RIGIDITY FOR PATTERSON–SULLIVAN SYSTEMS WITH APPLICATIONS TO RANDOM WALKS AND ENTROPY RIGIDITY

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Abstract. In this paper we introduce Patterson–Sullivan systems, which consist of a group action on a compact metrizable space and a quasi-invariant measure which behaves like a classical Patterson–Sullivan measure. For such systems we prove a generalization of Tukia's measurable boundary rigidity theorem. We then apply this generalization to (1) study the singularity conjecture for Patterson–Sullivan measures (or, conformal densities) and stationary measures of random walks on isometry groups of Gromov hyperbolic spaces, mapping class groups, and discrete subgroups of semisimple Lie groups; (2) prove versions of Tukia's theorem for word hyperbolic groups, Teichmüller spaces, and higher rank symmetric spaces; and (3) prove an entropy rigidity result for pseudo-Riemannian hyperbolic spaces.

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1. Introduction

Let $\mathbb{H}^n$ denote real hyperbolic n-space and let $\partial_\infty \mathbb{H}^n$ denote its boundary at infinity. Given a discrete subgroup $\Gamma < \mathsf{Isom}(\mathbb{H}^n)$ and $\delta \geq 0$, a Borel probability measure μ on $\partial_\infty \mathbb{H}^n$ is called a Patterson–Sullivan measure (or conformal measure) for Γ of dimension δ if for any $\gamma \in \Gamma$ and Borel subset $E \subset \partial_\infty \mathbb{H}^n$,

$$\mu(\gamma E) = \int_E |\gamma'|^\delta d\mu. \tag{1}$$

These measures play a fundamental role in the study of geometry and dynamics of discrete subgroups of $\mathsf{Isom}(\mathbb{H}^n)$, or equivalently, of hyperbolic n-manifolds.

The celebrated rigidity theorem of Mostow [Mos75, Mos73] asserts that the geometry of a finite-volume hyperbolic n-manifold, $n \geq 3$, is determined by its fundamental group (see also [Pra73]). By considering Patterson–Sullivan measures, Tukia generalized Mostow's rigidity theorem to infinite-volume hyperbolic manifolds, as in the following theorem (which implies Mostow's rigidity).

Theorem 1.1. [Tuk89, Thm. 3C] *For $i = 1, 2$ let $\Gamma_i < \mathsf{Isom}(\mathbb{H}^{n_i})$ be a Zariski dense discrete subgroup and let μ_i be a Patterson–Sullivan measure for Γ_i of dimension δ_i. Suppose*

- $\sum_{\gamma \in \Gamma_1} e^{-\delta_1 \mathrm{d}(o, \gamma o)} = +\infty$ *for some $o \in \mathbb{H}^{n_1}$.*
- *There exists an onto homomorphism $\rho : \Gamma_1 \to \Gamma_2$ and a μ_1-a.e. defined measurable ρ-equivariant injective boundary map $f : \partial_\infty \mathbb{H}^{n_1} \to \partial_\infty \mathbb{H}^{n_2}$.*

If the measures $f_*\mu_1$ and μ_2 are not singular, then $n_1 = n_2$ and ρ extends to an isomorphism $\mathsf{Isom}(\mathbb{H}^{n_1}) \to \mathsf{Isom}(\mathbb{H}^{n_2})$.

Prior to Tukia's work, Sullivan [Sul82, Thm. 5] proved the above theorem in the special case when $\delta_1 = \delta_2$ and $n_1 = n_2$. Later Yue [Yue96] extended Tukia's theorem to discrete subgroups in isometry groups of negatively curved symmetric spaces.

In this paper, we define "Patterson–Sullivan systems" which consist of a group action and a quasi-invariant measure which behaves like a classical Patterson–Sullivan measure. More precisely, given a compact metrizable space M and a subgroup $\Gamma < \mathsf{Homeo}(M)$, a function $\sigma : \Gamma \times M \to \mathbb{R}$ is called a *κ-coarse-cocycle* if

$$|\sigma(\gamma_1\gamma_2, x) - (\sigma(\gamma_1, \gamma_2 x) + \sigma(\gamma_2, x))| \leq \kappa \tag{2}$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$. Given such a coarse-cocycle and $\delta \geq 0$, a Borel probability measure μ on M is called *coarse σ-Patterson–Sullivan measure of dimension δ* if there exists $C \geq 1$ such that for any $\gamma \in \Gamma$ the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$C^{-1} e^{-\delta\sigma(\gamma^{-1}, x)} \leq \frac{d\gamma_*\mu}{d\mu}(x) \leq C e^{-\delta\sigma(\gamma^{-1}, x)} \quad \text{for } \mu\text{-a.e. } x \in M. \tag{3}$$

When $C = 1$ and hence equality holds in Equation (3), we call μ a *σ-Patterson–Sullivan measure.*

Remark 1.2. We note that we *do not assume anything on the support* of a Patterson–Sullivan measure (e.g. supported on a minimal set). Further, in specific settings these measures are sometimes called *(quasi-)conformal densities.*

A Patterson–Sullivan system consists of a coarse Patterson–Sullivan measure, a collection of open sets called shadows, and a choice of magnitude function all of which satisfy certain properties (see Section 1.4 for the precise definition). The definition is quite robust and in Example 1.30 below we list a number of examples of Patterson–Sullivan systems.

In a classical setting, M is the boundary of real hyperbolic space, the coarse-cocycle is an actual cocycle (implicit in Equation (1)), the shadows are the geodesic shadows, and the magnitude of an element is the distance it translates a fixed basepoint.

For Patterson–Sullivan systems we prove a version of Tukia's measurable boundary rigidity theorem (Theorem 1.1). Before stating our general theorem in Section 1.4 below, we describe a number of applications.

1.1. **Random walks.**

In this section we describe applications of our main theorem towards the singularity conjecture for Patterson–Sullivan measures and stationary measures of random walks in a variety of settings.

One novelty in this work is the observation that the singularity conjecture can be studied via Tukia-type measurable boundary rigidity theorems.

1.1.1. *Random walks on Gromov hyperbolic spaces.* Suppose (X, d_X) is a proper geodesic Gromov hyperbolic metric space and $\Gamma < \mathsf{Isom}(X)$ is a non-elementary discrete subgroup. Let m be a probability measure on Γ whose support generates Γ as a semigroup, i.e.

$$\bigcup_{n\geq 1} [\operatorname{supp} \mathsf{m}]^n = \Gamma. \tag{4}$$

Consider the random walk $W_n = \gamma_1 \cdots \gamma_n$ where the γ_i's are independent identically distributed elements of Γ each with distribution m. Then, given $o \in X$, almost every sample path $W_n o \in X$ converges to a point in the Gromov boundary $\partial_\infty X$ [Kai00, Remark following Thm. 7.7] (see also [MT18]). Further,

$$\nu(A) := \operatorname{Prob}\left(\lim_{n\to\infty} W_n o \in A\right) \tag{5}$$

defines a Borel probability measure ν on $\partial_\infty X$ called the *hitting measure* (or *harmonic measure*) for the random walk associated to m, and is the unique m-*stationary measure* on $\partial_\infty X$, that is $\mathsf{m} * \nu = \nu$.

Fixing a basepoint $o \in X$, the *coarse Busemann cocycle* $\beta : \Gamma \times \partial_\infty X \to \mathbb{R}$ is the coarse-cocycle defined by

$$\beta(g, x) = \limsup_{p\to x} \mathrm{d}_X(p, g^{-1}o) - \mathrm{d}_X(p, o). \tag{6}$$

A *coarse Busemann Patterson–Sullivan measure* on $\partial_\infty X$ is a coarse β-Patterson–Sullivan measure in the sense of Equation (3).

We will apply our generalization of Theorem 1.1 to the following well-studied problem.

Problem 1.3 (Singularity Problem)**.** If m has finite support, determine when the m-stationary measure ν is singular to some/any coarse Busemann Patterson–Sullivan measure for Γ on $\partial_\infty X$.

In what follows, we will consider a slightly more general class of probability measures: The probability measure m has *finite superexponential moment* if

$$\sum_{\gamma\in\Gamma} c^{|\gamma|}\mathsf{m}(\gamma) < +\infty \tag{7}$$

for any $c > 1$, where $|\cdot|$ is the distance from the identity with respect to a word metric on Γ.

We first present some applications of one of our main results (Theorem 1.9) towards Problem 1.3. For *any finitely generated Kleinian group*, we obtain the following, which was previously known only for geometrically finite groups [GT20].

Corollary 1.4 (corollary of Theorem 1.5 and Corollary 1.7)**.** *Suppose $X = \mathbb{H}^3$, $\Gamma < \mathsf{Isom}(\mathbb{H}^3)$ is a non-elementary finitely generated discrete subgroup, and m has finite superexponential moment. If Γ is not convex cocompact, then the m-stationary measure ν is singular to every coarse Busemann Patterson–Sullivan measure of Γ on $\partial_\infty \mathbb{H}^3$. In particular, if Γ is not a cocompact lattice, then ν is singular to the Lebesgue measure class on $\partial_\infty \mathbb{H}^3$.*

Our results for general X involve relatively hyperbolic groups which is a class of finitely generated groups including word hyperbolic groups, whose definition we delay to Definition 2.1, and quasi-convex subgroups of $\mathsf{Isom}(X)$ which are discrete subgroups whose orbits are quasi-convex in X.

Theorem 1.5 (corollary of Theorem 1.9)**.** *Suppose Γ is relatively hyperbolic (as an abstract group) and m has finite superexponential moment. If Γ is not a quasi-convex subgroup of $\mathsf{Isom}(X)$, then the m-stationary measure ν is singular to every coarse Busemann Patterson–Sullivan measure on $\partial_\infty X$.*

Remark 1.6. In the special case when Γ is word hyperbolic, X admits a geometric group action, and m is symmetric, Theorem 1.5 is due to Blachère–Haïssinsky–Mathieu [BHM11, Prop. 5.5]. In the special case when Γ acts geometrically finitely on X (which implies it is relatively hyperbolic), Theorem 1.5 is due to Gekhtman–Tiozzo [GT20, Coro. 4.2].

Theorem 1.5, in full generality, is new even for negatively curved symmetric spaces. In this case, quasi-convex subgroups are convex cocompact subgroups, $\partial_\infty X$ has a smooth structure, and there is always a Busemann Patterson–Sullivan measure in the Lebesgue measure class. Using these facts, we will prove the following.

Corollary 1.7 (see Corollary 12.2 below)**.** *Suppose X is a negatively curved symmetric space, Γ is relatively hyperbolic (as an abstract group), and m has finite superexponential moment. If Γ is not a cocompact lattice in $\mathsf{Isom}(X)$, then the m-stationary measure ν is singular to the Lebesgue measure class on $\partial_\infty X$.*

Corollary 1.4 follows from Theorem 1.5 and Corollary 1.7. Indeed, when $X = \mathbb{H}^3$, every finitely generated non-elementary discrete subgroup of $\mathsf{Isom}(\mathbb{H}^3)$ is relatively hyperbolic relative to some (possibly empty) collection of peripheral subgroups which are virtually abelian. This can be deduced by Scott core theorem [Sco73] and Thurston's hyperbolization [Thu82] (see also [MT98, Thm. 4.10]).

Remark 1.8. In the special case when $X = \mathbb{H}^n$ is real hyperbolic space, $n \geq 3$, and Γ is a non-uniform lattice in $\mathsf{Isom}(\mathbb{H}^n)$, Corollary 1.7 is due to Randecker–Tiozzo [RT21]. When $X = \mathbb{H}^2$, this was obtained in different contexts [GLJ90, DKN09, KLP11, GMT15]. Further, Kosenko–Tiozzo [KT22] explicitly constructed cocompact lattices of $\mathsf{Isom}(\mathbb{H}^2)$ such that hitting measures are singular to the Lebesgue measure class on $\partial_\infty \mathbb{H}^2$.

In fact, we show that the non-singularity occurs precisely when any Γ-orbit is roughly isometric to the Green metric associated to the random walk. The *Green metric* on Γ is defined by

$$\mathrm{d}_G(g,h) = -\log \frac{G_{\mathsf{m}}(g,h)}{G_{\mathsf{m}}(\mathrm{id},\mathrm{id})} \quad \text{for } g,h \in \Gamma \tag{8}$$

where $G_{\mathsf{m}}(g,h) = \sum_{n=0}^{\infty} \mathsf{m}^{*n}(g^{-1}h)$ is the Green function. When m has finite superexponential moment and Γ is finitely generated and non-amenable, the Green metric d_G on Γ is quasi-isometric to a word metric with respect to a finite generating set [GT20, Prop. 7.8]. So Theorem 1.5 is a consequence of the following.

Theorem 1.9 (see Theorem 12.1 below)**.** *Suppose Γ is relatively hyperbolic (as an abstract group), m has finite superexponential moment, ν is the m-stationary measure, and μ is a coarse Busemann Patterson–Sullivan measure for Γ on $\partial_\infty X$ of dimension δ. Then the following are equivalent:*

1. *The measures ν and μ are not singular.*
2. *The measures ν and μ are in the same measure class and the Radon–Nikodym derivatives are a.e. bounded from above and below by a positive number.*
3. *For any $o \in X$,*
$$\sup_{\gamma\in\Gamma} |\mathrm{d}_G(\mathrm{id},\gamma) - \delta\, \mathrm{d}_X(o,\gamma o)| < +\infty.$$
In particular, Γ is quasi-convex and δ is the critical exponent of Γ.

When Γ is assumed to be a quasi-convex subgroup of $\mathsf{Isom}(X)$ (in particular, word hyperbolic) and m is symmetric, Theorem 1.9 was obtained by Blachère–Haïssinsky–Mathieu [BHM11, Thm. 1.5]. In the special case when Γ acts geometrically finitely on X (which implies it is relatively hyperbolic), Theorem 1.9 is due to Gekhtman–Tiozzo [GT20, Thm. 4.1]. For relatively hyperbolic groups, Dussaule–Gekhtman [DG20] proved an analogous statement for Patterson–Sullivan measure coming from a word metric on Γ.

1.1.2. *Random walks on mapping class groups and Teichmüller spaces.* Let Σ be a closed connected orientable surface of genus at least two, $\mathrm{Mod}(\Sigma)$ denote the mapping class group of Σ, and $(\mathcal{T}, \mathrm{d}_{\mathcal{T}})$ denote the Teichmüller space of Σ endowed with Teichmüller metric $\mathrm{d}_{\mathcal{T}}$.

Thurston [Thu88] compactified $\mathcal{T}$ by the space $\mathcal{PMF}$ of projective measured foliations on Σ. This compactification is called Thurston's compactification and $\mathcal{PMF}$ is also referred to as Thurston's boundary.

Let $\Gamma < \mathrm{Mod}(\Sigma)$ be a non-elementary subgroup (i.e. Γ is not virtually cyclic and contains a pseudo-Anosov element) and m a probability measure on Γ whose support generates Γ as a semigroup. Kaimanovich–Masur [KM96] showed that there exists a unique m-stationary measure ν on $\mathcal{PMF}$ and the subset $\mathcal{UE} \subset \mathcal{PMF}$ of uniquely ergodic foliations has full ν-measure. Further, for any $o \in \mathcal{T}$ the measure ν is the hitting measure for the associated random walk on the orbit $\Gamma(o) \subset \mathcal{T}$.

Analogous to Problem 1.3, Kaimanovich–Masur suggested the following.

Conjecture 1.10 (Kaimanovich–Masur [KM96, pg. 9])**.** If m has finite support, then the m-stationary measure ν is singular to every Busemann Patterson–Sullivan measure for Γ.

For a special type of Patterson–Sullivan measure which is of Lebesgue measure class on $\mathcal{PMF}$, Gadre [Gad14] proved the singularity of m-stationary measure for finitely supported m. Later, Gadre–Maher–Tiozzo [GMT17] extended this result to m with finite first moment with respect to a word metric as well.

To the best of our knowledge, Conjecture 1.10 is only known for the Lebesgue measure class. We also note that many subgroups of $\mathrm{Mod}(\Sigma)$ have limit sets with Lebesgue measure zero (e.g. handlebody groups [Mas86, Ker90]), which automatically implies that the stationary measure is singular to the Lebesgue measure class. See ([Mas82], [KL07, Sect. 3.3]) for more discussion on the nullity of limit sets.

As an application of our generalization of Tukia's theorem, we prove Conjecture 1.10 for a certain class of subgroups of $\mathrm{Mod}(\Sigma)$, showing the singularity of m-stationary measure and *any* Busemann Patterson–Sullivan measure. Before presenting the theorem, we first define Patterson–Sullivan measures in this context.

Gardiner–Masur [GM91] introduced another compactification by $\partial_{GM}\mathcal{T}$, called *Gardiner–Masur boundary* of $\mathcal{T}$, and proved that $\mathcal{PMF}$ is a proper subset of $\partial_{GM}\mathcal{T}$. Liu–Su [LS14] showed that $\partial_{GM}\mathcal{T}$ is the horofunction boundary of $(\mathcal{T}, \mathrm{d}_{\mathcal{T}})$. Hence, after fixing $o \in \mathcal{T}$, one can define a cocycle $\beta : \mathrm{Mod}(\Sigma) \times \partial_{GM}\mathcal{T} \to \mathbb{R}$ by

$$\beta(g, x) = \lim_{p \to x} \mathrm{d}_{\mathcal{T}}(p, g^{-1}o) - \mathrm{d}_{\mathcal{T}}(p, o)$$

where $p \in \mathcal{T}$ converges to $x \in \partial_{GM}\mathcal{T}$. A *Busemann Patterson–Sullivan measure* on $\partial_{GM}\mathcal{T}$ is a β-Patterson–Sullivan measure in the sense of Equation (3). These measures have been constructed and studied by several authors, including Coulon [Cou24] and Yang [Yan22].

We also note that Athreya–Bufetov–Eskin–Mirzakhani [ABEM12] constructed a Patterson–Sullivan measure for $\mathrm{Mod}(\Sigma)$ on $\mathcal{PMF}$ using Thurston measure, and Gekhtman [Gek12] constructed Patterson–Sullivan measures for convex cocompact subgroups of $\mathrm{Mod}(\Sigma)$ on $\mathcal{UE}$. Since the identity map $\mathcal{T} \to \mathcal{T}$ continuously extends to a topological embedding $\mathcal{UE} \hookrightarrow \partial_{GM}\mathcal{T}$ [Miy13], the Patterson–Sullivan measures constructed in [Gek12] are Patterson–Sullivan measures on $\partial_{GM}\mathcal{T}$. Further, by works of Masur [Mas82] and Veech [Vee82], the Patterson–Sullivan measure constructed in [ABEM12] gives a full measure on $\mathcal{UE}$, and therefore can be identified with a Busemann Patterson–Sullivan measure on $\partial_{GM}\mathcal{T}$.

Finally, since the m-stationary measure ν also gives a full measure on $\mathcal{UE}$, we can view ν as a measure on $\partial_{GM}\mathcal{T}$. Moreover, any measure on $\mathcal{PMF}$ is non-singular to ν on $\mathcal{PMF}$ if and only if its restriction on $\mathcal{UE}$ is non-singular to ν viewed as measures on $\partial_{GM}\mathcal{T}$.

We now state our contribution towards Conjecture 1.10.

Theorem 1.11 (see Corollary 12.4 below)**.** *Suppose Γ is relatively hyperbolic (as an abstract group) and m has finite superexponential moment. If Γ contains a multitwist, then the m-stationary measure ν is singular to every Busemann Patterson–Sullivan measures on $\partial_{GM}\mathcal{T}$.*

As explained above, Theorem 1.11 implies the same statement for Patterson–Sullivan measures on $\mathcal{PMF}$, such as the measures constructed in [ABEM12, Gek12]. Note also that Patterson–Sullivan measures under consideration do not have any assumptions on their supports. We also remark that in Theorem 1.11, the multitwist in Γ does not necessarily belong to a peripheral subgroup of Γ.

There are many examples of subgroups of $\mathrm{Mod}(\Sigma)$ which are relatively hyperbolic and containing multitwists, so Theorem 1.11 applies to. For instance, the combination theorem for Veech subgroups by Leininger–Reid [LR06] produces closed surface subgroups in $\mathrm{Mod}(\Sigma)$ with multitwists, and so-called parabolically geometrically finite subgroups introduced by Dowdall–Durham–Leininger–Sisto [DDLS24] are relatively hyperbolic and contain multitwists in their peripheral subgroups. Many examples of parabolically geometrically finite subgroups were also constructed by Udall [Uda25], Aougab–Bray–Dowdall–Hoganson–Maloni–Whitfield [ABD$^+$25], and Loa [Loa21]. Finally, in their proof of the purely pseudo-Anosov surface subgroup conjecture, Kent–Leininger [KL24] constructed a type-preserving homomorphism from a finite index subgroup of the fundamental group of the figure-8 knot complement into $\mathrm{Mod}(\Sigma)$ when Σ has genus at least 4. The image of such a homomorphism is relatively hyperbolic and contains a multitwist.

Theorem 1.11 will be a consequence of the following.

Theorem 1.12 (see Theorem 12.3 below)**.** *Suppose Γ is relatively hyperbolic (as an abstract group), m has a finite superexponential moment with the m-stationary measure ν, and μ is a Busemann Patterson–Sullivan measure for Γ on $\partial_{GM}\mathcal{T}$ of dimension δ. If the measures ν and μ are not singular, then:*

1. *For any $o \in \mathcal{T}$,*
$$\sup_{\gamma\in\Gamma} |\mathrm{d}_G(\mathrm{id},\gamma) - \delta\, \mathrm{d}_{\mathcal{T}}(o,\gamma o)| < +\infty.$$
In particular, δ is the critical exponent of Γ and $\sum_{\gamma\in\Gamma} e^{-\delta\, \mathrm{d}_{\mathcal{T}}(o,\gamma o)} = +\infty$.
2. *If d_w is a word metric on Γ with respect to a finite generating set, then the map*
$$\gamma \in (\Gamma, \mathrm{d}_w) \mapsto \gamma o \in (\mathcal{T}, \mathrm{d}_{\mathcal{T}})$$
is a quasi-isometric embedding.

For parabolically geometrically finite subgroups, we will also prove the converse of Theorem 1.12, see Theorem 12.5 below.

1.1.3. *Random walks on discrete subgroups of Lie groups.* Let G be a connected semisimple Lie group without compact factors and with finite center. Suppose $\Gamma < \mathsf{G}$ is a Zariski dense discrete subgroup, and m is a probability measure on Γ whose support generates Γ as a semigroup. Fix a minimal parabolic subgroup P and let $\mathcal{F} := \mathsf{G}/\mathsf{P}$ denote the Furstenberg boundary. Then there is a unique m-*stationary measure* ν on $\mathcal{F}$ [Fur73, GR85]. The measure ν is also referred to as the *Furstenberg measure.*

In this section we consider the following well-known conjecture (cf. Kaimanovich–Le Prince [KLP11]).

Conjecture 1.13 (Singularity conjecture)**.** If m has finite support, then the m-stationary measure ν is singular to the Lebesgue measure class on $\mathcal{F}$.

In [KZ25], we give an affirmative answer to the singularity conjecture when G has Kazhdan's property (T) and Γ is not a lattice. In this case, it is not necessary to assume any moment condition on m, and it suffices to have that $\operatorname{supp}\mathsf{m}$ generates Γ as a group, not necessarily as a semigroup.

In this paper, we consider the singularity conjecture for a more general class of measures, the "Iwasawa Patterson–Sullivan measures" introduced by Quint [Qui02a], and for general semisimple Lie groups.

Delaying precise definitions until Section 9, we fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of G, a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$, and a positive closed Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Then let $\Delta \subset \mathfrak{a}^*$ be the corresponding system of simple restricted roots, and let $\kappa : \mathsf{G} \to \mathfrak{a}^+$ denote the associated Cartan projection.

For the usage in later sections, we consider general flag manifolds. Given a non-empty subset $\theta \subset \Delta$, we let $\mathsf{P}_\theta < \mathsf{G}$ denote the associated parabolic subgroup and let $\mathcal{F}_\theta = \mathsf{G} / \mathsf{P}_\theta$ denote the associated partial flag manifold. We denote by $B_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ the *partial Iwasawa cocycle*, a vector valued cocycle whose image lies in a subspace $\mathfrak{a}_\theta \subset \mathfrak{a}$ associated to θ.

Given a functional $\phi \in \mathfrak{a}_\theta^*$ and a subgroup $\Gamma < \mathsf{G}$, a Borel probability measure μ on $\mathcal{F}_\theta$ is called a *coarse ϕ-Patterson–Sullivan measure for* Γ if it is a coarse $(\phi \circ B_\theta)$-Patterson–Sullivan measure for Γ in the sense of Equation (3). We refer to these measures as *coarse Iwasawa Patterson–Sullivan measures*.

In the case when $\mathsf{G} = \mathsf{Isom}_0(\mathbb{H}^n)$, $\Delta = \{\alpha\}$ consists of a single simple restricted root and $\mathcal{F}_\alpha$ naturally identifies with $\partial_\infty \mathbb{H}^n$. Employing the ball model for $\mathbb{H}^n$ with $o \in \mathbb{H}^n$ as the center of the ball so that $\partial_\infty \mathbb{H}^n = \mathbb{S}^{n-1}$,

$$|g'(x)|_{\partial_\infty \mathbb{H}^n} = e^{-(\alpha \circ B_\alpha)(g,x)}$$

for all $g \in \mathsf{G}$ and $x \in \partial_\infty \mathbb{H}^n$. So the above definitions encompasses the classical case described in Equation (1).

As $\mathcal{F} = \mathcal{F}_\Delta$ always supports a Iwasawa Patterson–Sullivan measure in the Lebesgue measure class [Qui02a, Lem. 6.3], it is natural to consider the following generalization of Conjecture 1.13.

Conjecture 1.14 (generalized Singularity conjecture)**.** If m has finite support, then the m-stationary measure ν is singular to every coarse Iwasawa Patterson–Sullivan measure on $\mathcal{F}$.

We prove that non-singularity implies strong restrictions on how a discrete subgroup embeds in G.

Theorem 1.15 (see Theorem 12.7 below)**.** *Suppose Γ is relatively hyperbolic (as an abstract group), m has finite superexponential moment, and μ is a coarse ϕ-Patterson–Sullivan measure on $\mathcal{F}$ of dimension δ. If the measures ν and μ are not singular, then:*

(1) $\sup_{\gamma \in \Gamma} |\mathrm{d}_G(\mathrm{id}, \gamma) - \delta\phi(\kappa(\gamma))| < +\infty$. *In particular,* $\sum_{\gamma \in \Gamma} e^{-\delta\phi(\kappa(\gamma))} = +\infty$ *and $\delta\phi \in \mathfrak{a}^*$ is tangent to the growth indicator of Γ.*

(2) If d_w is a word metric on Γ with respect to a finite generating set, (X, d_X) is the symmetric space associated to G, and $x_0 \in X$, then the map

$$\gamma \in (\Gamma, \mathrm{d}_w) \mapsto \gamma x_0 \in (X, \mathrm{d}_X)$$

is a quasi-isometric embedding.

For some classes of groups, it is easy to verify that the map in part (2) cannot be a quasi-isometric embedding.

Corollary 1.16 (see Corollary 12.8 below)**.** *Suppose Γ is word hyperbolic (as an abstract group) and m has finite superexponential moment. If Γ contains a unipotent element of G, then the m-stationary measure ν is singular to every coarse Iwasawa Patterson–Sullivan measure on $\mathcal{F}$.*

More generally, Corollary 1.16 holds when Γ is relatively hyperbolic and contains an element u which is unipotent (as an element of G) and the stable translation length of u is positive on a Cayley graph of Γ (e.g. u is loxodromic [DG20, Prop. 7.8]).

1.2. **Tukia's measurable boundary rigidity theorem.** In this section we describe special cases of our main theorem in a variety of settings.

1.2.1. *Tukia's theorem for word hyperbolic groups.* We establish a version of Tukia's theorem for word metrics on word hyperbolic groups, which implies that any measurable isomorphism between Gromov boundaries with respect to coarse Patterson–Sullivan measures always extends to a *homeomorphism*.

Theorem 1.17 (see Theorem 8.8 below). *For $i = 1, 2$ suppose Γ_i is a non-elementary word hyperbolic group endowed with a word metric d_i with respect to a finite generating set and μ_i is a coarse Busemann Patterson–Sullivan measure for Γ_i of dimension δ_i on $\partial_\infty\Gamma_i$. Assume there exist*

- *a homomorphism $\rho : \Gamma_1 \to \Gamma_2$ with non-elementary image and*
- *a μ_1-almost everywhere defined measurable ρ-equivariant injective map $f : \partial_\infty\Gamma_1 \to \partial_\infty\Gamma_2$.*

If $f_*\mu_1$ and μ_2 are not singular, then $\ker\rho$ is finite, $\rho(\Gamma_1) < \Gamma_2$ has finite index,

$$\sup_{\gamma_1,\gamma_2\in\Gamma_1} |\delta_1\,\mathrm{d}_1(\gamma_1,\gamma_2) - \delta_2\,\mathrm{d}_2(\rho(\gamma_1),\rho(\gamma_2))| < +\infty,$$

and there exists a ρ-equivariant homeomorphism $\tilde f : \partial_\infty\Gamma_1 \to \partial_\infty\Gamma_2$ such that

(1) $\tilde f = f$ μ_1-a.e.,
(2) $\tilde f_*\mu_1$, μ_2 are in the same measure class and the Radon–Nikodym derivatives are a.e. bounded from above and below by a positive number.

In fact we prove Theorem 1.17 for Patterson–Sullivan measures associated to a more general class of cocycles introduced in [BCZZ24b], see Definition 8.3 and Theorem 8.8.

Remark 1.18. Given two minimal convergence group actions $\Gamma_1 \curvearrowright M_1$ and $\Gamma_2 \curvearrowright M_2$ and an onto homomorphism $\rho : \Gamma_1 \to \Gamma_2$, it is known that any continuous ρ-equivariant map $f : M_1 \to M_2$ is injective on the so-called Myrberg limit set of Γ_1 [Ger12, Prop. 7.5.2] (see also [Yan22, Lem. 10.5]). Moreover, for a word hyperbolic group, the Myrberg limit set on its Gromov boundary is of full measure with respect to any coarse Busemann Patterson–Sullivan measure [Yan22, Thm. 1.14] (see also [Coo93, Cor. 7.3]). Hence, any continuous equivariant maps between Gromov boundaries of word hyperbolic groups satisfies the condition in Theorem 1.17.

1.2.2. *Tukia's theorem for Teichmüller spaces.* We establish a version of Tukia's theorem for Teichmüller spaces.

Theorem 1.19 (corollary to Theorems 1.28 and 10.1). *For $i = 1, 2$, let Σ_i be a closed connected orientable surface of genus at least two and $\mathcal{T}_i$ its Teichmüller space. Let $\Gamma_i < \mathrm{Mod}(\Sigma_i)$ be a non-elementary subgroup and μ_i a Busemann Patterson–Sullivan measure for Γ_i of dimension δ_i on $\partial_{GM}\mathcal{T}_i$. Suppose*

- $\displaystyle\sum_{\gamma\in\Gamma_1} e^{-\delta_1\,\mathrm{d}_{\mathcal{T}_1}(o_1,\gamma o_1)} = +\infty$ *for $o_1 \in \mathcal{T}_1$.*

- *There exists an onto homomorphism $\rho : \Gamma_1 \to \Gamma_2$ and a μ_1-almost everywhere defined measurable ρ-equivariant injective map $f : \partial_{GM} \mathcal{T}_1 \to \partial_{GM} \mathcal{T}_2$.*

If $f_*\mu_1$ and μ_2 are not singular, then for any $o_2 \in \mathcal{T}_2$, the orbit map $\gamma o_1 \mapsto \rho(\gamma) o_2$ is a rough isometry after scaling, i.e.,

$$\sup_{\gamma \in \Gamma_1} |\delta_1 \, \mathrm{d}_{\mathcal{T}_1}(o_1, \gamma o_1) - \delta_2 \, \mathrm{d}_{\mathcal{T}_2}(o_2, \rho(\gamma) o_2)| < +\infty.$$

Remark 1.20. As shown by Yang [Yan22], $\sum_{\gamma \in \Gamma_1} e^{-\delta_1 \, \mathrm{d}_{\mathcal{T}_1}(o_1, \gamma o_1)} = +\infty$ implies that $\mathcal{UE}(\Sigma_1) \subset \partial_{GM} \mathcal{T}_1$ has a full μ_1-measure. Hence, the boundary map f and measure μ_1 can be regarded to be defined on $\mathcal{PMF}(\Sigma_1)$, i.e. Thurston's boundary.

For a convex cocompact $\Gamma < \mathrm{Mod}(\Sigma)$, there exists a unique Γ-minimal subset of $\mathcal{PMF}$, called the limit set of Γ, and is the image of a Γ-equivariant embedding of $\partial_\infty \Gamma$ into $\mathcal{UE}$ [FM02, Prop. 3.2]. Moreover, if μ is a Patterson–Sullivan measure for Γ of dimension δ and $\sum_{\gamma \in \Gamma} e^{-\delta \, \mathrm{d}_{\mathcal{T}}(o, \gamma o)} = +\infty$, then μ is supported on the limit set of Γ [Gek12] (see also [Cou24, Yan22]). Hence, the boundary map f as in Theorem 1.19 always exists for two isomorphic convex cocompact subgroups. See also Remark 1.18.

1.2.3. *Tukia's theorem in higher rank.* Using the Iwasawa Patterson–Sullivan measures introduced in Section 1.1.3, we extend Tukia's theorem to a class of discrete subgroups in higher rank semisimple Lie groups called transverse groups, which can be viewed as a higher rank analogue of Kleinian groups. This class is defined in Section 9 and includes the Anosov and relatively Anosov subgroups and their subgroups. Further, any discrete subgroup of a rank one non-compact simple Lie group is transverse.

Theorem 1.21 (see Corollary 9.14 below)**.** *Let $\mathsf{G}_1, \mathsf{G}_2$ be non-compact simple Lie groups with trivial centers. Let $\Gamma < \mathsf{G}_1$ be a Zariski dense P_{θ_1}-transverse subgroup, μ a coarse ϕ-Patterson–Sullivan measure for Γ of dimension $\delta \geq 0$ on $\mathcal{F}_{\theta_1}$, and $\rho : \Gamma \to \mathsf{G}_2$ a representation with Zariski dense image. Suppose*

- $\displaystyle\sum_{\gamma \in \Gamma} e^{-\delta \phi(\kappa(\gamma))} = +\infty.$
- *There exists a μ-almost everywhere defined measurable ρ-equivariant injective map $f : \mathcal{F}_{\theta_1} \to \mathcal{F}_{\theta_2}$.*

If $f_*\mu$ is not singular to some coarse Iwasawa Patterson–Sullivan measure for $\rho(\Gamma)$, then ρ extends to a Lie group isomorphism $\mathsf{G}_1 \to \mathsf{G}_2$.

Remark 1.22.

(1) As in Margulis' superrigidity theorem, the representation ρ is not assumed to be discrete in Theorem 1.21, in contrast to Theorem 1.1 and Yue's generalization [Yue96].
(2) Theorem 1.21 follows from a more general statement (Corollary 9.14) about a non-elementary transverse subgroup of a semisimple Lie group and its irreducible representation into a semisimple Lie group.
(3) See Remark 9.15 for a version of the theorem for non-transverse Zariski dense discrete subgroups.

Remark 1.23. Theorem 1.21 was previously established in a variety of special cases. In all of these previous works, the representation ρ was assumed to be discrete faithful and the boundary map was assumed to be a topological embedding.

- Kim–Oh [KO24] considered the cases when either
  (1) G_1 is rank one, ρ is faithful, and $\rho(\Gamma)$ is P_{Δ_2}-divergent.
  (2) Γ is P_{Δ_1}-Anosov, ρ is faithful, and $\rho(\Gamma)$ is P_{Δ_2}-Anosov.
- Kim [Kim24] considered the case where Γ is P_{θ_1}-hypertransverse (P_{θ_1}-transverse with an extra assumption), ρ is faithful, and $\rho(\Gamma)$ is P_{θ_2}-divergent.
- Blayac–Canary–Zhu–Zimmer [BCZZ24b] considered the case where Γ is P_{θ_1}-transverse, ρ is faithful, and $\rho(\Gamma)$ is P_{θ_2}-transverse.

In contrast to these previous works, in Theorem 1.21, ρ does not need to be discrete or faithful, and the boundary map *does not even need to be continuous*. Further, in many natural settings the boundary maps will not be a topological embedding (e.g. Cannon–Thurston maps [CT07]), continuous, or even defined everywhere (e.g. maps between limit sets of isomorphic geometrically finite groups [Tuk95]).

1.3. **Entropy rigidity in pseudo-Riemannian hyperbolic geometry.** Delaying more definitions until Section 13, let $\mathbb{H}^{p,q}$ be pseudo-Riemannian hyperbolic space of signature (p,q). The group $\mathsf{PO}(p,q+1)$ acts by isometries on this pseudo-metric space and using this action Danciger–Guéritaud–Kassel [DGK18] introduced $\mathbb{H}^{p,q}$-convex cocompact subgroups of $\mathsf{PO}(p,q+1)$. Glorieux–Monclair [GM21] introduced a *critical exponent* $\delta_{\mathbb{H}^{p,q}}(\Gamma)$ for a convex cocompact subgroup $\Gamma < \mathsf{PO}(p,q)$ and proved that

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) \leq p-1.$$

The critical exponent $\delta_{\mathbb{H}^{p,q}}(\Gamma)$ is also referred to as *entropy* of Γ.

Using our version of Tukia's theorem for higher rank Lie groups (Theorem 1.21), we characterize the equality case.

Theorem 1.24 (see Theorem 13.2 below)**.** *If $\Gamma < \mathsf{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact and $\delta_{\mathbb{H}^{p,q}}(\Gamma) = p-1$, then Γ preserves and acts cocompactly on a totally geodesic copy of $\mathbb{H}^p$ in $\mathbb{H}^{p,q}$.*

Remark 1.25. A totally geodesic copy of $\mathbb{H}^k$ in $\mathbb{H}^{p,q}$ is a subset of the form $\mathbb{P}(V) \cap \mathbb{H}^{p,q}$ where $V \subset \mathbb{R}^{p+q+1}$ is a $(k+1)$-dimensional linear subspace and the associated bilinear form $[\cdot,\cdot]_{p,q+1}$ restricted to V has signature $(k,1)$.

A number of special cases of Theorem 1.24 have been previously established:

(1) $\mathbb{H}^{p,0}$ is real hyperbolic p-space and $\mathbb{H}^{p,0}$-convex cocompact coincides with the usual definition in real hyperbolic geometry. In this case, the above theorem follows from a result of Tukia [Tuk84], which also shows that a non-lattice geometrically finite group has critical exponent strictly less than $p-1$.
(2) Collier–Tholozan–Toulisse [CTT19] proved the above theorem when $p=2$ and Γ is the fundamental group of a closed surface.
(3) Mazzoli–Viaggi [MV23] proved the above theorem when Γ is the fundamental group of a closed p-manifold.

The techniques used in [CTT19, MV23] strongly use the fact that Γ is the fundamental group of a closed manifold and are very different than the approach taken here. In the proof of Theorem 1.24 we construct coarse Iwasawa Patterson–Sullivan measures on two different flag manifolds and show that there is a measurable map so that the push-forward of one of the measures is non-singular to the other. Then we use Theorem 1.21 to constrain the eigenvalues of elements in the group, which in turn constrains the Zariski closure of the group.

1.4. **Patterson–Sullivan systems.** We now define Patterson–Sullivan systems and then state our generalization of Tukia's theorem. In the classical setting of real hyperbolic geometry, "geodesic shadows" play a fundamental role in the study of Patterson–Sullivan measures and our definition of Patterson–Sullivan systems attempts to extract the key properties of these sets.

As in the beginning of the introduction, let M be a compact metric space and let $\Gamma < \mathsf{Homeo}(M)$ be a subgroup. Recall that coarse-cocycles and coarse Patterson–Sullivan measures were introduced in Equations (2) and (3).

Definition 1.26. A *Patterson–Sullivan-system (PS-system) of dimension* δ consists of

- a coarse-cocycle $\sigma : \Gamma \times M \to \mathbb{R}$,
- coarse σ-Patterson–Sullivan measure (PS-measure) μ of dimension δ,
- for each $\gamma \in \Gamma$, a number $\|\gamma\|_\sigma \in \mathbb{R}$ called the *σ-magnitude of* γ, and
- for each $\gamma \in \Gamma$ and $R > 0$, a non-empty open set $\mathcal{O}_R(\gamma) \subset M$ called the *R-shadow of* γ

such that:

(PS1) For any $\gamma \in \Gamma$, there exists $c = c(\gamma) > 0$ such that $|\sigma(\gamma, x)| \leq c(\gamma)$ for μ-a.e. $x \in M$.

(PS2) For every $R > 0$ there is a constant $C = C(R) > 0$ such that

$$\|\gamma\|_\sigma - C \leq \sigma(\gamma, x) \leq \|\gamma\|_\sigma + C$$

for all $\gamma \in \Gamma$ and μ-a.e. $x \in \gamma^{-1}\,\mathcal{O}_R(\gamma)$.

(PS3) If $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, $Z \subset M$ is compact, and $[M \smallsetminus \gamma_n^{-1}\,\mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance, then for any $x \in Z$, there exists $g \in \Gamma$ such that

$$gx \notin Z.$$

We call the PS-system *well-behaved* with respect to a collection

$$\mathcal{H} := \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$$

of non-increasing subsets of Γ if the following additional properties hold:

(PS4) Γ is countable and for any $T > 0$, the set $\{\gamma \in \mathcal{H}(0) : \|\gamma\|_\sigma \leq T\}$ is finite.

(PS5) If $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, $Z \subset M$ is compact, and $[M \smallsetminus \gamma_n^{-1}\,\mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance, then for any $h_1, \dots, h_m \in \Gamma$ and $x \in Z$, there exists $g \in \Gamma$ such that

$$gx \notin \bigcup_{i=1}^m h_i Z.$$

(PS6) If $R_1 \leq R_2$ and $\gamma \in \mathcal{H}(0)$, then $\mathcal{O}_{R_1}(\gamma) \subset \mathcal{O}_{R_2}(\gamma)$.

(PS7) For any $R > 0$ there exist $C > 0$ and $R' > 0$ such that: if $\alpha, \beta \in \mathcal{H}(R)$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$, and $\mathcal{O}_R(\alpha) \cap \mathcal{O}_R(\beta) \neq \emptyset$, then

$$\mathcal{O}_R(\beta) \subset \mathcal{O}_{R'}(\alpha)$$

and

$$\left| \|\beta\|_\sigma - (\|\alpha\|_\sigma + \|\alpha^{-1}\beta\|_\sigma) \right| \leq C.$$

(PS8) For every $R > 0$, there exists a set $M' \subset M$ of full μ-measure such that

$$\lim_{n \to \infty} \operatorname{diam} \mathcal{O}_R(\gamma_n) = 0$$

whenever $\{\gamma_n\} \subset \mathcal{H}(R)$ is an escaping sequence and

$$x \in M' \cap \bigcap_{n=1}^{\infty} \mathcal{O}_R(\gamma_n).$$

We call the collection $\mathcal{H}$ the *hierarchy* of the Patterson–Sullivan system.

Remark 1.27. Property (PS3) and the stronger Property (PS5) can be viewed as saying the action of Γ on Z is "irreducible" and "strongly irreducible" respectively.

For a well-behaved Patterson–Sullivan system (M, Γ, σ, μ) with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$, we consider the following analogue of the conical limit set:

$$\Lambda^{\mathrm{con}}(\mathcal{H}) := \left\{ x \in M : \begin{array}{c} \exists\, R > 0,\ \gamma \in \Gamma,\ \text{an escaping sequence } \{\gamma_n \in \mathcal{H}(n)\} \\ \text{s.t. } x \in \gamma\, \mathcal{O}_R(\gamma_n) \text{ for all } n \geq 1 \end{array} \right\}. \tag{9}$$

We now state our generalization of Tukia's rigidity theorem (Theorem 1.1) to PS-systems.

Theorem 1.28 (see Theorem 7.1 below)**.** *Suppose*

- $(M_1, \Gamma_1, \sigma_1, \mu_1)$ *is a well-behaved PS-system of dimension* δ_1 *with respect to a hierarchy* $\mathcal{H}_1 = \{\mathcal{H}_1(R) \subset \Gamma_1 : R \geq 0\}$ *and*
$$\mu_1(\Lambda^{\mathrm{con}}(\mathcal{H}_1)) = 1.$$
- $(M_2, \Gamma_2, \sigma_2, \mu_2)$ *is a PS-system of dimension* δ_2*.*
- *There exists an onto homomorphism* $\rho : \Gamma_1 \to \Gamma_2$ *and a* μ_1*-a.e. defined measurable* ρ*-equivariant injective map* $f : M_1 \to M_2$*.*

If the measures $f_*\mu_1$ *and* μ_2 *are not singular, then*

$$\sup_{\gamma \in \Gamma_1} \left| \delta_1 \left\| \gamma \right\|_{\sigma_1} - \delta_2 \left\| \rho(\gamma) \right\|_{\sigma_2} \right| < +\infty.$$

Remark 1.29. Although formulated differently, Theorem 1.28 contains Tukia's theorem as a special case. Under the hypothesis of Theorem 1.1, the Patterson–Sullivan measures μ_i are part of a well-behaved PS-system with respect to a trivial hierarchy $\mathcal{H}_i(R) \equiv \Gamma_i$ and with magnitude function

$$\gamma \mapsto \mathrm{d}_{\mathbb{H}^{n_i}}(o_i, \gamma o_i)$$

where $o_i \in \mathbb{H}^{n_i}$ is a basepoint. Further, the conical limit set defined in Equation (9) coincides with the classical conical limit set in hyperbolic geometry. The classical Hopf–Tsuji–Sullivan dichotomy then implies that $\mu_1(\Lambda^{\mathrm{con}}(\mathcal{H}_1)) = 1$ and hence Theorem 1.28 implies that

$$\sup_{\gamma \in \Gamma_1} |\delta_1\, \mathrm{d}_{\mathbb{H}^{n_1}}(o_1, \gamma o_1) - \delta_2\, \mathrm{d}_{\mathbb{H}^{n_2}}(o_2, \rho(\gamma) o_2)| < +\infty.$$

It then follows from marked length spectrum rigidity that $n_1 = n_2$ and ρ extends to an isomorphism $\mathsf{Isom}(\mathbb{H}^{n_1}) \to \mathsf{Isom}(\mathbb{H}^{n_2})$, as in Theorem 1.1. Similarly, Theorem 1.17, Theorem 1.19, and Theorem 1.21 are consequences of Theorem 1.28.

Example 1.30 (PS-systems)**.** Our abstract setting encompasses the following:

(1) Stationary measures on the Bowditch boundary of a relatively hyperbolic group associated to random walks with finite superexponential moments are contained in well-behaved PS-systems (see Section 11).

(2) Coarse Busemann PS-measures on the Gromov boundary of a proper geodesic Gromov hyperbolic space are contained in well-behaved PS-systems. More generally, coarse PS-measures associated to expanding coarse-cocycles (introduced in [BCZZ24b]) are contained in well-behaved PS-systems (see Section 8).
(3) Coarse Iwasawa PS-measures on a partial flag manifold associated to Zariski dense subgroups (more generally "P_θ-irreducible" subgroups) are always contained in PS-systems. When the subgroup is transverse and the measure is supported on the limit set, they are contained in well-behaved PS-systems (see Section 9; see also Theorem 9.12 for general Zariski dense discrete subgroups).
(4) Busemann PS-measures
    - on the Gardiner–Masur boundary $\partial_{GM}\mathcal{T}$ of Teichmüller space for non-elementary subgroups of a mapping class group,
    - on the geodesic boundary of a CAT(0)-space for discrete groups of isometries with rank one elements,

    are contained in well-behaved PS-systems (see Section 10 for a general discussion on group actions with contracting isometries).

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2. Preliminaries

2.1. **Possibly ambiguous notation/terminology.** We briefly define any possible ambiguous notation and terminology.

(1) A sequence $\{y_n\}$ in a countable set Y is *escaping* if it eventually leaves every finite set, i.e. if $F \subset Y$ is finite, then $\#\{n : y_n \in F\}$ is finite.
(2) Any connected semisimple Lie group G with trivial center is real algebraic [Zim84, Prop. 3.1.6]. Hence, Zariski density is defined for $\mathsf{H} < \mathsf{G}$, in the sense that no finite index subgroup of H is contained in a proper connected closed subgroup of G.
(3) Given a proper metric space X we endow the isometry group $\mathsf{Isom}(X)$ with the compact open topology. Then a subgroup $\Gamma < \mathsf{Isom}(X)$ is discrete if and only if it is countable and acts properly on X.

2.2. **The Hausdorff distance.** Suppose (M, d) is a compact metric space. Given a subset $C \subset M$ and $\epsilon > 0$, let $\mathcal{N}_\epsilon(C)$ denote the open ϵ-neighborhood of C with respect to d. The *Hausdorff distance* between two compact subsets $C_1, C_2 \subset M$ is

$$\mathrm{d}^{\mathrm{Haus}}(C_1, C_2) := \inf\{\epsilon : C_1 \subset \mathcal{N}_\epsilon(C_2) \text{ and } C_2 \subset \mathcal{N}_\epsilon(C_1)\}.$$

Notice that for the empty set we have

$$\mathrm{d}^{\mathrm{Haus}}(\emptyset, C) = \mathrm{d}^{\mathrm{Haus}}(C, \emptyset) = \begin{cases} 0 & \text{if } C = \emptyset \\ +\infty & \text{if } C \neq \emptyset \end{cases}.$$

This metric induces a compact topology on the space of compact subsets of M where $C_n \to C$ if and only if

$$\lim_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}(C_n, C) = 0.$$

Notice that the empty set is an isolated point: $C_n \to \emptyset$ if and only if $C_n = \emptyset$ for all n sufficiently large.

2.3. **Relatively hyperbolic groups.** There are several equivalent definitions of relatively hyperbolic groups and we state the definition we use in this paper.

Suppose $\Gamma < \mathsf{Homeo}(M)$ is a convergence group.

- A point $x \in M$ is a *conical limit point* of Γ if there are $a, b \in M$ distinct and $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n(x) \to a$ and $\gamma_n(y) \to b$ for all $y \in M \smallsetminus \{x\}$.
- An element $\gamma \in \Gamma$ is *parabolic* if it has infinite order and fixes exactly one point in M.
- A point $x \in M$ is a *parabolic fixed point* of Γ if the stabilizer $\mathrm{Stab}_\Gamma(x)$ is infinite and every infinite order element in $\mathrm{Stab}_\Gamma(x)$ is parabolic. A *bounded parabolic fixed point* $x \in M$ is a parabolic fixed point where the quotient $\mathrm{Stab}_\Gamma(x)\backslash(M - \{x\})$ is compact.
- Γ is a *geometrically finite convergence group* if every point in M is either a conical limit point or a bounded parabolic fixed point of Γ.

Definition 2.1. Given a finitely generated group Γ and a collection $\mathcal{P}$ of finitely generated infinite subgroups, we say that $(\Gamma, \mathcal{P})$ is *relatively hyperbolic*, if Γ acts on a compact perfect metrizable space M as a geometrically finite convergence group and the maximal parabolic subgroups are exactly the set

$$\{\gamma P \gamma^{-1} : P \in \mathcal{P}, \gamma \in \Gamma\}.$$

Given a relatively hyperbolic group $(\Gamma, \mathcal{P})$, any two compact perfect metrizable spaces satisfying Definition 2.1 are Γ-equivariantly homeomorphic (see [Bow12, Thm. 9.4]). This unique topological space is then denoted by $\partial(\Gamma, \mathcal{P})$ and called the *Bowditch boundary of* $(\Gamma, \mathcal{P})$.

Remark 2.2. Note that by definition we assume that a relatively hyperbolic group is non-elementary, finitely generated, and has finitely generated peripheral subgroups.

Part 1. Abstract PS-systems

3. Basic properties of PS-systems

In this section we observe some immediate consequences of the definitions introduced in Section 1.4.

Proposition 3.1 (Shadow Lemma)**.** *Let (M, Γ, σ, μ) be a PS-system of dimension $\delta \geq 0$. For any $R > 0$ sufficiently large there exists $C = C(R) > 1$ such that*

$$\frac{1}{C} e^{-\delta \|\gamma\|_\sigma} \leq \mu(\mathcal{O}_R(\gamma)) \leq C e^{-\delta \|\gamma\|_\sigma}$$

for all $\gamma \in \Gamma$ and

$$\inf_{\gamma \in \Gamma} \mu(\gamma^{-1}\, \mathcal{O}_R(\gamma)) > 0.$$

Proof. We first show that for any $R > 0$ sufficiently large,

$$\inf_{\gamma \in \Gamma} \mu(\gamma^{-1} \mathcal{O}_R(\gamma)) > 0. \tag{10}$$

Suppose not. Then for every $n \geq 1$ there exists $\gamma_n \in \Gamma$ with

$$\mu(\gamma_n^{-1} \mathcal{O}_n(\gamma_n)) < \frac{1}{n}.$$

Fix a metric on M which generates the topology. Passing to a subsequence, we can suppose that $M \smallsetminus \gamma_n^{-1} \mathcal{O}_n(\gamma_n)$ converges to a compact set Z with respect to the Hausdorff distance (note it is possible for Z to be the empty set, in which case $M \smallsetminus \gamma_n^{-1} \mathcal{O}_n(\gamma_n)$ is also empty for n sufficiently large).

Fix $\epsilon > 0$. Then $M \smallsetminus \gamma_n^{-1} \mathcal{O}_n(\gamma_n) \subset \mathcal{N}_\epsilon(Z)$ for n sufficiently large and hence

$$\mu(\mathcal{N}_\epsilon(Z)) \geq \lim_{n \to \infty} \mu(M \smallsetminus \gamma_n^{-1} \mathcal{O}_n(\gamma_n)) = 1.$$

Since $\epsilon > 0$ is arbitrary and Z is closed,

$$\mu(Z) = \lim_{n \to \infty} \mu(\mathcal{N}_{1/n}(Z)) = 1.$$

On the other hand, by Property (PS3), $\bigcap_{\gamma \in \Gamma} \gamma Z = \emptyset$. Since M is compact, there exist finitely many $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\bigcap_{i=1}^n \gamma_i Z = \emptyset$, which is a contradiction to $\mu(Z) = 1$ and the Γ-quasi-invariance of μ. Thus Equation (10) is true for sufficiently large $R > 0$.

Fix $R > 0$ satisfying Equation (10) and let $\epsilon_0 := \inf_{\gamma \in \Gamma} \mu(\gamma^{-1} \mathcal{O}_R(\gamma))$. Since

$$\mu(\mathcal{O}_R(\gamma)) = \int_{\gamma^{-1} \mathcal{O}_R(\gamma)} \frac{d\gamma_*^{-1}\mu}{d\mu} d\mu,$$

by Property (PS2), there exists $C = C(R) > 1$ such that

$$\frac{\epsilon_0}{C} e^{-\delta \|\gamma\|_\sigma} \leq \mu(\mathcal{O}_R(\gamma)) \leq C e^{-\delta \|\gamma\|_\sigma}. \qquad \square$$

We will use the following version of the Vitali covering lemma.

Lemma 3.2. *Let (M, Γ, σ, μ) be a well-behaved PS-system with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$. Let $R > 0$ and let $R' > 0$ be the constant satisfying Property (PS7) for R. Then for any $I \subset \mathcal{H}(R)$, there exists $J \subset I$ such that*

$$\bigcup_{\gamma \in I} \mathcal{O}_R(\gamma) \subset \bigcup_{\gamma \in J} \mathcal{O}_{R'}(\gamma)$$

and the shadows $\{\mathcal{O}_R(\gamma) : \gamma \in J\}$ are pairwise disjoint.

Proof. By Property (PS4) we can enumerate $I = \{\gamma_n\}$ so that

$$\|\gamma_1\|_\sigma \leq \|\gamma_2\|_\sigma \leq \|\gamma_3\|_\sigma \leq \cdots$$

Now we define indices $j_1 < j_2 < \cdots$ as follows. First let $j_1 = 1$. Then supposing $j_1, \ldots, j_k$ have been selected, let j_{k+1} be the smallest index greater than j_k such that

$$\mathcal{O}_R(\gamma_{j_{k+1}}) \cap \bigcup_{i=1}^{k} \mathcal{O}_R(\gamma_{j_i}) = \emptyset.$$

(This process could terminate after finitely many steps).

We claim that $J = \{\gamma_{j_k}\}$ has the desired properties. By construction, the shadows $\{\mathcal{O}_R(\gamma) : \gamma \in J\}$ are pairwise disjoint. For any $\gamma_n \in I \smallsetminus J$, we can pick k such that $j_k < n$ and that k is the maximal index with this property. Since $\gamma_n \notin J$, we must have

$$\mathcal{O}_R(\gamma_n) \cap \bigcup_{i=1}^{k} \mathcal{O}_R(\gamma_{j_i}) \neq \emptyset$$

and so

$$\mathcal{O}_R(\gamma_n) \subset \bigcup_{i=1}^{k} \mathcal{O}_{R'}(\gamma_{j_i})$$

by Property (PS7). Thus

$$\bigcup_{\gamma \in I} \mathcal{O}_R(\gamma) \subset \bigcup_{\gamma \in J} \mathcal{O}_{R'}(\gamma). \qquad \square$$

We will crucially use the following diagonal covering lemma several times in the arguments that follow. It applies in the case when $\Gamma < \mathsf{Homeo}(M_1)$ is part of a well-behaved PS-system and $\rho(\Gamma) < \mathsf{Homeo}(M_2)$ is part of a PS-system.

Lemma 3.3. *Let M_1, M_2 be compact metrizable spaces. Suppose $\Gamma < \mathsf{Homeo}(M_1)$ and $\rho : \Gamma \to \mathsf{Homeo}(M_2)$ is a homomorphism. If*

- *$Z_1 \subset M_1$, $Z_2 \subset M_2$ are compact,*
- *for any finitely many $h_1, \dots, h_m \in \Gamma$ and $x \in Z_1$, there exists $g \in \Gamma$ such that*
$$gx \notin \bigcup_{i=1}^{m} h_i Z_1,$$
and
- *for any $y \in Z_2$, there exists $h \in \Gamma$ such that $\rho(h)y \notin Z_2$,*

then we have

$$M_1 \times M_2 = \bigcup_{\gamma \in \Gamma} (M_1 \smallsetminus \gamma Z_1) \times (M_2 \smallsetminus \rho(\gamma) Z_2).$$

Proof. The third hypothesis implies that $\bigcap_{\gamma \in \Gamma} \rho(\gamma) Z_2 = \emptyset$. Since Z_2 is compact, there exist finitely many elements $h_1, \dots, h_m \in \Gamma$ such that

$$\rho(h_1) Z_2 \cap \dots \cap \rho(h_m) Z_2 = \emptyset.$$

Now suppose to the contrary that

$$\begin{aligned} C &:= M_1 \times M_2 \smallsetminus \bigcup_{\gamma \in \Gamma} (M_1 \smallsetminus \gamma Z_1) \times (M_2 \smallsetminus \rho(\gamma) Z_2) \\ &= \bigcap_{\gamma \in \Gamma} \big(M_1 \times \rho(\gamma) Z_2\big) \cup \big(\gamma Z_1 \times M_2\big) \end{aligned}$$

is non-empty. Let $(x, y) \in C$. Since C is invariant under the action of $\{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\}$, we have

$$(\gamma x, \rho(\gamma) y) \in C \quad \text{for all } \gamma \in \Gamma.$$

By the choice of $\{\rho(h_1), \dots, \rho(h_m)\}$, we have for some $j \in \{1, \dots, m\}$ that $y \notin \rho(h_j) Z_2$, and hence

$$(x, y) \in h_j Z_1 \times M_2.$$

In other words,

$$(h_j^{-1} x, \rho(h_j)^{-1} y) \in Z_1 \times M_2.$$

By the second hypothesis, there exists $g \in \Gamma$ such that

$$g(h_j^{-1}x) \notin \bigcup_{i=1}^{m} h_i Z_1.$$

On the other hand, there exists $i \in \{1, \dots, m\}$ such that $\rho(g)\rho(h_j)^{-1}y \notin \rho(h_i)Z_2$. Since $(gh_j^{-1}x, \rho(g)\rho(h_j)^{-1}y) \in C$, we have

$$(gh_j^{-1}x, \rho(g)\rho(h_j)^{-1}y) \in h_i Z_1 \times M_2.$$

In particular,

$$g(h_j^{-1}x) \in h_i Z_1.$$

This is a contradiction to the choice of $g \in \Gamma$. $\square$

4. An analogue of the conical limit set

Let (M, Γ, σ, μ) be a PS-system of dimension $\delta \geq 0$. In this section we introduce an analogue of the conical limit set and relate its measure to the divergence of the Poincaré series.

Given a subset $H \subset \Gamma$, let $\Lambda_R(H) \subset M$ be the set of points $x \in M$ where there exists an escaping sequence $\{\gamma_n\} \subset H$ and $R > 0$ such that

$$x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n).$$

Using this notation, the conical limit set of a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$ defined in Equation (9) can be rewritten as

$$\Lambda^{\mathrm{con}}(\mathcal{H}) = \Gamma \cdot \bigcup_{R>0} \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}(n)).$$

For simplicity, we denote by $\Lambda^{\mathrm{con}}(\Gamma)$ the conical limit set of the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$.

Theorem 4.1.

(1) If $\mu(\Lambda_R(H)) > 0$ for some $H \subset \Gamma$ and $R > 0$, then $\sum_{\gamma \in H} e^{-\delta \|\gamma\|_\sigma} = +\infty$.

(2) If (M, Γ, σ, μ) is well-behaved with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$ and $\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty$, then $\mu\left(\Lambda^{\mathrm{con}}(\Gamma)\right) = 1$.

Remark 4.2. In many examples, the shadows have the following additional property: for any $\alpha \in \Gamma$ and $R > 0$, there exists $R' > 0$ such that

$$\alpha\, \mathcal{O}_R(\gamma) \subset \mathcal{O}_{R'}(\alpha\gamma)$$

for all $\gamma \in \Gamma$. In this case, one has $\Lambda^{\mathrm{con}}(\Gamma) = \bigcup_{R>0} \Lambda_R(\Gamma)$.

4.1. **Proof of Theorem 4.1 part (1).** By Property (PS2), there exists $C = C(R) > 0$ such that for any $\gamma \in \Gamma$,

$$\mu(\mathcal{O}_R(\gamma)) = \gamma_*^{-1}\mu(\gamma^{-1}\, \mathcal{O}_R(\gamma)) \leq C e^{-\delta \|\gamma\|_\sigma}.$$

Now suppose $\sum_{\gamma \in H} e^{-\delta \|\gamma\|_\sigma} < +\infty$. Then H is countable and enumerating $H = \{\gamma_n\}$, we have

$$\Lambda_R(H) \subset \bigcup_{n \geq N} \mathcal{O}_R(\gamma_n) \quad \text{for all } N > 0.$$

Therefore, $\mu(\Lambda_R(H)) = 0$, which is a contradiction.

4.2. **Proof of Theorem 4.1 part (2).** The proof is exactly the same as the proof of [BCZZ24b, Prop. 7.1], which itself is similar to an earlier argument of Roblin [Rob03]. Since the proof is short, we include it here.

We use the following variant of Borel–Cantelli Lemma.

Lemma 4.3 (Kochen–Stone Lemma [KS64])**.** *Let (X, ν) be a finite measure space. If $\{A_n\} \subset X$ is a sequence of measurable sets where*

$$\sum_{n=1}^{\infty} \nu(A_n) = +\infty \quad \textit{and} \quad \liminf_{N\to\infty} \frac{\sum_{n,m=1}^{N} \nu(A_n \cap A_m)}{\left(\sum_{n=1}^{N} \nu(A_n)\right)^2} < +\infty,$$

then

$$\nu\left(\{x \in X : x \text{ is contained in infinitely many of } A_1, A_2, \dots\}\right) > 0.$$

Using the Shadow Lemma (Proposition 3.1), fix $R > 0$ and $C_1 > 1$ such that

$$\frac{1}{C_1} e^{-\delta\|\gamma\|_\sigma} \le \mu\Big(\mathcal{O}_R(\gamma)\Big) \le C_1 e^{-\delta\|\gamma\|_\sigma} \tag{11}$$

for all $\gamma \in \Gamma$. Using Property (PS4), we can fix an enumeration $\Gamma = \{\gamma_n\}$ such that

$$\|\gamma_1\|_\sigma \le \|\gamma_2\|_\sigma \le \cdots .$$

We will show that the sets $A_n := \mathcal{O}_R(\gamma_n)$ satisfy the hypothesis of the Kochen–Stone Lemma.

The first estimate follows immediately from the divergence of the Poincaré series

$$\sum_{n=1}^{\infty} \mu(A_n) \ge \frac{1}{C_1} \sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty.$$

The other estimate is only slightly more involved. Using Property (PS7), there exists $C_2' > 0$ such that: if $1 \le n \le m$ and $A_n \cap A_m \neq \emptyset$, then

$$\|\gamma_n\|_\sigma + \left\|\gamma_n^{-1}\gamma_m\right\|_\sigma \le \|\gamma_m\|_\sigma + C_2'.$$

Hence, in this case, $\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma \le \|\gamma_m\|_\sigma + C_2$ where $C_2 = C_2' - \|\gamma_1\|_\sigma$ and

$$\mu(A_n \cap A_m) \le \mu(A_m) \le C_1 e^{-\delta\|\gamma_m\|_\sigma} \le C_3 e^{-\delta\|\gamma_n\|_\sigma} e^{-\delta\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma}$$

where $C_3 := C_1 e^{\delta C_2'}$.

Let $f(N) := \max\{n : \|\gamma_n\|_\sigma \le \|\gamma_N\|_\sigma + C_2\}$, which is finite by Property (PS4). Then

$$\sum_{m,n=1}^{N} \mu(A_n \cap A_m) \le 2 \sum_{1\le n\le m\le N} \mu(A_n \cap A_m) \le 2C_3 \sum_{1\le n\le m\le N} e^{-\delta\|\gamma_n\|_\sigma} e^{-\delta\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma}$$
$$\le 2C_3 \sum_{n=1}^{N} e^{-\delta\|\gamma_n\|_\sigma} \sum_{n=1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma}.$$

Thus to apply the Kochen–Stone lemma, it suffices to observe the following.

Lemma 4.4. *There exists $C_4 > 0$ such that:*

$$\sum_{n=1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma} \le C_4 \sum_{n=1}^{N} e^{-\delta\|\gamma_n\|_\sigma}$$

for all $N \ge 1$.

Proof. Notice if $N < n \le m \le f(N)$ and $A_n \cap A_m \neq \emptyset$, then

$$\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma \le \|\gamma_m\|_\sigma - \|\gamma_n\|_\sigma + C_2' \le C_2 + C_2'.$$

Let $D := \#\{\gamma \in \Gamma : \|\gamma\|_\sigma \le C_2 + C_2'\}$, which is finite by Property (PS4). Then

$$\sum_{n=N+1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma} \le C_1 \sum_{n=N+1}^{f(N)} \mu(A_n) \le C_1 D \mu\left(\bigcup_{n=N+1}^{f(N)} A_n\right) \le C_1 D$$

where Equation (11) is applied in the first inequality. Hence

$$\sum_{n=1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma} \le \left(1 + C_1 D e^{\delta\|\gamma_1\|_\sigma}\right) \sum_{n=1}^{N} e^{-\delta\|\gamma_n\|_\sigma}. \qquad \square$$

So by the Kochen–Stone lemma, the set

$$\Lambda_R(\Gamma) = \{x \in M : x \text{ is contained in infinitely many of } A_1, A_2, \dots\}$$

has positive μ-measure. Hence $\mu(\Lambda^{\mathrm{con}}(\Gamma)) > 0$.

Suppose for a contradiction that $\mu(\Lambda^{\mathrm{con}}(\Gamma)) < 1$. Then

$$\mu'(\cdot) := \frac{1}{\mu(\Lambda^{\mathrm{con}}(\Gamma)^c)} \mu\left(\Lambda^{\mathrm{con}}(\Gamma)^c \cap \cdot\right)$$

is a σ-PS measure of dimension δ, and so by the argument above we must have $\mu'(\Lambda^{\mathrm{con}}(\Gamma)) > 0$, which is impossible. Hence $\mu(\Lambda^{\mathrm{con}}(\Gamma)) = 1$. $\square$

5. An analogue of the Lebesgue differentiation theorem

Let (M, Γ, σ, μ) be a well-behaved PS-system of dimension $\delta \ge 0$ with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \ge 0\}$. Fix $R_0 > 0$ such that any $R \ge R_0$ satisfies the Shadow Lemma (Proposition 3.1).

In this section we prove the following analogue of the Lebesgue differentiation theorem (which is known to hold for many particular PS-systems).

Theorem 5.1. *Fix $R \ge R_0$. If $h \in L^1(M, \mu)$, then for μ-a.e. $x \in M$ we have*

$$0 = \lim_{n \to \infty} \frac{1}{\mu(\gamma\, \mathcal{O}_R(\gamma_n))} \int_{\gamma\, \mathcal{O}_R(\gamma_n)} |h(y) - h(x)| d\mu(y),$$

and hence

$$h(x) = \lim_{n \to \infty} \frac{1}{\mu(\gamma\, \mathcal{O}_R(\gamma_n))} \int_{\gamma\, \mathcal{O}_R(\gamma_n)} h(y) d\mu(y),$$

whenever $x \in \bigcap_{n \ge 1} \gamma\, \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$.

Delaying the proof of the theorem, we state several corollaries. We will use Theorem 5.1 to prove that Γ acts ergodically.

Corollary 5.2. *If $\mu(\Lambda^{\mathrm{con}}(\mathcal{H})) = 1$, then the Γ-action on (M, μ) is ergodic. In particular, if the hierarchy is trivial (i.e. $\mathcal{H}(R) \equiv \Gamma$) and $\sum_{\gamma \in \Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty$, then the Γ-action on (M, μ) is ergodic.*

Corollary 5.2 is a consequence of Theorem 5.1 and the following lemma (which is itself a corollary of Theorem 5.1).

Lemma 5.3. *Fix $R \geq R_0$. If $E \subset M$ is measurable, then for μ-a.e. $x \in E$ we have*

$$0 = \lim_{n \to \infty} \mu \left(\gamma_n^{-1} \mathcal{O}_R(\gamma_n) \smallsetminus \gamma_n^{-1} \gamma^{-1} E \right)$$

whenever $x \in \bigcap_{n \geq 1} \gamma \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$.

Remark 5.4. Lemma 5.3 can be viewed as an analogue of the Lebesgue density theorem.

For use in Section 8 we also record the following corollary about approximate continuity of maps into separable metric spaces.

Corollary 5.5. *Fix $R \geq R_0$. If $F : M \to (Y, \mathrm{d}_Y)$ is a Borel measurable map into a separable metric space, then for μ-a.e. $x \in M$ we have*

$$0 = \lim_{n \to \infty} \frac{1}{\mu(\gamma \mathcal{O}_R(\gamma_n))} \mu \left(\{ y \in \gamma \mathcal{O}_R(\gamma_n) : \mathrm{d}_Y(F(x), F(y)) > \epsilon \} \right)$$

for all $\epsilon > 0$ whenever $x \in \bigcap_{n \geq 1} \gamma \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$.

The rest of the section is devoted to the proof of the theorem and the three corollaries.

5.1. **Proof of Theorem 5.1.**

Recall that any $R \geq R_0$ satisfies the Shadow Lemma (Proposition 3.1) and recall that $\Lambda_R(\mathcal{H}(R))$ is the set of points $x \in M$ such that $x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n)$ for some escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$.

Fix $R \geq R_0$ and $h \in L^1(M, \mu)$. For $\alpha \in \Gamma$, define functions $\mathcal{A}_\alpha h, \mathcal{B}_\alpha h : M \to [0, +\infty]$ by

$$\mathcal{A}_\alpha h(x) = \begin{cases} \lim\limits_{T \to \infty} \sup\limits_{\substack{\gamma \in \mathcal{H}(R) \\ \|\gamma\|_\sigma \geq T \\ x \in \alpha \mathcal{O}_R(\gamma)}} \frac{1}{\mu(\alpha \mathcal{O}_R(\gamma))} \int_{\alpha \mathcal{O}_R(\gamma)} |h(y) - h(x)| \, d\mu(y) & \text{if } x \in \alpha \Lambda_R(\mathcal{H}(R)) \\ 0 & \text{else} \end{cases}$$

and

$$\mathcal{B}_\alpha h(x) = \begin{cases} \lim\limits_{T \to \infty} \sup\limits_{\substack{\gamma \in \mathcal{H}(R) \\ \|\gamma\|_\sigma \geq T \\ x \in \alpha \mathcal{O}_R(\gamma)}} \frac{1}{\mu(\alpha \mathcal{O}_R(\gamma))} \int_{\alpha \mathcal{O}_R(\gamma)} |h(y)| \, d\mu(y) & \text{if } x \in \alpha \Lambda_R(\mathcal{H}(R)) \\ 0 & \text{else} \end{cases}.$$

Lemma 5.6. *If $\alpha \in \Gamma$, then $\mathcal{A}_\alpha h = 0$ μ-a.e.*

Proof. It suffices to show that $\mu(\{x : \mathcal{A}_\alpha h(x) > c\}) = 0$ for any $c > 0$. To that end, fix $c, \epsilon > 0$ and a continuous function $g : M \to \mathbb{R}$ with

$$\int_M |h - g| \, d\mu < \epsilon.$$

Then

$$\mathcal{A}_\alpha h(x) \leq \mathcal{B}_\alpha (h - g)(x) + |h(x) - g(x)| + \mathcal{A}_\alpha g(x).$$

Hence

$$\{x : \mathcal{A}_\alpha h(x) > c\} \subset N_1 \cup N_2 \cup N_3$$

where

$$N_1 := \{x : \mathcal{B}_\alpha(h-g)(x) > c/3\};$$
$$N_2 := \{x : |h(x) - g(x)| > c/3\};$$
$$N_3 := \{x : \mathcal{A}_\alpha g(x) > c/3\}.$$

Since g is continuous, Property (PS8) implies that $\mu(N_3) = 0$. Further,

$$\mu(N_2) \leq \frac{3}{c} \int_M |h-g| \, d\mu < \frac{3}{c}\epsilon.$$

To bound $\mu(N_1)$, we use Lemma 3.2. For any $x \in N_1$ there exists $\gamma_x \in \mathcal{H}(R)$ such that $x \in \alpha\, \mathcal{O}_R(\gamma_x)$ and

$$\int_{\alpha\, \mathcal{O}_R(\gamma_x)} |h-g| \, d\mu > \frac{c}{4}\mu(\alpha\, \mathcal{O}_R(\gamma_x)).$$

By Lemma 3.2 there exist $N_1' \subset N_1$ and $R' > R$ such that

$$N_1 \subset \bigcup_{x \in N_1} \alpha\, \mathcal{O}_R(\gamma_x) \subset \bigcup_{x \in N_1'} \alpha\, \mathcal{O}_{R'}(\gamma_x)$$

and the shadows $\{\alpha\, \mathcal{O}_R(\gamma_x) : x \in N_1'\}$ are disjoint. By Property (PS1), there exists $C_\alpha > 1$ such that

$$C_\alpha^{-1}\mu \leq \alpha_*^{-1}\mu \leq C_\alpha \mu.$$

Then by the Shadow Lemma (Proposition 3.1), there exists $C = C(\alpha, R, R') > 1$ such that

$$\mu(\alpha\, \mathcal{O}_{R'}(\gamma)) \leq C\mu(\alpha\, \mathcal{O}_R(\gamma))$$

for all $\gamma \in \Gamma$. Then

$$\begin{aligned}\mu(N_1) &\leq \sum_{x \in N_1'} \mu(\alpha\, \mathcal{O}_{R'}(\gamma_x)) \leq C \sum_{x \in N_1'} \mu(\alpha\, \mathcal{O}_R(\gamma_x)) < \frac{4C}{c} \sum_{x \in N_1'} \int_{\alpha\, \mathcal{O}_R(\gamma_x)} |h-g| \, d\mu \\ &\leq \frac{4C}{c} \int_M |h-g| \, d\mu < \frac{4C}{c}\epsilon.\end{aligned}$$

Thus

$$\mu(\{x : \mathcal{A}_\alpha h(x) > c\}) \leq \mu(N_1) + \mu(N_2) + \mu(N_3) < \frac{4C}{c}\epsilon + \frac{3}{c}\epsilon + 0.$$

Since $\epsilon > 0$ was arbitrary, we see that $\{x : \mathcal{A}_\alpha h(x) > c\}$ is μ-null. Then since $c > 0$ was arbitrary, $\mathcal{A}_\alpha h = 0$ μ-a.e. □

We now finish the proof of Theorem 5.1. Fix $h \in L^1(M, \mu)$ and set

$$M' := \bigcap_{\alpha \in \Gamma} \{x : \mathcal{A}_\alpha h(x) = 0\}.$$

Then $\mu(M') = 1$ by Lemma 5.6.

Fix $x \in M'$ and suppose that

$$x \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_R(\gamma_n)$$

for some $\gamma \in \Gamma$ and an escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$. Then

$$\limsup_{n \to \infty} \frac{1}{\mu(\gamma\, \mathcal{O}_R(\gamma_n))} \int_{\gamma\, \mathcal{O}_R(\gamma_n)} |h(y) - h(x)| \, d\mu(y) \leq \mathcal{A}_\gamma h(x) = 0,$$

completing the proof. $\square$

5.2. **Proof of Lemma 5.3.** Fix $R \geq R_0$ and a measurable set $E \subset M$.

For each $g \in \Gamma$, consider the function $\mathbf{1}_{g^{-1}E}$. Then by Theorem 5.1, we have a measurable subset $M_g \subset M$ such that $\mu(M_g) = 1$ and for any $y \in M_g \cap g^{-1}E$,

$$1 = \lim_{n \to \infty} \frac{\mu(g^{-1}E \cap \gamma\, \mathcal{O}_R(\gamma_n))}{\mu(\gamma\, \mathcal{O}_R(\gamma_n))}$$

whenever $y \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$. Set

$$M_E := \bigcap_{g \in \Gamma} gM_g.$$

Since μ is Γ-quasi-invariant, $\mu(M_E) = 1$.

Fix $x \in E \cap M_E$ and suppose that $x \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$. We then have $\gamma^{-1}x \in M_\gamma \cap \gamma^{-1}E$ and moreover $\gamma^{-1}x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n)$. Therefore

$$1 = \lim_{n \to \infty} \frac{\mu(\gamma^{-1}E \cap \mathcal{O}_R(\gamma_n))}{\mu(\mathcal{O}_R(\gamma_n))} = \lim_{n \to \infty} \frac{(\gamma_n^{-1}{}_*\mu)(\gamma_n^{-1}\gamma^{-1}E \cap \gamma_n^{-1}\, \mathcal{O}_R(\gamma_n))}{(\gamma_n^{-1}{}_*\mu)(\gamma_n^{-1}\, \mathcal{O}_R(\gamma_n))}.$$

In particular,

$$\lim_{n \to \infty} \frac{(\gamma_n^{-1}{}_*\mu)(\gamma_n^{-1}\gamma^{-1}E^c \cap \gamma_n^{-1}\, \mathcal{O}_R(\gamma_n))}{(\gamma_n^{-1}{}_*\mu)(\gamma_n^{-1}\, \mathcal{O}_R(\gamma_n))} = 0.$$

By Property (PS2), there exists $C = C(R) > 1$ such that

$$Ce^{-\delta\|\gamma_n\|} \leq \frac{d\gamma_n^{-1}{}_*\mu}{d\mu} \leq Ce^{-\delta\|\gamma_n\|} \quad \mu\text{-a.e.}$$

on $\gamma_n^{-1}\, \mathcal{O}_R(\gamma_n)$. So

$$\lim_{n \to \infty} \frac{\mu(\gamma_n^{-1}\gamma^{-1}E^c \cap \gamma_n^{-1}\, \mathcal{O}_R(\gamma_n))}{\mu(\gamma_n^{-1}\, \mathcal{O}_R(\gamma_n))} = 0.$$

Since $\mu(\gamma_n^{-1}\, \mathcal{O}_R(\gamma_n)) \leq 1$, we then have

$$\lim_{n \to \infty} \mu(\gamma_n^{-1}\gamma^{-1}E^c \cap \gamma_n^{-1}\, \mathcal{O}_R(\gamma_n)) = 0,$$

which implies that

$$0 = \lim_{n \to \infty} \mu(\gamma_n^{-1}\, \mathcal{O}_R(\gamma_n) \smallsetminus \gamma_n^{-1}\gamma^{-1}E).$$

$\square$

5.3. **Proof of Corollary 5.2.** Once we show the first statement, the second follows from Theorem 4.1.

Recall that $\Lambda^{\mathrm{con}}(\mathcal{H}) = \Gamma \cdot \bigcup_{R>0} \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}(n))$, which is assumed to have full μ-measure. We show that the Γ-action on (M, μ) is ergodic using Lemma 5.3. Let $E \subset M$ be a Γ-invariant measurable set with $\mu(E) > 0$. Since the sequence $\Gamma \cdot \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}(n))$ is non-decreasing in R by Property (PS6), there exists $R \geq R_0$ such that $\mu(E \cap \Gamma \cdot \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}(n))) > 0$.

Fix a sequence $R_k \to +\infty$. For each $k \geq 1$, let $M_k \subset M$ a full measure set satisfying Lemma 5.3. We then set $M_E := \bigcap_{k \geq 1} M_k$ which is of μ-full measure.

Fix $x \in E \cap M_E \cap \Gamma \cdot \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}(n))$. Then there exist $\gamma \in \Gamma$ and an escaping sequence $\{\gamma_n \in \mathcal{H}(n)\}$ such that

$$x \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_R(\gamma_n).$$

Since the hierarchy $\mathcal{H}$ consists of a non-increasing sequence of subsets of Γ, for each $k \geq 1$, we have $\gamma_n \in \mathcal{H}(R_k)$ for all large $n \geq 1$. Then by Property (PS6), Lemma 5.3, and the Γ-invariance of E,

$$0 = \lim_{n \to \infty} \mu(\gamma_n^{-1}\, \mathcal{O}_{R_k}(\gamma_n) \smallsetminus E).$$

Hence, after passing to a subsequence of $\{\gamma_n\}$, we have

$$0 = \lim_{n \to \infty} \mu(\gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n) \smallsetminus E).$$

Fix a metric on M which generates the topology. Passing to a subsequence, we can suppose that $M \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n)$ converges to some compact set $Z \subset M$ with respect to the Hausdorff distance (it is possible for $Z = \emptyset$, in which case $M \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n) = \emptyset$ for n sufficiently large).

Then for each $j \geq 1$,

$$M \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n) \subset \mathcal{N}_{1/j}(Z)$$

when n is sufficiently large. Therefore

$$\begin{aligned}
\mu((M \smallsetminus Z) \smallsetminus E) &\leq \mu((M \smallsetminus \mathcal{N}_{1/j}(Z)) \smallsetminus E) + \mu(\mathcal{N}_{1/j}(Z) \smallsetminus Z) \\
&\leq \lim_{n \to \infty} \mu(\gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n) \smallsetminus E) + \mu(\mathcal{N}_{1/j}(Z) \smallsetminus Z) \\
&= \mu(\mathcal{N}_{1/j}(Z) \smallsetminus Z).
\end{aligned}$$

Since Z is closed, $\mathcal{N}_{1/j}(Z) \smallsetminus Z$ is a decreasing sequence of sets whose limit is the empty set. Therefore, taking $j \to +\infty$, we have

$$\mu((M \smallsetminus Z) \smallsetminus E) = 0.$$

By Property (PS3), $M = \bigcup_{\gamma \in \Gamma} M \smallsetminus \gamma Z$. Therefore, it follows from the Γ-invariance of E and the Γ-quasi-invariance of μ that

$$\mu(M \smallsetminus E) \leq \sum_{\gamma \in \Gamma} \mu((M \smallsetminus \gamma Z) \smallsetminus E) = \sum_{\gamma \in \Gamma} \gamma_*^{-1} \mu((M \smallsetminus Z) \smallsetminus E) = 0.$$

This shows $\mu(E) = 1$, finishing the proof. $\square$

5.4. **Proof of Corollary 5.5.** Fix $R \geq R_0$ and fix a countable dense subset $D = \{z_n\} \subset Y$. For $k \in \mathbb{N}$ define $f_k : M \to \mathbb{N}$ by letting

$$f_k(x) = \min\{n : \mathrm{d}_Y(F(x), z_n) < 1/k\}.$$

Then for $K \in \mathbb{N}$ let $h_{k,K}(x) = \min\{f_k(x), K\}$. Each $h_{k,K}$ is bounded and hence in $L^1(M, \mu)$. Then there exists a full measure set M' such that Theorem 5.1 holds for every $x \in M'$ and every $h_{k,K}$, for our given $R \geq R_0$.

Now fix $x \in M'$ and $\epsilon > 0$. Then fix $k \in \mathbb{N}$ with $\frac{1}{2k} < \epsilon$ and fix $K \in \mathbb{N}$ with $f_k(x) < K$. Then for $y \in M$,

$$\mathrm{d}_Y(F(x), F(y)) > \epsilon \Rightarrow |h_{k,K}(x) - h_{k,K}(y)| \geq 1.$$

So whenever $x \in \bigcap_{n\geq 1} \gamma\, \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$, we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \frac{1}{\mu(\gamma\, \mathcal{O}_R(\gamma_n))} \mu\left(\{y \in \gamma\, \mathcal{O}_R(\gamma_n) : \mathrm{d}_Y(F(x), F(y)) > \epsilon\}\right) \\ &\leq \lim_{n\to\infty} \frac{1}{\mu(\gamma\, \mathcal{O}_R(\gamma_n))} \int_{\gamma\, \mathcal{O}_R(\gamma_n)} |h_{k,K}(x) - h_{k,K}(y)|\, d\mu(y) = 0. \end{aligned}$$

□

6. Mixed Shadows and a Shadow Lemma

For the rest of the section suppose

- $(M_1, \Gamma_1, \sigma_1, \mu_1)$ is a well-behaved PS-system of dimension δ_1 with respect to a hierarchy $\mathcal{H}_1 = \{\mathcal{H}_1(R) \subset \Gamma_1 : R \geq 0\}$.
- $(M_2, \Gamma_2, \sigma_2, \mu_2)$ is a PS-system of dimension δ_2.
- There exists an onto homomorphism $\rho : \Gamma_1 \to \Gamma_2$ and a measurable ρ-equivariant map $f : Y \to M_2$ where $Y \subset M_1$ is a Γ_1-invariant subset of full μ_1-measure.

In this section we introduce mixed shadows, which play a key role in our main rigidity result, and prove a version of the Shadow Lemma.

Definition 6.1. For $R > 0$ and $\gamma \in \Gamma$, the associated *mixed shadow* is

$$\mathcal{O}_R^f(\gamma) := \mathcal{O}_R(\gamma) \cap f^{-1}(\mathcal{O}_R(\rho(\gamma))) \cap Y \subset M_1.$$

Theorem 6.2 (Mixed Shadow Lemma)**.**

1. *For any sufficiently large $R > 0$, there exists $C = C(R) > 1$ such that*
$$\frac{1}{C} e^{-\delta_1 \|\gamma\|_{\sigma_1}} \leq \mu_1\left(\mathcal{O}_R^f(\gamma)\right) \leq C e^{-\delta_1 \|\gamma\|_{\sigma_1}}$$
for all $\gamma \in \Gamma$.
2. *Suppose, in addition, that f maps Borel subsets of Y to Borel subsets of M_2 and $\mu_2(f(Y)) > 0$. Then for any sufficiently large $R > 0$, there exists $C = C(R) > 1$ such that*
$$\frac{1}{C} e^{-\delta_2 \|\rho(\gamma)\|_{\sigma_2}} \leq \mu_2\left(f\left(\mathcal{O}_R^f(\gamma)\right)\right) \leq C e^{-\delta_2 \|\rho(\gamma)\|_{\sigma_2}}$$
for all $\gamma \in \Gamma$.

Delaying the proof of the theorem for a moment, we establish the following corollary.

Theorem 6.3. *There exists $R_0 > 0$ such that: if $R \geq R_0$ and $h \in L^1(M_1, \mu_1)$, then for μ_1-a.e. $x \in M_1$ we have*

$$h(x) = \lim_{n\to\infty} \frac{1}{\mu_1\left(\mathcal{O}_R^f(\gamma_n)\right)} \int_{\mathcal{O}_R^f(\gamma_n)} h(y) d\mu_1(y)$$

whenever $x \in \bigcap_{n\geq 1} \mathcal{O}_R(\gamma_n)$ for some escaping sequence $\{\gamma_n\} \subset \mathcal{H}_1(R)$.

Proof. Fix $R_0 > 0$ such that any $R \geq R_0$ satisfies Proposition 3.1 for $(M_1, \Gamma_1, \sigma_1, \mu_1)$ and Theorem 6.2 part (1). Fix $R \geq R_0$ and $h \in L^1(M_1, \mu_1)$. Let $M_1' \subset M_1$ be a full μ_1-measure set satisfying Theorem 5.1 for h and R.

Now fix $x \in M_1'$ and an escaping sequence $\{\gamma_n\} \subset \mathcal{H}_1(R)$ where $x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n)$. By Theorem 5.1,

$$0 = \lim_{n \to \infty} \frac{1}{\mu_1(\mathcal{O}_R(\gamma_n))} \int_{\mathcal{O}_R(\gamma_n)} |h(y) - h(x)| d\mu_1(y).$$

By our choice of R_0, there exists $C = C(R) > 1$ such that

$$\mu_1\left(\mathcal{O}_R^f(\gamma_n)\right) \geq C \mu_1(\mathcal{O}_R(\gamma_n)).$$

Then, since $\mathcal{O}_R^f(\gamma_n) \subset \mathcal{O}_R(\gamma_n)$,

$$\begin{aligned}
0 \leq \frac{1}{\mu_1\left(\mathcal{O}_R^f(\gamma_n)\right)} & \int_{\mathcal{O}_R^f(\gamma_n)} |h(y) - h(x)| d\mu_1(y) \\
& \leq \frac{1}{\mu_1\left(\mathcal{O}_R^f(\gamma_n)\right)} \int_{\mathcal{O}_R(\gamma_n)} |h(y) - h(x)| d\mu_1(y) \\
& \leq \frac{C}{\mu_1(\mathcal{O}_R(\gamma_n))} \int_{\mathcal{O}_R(\gamma_n)} |h(y) - h(x)| d\mu_1(y) \to 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left| h(x) - \lim_{n \to \infty} \frac{1}{\mu_1\left(\mathcal{O}_R^f(\gamma_n)\right)} \int_{\mathcal{O}_R^f(\gamma_n)} h(y) d\mu_1(y) \right| \\
&\qquad \leq \lim_{n \to \infty} \frac{1}{\mu_1\left(\mathcal{O}_R^f(\gamma_n)\right)} \int_{\mathcal{O}_R^f(\gamma_n)} |h(y) - h(x)| d\mu_1(y) = 0.
\end{aligned}$$

□

6.1. **Proof of Theorem 6.2.** Fix metrics on M_1, M_2 which induce their topologies. As in the proof of the classical Shadow Lemma, we start by proving lower bounds for translates of shadows.

Lemma 6.4. *For any sufficiently large $R > 0$,*

$$\inf_{\gamma \in \Gamma_1} \mu_1\left(\gamma^{-1} \mathcal{O}_R^f(\gamma)\right) > 0.$$

Proof. Suppose not. Then there exist sequences $R_n \to +\infty$ and $\{\gamma_n\} \subset \Gamma_1$ such that

$$\mu_1\left(\gamma_n^{-1} \mathcal{O}_{R_n}^f(\gamma_n)\right) < \frac{1}{n} \quad \text{for all} \quad n \geq 1.$$

Since $\mu_1(Y) = 1$ and f is ρ-equivariant,

$$\mu_1\Big(\gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n) \cap f^{-1}\big(\rho(\gamma_n)^{-1} \mathcal{O}_{R_n}(\rho(\gamma_n))\big)\Big) < \frac{1}{n}.$$

Note that

$$\begin{aligned}
M_1 \smallsetminus & \Big(\gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n) \cap f^{-1}\big(\rho(\gamma_n)^{-1} \mathcal{O}_{R_n}(\rho(\gamma_n))\big)\Big) \\
&= \Big(M_1 \smallsetminus \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n)\Big) \cup \Big(M_1 \smallsetminus f^{-1}\big(\rho(\gamma_n)^{-1} \mathcal{O}_{R_n}(\rho(\gamma_n))\big)\Big) \\
&= \Big(M_1 \smallsetminus \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n)\Big) \cup f^{-1}\Big(M_2 \smallsetminus \rho(\gamma_n)^{-1} \mathcal{O}_{R_n}(\rho(\gamma_n))\Big).
\end{aligned}$$

After passing to a subsequence, we can assume that

$$[M_1 \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n)] \to Z_1$$

for some (possibly empty) compact subset $Z_1 \subset M_1$ with respect to the Hausdorff distance and

$$[M_2 \smallsetminus \rho(\gamma_n)^{-1}\, \mathcal{O}_{R_n}(\rho(\gamma_n))] \to Z_2$$

for some (possibly empty) compact subset $Z_2 \subset M_2$ with respect to the Hausdorff distance.

For any $\epsilon > 0$ and $n \geq 1$ sufficiently large (depending on ϵ),

$$M_1 \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n) \subset \mathcal{N}_\epsilon(Z_1) \quad \text{and} \quad M_2 \smallsetminus \rho(\gamma_n)^{-1}\, \mathcal{O}_{R_n}(\rho(\gamma_n)) \subset \mathcal{N}_\epsilon(Z_2).$$

Hence

$$\mu_1\Big(\mathcal{N}_\epsilon(Z_1) \cup f^{-1}(\mathcal{N}_\epsilon(Z_2))\Big) > 1 - 1/n$$

for all large $n \geq 1$. Taking the limit $n \to \infty$, we have

$$\mu_1\Big(\mathcal{N}_\epsilon(Z_1) \cup f^{-1}(\mathcal{N}_\epsilon(Z_2))\Big) = 1.$$

Since Z_1 and Z_2 are closed,

$$Z_1 \cup f^{-1}(Z_2) = \bigcap_{k \geq 1} \mathcal{N}_{1/k}(Z_1) \cup f^{-1}(\mathcal{N}_{1/k}(Z_2)).$$

We therefore have $\mu_1(Z_1 \cup f^{-1}(Z_2)) = 1$. In other words,

$$\mu_1\left((M_1 \smallsetminus Z_1) \cap f^{-1}(M_2 \smallsetminus Z_2)\right) = 0, \tag{12}$$

and hence

$$\mu_1\left(\bigcup_{\gamma \in \Gamma_1} (M_1 \smallsetminus \gamma Z_1) \cap f^{-1}(M_2 \smallsetminus \rho(\gamma) Z_2)\right) = 0$$

by the Γ_1-quasi-invariance of μ_1. However then Lemma 3.3 implies $\mu_1(M_1) = 0$, contradiction. $\square$

Lemma 6.5. *Suppose that f maps Borel subsets of Y to Borel subsets of M_2 and $\mu_2(f(Y)) > 0$. For any sufficiently large $R > 0$,*

$$\inf_{\gamma \in \Gamma_1} \mu_2\left(\rho(\gamma)^{-1} f\left(\mathcal{O}_R^f(\gamma)\right)\right) > 0.$$

Proof. Suppose not. Then there exist sequences $R_n \to +\infty$ and $\{\gamma_n\} \subset \Gamma_1$ such that

$$\mu_2\left(\rho(\gamma_n)^{-1} f\left(\mathcal{O}_{R_n}^f(\gamma_n)\right)\right) < \frac{1}{n}.$$

Then, since f is ρ-equivariant,

$$\mu_2\left(f\left(\gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n) \cap Y\right) \cap \rho(\gamma_n)^{-1}\, \mathcal{O}_{R_n}(\rho(\gamma_n))\right) \to 0. \tag{13}$$

After passing to a subsequence, we can assume that

$$[M_1 \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n)] \to Z_1$$

for some (possibly empty) compact subset $Z_1 \subset M_1$ with respect to the Hausdorff distance and

$$[M_2 \smallsetminus \rho(\gamma_n)^{-1}\, \mathcal{O}_{R_n}(\rho(\gamma_n))] \to Z_2$$

for some (possibly empty) compact subset $Z_2 \subset M_2$ with respect to the Hausdorff distance.

By Lemma 3.3,

$$M_1 \times M_2 = \bigcup_{\gamma \in \Gamma_1} (M_1 \smallsetminus \gamma Z_1) \times (M_2 \smallsetminus \rho(\gamma) Z_2)$$

and hence

$$M_1 \times M_2 = \bigcup_{\epsilon > 0} \bigcup_{\gamma \in \Gamma_1} \left(M_1 \smallsetminus \gamma \overline{\mathcal{N}_\epsilon(Z_1)} \right) \times \left(M_2 \smallsetminus \rho(\gamma) \overline{\mathcal{N}_\epsilon(Z_2)} \right).$$

By compactness, we can fix $\epsilon > 0$ and a finite set $F \subset \Gamma$ such that

$$M_1 \times M_2 = \bigcup_{\gamma \in F} \left(M_1 \smallsetminus \gamma \overline{\mathcal{N}_{2\epsilon}(Z_1)} \right) \times \left(M_2 \smallsetminus \rho(\gamma) \overline{\mathcal{N}_{2\epsilon}(Z_2)} \right). \tag{14}$$

Now for $n \geq 1$ sufficiently large,

$$M_1 \smallsetminus \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n) \subset \mathcal{N}_\epsilon(Z_1) \quad \text{and} \quad M_2 \smallsetminus \rho(\gamma_n)^{-1} \mathcal{O}_{R_n}(\rho(\gamma_n)) \subset \mathcal{N}_\epsilon(Z_2)$$

and hence

$$M_1 \smallsetminus \mathcal{N}_\epsilon(Z_1) \subset \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n) \quad \text{and} \quad M_2 \smallsetminus \mathcal{N}_\epsilon(Z_2) \subset \rho(\gamma_n)^{-1} \mathcal{O}_{R_n}(\rho(\gamma_n)).$$

So, by Equation (13),

$$\mu_2 \left(f \left(Y \smallsetminus \mathcal{N}_\epsilon(Z_1) \right) \cap \left(M_2 \smallsetminus \mathcal{N}_\epsilon(Z_2) \right) \right) = 0.$$

Then, since μ_2 is $\rho(\Gamma)$-quasi-invariant,

$$\mu_2 \left(\bigcup_{\gamma \in F} f \left(Y \smallsetminus \gamma \mathcal{N}_\epsilon(Z_1) \right) \cap \left(M_2 \smallsetminus \rho(\gamma) \mathcal{N}_\epsilon(Z_2) \right) \right) = 0.$$

Then Equation (14) implies that $\mu_2(f(Y)) = 0$, which is a contradiction. $\square$

With the lower bounds in Lemmas 6.4 and 6.5, one can complete the proof of Theorem 6.2 by arguing exactly same as in Proposition 3.1. $\square$

7. The Main Theorem

In this section we prove Theorem 1.28, which we restate here.

Theorem 7.1. *Suppose*

- $(M_1, \Gamma_1, \sigma_1, \mu_1)$ *is a well-behaved PS-system of dimension* δ_1 *with respect to a hierarchy* $\mathcal{H}_1 = \{\mathcal{H}_1(R) \subset \Gamma_1 : R \geq 0\}$ *and*
$$\mu_1(\Lambda^{\mathrm{con}}(\mathcal{H}_1)) = 1.$$
- $(M_2, \Gamma_2, \sigma_2, \mu_2)$ *is a PS-system of dimension* δ_2.
- *There exists an onto homomorphism* $\rho : \Gamma_1 \to \Gamma_2$, *a measurable* Γ_1*-invariant set* Y *with full* μ_1*-measure, and a measurable* ρ*-equivariant injective map* $f : Y \to M_2$.

If the measures $f_* \mu_1$ *and* μ_2 *are not singular, then*

$$\sup_{\gamma \in \Gamma_1} \left| \delta_1 \left\| \gamma \right\|_{\sigma_1} - \delta_2 \left\| \rho(\gamma) \right\|_{\sigma_2} \right| < +\infty.$$

Remark 7.2. By Theorem 4.1, when we have the trivial hierarchy $\mathcal{H}_1(R) \equiv \Gamma_1$, the condition $\mu_1(\Lambda^{\mathrm{con}}(\mathcal{H}_1)) = 1$ in Theorem 7.1 is equivalent to

$$\sum_{\gamma \in \Gamma_1} e^{-\delta_1 \left\| \gamma \right\|_{\sigma_1}} = +\infty.$$

7.1. **Proof of Theorem 7.1.** The rest of the section is devoted to the proof the theorem. For notational convenience, we write $\|\cdot\|_i = \|\cdot\|_{\sigma_i}$ for $i = 1, 2$.

Suppose that $f_*\mu_1$ and μ_2 are not singular. Since $f|_Y$ is injective and M_1, M_2 are compact and metrizable, f maps Borel subsets of Y to Borel subsets of M_2 [Kec95, Coro. 15.2]. Hence

$$\tilde{\mu}_2 := \mu_2(f(Y \cap \cdot)) \tag{15}$$

defines a finite Borel measure on M_1.

Lemma 7.3. *The Borel measure $\tilde{\mu}_2$ is non-zero, and after possibly replacing Y with a subset, we can assume that $\tilde{\mu}_2 \asymp \mu_1$ (i.e., $\tilde{\mu}_2 \ll \mu_1$ and $\tilde{\mu}_2 \gg \mu_1$).*

Proof. Decompose

$$\tilde{\mu}_2 = \tilde{\mu}_2' + \tilde{\mu}_2''$$

where $\tilde{\mu}_2' \ll \mu_1$ and $\tilde{\mu}_2''$ is singular to μ_1.

Suppose for a contradiction that $\tilde{\mu}_2'$ is the zero measure. Then $\tilde{\mu}_2$ is singular to μ_1. Then there exists a measurable subset $Y' \subset Y \subset M_1$ such that $\mu_1(Y') = 1$ and $\tilde{\mu}_2(Y') = 0$. Then

$$f_*\mu_1(f(Y')) \geq \mu_1(Y') = 1$$

and

$$\mu_2(f(Y')) = \tilde{\mu}_2(Y') = 0.$$

Hence μ_2 and $f_*\mu_1$ are singular, which is a contradiction. So $\tilde{\mu}_2'$ is not the zero measure. In particular, $\tilde{\mu}_2$ is non-zero.

Now fix a measurable subset $A \subset Y$ such that $\mu_1(A) = 1$ and $\tilde{\mu}_2''(A) = 0$. Since μ_1 is Γ_1-quasi-invariant, $A' := \bigcap_{\gamma \in \Gamma_1} \gamma A$ also has full μ_1-measure and so by replacing Y with A' we can assume that $\tilde{\mu}_2 \ll \mu_1$.

Suppose for a contradiction that $\mu_1 \not\ll \tilde{\mu}_2$. Then there exists a measurable subset $B \subset Y$ where $\mu_1(B) > 0$ and $\tilde{\mu}_2(B) = 0$. Since the Γ_1-action on (M_1, μ_1) is ergodic (Corollary 5.2), $\mu_1(\Gamma_1 \cdot B) = 1$. Since μ_2 is Γ_2-quasi-invariant and $\mu_2(f(Y \cap B)) = \tilde{\mu}_2(B) = 0$,

$$\tilde{\mu}_2(\Gamma_1 \cdot B) \leq \sum_{\gamma \in \Gamma_1} \mu_2(\rho(\gamma) f(Y \cap B)) = 0.$$

Hence μ_1 and $\tilde{\mu}_2$ are singular, which contradicts the fact that $\tilde{\mu}_2 \ll \mu_1$. So $\mu_1 \ll \tilde{\mu}_2$ and thus $\mu_1 \asymp \tilde{\mu}_2$. $\square$

By Lemma 7.3, we can consider the following Radon–Nykodim derivative:

$$h := \frac{d\tilde{\mu}_2}{d\mu_1} \in L^1(M_1, \mu_1).$$

Since μ_1, μ_2 are PS-measures, h satisfies the following.

Lemma 7.4. *There exists $C_1 \geq 0$ such that for any $\gamma \in \Gamma_1$ and μ_1-a.e. $x \in M_1$,*

$$e^{-C_1 + \delta_1 \sigma_1(\gamma, x) - \delta_2 \sigma_2(\rho(\gamma), f(x))} \cdot h(x) \leq h(\gamma x) \leq e^{C_1 + \delta_1 \sigma_1(\gamma, x) - \delta_2 \sigma_2(\rho(\gamma), f(x))} \cdot h(x).$$

Proof. Since μ_2 is a coarse σ_2-PS measure of dimension δ_2, there exists $c_1 \geq 0$ such that

$$e^{-c_1 - \delta_2 \sigma_2(\rho(\gamma), y)} \leq \frac{d\rho(\gamma^{-1})_*\mu_2}{d\mu_2}(y) \leq e^{c_1 - \delta_2 \sigma_2(\rho(\gamma), y)}$$

for all $\gamma \in \Gamma_1$ and μ_2-a.e. $y \in M_2$. Since Y is Γ-invariant and f is ρ-equivariant, we have for a measurable $A \subset M_1$ that

$$\gamma_*^{-1}\tilde{\mu}_2(A) = \mu_2(f(Y \cap \gamma A)) = \mu_2(\rho(\gamma)f(Y \cap A)) = \rho(\gamma)_*^{-1}\mu_2(f(Y \cap A)).$$

Hence

$$e^{-c_1-\delta_2\sigma_2(\rho(\gamma),f(x))} \leq \frac{d\gamma_*^{-1}\tilde{\mu}_2}{d\tilde{\mu}_2}(x) \leq e^{c_1-\delta_2\sigma_2(\rho(\gamma),f(x))} \tag{16}$$

for all $\gamma \in \Gamma_1$ and $\tilde{\mu}_2$-a.e. $x \in M_1$. Since $\mu_1 \asymp \tilde{\mu}_2$, this equation holds for μ_1-a.e. $x \in M_1$.

Since μ_1 is a coarse σ_1-PS measure of dimension δ_1, there exists $c_2 \geq 0$ such that

$$e^{-c_2-\delta_1\sigma_1(\gamma,x)} \leq \frac{d\gamma_*^{-1}\mu_1}{d\mu_1}(x) \leq e^{c_2-\delta_1\sigma_1(\gamma,x)} \tag{17}$$

for all $\gamma \in \Gamma_1$ and μ_1-a.e. $x \in M_1$.

Finally, for any $\gamma \in \Gamma_1$,

$$\frac{d\gamma_*^{-1}\tilde{\mu}_2}{d\tilde{\mu}_2} h d\mu_1 = d\gamma_*^{-1}\tilde{\mu}_2 = d\gamma_*^{-1}(h\mu_1) = (h \circ \gamma) d\gamma_*^{-1}\mu_1.$$

Combining with Equations (16) and (17), we get the desired inequalities with $C_1 := c_1 + c_2$. $\square$

Since $\mu_1 \asymp \tilde{\mu}_2$ and Γ_1 is countable, using Property (PS1) we can replace Y with a Γ-invariant full μ_1-measure subset such that for all $\gamma \in \Gamma_1$,

$$\sup_{x \in Y} |\sigma_1(\gamma, x)| < +\infty \quad \text{and} \quad \sup_{x \in Y} |\sigma_2(\rho(\gamma), f(x))| < +\infty. \tag{18}$$

Since

$$1 = \mu_1(\Lambda^{\rm con}(\mathcal{H}_1)) = \mu_1\left(\Gamma \cdot \bigcup_{R>0} \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}_1(n))\right) > 0$$

and μ_1 is Γ_1-quasi-invariant, we can fix $R > 0$ such that $\bigcap_{n\geq 1} \Lambda_R(\mathcal{H}_1(n))$ has positive μ_1-measure. Since $\mu_1 \asymp \tilde{\mu}_2$, h is positive and finite μ_1-a.e. and thus we can fix $n_0 \geq 1$ sufficiently large so that the set

$$E := \{x \in Y : n_0^{-1} \leq h(x) \leq n_0\} \cap \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}_1(n))$$

has positive μ_1-measure.

Fix a sequence $R_n \to +\infty$ with $R_n \geq R$ for all n. After possibly increasing $R > 0$, we can assume that

- R satisfies Theorem 6.2,
- there is a subset $M_1' \subset M_1$ of full μ_1-measure that satisfies Theorem 6.3 for h and R, and satisfies Lemma 5.3 for E and all R_n.

Fix

$$x_0 \in E \cap M_1'.$$

Since $x_0 \in E \subset \bigcap_{n\geq 1} \Lambda_R(\mathcal{H}_1(n))$, there exists an escaping sequence $\{\gamma_n \in \mathcal{H}_1(n)\}$ such that

$$x_0 \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n).$$

Since $\{\gamma_n\}_{n>R} \subset \mathcal{H}_1(R)$ and $x_0 \in M_1'$, we have

$$h(x_0) = \lim_{n\to\infty} \frac{1}{\mu_1(\mathcal{O}_R^f(\gamma_n))} \int_{\mathcal{O}_R^f(\gamma_n)} h d\mu_1 = \lim_{n\to\infty} \frac{\tilde{\mu}_2(\mathcal{O}_R^f(\gamma_n))}{\mu_1(\mathcal{O}_R^f(\gamma_n))}.$$

Since $x_0 \in E$, we have $h(x_0) \in [n_0^{-1}, n_0]$. Further, since R satisfies Theorem 6.2, there exists $C_2 = C_2(R) > 1$ such that

$$\begin{aligned} &\frac{1}{C_2} e^{-\delta_1 \|\gamma\|_1} \leq \mu_1(\mathcal{O}_R^f(\gamma)) \leq C_2 e^{-\delta_1\|\gamma\|_1} \quad \text{and} \\ &\frac{1}{C_2} e^{-\delta_2 \|\rho(\gamma)\|_2} \leq \tilde{\mu}_2(\mathcal{O}_R^f(\gamma)) \leq C_2 e^{-\delta_2\|\rho(\gamma)\|_2} \end{aligned} \tag{19}$$

for all $\gamma \in \Gamma_1$. Thus

$$C_3 := \sup_{n\geq 1} \left|\delta_1 \left\|\gamma_n\right\|_1 - \delta_2 \left\|\rho(\gamma_n)\right\|_2\right| \tag{20}$$

is finite.

Using Lemma 3.3 we will prove the following covering lemma.

Proposition 7.5. *There exist $R' > 0$, $\alpha_1, \dots, \alpha_m \in \Gamma_1$, and $M_1'' \subset M_1$ with full μ_1-measure with the following property: for any $x \in M_1''$ there exist $1 \leq i \leq m$ and $n \in \mathbb{N}$ such that*

$$x \in \alpha_i \gamma_n^{-1} E$$

and

$$(x, f(x)) \in \left(\alpha_i \gamma_n^{-1} \, \mathcal{O}_{R'}(\gamma_n)\right) \times \left(\rho(\alpha_i)\rho(\gamma_n)^{-1} \, \mathcal{O}_{R'}(\rho(\gamma_n))\right).$$

Delaying the proof of the proposition, we complete the proof of Theorem 7.1.

Lemma 7.6. *There exists $D > 1$ such that*

$$D^{-1} \leq h(x) \leq D$$

for μ_1-a.e. $x \in M_1$.

Proof. Let $R' > 0$, $\alpha_1, \dots, \alpha_m \in \Gamma_1$, and $M_1'' \subset M_1$ be as in Proposition 7.5.

We start by fixing some constants. Fix $\kappa > 0$ such that σ_1, σ_2 are both κ-coarse-cocycles. Since Γ is countable and $\mu_1 \asymp \tilde{\mu}_2$, using Property (PS2) we can fix $C_4 > 0$ and replace M_1'' with a full μ_1-measure subset such that: if $x \in M_1''$ and $\gamma \in \Gamma_1$, then

$$\left|\sigma_1(\gamma, x) - \|\gamma\|_1\right| \leq C_4$$

whenever $x \in \gamma^{-1} \, \mathcal{O}_{R'}(\gamma)$ and

$$\left|\sigma_2(\rho(\gamma), f(x)) - \|\rho(\gamma)\|_2\right| \leq C_4$$

whenever $f(x) \in \rho(\gamma)^{-1} \, \mathcal{O}_{R'}(\rho(\gamma))$. Replacing M_1'' by $M_1'' \cap Y$, we can also assume that $M_1'' \subset Y$ and hence

$$C_5 := \max_{1\leq i \leq m} \max \left\{ \sup_{y\in M_1''} \left|\sigma_1(\alpha_i^{-1}, y)\right|, \sup_{y \in M_1''} \left|\sigma_2(\rho(\alpha_i)^{-1}, f(y))\right| \right\} < +\infty$$

is finite, see Equation (18). Again replacing M_1'' with a full μ_1-measure subset we can also assume that the estimate in Lemma 7.4 holds for all $x \in M_1''$ and all $\gamma \in \Gamma_1$. Finally, since μ_1 is Γ_1-quasi-invariant and Γ_1 is countable, we can replace M_1'' by $\bigcap_{\gamma\in\Gamma} \gamma M_1''$ and assume that M_1'' is Γ_1-invariant.

Fix $x \in M_1''$. By Proposition 7.5, there exist $1 \le i \le m$ and $n \in \mathbb{N}$ such that

$$x \in \alpha_i \gamma_n^{-1} E \cap \alpha_i \gamma_n^{-1} \mathcal{O}_{R'}(\gamma_n) \quad \text{and} \quad f(x) \in \rho(\alpha_i)\rho(\gamma_n)^{-1} \mathcal{O}_{R'}(\rho(\gamma_n)).$$

By Lemma 7.4,

$$\begin{aligned} & e^{-C_1 - \delta_1 \sigma_1(\gamma_n \alpha_i^{-1}, x) + \delta_2 \sigma_2(\rho(\gamma_n)\rho(\alpha_i)^{-1}, f(x))} h(\gamma_n \alpha_i^{-1} x) \\ & \le h(x) \le e^{C_1 - \delta_1 \sigma_1(\gamma_n \alpha_i^{-1}, x) + \delta_2 \sigma_2(\rho(\gamma_n)\rho(\alpha_i)^{-1}, f(x))} h(\gamma_n \alpha_i^{-1} x). \end{aligned}$$

Further, since $\alpha_i^{-1} x \in \gamma_n^{-1} \mathcal{O}_{R'}(\gamma_n) \cap M_1''$, we have

$$\begin{aligned} \left| \sigma_1(\gamma_n \alpha_i^{-1}, x) - \|\gamma_n\|_1 \right| & \le \kappa + \left| \sigma_1(\gamma_n, \alpha_i^{-1} x) + \sigma_1(\alpha_i^{-1}, x) - \|\gamma_n\|_1 \right| \\ & \le \kappa + C_4 + C_5. \end{aligned}$$

Likewise,

$$\left| \sigma_2(\rho(\gamma_n)\rho(\alpha_i)^{-1}, f(x)) - \|\rho(\gamma_n)\|_2 \right| \le \kappa + C_4 + C_5.$$

Since $\gamma_n \alpha_i^{-1} x \in E$,

$$n_0^{-1} \le h(\gamma_n \alpha_i^{-1} x) \le n_0.$$

Finally notice that

$$|\delta_1 \|\gamma_n\|_1 - \delta_2 \|\rho(\gamma_n)\|_2| \le C_3 < +\infty$$

by Equation (20). Thus

$$D^{-1} \le h(x) \le D,$$

where $D := e^{C_1 + C_3 + (\delta_1 + \delta_2)(\kappa + C_4 + C_5)} n_0$. □

Recalling that $h = \frac{d\tilde{\mu}_2}{d\mu_1}$, it follows from Lemma 7.6 that

$$D^{-1} \mu_1(\mathcal{O}_R^f(\gamma)) \le \tilde{\mu}_2(\mathcal{O}_R^f(\gamma)) \le D \mu_1(\mathcal{O}_R^f(\gamma))$$

for all $\gamma \in \Gamma_1$. Therefore, by Equation (19), we have the desired estimate:

$$\sup_{\gamma \in \Gamma} \left| \delta_1 \|\gamma\|_{\sigma_1} - \delta_2 \|\rho(\gamma)\|_{\sigma_2} \right| < +\infty.$$

Now the proof of Theorem 7.1 is complete once we show Proposition 7.5.

7.2. **Proof of Proposition 7.5.** Fix metrics on M_1 and M_2 inducing their topologies. For each $j \ge 1$ fix a subsequence $\{\tilde{\gamma}_{j,n}\} \subset \{\gamma_n\}$ so that

$$[M_1 \smallsetminus \tilde{\gamma}_{j,n}^{-1} \mathcal{O}_{R_j}(\tilde{\gamma}_{j,n})] \to Z_j \quad \text{and} \quad [M_2 \smallsetminus \rho(\tilde{\gamma}_{j,n})^{-1} \mathcal{O}_{R_j}(\rho(\tilde{\gamma}_{j,n}))] \to Z_j'$$

for some (possibly empty) compact subsets $Z_j \subset M_1$ and $Z_j' \subset M_2$ with respect to the Hausdorff distance. Then passing to a subsequence of $\{R_j\}$, we can assume that

$$Z_j \to Z \quad \text{and} \quad Z_j' \to Z'$$

for some (possibly empty) compact subsets $Z \subset M_1$ and $Z' \subset M_2$ with respect to the Hausdorff distance.

By a diagonal argument, we can extract a subsequence $\{\gamma_{n_j}\} \subset \{\gamma_n\}$ so that

$$[M_1 \smallsetminus \gamma_{n_j}^{-1} \mathcal{O}_{R_j}(\gamma_{n_j})] \to Z \quad \text{and} \quad [M_2 \smallsetminus \rho(\gamma_{n_j}^{-1}) \mathcal{O}_{R_j}(\rho(\gamma_{n_j}))] \to Z'$$

with respect to the Hausdorff distance. Since $(M_1, \Gamma_1, \sigma_1, \mu_1)$ and $(M_2, \Gamma_2, \sigma_2, \mu_2)$ are PS-systems and $(M_1, \Gamma_1, \sigma_1, \mu_1)$ is well-behaved (with respect to the hierarchy $\mathcal{H}_1 = \{\mathcal{H}_1(R) \subset \Gamma_1 : R \ge 0\}$), it then follows from Lemma 3.3 that

$$M_1 \times M_2 = \bigcup_{\gamma \in \Gamma_1} (M_1 \smallsetminus \gamma Z) \times (M_2 \smallsetminus \rho(\gamma) Z').$$

This implies that

$$M_1 \times M_2 = \bigcup_{\epsilon>0} \bigcup_{\gamma \in \Gamma_1} (M_1 \smallsetminus \gamma \overline{\mathcal{N}_\epsilon(Z)}) \times (M_2 \smallsetminus \rho(\gamma) \overline{\mathcal{N}_\epsilon(Z')}).$$

By the compactness, there exist $\epsilon > 0$ and $\alpha_1, \dots, \alpha_m \in \Gamma_1$ such that

$$M_1 \times M_2 = \bigcup_{i=1}^{m} (M_1 \smallsetminus \alpha_i \overline{\mathcal{N}_\epsilon(Z)}) \times (M_2 \smallsetminus \rho(\alpha_i) \overline{\mathcal{N}_\epsilon(Z')}).$$

We then fix $j_0 \geq 1$ such that

$$Z_{j_0} \subset \mathcal{N}_{\epsilon/2}(Z) \quad \text{and} \quad Z'_{j_0} \subset \mathcal{N}_{\epsilon/2}(Z').$$

Let $\{\tilde{\gamma}_n\} = \{\tilde{\gamma}_{j_0,n}\}$. Then there exists $N \geq 1$ such that for any $n \geq N$,

$$M_1 \smallsetminus \tilde{\gamma}_n^{-1} \mathcal{O}_{R_{j_0}}(\tilde{\gamma}_n) \subset \mathcal{N}_{\epsilon/2}(Z_{j_0}) \quad \text{and} \quad M_2 \smallsetminus \rho(\tilde{\gamma}_n)^{-1} \mathcal{O}_{R_{j_0}}(\rho(\tilde{\gamma}_n)) \subset \mathcal{N}_{\epsilon/2}(Z'_{j_0}).$$

Therefore,

$$M_1 \times M_2 = \bigcup_{i=1}^{m} \left(\alpha_i \tilde{\gamma}_n^{-1} \mathcal{O}_{R_{j_0}}(\tilde{\gamma}_n)\right) \times \left(\rho(\alpha_i)\rho(\tilde{\gamma}_n)^{-1} \mathcal{O}_{R_{j_0}}(\rho(\tilde{\gamma}_n))\right) \tag{21}$$

for all $n \geq N$.

Recall that M_1' satisfies Lemma 5.3 for E and all R_n. Also, since $n \mapsto \mathcal{H}(n)$ is a non-increasing sequence of sets and $\tilde{\gamma}_n \in \mathcal{H}(n)$ for all n, we have $\{\tilde{\gamma}_n\}_{n > R_{j_0}} \subset \mathcal{H}(R_{j_0})$. So by Lemma 5.3,

$$\lim_{n\to\infty} \mu_1(\tilde{\gamma}_n^{-1} \mathcal{O}_{R_{j_0}}(\tilde{\gamma}_n) \smallsetminus \tilde{\gamma}_n^{-1} E) = 0.$$

Hence, since μ_1 is Γ_1-quasi-invariant,

$$\lim_{n\to\infty} \mu_1(\alpha_i \tilde{\gamma}_n^{-1} \mathcal{O}_{R_{j_0}}(\tilde{\gamma}_n) \smallsetminus \alpha_i \tilde{\gamma}_n^{-1} E) = 0$$

for all $i = 1, \dots, m$. We set

$$M_1'' := M_1 \smallsetminus \bigcap_{n \geq N} \bigcup_{i=1}^{m} \left(\alpha_i \tilde{\gamma}_n^{-1} \mathcal{O}_{R_{j_0}}(\tilde{\gamma}_n) \smallsetminus \alpha_i \tilde{\gamma}_n^{-1} E\right),$$

which is of full μ_1-measure.

For $x \in M_1''$, there exists $n \geq N$ such that

$$x \notin \bigcup_{i=1}^{m} \left(\alpha_i \tilde{\gamma}_n^{-1} \mathcal{O}_{R_{j_0}}(\tilde{\gamma}_n) \smallsetminus \alpha_i \tilde{\gamma}_n^{-1} E\right).$$

On the other hand, by Equation (21), there exists $1 \leq i \leq m$ such that

$$(x, f(x)) \in \left(\alpha_i \tilde{\gamma}_n^{-1} \mathcal{O}_{R_{j_0}}(\tilde{\gamma}_n)\right) \times \left(\rho(\alpha_i)\rho(\tilde{\gamma}_n)^{-1} \mathcal{O}_{R_{j_0}}(\rho(\tilde{\gamma}_n))\right),$$

and therefore we must have

$$x \in \alpha_i \tilde{\gamma}_n^{-1} E$$

as well. This completes the proof of Proposition 7.5 with $R' := R_{j_0}$, and hence the proof of Theorem 7.1. $\square$

Part 2. Examples and Applications

8. Convergence groups and expanding coarse-cocycles

In [BCZZ24b], Blayac–Canary–Zhu–Zimmer developed Patterson–Sullivan theory for coarse-cocycles of convergence groups. In this section we show that this theory is a special case of the definitions developed in the current paper.

Let M be a compact metrizable space and let $\Gamma < \mathsf{Homeo}(M)$ be a non-elementary convergence group. In [BCZZ24b, Prop. 2.3] it was observed that the set $\Gamma \sqcup M$ has a unique topology such that

- $\Gamma \sqcup M$ is a compact metrizable space.
- The inclusions $\Gamma \hookrightarrow \Gamma \sqcup M$ and $M \hookrightarrow \Gamma \sqcup M$ are embeddings (where in the first embedding Γ has the discrete topology).
- the Γ-action on $\Gamma \sqcup M$, induced by the left-multiplication on Γ and the given Γ-action on M, is a convergence action.

Moreover,

- $\gamma_n \to a \in M$ and $\gamma_n^{-1} \to b \in M$ if and only if $\gamma_n|_{M \smallsetminus \{b\}} \to a$ locally uniformly.

For the rest of the section fix a metric d on $\Gamma \sqcup M$ which generates this topology.

In this setting, shadows can be defined as follows: for $\gamma \in \Gamma$ and $R > 0$ let

$$\mathcal{O}_R(\gamma) := \gamma \left(M \smallsetminus \overline{B_{1/R}(\gamma^{-1})} \right) \tag{22}$$

where $B_{1/R}(\gamma^{-1})$ denotes the open ball of radius $1/R$ centered at γ^{-1} with respect to d.

Remark 8.1. In [BCZZ24b], shadows are defined to be the closed sets

$$\gamma \left(M \smallsetminus B_{1/R}(\gamma^{-1}) \right).$$

For the results cited below the difference between the two definitions is immaterial.

Observation 8.2. [BCZZ24b, proof of Lem. 5.4] With shadows as in Equation (22), the set $\Lambda^{\mathrm{con}}(\Gamma)$ defined in Section 4 coincides with the set of conical limit points in the usual convergence group sense. Moreover, if $\mathrm{d}(a,b) > 1/R$, $\gamma_n^{-1} x \to a$, and $\gamma_n^{-1} y \to b$ for all $y \in M \smallsetminus \{x\}$, then

$$x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n).$$

In [BCZZ24b, Def. 1.2, Prop. 3.2 and 3.3] the following special class of coarse-cocycles where introduced.

Definition 8.3. A coarse-cocycle $\sigma : \Gamma \times M \to \mathbb{R}$ is called *expanding* if:

1. There exists $\kappa > 0$ such that for any $\gamma \in \Gamma$, the function $\sigma(\gamma, \cdot)$ is *κ-coarsely-continuous*: for $x_0 \in M$,
$$\limsup_{x \to x_0} |\sigma(\gamma, x) - \sigma(\gamma, x_0)| \leq \kappa.$$
2. For every $\gamma \in \Gamma$, there is a number $\|\gamma\|_\sigma \in \mathbb{R}$, called the *$\sigma$-magnitude of* γ, with the following properties:
   (a) $\lim_{n \to \infty} \|\gamma_n\|_\sigma = +\infty$ for any escaping sequence $\{\gamma_n\} \subset \Gamma$.

(b) For any $\epsilon > 0$, there exists $C > 0$ such that

$$\|\gamma\|_\sigma - C \leq \sigma(\gamma, x) \leq \|\gamma\|_\sigma + C$$

whenever $x \in M \smallsetminus B_\epsilon(\gamma^{-1})$.

Part of [BCZZ24b] was devoted to developing a theory of PS-measures for expanding coarse-cocycles and using these results we show that this theory is a special case of our well-behaved PS-systems.

Theorem 8.4. *Let $\sigma : \Gamma \times M \to \mathbb{R}$ be an expanding coarse-cocycle and μ a coarse σ-PS measure, then (M, Γ, σ, μ) is a well-behaved PS-system with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$, with shadows as in Equation* (22).

Proof. Since each $\sigma(\gamma, \cdot)$ is coarsely-continuous, Property (PS1) is satisfied. Property (PS2) follows from the defining property of the σ-magnitude and the definition of the shadows. Property (PS6) follows from the definition of the shadows.

Property (PS4) is a consequence of [BCZZ24b, Prop. 3.3 part (2)], Property (PS7) is a consequence of [BCZZ24b, Prop. 5.1 parts (3) and (4)], and Property (PS8) is a consequence of [BCZZ24b, Prop. 5.1 part (2)].

To verify Property (PS3) and Property (PS5), assume $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, and $[M \smallsetminus \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance. Then Z must be singleton or empty. Then, since Γ is a non-elementary convergence group, Property (PS3) and Property (PS5) are true. ☐

8.1. **Examples.** We will describe one class of examples of expanding coarse-cocycle, for more see [BCZZ24b, Sect. 1.2]. For the rest of this subsection suppose (X, d_X) is a proper geodesic Gromov hyperbolic metric space and $\Gamma < \mathsf{Isom}(X)$ is discrete.

Following [BCZZ24b, Def. 1.9] (which is similar to [CT24, Def. 2.2]), a function $\psi\colon X \times X \to \mathbb{R}$ is a *coarsely additive potential* if

(1) $\lim_{r\to\infty} \inf_{\mathrm{d}_X(p,q)\geq r} \psi(p, q) = +\infty$,
(2) for any $r > 0$,
$$\sup_{\mathrm{d}_X(p,q)\leq r} |\psi(p, q)| < +\infty,$$
(3) for every $r > 0$ there exists $\kappa = \kappa(r) > 0$ such that: if u is contained in the r-neighborhood of a geodesic in (X, d_X) joining p to q, then
$$\big|\psi(p, q) - \big(\psi(p, u) + \psi(u, q)\big)\big| \leq \kappa.$$

Theorem 8.5. [BCZZ24b, Thm. 1.11 and 1.13]

(1) If ψ is a Γ-invariant coarsely additive potential, then
$$\sigma_\psi(\gamma, x) := \limsup_{p\to x} \psi(\gamma^{-1}o, p) - \psi(o, p)$$
is an expanding coarse-cocycle on $\partial_\infty X$ and one can choose
$$\|\gamma\|_{\sigma_\psi} = \psi(o, \gamma o).$$
(2) If Γ acts cocompactly on X and $\sigma : \Gamma \times \partial_\infty X \to \mathbb{R}$ is an expanding coarse-cocyle, then there exists a Γ-invariant coarsely additive potential ψ such that
$$\sup_{\gamma\in\Gamma, x\in\partial_\infty X} |\sigma(\gamma, x) - \sigma_\psi(\gamma, x)| < +\infty.$$

Example 8.6. The distance function d_X is a $\mathsf{Isom}(X)$-invariant coarsely additive potential and the associated expanding coarse-cocycle is just the coarse Busemann cocycle.

Example 8.7 (see [BCZZ24b, Sect. 1.2.5])**.** Suppose Γ is word hyperbolic, X is a Cayley graph of Γ, and m is a probability measure on Γ with finite superexponential moment and whose support generates Γ as a semigroup. Then the Green metric d_G is a Γ-invariant coarsely additive potential and the unique m-stationary measure on $\partial_\infty \Gamma$ is a σ_{d_G}-PS measure of dimension 1. Note: in [BCZZ24b, Sect. 1.2.5] it is assumed that m has finite support, but using [Gou15] the same discussion is valid when m has finite superexponential moment.

In Section 11 we consider stationary measures on the Bowditch boundary of a relatively hyperbolic group.

8.2. **Measurable isomorphisms.** As an application of Theorem 1.28, we show that for word hyperbolic groups a measurable isomorphism between boundaries endowed with PS-measures is always induced by a homeomorphism.

Theorem 8.8. *For $i = 1, 2$ suppose Γ_i is non-elementary word hyperbolic, $\sigma_i : \Gamma_i \times \partial_\infty \Gamma_i \to \mathbb{R}$ is an expanding coarse-cocycle, and μ is a coarse σ_i-PS measure for Γ_i of dimension δ_i on $\partial_\infty \Gamma_i$. Assume there exist*

- *a homomorphism $\rho : \Gamma_1 \to \Gamma_2$ with non-elementary image and*
- *a μ_1-almost everywhere defined measurable ρ-equivariant injective map $f : \partial_\infty \Gamma_1 \to \partial_\infty \Gamma_2$.*

If $f_* \mu_1$ and μ_2 are not singular, then $\ker \rho$ is finite, $\rho(\Gamma_1) < \Gamma_2$ has finite index,

$$\sup_{\gamma \in \Gamma_1} \left| \delta_1 \left\| \gamma \right\|_{\sigma_1} - \delta_2 \left\| \rho(\gamma) \right\|_{\sigma_2} \right| < +\infty,$$

and there exists a ρ-equivariant homeomorphism $\tilde{f} : \partial_\infty \Gamma_1 \to \partial_\infty \Gamma_2$ such that

(1) $\tilde{f} = f$ μ_1-a.e.,

(2) $\sup_{(\gamma, x) \in \Gamma_1 \times \partial_\infty \Gamma_1} \left| \delta_1 \sigma_1(\gamma, x) - \delta_2 \sigma_2(\rho(\gamma), \tilde{f}(x)) \right| < +\infty$,

(3) $\tilde{f}_* \mu_1$, μ_2 are in the same measure class and the Radon–Nikodym derivatives are a.e. bounded from above and below by a positive number.

8.3. **Proof of Theorem 8.8.** For notational convenience, we let $\left\| \cdot \right\|_i := \left\| \cdot \right\|_{\sigma_i}$.

By Theorem 8.5 we can assume that each σ_i corresponds to a coarsely additive potential on a Cayley graph. Then the third defining property for coarsely additive potentials implies that there exist $c > 1$ such that

$$c^{-1} \left| \gamma \right|_i - c \leq \left\| \gamma \right\|_i \leq c \left| \gamma \right|_i + c \tag{23}$$

for all $\gamma \in \Gamma_i$, where $\left| \cdot \right|_i$ is the distance from $\mathrm{id} \in \Gamma_i$ with respect to a word metric on Γ_i with respect to a finite generating set.

By Theorem 1.28,

$$\sup_{\gamma \in \Gamma_1} \left| \delta_1 \left\| \gamma \right\|_1 - \delta_2 \left\| \rho(\gamma) \right\|_2 \right| < +\infty. \tag{24}$$

Then Property (PS4) implies that $\ker \rho$ is finite and Equation (23) implies that ρ induces a quasi-isometric embedding $\Gamma_1 \to \Gamma_2$. So there exists a ρ-equivariant embedding $\tilde{f} : \partial_\infty \Gamma_1 \to \partial_\infty \Gamma_2$.

For a subgroup $\mathsf{H} < \Gamma_2$, let $\delta_{\sigma_2}(\mathsf{H})$ be the critical exponent of the Poincaré series $s \mapsto \sum_{g \in \Gamma_2} e^{-s\|g\|_{\sigma_2}}$. Since μ_2 is a coarse σ_2-PS measure for Γ_2 of dimension δ_2, [BCZZ24b, Prop. 6.2] implies that $\delta_2 \geq \delta_{\sigma_2}(\Gamma_2)$. Moreover, since every point in $\partial_\infty \Gamma_i$ is conical, [BCZZ24b, Prop. 6.3] implies that for $i = 1, 2$,

$$\sum_{\gamma \in \Gamma_i} e^{-\delta_i \|\gamma\|_i} = +\infty.$$

This, together with Equation (24), implies that

$$\delta_{\sigma_2}(\Gamma_2) = \delta_{\sigma_2}(\rho(\Gamma_1)) = \delta_2.$$

Then [BCZZ24b, Thm. 4.3] implies that $\tilde{f}(\partial_\infty \Gamma_1) = \partial_\infty \Gamma_2$. Since $\rho(\Gamma_1)$ is quasi-convex in Γ_2, this implies that $\rho(\Gamma_1) < \Gamma_2$ has finite index.

Now by replacing Γ_1 with $\Gamma_1 / \ker \rho$ and Γ_2 with $\rho(\Gamma_2)$, it suffices to consider the case where $\Gamma := \Gamma_1 = \Gamma_2$, $\rho : \Gamma \to \Gamma$ is the identity representation, and $f : \partial_\infty \Gamma \to \partial_\infty \Gamma$ commutes with the Γ action, then show that

(1) $f = \mathrm{id}_{\partial_\infty \Gamma}$ μ_1-a.e.,
(2) $\sup_{(\gamma, x) \in \Gamma \times \partial_\infty \Gamma} |\delta_1 \sigma_1(\gamma, x) - \delta_2 \sigma_2(\gamma, x)| < +\infty$,
(3) μ_1, μ_2 are in the same measure class and the Radon–Nikodym derivatives are a.e. bounded from above and below by a positive number.

Assertions (2) and (3) are an immediate consequence of [BCZZ24b, Prop. 13.1 and 13.2].

We now show (1). Fix $R_j \to +\infty$. After possibly passing to a tail of $\{R_j\}$, by Corollary 5.5 and the fact that $\mathcal{H}_1(R) \equiv \Gamma$, there exists a μ_1-full measure set M' such that whenever $x \in M' \cap \bigcap_{n \geq 1} \gamma\, \mathcal{O}_{R_j}(\gamma_n)$ for some $j \geq 1$, $\gamma \in \Gamma$, and an escaping sequence $\{\gamma_n\} \subset \Gamma$, we have

$$0 = \lim_{n \to \infty} \frac{1}{\mu(\gamma\, \mathcal{O}_{R_j}(\gamma_n))} \mu\left(\left\{ y \in \gamma\, \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}(f(x), f(y)) > \epsilon \right\}\right)$$

for all $\epsilon > 0$.

Fix $x \in M'$. Since Γ acts on $\partial_\infty \Gamma$ as a uniform convergence group, x is a conical limit point. So there exist $\{\gamma_n\}$ and distinct $a, b \in \partial_\infty \Gamma$ such that $\gamma_n^{-1} x \to a$ and $\gamma_n^{-1} y \to b$ for all $y \in \partial_\infty \Gamma \smallsetminus \{x\}$. Then $\gamma_n \to x$ and $\gamma_n^{-1} \to b$ in $\Gamma \sqcup \partial_\infty \Gamma$. So $\gamma_n|_{\partial_\infty \Gamma \smallsetminus \{b\}} \to x$ locally uniformly. Further, by Observation 8.2,

$$x \in \bigcap_{n \geq 1} \mathcal{O}_{R'}(\gamma_n)$$

where $R' := \frac{2}{\mathrm{d}(a,b)}$.

Lemma 8.9. *After replacing $\{\gamma_n\}$ with a subsequence we can find a μ_1-full measure set E where $\gamma_n f(y) \to f(x)$ for all $y \in E$.*

Assuming the lemma for a moment we finish the proof. By [BCZZ24b, Prop. 6.3 and 7.1], μ_1 has no atoms and by assumption f is injective on a full measure set. Thus $f(E)$ has at least two points. Then, since $\gamma_n|_{\partial_\infty \Gamma \smallsetminus \{b\}} \to x$ locally uniformly, we must have $f(x) = x$. Since $x \in M'$ was arbitrary, we see that $f = \mathrm{id}_{\partial_\infty \Gamma}$ μ_1-a.e.

Proof of Lemma 8.9. For $R_j \geq R'$, notice that

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{1}{\mu(\mathcal{O}_{R_j}(\gamma_n))}\mu\left(\left\{y \in \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}(f(x), f(y)) > \epsilon\right\}\right)\\ &= \lim_{n\to\infty} \frac{1}{(\gamma_n^{-1})_*\mu(\gamma_n^{-1}\,\mathcal{O}_{R_j}(\gamma_n))}(\gamma_n^{-1})_*\mu\left(\left\{y \in \gamma_n^{-1}\,\mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}(f(x), \gamma_n f(y)) > \epsilon\right\}\right)\end{aligned}$$

for all $\epsilon > 0$.

By Property (PS2), there exists $C_j = C_j(R_j) > 1$ such that

$$\frac{1}{C_j}e^{-\delta\|\gamma_n\|} \leq \frac{d\gamma_n^{-1}{}_*\mu}{d\mu} \leq C_j e^{-\delta\|\gamma_n\|} \quad \mu\text{-a.e.}$$

on $\gamma_n^{-1}\,\mathcal{O}_{R_j}(\gamma_n)$. Hence

$$0 = \lim_{n\to\infty} \frac{1}{\mu(\gamma_n^{-1}\,\mathcal{O}_{R_j}(\gamma_n))}\mu\left(\left\{y \in \gamma_n^{-1}\,\mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}(f(x), \gamma_n f(y)) > \epsilon\right\}\right)$$

for all $R_j \geq R'$ and $\epsilon > 0$. Since

$$\mu(\gamma^{-1}\,\mathcal{O}_{R_j}(\gamma)) \leq 1,$$

we have

$$0 = \lim_{n\to\infty} \mu\left(\left\{y \in \gamma_n^{-1}\,\mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}(f(x), \gamma_n f(y)) > \epsilon\right\}\right)$$

for all $R_j \geq R'$ and $\epsilon > 0$.

After passing to a subsequence of $\{\gamma_n\}$, we can fix $\epsilon_n \searrow 0$ such that

$$\sum_{n=1}^{\infty} \mu\left(\left\{y \in \gamma_n^{-1}\,\mathcal{O}_{R_n}(\gamma_n) : \mathrm{d}(f(x), \gamma_n f(y)) > \epsilon_n\right\}\right) < +\infty. \tag{25}$$

Recall that $\gamma_n^{-1} \to b$ in $\Gamma \sqcup M$. Then let

$$E_n := \left\{y \in \gamma_n^{-1}\,\mathcal{O}_{R_n}(\gamma_n) : \mathrm{d}(f(x), \gamma_n f(y)) > \epsilon_n\right\}$$

and

$$E := (\partial_\infty\Gamma - \{b\}) \smallsetminus \bigcap_{N=1}^{\infty}\bigcup_{n\geq N} E_n.$$

By [BCZZ24b, Prop. 6.3 and 7.1], μ_1 has no atoms and hence Equation (25) implies that E has full μ_1-measure. Further, if $y \in E \subset \partial_\infty\Gamma \smallsetminus \{b\}$, then

$$y \in \gamma_n^{-1}\,\mathcal{O}_{R_n}(\gamma_n) = \partial_\infty\Gamma \smallsetminus \overline{B_{1/R_n}(\gamma_n^{-1})}$$

for n sufficiently large and there exists $N \geq 1$ such that $y \notin \bigcup_{n\geq N} E_n$. Thus $\gamma_n f(y) \to f(x)$. $\square$

9. Discrete subgroups of Lie groups

Let G be a connected semisimple Lie group without compact factors and with finite center. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of G, a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$, and a positive closed Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Then let

$$\kappa : \mathsf{G} \to \mathfrak{a}^+$$

denote the associated Cartan projection. Denoting by $\mathsf{A} = \exp \mathfrak{a}$ and $\mathsf{A}^+ = \exp \mathfrak{a}^+$, we have $\mathsf{G} = \mathsf{K}\mathsf{A}^+\mathsf{K}$ for a maximal compact subgroup $\mathsf{K} < \mathsf{G}$. The Jordan projection $\lambda : \mathsf{G} \to \mathfrak{a}^+$ is given by

$$\lambda(g) = \lim_{n\to\infty} \frac{\kappa(g^n)}{n}.$$

We also let $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ denote the opposition involution, which is defined as $\mathrm{i}(\cdot) = -\operatorname{Ad}_{w_0}(\cdot)$ where w_0 is the longest Weyl element. We then have $\kappa(g^{-1}) = \mathrm{i}(\kappa(g))$ for all $g \in \mathsf{G}$.

Let $X := \mathsf{G}/\mathsf{K}$ and fix a basepoint $o = [e] \in \mathsf{G}/\mathsf{K}$. Fix a K-invariant norm $\|\cdot\|$ on $\mathfrak{a}$ induced from the Killing form, and let d_X denote the G-invariant symmetric Riemannian metric on X defined by $\mathrm{d}_X(go, ho) = \|\kappa(g^{-1}h)\|$ for $g, h \in \mathsf{G}$.

Let $\mathsf{M} < \mathsf{K}$ be the centralizer of A, and Δ the set of all simple roots associated to $\mathfrak{a}^+$. For a non-empty subset $\theta \subset \Delta$, let P_θ be the standard parabolic subgroup corresponding to θ. That is, P_θ is generated by $\mathsf{M}\mathsf{A}$ and all root subgroups U_α, where α ranges over all positive roots and any negative root which is a $\mathbb{Z}$-linear combination of $\Delta \smallsetminus \theta$. We denote by N_θ the unipotent radical of P_θ. We simply write $\mathsf{P} = \mathsf{P}_\Delta$ and $\mathsf{N} = \mathsf{N}_\Delta$.

Let $\mathfrak{a}_\theta := \bigcap_{\alpha \in \Delta \smallsetminus \theta} \ker \alpha$ and let $\mathfrak{a}_\theta^*$ denote the space of $\mathbb{R}$-linear forms on $\mathfrak{a}_\theta$. Let $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ be the unique projection which is invariant under all Weyl elements fixing $\mathfrak{a}_\theta$ pointwise. We can identify $\mathfrak{a}_\theta^*$ with the subspace of p_θ-invariant linear forms on $\mathfrak{a}$.

The Furstenberg boundary and general θ-boundary are defined as

$$\mathcal{F} := \mathsf{G}/\mathsf{P} = \mathsf{K}/\mathsf{M} \quad \text{and} \quad \mathcal{F}_\theta := \mathsf{G}/\mathsf{P}_\theta$$

respectively. We denote by $\pi_\theta : \mathcal{F} \to \mathcal{F}_\theta$ the quotient map.

Let $\mathsf{P}_\theta^{\mathrm{opp}} := w_0 \, \mathsf{P}_{\mathrm{i}(\theta)} \, w_0^{-1}$ which is a parabolic subgroup opposite to P_θ, and denote by $\mathsf{N}_\theta^{\mathrm{opp}}$ the unipotent radical of $\mathsf{P}_\theta^{\mathrm{opp}}$. Two points $x \in \mathcal{F}_\theta$ and $y \in \mathcal{F}_{\mathrm{i}(\theta)}$ are called *transverse* if there exists $g \in \mathsf{G}$ such that

$$x = g\,\mathsf{P}_\theta \quad \text{and} \quad y = gw_0\,\mathsf{P}_{\mathrm{i}(\theta)}\,.$$

One can see that $x \in \mathcal{F}_\theta$ is transverse to $w_0\,\mathsf{P}_{\mathrm{i}(\theta)}$ if and only if $x \in \mathsf{N}_\theta^{\mathrm{opp}}\,\mathsf{P}_\theta$.

9.1. **Iwasawa cocycles and Patterson–Sullivan measures.** The *Iwasawa cocycle* $B : \mathsf{G} \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $g \in \mathsf{G}$ and $x \in \mathcal{F}$, fix $k \in \mathsf{K}$ such that $k\mathsf{M} = x$ and let $B(g, x) \in \mathfrak{a}$ be the unique element such that

$$gk \in \mathsf{K}\,(\exp B(g, x))\,\mathsf{N}.$$

For general $\theta \subset \Delta$, the *partial Iwasawa cocycle* $B_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ is defined as

$$B_\theta(g, x) = p_\theta\left(B(g, \tilde{x})\right)$$

for some (any) $\tilde{x} \in \pi_\theta^{-1}(x) \in \mathcal{F}$. This does not depend on the choice of $\tilde{x}$ [Qui02a, Lem. 6.1]. Then B_θ satisfies the cocycle relation: for any $x \in \mathcal{F}_\theta$ and $g_1, g_2 \in \mathsf{G}$,

$$B_\theta(g_1 g_2, x) = B_\theta(g_1, g_2 x) + B_\theta(g_2, x).$$

Let $\mathsf{H} < \mathsf{G}$ be a subgroup. Recall from the introduction that for $\delta \geq 0$ and $\phi \in \mathfrak{a}_\theta^*$, a Borel probability measure μ on $\mathcal{F}_\theta$ is called a *coarse ϕ-Patterson–Sullivan measure (coarse ϕ-PS measure) for* H *of dimension* δ if there exists $C \geq 1$ such that for any $\gamma \in \mathsf{H}$ the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$C^{-1} e^{-\delta\phi(B_\theta(g^{-1}, x))} \leq \frac{d\gamma_*\mu}{d\mu}(x) \leq C e^{-\delta\phi(B_\theta(g^{-1}, x))} \quad \text{for } \mu\text{-a.e. } x \in \mathcal{F}_\theta\,.$$

If $C = 1$, then μ is a *ϕ-Patterson–Sullivan measure (ϕ-PS measure) for* H *of dimension* δ.

9.2. **Limit sets.** We say that a sequence $\{g_n\} \subset \mathsf{G}$ *converges to* $x \in \mathcal{F}_\theta$ if

- $\alpha(\kappa(g_n)) \to +\infty$ for all $\alpha \in \theta$ and
- a Cartan decomposition $g_n = k_n(\exp \kappa(g_n))\ell_n \in \mathsf{K}\mathsf{A}^+\mathsf{K}$ satisfies

$$k_n \, \mathsf{P}_\theta \to x \quad \text{in } \mathcal{F}_\theta.$$

We say that the sequence $g_n o \in X$ *converges to* x *if* $g_n \to x$. This notion of convergence leads us to define the limit set of a discrete subgroup.

Definition 9.1. Let $\Gamma < \mathsf{G}$ be a discrete subgroup. The *limit set of* Γ *in* $\mathcal{F}_\theta$ is defined as

$$\Lambda_\theta(\Gamma) := \{x \in \mathcal{F}_\theta : \gamma_n \to x \text{ for some sequence } \{\gamma_n\} \subset \Gamma\}.$$

When $\Gamma < \mathsf{G}$ is Zariski dense, then $\Lambda_\theta(\Gamma)$ is the unique Γ-minimal set in $\mathcal{F}_\theta$ as shown by Benoist [Ben97]. Note that if $\{g_n\} \subset \mathsf{G}$ is a sequence converging to a point in $\mathcal{F}_\theta$, then $\{g_n^{-1}\} \subset \mathsf{G}$ has a subsequence converging to a point in $\mathcal{F}_{\mathrm{i}(\theta)}$. The following well-known lemma asserts that such a sequence $\{g_n\} \subset \mathsf{G}$ exhibits a source-sink dynamics, giving the motivation for the definitions above.

Lemma 9.2. *Let* $\{g_n\} \subset \mathsf{G}$ *be a sequence such that* $g_n \to x \in \mathcal{F}_\theta$ *and* $g_n^{-1} \to y \in \mathcal{F}_{\mathrm{i}(\theta)}$ *as* $n \to \infty$. *Then for any* $z \in \mathcal{F}_\theta$ *transverse to* $y \in \mathcal{F}_{\mathrm{i}(\theta)}$, *we have*

$$g_n z \to x \quad \text{as } n \to \infty.$$

For a proof see [LO23, Lem. 2.9] (for $\theta = \Delta$), [KOW23, Lem. 2.4], [CZZ24, Prop. 2.3], or [KLP17, Sect. 4].

9.3. **Transverse subgroups.** The class of transverse subgroups of G provides well-behaved PS-systems.

Definition 9.3. A discrete subgroup $\Gamma < \mathsf{G}$ is P_θ*-transverse* if

- $\alpha(\kappa(g_n)) \to +\infty$ for all $\alpha \in \theta$ and
- any distinct $x, y \in \Lambda_{\theta \cup \mathrm{i}(\theta)}(\Gamma)$ are transverse.

A P_θ-transverse subgroup $\Gamma < \mathsf{G}$ is called *non-elementary* if $\#\Lambda_{\theta \cup \mathrm{i}(\theta)}(\Gamma) > 2$.

Remark 9.4. In the literature, transverse groups are sometimes called antipodal groups (e.g. [KLP17]).

It is easy to see that for a P_θ-transverse $\Gamma < \mathsf{G}$, the canonical projection $\Lambda_{\theta \cup \mathrm{i}(\theta)}(\Gamma) \to \Lambda_\theta(\Gamma)$ is a Γ-equivariant homeomorphism (cf. [KOW23, Lem. 9.5]). An important feature of a P_θ-transverse subgroup $\Gamma < \mathsf{G}$ is that the Γ-action on $\Lambda_\theta(\Gamma)$ is a convergence action ([KLP17, Thm. 4.16], [CZZ24, Prop. 2.8]) and that there is a natural class of expanding cocycles.

Proposition 9.5. [BCZZ24a, Prop. 10.3] *Let* $\Gamma < \mathsf{G}$ *be a non-elementary* P_θ*-transverse subgroup and* $\phi \in \mathfrak{a}_\theta^*$. *If* $\phi(\kappa(\gamma_n)) \to +\infty$ *for any sequence* $\{\gamma_n\} \subset \Gamma$ *of distinct elements, then* $\sigma_\phi := \phi \circ B_\theta|_{\Gamma \times \Lambda_\theta(\Gamma)}$ *is an expanding coarse-cocycle with magnitude* $\gamma \mapsto \phi(\kappa(\gamma))$.

Hence, if μ *is a coarse* ϕ*-PS measure for* Γ *supported on* $\Lambda_\theta(\Gamma)$ *of dimension* δ, *then* $(\Lambda_\theta(\Gamma), \Gamma, \sigma_\phi, \mu)$ *is a well-behaved PS-system of dimension* δ *with resepct to the trivial hierarchy* $\mathcal{H}(R) \equiv \Gamma$, *with shadows as in Equation* (22).

Given a subgroup $\Gamma < \mathsf{G}$ and a functional $\phi \in \mathfrak{a}_\theta^*$, let $\delta_\phi(\Gamma) \in [0, +\infty]$ denote the critical exponent of the Poincaré series

$$\sum_{\gamma \in \Gamma} e^{-s\phi(\kappa(\gamma))},$$

i.e. the series diverges for $s \in [0, \delta_\phi(\Gamma))$ and converges for $s \in (\delta_\phi(\Gamma), +\infty)$. For transverse groups, we have the following existence/uniqueness results.

Theorem 9.6. *Suppose $\Gamma < \mathsf{G}$ is a non-elementary P_θ-transverse subgroup and $\phi \in \mathfrak{a}_\theta^*$ satisfies $\delta_\phi(\Gamma) < +\infty$.*

(1) [CZZ24] *There exists a ϕ-PS measure for Γ of dimension $\delta_\phi(\Gamma)$ supported on $\Lambda_\theta(\Gamma)$.*
(2) [CZZ24] *If $\sum_{\gamma \in \Gamma} e^{-\delta_\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$, then there is a unique ϕ-PS measure for Γ of dimension $\delta_\phi(\Gamma)$ supported on $\Lambda_\theta(\Gamma)$.*
(3) [KOW23] *If Γ is Zariski dense and $\sum_{\gamma \in \Gamma} e^{-\delta_\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$, then any ϕ-PS measure for Γ of dimension $\delta_\phi(\Gamma)$ is supported on $\Lambda_\theta(\Gamma)$.*

9.4. **Anosov and relatively Anosov groups.** A non-elementary P_θ-transverse group Γ is P_θ*-Anosov* if it is word hyperbolic (as an abstract group) and there is an equivariant homeomorphism between the Gromov boundary $\partial_\infty \Gamma$ and the limit set $\Lambda_\theta(\Gamma)$. More generally, a non-elementary P_θ-transverse group Γ is *relatively* P_θ*-Anosov with respect to a collection $\mathcal{P}$ of subgroups* if it is relatively hyperbolic with respect to $\mathcal{P}$ (as an abstract group) and there is an equivariant homeomorphism between the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ and the limit set $\Lambda_\theta(\Gamma)$.

For relatively Anosov groups, the Poincaré series diverges at its critical exponent.

Theorem 9.7. [CZZ25] *If $\Gamma < \mathsf{G}$ is relatively P_θ-Anosov, $\phi \in \mathfrak{a}_\theta^*$, and $\delta_\phi(\Gamma) < +\infty$, then $\sum_{\gamma \in \Gamma} e^{-\delta_\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$.*

9.5. **Irreducible subgroups.** We now consider a more general class of subgroups.

Definition 9.8. A subgroup $\Gamma < \mathsf{G}$ is called P_θ*-irreducible* if for any $x \in \mathcal{F}_\theta$ and $y \in \mathcal{F}_{\mathrm{i}(\theta)}$, there exists $\gamma \in \Gamma$ such that γx is transverse to y. We say that Γ is *strongly* P_θ*-irreducible* if any finite index subgroup of Γ is P_θ-irreducible.

It is easy to see that any Zariski dense subgroup of G is strongly P_θ-irreducible. We will show that irreducible subgroups form PS-systems, with higher rank shadows defined as follows. First, for $p \in X$ and $R > 0$, let $B_X(p, R)$ denote the metric ball $\{x \in X : \mathrm{d}_X(x, p) < R\}$. Then, for $q \in X$, the θ*-shadow* $O_R^\theta(q, p) \subset \mathcal{F}_\theta$ of $B_X(p, R)$ viewed from q is defined as

$$O_R^\theta(q, p) := \{g\,\mathsf{P}_\theta \in \mathcal{F}_\theta : g \in \mathsf{G},\ go = q,\ gA^+o \cap B_X(p, R) \neq \emptyset\}.$$

Note that for any $g \in \mathsf{G}$, $q, p \in X$, and $R > 0$,

$$gO_R^\theta(q, p) = O_R^\theta(gq, gp).$$

We will use the following observations.

Lemma 9.9 ([LO23, Lem. 5.7], [KOW23, Lem. 5.7])**.** *For any $R > 0$ there exists $C > 0$ such that: if $g \in \mathsf{G}$ and $x \in O_R^\theta(g^{-1}o, o)$, then*

$$\|p_\theta(\kappa(g)) - B_\theta(g, x)\| \leq C.$$

Lemma 9.10. *For any relatively compact subset $V \subset \mathsf{N}_\theta^{\mathrm{opp}}$ there exists $R_0 > 0$ such that: if $g \in \mathsf{G}$ has a Cartan decomposition $g = ka\ell \in \mathsf{KA}^+\mathsf{K}$, then*

$$\ell^{-1} V \, \mathsf{P}_\theta \subset O^\theta_{R_0}(g^{-1}o, o).$$

Proof. Notice that the desired inclusion is equivalent to $V \, \mathsf{P}_\theta \subset O^\theta_{R_0}(a^{-1}o, o)$.

Fix $h \in V$ and let

$$ah = k'bn \in \mathsf{KAN}.$$

denote the Iwasawa decomposition of ah. Notice that

$$aha^{-1} = k'(ba^{-1})(ana^{-1}) \in \mathsf{KAN},$$

is the Iwasawa decomposition of aha^{-1}. Since $V \subset \mathsf{N}_\theta^{\mathrm{opp}}$ is relatively compact and $a \in \mathsf{A}^+$, there exists a relatively compact subset $V' \subset \mathsf{N}_\theta^{\mathrm{opp}}$, which only depends on V, such that $aha^{-1} \in V'$. Then, since the Iwasawa decomposition induces a diffeomorphism $\mathsf{K} \times \mathsf{A} \times \mathsf{N} \to \mathsf{G}$, there exists a relatively compact subset $W \subset \mathsf{G}$, which only depends on V, such that

$$ba^{-1}, ana^{-1} \in W.$$

Since $n \in \mathsf{N}$ and $a \in \mathsf{A}^+$, there exists a relatively compact subset $W' \subset \mathsf{G}$, which only depends on V, such that $n \in W'$.

Then

$$hn^{-1}b^{-1}a \in V \cdot W'^{-1} \cdot W^{-1}$$

is uniformly bounded. Thus there exists $R_0 > 0$, which only depends on V, such that

$$hn^{-1}b^{-1}ao \in B_X(o, R_0).$$

Therefore, $h \, \mathsf{P}_\theta = h(n^{-1}b^{-1}) \, \mathsf{P}_\theta \in O^\theta_{R_0}(hn^{-1}b^{-1}o, o)$. Since $hn^{-1}b^{-1} = a^{-1}k'$, we have $h \, \mathsf{P}_\theta \in O^\theta_{R_0}(a^{-1}o, o)$. This finishes the proof. $\square$

We now verify that irreducible subgroups give PS-systems. We emphasize that Γ is not assumed to be discrete in the following.

Theorem 9.11. *Let $\Gamma < \mathsf{G}$ be a P_θ-irreducible subgroup. If $\phi \in \mathfrak{a}_\theta^*$ and μ is a coarse ϕ-PS measure on $\mathcal{F}_\theta$, then $(\mathcal{F}_\theta, \Gamma, \sigma_\phi, \mu)$ is a PS-system with magnitude $\gamma \mapsto \phi(\kappa(\gamma))$ and shadows $\{\mathcal{O}_R(\gamma) := O^\theta_R(o, \gamma o) : \gamma \in \Gamma, R > 0\}$.*

Proof. Since B_θ is continuous and $\mathcal{F}_\theta$ is compact, Property (PS1) holds. Property (PS2) follows from Lemma 9.9. We now show Property (PS3).

Suppose $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, and $[M \smallsetminus \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance. Since

$$\gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n) = O^\theta_R(\gamma_n^{-1}o, o),$$

Lemma 9.10 implies that $Z \subset \mathcal{F}_\theta \smallsetminus k\mathsf{N}_\theta^{\mathrm{opp}} \, \mathsf{P}_\theta$ for some $k \in \mathsf{K}$. Since $k\mathsf{N}_\theta^{\mathrm{opp}} \, \mathsf{P}_\theta$ consists of points transverse to $kw_0 \, \mathsf{P}_{\mathrm{i}(\theta)}$, Property (PS3) follows from the definition of P_θ-irreducibility. $\square$

9.6. **Zariski dense discrete subgroups.** In this section, we show that Zariski dense discrete subgroups give rise to well-behaved PS-systems with respect to some natural subsets.

Let $\Gamma < \mathsf{G}$ be a Zariski dense discrete subgroup. For $R > 0$ and $\gamma \in \Gamma$, we consider the shadow

$$\mathcal{O}_R(\gamma) := O_R^{\Delta}(o, \gamma o) \subset \mathcal{F}. \tag{26}$$

For $u \in \operatorname{int} \mathfrak{a}^+$ and $r > 0$, we collect elements of Γ along the direction u:

$$\Gamma_{u,r} := \{\gamma \in \Gamma : \|\kappa(\gamma) - tu\| < r \text{ for some } t > 0\}.$$

Theorem 9.12. *Let $\Gamma < \mathsf{G}$ be a Zariski dense discrete subgroup and $u \in \operatorname{int} \mathfrak{a}^+$. Let $\phi \in \mathfrak{a}^*$ be such that $\phi(u) > 0$ and let μ be a ϕ-PS measure for Γ on $\mathcal{F}$. Then for any $r > 0$, the PS-system $(\mathcal{F}, \Gamma, \sigma_\phi, \mu)$ is well-behaved with respect to the constant hierarchy $\mathcal{H}(R) \equiv \Gamma_{u,r}$, with magnitude $\gamma \mapsto \phi(\kappa(\gamma))$ and shadows as in Equation (26).*

Proof. By Theorem 9.11, $(\mathcal{F}, \Gamma, \sigma_\phi, \mu)$ is a PS-system. To see Property (PS5), let $\{\gamma_n\} \subset \Gamma$ and $R_n \to +\infty$ be sequences so that $[M \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance. Since

$$\gamma_n^{-1}\, \mathcal{O}_{R_n}(\gamma_n) = O_{R_n}^{\Delta}(\gamma_n^{-1} o, o),$$

Lemma 9.10 implies that $Z \subset \mathcal{F} \smallsetminus k\mathsf{N}^{\mathrm{opp}}\,\mathsf{P}$ for some $k \in \mathsf{K}$.

Thus Z is contained in a proper subvariety of $\mathcal{F}$. Hence, Property (PS5) follows from the Zariski density of Γ. Property (PS4) and Property (PS6) are straightforward. By [BLLO23, Lem. 3.6 and its proof], Property (PS7) holds. Property (PS8) is a consequence of $u \in \operatorname{int} \mathfrak{a}^+$. ☐

Remark 9.13. The set $\Lambda^{\mathrm{con}}(\Gamma_{u,r}) = \Lambda^{\mathrm{con}}(\mathcal{H})$ above is related to the notion of "u-directional limit set" discussed in [Lin06, BLLO23, Sam24, KOW25]. When Γ is an irreducible lattice and μ is a K-invariant measure on $\mathcal{F}$, it follows from the work of Link [Lin06] that $\mu(\Lambda^{\mathrm{con}}(\Gamma_{u,r})) = 1$ for all large $r > 0$. For general Γ and μ, it was shown by Burger–Landesberg–Lee–Oh [BLLO23] that $\mu(\Lambda^{\mathrm{con}}(\Gamma_{u,r})) = 1$ holds for large $r > 0$ if and only if the right-multiplication of $\exp(u\mathbb{R})$ on $\Gamma\backslash\mathsf{G}/\mathsf{M}$ is ergodic with respect to a Bowen–Margulis–Sullivan measure associated to μ (see also [KOW25]). It was also shown in [BLLO23] that if $\Gamma < \mathsf{G}$ is P_Δ-Anosov and $\operatorname{rank} G \leq 3$, $\mu(\Lambda^{\mathrm{con}}(\Gamma_{u,r})) = 1$ for some $u \in \operatorname{int} \mathfrak{a}^+$ and all large $r > 0$.

9.7. **Tukia's theorem in higher rank.** Let $\mathsf{G}_1, \mathsf{G}_2$ be connected semisimple Lie groups without compact factors and with finite centers. For $i = 1, 2$, let θ_i be a non-empty subset of simple roots for G_i. Combining Proposition 9.5 and Theorem 7.1, we obtain the following.

Corollary 9.14. *For $i = 1, 2$, let $\Gamma_i < \mathsf{G}_i$, $\phi_i \in \mathfrak{a}_{\theta_i}^*$, and μ_i a coarse ϕ_i-PS measure for Γ_i of dimension δ_i on $\mathcal{F}_{\theta_i}$. Suppose*

- *Γ_1 is non-elementary P_{θ_1}-transverse and $\sum_{\gamma \in \Gamma_1} e^{-\delta_1 \phi_1(\kappa(\gamma))} = +\infty$.*
- *Γ_2 is P_{θ_2}-irreducible.*
- *There exists an onto homomorphism $\rho : \Gamma_1 \to \Gamma_2$ and a μ_1-almost everywhere defined measurable ρ-equivariant injective map $f : \mathcal{F}_{\theta_1} \to \mathcal{F}_{\theta_2}$.*

If $f_*\mu_1$ and μ_2 are not singular, then

$$\sup_{\gamma \in \Gamma_1} |\delta_1 \phi_1(\kappa(\gamma)) - \delta_2 \phi_2(\kappa(\rho(\gamma)))| < +\infty.$$

Remark 9.15. Note that Γ_2 is not assumed to be discrete. When θ_1 is the set of all simple roots for G_1, by Theorem 9.12, we can replace the first condition in Corollary 9.14 with Γ_1 being Zariski dense discrete and $\mu_1(\Lambda^{\mathrm{con}}(\Gamma_{1,u,r})) = 1$ for some $u \in \operatorname{int} \mathfrak{a}_1^+$ with $\phi_1(u) > 0$ and $r > 0$.

To complete the proof of Theorem 1.21 from the introduction, we use the following result of Dal'Bo–Kim.

Theorem 9.16. [DK00] *For $i = 1, 2$, suppose that G_i is simple and has a trivial center and let $\Gamma_i < \mathsf{G}_i$ be a Zariski dense subgroup and $\phi_i \in \mathfrak{a}_i^* \smallsetminus \{0\}$. If $\rho : \Gamma_1 \to \Gamma_2$ is an onto homomorphism and*

$$\sup_{\gamma \in \Gamma_1} |\phi_1(\lambda(\gamma)) - \phi_2(\lambda(\rho(\gamma)))| < +\infty,$$

then ρ extends to a Lie group isomorphism $\mathsf{G}_1 \to \mathsf{G}_2$.

9.8. **The Linear Case.** For use in Section 13 we specialize some of the above discussion to the case when $\mathsf{G} = \mathsf{PGL}(d, \mathbb{R})$. In this case, we can let

$$\mathfrak{a} = \{\mathrm{diag}(a_1, \dots, a_d) : a_1 + \dots + a_d = 0\}$$

and

$$\mathfrak{a}^+ = \{\mathrm{diag}(a_1, \dots, a_d) \in \mathfrak{a} : a_1 \geq \dots \geq a_d\}.$$

Then the Cartan and Jordan projections are given by

$$\kappa(g) = (\log \sigma_1(g), \dots, \log \sigma_d(g)) \quad \text{and} \quad \lambda(g) = (\log \lambda_1(g), \dots, \log \lambda_d(g))$$

where $\sigma_1(g) \geq \dots \geq \sigma_d(g)$ are the singular values and $\lambda_1(g) \geq \dots \geq \lambda_d(g)$ are the absolute values of the generalized eigenvalues of some (any) representative of g in $\mathsf{GL}(d, \mathbb{R})$ with determinant ± 1.

With this choice of $\mathfrak{a}^+$, $\Delta = \{\alpha_1, \dots, \alpha_{d-1}\}$ where

$$\alpha_j(\mathrm{diag}(a_1, \dots, a_d)) = a_j - a_{j+1}$$

and the opposition involution satisfies $\mathrm{i}(\alpha_j) = \alpha_{d-j}$.

We also let $\omega_j \in \mathfrak{a}^*$ denote the fundamental weight associated to α_j, which satisfies

$$\omega_j(\mathrm{diag}(a_1, \dots, a_d)) = a_1 + \dots + a_j.$$

Notice that when $\theta \subset \Delta$,

$$\mathfrak{a}_\theta^* = \langle \omega_j|_{\mathfrak{a}_\theta} : \alpha_j \in \theta \rangle .$$

Given $\theta = \{\alpha_{j_1}, \dots, \alpha_{j_k}\} \subset \Delta$ with $j_1 < \dots < j_k$, the parabolic subgroup P_θ is the stabilizer of the partial flag

$$\langle e_1, \dots, e_{j_1} \rangle \subset \langle e_1, \dots, e_{j_2} \rangle \subset \dots \subset \langle e_1, \dots, e_{j_k} \rangle$$

where $e_1, \dots, e_d$ is the standard basis of $\mathbb{R}^d$. So we can identify $\mathcal{F}_\theta$ with the partial flag manifold $\mathcal{F}_{j_1,\dots,j_k}(\mathbb{R}^d)$ and $\mathcal{F}_{\mathrm{i}(\theta)}$ with the partial flag manifold $\mathcal{F}_{d-j_k,\dots,d-j_1}(\mathbb{R}^d)$. Using these identifications, two flags $x = (x^{j_i})_{i=1}^k \in \mathcal{F}_\theta$ and $y = (y^{d-j_i})_{i=1}^k \in \mathcal{F}_{\mathrm{i}(\theta)}$ are transverse if and only if x^{j_i} and y^{d-j_i} are transverse for all $i = 1, \dots, k$.

To avoid cumbersome notation, in this setting we often replace θ subscripts with the indices appearing in θ, e.g. if $\theta = \{\alpha_1, \alpha_{d-1}\}$, then

$$\mathsf{P}_{1,d-1} = \mathsf{P}_\theta, \quad \mathcal{F}_{1,d-1} = \mathcal{F}_\theta, \quad \text{and} \quad \Lambda_{1,d-1}(\Gamma) = \Lambda_\theta(\Gamma).$$

The standard inner product on $\mathbb{R}^d$ induces an inner product on $\wedge^j \mathbb{R}^d$ where $\{e_{i_1} \wedge \dots \wedge e_{i_j}\}$ is an orthonormal basis. Given $v \in \wedge^j \mathbb{R}^d$, we let $\|v\|$ denote the norm induced by this inner product. Then when $\alpha_j \in \theta$, the partial Iwasawa cocycle satisfies

$$\omega_j\left(B_\theta(g,x)\right) = \log \frac{\|\tilde{g}(v_1 \wedge \cdots \wedge v_j)\|}{\|v_1 \wedge \cdots \wedge v_j\|} \tag{27}$$

where $x^j = \langle v_1, \ldots, v_j\rangle$ and $\tilde{g}$ is some (any) representative of g in $\mathsf{GL}(d,\mathbb{R})$ with determinant ± 1.

Recall that a subgroup $\Gamma < \mathsf{PGL}(d,\mathbb{R})$ is *irreducible* if there are no Γ-invariant proper linear subspaces and *strongly irreducible* if every finite index subgroup is irreducible. We will use the following result of Labourie.

Proposition 9.17. [Lab06, Prop. 10.3] *If $\Gamma < \mathsf{PGL}(d,\mathbb{R})$ is strongly irreducible, then Γ is strongly P_θ-irreducible for every non-empty $\theta \subset \Delta$.*

10. Group actions with contracting isometries

In this section we use the theory of contracting isometries on general metric spaces developed by Coulon [Cou24] and Yang [Yan22], to verify that Busemann PS-measures on the Gardiner–Masur boundary of Teichmüller space are part of PS-systems. Let Σ, $(\mathcal{T}, \mathrm{d}_{\mathcal{T}})$, and $\mathrm{Mod}(\Sigma)$ be as in Section 1.1.2.

Theorem 10.1 (Teichmüller space)**.** *Suppose $\Gamma < \mathrm{Mod}(\Sigma)$ is non-elementary and μ is a Busemann PS-measure for Γ of dimension δ on $\partial_{GM}\mathcal{T}$. Then μ is part of a well-behaved PS-system with respect to some hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$ and with magnitude function $\gamma \mapsto \mathrm{d}_{\mathcal{T}}(o, \gamma o)$ for a fixed $o \in \mathcal{T}$. Moreover, if $\sum_{\gamma\in\Gamma} e^{-\delta\, \mathrm{d}_{\mathcal{T}}(o,\gamma o)} = +\infty$, then*

$$\mu(\Lambda^{\mathrm{con}}(\mathcal{H})) = 1.$$

In fact, we show a more general result about isometric actions on general metric spaces which have a contracting isometry (see Theorems 10.11 and 10.13 below).

Remark 10.2. Let X be a proper geodesic CAT(0) space. The same statement as in Theorem 10.1 holds for a non-elementary discrete subgroup of $\mathsf{Isom}(X)$ with a rank one isometry and a Busemann PS-measure on the visual boundary (which coincides in this case with the horofunction boundary) (see Examples 10.5, 10.7, and 10.14).

10.1. Contracting isometries.

Let (X, d) be a proper geodesic metric space. For a closed subset $Y \subset X$ and $x \in X$, a point $y \in Y$ is called a nearest-point projection of x on Y if $\mathrm{d}(x,y) = \mathrm{d}(x,Y)$. This defines a set-valued map π_Y as follows: for a subset $Z \subset X$,

$$\pi_Y(Z) = \{y \in Y : y \text{ is a nearest-point projection of some } z \in Z\}.$$

Definition 10.3. For $\alpha \geq 0$, a closed subset $Y \subset X$ is called *α-contracting* if for any geodesic $L \subset X$ with $\mathrm{d}(L,Y) \geq \alpha$,

$$\operatorname{diam} \pi_Y(L) \leq \alpha.$$

We call Y *contracting* if Y is α-contracting for some $\alpha \geq 0$.

Definition 10.4. An isometry $g \in \mathsf{Isom}(X)$ is called *(α-)contracting* if an orbit map $\mathbb{Z} \to X$, $n \mapsto g^n x$, is a quasi-isometric embedding and the image is (α-)contracting, for some (hence any) $x \in X$.

Note that conjugates of contracting elements are contracting. In this section, we consider the assumption:

(CTG) $\Gamma < \mathsf{Isom}(X)$ discrete with a contracting isometry.

Such Γ is acylindrically hyperbolic [Sis18]. We also call Γ *non-elementary* if Γ is not virtually cyclic.

Example 10.5. The following are examples of metric spaces and contracting isometries:

(1) When X is Gromov hyperbolic space, any loxodromic isometry on X is contracting [Gro87].
(2) Let Γ be a relatively hyperbolic group acting properly and cocompactly on a metric space X by isometries (e.g. X is a Cayley graph of Γ). Then any infinite order element of Γ which is not conjugated into a peripheral subgroup is contracting [GP13, GP16].
(3) If X is CAT(0), any rank one isometry of X is contracting [BF09].
(4) Let Σ be a closed connected orientable surface of genus at least two. Consider the action of its mapping class group $\mathrm{Mod}(\Sigma)$ on its Teichmüller space $\mathcal{T}$ equipped with the Teichmüller metric. Then pseudo-Anosov mapping classes are contracting [Min96].

10.2. **Horofunction compactification.** We recall the horofunction compactification of X. Fix a basepoint $o \in X$ and let

$$C_*(X) := \{h : X \to \mathbb{R} : h(o) = 0\}$$

which is equipped with the topology of uniform convergence on compact subsets.

We embed $X \hookrightarrow C_*(X)$ via the map

$$x \mapsto \mathrm{d}(x, \cdot) - \mathrm{d}(x, o).$$

Then by Arzelà–Ascoli theorem, its image has the compact closure. This gives the horofunction compactification.

Definition 10.6. The *horofunction compactification* $\overline{X}$ of X is the closure of X in $C_*(X)$. The *horofunction boundary* of X is $\partial_H X := \overline{X} \smallsetminus X$.

Note that every $h \in \overline{X}$ is 1-Lipschitz. Since uniform convergence on compact subsets is equivalent to pointwise convergence for 1-Lipschitz functions, it follows from the separability of X that $\overline{X}$ is metrizable.

Example 10.7. The following examples are horofunction boundaries. See [Yan22] for further discussion on each of them.

(1) When X is CAT(0), it is well-known that the visual boundary is the same as the horofunction boundary [BH99, II.8].
(2) As mentioned in the introduction, the horofunction boundary of a Teichmüller space $\mathcal{T}$ equipped with its Teichmüller metric is the same as Gardiner–Masur boundary $\partial_{GM} \mathcal{T}$ of $\mathcal{T}$ [LS14].

We employ a slightly different point of view on the horofunction compactification, which is more suitable to our purpose. For $h \in C_*(X)$, the function $c_h : X \times X \to \mathbb{R}$ defined as

$$c_h(x, y) := h(x) - h(y)$$

is a cocycle, i.e. $c_h(x,z) = c_h(x,y)+c_h(y,z)$. Conversely, given a continuous cocycle $c : X \times X \to \mathbb{R}$, we have $c(\cdot, o) \in C_*(X)$. This gives another characterization of $C_*(X)$ as the space of all continuous cocycles.

In this perspective, each point $x \in X$ corresponds to the Busemann cocycle $b_x : X \times X \to \mathbb{R}$ defined as

$$b_x(y,z) = \mathrm{d}(x,y) - \mathrm{d}(x,z).$$

In the rest of this section, we regard each point of $\overline{X}$ as a cocycle. It is easy to see that for $c \in \overline{X}$,

$$|c(x,y)| \le \mathrm{d}(x,y) \quad \text{for all } x,y \in X.$$

For $g \in \mathsf{Isom}(X)$, its action on X extends to a homeomorphism of $\overline{X}$, by

$$(gc)(x,y) = c(g^{-1}x, g^{-1}y).$$

In particular, $(gc)(gx,gy) = c(x,y)$.

10.3. **Shadows.** Given $x,y \in X$ and $c \in \overline{X}$, the *Gromov product* is

$$\langle x, c\rangle_y = \frac{1}{2}(\mathrm{d}(y,x) + c(y,x)),$$

which is equal to the usual Gromov product when $c \in X$.

Definition 10.8. Let $x,y \in X$ and $R > 0$. The *R-shadow* of y seen from x is

$$O_R(x,y) := \{c \in \overline{X} : \langle x, c\rangle_y < R\}.$$

Note that for $g \in \mathsf{Isom}(X)$,

$$gO_R(x,y) = O_R(gx,gy).$$

The following is direct from the definition:

Observation 10.9. Let $x,y \in X$ and $R > 0$. If $c \in O_R(x,y)$, then

$$\mathrm{d}(x,y) - 2R < c(x,y) \le \mathrm{d}(x,y)$$

10.4. **Patterson–Sullivan measures.** For $\Gamma < \mathsf{Isom}(X)$, the *Busemann cocycle* $\beta : \Gamma \times \overline{X} \to \mathbb{R}$ is

$$\beta(\gamma, c) = c(\gamma^{-1}o, o).$$

Recall from Equation (3) that a probability measure μ is a β-PS measure for Γ of dimension $\delta \ge 0$ on $\overline{X}$ if for every $\gamma \in \Gamma$,

$$\frac{d\gamma_*\mu}{d\mu}(c) = e^{\delta c(o,\gamma o)} \quad \text{for } \mu\text{-a.e. } c \in \overline{X}$$

(in this setting we do not consider coarse PS-measures). We denote by $\delta_\Gamma \ge 0$ the *critical exponent* of the Poincaré series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-s\,\mathrm{d}(o,\gamma o)}.$$

Following Patterson [Pat76] and Sullivan [Sul79]'s construction, Coulon and Yang showed the existence of PS-measures in the critical dimension.

Proposition 10.10 ([Cou24, Prop. 4.3, Cor. 4.25], [Yan22, Lem. 6.3, Prop. 6.8])**.** *Let $\Gamma < \mathsf{Isom}(X)$ be a non-elementary subgroup satisfying* (CTG)*. If $\delta_\Gamma < +\infty$, then there exists a β-PS measure of dimension δ_Γ, which is supported on $\partial_H X$. Moreover, if a β-PS measure for Γ of dimension δ exists, then $\delta \ge \delta_\Gamma$.*

10.5. **Verification of PS-system.** In the rest of this section, let $\Gamma < \mathsf{Isom}(X)$ be a non-elementary subgroup satisfying (CTG). We verify that the Γ-action on $\partial_H X$ gives a PS-system. For $\gamma \in \Gamma$, we define the β-magnitude by

$$\|\gamma\|_\beta := \mathrm{d}(o, \gamma o) \tag{28}$$

and the R-shadow of γ to be

$$\mathcal{O}_R(\gamma) := \partial_H X \cap O_R(o, \gamma o). \tag{29}$$

Theorem 10.11. *Let $\Gamma < \mathsf{Isom}(X)$ be non-elementary and satisfying* (CTG)*. If μ is a β-PS measure for Γ, then $(\overline{X}, \Gamma, \beta, \mu)$ is a PS-system with magnitude and shadows as in Equations* (28) *and* (29)*. Moreover, Properties (PS5)–(PS6) hold with any choice of the hierarchy $\mathcal{H}$.*

We further show that the PS-system in Theorem 10.11 is well-behaved under some condition related to a saturation of $\partial_H X$; w call $c \in \partial_H X$ *saturated* if for any $c' \in \overline{X} \smallsetminus \{c\}$, $\|c - c'\|_\infty = +\infty$.

To make an appropriate choice of the hierarchy $\{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$, we use the notion of contracting tails, following [Cou24].

Definition 10.12. Let $\alpha, L \geq 0$. For $x, y \in X$, we say that the pair (x, y) has an (α, L)*-contracting tail* if there exists an α-contracting geodesic τ ending at y and a projection $p \in \tau$ of x such that $\mathrm{d}(p, y) \geq L$.

We then consider the following subset of Γ:

$$\mathcal{C}(\alpha, L) := \{\gamma \in \Gamma : (o, \gamma o) \text{ has an } (\alpha, L)\text{-contracting tail}\}.$$

Note that for a fixed $\alpha \geq 0$, the set $\mathcal{C}(\alpha, L)$ is non-increasing in $L \geq 0$.

Theorem 10.13. *Let $\Gamma < \mathsf{Isom}(X)$ be non-elementary and satisfying* (CTG)*. If μ is a β-PS measure for Γ and μ-a.e. point in $\Lambda^{\mathrm{con}}(\Gamma)$ is saturated, then the PS-system $(\partial_H X, \Gamma, \beta, \mu)$ is well-behaved with respect to the hierarchy $\{\mathcal{H}(R) = \mathcal{C}(\alpha, R + 16\alpha + 1) : R \geq 0\}$ for some $\alpha \geq 0$, with magnitude and shadows as in Equations* (28) *and* (29)*.*

Example 10.14. The following are examples that almost every point is saturated:

(1) Suppose that (X, d) is CAT(0). Then its horofunction boundary $\partial_H X$ is the same as its visual boundary, and every single point of $\partial_H X$ is saturated.
(2) Suppose that (X, d) is the Teichmüller space $\mathcal{T}$ of a closed connected orientable sufrace Σ of genus at least two, equipped with the Teichmüller metric. Then its horofunction boundary $\partial_{GM} \mathcal{T}$ contains the space $\mathcal{PMF}$ of projective measured foliations on Σ as a proper subset [GM91]. Moreover, the subset $\mathcal{UE} \subset \mathcal{PMF}$ of uniquely ergodic ones is topologically embedded in $\partial_{GM} \mathcal{T}$ [Miy13, Coro. 1], and every point in $\mathcal{UE}$ is saturated [Yan22, Lem. 12.6].

    Let $\Gamma < \mathrm{Mod}(\Sigma)$ be non-elementary and μ its PS-measure of dimension δ on $\partial_{GM} \mathcal{T}$. If $\sum_{\gamma \in \Gamma} e^{-\delta \,\mathrm{d}(o, \gamma o)} < +\infty$, $\mu(\Lambda^{\mathrm{con}}(\Gamma)) = 0$ by Theorem 4.1. If $\sum_{\gamma \in \Gamma} e^{-\delta \,\mathrm{d}(o, \gamma o)} = +\infty$, we have $\mu(\mathcal{UE}) = 1$ [Yan22, Thm. 1.14, Lem. 12.6]. Therefore, in any case, the condition in Theorem 10.13 is verified.

In general, points in $\partial_H X$ may not be saturated, even in contracting limit sets. On the other hand, one can proceed the same argument as in our proof of the rigidity theorm (Theorem 7.1) in the so-called reduced horofunction boundary of X, which is obtained as the quotient of $\partial_H X$ under the equivalence relation $c \sim c'$ if and only if $\|c - c'\|_\infty < +\infty$. When the reduced horofunction boundary is metrizable (e.g. X is a proper geodesic Gromov hyperbolic space), the same argument can be proceeded. In general, one should employ [Cou23, Prop. 5.1]. We omit this discussion in the current paper.

10.6. **Boundary of contracting subsets.** To prove Theorem 10.11 and Theorem 10.13, we need to introduce more notation. Let $Y \subset X$ be a closed subset and $c \in \overline{X}$. A point $p \in Y$ is called a *projection* of c on Y if

$$c(p, y) \leq 0 \quad \text{for all } y \in Y.$$

When $c \in X$, a point $y \in Y$ is a projection if and only if it is a nearest-point projection. The *boundary at infinity* $\partial^+ Y$ is the set of all $c \in \partial_H X$ such that there is no projection of c on Y.

Going back to a classical setting for a moment, a non-elementary discrete subgroup of $\mathsf{Isom}(\mathbb{H}^n)$ has infinitely many loxodromic elements with disjoint fixed points on $\partial_\infty \mathbb{H}^n$. The following is a similar phenomenon in this current setting.

Proposition 10.15 ([Yan19, Lem. 2.12], [Cou24, Prop. 3.15])**.** *Let $\Gamma < \mathsf{Isom}(X)$ be a non-elementary discrete subgroup. If $\gamma \in \Gamma$ is a contracting isometry, then there exist infinitely many $g_i \in \Gamma$ such that*

$$\partial^+(g_i \langle \gamma \rangle o) \cap \partial^+(g_j \langle \gamma \rangle o) = \emptyset \quad \textit{for all } i \neq j.$$

In particular, $\partial^+(\langle g_i \gamma g_i^{-1} \rangle o) \cap \partial^+(\langle g_j \gamma g_j^{-1} \rangle o) = \emptyset$ for all $i \neq j$.

10.7. **Invisible locus.** We describe the locus which cannot be seen from a sequence of shadows. The following two lemmas can be proved by a slight modification of [Cou24, Proof of Prop. 4.9].

Lemma 10.16. [Cou24, Proof of Prop. 4.9] *Let $\{z_n\} \subset X$ be a sequence converging to $z \in \partial_H X$. Let $g \in \mathsf{Isom}(X)$ be an α-contracting isometry such that $z \notin \partial^+(\langle g \rangle o)$. Suppose that $\{p_n\} \subset \langle g \rangle o$ is a sequence of projections of z_n and that $p_n \to p \in \langle g \rangle o$. Then for any $R_n \to +\infty$, we have*

$$\overline{X} \smallsetminus O_{R_n}(z_n, o) \subset \{c \in \overline{X} : c(p, g^k o) \leq 4\alpha \textit{ for all } k \in \mathbb{Z}\} \quad \textit{for all large } n.$$

Lemma 10.17. [Cou24, Proof of Prop. 4.9] *Let $g \in \mathsf{Isom}(X)$ be an α-contracting isometry. For $i = 1, \dots, m$, let $p_i \in \langle g \rangle o$ and set*

$$\tilde{Z}_i := \{c \in \overline{X} : c(p_i, g^k o) \leq 4\alpha \textit{ for all } k \in \mathbb{Z}\}.$$

Then there exists $N > 0$ such that for all $n \in \mathbb{Z}$ with $|n| > N$, we have

$$\left(\bigcup_{i=1}^m \tilde{Z}_i \right) \cap g^n \left(\bigcup_{i=1}^m \tilde{Z}_i \right) = \emptyset.$$

10.8. **Proof of Theorem 10.11.** As we observed above, $|c(x, y)| \leq \mathrm{d}(x, y)$ for all $c \in \overline{X}$ and $x, y \in X$. Hence, Property (PS1) follows. Property (PS2) follows from Observation 10.9. Property (PS4) and Property (PS6) are straightforward.

Fix a metric on $\overline{X}$ which generates the topology. Property (PS3) is implied by Property (PS5). To see Property (PS5), let $\{\gamma_n\} \subset \Gamma$ and $R_n \to +\infty$ be sequences such that $[\partial_H X \smallsetminus \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance. After passing to a subsequence, we may assume that $\gamma_n^{-1} o \to z \in \partial_H X$. By Proposition 10.15, for any $h_1, \dots, h_m \in \Gamma$, there exists an α-contracting isometry $g \in \Gamma$ such that

$z, h_1 z, \ldots, h_m z \notin \partial^+(\langle g \rangle o)$, for some $\alpha \geq 0$. Then Property (PS5) is a conseqeunce of Lemma 10.16 and Lemma 10.17. $\square$

10.9. **Properties of contracting tails.** To show the well-behavedness, we employ some propeties of contracting tails obtained in [Cou24].

Proposition 10.18. [Cou24, Lem. 4.15, Lem. 5.2] *Let $\alpha, L, R \geq 0$ with $L > R + 16\alpha$. If $\|\gamma_1\|_\beta \leq \|\gamma_2\|_\beta$ and $O_R(o, \gamma_1 o) \cap O_R(o, \gamma_2 o) \neq \emptyset$ for $\gamma_1, \gamma_2 \in \mathcal{C}(\alpha, L)$, then*

1. $\left| \|\gamma_2\|_\beta - (\|\gamma_1\|_\beta + \|\gamma_1^{-1}\gamma_2\|_\beta) \right| \leq 4R + 44\alpha$;
2. $O_R(o, \gamma_2 o) \subset O_{R+42\alpha}(o, \gamma_1 o)$.

Recall from Section 4 the notion of conical limit set for a subset of Γ. As a generalization of Hopf–Tsuji–Sullivan dichotomy, the following was obtained by Coulon [Cou24] (see also Yang [Yan22]).

Theorem 10.19. [Cou24, Coro. 5.19] *If $\sum_{\gamma \in \Gamma} e^{-\delta_\Gamma \|\gamma\|_\beta} = +\infty$, then there exists $\alpha_0, R_0 \geq 0$ such that for any β-PS measure μ of dimension δ_Γ,*

$$\mu\left(\Lambda_{R_0}(\mathcal{C}(\alpha_0, L))\right) = 1 \quad \textit{for all } L \geq 0.$$

Property (PS8) says that shadows converging to a generic point have diameter decaying to 0. This can be observed from contracting tails. Recall that $c \in \partial_H X$ is saturated if for any $c' \in \overline{X} \smallsetminus \{c\}$, $\|c - c'\|_\infty = +\infty$.

Lemma 10.20. [Cou24, Coro. 5.14] *Let $\alpha, L, R \geq 0$ with $L > R + 13\alpha$. Let $c \in \Lambda_R(\mathcal{C}(\alpha, L))$ be saturated. For any open neighborhood $U \subset \overline{X}$ of c, there exists $T \geq 0$ such that for any $\gamma \in \mathcal{C}(\alpha, L)$ with $\mathrm{d}(o, \gamma o) \geq T$,*

$$c \in O_R(o, \gamma o) \Longrightarrow O_R(o, \gamma o) \subset U.$$

10.10. **Proof of Theorem 10.13.** By Thoerem 10.11, it suffices to verify Properties (PS7) and (PS8). First, note that for any $\alpha \geq 0$, the hierarchy $\{\mathcal{H}(R) = \mathcal{C}(\alpha, R + 16\alpha + 1) : R \geq 0\}$ satisfies Property (PS7) by Proposition 10.18.

Hence, it suffices to show that Property (PS8) holds for some $\alpha \geq 0$. We consider two cases separately. Suppose first that $\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\beta} < +\infty$. Then by Theorem 10.11 and Theorem 4.1,

$$\mu(\Lambda^{\mathrm{con}}(\Gamma)) = 0.$$

Setting $M' := \partial_H X \smallsetminus \Lambda^{\mathrm{con}}(\Gamma)$, Property (PS8) is vacuously true for any $\alpha \geq 0$.

Now suppose that $\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\beta} = +\infty$. By Theorem 10.19, there exist $\alpha_0, R_0 \geq 0$ such that

$$\mu(\Lambda_{R_0}(\mathcal{C}(\alpha_0, L))) = 1 \quad \text{for all } L \geq 0.$$

We then consider the hierarchy $\{\mathcal{H}(R) = \mathcal{C}(\alpha_0, R + 16\alpha_0 + 1) : R \geq 0\}$ and set

$$M' := \left\{ c \in \bigcap_{L \geq 0} \Lambda_{R_0}(\mathcal{C}(\alpha_0, L)) : c \text{ is saturated} \right\}$$

which has the full μ-measure by the hypothesis. To see Property (PS8), fix $R > 0$. Then for any $c \in M'$, if $c \in \bigcap_{n=1}^\infty \mathcal{O}_R(\gamma_n)$ for some escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$, then $\lim_{n \to \infty} \operatorname{diam} \mathcal{O}_R(\gamma_n) = 0$ by Lemma 10.20. This completes the proof. $\square$

10.11. **Proof of Theorem 10.1.** This follows immediately from Example 10.14, Theorem 10.13, and Theorem 10.19. $\square$

11. Random walks on relatively hyperbolic groups

In this section we use results in [GGPY21] to show that the stationary measures on the Bowditch boundary of a relatively hyperbolic group (Definition 2.1) are examples of PS-measures on well-behaved PS-systems. For word hyperbolic groups see the discussion in Section 8.7.

For the rest of the section suppose $(\Gamma, \mathcal{P})$ is relatively hyperbolic and suppose m is a probability measure on Γ such that:

(1) The support of m generates Γ as a semigroup, see Equation (4).
(2) m has finite superexponential moment, see Equation (7).

By the work of Maher–Tiozzo [MT18, Thm. 1.1], there exists a unique m-stationary measure ν on $\partial(\Gamma, \mathcal{P})$ and this measure has no atoms. Moreover, it is realized as the hitting measure for a sample path in a Gromov model for $(\Gamma, \mathcal{P})$. In particular, ν is Γ-quasi-invariant. We consider the measurable cocycle defined by

$$\sigma_{\mathsf{m}}(\gamma, \cdot) = -\log \frac{d\gamma_*^{-1}\nu}{d\nu}$$

so that ν is a σ_{m}-PS measure of dimension 1. More precisely, let $M' \subset \partial(\Gamma, \mathcal{P})$ be a Γ-invariant subset of full ν-measure on which the Radon–Nykodim derivative $\frac{d\gamma_*^{-1}\nu}{d\nu}$ is defined for all $\gamma \in \Gamma$. Then we set $\sigma_{\mathsf{m}}(\gamma, x) = -\log \frac{d\gamma_*^{-1}\nu}{d\nu}$ for $x \in M'$ and $\sigma_{\mathsf{m}}(\gamma, x) = 0$ for $x \notin M'$. Since the set of bounded parabolic points is countable and ν has no atoms, ν assigns full measure to the set of conical limit points.

In the rest of the section, fix a metric d on $\Gamma \sqcup \partial(\Gamma, \mathcal{P})$ that generates the topology described at the start of Section 8. Also let d_G be the Green metric on Γ associated to m, which is a left Γ-invariant asymmetric metric on Γ, see Equation (8).

Theorem 11.1. *With the notation above, $(\partial(\Gamma, \mathcal{P}), \Gamma, \sigma_{\mathsf{m}}, \nu)$ is a well-behaved PS-system of dimension 1 with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$, with magnitude function $\|\cdot\|_{\mathsf{m}} := \mathrm{d}_G(\mathrm{id}, \cdot)$ and shadows as in Equation (22). Moreover,*

$$\sum_{\gamma \in \Gamma} e^{-\|\gamma\|_{\mathsf{m}}} = +\infty.$$

11.1. **Proof of Theorem 11.1.** As described above, the conical limit set has full ν-measure. Then by Theorem 4.1 and Observation 8.2, it suffices to prove the first assertion in Theorem 11.1.

For notational convenience, we write

$$\| \cdot \| := \| \cdot \|_{\mathsf{m}}.$$

Properties (PS3), (PS5), (PS6), and (PS8) can be verified as in the proof of Theorem 8.4. By [GT20, Prop. 7.8], the Green metric d_G is quasi-isometric to any word metric on Γ with respect to a finite generating set and hence Property (PS4) holds.

Property (PS1) follows from the fact that ν is a stationary measure and $\operatorname{supp} \mathsf{m}$ generates Γ as a semigroup. In particular, since

$$\nu = \mathsf{m}^{*k} * \nu = \sum_{\gamma \in \Gamma} \mathsf{m}^{*k}(\gamma) \gamma_* \nu,$$

we have

$$\nu \geq \left(\max_{k \geq 1} \mathsf{m}^{*k}(\gamma) \right) \gamma_* \nu$$

and

$$\gamma_*\nu \geq \gamma_*\left(\max_{k\geq 1} \mathsf{m}^{*k}(\gamma^{-1})\right)(\gamma^{-1})_*\nu = \left(\max_{k\geq 1} \mathsf{m}^{*k}(\gamma^{-1})\right)\nu.$$

It remains to verify Properties (PS2) and Property (PS7). The following can be deduced from [GGPY21, Coro. 1.8].

Theorem 11.2. [GGPY21] *For every $\epsilon > 0$ there exists $C = C(\epsilon) > 0$ such that: if $\mathrm{d}(\alpha, \beta) > \epsilon$, then*

$$\mathrm{d}_G(\alpha, \beta) \leq \mathrm{d}_G(\alpha, \mathrm{id}) + \mathrm{d}_G(\mathrm{id}, \beta) \leq \mathrm{d}_G(\alpha, \beta) + C. \tag{30}$$

Remark 11.3. One always has $\mathrm{d}_G(\alpha, \beta) \leq \mathrm{d}_G(\alpha, \mathrm{id}) + \mathrm{d}_G(\mathrm{id}, \beta)$ and so the non-trivial part of the above statement is the second inequality.

We first prove Property (PS2).

Proposition 11.4. *There exists a Γ-invariant full ν-measure subset $Y \subset \partial(\Gamma, \mathcal{P})$ where for any $R > 0$, there exists $C = C(R) > 0$ such that: if $x \in \gamma^{-1}\,\mathcal{O}_R(\gamma) \cap Y$ for some $\gamma \in \Gamma$, then*

$$|\|\gamma\| - \sigma_{\mathsf{m}}(\gamma, x)| \leq C.$$

Proof. We consider the Martin boundary $\partial_M(\Gamma, \mathsf{m})$, which is the horofunction boundary for the Green metric d_G. First, for $\gamma \in \Gamma$, define $K_\gamma : \Gamma \to \mathbb{R}$ by $K_\gamma(g) = \frac{G_{\mathsf{m}}(g,\gamma)}{G_{\mathsf{m}}(\mathrm{id},\gamma)}$, where G_{m} is the Green function for m (Equation (8)). Then the *Martin boundary* $\partial_M(\Gamma, \mathsf{m})$ consists of functions $K : \Gamma \to \mathbb{R}$ where $K = \lim_{n\to\infty} K_{\gamma_n}$ for some escaping sequence $\{\gamma_n\} \subset \Gamma$. Then the set $\Gamma \sqcup \partial_M(\Gamma, \mathsf{m})$ has a topology making it a compact metrizable space and where an escaping sequence $\{\gamma_n\}$ converges to $K \in \partial_M(\Gamma, \mathsf{m})$ if and only if $K_{\gamma_n} \to K$ pointwise (see [Woe00, Sect. 24]). Further the left action of Γ on Γ extends to a continuous action on $\Gamma \sqcup \partial_M(\Gamma, \mathsf{m})$ where $\gamma \cdot K = \frac{K\circ\gamma^{-1}}{K(\gamma^{-1})}$.

By [GGPY21, Coro. 1.7], the identity map $\Gamma \to \Gamma$ extends to a continuous surjective equivariant map

$$\pi : \Gamma \sqcup \partial_M(\Gamma, \mathsf{m}) \to \Gamma \sqcup \partial(\Gamma, \mathcal{P})$$

where the pre-image of each conical limit point $x \in \partial(\Gamma, \mathcal{P})$ is a singleton $\{K_x\}$ and

$$K_x = \lim_{\gamma\to x} K_\gamma.$$

There exists a m-stationary measure ν_0 on $\partial_M(\Gamma, \mathsf{m})$ such that

$$\frac{d\gamma_*\nu_0}{d\nu_0}(K) = K(\gamma)$$

for ν_0-a.e. K (see [Woe00, Thm. 24.10]). Since π is equivariant, $\pi_*\nu_0$ is a stationary measure on $\partial(\Gamma, \mathcal{P})$ and so, by uniqueness, $\nu = \pi_*\nu_0$. Then

$$\sigma_{\mathsf{m}}(\gamma, x) = -\log \frac{d\gamma_*^{-1}\nu}{d\nu}(x) = -\log K_x(\gamma^{-1}) \tag{31}$$

for ν-a.e. conical limit point x. Let Y be a ν-full measure set where every $x \in Y$ is conical and satisfies Equation (31). Since ν is Γ-quasi-invariant, replacing Y with $\bigcap_{\gamma\in\Gamma} \gamma Y$, we may assume that Y is Γ-invariant.

Now fix $R > 0$. Fix $\gamma \in \Gamma$ and

$$x \in \gamma^{-1}\,\mathcal{O}_R(\gamma) \cap Y = \left(\partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/R}(\gamma^{-1})}\right) \cap Y.$$

Then x is conical and $\sigma_{\mathsf{m}}(\gamma, x) = -\log K_x(\gamma^{-1})$. Fix a sequence $\{\alpha_n\} \subset \Gamma$ converging to x. Then $\mathrm{d}(\gamma^{-1}, \alpha_n) > 1/(2R)$ for n large. Hence by Theorem 11.2

$$\begin{aligned} \left| \|\gamma\| - \sigma_{\mathsf{m}}(\gamma, x) \right| &= \left| \mathrm{d}_G(\mathrm{id}, \gamma) + \log K_x(\gamma^{-1}) \right| \\ &= \lim_{n\to\infty} \left| \mathrm{d}_G(\gamma^{-1}, \mathrm{id}) + \mathrm{d}_G(\mathrm{id}, \alpha_n) - \mathrm{d}_G(\gamma^{-1}, \alpha_n) \right| \end{aligned}$$

is bounded by a constant which only depends on R. □

To verify Property (PS7), we will use the following lemma whose proof follows [BCZZ24b, Prop. 3.3 part (7)].

Lemma 11.5. *For any $\epsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that: if $\alpha, \beta \in \Gamma$, $\|\alpha\| \le \|\beta\|$, and $\beta^{-1}\alpha \notin F$, then*

$$\mathrm{d}(\beta^{-1}, \beta^{-1}\alpha) \le \epsilon.$$

Proof. By Theorem 11.2, there exists $C = C(\epsilon) > 0$ such that if $\alpha, \beta \in \Gamma$ and $\mathrm{d}(\alpha^{-1}, \beta) > \epsilon$, then

$$\mathrm{d}_G(\alpha^{-1}, \mathrm{id}) + \mathrm{d}_G(\mathrm{id}, \beta) \le \mathrm{d}_G(\alpha^{-1}, \beta) + C,$$

which is equivalent to

$$\|\alpha\| + \|\beta\| - C \le \|\alpha\beta\|.$$

Let $F := \{\gamma \in \Gamma : \|\gamma\| \le C\}$, which is finite by Property (PS4) shown above. Now if $\alpha, \beta \in \Gamma$ satisfy $\|\alpha\| \le \|\beta\|$ and $\beta^{-1}\alpha \notin F$, then

$$\|\beta\| + \|\beta^{-1}\alpha\| - C > \|\beta\| \ge \|\alpha\| = \|\beta\beta^{-1}\alpha\|.$$

Therefore, $\mathrm{d}(\beta^{-1}, \beta^{-1}\alpha) \le \epsilon$ as desired. □

We now prove the first half of Property (PS7).

Proposition 11.6. *For any $R > 0$, there exists $R' > 0$ such that: if $\alpha, \beta \in \Gamma$, $\|\alpha\| \le \|\beta\|$, and $\mathcal{O}_R(\alpha) \cap \mathcal{O}_R(\beta) \ne \emptyset$, then*

$$\mathcal{O}_R(\beta) \subset \mathcal{O}_{R'}(\alpha).$$

Proof. Suppose to the contrary that there exist $R > 0$ and sequences $\alpha_n, \beta_n \in \Gamma$ such that $\|\alpha_n\| \le \|\beta_n\|$, $\mathcal{O}_R(\alpha_n) \cap \mathcal{O}_R(\beta_n) \ne \emptyset$, and $\mathcal{O}_R(\beta_n) \not\subset \mathcal{O}_n(\alpha_n)$ for all $n \ge 1$. This implies that for all $n \ge 1$,

$$\alpha_n^{-1}\beta_n \left(\partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/R}(\beta_n^{-1})} \right) \not\subset \partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/n}(\alpha_n^{-1})}.$$

Then the sequence $\{\beta_n^{-1}\alpha_n\}$ is escaping; otherwise, $\beta_n^{-1}\alpha_n \overline{B_{1/n}(\alpha_n^{-1})} \subset \overline{B_{1/R}(\beta_n^{-1})}$ for all large $n \ge 1$, which contradicts our assumptions.

By Lemma 11.5, $\mathrm{d}(\beta_n^{-1}, \beta_n^{-1}\alpha_n) \to 0$ as $n \to \infty$. Hence, for $n \ge 1$ sufficiently large we have

$$\begin{aligned} \alpha_n^{-1}\,\mathcal{O}_R(\beta_n) &= \alpha_n^{-1}\beta_n \left(\partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/R}(\beta_n^{-1})} \right) \\ &\subset \alpha_n^{-1}\beta_n \left(\partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/(2R)}(\beta_n^{-1}\alpha_n)} \right) \subset \mathcal{O}_{2R}(\alpha_n^{-1}\beta_n). \end{aligned}$$

Since $\{\alpha_n^{-1}\beta_n\}$ is escaping as well, it follows from [BCZZ24b, Prop. 5.1 part (2)] that $\operatorname{diam} \mathcal{O}_{2R}(\alpha_n^{-1}\beta_n) \to 0$ as $n \to \infty$, and hence $\operatorname{diam} \alpha_n^{-1}\,\mathcal{O}_R(\beta_n) \to 0$ as $n \to \infty$.

Since $\mathcal{O}_R(\alpha_n) \cap \mathcal{O}_R(\beta_n) \ne \emptyset$ and $\alpha_n^{-1}\,\mathcal{O}_R(\alpha_n) = \partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/R}(\alpha_n^{-1})}$, it follows from $\lim_{n\to\infty} \operatorname{diam} \alpha_n^{-1}\,\mathcal{O}_R(\beta_n) = 0$ that

$$\alpha_n^{-1}\,\mathcal{O}_R(\beta_n) \subset \partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/(2R)}(\alpha_n^{-1})} \quad \text{for all large } n \ge 1.$$

Therefore, $\mathcal{O}_R(\beta_n) \subset \mathcal{O}_{2R}(\alpha_n)$ for all large $n \geq 1$, which is a contradiction. This finishes the proof. $\square$

We prove the second half of Property (PS7).

Proposition 11.7. *For any $R > 0$, there exists $C > 0$ such that if $\alpha, \beta \in \Gamma$, $\|\alpha\| \leq \|\beta\|$, and $\mathcal{O}_R(\alpha) \cap \mathcal{O}_R(\beta) \neq \emptyset$, then*

$$\left| \|\beta\| - (\|\alpha\| + \left\|\alpha^{-1}\beta\right\|) \right| \leq C.$$

Proof. Suppose to the contrary that there exist $R > 0$ and sequences $\alpha_n, \beta_n \in \Gamma$ such that $\|\alpha_n\| \leq \|\beta_n\|$, $\mathcal{O}_R(\alpha_n) \cap \mathcal{O}_R(\beta_n) \neq \emptyset$, and

$$\left| \|\beta_n\| - (\|\alpha_n\| + \left\|\alpha_n^{-1}\beta_n\right\|) \right| \geq n \quad \text{for all } n \geq 1.$$

By Theorem 11.2, we have

$$\|\beta_n\| \leq \|\alpha_n\| + \|\alpha_n^{-1}\beta_n\|$$

for all $n \geq 1$. Hence, by assumption, the sequence $\{\alpha_n^{-1}\beta_n\}$ is escaping. Similarly, for all $n \geq 1$,

$$\|\alpha_n^{-1}\beta_n\| - \|\alpha_n^{-1}\| \leq \|\beta_n\| \leq \|\alpha_n\| + \|\alpha_n^{-1}\beta_n\|,$$

and hence the sequence $\{\alpha_n\}$ is also escaping. Since $\|\alpha_n\| \leq \|\beta_n\|$, $\{\beta_n\}$ is an escaping sequence as well. Since $\{\alpha_n^{-1}\beta_n\}$ is escaping, Lemma 11.5 implies that

$$\lim_{n\to\infty} \mathrm{d}(\beta_n^{-1}, \beta_n^{-1}\alpha_n) = 0.$$

As Properties (PS1)–(PS3) have been verified, $(\partial(\Gamma, \mathcal{P}), \Gamma, \sigma_{\mathsf{m}}, \nu)$ is a PS-system. Hence, by Proposition 3.1, there exists $R_0 > 0$ such that $\nu(\mathcal{O}_{R_0}(\gamma)) > 0$ for all $\gamma \in \Gamma$. Now by increasing $R > 0$, we may assume that $R > R_0$. By Proposition 11.6, we can fix $R' > 0$ such that

$$\mathcal{O}_R(\beta_n) \subset \mathcal{O}_{R'}(\alpha_n) \quad \text{for all } n \geq 1.$$

Let $Y \subset \partial(\Gamma, \mathcal{P})$ be the subset in Proposition 11.4. Since each $\mathcal{O}_R(\beta_n)$ has positive measure, for each $n \geq 1$ there exists a point

$$x_n \in \mathcal{O}_R(\beta_n) \cap Y \subset \mathcal{O}_{R'}(\alpha_n) \cap Y.$$

Moreover, since $\mathcal{O}_R(\beta_n) = \beta_n \left(\partial(\Gamma, \mathcal{P}) \smallsetminus \overline{B_{1/R}(\beta_n^{-1})} \right)$ and $\lim_{n\to\infty} \mathrm{d}(\beta_n^{-1}, \beta_n^{-1}\alpha_n) = 0$, we have

$$\mathrm{d}(\beta_n^{-1}x_n, \beta_n^{-1}\alpha_n) \geq \mathrm{d}(\beta_n^{-1}x_n, \beta_n^{-1}) - \mathrm{d}(\beta_n^{-1}\alpha_n, \beta_n^{-1}) \geq \frac{1}{2R}$$

for n sufficiently large. Hence,

$$\alpha_n^{-1}x_n \in \mathcal{O}_{2R}(\alpha_n^{-1}\beta_n) \cap Y.$$

Now if $C = C(\max\{R', 2R\}) > 0$ satisfies Proposition 11.4, then

$$\left| \|\alpha_n\| - \sigma_{\mathsf{m}}(\alpha_n, \alpha_n^{-1}x_n) \right| \leq C, \quad \left| \|\beta_n\| - \sigma_{\mathsf{m}}(\beta_n, \beta_n^{-1}x_n) \right| \leq C, \text{ and}$$
$$\left| \|\alpha_n^{-1}\beta_n\| - \sigma_{\mathsf{m}}(\alpha_n^{-1}\beta_n, \beta_n^{-1}x_n) \right| \leq C.$$

Further, by the cocycle property

$$\sigma_{\mathsf{m}}(\beta_n, \beta_n^{-1}x_n) = \sigma_{\mathsf{m}}(\alpha_n\alpha_n^{-1}\beta_n, \beta_n^{-1}x_n) = \sigma_{\mathsf{m}}(\alpha_n, \alpha_n^{-1}x_n) + \sigma_{\mathsf{m}}(\alpha_n^{-1}\beta_n, \beta_n^{-1}x_n).$$

Combining altogether,

$$\left| \|\beta_n\| - (\|\alpha_n\| + \left\|\alpha_n^{-1}\beta_n\right\|) \right| \leq 3C,$$

which is a contradiction, finishing the proof. $\square$

The proof of Theorem 11.1 is now complete. □

12. Rigidity results for random walks

In the following subsections we suppose that

- $(\Gamma, \mathcal{P})$ is a relatively hyperbolic group and
- m is probability measure on Γ with finite superexponential moment as in Equation (7) and whose support generates Γ as a semigroup.

Let ν_0 be the unique m-stationary measure on $\partial(\Gamma, \mathcal{P})$ and let $(\partial(\Gamma, \mathcal{P}), \Gamma, \sigma_{\mathsf{m}}, \nu_0)$ be the well-behaved PS-system in Theorem 11.1.

In the subsections that follow we will assume that Γ is a subgroup of either the isometry group of a Gromov hyperbolic space, the mapping class group of a surface, or a semisimple Lie group.

12.1. **Random walks on the isometry group of a Gromov hyperbolic space.**

In this subsection we further suppose that

- (X, d_X) is a proper geodesic Gromov hyperbolic space, and
- $\Gamma < \mathsf{Isom}(X)$ is a non-elementary discrete subgroup.

In this setting, Kaimanovich proved that there exists a unique m-stationary measure ν on the Gromov boundary $\partial_\infty X$, and is the hitting measure for a sample path [Kai00, Remark following Thm. 7.7].

A subset $Y \subset X$ is *quasi-convex* if there exists $R > 0$ such that any geodesic joining two points in Y is contained in the R-neighborhood of Y. Then a discrete subgroup $\Gamma' < \mathsf{Isom}(X)$ is *quasi-convex* if for any $o \in X$ the orbit $\Gamma'(o) \subset X$ is quasi-convex (see [Swe01] for properties of such groups). Using the Morse Lemma, it is easy to see that a subgroup is quasi-convex if and only if any orbit map is a quasi-isometric embedding with respect to a word metric on the group with respect to a finite generating set.

Theorem 12.1. *If μ is a coarse Busemann PS-measure for Γ on $\partial_\infty X$ of dimension δ, then the following are equivalent:*

(1) The measures ν and μ are not singular.
(2) The measures ν and μ are in the same measure class and the Radon–Nikodym derivatives are a.e. bounded from above and below by a positive number.
(3) For any $o \in X$,

$$\sup_{\gamma \in \Gamma} |\mathrm{d}_G(\mathrm{id}, \gamma) - \delta\, \mathrm{d}_X(o, \gamma o)| < +\infty.$$

In particular, Γ is quasi-convex, δ is equal the critical exponent of Γ, and $\sum_{\gamma \in \Gamma} e^{-\delta\, \mathrm{d}_X(o, \gamma o)} = +\infty$.

Proof. The implication (2) ⇒ (1) is clear. We now prove (1) ⇒ (3). By [Kai00, Remark following Thm. 7.7], the spaces $(\partial(\Gamma, \mathcal{P}), \nu_0)$ and $(\partial_\infty X, \nu)$ are both Poisson boundaries for (Γ, m). Hence, there is a Γ-equivariant isomorphism

$$f : (\partial(\Gamma, \mathcal{P}), \nu_0) \to (\partial_\infty X, \nu).$$

By assumption $\nu = f_* \nu_0$ is not singular with respect to μ. As explained in Example 8.6 and Theorem 8.4, μ is a coarse PS-measure in a PS-system which has magnitude function

$$\gamma \mapsto \mathrm{d}_X(o, \gamma o).$$

Then by Theorem 7.1,

$$\sup_{\gamma\in\Gamma} |\mathrm{d}_G(\mathrm{id},\gamma) - \delta\, \mathrm{d}_X(o,\gamma o)| < +\infty.$$

Moreover, since d_G is quasi-isometric to a word metric on Γ with respect to a finite generating set by [GT20, Prop. 7.8], Γ is quasi-convex. Since μ is of dimension δ, δ is at least the critical exponent of the Poincaré series [Coo93, Coro. 6.6]. Together with $\sum_{\gamma\in\Gamma} e^{-\mathrm{d}_G(\mathrm{id},\gamma)} = +\infty$ (Theorem 11.1), we have that δ is equal to the critical exponent and the Poincaré series diverges at δ.

It remains to show (3) $\Rightarrow$ (2). Assuming (3), Γ is a word hyperbolic group and the orbit map $\gamma \in \Gamma \mapsto \gamma o \in X$ is a quasi-isometric embedding with respect to a word metric on Γ as mentioned above. Hence we can assume that $\mathcal{P} = \emptyset$ and so $\partial(\Gamma,\mathcal{P})$ coincides with the Gromov boundary $\partial_\infty\Gamma$. Further, the orbit map continuously extends to $f : \partial_\infty\Gamma \to \partial_\infty X$ which is a Γ-equivariant homeomorphism onto its image. Since both ν and ν_0 are hitting measures, we have $f_*\nu_0 = \nu$. Since $\sum_{\gamma\in\Gamma} e^{-\delta\, \mathrm{d}_X(o,\gamma o)} = +\infty$, Theorem 8.4, Observation 8.2, and Theorem 4.1, imply that $\mu(f(\partial_\infty\Gamma)) = 1$. Hence, we can take a pull-back of the Busmann cocycle on $\partial_\infty X$ and μ to $\partial_\infty\Gamma$. Since the Busemann cocycle on $\partial_\infty X$ is expanding (Example 8.6), the same is true for the pull-back. Therefore, (2) follows from [BCZZ24b, Prop. 13.1 and 13.2]. □

We now restate and prove Corollary 1.7.

Corollary 12.2. *Suppose X is a negatively curved symmetric space. If Γ is not a cocompact lattice in $\mathsf{Isom}(X)$, then ν is singular to the Lebesgue measure class on $\partial_\infty X$.*

Proof. Suppose that ν is non-singular to the Lebesgue measure class on $\partial_\infty X$. Since the Lebesgue measure class contains a Busemann PS-measure for Γ (cf. [Qui02a, Lem. 6.3]), it follows from Theorem 12.1 that Γ is convex cocompact. Since ν is supported on the limit set on Γ, the limit set has a positive Lebesgue measure class. By the classical Hopf–Tsuji–Sullivan dichotomy [Rob03], the Lebesgue measure class gives a unique PS-measure supported on the limit set. Therefore, $\partial_\infty X$ is the limit set of Γ, and hence Γ must be a cocompact lattice. □

12.2. **Random walks on mapping class groups and Teichmüller spaces.** Let $\mathrm{Mod}(\Sigma)$ denote the mapping class group of a closed connected orientable surface Σ of genus at least two and let $(\mathcal{T}, \mathrm{d}_{\mathcal{T}})$ is the Teichmüller space of Σ equipped with the Teichmüller metric.

We continue to assume that Γ and m satisfy the assumptions at the start of the section. In this subsection we further suppose that

- $\Gamma < \mathrm{Mod}(\Sigma)$ is a non-elementary subgroup.

Thurston compactified $\mathcal{T}$ with the space $\mathcal{PMF}$ of projective measured foliations on Σ [Thu88]. In this setting, Kaimanovich–Masur showed that there exists a unique m-stationary measure ν on $\mathcal{PMF}$, and is the hitting measure for a sample path in $\mathcal{T}$ and supported on the subset $\mathcal{UE} \subset \mathcal{PMF}$ of uniquely ergodic foliations [KM96, Thm. 2.2.4]. Since $\mathcal{UE}$ is topologically embedded in the Gardiner–Masur boundary $\partial_{GM}\mathcal{T}$ [Miy13], ν can also be regarded as a measure on $\partial_{GM}\mathcal{T}$, where PS-measures are defined.

Theorem 12.3. *If μ is a Busemann PS-measure for Γ on $\partial_{GM}\mathcal{T}$ of dimension δ and the measures ν, μ are not singular, then:*

(1) *For any $o \in \mathcal{T}$,*
$$\sup_{\gamma\in\Gamma} |\mathrm{d}_G(\mathrm{id},\gamma) - \delta\,\mathrm{d}_{\mathcal{T}}(o,\gamma o)| < +\infty.$$
In particular, δ is the critical exponent of Γ and $\sum_{\gamma\in\Gamma} e^{-\delta\,\mathrm{d}_{\mathcal{T}}(o,\gamma o)} = +\infty$.

(2) *If d_w is a word metric on Γ with respect to a finite generating set, then the map*
$$\gamma \in (\Gamma, \mathrm{d}_w) \mapsto \gamma o \in (\mathcal{T}, \mathrm{d}_{\mathcal{T}})$$
is a quasi-isometric embedding.

Proof. By [Kai00, Thm. 2.2.4] the space $(\mathcal{PMF}, \nu)$ is a Poisson boundary for (Γ, m) and by [Kai00, Remark following Thm. 7.7], the space $(\partial(\Gamma,\mathcal{P}), \nu_0)$ is a Poisson boundary for (Γ, m). Hence there is an isomorphism
$$f : (\partial(\Gamma,\mathcal{P}), \nu_0) \to (\mathcal{PMF}, \nu).$$
Since $\nu(\mathcal{UE}) = 1$, we can view f as a map into $\mathcal{UE} \subset \partial_{GM}\mathcal{T}$.

By assumption $\nu = f_*\nu_0$ is not singular with respect to μ. By Theorem 10.11, μ is a PS-measure in a PS-system which has magnitude function
$$\gamma \mapsto \mathrm{d}_{\mathcal{T}}(o,\gamma o).$$
Then by Theorem 7.1,
$$\sup_{\gamma\in\Gamma} |\mathrm{d}_G(\mathrm{id},\gamma) - \delta\,\mathrm{d}_{\mathcal{T}}(o,\gamma o)| < +\infty.$$
Since μ is of dimension $\delta \geq 0$, δ is at least the critical exponent of the Poincaré series ([Cou24, Prop. 4.23], [Yan22, Prop. 6.8]). Since $\sum_{\gamma\in\Gamma} e^{-\mathrm{d}_G(\mathrm{id},\gamma)} = +\infty$ by Theorem 11.1, we have that δ is equal to the critical exponent and the Poincaré series diverges at δ, showing (1).

By [GT20, Prop. 7.8] the Green metric is quasi-isometric to d_w. Therefore, (2) follows. □

We can now restate (as a corollary) and prove Theorem 1.11.

Corollary 12.4. *If Γ contains a multitwist, then the m-stationary measure ν is singular to every Busemann Patterson–Sullivan measures on $\partial_{GM}\mathcal{T}$.*

Proof. By Farb–Lubotzky–Minsky [FLM01], every infinite order element $g \in \mathrm{Mod}(\Sigma)$ has positive stable translation length on its Cayley graph, i.e.,
$$\limsup_{n\to\infty} \frac{\mathrm{d}_w(\mathrm{id}, g^n)}{n} > 0$$
for any word metric d_w on $\mathrm{Mod}(\Sigma)$ with respect to a finite generating set. On the other hand, an infinite order mapping class has zero stable translation length on $\mathcal{T}$ if and only if one of its power is a multitwist. So the result follows from Theorem 12.3. □

For a special class of subgroups, we prove the converse of Theorem 12.3. A subgroup $\Gamma' < \mathrm{Mod}(\Sigma)$ is *parabolically geometrically finite (PGF)* if

- $(\Gamma', \mathcal{P}')$ is relatively hyperbolic for some $\mathcal{P}' = \{P_1, \dots, P_n\}$ where each $P_i < \Gamma'$ contains a finite index, abelian subgroup consisting entirely of multitwists;
- the coned off Cayley graph of $(\Gamma', \mathcal{P}')$ embeds Γ'-equivariantly and quasi-isometrically into the curve complex of Σ.

See [DDLS24, Def. 1.10] for details. When $\mathcal{P}' = \emptyset$, the group Γ' is convex cocompact. This is equivalent to the original definition of [FM02] as shown by [KL08, Ham05].

Theorem 12.5. *Suppose Γ is PGF. If μ is a Busemann PS-measure for Γ on $\partial_{GM}\mathcal{T}$ of dimension δ, then the following are equivalent:*

1. *(1) The measures ν and μ are not singular,*
2. *(2) The measures ν and μ are in the same measure class and the Radon–Nikodym derivatives are a.e. bounded from above and below by a positive number,*
3. *(3) For any $o \in \mathcal{T}$,*

$$\sup_{\gamma\in\Gamma} |\mathrm{d}_G(\mathrm{id},\gamma) - \delta\, \mathrm{d}_{\mathcal{T}}(o,\gamma o)| < +\infty.$$

In particular, Γ is convex cocompact, δ is the critical exponent of Γ, and $\sum_{\gamma\in\Gamma} e^{-\delta\, \mathrm{d}_{\mathcal{T}}(o,\gamma o)} = +\infty$.

Proof. The implication (2) $\Rightarrow$ (1) is clear and (1) $\Rightarrow$ (3) follows from Theorem 12.3. Now suppose (3). Then Γ is word hyperbolic and the orbit map $\gamma \mapsto \gamma o$ continuously extends to a Γ-equivariant map $f : \partial_\infty\Gamma \to \mathcal{UE}$ which is a homeomorphism onto its image, after replacing $o \in \mathcal{T}$ with another point if necessary [FM02, Thm. 1.1, Prop. 3.2]. Hence, $\nu = f_*\nu_0$ since both ν and ν_0 are hitting measures. Since $\sum_{\gamma\in\Gamma} e^{-\delta\, \mathrm{d}_{\mathcal{T}}(o,\gamma o)} = +\infty$, Theorem 10.19 implies that $\mu(f(\partial_\infty\Gamma)) = 1$. Hence, we can take the pull-back of the measure μ to $\partial_\infty\Gamma$ via f, which is a PS-measure for the cocycle $\sigma_{\mathcal{T}}$ given in Proposition 12.6 below. In Proposition 12.6 below we will verify that $\sigma_{\mathcal{T}}$ is an expanding cocycle. Therefore, (2) follows from [BCZZ24b, Prop. 13.1 and 13.2]. ☐

Proposition 12.6. *Suppose $\Gamma < \mathrm{Mod}(\Sigma)$ is convex cocompact. Let $f : \partial_\infty\Gamma \to \mathcal{UE} \subset \partial_{GM}\mathcal{T}$ be the Γ-equivariant embedding induced from a quasi-isometric embedding $\gamma \in \Gamma \mapsto \gamma o \in \mathcal{T}$ for some $o \in \mathcal{T}$. Then the cocycle $\sigma_{\mathcal{T}} : \Gamma \times \partial_\infty\Gamma \to \mathbb{R}$ given by*

$$\sigma_{\mathcal{T}}(\gamma, x) := f(x)(\gamma^{-1}o, o)$$

is an expanding cocycle with magnitude $\gamma \mapsto \mathrm{d}_{\mathcal{T}}(o,\gamma o)$.

Proof. It is clear that $\sigma_{\mathcal{T}}$ is a cocycle and $\lim_{n\to\infty} \mathrm{d}_{\mathcal{T}}(o,\gamma_n o) = +\infty$ for any escaping sequence $\{\gamma_n\} \subset \Gamma$. Moreover, since $f(\partial_\infty\Gamma) \subset \mathcal{UE}$, $\sigma_{\mathcal{T}}$ is continuous [Miy13]. Recalling the metric d on $\Gamma \sqcup \partial_\infty\Gamma$ from Section 8, it remains to show that for any $\epsilon > 0$, there exists $C > 0$ such that

$$\mathrm{d}_{\mathcal{T}}(o,\gamma o) - C \le \sigma_{\mathcal{T}}(\gamma, \gamma^{-1}x) \le \mathrm{d}_{\mathcal{T}}(o,\gamma o) + C$$

whenever $x \in \gamma\left(\partial_\infty\Gamma \smallsetminus \overline{B_\epsilon(\gamma^{-1})}\right)$, where B_ϵ is the open d-ball of radius ϵ centered at γ^{-1}.

Let d_w be a word metric on Γ with respect to a finite generating set. Fix $\epsilon > 0$. It is easy to see that there exists $R_0 > 0$ such that for any $\gamma \in \Gamma$ and $x \in \gamma\left(\partial_\infty\Gamma \smallsetminus \overline{B_\epsilon(\gamma^{-1})}\right)$, any geodesic ray $[\mathrm{id}, x) \subset \Gamma$ with respect to d_w intersects the d_w-ball of radius R_0 centered at γ.

Let $\gamma \in \Gamma$ and $x \in \gamma\left(\partial_\infty \Gamma \smallsetminus \overline{B_\epsilon(\gamma^{-1})}\right)$. Fix a geodesic ray $[\mathrm{id}, x) \subset \Gamma$ and for each $n \geq 1$, let $\gamma_n \in [\mathrm{id}, x)$ be such that $\mathrm{d}_w(\mathrm{id}, \gamma_n) = n$. By [Miy13],

$$\sigma_{\mathcal{T}}(\gamma, \gamma^{-1}x) = \lim_{n \to \infty} \mathrm{d}_{\mathcal{T}}(\gamma_n o, o) - \mathrm{d}_{\mathcal{T}}(\gamma_n o, \gamma o).$$

Fix $k \geq 1$ with $\mathrm{d}_w(\gamma, \gamma_k) < R_0$. Since the orbit map $\Gamma \to \mathcal{T}$ is a quasi-isometric embedding, we have

$$\mathrm{d}_{\mathcal{T}}(\gamma o, \gamma_k o) < R$$

for some $R > 0$ determined by R_0.

For each $n \geq 1$, let $L_n \subset \mathcal{T}$ be the geodesic from o to $\gamma_n o$. Since $\Gamma(o) \subset \mathcal{T}$ is quasi-convex [FM02], there exists $C_0 > 0$ such that L_n is contained in the C_0-neighborhood of $\Gamma(o)$ for all $n \geq 1$. Hence, the nearest-point projection $L_n' \subset \Gamma(o)$ of L_n is a quasi-geodesic. Since the orbit map is a quasi-isometric embedding, it follows from the Morse Lemma for (Γ, d_w) that for some uniform $C_1 > 0$, the quasi-geodesic $\{\gamma_1 o, \dots, \gamma_n o\} \subset \mathcal{T}$ is contained in the C_1-neighborhood of L_n, for all $n \geq 1$.

Now for all $n \geq k$,

$$\mathrm{d}_{\mathcal{T}}(\gamma o, L_n) < R + C_1$$

and hence

$$|(\mathrm{d}_{\mathcal{T}}(\gamma_n o, o) - \mathrm{d}_{\mathcal{T}}(\gamma_n o, \gamma o)) - \mathrm{d}_{\mathcal{T}}(o, \gamma o)| < 2(R + C_1).$$

Taking $n \to \infty$, we have $\left|\sigma_{\mathcal{T}}(\gamma, \gamma^{-1}x) - \mathrm{d}_{\mathcal{T}}(o, \gamma o)\right| \leq 2(R + C_1)$, completing the proof with $C := 2(R + C_1)$. ☐

12.3. **Random walks on discrete subgroups of Lie groups.** We continue to assume that Γ and m satisfy the assumptions at the start of the section. In this subsection we suppose that

- G is connected semisimple Lie group without compact factors and with finite center, and
- $\Gamma < \mathsf{G}$ is a Zariski dense discrete subgroup.

Recall that $\mathcal{F} = \mathsf{G} / \mathsf{P}$ is the Furstenberg broundary. Guivarc'h and Raugi showed that there exists a unique m-stationary measure ν on $\mathcal{F}$, and it is the hitting measure for a sample path [GR85].

As a higher rank analogue of critical exponent, Quint introduced the notion of growth indicator on Γ [Qui02b]. Fixing any norm $\|\cdot\|$ on $\mathfrak{a}$, the *growth indicator of* Γ is the function $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ defined as follows: for $u \neq 0$,

$$\psi_\Gamma(u) := \|u\| \inf_{\mathcal{C} \ni u} \left\{ \text{critical exponent of } s \mapsto \sum_{\gamma \in \Gamma} e^{-s\|\kappa(\gamma)\|} \right\}$$

where the infimum is over all open cones in $\mathfrak{a}$ containing u, and $\psi_\Gamma(0) = 0$. A functional $\phi \in \mathfrak{a}^*$ is *tangent* to the growth indicator of Γ if $\phi \geq \psi_\Gamma$ on $\mathfrak{a}$ and there exists non-zero $u \in \mathfrak{a}^+$ with $\phi(u) = \psi_\Gamma(u)$.

Theorem 12.7. *If μ is a coarse ϕ-PS measure for Γ on $\mathcal{F}$ of dimension δ and the measures ν, μ are not singular, then:*

(1) $\sup_{\gamma \in \Gamma} |\mathrm{d}_G(\mathrm{id}, \gamma) - \delta\phi(\kappa(\gamma))| < +\infty$. In particular, $\sum_{\gamma \in \Gamma} e^{-\delta\phi(\kappa(\gamma))} = +\infty$ and $\delta\phi$ is tangent to the growth indicator of Γ.

(2) *If* d_w *is a word metric on* Γ *with respect to a finite generating set,* (X, d_X) *is the symmetric space associated to* G*, and* $x_0 \in X$*, then the map*

$$\gamma \in (\Gamma, \mathrm{d}_w) \mapsto \gamma x_0 \in (X, \mathrm{d}_X)$$

is a quasi-isometric embedding.

Proof. By [Kai00, Thm. 10.7] the space $(\mathcal{F}, \nu)$ is a Poisson boundary for (Γ, m) and by [Kai00, Remark following Thm. 7.7], the space $(\partial(\Gamma, \mathcal{P}), \nu_0)$ is a Poisson boundary for (Γ, m). Hence there is an isomorphism $f : (\partial(\Gamma, \mathcal{P}), \nu_0) \to (\mathcal{F}, \nu)$.

By assumption $\nu = f_* \nu_0$ is not singular with respect to μ. By Theorem 9.11, μ is a coarse PS-measure in a PS-system which has magnitude function

$$\gamma \mapsto \phi(\kappa(\gamma)).$$

Then by Theorem 7.1,

$$\sup_{\gamma \in \Gamma} |\mathrm{d}_G(\mathrm{id}, \gamma) - \delta \phi(\kappa(\gamma))| < +\infty,$$

showing the first part of (1). Since $\sum_{\gamma \in \Gamma} e^{-\mathrm{d}_G(\mathrm{id}, \gamma)} = +\infty$ by Theorem 11.1, we have

$$\sum_{\gamma \in \Gamma} e^{-\delta \phi(\kappa(\gamma))} = +\infty.$$

Then [Qui02b, Lem. 3.1.3] implies that $\delta \phi(u) \leq \psi_\Gamma(u)$ for some $u \neq 0$. Finally, the existence of the coarse ϕ-PS measure μ of dimension δ implies that $\delta \phi \geq \psi_\Gamma$ by [Qui02a, Thm. 8.1] and so $\delta \phi$ is tangent to the growth indicator of Γ. Note that while [Qui02a, Thm. 8.1] assumes the PS-measure is non-coarse, the same proof works for coarse PS-measures as well.

To show (2), let $S \subset \Gamma$ be the finite symmetric generating set which induces d_w. By [GT20, Prop. 7.8] the Green metric is quasi-isometric to d_w and so there exist $a > 1$ and $b > 0$ such that

$$a^{-1} \mathrm{d}_w(\gamma_1, \gamma_2) - b \leq \phi(\kappa(\gamma_1^{-1} \gamma_2)) \leq a \, \mathrm{d}_w(\gamma_1, \gamma_2) + b$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Then

$$\mathrm{d}_w(\gamma_1, \gamma_2) \leq a \phi(\kappa(\gamma_1^{-1} \gamma_2)) + b \leq a \|\phi\| \left\| \kappa(\gamma_1^{-1} \gamma_2) \right\| + b = a \|\phi\| \, \mathrm{d}_X(\gamma_1 o, \gamma_2 o) + b$$

and

$$\mathrm{d}_X(\gamma_1 x_0, \gamma_2 x_0) \leq C \, \mathrm{d}_w(\gamma_1, \gamma_2)$$

where $C := \max_{s \in S} \mathrm{d}_X(x_0, s x_0)$. So (2) follows. □

We now restate and prove Corollary 1.16.

Corollary 12.8. *If* Γ *is word hyperbolic (as an abstract group) and contains a unipotent element of* G*, then* ν *is singular with respect to every coarse Iwasawa PS-measure on* $\mathcal{F}$*.*

Proof. Suppose for a contradiction that ν is non-singular to some coarse ϕ-PS measure μ of dimension δ. Fix a word metric d_w on Γ with respect to a finite generating set and $x_0 \in X$. By Theorem 12.7, the map

$$\gamma \in (\Gamma, \mathrm{d}_w) \mapsto \gamma x_0 \in (X, \mathrm{d}_X)$$

is a quasi-isometry. However, if $u \in \Gamma$ is a unipotent element of G, then

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{d}_X(u^n x_0, x_0) = 0$$

while

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{d}_w(u^n, \mathrm{id}) > 0$$

since Γ is word hyperbolic and $u \in \Gamma$ has infinite order (hence is loxodromic). So we have a contradiction. $\square$

13. Pseudo-Riemannian hyperbolic spaces

In this section we prove Theorem 1.24 from the introduction. Throughout the section we will freely use the notation introduced in Section 9.8.

Let $[\cdot,\cdot]_{p,q+1}$ denote the symmetric bilinear form on $\mathbb{R}^{p+q+1}$ defined by

$$[v,w]_{p,q+1} = v_1w_1 + \cdots + v_pw_p - v_{p+1}w_{p+1} - \cdots - v_{p+q+1}w_{p+q+1}.$$

Then let $\mathsf{O}(p,q+1) < \mathsf{GL}(p+q+1,\mathbb{R})$ denote the group which preserves $[\cdot,\cdot]_{p,q+1}$ and let $\mathsf{PO}(p,q+1) < \mathsf{PGL}(p+q+1,\mathbb{R})$ denote its projectivization.

The associated pseudo-Riemannian hyperbolic space is

$$\mathbb{H}^{p,q} := \{[v] \in \mathbb{P}(\mathbb{R}^{p+q+1}) : [v,v]_{p,q+1} < 0\}.$$

By studying the action on $\mathbb{H}^{p,q}$, Danciger–Guéritaud–Kassel [DGK18] introduced convex cocompact subgroups of $\mathsf{PO}(p,q+1)$.

A subset of $\mathbb{P}(\mathbb{R}^{p+q+1})$ is *properly convex* if it is bounded and convex in some affine chart of $\mathbb{P}(\mathbb{R}^{p+q+1})$. A *non-trivial projective line segment* is a connected subset of a projective line that contains more than one point.

Definition 13.1. [DGK18] A discrete subgroup $\Gamma < \mathsf{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$*-convex cocompact* if there exists a convex subset $\mathcal{C} \subset \mathbb{H}^{p,q}$ such that

- $\mathcal{C}$ is closed in $\mathbb{H}^{p,q}$, has non-empty interior, and the set of accumulation points $\partial_{\mathrm{i}}\,\mathcal{C}$ in $\partial\,\mathbb{H}^{p,q}$ contains no non-trivial projective line segments,
- $\mathcal{C}$ is Γ-invariant and the quotient $\Gamma\backslash\,\mathcal{C}$ is compact.

As mentioned in the introduction, Glorieux–Monclair [GM21] introduced a *critical exponent* $\delta_{\mathbb{H}^{p,q}}(\Gamma)$ for a $\mathbb{H}^{p,q}$-convex cocompact subgroup $\Gamma < \mathsf{PO}(p,q+1)$ and proved that

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) \leq p-1. \tag{32}$$

In this section we prove Theorem 1.24, which we restate here.

Theorem 13.2. *If $\Gamma < \mathsf{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact and $\delta_{\mathbb{H}^{p,q}}(\Gamma) = p-1$, then Γ preserves and acts cocompactly on a totally geodesic copy of $\mathbb{H}^p$ in $\mathbb{H}^{p,q}$.*

When $\Gamma < \mathsf{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, results of Carvajales [Car20, Remarks 6.9 and 7.15] and Sambarino [Sam24, Prop. 3.3.2] imply that

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) = \delta_{\omega_1}(\Gamma).$$

where ω_1 is the fundamental weight associated to α_1 and $\delta_{\omega_1}(\Gamma)$ is the critical exponent associated to ω_1 (for definitions see Sections 9.8 and 9.3).

Let $d := p+q+1$. Since $\omega_1(\kappa(g)) = \omega_{d-1}(\kappa(g))$ for all $g \in \mathsf{PO}(p,q+1)$, we then have

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) = \delta_{\omega_1}(\Gamma) = \delta_{\psi}(\Gamma) \tag{33}$$

where $\psi := \frac{1}{2}(\omega_1 + \omega_{d-1})$.

13.1. **The Anosov property and negativity of the limit set.** In the arguments that follow we will need some results from [DGK24, DGK18] about convex cocompact subgroups in $\mathsf{PO}(p, q+1)$.

For the rest of the section suppose that $\Gamma < \mathsf{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact and suppose that $\mathcal{C}$ satisfies Definition 13.1. Let $d := p + q + 1$.

By [DGK24, Thm. 1.24], $\Gamma < \mathsf{PO}(p, q+1)$ is $\mathsf{P}_{1,d-1}$-Anosov and by [DGK24, Thm. 1.15 and Lem. 7.1]

$$\Lambda_{1,d-1}(\Gamma) = \{(x, x^{\perp}) : x \in \partial_{\mathrm{i}} \mathcal{C}\}$$

where $x^{\perp}$ is the orthogonal complement with respect to $[\cdot, \cdot]_{p,q+1}$.

Let $\tilde{\mathcal{C}} \subset \mathbb{R}^{p+q+1}$ be a convex cone above $\mathcal{C}$ and let $\tilde{\Lambda} \subset \mathbb{R}^{p+q+1}$ be the cone above $\partial_{\mathrm{i}} \mathcal{C}$ contained in the closure of $\tilde{\mathcal{C}}$. Any element $\gamma \in \Gamma$ lifts to a unique element

$$\tilde{\gamma} \in \mathsf{O}(p, q+1)$$

which preserves $\tilde{\mathcal{C}}$. By uniqueness, the map

$$\gamma \in \Gamma \mapsto \tilde{\tau}(\gamma) := \tilde{\gamma} \tag{34}$$

is a injective homomorphism.

Theorem 13.3. [DGK24] *If $x \in \tilde{\Lambda}$ and $y \in \tilde{\mathcal{C}} \cup \tilde{\Lambda}$ are not collinear, then $[x, y]_{p,q+1} < 0$.*

Proof. By [DGK24, Thm. 1.24] we have $[x, y]_{p,q+1} < 0$ when $x, y \in \tilde{\Lambda}$ are not collinear. In the case when $x \in \tilde{\Lambda}$ and $y \in \tilde{\mathcal{C}}$, [DGK24, Lem. 11.4] says that $[x, y]_{p,q+1} \neq 0$. Then, since y can be continuously deformed to a point in $\tilde{\Lambda} \smallsetminus \mathbb{R}^{+} \cdot x$, we must have $[x, y]_{p,q+1} < 0$. $\square$

13.2. **Patterson–Sullivan measures and Hausdorff dimension.** Suppose $\Gamma < \mathsf{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact. As before, let $d := p + q + 1$ and

$$\psi := \frac{1}{2}(\omega_1 + \omega_{d-1}).$$

Since Γ is $\mathsf{P}_{1,d-1}$-Anosov and $\psi \in \mathfrak{a}_{1,d-1}^{*}$, there is a unique ψ-PS measure $\tilde{\mu}_{\psi}$ for Γ supported on $\Lambda_{1,d-1}(\Gamma)$ of dimension $\delta := \delta_{\psi}(\Gamma)$, see Theorems 9.7 and 9.6. Let μ_{ψ} be the push-forward of $\tilde{\mu}_{\psi}$ under the homeomorphism $\Lambda_{1,d-1}(\Gamma) \to \Lambda_1(\Gamma)$.

Fix a distance $\mathrm{d}_{\mathbb{P}}$ on $\partial\, \mathbb{H}^{p,q}$ induced by a Riemannian metric and let $\mathcal{H}^{\delta}$ be the associated δ-dimensional Hausdorff measure.

Proposition 13.4. *There exists $C > 1$ such that $\mu_{\psi}(A) \leq C\, \mathcal{H}^{\delta}(A)$ for any Borel measurable set $A \subset \Lambda_1(\Gamma)$.*

The rest of the subsection is devoted to the proof of Proposition 13.4. We will use results in [DKO24] to prove the proposition. Alternatively, one could use results in [GM21] or [GMT23].

Define a distance-like function d_{Λ} on $\Lambda_1(\Gamma)$ by

$$\mathrm{d}_{\Lambda}(x, y) = \big(\sin \angle(x, y^{\perp})\big)^{1/4} \big(\sin \angle(y, x^{\perp})\big)^{1/4}$$

(in the notation of [DKO24] this is d_{ψ}, see [DKO24, Def. 5.1, Lem. 10.4]).

For $x \in \Lambda_1(\Gamma)$ and $r > 0$, let $B_{\Lambda}(x, r) := \{y \in \Lambda_1(\Gamma) : \mathrm{d}_{\Lambda}(x, y) < r\}$. Then by [DKO24, Thm. 8.2], there exist $C_1 > 1$, $r_0 > 0$ such that

$$C_1^{-1} r^{\delta} \leq \mu_{\psi}(B_{\Lambda}(x, r)) \leq C_1 r^{\delta} \tag{35}$$

for all $x \in \Lambda_1(\Gamma)$ and $r \in [0, r_0]$.

Lemma 13.5. *There exists* $C_2 > 0$ *such that* $\mathrm{d}_\Lambda \leq C_2 \, \mathrm{d}_{\mathbb{P}}$ *on* $\Lambda_1(\Gamma)$.

Proof. One can show that

$$\sin \angle([v],[w]^\perp) = \frac{\left|[v,w]_{p,q+1}\right|}{\|v\| \, \|w\|}$$

and so

$$\mathrm{d}_\Lambda([v],[w]) = \frac{\left|[v,w]_{p,q+1}\right|^{1/2}}{\|v\|^{1/2} \, \|w\|^{1/2}}.$$

Further, we can fix $\epsilon > 0$ such that

$$\mathrm{d}_{\mathbb{P}}([v],[w]) \geq \epsilon \min\{\|v-w\|, \|v-(-w)\|, \|(-v)-w\|, \|(-v)-(-w)\|\}$$

when $v, w \in \mathbb{S}^{p-1} \times \mathbb{S}^q$.

Now fix $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{S}^{p-1} \times \mathbb{S}^q$ with $[v],[w] \in \Lambda_1(\Gamma)$. Then

$$\begin{aligned} \mathrm{d}_\Lambda([v],[w]) &= \frac{\left|[v,w]_{p,q+1}\right|^{1/2}}{\|v\|^{1/2} \, \|w\|^{1/2}} = \frac{1}{2} \left|[v-w, v-w]_{p,q+1}\right|^{1/2} \\ &= \frac{1}{2} \sqrt{\left| \|v_1 - w_1\|^2 - \|v_2 - w_2\|^2 \right|} \leq \frac{1}{2} \|v - w\|. \end{aligned}$$

Since v, w were arbitrary lifts of $[v],[w]$ in $\mathbb{S}^{p-1} \times \mathbb{S}^q$, we have

$$\mathrm{d}_\Lambda([v],[w]) \leq \epsilon^{-1} \, \mathrm{d}_{\mathbb{P}}([v],[w]).$$

□

For $x \in \partial \, \mathbb{H}^{p,q}$ and $r > 0$, let $B_{\mathbb{P}}(x,r) := \{y \in \partial \, \mathbb{H}^{p,q} : \mathrm{d}_{\mathbb{P}}(x,y) < r\}$. Then the previous lemma implies that

$$B_{\mathbb{P}}(x,r) \cap \Lambda_1(\Gamma) \subset B_\Lambda(x, C_2 r)$$

for all $x \in \Lambda_1(\Gamma)$ and $r > 0$.

Now we are ready to prove Proposition 13.4.

Proof of Proposition 13.4. Suppose $A \subset \Lambda_1(\Gamma)$ is a Borel measurable set. Fix $\{x_n\}_{n \in I} \subset \partial \, \mathbb{H}^{p,q}$ and $\{r_n\}_{n \in I} \subset (0, \frac{1}{2C_2} r_0]$ such that

$$A \subset \bigcup_{n \in I} B_{\mathbb{P}}(x_n, r_n).$$

We can assume that for every $n \in I$ there exists $y_n \in \Lambda_1(\Gamma) \cap B_{\mathbb{P}}(x_n, r_n)$. Then

$$A \subset \bigcup_{n \in I} B_\Lambda(y_n, 2C_2 r_n).$$

Hence by Equation (35),

$$\mu_\psi(A) \leq \sum_{n \in I} \mu_\psi\left(B_\Lambda(y_n, 2C_2 r_n)\right) \leq \sum_{n \in I} C_1 (2C_2)^\delta r_n^\delta.$$

Thus $\mu_\psi \leq C_1 (2C_2)^\delta \, \mathcal{H}^\delta$. □

13.3. **A second Patterson–Sullivan measure.** As in the previous subsection, fix a distance $\mathrm{d}_{\mathbb{P}}$ on $\partial \mathbb{H}^{p,q}$ induced by a Riemannian metric and let $\mathcal{H}^{p-1}$ be the associated $(p-1)$-dimensional Hausdorff measure.

In the next result let $\mathrm{pr} : \mathcal{F}_{1,p}(\mathbb{R}^{p+q+1}) \to \mathbb{P}(\mathbb{R}^{p+q+1})$ denote the natural projection.

Proposition 13.6. *Suppose that* $\Gamma < \mathsf{PO}(p, q+1)$ *is* $\mathbb{H}^{p,q}$*-convex cocompact. Then* $\mathcal{H}^{p-1}(\Lambda_1(\Gamma)) < +\infty$*. Moreover, if* $\mathcal{H}^{p-1}(\Lambda_1(\Gamma)) > 0$*, then there exists an* $\mathcal{H}^{p-1}$*-a.e. defined measurable* Γ*-equivariant map* $\zeta : \Lambda_1(\Gamma) \to \mathcal{F}_{1,p}(\mathbb{R}^{p+q+1})$ *such that*

(1) $\mathrm{pr} \circ \zeta = \mathrm{id}_{\Lambda_1(\Gamma)}$*, and*
(2) $\frac{1}{\mathcal{H}^{p-1}(\Lambda_1(\Gamma))}\zeta_* \left(\mathcal{H}^{p-1}|_{\Lambda_1(\Gamma)}\right)$ *is a coarse* $(p\omega_1 - \omega_p)$*-PS measure for* Γ *of dimension* 1*.*

The first assertion is well-known and the "moreover" part is very similar to [PSW23, Prop. 6.4] (which considers Anosov groups whose limits are Lipschitz manifolds).

Proof. Suppose $\Gamma < \mathsf{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact. First observe that the map

$$\Phi : \mathbb{S}^{p-1} \times \mathbb{S}^q \to \partial \mathbb{H}^{p,q}$$
$$\Phi(v, w) = [(v, w)]$$

is a smooth 2-to-1 covering map. Let $\Lambda' := \Phi^{-1}(\Lambda_1(\Gamma))$. Theorem 13.3 implies that

$$\langle v_1, v_2\rangle \leq \langle w_1, w_2\rangle \tag{36}$$

for all $(v_1, w_1), (v_2, w_2) \in \Lambda'$.

Observation 13.7. The projection $(v, w) \mapsto v$ is 1-to-1 on Λ'.

Proof. If $(v, w_1), (v, w_2) \in \Lambda'$, then Equation (36) implies that

$$1 = \langle v, v\rangle \leq \langle w_1, w_2\rangle \leq \|w_1\| \, \|w_2\| = 1.$$

So by the equality case of Cauchy–Schwarz we must have $w_1 = w_2$. □

Then there exists a closed set $D \subset \mathbb{S}^{p-1}$ and a function $f : D \to \mathbb{S}^q$ such that

$$\Lambda' = \{(x, f(x)) : x \in D\}.$$

By Equation (36),

$$\langle x, y\rangle \leq \langle f(x), f(y)\rangle$$

for all $x, y \in D$. Hence

$$\|f(x) - f(y)\| \leq \|x - y\|$$

for all $x, y \in D$. This implies that Λ' is bi-Lipschitz to D. Since Φ is smooth, $\mathcal{H}^{p-1}(\Lambda_1(\Gamma)) < +\infty$.

Now suppose that $\mathcal{H}^{p-1}(\Lambda_1(\Gamma)) > 0$. Since there exists an onto Lipschitz map $D \to \Lambda_1(\Gamma)$, the set $\Lambda_1(\Gamma)$ is $(p-1)$-rectifiable. Then $\mathcal{H}^{p-1}$-a.e. $x \in \Lambda_1(\Gamma)$ has a well-defined tangent plane $T_x\Lambda_1(\Gamma)$, see Appendix A. For such x, let $V_x \subset \mathbb{R}^{p+q+1}$ be the p-dimensional linear subspace containing x with $T_x \mathbb{P}(V_x) = T_x\Lambda_1(\Gamma)$. Then define a $\mathcal{H}^{p-1}$-a.e. defined measurable map $\zeta : \Lambda_1(\Gamma) \to \mathcal{F}_{1,p}(\mathbb{R}^d)$ by

$$\zeta(x) = (x, V_x).$$

Since tangent planes are $\mathcal{H}^{p-1}$-a.e. unique, we can assume that ζ is Γ-equivariant.

Let $\nu := \frac{1}{\mathcal{H}^{p-1}(\Lambda_1(\Gamma))}\zeta_*\left(\mathcal{H}^{p-1}|_{\Lambda_1(\Gamma)}\right)$. By the coarea formula (Equation (40)) and Observation A.1 in Appendix A, there exists $C > 1$ such that for every $\gamma \in \Gamma$ the measures $\gamma_*\nu$, ν are absolutely continuous and

$$\frac{1}{C}e^{-(p\omega_1-\omega_p)(B_{1,p}(\gamma^{-1},F))} \leq \frac{d\gamma_*\nu}{d\nu}(F) \leq Ce^{-(p\omega_1-\omega_p)(B_{1,p}(\gamma^{-1},F))} \quad \nu\text{-a.e.}$$

Hence the measure ν is a coarse $(p\omega_1 - \omega_p)$-PS measure for Γ of dimension 1. $\square$

13.4. **The proof in the strongly irreducible case.** We prove the main theorem (Theorem 13.2) in the strongly irreducible case.

Proposition 13.8. *If $\Gamma < \mathsf{PO}(p, q + 1)$ is $\mathbb{H}^{p,q}$-convex cocompact, strongly irreducible, and $\delta_{\mathbb{H}^{p,q}}(\Gamma) = p-1$, then $q = 0$ and Γ is a uniform lattice in $\mathsf{PO}(p, q+1) = \mathsf{PO}(p, 1)$.*

Proof. Fix a distance $\mathrm{d}_{\mathbb{P}}$ on $\partial\,\mathbb{H}^{p,q}$ induced by a Riemannian metric and let $\mathcal{H}^{p-1}$ be the associated $(p-1)$-dimensional Hausdorff measure.

Let μ_ψ be as in Section 13.2. By Equation (33), $\delta_\psi(\Gamma) = \delta_{\mathbb{H}^{p,q}}(\Gamma) = p - 1$, so Proposition 13.4 implies that $\mathcal{H}^{p-1}(\Lambda_1(\Gamma)) > 0$. So by Proposition 13.6 there exists a $\mathcal{H}^{p-1}$-a.e. defined measurable Γ-equivariant map $\zeta : \Lambda_1(\Gamma) \to \mathcal{F}_{1,p}(\mathbb{R}^{p+q+1})$ such that

(1) $\mathrm{pr} \circ \zeta = \mathrm{id}_{\Lambda_1(\Gamma)}$, and
(2) $\nu := \frac{1}{\mathcal{H}^{p-1}(\Lambda_1(\Gamma))}\zeta_*\left(\mathcal{H}^{p-1}|_{\Lambda_1(\Gamma)}\right)$ is a coarse $(p\omega_1 - \omega_p)$-PS measure for Γ of dimension 1.

By Proposition 13.4, ζ is also μ_ψ-almost everywhere defined and

$$\zeta_*\mu_\psi \ll \nu.$$

By Proposition 9.5, μ_ψ is part of a well-behaved PS-system with magnitude function $g \mapsto \psi(\kappa(g)) = \omega_1(\kappa(g))$. By Proposition 9.17 and Theorem 9.11, ν is part of a PS-system with magnitude function $g \mapsto (p\omega_1 - \omega_p)(\kappa(g))$. Since Γ is $\mathsf{P}_{1,d-1}$-Anosov, it follows from Theorem 9.7 that

$$\sum_{g\in\Gamma} e^{-(p-1)\omega_1(\kappa(g))} = +\infty.$$

Thus by Theorem 7.1,

$$(p-1)\omega_1(\lambda(g)) = (p\omega_1 - \omega_p)(\lambda(g)) \tag{37}$$

for all $g \in \Gamma$, where $\lambda(g) = \lim_{n\to\infty} \kappa(g^n)/n$ is the Jordan projection of g.

Recall that

$$\lambda(g) = (\log \lambda_1(g), \cdots, \log \lambda_{p+q+1}(g))$$

where $\lambda_1(g) \geq \cdots \geq \lambda_d(g)$ are the absolute values of the generalized eigenvalues of some (any) representative of g in $\mathsf{GL}(d, \mathbb{R})$ with determinant ± 1.

Lemma 13.9. *If $\gamma \in \Gamma$, then $\lambda_j(\gamma) = 1$ for $j = 2, \ldots, p + q$.*

Proof. Let $r := \min\{p, q + 1\}$. Since $\gamma \in \mathsf{PO}(p, q + 1)$, the eigenvalues satisfy

$$\lambda_j(\gamma) = \lambda_{p+q+2-j}(\gamma)^{-1} \quad \text{for } j = 1, \ldots, r$$
$$\lambda_j(\gamma) = 1 \quad \text{for } j = r + 1, \ldots, d - r.$$

In particular, $\lambda_j(\gamma) \geq 1$ for $2 \leq j \leq p$. Then

$$\begin{aligned}(p\omega_1 - \omega_p)(\lambda(\gamma)) &= p \log \lambda_1(\gamma) - \log\big(\lambda_1(\gamma)\cdots\lambda_p(\gamma)\big)\\ &= (p-1)\log\lambda_1(\gamma) - \log\big(\lambda_2(\gamma)\cdots\lambda_p(\gamma)\big)\\ &\leq (p-1)\log\lambda_1(\gamma) = (p-1)\omega_1(\lambda(\gamma)).\end{aligned}$$

So Equation (37) implies that $\lambda_j(\gamma) = 1$ for $j = 2, \dots, p$. The same reasoning applied to γ^{-1} shows that $\lambda_j(\gamma) = 1$ for $j = p+q, \dots, q+2$. Since

$$\lambda_1(\gamma) \geq \cdots \geq \lambda_{p+q+1}(\gamma),$$

we have $\lambda_j(\gamma) = 1$ for $j = 2, \dots, p+q$ □

Hence for every $\gamma \in \Gamma$ we have

$$\lambda_2(\gamma) = \cdots = \lambda_{p+q}(\gamma) = 1.$$

Since Γ is strongly irreducible, this eigenvalue condition implies that Γ has a finite index subgroup which is conjugate to a Zariski dense subgroup of $\mathsf{PO}_0(p+q,1)$, see Observation B.1 in Appendix B. Since Γ is also a subgroup of $\mathsf{PO}(p,q+1)$, we must have $q = 0$.

Since $q = 0$, Γ is a convex cocompact subgroup of $\mathsf{PO}(p,1)$ in the classical real hyperbolic geometry sense. Since $\delta_{\mathbb{H}^{p,0}}(\Gamma) = p-1$ coincides with the classical critical exponent from real hyperbolic geometry (by definition, see [GM21]), a result of Tukia [Tuk84] implies that Γ is a uniform lattice in $\mathsf{PO}(p,1)$. □

13.5. **Reducing to the strongly irreducible case.** In this subsection we explain how to reduce to the strongly irreducible case.

Suppose $\Gamma_0 < \mathsf{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact and **has connected Zariski closure**. Let $\tilde{\tau} : \Gamma_0 \to \mathsf{SL}^{\pm}(p+q+1,\mathbb{R})$ be a lift as in Equation (34).

Let

$$U := \operatorname{Span} \Lambda_1(\Gamma_0)$$

and

$$V_1 := U \cap \bigcap_{x \in \Lambda_1(\Gamma_0)} x^{\perp} = U \cap \bigcap_{x \in \mathbb{P}(U)} x^{\perp}$$

(here $x^{\perp}$ is the orthogonal complement with respect to $[\cdot,\cdot]_{p,q+1}$). Then fix subspaces V_2, V_3 such that $U = V_1 \oplus V_2$ and

$$\mathbb{R}^{p+q+1} = U \oplus V_3 = V_1 \oplus V_2 \oplus V_3.$$

By construction, any element of Γ_0 is upper triangular relative to the decomposition $\mathbb{R}^{p+q+1} = V_1 \oplus V_2 \oplus V_3$ and so we can define representations

$$\tilde{\rho}_i : \Gamma_0 \to \mathsf{GL}(V_i)$$

such that for every $\gamma \in \Gamma_0$ we have

$$\tilde{\tau}(\gamma) = \begin{pmatrix} \tilde{\rho}_1(\gamma) & * & * \\ 0 & \tilde{\rho}_2(\gamma) & * \\ 0 & 0 & \tilde{\rho}_3(\gamma) \end{pmatrix}$$

relative to the decomposition $\mathbb{R}^{p+q+1} = V_1 \oplus V_2 \oplus V_3$.

Let $\rho_2 : \Gamma_0 \to \mathsf{PGL}(V_2)$ be the projectivization of $\tilde{\rho}_2$. It follows from [GGKW17, the proof of Prop. 4.13] that $\rho_2(\Gamma_0) \subset \mathsf{PGL}(V_2)$ is irreducible, P_1-Anosov, and

$$\frac{\lambda_1(\rho_2(\gamma))}{\lambda_{\dim V_2}(\rho_2(\gamma))} = \frac{\lambda_1(\gamma)}{\lambda_{p+q+1}(\gamma)} \tag{38}$$

for all $\gamma \in \Gamma_0$. Moreover, if $\pi : V_1 \oplus V_2 \to V_2$ is the projection, then the map

$$x \in \Lambda_1(\Gamma_0) \mapsto \pi(x) \in \mathbb{P}(V_2)$$

is a homeomorphism onto $\Lambda_1(\rho_2(\Gamma_0))$.

By the definition of V_1,

$$[\rho_2(\gamma)v, \rho_2(\gamma)w]_{p,q+1} = [v,w]_{p,q+1} \tag{39}$$

for all $v, w \in V_2$ and $\gamma \in \Gamma_0$.

Since Γ_0 has connected Zariski closure, so does $\rho_2(\Gamma_0)$. Thus any finite index subgroup of $\rho_2(\Gamma_0)$ has the same Zariski closure and hence is irreducible. So $\rho_2(\Gamma_0)$ is strongly irreducible.

Lemma 13.10. *The linear subspace*

$$V_2^{\mathrm{null}} := \{v \in V_2 : [v,w]_{p,q+1} = 0 \text{ for all } w \in V_2\}$$

is trivial.

Proof. Equation (39) implies that the linear subspace V_2^{null} is $\rho_2(\Gamma_0)$-invariant. Hence by irreducibility, $V_2^{\mathrm{null}} = \{0\}$ or $V_2^{\mathrm{null}} = V_2$.

Fix $x, y \in \Lambda_1(\Gamma_0)$ distinct. By Theorem 13.3, we can lift x, y to $\tilde{x}, \tilde{y} \in \mathbb{R}^{p+q+1}$ such that $[\tilde{x}, \tilde{y}]_{p,q+1} < 0$. We can also write $\tilde{x} = x_1 + x_2$ and $\tilde{y} = y_1 + y_2$ relative to the decomposition $U = V_1 \oplus V_2$. Then

$$0 > [\tilde{x}, \tilde{y}]_{p,q+1} = [x_1 + x_2, y_1 + y_2]_{p,q+1} = [x_2, y_2]_{p,q+1}.$$

Hence $x_2, y_2 \notin V_2^{\mathrm{null}}$ and so V_2^{null} is trivial. $\square$

Thus $[\cdot,\cdot]_{p,q+1}$ restricts to a non-degenerate symmetric 2-form on V_2. So there exist p', q' such that $0 \le p' \le p$ and $-1 \le q' \le q$, and an isomorphism

$$T : V_2 \to \mathbb{R}^{p'+q'+1}$$

such that

$$[T(v), T(w)]_{p',q'+1} = [v,w]_{p,q+1}$$

for all $v, w \in V_2$. Let $\Phi : \mathsf{PGL}(V_2) \to \mathsf{PGL}(p'+q'+1, \mathbb{R})$ be the representation $\Phi(g) = [T \circ g \circ T^{-1}]$. Then Equation (39) implies that

$$\Gamma' := \Phi(\rho_2(\Gamma_0)) < \mathsf{PO}(p', q'+1).$$

Since $\rho_2(\Gamma_0)$ is P_1-Anosov, Γ' is non-compact. Hence we must have $p' > 0$ and $q' > -1$.

Proposition 13.11. $\Gamma' := \Phi(\rho_2(\Gamma_0))$ *is* $\mathbb{H}^{p',q'}$*-convex cocompact, strongly irreducible, and*

$$\delta_{\mathbb{H}^{p',q'}}(\Gamma') = \delta_{\mathbb{H}^{p,q}}(\Gamma_0).$$

Moreover, $p' \le p$ *and*

$$\dim V_1 \le p - p'.$$

Proof. The strong irreducibility of Γ' follows from the strong irreducibility of $\rho_2(\Gamma_0)$. We first verify that Γ' is $\mathbb{H}^{p',q'}$-convex cocompact. By [DGK24, Thm. 1.24], it suffices to show that $\Lambda_1(\Gamma')$ lifts to a cone in $\mathbb{R}^{p'+q'+1}$ where $[\cdot,\cdot]_{p',q'+1}$ is negative for every pair of non-collinear points.

Recall that $\pi : V_1 \oplus V_2 \to V_2$ was the projection and $\Lambda_1(\rho_2(\Gamma_0)) = \pi(\Lambda_1(\Gamma_0))$. Fix a cone $\tilde{\Lambda} \subset \mathbb{R}^{p+q+1}$ above $\Lambda_1(\Gamma_0)$ as in Section 13.1. Then

$$\tilde{\Lambda}' := T\pi(\tilde{\Lambda})$$

is a cone above $\Lambda_1(\Gamma')$. Fix $x', y' \in \tilde{\Lambda}'$ non-collinear. Then $x' = T\pi(x)$ and $y' = T\pi(y)$ for some non-collinear $x, y \in \tilde{\Lambda}$. We can write $x = x_1 + x_2$ and $y = y_1 + y_2$ relative to the decomposition $U = V_1 \oplus V_2$. Then

$$\begin{aligned}[x', y']_{p',q'} &= [T\pi(x), T\pi(y)]_{p',q'} = [\pi(x), \pi(y)]_{p,q+1} = [x_2, y_2]_{p,q+1} \\ &= [x_1 + x_2, y_1 + y_2]_{p,q+1} = [x, y]_{p,q+1} < 0\end{aligned}$$

by Theorem 13.3. So Γ' is $\mathbb{H}^{p',q'}$-convex cocompact.

Let $[\Gamma]$ and $[\Gamma']$ denote the set of conjugacy classes in Γ and Γ'. Then by [Car20, Remarks 6.9, 7.15],

$$\delta_{\mathbb{H}^{p',q'}}(\Gamma') = \lim_{R\to\infty} \frac{1}{R} \log \# \left\{ [\gamma] \in [\Gamma'] : \frac{1}{2} \log \frac{\lambda_1(\gamma)}{\lambda_{p'+q'+1}(\gamma)} \leq R \right\}$$

and

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) = \lim_{R\to\infty} \frac{1}{R} \log \# \left\{ [\gamma] \in [\Gamma] : \frac{1}{2} \log \frac{\lambda_1(\gamma)}{\lambda_{p+q+1}(\gamma)} \leq R \right\}.$$

So by Equations (38),

$$\delta_{\mathbb{H}^{p',q'}}(\Gamma') = \delta_{\mathbb{H}^{p,q}}(\Gamma).$$

For the "moreover" part, notice that $[\cdot,\cdot]_{p,q+1}$ is positive semidefinite on $W := V_1 \oplus T^{-1}(\mathbb{R}^{p'} \oplus \{0_{\mathbb{R}^{q'+1}}\})$ and hence

$$W \cap (\{0_{\mathbb{R}^p}\} \times \mathbb{R}^{q+1}) = \emptyset.$$

Thus

$$\dim V_1 + p' = \dim W \leq (p + q + 1) - (q + 1) = p. \qquad \square$$

13.6. **Proof of Theorem 13.2.** We can now prove the theorem in full generality. Suppose $\Gamma < \mathsf{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact and $\delta_{\mathbb{H}^{p,q}}(\Gamma) = p-1$. Fix a finite index subgroup $\Gamma_0 < \Gamma$ with connected Zariski closure. Then $\Gamma_0 < \mathsf{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact,

$$\delta_{\mathbb{H}^{p,q}}(\Gamma_0) = \delta_{\mathbb{H}^{p,q}}(\Gamma) = p - 1,$$

and

$$\Lambda_1(\Gamma_0) = \Lambda_1(\Gamma).$$

Let $\mathbb{R}^{p+q+1} = V_1 \oplus V_2 \oplus V_3$ and $\Gamma' < \mathsf{PO}(p', q'+1)$ be as in Section 13.5. Then by Proposition 13.11 and Glorieux–Monclair's [GM21] upper bound on critical exponent,

$$p - 1 = \delta_{\mathbb{H}^{p',q'}}(\Gamma') \leq p' - 1.$$

So by the "moreover" part of Proposition 13.11, we have $p' = p$ and $V_1 = \{0\}$.

Since $\delta_{\mathbb{H}^{p',q'}}(\Gamma') = p'-1$, Proposition 13.8 implies that $q' = 0$ and $\Gamma' < \mathsf{PO}(p', 1)$ is a cocompact lattice. Since $V_1 = \{0\}$, we see that Γ_0 preserves $Y := \mathbb{P}(V_2) \cap \mathbb{H}^{p,q}$. Since $q' = 0$, we see that Y is a totally geodesic copy of $\mathbb{H}^p$. Since $\Gamma' < \mathsf{PO}(p', 1)$ is a cocompact lattice, Γ_0 acts cocompactly on Y. Since Γ_0 preserves $Y = \mathbb{P}(V_2) \cap \mathbb{H}^{p,q}$, we have

$$\Lambda_1(\Gamma) = \Lambda_1(\Gamma_0) \subset \mathbb{P}(V_2)$$

and hence Γ also preserves Y. $\square$

Part 3. Appendices

Appendix A. Rectifiable sets

In this appendix we record some basis properties of rectifiable sets that are used in the proof of Theorem 13.2. For more background see [Fed69, Sect. 3.2].

A.1. The Euclidean Case.

Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure induced by the Euclidean metric on $\mathbb{R}^d$. A subset $E \subset \mathbb{R}^d$ is *k-rectifiable* if $\mathcal{H}^k(E) < +\infty$ and there exists a countable collection of Lipschitz maps $f_i : U_i \to \mathbb{R}^d$ defined on bounded subsets $U_i \subset \mathbb{R}^k$ such that

$$\mathcal{H}^k\left(E \smallsetminus \bigcup_i f(U_i)\right) = 0.$$

(This is called $(\mathcal{H}^k, k)$-rectifiable in [Fed69].)

If $E \subset \mathbb{R}^d$ is k-rectifiable, then for $\mathcal{H}^k$-a.e. $x \in E$ there exists a unique k-dimensional subspace T_xE, called the *approximate tangent plane of E at x*, such that

$$\lim_{r\searrow 0} \frac{1}{r^k}\,\mathcal{H}^k\left(E \cap B_r(x) \smallsetminus \{y : \mathrm{d}_{\mathbb{R}^d}(y, x + T_xE) < \epsilon\,|y-x|\}\right) = 0$$

for all $\epsilon > 0$, see [Fed69, Thm. 3.2.19].

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$ and for $k \geq 1$ let $\|\cdot\|_{\wedge^k \mathbb{R}^d}$ be the norm induced by the inner product on $\wedge^k \mathbb{R}^d$ where $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ is an orthonormal basis. Given a linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ and a k-dimensional subspace $V = \langle v_1, \dots, v_k\rangle$, let

$$J(A, V) := \frac{\|A(v_1 \wedge \cdots \wedge v_k)\|_{\wedge^k \mathbb{R}^d}}{\|v_1 \wedge \cdots \wedge v_k\|_{\wedge^k \mathbb{R}^d}}.$$

Suppose $E \subset \mathbb{R}^d$ is k-rectifiable with $\mathcal{H}^k(E) > 0$, U is a neighborhood of E, and $\varphi : U \to \mathbb{R}^d$ is a diffeomorphism onto its image which induces a homeomorphism of $E \to E$. Let $\nu := \mathcal{H}^k|_E$. As a consequence of the coarea formula, see [Fed69, Cor. 3.2.20], the measure $\nu, \varphi_*\nu$ are absolutely continuous and

$$\frac{d\varphi_*\nu}{d\nu} = J(d(\varphi^{-1})(x), T_xE) \quad \nu\text{-a.e.} \tag{40}$$

A.2. The manifold case.

Next suppose that (M, d_M) is a Riemannian d-manifold and let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure induced by the Riemannian distance on M. One can define k-rectifiable subsets $E \subset M$ exactly as in the Euclidean case. Moreover, if $E \subset M$ is k-rectifiable, (U, ψ) is a coordinate chart, and $U' \subset U$ is a relatively compact set, then the set

$$\psi(U' \cap E) \subset \mathbb{R}^d$$

is a k-rectifiable subset of $\mathbb{R}^d$. Thus for $\mathcal{H}^k$-a.e. $x \in E$ there exists a unique k-dimensional subspace $T_xE \subset T_xM$, called the *approximate tangent plane of E at x*, such that

$$\lim_{r\searrow 0} \frac{1}{r^k}\,\mathcal{H}^k\left(E \cap B_r(x) \smallsetminus \{y : \mathrm{d}_M(y, \exp_x(T_xE \cap \mathcal{O})) < \epsilon\,|y-x|\}\right) = 0$$

for all sufficiently small $\epsilon > 0$ and sufficiently small neighborhood $\mathcal{O}$ of 0 in T_xM.

A.3. **The Iwasawa cocycle.** In this subsection we consider transformations of projective spaces and make a calculation that is used in the proof of Theorem 13.2. In this section we write $\|\cdot\| := \|\cdot\|_{\mathbb{R}^d}$ for the standard Euclidean norm on $\mathbb{R}^d$.

Let h_0 denote the $\mathsf{O}(d)$-invariant Riemannian metric on $\mathbb{P}(\mathbb{R}^d)$ scaled so that if $v \in \mathbb{R}^d$ is a unit vector and $w \in \mathbb{R}^d$ is orthogonal to v, then

$$\left\| \frac{d}{dt}\Big|_{t=0} [v+tw] \right\|_{h_0} = \|w\|. \tag{41}$$

The metric h_0 induces a metric h_k on the bundle $\wedge^k T\,\mathbb{P}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$ where $\{u_{i_1} \wedge \cdots \wedge u_{i_k}\}$ is an orthonormal basis of $\wedge^k T_x\,\mathbb{P}(\mathbb{R}^d)$ whenever $\{u_1, \dots, u_d\}$ is an orthonormal basis of $T_x\,\mathbb{P}(\mathbb{R}^d)$. Then, given a linear map $A : T_x\,\mathbb{P}(\mathbb{R}^d) \to T_y\,\mathbb{P}(\mathbb{R}^d)$ and a k-dimensional subspace $V = \langle v_1, \dots, v_k \rangle \subset T_x\,\mathbb{P}(\mathbb{R}^d)$, let

$$J(A, V) := \frac{\|A(v_1 \wedge \cdots \wedge v_k)\|_{h_k}}{\|v_1 \wedge \cdots \wedge v_k\|_{h_k}}.$$

Using the notation from Section 9.8 we have the following.

Observation A.1. If $\gamma \in \mathsf{PGL}(d, \mathbb{R})$, $x = [v_1] \in \mathbb{P}(\mathbb{R}^d)$, and $V = \langle v_1, \dots, v_{k+1} \rangle \in \mathrm{Gr}_{k+1}(\mathbb{R}^d)$, then

$$\log J\left(d(\gamma)_x, T_x\,\mathbb{P}(V)\right) = (\omega_{k+1} - (k+1)\omega_1)(B_{1,k+1}(\gamma, (x, V))).$$

Proof. Let $\tilde{\gamma}$ be a representative of γ in $\mathsf{GL}(d, \mathbb{R})$ with determinant ± 1. For each $y \in \mathbb{P}(\mathbb{R}^d)$, fix a unit vector v_y with $y = [v_y]$. Then define a linear isomorphism $\tau_y : y^\perp \to T_y\,\mathbb{P}(\mathbb{R}^d)$ by

$$\tau_y(w) = \frac{d}{dt}\Big|_{t=0} [v_y + tw].$$

By Equation (41),

$$\|\tau_y(w_1) \wedge \cdots \wedge \tau_y(w_k)\|_{h_k} = \|w_1 \wedge \cdots \wedge w_k\|_{\wedge^k \mathbb{R}^d} = \|v_y \wedge w_1 \wedge \cdots \wedge w_k\|_{\wedge^{k+1} \mathbb{R}^d}$$

where that last equality follows from the fact that v_y is a unit vector and $w_1, \dots, w_k \in y^\perp$. Also, notice that $\frac{\tilde{\gamma} v_y}{\|\tilde{\gamma} v_y\|} = \pm v_{\gamma y}$ and so

$$d(\gamma)_y \tau_y(w) = \pm \tau_{\gamma y} \left(\frac{1}{\|\tilde{\gamma} v_y\|} \tilde{\gamma} w - \langle \tilde{\gamma} w, \tilde{\gamma} v_y \rangle \frac{\tilde{\gamma} v_y}{\|\tilde{\gamma} v_y\|^3} \right).$$

Modifying $v_1, \dots, v_{k+1}$ we can assume that $v_1 = v_x$ and $v_2, \dots, v_{k+1} \in x^\perp$. Then

$$T_x\,\mathbb{P}(V) = \langle \tau_x(v_2), \cdots, \tau_x(v_{k+1}) \rangle.$$

Let

$$w_j := \frac{1}{\|\tilde{\gamma} v_x\|} \tilde{\gamma} v_j - \langle \tilde{\gamma} v_j, \tilde{\gamma} v_x \rangle \frac{\tilde{\gamma} v_x}{\|\tilde{\gamma} v_x\|^3}.$$

Then, by the above formulas,

$$\|\tau_x(v_2) \wedge \cdots \wedge \tau_x(v_{k+1})\|_{h_k} = \|v_1 \wedge v_2 \wedge \cdots \wedge v_{k+1}\|_{\wedge^{k+1} \mathbb{R}^d}$$

and

$$\begin{aligned} \left\| d(\gamma)_x \Big(\tau_x(v_2) \wedge \cdots \wedge \tau_x(v_{k+1}) \Big) \right\|_{h_k} &= \|\tau_{\gamma x}(w_2) \wedge \cdots \wedge \tau_{\gamma x}(w_{k+1})\|_{h_k} \\ &= \|v_{\gamma x} \wedge w_2 \wedge \cdots \wedge w_{k+1}\|_{\wedge^{k+1} \mathbb{R}^d} = \frac{\|\tilde{\gamma}(v_1 \wedge \cdots \wedge v_{k+1})\|_{\wedge^{k+1} \mathbb{R}^d}}{\|\tilde{\gamma} v_1\|^{k+1}}. \end{aligned}$$

Since $\|v_1\| = 1$, Equation (27) implies the desired equality. $\square$

Appendix B. Eigenvalues and conjugacy

In this appendix, we prove the following observation that was used in the proof of Theorem 13.2.

Observation B.1. If $d \geq 3$, $\Gamma < \mathsf{PGL}(d, \mathbb{R})$ is a strongly irreducible proximal subgroup, and

$$\lambda_2(\gamma) = \cdots = \lambda_{d-1}(\gamma) = 1$$

for all $\gamma \in \Gamma$, then Γ is conjugate to a Zariski dense subgroup of $\mathsf{PO}_0(d-1,1)$ or $\mathsf{PO}(d-1,1)$.

Proof. Let H denote the Zariski closure of Γ and let $\mathsf{H}^0 < \mathsf{H}$ denote the connected component of the identity. By [BCLS15, Lem. 2.18], H^0 is a connected semisimple Lie group with trivial center. By a theorem of Benoist [Ben97],

$$\lambda_2(h) = \cdots = \lambda_{d-1}(h) = 1$$

for all $h \in \mathsf{H}^0$. Thus H^0 is a rank one non-compact simple group. Let X be the symmetric space associated to H^0 and let $\rho : \mathsf{H}^0 \to \mathsf{Isom}(X)$ be the induced map. Since H^0 has trivial center, ρ induces an isomorphism between H^0 and $\mathsf{Isom}_0(X)$, the connected component of the identity in $\mathsf{Isom}(X)$. Further, X is a negatively curved symmetric space, the geodesic boundary has a $\mathsf{Isom}(X)$-invariant smooth structure, and there exists a ρ^{-1}-equivariant smooth embedding $\xi : \partial_\infty X \hookrightarrow \mathbb{P}(\mathbb{R}^d)$ (for details about the construction of ξ, see for instance [ZZ24b, Sect. 4]).

Lemma B.2. *$X = \mathbb{H}^m$ is real hyperbolic m-space, $m = \dim X$.*

Proof. Suppose $\gamma \in \mathsf{Isom}(X)$ is loxodromic, i.e. γ has no fixed points in X and has two fixed points $x^\pm$ in $\partial_\infty X$. Then the eigenvalue condition implies that all eigenvalues of the derivative $d(\gamma)_{x^\pm} : T_{x^\pm}\partial_\infty X \to T_{x^\pm}\partial_\infty X$ have the same modulus. From the description of the negatively curved symmetric spaces in [Mos73, Chap. 19], this is only possible if X is a real hyperbolic space. $\square$

Now we can identify $\mathsf{Isom}(X)$ with $\mathsf{PO}(m,1)$ and view ρ^{-1} as an irreducible representation of $\mathsf{PO}_0(m,1)$, the connected component of the identity in $\mathsf{PO}(m,1)$. It then follows from the eigenvalue condition and the theory of highest weights (see for instance [ZZ24a, Lem. 10.4]) that $m = d-1$ and $\mathsf{H}^0 = \rho^{-1}(\mathsf{PO}_0(d-1,1))$ is conjugate to $\mathsf{PO}_0(d-1,1)$. So, after conjugating, we can assume that $\mathsf{H}^0 = \mathsf{PO}_0(d-1,1)$.

Next let G be the normalizer of $\mathsf{PO}_0(d-1,1)$ in $\mathsf{PGL}(d,\mathbb{R})$ and let $\tau : \mathsf{G} \to \mathrm{Aut}(\mathsf{PO}_0(d-1,1))$ be the map induced by conjugation. By Schur's lemma, τ is injective. Further, $\tau|_{\mathsf{PO}(d-1,1)}$ is onto. Hence $\mathsf{H} \leq \mathsf{G} = \mathsf{PO}(d-1,1)$. $\square$

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