Investigation on the Relationship Between Location of Fretted Note on Guitar Neck and Note Intonation Error.

IB Physics Extended Essay

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Contents

1 Introduction

1.1 Personal observation

In the guitar community, when comparing the build quality of different electric guitars, one of the key determining points is its ability to keep good intonation. Intonation refers to the accuracy of the instrument's pitch across all frets. Slight deviations in intonation can have a significant impact on the overall sound quality, especially when playing chords, as multiple slightly-out-of-tune notes together can clash and produce muffled, discordant sounding chords, most notably when playing high up the neck (above $11 - 12^{th}$ fret). This problem has been a topic of debate in many guitar forums online, with a lot of hypotheses on why it happens and discussions on how to set a perfect intonation. There has been extremely limited formal studies of the subject, the two most notable ones came from the works of classical guitar luthiers [1] ([Bartolinis, 1982] and [Byers, 1996]), the former of which is considered seminal, yet seems to have been lost on the internet. In terms of physics investigation there have been general basic studies on intonation yet very little detailed analysis [Varieschi, 2010]. The model used in both [Byers, 1996] and [Varieschi, 2010] are derived from the original model by [Bartolinis, 1982], but I think they don't reflect the real world behavior really well. Therefore, in this investigation I want to develop a different model to explore how intonation works, what causes

[1] Craftsmen who build or repair stringed instruments

intonation problems, and from there make observations on some solutions to achieve good intonation.

1.2 Background information & Theory

Basic model of guitar strings and frets

Figure 1 shows the most simple model of an electric guitar, consisting of a steel string raised at two ends by the bridge and the nut. The end behind the nut is wound around a tuning peg used to adjust the string tension and tune the string to match a frequency. This is stretched over a fretboard with raised metal frets, and a pickup - an electromagnet to capture string vibrations turning them into electrical signals. The fret distances are carefully calculated and accurately placed along the length of the fretboard, so that when a string is pressed down on a fret, the fret itself will act as a stopper and change the length of the vibrating segment, creating different vibrating frequency to make other notes in a scale.

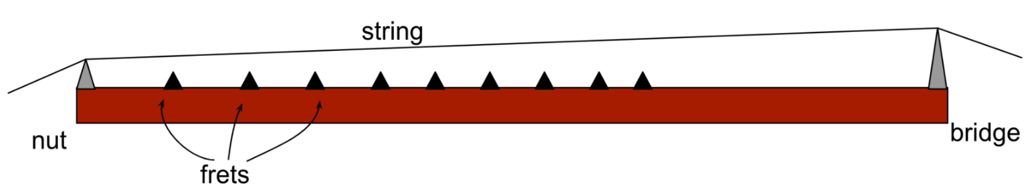

Figure 1: A simplified model of the electric guitar I will use. The pickup is not shown.

How the guitar string and frets work

When a string is plucked, it will create travelling waves on the string that reflect at both ends, creating standing waves. The frequency of the wave with the longest wavelength is the fundamental frequency, also called the first harmonic. This is the lowest frequency, and there will usually be other higher harmonics that when combined create the characteristic timbre of the electric guitar.

The fundamental frequency f_0 of a string can be determined by Mersenne's law:

$$f_0 = \frac{1}{2l_s}\sqrt{\frac{T}{\mu}} \tag{1}$$

where l_s is the vibrating length of the string, T is the tension, and μ is the linear density of the string (mass of string per unit length).
[Steinhaus, 1999]

When the string is pressed down on a fret and plucked, its vibrating length changes, changing the frequency. Figure 2 shows a model of the string pressed down on fret n. The distance from the bridge to the fret then is l_n

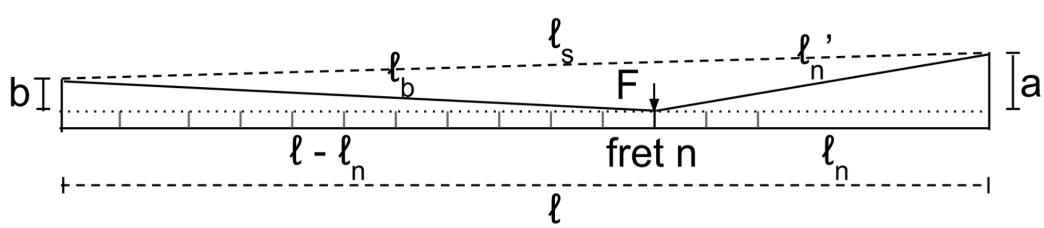

Figure 2: A simplified model of the fretted string

In Western music, the ubiquitous musical system is the twelve-tone equal temperament (12-TET) system, where an octave is divided into 12 equally spaced notes on a logarithmic scale [Nov]. The ratio is thus equal to $\sqrt[12]{2}$, therefore to position the frets correlating to the notes, the scale length needs to be divided into powers of $\sqrt[12]{2}$. To do this, luthiers traditionally use a formula to calculate the position of each fret. [Mottola]

$$l_n = \frac{l}{2^{\frac{n}{12}}} \tag{2}$$

Cause of intonation issues

Intonation deviation happens when there is a difference between the frequency that we expect when we fret down on a note and the actual frequency that we get.

Intonation deviations happen because luthiers calculated the positioning of the frets with a basic, simplified model, assuming the length of the string is equal to the length of the fretboard, the string doesn't stretch, and ignoring the effects of the nut and bridge; whereas in reality, the length of the string is slightly longer than the fretboard (due to the nut and bridge's height difference), and when the string is pressed down the length will stretch a little as well as the tension will slightly increase.

From Figure 2, using Mersenne's Law, the fundamental frequency of the string f_0 according to the luthier's model is:

$$f_0 = \frac{1}{2l}\sqrt{\frac{T}{\mu}} \tag{3}$$

Where l is the "scale length" of the guitar (the distance between the bridge and the nut). [Steinhaus, 1999]

And the expected frequency f_n at the n^{th} fret is:

$$f_n = \frac{1}{2l_n}\sqrt{\frac{T}{\mu}} \tag{4}$$

l_n is the length between the n^{th} fret and the bridge. However, in real life the string doesn't follow this perfect simplified model, so if we take into account the assumptions luthiers make, the initial frequency f_0 is governed by:

$$f_0 = \frac{1}{2l_s}\sqrt{\frac{T_a}{\mu}} \tag{5}$$

Where l_s is the actual length of the whole vibrating string, and T_a is the actual tension on the string. Notice this f_0 is the same as in the luthier's model because this is the fundamental tuning frequency of the string. However, once the string is fretted the intonation will deviate. The observed frequency f'_n at the n^{th} fret is:

$$f'_n = \frac{1}{2l'_n}\sqrt{\frac{T_a + \Delta T}{\mu}} \tag{6}$$

Here l'_n is the length of the vibrating segment, and there is a ΔT term added to the tension because the fretting force causes the total tension to increase slightly.

This difference between f_n and f'_n is what causing the intonation deviation. We call this Δf:

$$\begin{aligned}
\Delta f &= f'_n - f_n && (7)\\
&= \frac{1}{2\sqrt{\mu}}\left(\frac{\sqrt{T_a + \Delta T}}{l'_n} - \frac{\sqrt{T}}{l_n}\right) && (8)
\end{aligned}$$

From Figure 2 it is easy to see $l'_n = \sqrt{a^2 + l_n^2}$:

$$\begin{aligned}
\Delta f &= \frac{1}{2\sqrt{\mu}}\left(\frac{\sqrt{T_a + \Delta T}}{\sqrt{a^2 + l_n^2}} - \frac{\sqrt{T}}{l_n}\right) && (9)\\
&= \frac{1}{2\sqrt{\mu}}\left(\frac{\sqrt{T_a}\sqrt{1 + \frac{\Delta T}{T_a}}}{l_n\sqrt{1 + (\frac{a}{l_n})^2}} - \frac{\sqrt{T}}{l_n}\right) && (10)
\end{aligned}$$

Also from (3) and (5):

$$\sqrt{T_a} = \sqrt{T}\left(\frac{l_s}{l}\right) \tag{11}$$

Therefore:

$$\begin{aligned}
\Delta f &= \frac{1}{2\sqrt{\mu}}\left(\frac{\sqrt{T}(\frac{l_s}{l})\sqrt{1 + \frac{\Delta T}{T_a}}}{l_n\sqrt{1 + (\frac{a}{l_n})^2}} - \frac{\sqrt{T}}{l_n}\right) && (12)\\
&= \frac{1}{2l_n}\sqrt{\frac{T}{\mu}}\left(\left(\frac{l_s}{l}\right)\left(1 + \frac{\Delta T}{T_a}\right)^{\frac{1}{2}}\left(1 + \left(\frac{a}{l_n}\right)^2\right)^{-\frac{1}{2}} - 1\right) && (13)
\end{aligned}$$

From Figure 2: $l_s = \sqrt{(a-b)^2 + l^2}$, so:

$$\frac{l_s}{l} = \sqrt{1 + \left(\frac{a-b}{l}\right)^2}$$

And we get:

$$\Delta f = \frac{1}{2l_n}\sqrt{\frac{T}{\mu}}\left(\left(1+\left(\frac{a-b}{l}\right)^2\right)^{\frac{1}{2}}\left(1+\frac{\Delta T}{T_a}\right)^{\frac{1}{2}}\left(1+\left(\frac{a}{l_n}\right)^2\right)^{-\frac{1}{2}} - 1\right) \tag{14}$$

Since $a - b << l$, $\Delta T << T_a$, $a << l_n$, we can approximate the expression to first order using the binomial approximation [Špakula, 2011], expand and simplify, only keeping terms up to first order overall.

$$\Delta f \approx \frac{1}{2l_n}\sqrt{\frac{T}{\mu}}\left(\left(1+\frac{1}{2}\left(\frac{a-b}{l}\right)^2\right)\left(1+\frac{\Delta T}{2T_a}\right)\left(1-\frac{1}{2}\left(\frac{a}{l_n}\right)^2\right) - 1\right) \tag{15}$$

$$= \frac{1}{2l_n}\sqrt{\frac{T}{\mu}}\left(1+\frac{1}{2}\left(\frac{a-b}{l}\right)^2 + \frac{\Delta T}{2T_a} - \frac{1}{2}\left(\frac{a}{l_n}\right)^2 - 1\right) \tag{16}$$

$$= \frac{1}{4l_n}\sqrt{\frac{T}{\mu}}\left(\left(\frac{a-b}{l}\right)^2 + \frac{\Delta T}{T_a} - \left(\frac{a}{l_n}\right)^2\right) \tag{17}$$

Tension of the guitar string

When the guitar string is under tension T_a, it causes the string to stretch an amount Δl_0 as compared to the unstretched length, which is the same as the effective vibrating strength of the string l_s. This relationship can be determined as:

$$T_a = \frac{AY\Delta l_0}{l_s} \tag{18}$$

Where A is the cross-sectional area of the string, and Y is the Young's modulus of the material. [Polak et al., 2018]

When we press down on it, the whole string stretches accordingly by a small amount Δl. Assuming there is no friction on the point of contact, the tension in the whole string increases by an amount ΔT:

$$T_a + \Delta T = \frac{AY(\Delta l_0 + \Delta l)}{l_s} \tag{19}$$

Thus the relationship between ΔT and Δl can be determined:

$$\Delta T = \frac{AY\Delta l}{l_s} \tag{20}$$

It is very hard to observe and measure Δl directly, so we can try rewriting it in terms of other variables that can be determined. From Figure 2,

$$\begin{aligned}
\Delta l &= (l_b + l_n') - l_s && (21)\\
&= \sqrt{(l-l_n)^2 + b^2} + \sqrt{l_n^2 + a^2} - l_s && (22)\\
&= (l - l_n)\left(1 + \left(\frac{b}{l-l_n}\right)^2\right)^{\frac{1}{2}} + l_n\left(1 + \left(\frac{a}{l_n}\right)^2\right)^{\frac{1}{2}} - l_s && (23)
\end{aligned}$$

Once again, we can approximate this to the first order since $a << l_n$ and

$b << l - l_n$:

$$\Delta l \approx (l - l_n)\left(1 + \frac{b^2}{2(l - l_n)^2}\right) + l_n\left(1 + \frac{a^2}{2l_n^2}\right) - l_s \tag{24}$$

$$= (l - l_n) + \frac{b^2}{2(l - l_n)} + l_n + \frac{a^2}{2l_n} - l_s \tag{25}$$

$$= \frac{b^2}{2(l - l_n)} + \frac{a^2}{2l_n} - (l_s - l) \tag{26}$$

From Figure 2, we can expand and approximate l_s up to first order $(a - b << l)$:

$$l_s = \sqrt{l^2 + (a - b)^2} \tag{27}$$

$$= l\left(1 + \left(\frac{a - b}{l}\right)^2\right)^{\frac{1}{2}} \tag{28}$$

$$\approx l\left(1 + \frac{1}{2}\left(\frac{a - b}{l}\right)^2\right) \tag{29}$$

$$= l + \frac{(a - b)^2}{2l} \tag{30}$$

Therefore from (26):

$$\Delta l = \frac{b^2}{2(l - l_n)} + \frac{a^2}{2l_n} - \frac{(a - b)^2}{2l} \tag{31}$$

Substituting this into (20) we get

$$\Delta T = \frac{AY}{2l_s}\left(\frac{b^2}{l - l_n} + \frac{a^2}{l_n} - \frac{(a - b)^2}{l}\right) \tag{32}$$

Which can be substituted into (17):

$$\Delta f = \frac{1}{4l_n}\sqrt{\frac{T}{\mu}}\left(\frac{AY}{2l_sT_a}\left(\frac{b^2}{l-l_n}+\frac{a^2}{l_n}-\frac{(a-b)^2}{l}\right)+\left(\frac{a-b}{l}\right)^2-\left(\frac{a}{l_n}\right)^2\right) \tag{33}$$

We need to write l_sT_a in terms of l, T and other variables. From (11) and (28), once again using a first order approximation since $a-b << l$:

$$T_a = T\left(\frac{l_s}{l}\right)^2 \tag{34}$$

$$\implies l_sT_a = T\frac{l_s^3}{l^2} \tag{35}$$

$$= Tl\left(1+\left(\frac{a-b}{l}\right)^2\right)^{\frac{3}{2}} \tag{36}$$

$$\approx Tl\left(1+\frac{3}{2}\left(\frac{a-b}{l}\right)^2\right) \tag{37}$$

Final expression relating frequency change and position of fret

Finally, we can substitute (37) into (33) to get the expression between the intonation shift Δf and the fret position l_n

$$\Delta f = \frac{1}{4l_n}\sqrt{\frac{T}{\mu}}\left(\frac{AY}{2Tl\big(1+\frac{3}{2}(\frac{a-b}{l})^2\big)}\left(\frac{b^2}{l-l_n}+\frac{a^2}{l_n}-\frac{(a-b)^2}{l}\right)+\left(\frac{a-b}{l}\right)^2-\left(\frac{a}{l_n}\right)^2\right) \tag{38}$$

From (3) we get

$$\sqrt{\frac{T}{\mu}} = 2lf_0$$
$$\implies T = 4\mu l^2 {f_0}^2$$

μ, the linear density of the string, is simply the mass per unit length of the string:

$$\mu = \frac{m}{l_s} = \frac{\rho V}{l_s} = \frac{\rho l_s A}{l_s} = \rho A$$

where m is the mass of the vibrating string section, V is its volume, and ρ is the volumetric density of the material. Therefore:

$$T = 4\rho A l^2 {f_0}^2$$

Substituting all this into (38) we get the final expression between the intonation shift Δf and the fret position l_n relating all our known variables:

$$\Delta f = \frac{lf_0}{2l_n}\left(\frac{Y}{8\rho l^3 {f_0}^2(1+\frac{3}{2}(\frac{a-b}{l})^2)}\left(\frac{b^2}{l-l_n}+\frac{a^2}{l_n}-\frac{(a-b)^2}{l}\right)+\frac{(a-b)^2}{l^2}-\frac{a^2}{{l_n}^2}\right) \tag{39}$$

2 Experiment

From the final expression relating Δf and l_n, knowing all other constants I can plot the graph for this. Δf is the dependent variable, and l_n is the independent variable. I expect the intonation difference Δf to be higher when the fretting distance is closer to the bridge (l_n closer to 0), and lower closer to the nut, in the first few frets (larger values of l_n). The general shape of the graph can be compared with my expectation as a sanity check. It is impossible to linearize the equation, therefore I will have to resort to collecting the data, plotting the graph, then confirming the relationship using a regression method to see how well the data fits the model.

The choice of guitar I will use for this experiment is a Fender Stratocaster. This is arguably the most iconic and popular guitar in history, known for its versatility and playability. Therefore, I pick this guitar because there is a wealth of information available on it, making it easier to find data to support my research or to compare my results to other studies. Another benefit is to ensure ease of replicability of the experiment.

The choice of string I will use is a G string from D'Addario's set of Nickel Wound Regular Light Gauge - EXL110-10P that I have. I choose the G string because it is the thickest plain unwound string in the set. This is because from my experience the thicker strings will make the effects of the intonation shift more noticeable and easier to measure for the experiment.

2.1 Determining the constants

From the equation (39), the constants I need to determine are f_0, ρ, Y, a, b and l.

- l is the scale length of the guitar. For a Fender Stratocaster, it is 25.5 inches (64.77 cm). [Nemeroff, 2023] I round this up to 64.8 cm (3s.f). Then I can adjust the bridge position so that the scale length matches 64.8 cm (0.648 m)

- Initial frequency f_0 can be measured directly in the experiment. The frequency we aim for is the frequency of G string on the guitar for a standard A440 tuning system, G_3 at 196 Hz [Suits, 1998]

- Young's modulus Y is dependent on the specifications of the string material. Nearly all electric guitar strings follow the ASTM-A228 manufacturing standards for steel music wire, and the value of Y is 210 GPa (2.10×10^{11} Pa). [ASTM A228 Steel (UNS K08500)]

- ρ, the density of the string material, is determined according to the ASTM-A228 standards: $\rho = 7.80 \times 10^3\,\mathrm{kg\,m^{-3}}$. [ASTM A228 Steel (UNS K08500)]

- a and b: it is very hard to measure these directly, as they are the distance from the top of the nut and bridge to the top of the frets, not the height itself. Therefore, I can only set up the guitar indirectly according to recommended values and calculate them afterwards. There are a lot of resources online on how to set up the guitar. I choose to follow the instructions by Stewmac [StewMac.com], a reputable online guitar retailer, and take the average value of the action (distance between string and top of fret) for the 1st fret to be

0.016” (0.406 mm) and 12^{th} fret to be 0.070” (1.78 mm). I change the adjustable bridge height and file down/shim up the nut height to match these values. From there I can calculate the values of a and b as illustrated by Figure 3:

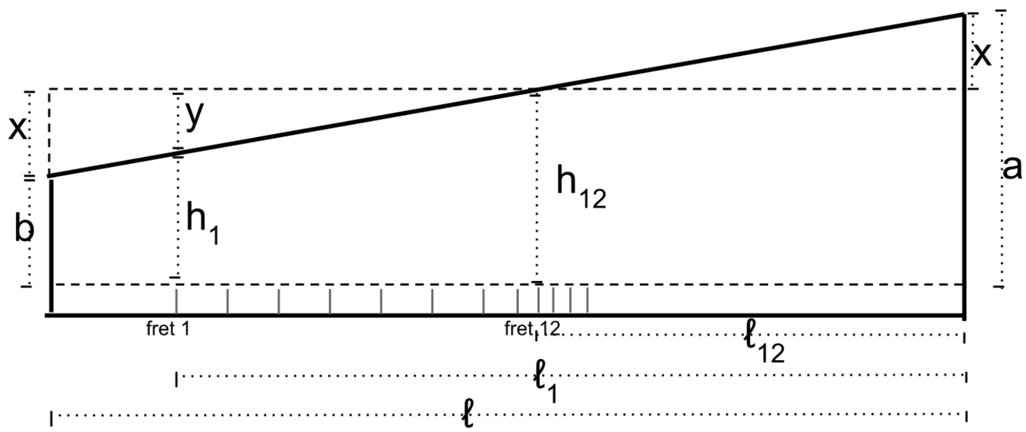

Figure 3: Diagram to calculate values of a and b

Since fret 12^{th} is exactly in the middle of l ($l_{12} = \frac{l}{2} = \frac{.648}{2} = .324$ m) we get:

$$a = h_{12} + x$$
$$b = h_{12} - x$$

From the luthier formula (2):

$$\begin{aligned} l_1 &= \frac{l}{2^{\frac{1}{12}}} \\ &= \frac{0.648}{2^{\frac{1}{12}}} \\ &= 0.612\,\text{m} \end{aligned}$$

and we also get the ratio in Figure 3

$$
\begin{aligned}
\frac{x}{y} &= \frac{l - l_{12}}{l_1 - l_{12}} \\
x &= y\frac{l - l_{12}}{l_1 - l_{12}} \\
&= (h_{12} - h_1)\frac{l - l_{12}}{l_1 - l_{12}} \\
&= (1.78 - 0.406) \cdot 10^{-3} \cdot \frac{0.648 - 0.324}{0.612 - 0.324} \\
&= 1.55 \times 10^{-3}\,\mathrm{m}
\end{aligned}
$$

Therefore

$$a = 1.78 + 1.55 = 3.33 \times 10^{-3}\,\mathrm{m}$$

$$b = 1.78 - 1.55 = 1.30 \times 10^{-4}\,\mathrm{m}$$

2.2 Apparatus

- 1 Fender Stratocaster style guitar
- 1 17-gauge plain unwound steel G string
- 1 guitar capo
- 1 guitar pick

2.3 Process

1. Setting up the guitar. I used the instructions by Stewmac [StewMac.com]

2. Connect the guitar to frequency measuring software. I used my audio interface to connect the guitar to my laptop, and the software I used is Visual Analyzer 2020 [VA2020 Website]. I choose this because it supports Fast Fourier Transform in real-time to convert guitar signals into frequency spectrum, and can be configured to provide results with high accuracy.

3. Measure f_0 by plucking the string with no capo on.

4. Pluck it 5 times with the pick, changing the plucking position each time as shown in Figure 4 (trial 1: above the neck pickup, trial 2: between neck and middle, trial 3: above middle, trial 4: between middle and bridge, trial 5: above bridge pickup). Record the peak frequency (highest dB) for each pick.

5. Put capo on 7^{th} fret and repeat step 4. Ensure that the edge of the capo must be right on top of the fret and the capo is pushing the string straight downwards, not sideways. (Figure 5)

6. Repeat step 5 from fret 8 up to fret 16. I cannot go higher than this because this is where the neck meets the body, it is impossible to put the capo.

Photos of experimental set up

Figure 4: My experimental setup. I position the guitar neck outside the table for ease of access to the capo

Figure 5: Close up of capo placement. This ensures consistent pressure and contact with the string and fret

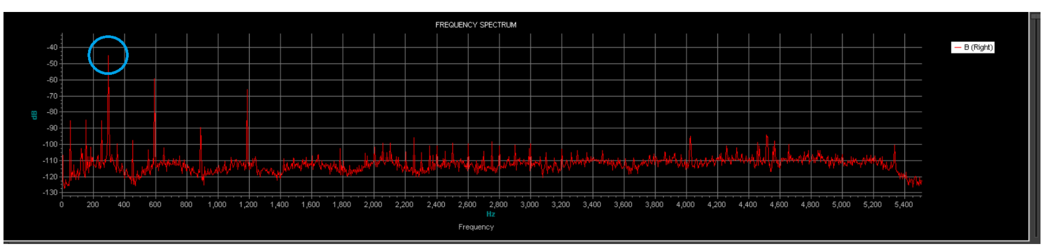

Figure 6: Frequency spectrum of a note in Visual Analyzer. The peak frequency is circled. This can be zoomed in to accurately read peak value

2.4 Tables of results

Table of raw data

Fret number	Frequency (Hz)				
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
0	195.82	195.82	196.49	196.49	196.47
7	294.06	294.06	294.07	294.06	294.07
8	310.96	310.96	312.31	311.67	312.31
9	330.41	330.41	330.40	329.74	330.41
10	349.81	350.43	349.81	349.81	350.43
11	370.73	371.39	371.39	370.76	370.72
12	393.20	393.21	393.20	393.20	393.20
13	416.49	416.50	416.49	416.50	417.15
14	440.87	441.54	442.21	441.54	442.22
15	468.18	468.18	468.16	468.18	468.18
16	496.38	495.98	495.99	496.58	496.38

Table 1: Raw collected data

The frequencies are taken up to 2d.p because this is the smallest the software can measure.

Table of processed data

Fret number	Fret distance (m)	Avg frequency (Hz)	Note	Standard frequency (Hz)	Δf (Hz)	Abs. unc. Δf (Hz)
0	0.648	196.22	G3	196.00		
7	0.432	294.06	D4	293.66	0.40	0.005
8	0.408	311.64	D#4	311.13	0.51	0.7
9	0.385	330.27	E4	329.63	0.65	0.3
10	0.364	350.05	F4	349.23	0.83	0.3
11	0.343	371.00	F#4	369.99	1.01	0.3
12	0.324	393.20	G4	392.00	1.20	0.005
13	0.306	416.63	G#4	415.30	1.32	0.3
14	0.289	441.67	A4	440.00	1.67	0.7
15	0.272	468.17	A#4	466.16	2.01	0.01
16	0.257	496.27	B4	493.88	2.38	0.3

Table 2: Table of processed data

Fret distance is l_n, calculated from fret number with the luthier equation (2)

2.5 Analysis of results

Graphs

After collecting the data, I can take the average value of $f_0 = 196.22\,\text{Hz}$ as the constant value of f_0 for Equation (39). Now I can plot the graph relating Δf and l_n. The range of the graph corresponds with the distance between the nut and the highest fret of the guitar to the bridge, where the fretboard is. My guitar has 22 frets, so the position of the final fret is

$$l_{22} = \frac{.648}{2^{\frac{22}{12}}} = 0.182\,\text{m}$$

Therefore the range I will be graphing is $\{0.182 \leq l_n \leq 0.648\}$. Below is the general form of the graph.

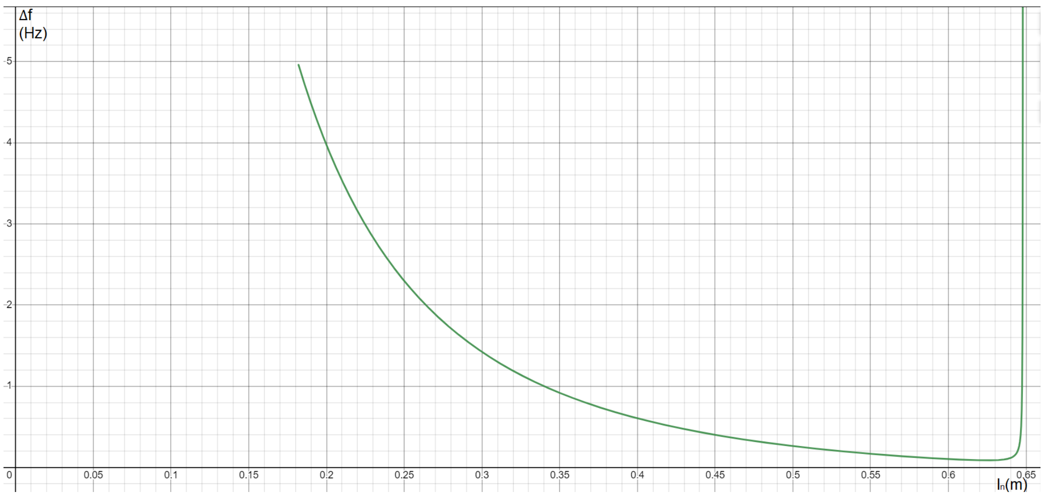

Figure 7: General form of the graph without data points

From the graph, we can see it matches with my expectation. Overall Δf is lower for larger l_n (further from the bridge, lower frets), and increases when l_n is smaller (higher frets, closer to the bridge). A noticeable thing is the function has no zero in this range, meaning everywhere you fret on the fretboard there will still be a bit of intonation error making it impossible to achieve perfect intonation.

Observing the graph, we can see there is a vertical asymptote at $l_n = l$. Mathematically this comes from equation (39) when the denominator of the terms inside brackets is equal to 0 ($l - l_n = 0$) (there is another asymptote at $l = 0$, but this is outside our range). At $\lim_{l_n \to l^-} \Delta f$ approaches $+\infty$. This is physically impossible. My explanation of this is because of the error terms in our first order approximation. When $l_n \approx l$ (very close to the nut), $l - l_n$ is very small and on a comparable order to the nut height b, the approximation at (24) no longer holds because the higher order error terms will take over. This means the function will no longer accurately model the behavior of the string at that point. My physical interpretation of this phenomenon $\Delta f \to +\infty$ is because, if you try to fret very close to the nut, the "breaking angle" (angle between the fret and the string) θ of the string in Figure 8 gets very large. Therefore, the amount of force downwards needed to counteract the tension to depress the string to touch the fret is much higher, which also increases the tension of the string accordingly, which then in turn requires an even larger force to fret down, increasing the tension even more, etc. This behavior of the tension causes the frequency to rise asymptotically the closer you get to the nut. This will continue until the tension in the string is too large and the string breaks. However, this doesn't happen in reality because the first fret is at $l_n = 0.612$, and you cannot fret closer to the nut than this (albeit for fretless instruments you can go as close as physically possible until the string breaks).

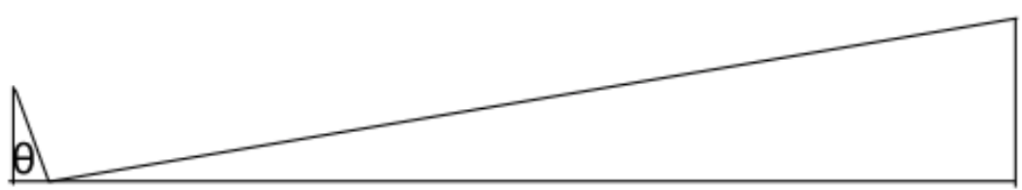

Figure 8: Breaking angle θ of string when fretting near the nut

To analyze the graph further I can take the first derivative of this with respect to l_n. The first derivative has a zero at $l_n = 0.625$ m, and substituting this back into the original function we get $\Delta f = 0.087$ Hz. This is a minimum on the graph, indicating that at this point there is the least intonation error. The closest fret to this is fret 1, at $l_n = 0.612$ m. This shows that on the whole fretboard, the intonation error is the smallest at fret 1, and increases as you go higher up the fretboard. The local maximum is at the last fret, fret 22 at $l_n = 0.182$m. Here the Δf is 4.96 Hz, which is a relatively large intonation error.

Also from the graph, the predicted Δf at fret 12 ($l_n = 0.324$m) is 1.15 Hz. This deviation can be put in terms of cents, which is defined as the difference between the frequencies of two consecutive notes divided into 100 equal parts [Suits, 1998]. Therefore:

$$\begin{aligned} \text{1 cent above D4} &= \frac{\text{D\#4 - D4}}{100} \\ &= \frac{311.33 - 293.66}{100} = 0.177\,\text{Hz} \end{aligned}$$

(Note values taken from [Suits, 1998])

So 1.15 Hz is equivalent to $\frac{1.15}{0.177} \approx$ 6-7 cents sharp. The smallest difference in pitch human can discern is around 5-6 cents [Loeffler, 2006], and this possibly explains why the intonation error gets noticeable from fret 12 and upwards.

Now I can plot a graph with the data points and perform a goodness-of-fit test to determine the correlation.

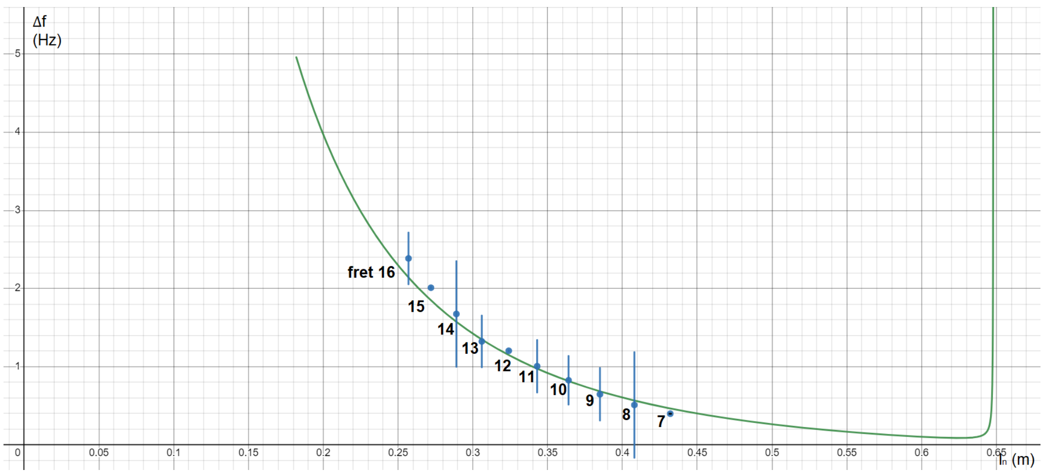

Figure 9: The original curve and the data points with error bars

Observation & Analysis

From Figure 9 we see the data points fit the function quite well and follow the general shape of the curve. The curve passes through all the error bars except for the data point at fret 7, which is still close to the curve but error bars are too small. A noticeable thing is that, although data points of lower frets stay pretty close to the curve, for higher frets (lower l_n) Δf are quite higher than the predicted values (around frets 13-16). One possible cause of this I can think of is due to the capo placement. Because of the neck shape around this position where it connects with the body of the guitar, it is quite difficult to place the capo squarely on the fret and pressing directly down on the string. This might cause it to pull the string a bit sideways or push it down behind the fret, increasing the tension and making the frequency go up.

The goodness-of-fit test I will perform on the data is the Standard Error of the Estimate (SEOE).

$$\text{SEOE} = \sqrt{\frac{\sum (Y - Y')^2}{n}}$$

Where Y is the actual value of the data, Y' is the value from the function, and n is the number of data points. [Lane]

The correlation value of SEOE is essentially the standard deviation of the residual values. The closer this standard deviation is to 0, the lower the errors and the better the function fits the data (higher correlation). I choose this test because it returns a normalized result and provides a clear measure on the quality of the correlation. Performing the test on the processed data points of Δf and l_n, I get a value of SEOE $= 0.105$. This is quite close to 0, indicating a low deviation of errors, so I believe the data fits the model well.

3 Conclusion & Evaluation

In conclusion, the higher the fretting position, the more intonation error there is. The relationship is not a linear one but quite complicated and can be modelled with the equation (39). By plotting the collected data and performing the goodness-of-fit test, it confirms the validity of the equation.

By choosing this simplified model, it makes the calculations easier and creates a useful approximation model. However, this doesn't take into account several factors in reality that can affect the intonation. For example, using a capo to simulate the fretting action of a human finger. The contact surface of a human finger on the string is very small, but that of a capo is much bigger. This can lead to more deformation of the string when using the capo and consequently an increase in tension & frequency. Also, normally guitarists don't press down directly on the fret like the capo, but a bit behind. This can depress the string more and lead to a higher intonation error. Another factor is the guitar neck isn't always rigid and straight but sometimes slightly bowed, either from the tension of the strings or from the player's setup, which can also affect the intonation. An assumption I made is the fretting point on the string would have no friction, therefore the tension increase would be distributed evenly throughout the string, but in reality the finger or capo would have a small amount of friction that can affect the result. Another limitation of the model is it only applies to plain unwound steel strings, so it can only be used for the highest 3 strings on an electric guitar, but for the other 3 wound strings we would need a different model.

An observation I can make from equation (39) is that it is not dependent on the gauge (thickness) of the string I use. However, this conflicts with my experience, because when playing I would notice the thicker G string would exhibit more intonation shift than the thinner high E string. To investigate this, I repeat the experiment with my thinner B string and E string, tuning them to their standard frequencies (B_3 : 246.94Hz and E_4 : 329.63Hz). The collected and processed data for these are in the Appendix.

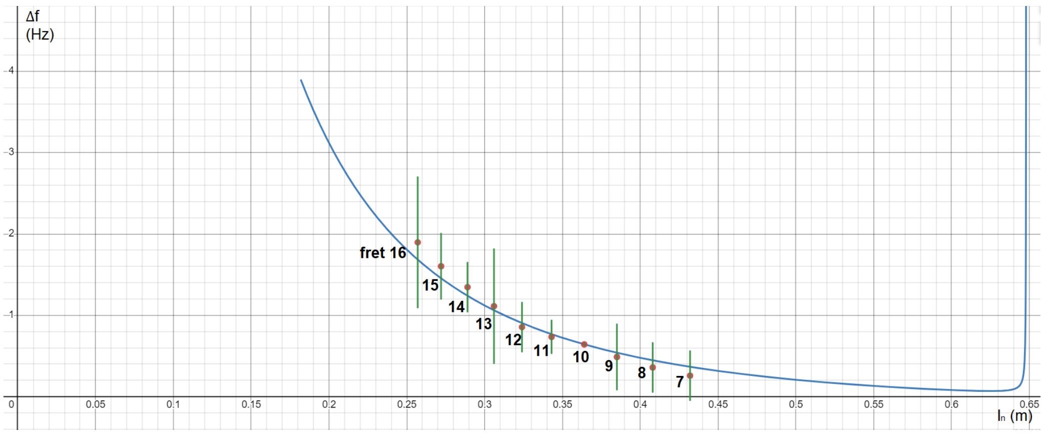

Figure 10: Curve for B string with data points and error bars

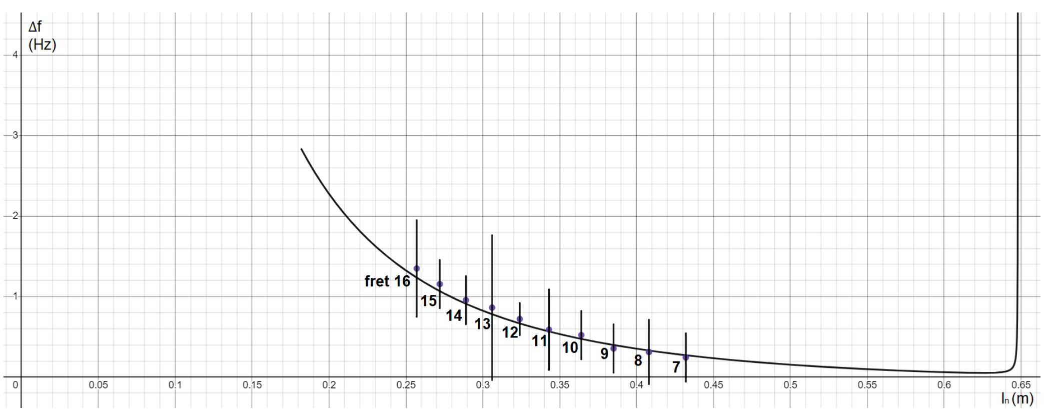

Figure 11: Curve for E string with data points and error bars

Overall we see that the graphs of B and E strings are successively lower, meaning the intonation deviation on those strings are less, showing that my observation is correct. I believe this is because the strings have different f_0

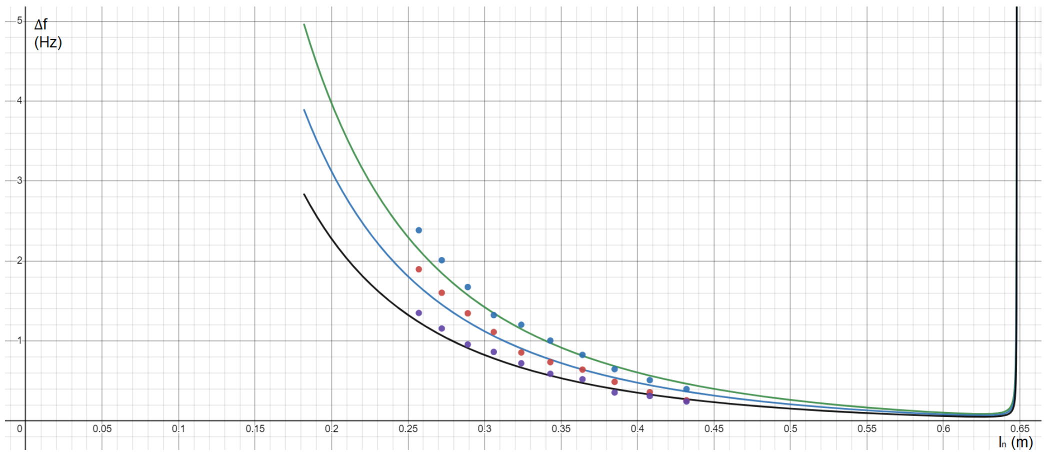

Figure 12: Curve for all 3 strings with data points, error bars have been hidden for clarity

which contribute to the differences in the curves. We can also make some observations from the data. Most noticeably, data points for higher frets are slightly higher than predicted by the curve. This is the same trend observed with the original data of the G string, and I believe the cause is the same: the capo not fretting directly down on the string but slightly pushing the string sideways. Also, we can see the error bars generally gets larger for the thinner strings (error bars for E string are generally bigger than B string, and error bars for B string are also bigger than G string). This means there is an increase in variance of the frequency for trials at each fret position for thinner strings. This might be because thin strings might slip slightly sideways under the capo each time it's plucked, meaning for the same fret, there will be a slightly different frequency for each pick, even though the capo is never moved. It is also noticeable that there is a large error bar at fret 13 and 16 for the B and E strings. Upon closer inspection of the guitar, it seems like the frets at these positions are quite worn down. This might cause the string to slip more under the capo, explaining the large error bar. Some improvements I can suggest is to use a stronger capo to minimize slipping, level the guitar frets before carrying out the experiment. We can also try to control the plucking with some mechanism that can consistently deliver the same force to the string to avoid pushing the string sideways too much.

Therefore, overall in order to minimize intonation error, it is recommended to use thinner gauge strings. However, players need to take into account other factors as well when changing strings. For example, it's not recommended for players with a slightly heavier playing style, because the lower resistance of thinner strings will make it more likely to be pushed sideways when fretted rather than directly down on the fret, which will also cause intonation shifts.

From the graph of the equation relating Δf and l_n we can also evaluate the effects of changing other variables on the shape of the curve. It is impossible to have zero intonation deviation through the whole graph, but I can try ways to minimize this error. I notice the variable that has the most effect on the graph is a, the vertical distance between top of the bridge and the frets. Simply reducing the value of a by around 1 mm, bringing it from 3.3 mm down to 2 mm (2×10^{-3} m) is enough to reduce the intonation shift at fret 16 down to around 1.8 Hz (Figure 13), which would be equivalent to $\frac{1.8}{C5-B4} \cdot 100 = \frac{1.8}{523.25-493.88} \cdot 100 \approx 6$ cents, which is almost unnoticeable to the human ear. [Loeffler, 2006]

Figure 13: Comparing before (green) and after changing value of a (purple)

So a lower string action would be better for intonation. Guitar manufacturers as well as guitarists can take this into account when setting up their guitars to reduce the intonation error. However, reducing the bridge height too much will cause buzzing of the string, where the string hit against other frets when it is plucked and vibrating. This will reduce the quality of the tone and might even lead to more intonation error. Therefore, a careful trial and error approach is needed when adjusting for a good intonation.

Overall from the investigation we see that the intonation of a guitar string is very complicated and affected by a lot of different factors. We cannot eliminate it completely, but there are some things we can do to control it to try and minimize the intonation shifts, such as using low gauge strings and having a low action.

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Appendix A Raw and Processed Data for B and E Strings

Fret number	Frequency (Hz)				
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
0	247.03	247.03	247.20	247.14	247.03
7	391.87	392.50	391.87	392.50	392.50
8	415.57	415.57	416.03	415.33	415.81
9	441.02	440.49	440.49	440.23	440.23
10	466.81	466.81	466.81	466.81	466.83
11	494.75	494.75	494.42	494.42	494.75
12	524.00	524.55	524.00	524.01	524.00
13	555.56	555.56	554.98	554.98	556.31
14	588.61	588.48	589.07	588.61	588.62
15	623.56	623.56	624.35	623.56	624.28
16	660.56	660.56	660.56	661.95	662.12

Figure 14: Raw data for B string

Fret number	Fret distance (m)	Avg frequency (Hz)	Note	Standard frequency (Hz)	Δf (Hz)	Abs. unc. Δf (Hz)
0	0.648	247.09	B3	246.94		
7	0.432	392.25	G4	392.00	0.258	0.3
8	0.408	415.66	G#4	415.30	0.359	0.3
9	0.385	440.49	A4	440.00	0.488	0.4
10	0.364	466.81	A#4	466.16	0.642	0.01
11	0.343	494.62	B4	493.88	0.735	0.2
12	0.324	524.11	C5	523.25	0.855	0.3
13	0.306	555.48	C#5	554.37	1.112	0.7
14	0.289	588.68	D5	587.33	1.346	0.3
15	0.272	623.86	D#5	622.25	1.603	0.4
16	0.257	661.15	E5	659.26	1.897	0.8

Figure 15: Processed data for B string

Fret number	Frequency (Hz)				
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
0	329.70	329.70	329.86	329.86	329.86
7	493.87	493.88	494.23	494.23	494.45
8	523.27	523.27	524.00	523.27	524.00
9	555.04	555.03	554.80	554.37	554.37
10	587.95	587.53	587.53	588.11	588.11
11	622.19	622.21	623.15	623.15	622.21
12	659.78	660.12	660.12	660.12	659.78
13	698.57	700.45	698.57	698.57	700.45
14	740.68	740.68	740.67	741.34	741.34
15	785.43	785.55	785.00	784.92	784.87
16	831.44	831.56	831.56	832.62	832.60

Figure 16: Raw data for E string

Fret number	Fret distance (m)	Avg frequency (Hz)	Note	Standard frequency (Hz)	Δf (Hz)	Abs. unc. Δf (Hz)
0	0.648	329.8	E4	329.63		
7	0.432	494.13	B4	493.88	0.243	0.3
8	0.408	523.56	C5	523.25	0.312	0.4
9	0.385	554.72	C#5	554.37	0.355	0.3
10	0.364	587.85	D5	587.33	0.521	0.3
11	0.343	622.84	D#5	622.25	0.589	0.5
12	0.324	659.98	E5	659.26	0.721	0.2
13	0.306	699.32	F5	698.46	0.863	0.9
14	0.289	740.94	F#5	739.99	0.956	0.3
15	0.272	785.15	G5	783.99	1.156	0.3
16	0.257	831.96	G#5	830.61	1.35	0.6

Figure 17: Processed data for E string