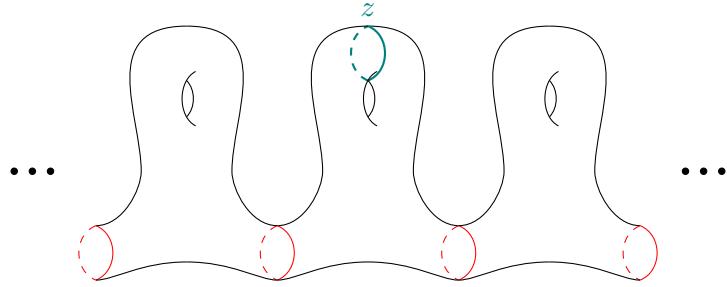


FIGURE 2. A handlebody of genus 2

Now consider the homomorphism $H_1(\partial V_2) \rightarrow \mathbb{Z}$ given by the algebraic intersection number with $[a] \in H_1(\partial V_2)$ in Figure 2. This induces \mathbb{Z} -covers $\tilde{V} \rightarrow V_2$ and $\partial \tilde{V} \rightarrow \partial V_2$ as in Figure 3. Since $\varphi \in \mathcal{H}_2$ is Torelli, it can be lifted to both covers.

FIGURE 3. Cyclic cover \tilde{V}

Let N be the mapping torus of ∂V_2 with the monodromy φ . Since φ is pseudo-Anosov, N is hyperbolic [Thu98]. Using the \mathbb{Z} -cover above, we can consider the generalized fibered cone $\mathcal{C} \subset H^1(N)$ which is two-dimensional. For each primitive integral $\alpha \in \mathcal{C}$, we know from Proposition 2.6 that there is a fibration $N \rightarrow S^1$ with the monodromy φ_α and the fiber S_α which is the quotient of ∂V . By [Hir11], it follows that φ_α extends to the corresponding quotient of \tilde{V} which is a handlebody.

Let $\mathcal{R} \subset \mathcal{C}$ be a two-dimensional proper subcone. For all but finitely many primitive integral element $\alpha \in \mathcal{R}$, as in the proof of Theorem 3.1, we can choose an element h in the Deck group and $k_\alpha \in \mathbb{N}$ comparable to $\|\alpha\|^2$ so that the curve z in Figure 3 and its translate $h \cdot z$ inject into the fiber S_α , and moreover z has the image disjoint from images of both $h \cdot z$ and $\varphi_\alpha^{k_\alpha}(h \cdot z)$ in the quotient S_α . We identify them with their images in S_α . Note that z , $h \cdot z$, and $\varphi_\alpha^{k_\alpha}(h \cdot z)$ bound disks in the handlebody. By [Hen18, Lemma 2.2], z and $h \cdot z$ bound disjoint disks and similarly z and $\varphi_\alpha^{k_\alpha}(h \cdot z)$ bound disjoint disks. Hence $h \cdot z$ and $\varphi_\alpha^{k_\alpha}(h \cdot z)$ represent vertices in the disk graph with distance at most 2. Since $\|\alpha\|$ is comparable to the genus g_α of S_α , we conclude

$$l_{\mathcal{D}_{g_\alpha}}(\varphi_\alpha) \lesssim \frac{1}{g_\alpha^2}$$

for all primitive integral $\alpha \in \mathcal{R}$. Since N is hyperbolic, φ_α is pseudo-Anosov by [Thu98]. Since the monodromy φ we started from is on the genus 2 surface, it is a simple computation using Thurston norm that all but finitely many natural numbers arise as g_α for primitive integral $\alpha \in \mathcal{R}$ (e.g. [Hir11, Proposition 6.10]). This completes the proof. \square

5. FOLDING SEQUENCES OF GRAPH MAPS AND (DOUBLED) HANDLEBODIES

The results in this section are not completely new and follow from work of various authors. For instance, see [Lau74], [HV04], [Luf78]. Nevertheless, we provide a direct construction of the relevant diffeomorphisms for the sake of the readers.

Let G be a finite connected graph and $\psi : G \rightarrow G$ a homotopy equivalence. We assume that f is *combinatorial*, that is, f maps vertices to vertices and edges to non-trivial edge-paths. The goal of this section is to provide one way of lifting a combinatorial homotopy equivalence of a graph to a diffeomorphism of a doubled handlebody and one of a handlebody. This liftings are obviously not unique and may come from other kinds of constructions.

Denoting by $V(\cdot)$ the set of vertices, $\psi : G \rightarrow G$ gives a subdivision G_Δ of G by setting $V(G_\Delta)$ to be a union of $V(G)$ and the preimage of $V(G)$ under ψ . The graph G_Δ is topologically identical to G . Then ψ is a composition $G \xrightarrow{i} G_\Delta \xrightarrow{\phi} G$ where $i : G \rightarrow G_\Delta$ is a subdivision map and $\phi : G_\Delta \rightarrow G$ is defined by $\phi(e) = \psi(i^{-1}(e))$ for an edge e of G_Δ . From the construction, ϕ is a graph map that sends an edge to an edge. By [Sta83], there is a finite sequence of foldings (or, folding sequence) so that ϕ is a composition of those foldings. Here, folding on a graph is identifying two edges with a common endpoint. For details, see [DKL15] and [Sta83].

In this section, we explicitly construct a 3-manifold M_G from G and a diffeomorphism $\varphi : M_G \rightarrow M_G$ from a folding sequence of $\psi : G \rightarrow G$. More precisely, we fix a folding sequence

$$G \xrightarrow{i} G_\Delta = G_0 \xrightarrow{\psi_0} G_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_n} G_{n+1} = G_\Delta \hookrightarrow G$$

where $\psi_i : G_i \rightarrow G_{i+1}$ is a folding, and then construct a diffeomorphism $\varphi_i : M_{G_i} \rightarrow M_{G_{i+1}}$ associated to ψ_i , with a canonical identification of M_{G_Δ} and M_G . As a result, composition of φ_i 's gives the desired diffeomorphism $\varphi : M_G \rightarrow M_G$. That is,

$$(5.1) \quad \begin{array}{ccc} M_G & \xrightarrow{\varphi} & M_G \\ \parallel & & \parallel \\ M_{G_0} & \xrightarrow{\varphi_0} & \cdots \xrightarrow{\varphi_n} M_{G_{n+1}} \end{array}$$

Note here that a folding sequence for ψ may not be unique; what we construct is a diffeomorphism $\varphi : M_G \rightarrow M_G$ respecting a fixed folding sequence for ψ . Throughout the paper, *graph* is finite and connected.

5.1. Doubled handlebody. For a graph G , we construct a corresponding doubled handlebody M_G as follows: we first replace each vertex of G with S^3 and each edge of G with $S^2 \times I$, where I is a compact interval. Then attachment of an edge to a vertex amounts to drilling out a 3-ball D^3 from S^3 and then gluing $S^2 \times I$ along one component of its boundary, as depicted in Figure 4.

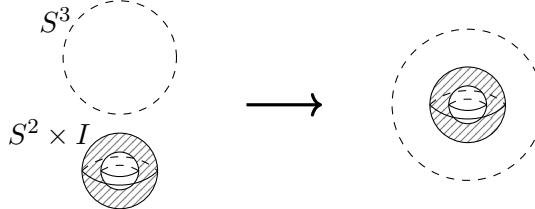


FIGURE 4. The gluing corresponding to an edge attached to a vertex.

Figure 5 demonstrates two examples of induced 3-manifolds. Note that Figure 5b will be used again in order to describe a folding map on M_G . Furthermore, there is a map $\mathcal{P} : M_G \rightarrow G$ which sends the S^3 corresponding to each vertex to the vertex itself, and sends $S^2 \times I$ to the corresponding edge by projection to the second component. It is easy to see that \mathcal{P} induces an isomorphism between fundamental groups.

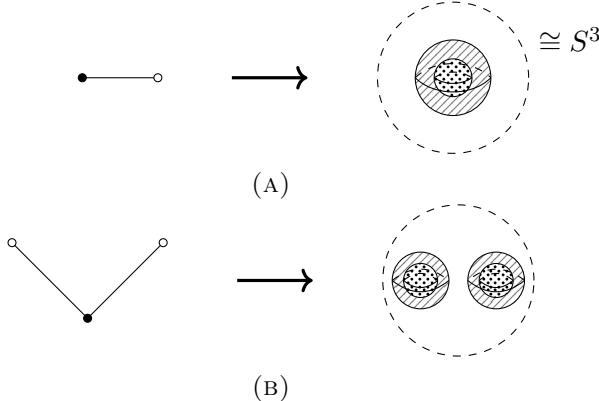


FIGURE 5. 3-manifolds obtained from the graphs. Light vertices in the graphs correspond to the dotted regions. Dark vertices in the graphs correspond to the outermost regions in the right figures. Edges of the graphs correspond to the hatched regions diffeomorphic to $S^2 \times I$.

For each of such foldings $\psi_i : G_i \rightarrow G_{i+1}$, let vertex v of G_i be the vertex on which the two edges to be folded are attached. Then the corresponding part in M_{G_i} is described in Figure 5b. Denote by S , S_1 and S_2 each part of the 3-manifold, induced from v and other two vertices, respectively. In other words, S is the outermost region in Figure 5b, and S_1 and S_2 are dotted ones.

To get the folding on M_{G_i} , we extract some pieces as follows:

- (1) We drill out a solid cylinder $D^2 \times I \subseteq S^2 \times I$ from each of the two $S^2 \times I$ corresponding to an edge to be folded.
- (2) Next, in S_1 , we delete a small 3-ball D^3 whose boundary contains $S_1 \cap (D^2 \times \partial I)$ where $D^2 \times I$ is the cylinder removed in (1). Similarly, we drill out a small 3-ball in S_2 .
- (3) Finally we delete a cylinder $D^2 \times I$ in S that connects two cylinders removed in (1).

The union of deleted pieces is a 3-ball. As a result, we obtain $M_{G_i} \setminus D^3$ as in Figure 6a.

To “fold” the manifold according to the folding of two edges in the graph, we make two corresponding $S^2 \times I$ ’s be contained in a single new $S^2 \times I$. Note that $(S^2 \times I) \setminus (D^2 \times I)$ has an annular face $\partial D^2 \times I$. Gluing two copies of them onto two opposite faces of $S^1 \times I \times I$, it results in $S^2 \times I$. In this regard, we glue $S^1 \times I \times I$ as indicated by patterns in Figure 6b. Then two copies of $(S^2 \times I) \setminus (D^2 \times I)$ corresponding to two edges get into a single $S^2 \times I$, representing the “folding” of the manifold according to the folding of the edges.

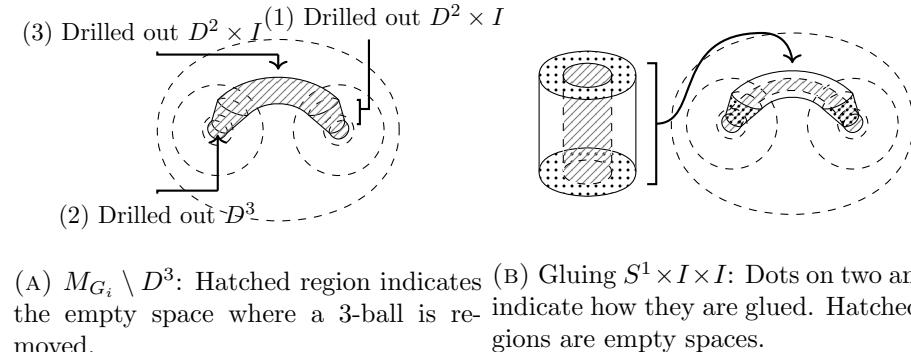


FIGURE 6. Folding of M_{G_i}

So far, we have seen how we “fold” the manifold by gluing $S^2 \times I \times I$. After gluing as in Figure 6b, the remaining boundary is S^2 : one annular face of $S^2 \times I \times I$ not glued and two 2-disks on the boundary of removed 3-balls in (3) of Figure 6a. See figure 7.

Hence, we can glue a 3-ball along this boundary diffeomorphic to S^2 . Gluing the 3-ball in this way represents the identifying endpoints of two folded

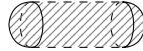


FIGURE 7. Empty (hatched) region in Figure 6b

edges (Figure 4). This whole procedure defines a map on M_{G_i} corresponding to folding two edges on the graph. Moreover, adding the solid torus as in Figure 6b and then gluing a 3-ball along the boundary in Figure 7 is just an adding a 3-ball to one in Figure 6a. Consequently, we get a diffeomorphism $\varphi_i : M_{G_i} \rightarrow M_{G_{i+1}}$ by folding two $S^2 \times I$ into one $S^2 \times I$.

Now as in (5.1), we define $\varphi : M_G \rightarrow M_G$ to be a composition of maps $\varphi_i : M_{G_i} \rightarrow M_{G_{i+1}}$. Since each folding map $\varphi_i : M_{G_i} \rightarrow M_{G_{i+1}}$ is a diffeomorphism, $\varphi : M_G \rightarrow M_G$ is the desired diffeomorphism respecting the folding sequence of $\psi : G \rightarrow G$. As a consequence, we now have:

Proposition 5.1. *Let G be a finite connected graph and $\psi : G \rightarrow G$ be a combinatorial homotopy equivalence. There is a doubled handlebody M_G , a map $\mathcal{P} : M_G \rightarrow G$ which induces an isomorphism on fundamental groups, and a diffeomorphism $\varphi : M_G \rightarrow M_G$ such that $\mathcal{P} \circ \varphi$ and $\psi \circ \mathcal{P}$ induces the same map on π_1 .*

Note that the choice of diffeomorphism φ is defined up to isotopy. In each of discussion in the next section, for a folding sequence of $\psi : G \rightarrow G$, we make a choice of specific *diffeomorphism* $\varphi : M_G \rightarrow M_G$ constructed in the above way to consider its generalized fibered cone.

5.2. Handlebody. We can also carry out the construction above with “half” of every pieces. More precisely, for a graph G , we get pieces by assigning each vertex to a 3-ball and each edge to a solid cylinder $D^2 \times I$ where $I = [0, 1]$. Then for each $D^2 \times I$, we attach $D^2 \times \{0\}$ to the ball assigned to the corresponding endpoint and similarly attach $D^2 \times \{1\}$ to the ball corresponding to the other endpoint. Now we get the handlebody H_G with genus rank $\pi_1(G)$. See Figure 8.

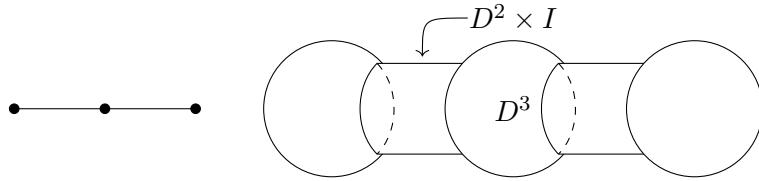


FIGURE 8. Attachment of 3-balls and solid cylinders according to a graph

Now it remains to get a diffeomorphism corresponding to a folding on G (or G_Δ). To get the folding on the handlebody, we just proceed as in the previous subsection with half of the pieces. First, we remove $(\text{half-disk}) \times I$ from each $D^2 \times I$, assigned to an edge supposed to be folded. Here, we take

the half-disk in D^2 so that the diameter of the half-disk is contained in ∂D^2 . We similarly eliminate $(\text{half-disk}) \times I$ from the 3-ball corresponding to the common endpoint of the edges to be folded. We also remove a small half-ball from each of two balls corresponding to vertices supposed to be identified via the folding on G (or G_Δ). See Figure 9.

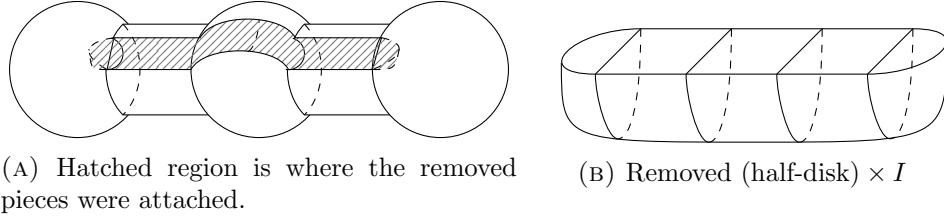


FIGURE 9. Drilling out from 3-balls and solid cylinders

Then gluing $(\text{half of } S^1) \times I \times I$ along some of its faces onto the removed regions, we finally get a handlebody removed a half-ball. See Figure 10.



FIGURE 10. Attaching $(\text{half of } S^1) \times I \times I$ to the removed region. The hatched region is empty due to the previous elimination, and the numbers indicate the way of gluing. Lines just stand for marking which part it comes from in Figure 9, so there is no membrane or wall within the hatched region.

Gluing the half-ball as in Figure 11 again, we finally get the desired folding on the handlebody, and it is indeed a diffeomorphism.

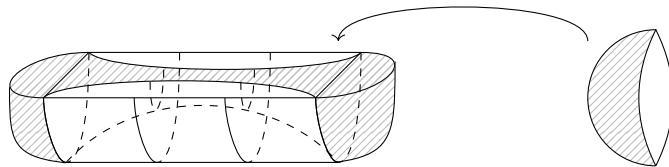


FIGURE 11. Regluing the half-ball. Left one is a result of the gluing in Figure 10. The hatched regions are 2-dimensional faces that two pieces are glued.

Hence, as a consequence, we have:

Proposition 5.2. *Let G be a finite connected graph and $\psi : G \rightarrow G$ be a combinatorial homotopy equivalence. There is a handlebody H_G , a map*

$\mathcal{P} : H_G \rightarrow G$ which induces an isomorphism on fundamental groups, and a diffeomorphism $\varphi : H_G \rightarrow H_G$ such that $\mathcal{P} \circ \varphi$ and $\psi \circ \mathcal{P}$ induces the same map on π_1 .

6. FREE-SPLITTING COMPLEXES AND FREE-FACTOR COMPLEXES

An immediate consequence of Theorem B and Proposition 5.1 is the following:

Corollary 6.1. *Let G be a finite connected graph and $\psi : G \rightarrow G$ be a combinatorial homotopy equivalence. Let M_G be the doubled handlebody and $\varphi : M_G \rightarrow M_G$ the induced diffeomorphism in Proposition 5.1. Let N be the mapping torus of φ . Let \mathcal{R} be an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone for φ . Then*

$$l(\varphi_\alpha) \lesssim \|\alpha\|^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R}$, where φ_α is the monodromy of the fibration $N \rightarrow S^1$ corresponding to α , $l(\varphi_\alpha)$ is the asymptotic translation length on the sphere complex of the fiber M_α , and $\|\cdot\|$ be any norm on $H^1(N)$.

6.1. Positive cone. We can obtain more results when we look at certain subcones of the generalized fibered cone.

Definition 6.2 (Train track map). *Let G be a finite connected graph, and consider a combinatorial homotopy equivalence $\psi : G \rightarrow G$. The map $\psi : G \rightarrow G$ is called a train track map if for each edge e and $n \geq 1$ the restriction $\psi^n|_e$ of ψ^n to e is an immersion, i.e. no back-tracking condition holds.*

- A train track map is irreducible if its transition matrix is irreducible.
- A train track map ψ is said to be expanding if the length of $\psi^n(e)$ diverges as $n \rightarrow \infty$ for each edge e .

Remark 6.3. As above, some literature defines the train track map to be a homotopy equivalence. For instance, see [DKL15, Definition 2.11]. In contrast, the train track map has also been defined as a map that is not necessarily a homotopy equivalence. For example, see [DKL17, Section 2.1].

Let G be a finite connected graph and $\psi : G \rightarrow G$ an expanding irreducible train track map. According to [DKL15, DKL17], there is a proper subcone of a component of the symmetrized BNS-invariant Σ_s , called *positive cone* and denoted by \mathcal{A} , containing the cohomology class corresponding to ψ . In the positive cone, each primitive integral class α corresponds to a fibration of the (folded) mapping torus of $\psi : G \rightarrow G$ over the circle whose monodromy map $\psi_\alpha : G_\alpha \rightarrow G_\alpha$ is an expanding irreducible train track map.

By picking the fold in Section 5 to correspond to the folds in the folded mapping torus of ψ , let $\varphi : M_G \rightarrow M_G$ be a diffeomorphism of a doubled handlebody M_G constructed in Section 5. Let N be the mapping torus of φ and \mathcal{C} be the generalized fibered cone of φ .

Theorem 6.4. *Let $\psi : G \rightarrow G$ be an expanding irreducible train track map and $\varphi : M_G \rightarrow M_G$ be an induced map on the doubled handlebody M_G in Section 5. Let \mathcal{R} be an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone of φ . Then we have*

$$l(\varphi_\alpha) \lesssim g_\alpha^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R} \cap \mathcal{A}$, where φ_α is the monodromy of the fibration $N \rightarrow S^1$ corresponding to α , $l(\varphi_\alpha)$ is the asymptotic translation length on the sphere complex of the fiber M_α , and $g_\alpha = \text{rank } \pi_1(G_\alpha)$.

Proof. From Corollary 6.1, we have already seen that

$$(6.1) \quad l(\varphi_\alpha) \lesssim \|\alpha\|^{-1-1/d}$$

for a norm $\|\cdot\|$ on $H^1(N; \mathbb{R})$, where N is the mapping torus of $\varphi : M_G \rightarrow M_G$. Now it remains to see how $\|\alpha\|$ is related to g_α . Since all norms on $H^1(N; \mathbb{R})$, a finite-dimensional \mathbb{R} -vector space, are equivalent, we are free to choose the norm $\|\cdot\|$.

In this line of thought, we introduce the Alexander norm on $H^1(N; \mathbb{R})$ in a similar spirit of [DKL15] to rewrite (6.1) in terms of the genus of each fiber. Similar to the Thurston norm, the Alexander norm ball is the dual of Newton polytope of the Alexander polynomial Δ . For details, see [McM02].

Denote $\|\alpha\|_A$ the Alexander norm of α . As in [DKL15], it follows from [McM02, Theorem 4.1] together with [But07, Theorem 3.1] that

$$\|\alpha\|_A = g_\alpha - 1$$

when α belongs to the cone on the open faces of the Alexander norm ball. This equality is obtained in the following way (cf. [McM02, Theorem 4.1]): Let $\alpha(\Delta)$ be the Laurant polynomial induced by α and Δ . Writing Δ as the sum of distinct terms, there is only one summand which yields the highest degree term in $\alpha(\Delta)$ and similarly for the lowest degree term, since α is inside the cone. It means that $\deg \alpha(\Delta)$ is exactly the difference of the highest and the lowest degrees of induced terms from the summand of Δ . On the other hand, the difference equals to $\|\alpha\|_A$, and thus $\deg \alpha(\Delta) = \|\alpha\|_A$. Combining with the fact that $g_\alpha = 1 + \deg \alpha(\Delta)$, we conclude the above equality.

Even if α is not contained in the cone on an open face, $\|\alpha\|_A$ has something to do with g_α . As one can see in the previous argument, the assumption of belonging to the cone is only for showing $\deg \alpha(\Delta) = \|\alpha\|_A$. Instead, if the assumption does not hold, then there can be two distinct summands of Δ deducing the highest (or the lowest) degree terms in $\alpha(\Delta)$ and thus cancellation may occur. As such, we obtain $\deg \alpha(\Delta) \leq \|\alpha\|_A$ rather than the equality. Then again from $g_\alpha = 1 + \deg \alpha(\Delta)$, we now conclude

$$\|\alpha\|_A \geq g_\alpha - 1.$$

Going back to the estimation (6.1), we can now relate $\|\alpha\|$ (or $\|\alpha\|_A$) and g_α by $\|\alpha\|_A \geq g_\alpha - 1$, regardless of the position of α relative to the cones on

the open faces of the Alexander norm ball. Consequently, we conclude that

$$l(\varphi_\alpha) \lesssim g_\alpha^{-1-1/d}$$

as desired. \square

Note here that the proper subcone \mathcal{A} depends on the choice of a folding sequence of ψ . However, our argument does not depend on which folding sequence we choose. Indeed, the estimate in Theorem 6.4 holds for any choice of a folding sequence.

6.2. Free-splitting and free-factor complexes. The *free-splitting complex* \mathcal{FS}_K of a group K is a simplicial complex consisting of free splittings of K . More precisely, its vertices are equivalence classes of free splittings of K whose corresponding graph of groups have a single edge, and two vertices are connected by an edge of length 1 if they are represented by free splittings with a common refinement. For instance, two free splittings $A * (B * C)$ and $(A * B) * C$ are connected by an edge. For higher dimensional simplices and the equivalence relation among free splittings, see [KR14].

We continue the discussion from the previous subsection. As in [AS11], the sphere complex of the fiber M_α is equivalent to the free splitting complex of its fundamental group. Accordingly we can restate Corollary 6.1 and Theorem 6.4 in terms of free-splitting complexes as follows. We simply write \mathcal{FS}_g the free-splitting complex of the free group F_g of rank g :

Corollary 6.5. *Let $\psi : G \rightarrow G$ be an expanding irreducible train track map and $\varphi : M_G \rightarrow M_G$ be the induced map on the doubled handlebody M_G in Proposition 5.1. Let N be the mapping torus of φ . Let \mathcal{R} be an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone of φ . Then*

$$l_{\mathcal{FS}_{\pi_1(M_\alpha)}}(\varphi_\alpha) \lesssim \|\alpha\|^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R}$, where φ_α is the monodromy of the fibration $N \rightarrow S^1$ corresponding to α with the fiber M_α and $\|\cdot\|$ be any norm on $H^1(N)$.

Moreover,

$$l_{\mathcal{FS}_{g_\alpha}}(\varphi_\alpha) \lesssim g_\alpha^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R} \cap \mathcal{A}$ where $g_\alpha = \text{rank } \pi_1(G_\alpha)$.

Similar to the free-splitting complex, the *free-factor complex* \mathcal{FF}_g of F_g is a simplicial complex whose vertices are conjugacy classes of proper free factors of F_g , and $k+1$ vertices form a k -simplex if they can be represented by proper free factors $A_0 \leq A_1 \leq \dots \leq A_k$ of F_g . Again, we set all edges of the free-factor complex to be of length 1.

As in [KR14], there is a coarsely $\text{Out}(F_g)$ -equivariant (coarseness independent of g) Lipschitz map Ψ from the vertices of the free-splitting complex to the vertices of the free-factor complex. Hence, the following analogous result for the free-factor complex is deduced from Corollary 6.5:

Corollary 6.6. *Let $\psi : G \rightarrow G$ be an expanding irreducible train track map and $\varphi : M_G \rightarrow M_G$ be the induced map on the doubled handlebody M_G in Proposition 5.1. Let N be the mapping torus of φ . Let \mathcal{R} be an intersection of a $d+1$ -dimensional rational subspace and a proper subcone of the generalized fibered cone of φ . Then*

$$l_{\mathcal{FF}_{g_\alpha}}(\varphi_\alpha) \lesssim g_\alpha^{-1-1/d}$$

for all primitive integral element $\alpha \in \mathcal{R} \cap \mathcal{A}$ where φ_α is the monodromy of the fibration $N \rightarrow S^1$ corresponding to α with the fiber M_α and $g_\alpha = \text{rank } \pi_1(G_\alpha)$.

For more relations among complexes defined on a free group, one can refer to [GH22] and [KL09].

Remark 6.7. *It was studied by Hironaka [Hir11] and Hensel–Kielak [HK21] that when monodromies in a given Thurston’s fibered cone can be extended to associated handlebodies. Indeed, in [HK21], they proved that for any free group automorphism $f : F_g \rightarrow F_g$, there are infinitely many pseudo-Anosov $\varphi \in \mathcal{H}_g$ such that every monodromy in Thurston’s fibered cone containing φ extends to the associated handlebody. Also, Laudenbach [Lau74] showed that for a doubled handlebody \mathcal{M}_g of genus g , the natural surjection $\text{Mod}(\mathcal{M}_g) \rightarrow \text{Out}(F_g)$ has the kernel acting trivially on the sphere complex of \mathcal{M}_g . Together with these results, Corollary 6.5 and Corollary 6.6 can be interpreted in terms of asymptotic translation lengths of $\text{Out}(F_g)$ -elements.*

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