Investigation on Relationship Between Location of Fretted Note on Guitar Neck and Note Intonation.

IB Physics Extended Essay

March 10, 2023

Contents

1 Introduction

1.1 Personal observation

Music and sound have always held a special place in my heart, not only as a form of entertainment but also as a means of self-expression and creativity. Since a very young age, I have been fascinated with and have pursued stringed instruments like basses and guitars. Through playing these instruments, I have been able to explore the different components of music and sound, and have developed a keen ear for tone and texture. However, recently I have become increasingly aware of the importance of intonation, which refers to the accuracy of the instrument's pitch across all frets. While playing the guitar, I have noticed that even slight deviations in intonation can have a significant impact on the overall sound quality. This change in intonation often starts around fret 7-8 and above, and most noticeable especially when I play chords high up the neck with a distortion pedal. This is because the higher harmonic frequencies from the distortion will clash with the other notes in the chord when the intonation is not perfect. This will result in a very muffled, discordant sounding chord. This intonation problem has been a topic of debate in many guitar forums online, with a lot of hypotheses on why it happens and discussions on how to set a perfect intonation. Therefore this motivated me to combine it with my love of physics and investigate the principles behind this problem. I want to explore how intonation works, why the note intonation is "off" when you fret a string on high frets, and from there propose solutions to achieve perfect intonation. This investigation is not only significant for my hobby of playing the guitar, but it is also a topic of interest for many musicians and music lovers worldwide. By delving deeper into the science behind the intonation, I hope to not only improve my skills as a guitarist but also contribute to the broader community of music enthusiasts.

1.2 Background information

Basic model of guitar strings and frets

The most simple model of an electric guitar basically consists of a piece of steel string anchored at two ends by the bridge saddle and the nut (closer to the fretboard). One end is usually wrapped around a tuning peg to make the string tension adjustable to tune the string to a fixed frequency. This is stretched over a pickup - to capture the string vibrations and turn it into electrical signals - and a fretboard - a long piece of wood with raised metal frets. The fret distances are carefully calculated and accurately placed along the length of the string, so that when the string is pressed down in a fret, the fret itself will act as a stopper and change the length of the vibrating string, thereby changing the vibrating frequency to make other notes in a scale.

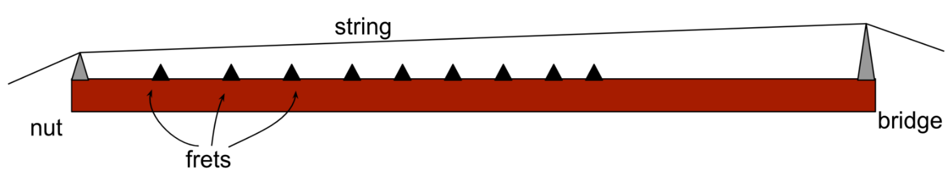

Figure 1: A simplified model of the electric guitar I will use. The pickup has been removed.

How the guitar string produces sound

When the string is plucked, it will create travelling waves on the string that reflect at both ends, creating standing waves. The frequency of the standing waves with the longest wavelength is the fundamental frequency, also called the first harmonic. This is the lowest frequency, and there will usually be other higher harmonics that when combined together create the characteristic timbre of the electric guitar.

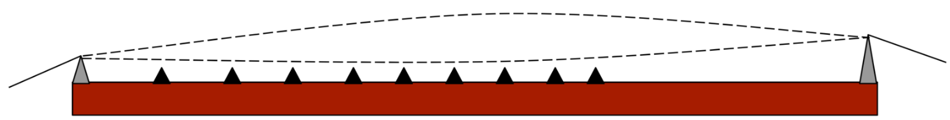

Figure 2: The "open" string (not fretted) is plucked and vibrating at its fundamental frequency.

The fundamental frequency f_0 of a string is determined by this formula

$$f_0 = \frac{v}{\lambda} \tag{1}$$

where v is the speed of the wave on the string and λ is the wavelength. From Figure 2 we see the wavelength is double the length of the guitar - the "scale length" l . Therefore

$$\lambda = 2l \tag{2}$$

The speed of the wave v on a stretched string with tension T can be determined by the equation:

$$v = \sqrt{\frac{T}{\mu}} \tag{3}$$

Where μ is the linear density of the string (mass of string per unit length) Substitute (2) and (3) into (1) we get the expression for the frequency of an open string

$$f_0 = \frac{1}{2l}\sqrt{\frac{T}{\mu}} \tag{4}$$

How frets work

When the string is pressed down on a fret and plucked, its vibrating length changes which changes the frequency as well. Let's say we press down on fret n. The distance from the saddle to the fret then is l_n

Figure 3: A simplified model of the fretted string

Theoretically, the frequency of this note is

$$f_n = \frac{1}{2l_n}\sqrt{\frac{T}{\mu}} \tag{5}$$

In order to make the frequency match with a specific note in the Western twelve-tone equal temperament (12-TET) system, the scale length needs to be divided into powers of $\sqrt[12]{2}$. To do this luthiers traditionally use a formula to calculate the position of each fret.

$$l_n = \frac{l}{2^{\frac{n}{12}}} \tag{6}$$

Cause of intonation issues

When pressed down, due to the distance between the nut and the saddle to top of the frets (b and a respectively in Figure 3), the vibrating string length also stretches. The effective vibrating length is now l'_n. The fretting force F also causes an increase in tension of the whole string, so the tension now is $T + \Delta T$.

Therefore, taking into account the fretting action, the frequency now will no longer be the same as the intended frequency f_n. This explains why we observe the intonation problem when fretting down on the string. We will call this new frequency f'_n

$$f'_n = \frac{1}{2l'_n}\sqrt{\frac{T+\Delta T}{\mu}} \tag{7}$$

The intonation deviation is then simply

$$\begin{aligned} \Delta f &= f'_n - f_n & (8) \\ &= \frac{1}{2\sqrt{\mu}}\left(\frac{\sqrt{T+\Delta T}}{l'_n} - \frac{\sqrt{T}}{l_n}\right) & (9) \end{aligned}$$

From Figure 3, using Pythagoras' Theorem for l_n' we get

$$l_n' = \sqrt{l_n^2 + a^2} \tag{10}$$

$$= \sqrt{l_n^2 \left(1 + \left(\frac{a}{l_n}\right)^2\right)} \tag{11}$$

$$= l_n \sqrt{1 + \left(\frac{a}{l_n}\right)^2} \tag{12}$$

$$= l_n \left(1 + \left(\frac{a}{l_n}\right)^2\right)^{\frac{1}{2}} \tag{13}$$

Subbing this into (9):

$$\Delta f = \frac{1}{2\sqrt{\mu}} \left(\frac{\sqrt{T + \Delta T}}{l_n \sqrt{1 + \left(\frac{a}{l_n}\right)^2}} - \frac{\sqrt{T}}{l_n} \right) \tag{14}$$

$$= \frac{1}{2 l_n \sqrt{\mu}} \left((T + \Delta T)^{\frac{1}{2}} \left(1 + \left(\frac{a}{l_n}\right)^2\right)^{-\frac{1}{2}} - \sqrt{T} \right) \tag{15}$$

$$= \frac{1}{2 l_n \sqrt{\mu}} \left(T^{\frac{1}{2}} \left(1 + \frac{\Delta T}{T}\right)^{\frac{1}{2}} \left(1 + \left(\frac{a}{l_n}\right)^2\right)^{-\frac{1}{2}} - \sqrt{T} \right) \tag{16}$$

$$= \frac{1}{2 l_n} \sqrt{\frac{T}{\mu}} \left(\left(1 + \frac{\Delta T}{T}\right)^{\frac{1}{2}} \left(1 + \left(\frac{a}{l_n}\right)^2\right)^{-\frac{1}{2}} - 1 \right) \tag{17}$$

Since ΔT is much smaller than T, and similarly a is much smaller than l_n, we can approximate this to the first order

$$\Delta f \approx \frac{1}{2 l_n} \sqrt{\frac{T}{\mu}} \left(\left(1 + \frac{\Delta T}{2T}\right) \left(1 - \frac{a^2}{2{l_n}^2}\right) - 1 \right) \tag{18}$$

$$= \frac{1}{2 l_n} \sqrt{\frac{T}{\mu}} \left(-\frac{a^2}{2{l_n}^2} + \frac{\Delta T}{2T} - \frac{a^2 \Delta T}{4T{l_n}^2} \right) \tag{19}$$

$$\approx \frac{1}{4 l_n} \sqrt{\frac{T}{\mu}} \left(\frac{\Delta T}{T} - \frac{a^2}{{l_n}^2} \right) \tag{20}$$

Tension of the guitar string

A stretched guitar string is under tension T. When we press down on it, the whole string stretches accordingly by a small amount Δl. Assuming there is no friction on the point of contact, the tension in the whole string increases by an amount ΔT

$$\Delta T = \frac{AY\Delta l}{l} \tag{21}$$

Where A is the cross sectional area of the string, and Y is the Young's modulus of the material. The total amount of stretch of the string Δl can be calculated from Figure 3.

$$\begin{aligned}
\Delta l &= (l_b + l_n') - l && (22)\\
&= \sqrt{(l-l_n)^2 + b^2} + \sqrt{l_n^2 + a^2} - l && (23)\\
&= (l-l_n)\sqrt{1 + \left(\frac{b}{l-l_n}\right)^2} + l_n\sqrt{1 + \left(\frac{a}{l_n}\right)^2} - l && (24)\\
&= (l-l_n)\left(1 + \left(\frac{b}{l-l_n}\right)^2\right)^{\frac{1}{2}} + l_n\left(1 + \left(\frac{a}{l_n}\right)^2\right)^{\frac{1}{2}} - l && (25)
\end{aligned}$$

Once again, we can approximate this to the first order since both a and b are much smaller than l_n and $l - l_n$. Therefore,

$$\begin{aligned}
\Delta l &\approx (l-l_n)\left(1 + \frac{b^2}{2(l-l_n)^2}\right) + l_n\left(1 + \frac{a^2}{2l_n^2}\right) - l && (26)\\
&= (l-l_n) + \frac{b^2}{2(l-l_n)} + l_n + \frac{a^2}{2l_n} - l && (27)\\
&= \frac{b^2}{2(l-l_n)} + \frac{a^2}{2l_n} && (28)
\end{aligned}$$

Substituting this into (21) we get

$$\Delta T = \frac{AY}{2l}\left(\frac{b^2}{l-l_n} + \frac{a^2}{l_n}\right) \tag{29}$$

Final expression relating frequency change and position of fret

Finally, we can substitute (29) into (20) to get the expression between the intonation shift Δf and the fret position l_n

$$\Delta f = \frac{1}{4l_n}\sqrt{\frac{T}{\mu}}\left(\frac{AY}{2Tl}\left(\frac{b^2}{l-l_n} + \frac{a^2}{l_n}\right) - \frac{a^2}{{l_n}^2}\right) \tag{30}$$

From (4) we get

$$\sqrt{\frac{T}{\mu}} = 2lf_0$$

and

$$T = 4\mu l^2 {f_0}^2$$

Subbing into (30)

$$\Delta f = \frac{lf_0}{2l_n}\left(\frac{AY}{8\mu l^3 {f_0}^2}\left(\frac{b^2}{l - l_n} + \frac{a^2}{l_n}\right) - \frac{a^2}{{l_n}^2}\right) \tag{31}$$

2 Experiment

From the final expression relating Δf and l_n, knowing all other constants I can plot the graph for this. Δf is the dependent variable, and l_n is the independent variable. I expect the intonation difference Δf to be higher when the fretting distance is closer to the bridge (l_n closer to 0), and lower closer to the nut, in the first few frets (larger values of l_n). The general shape of the graph can be compared with my expectation as a sanity check. It is impossible to linearize the equation, therefore I will have to resort to collecting the data, plotting the graph, then confirming the relationship using a regression method to see how well the data fits the model. The choice of guitar I will use for this experiment is a Fender Stratocaster. This is arguably the most iconic and popular guitar in history, known for its versatility and playability. Therefore I pick this guitar because there is a wealth of information available on it, to make it easier to find data to support my research or to compare my results to other studies. Also due to its popularity, the Stratocaster is a well-known and well-understood instrument. This can make it easier to control variables in my experiment and ensure consistent results. The choice of string I will use is a G string from D'Addario's set of Nickel Wound Regular Light Gauge - EXL110-10P that I have at home. I choose the G string because it is the thickest string in the set that is a plain unwound string. This is because the thicker string will make the effects of the intonation shift more noticeable.

2.1 Determining the constants

From the equation (31), the constants I need to determine are f_0, μ, A, Y, a, b and l.

- l is the scale length of the guitar. For a Fender Stratocaster, it is 25.5 inches (64.77cm). I round this up to 64.8cm (3s.f). From this information, I can adjust the bridge saddle position so that the scale length matches 64.8cm (0.648 m)

- Initial frequency f_0 can be measured directly in the experiment. Frequency we aim for is the frequency of G string on the guitar, G_3 at 196 Hz

- Young's modulus Y is dependent on the specifications of the string material. The string I use follows the ATSM-A228 standard, and the accepted literature value of Y is 210 GPa (2.10×10^{11} Pa)

- A - the cross-sectional area of the string. This can be calculated based on the diameter of the string. For this string set, the G string gauge is 17-gauge, which is equivalent to a diameter of 0.017 inches (0.4318 mm). Using the formula for area of a circle from diameter:

$$\begin{aligned} A &= \frac{\pi d^2}{4} \\ &= 1.464 \times 10^{-7}\,\mathrm{m} \end{aligned}$$

- μ, the linear density of the string, is determined as

$$\mu = \frac{m}{l} = \frac{\rho V}{l} = \frac{\rho l A}{l} = \rho A$$

  where m is the mass of the vibrating string section, V is its volume, and ρ is the volumetric density of the material. According to ATSM-A228 standards, $\rho = 7.80 \times 10^3\,\mathrm{kg\,m^{-3}}$.

$$\mu = 7.80 \times 10^3 \cdot 1.46 \times 10^{-7} = 1.14 \times 10^{-3}\,\mathrm{kg\,m^{-1}}$$

- a and b: it is very hard to measure these directly, as they are distance from the nut and saddle to the top of the fret, not the height itself. Therefore I can only set up the guitar indirectly according to recommended values and calculate them afterwards. I follow the instructions by Stewmac, and take the average value of the action (distance between string and top of fret) for the 1$^{\text{st}}$ fret to be 0.016" (0.406mm) and 12$^{\text{th}}$ fret to be 0.070" (1.78mm). I change the adjustable saddle height and file down/shim up the nut height to match these values. From there I can calculate the values of a and b:

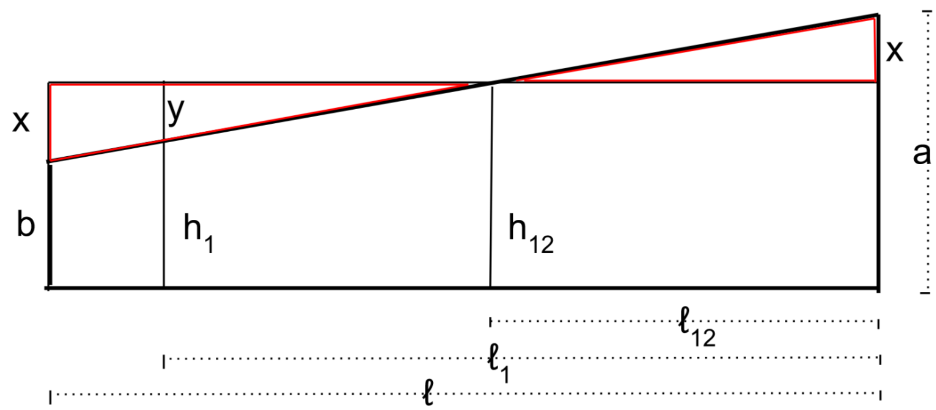

Figure 4: Diagram to calculate values of a and b (Exaggerated)

Since fret 12$^{\text{th}}$ is exactly in the middle of l ($l_{12} = \frac{l}{2} = \frac{.648}{2} = .324$ m) we get a pair of congruent triangles as highlighted. Therefore:

$$a = h_{12} + x$$
$$b = h_{12} - x$$

From the luthier formula (6) we get

$$\begin{aligned} l_1 &= \frac{l}{2^{\frac{1}{12}}} \\ &= \frac{.648}{2^{\frac{1}{12}}} \\ &= 0.612\,\text{m} \end{aligned}$$

and we also get the ratio in Figure 4

$$\begin{aligned} \frac{x}{y} &= \frac{l - l_{12}}{l_1 - l_{12}} \\ x &= y\frac{l - l_{12}}{l_1 - l_{12}} \\ &= (h_{12} - h_1)\frac{l - l_{12}}{l_1 - l_{12}} \\ &= (1.78 - .406) \cdot 10^{-3} \cdot \frac{.648 - .324}{.612 - .324} \\ &= 1.55 \times 10^{-3}\,\text{m} \end{aligned}$$

Therefore

$$a = 1.78 + 1.55 = 3.33 \times 10^{-3}\,\mathrm{m}$$
$$b = 1.78 - 1.55 = 1.30 \times 10^{-4}\,\mathrm{m}$$

2.2 Apparatus

- 1 Fender Stratocaster style guitar
- 1 17-gauge plain unwound steel G string
- 1 guitar capo
- 1 guitar pick

2.3 Process

1. Setting up the guitar. I used the instructions by Stewmac
2. Connect the guitar to frequency measuring software. I used my audio interface to connect the guitar to my laptop, and the software I used is Visual Analyzer 2020. I chose this because it supports FFT (Fast Fourier Transform) in real-time and can be configured to provide results with high accuracy.
3. Measure f_0 by plucking the string with no capo on.
4. Pluck it 5 times with the pick, changing the plucking position each time, from above the neck pickup, to between neck and middle pickups, above middle, between middle and bridge, and above bridge pickup. Record the peak frequency (highest dB) for each pick.
5. Put capo on 7^{th} fret and repeat step 4. Note that the edge of the capo must be right on top of the fret (Figure 5)
6. Repeat step 5 from fret 8 up to fret 16. I cannot go higher than this because this is where the neck meets the body, it is impossible to put the capo.

Figure 5: My experimental setup. I position the guitar neck outside the table for ease of access to the capo

Figure 6: Close up of capo placement. This ensures consistent pressure and contact with the string and fret

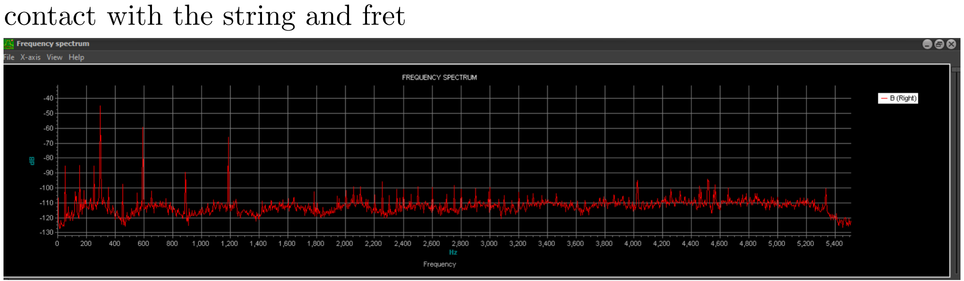

Figure 7: Frequency spectrum of a note in Visual Analyzer. This can be zommed in further to accurately read peak frequency value.

2.4 Tables of results

Table of raw data

Fret number	Frequency (Hz)				
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
0	195.82	195.82	196.49	196.49	196.47
7	294.76	294.76	294.77	294.76	294.77
8	311.74	311.74	313.09	312.45	313.09
9	331.15	331.15	331.14	330.48	331.15
10	350.66	351.28	350.66	350.66	351.28
11	371.63	372.29	372.29	371.66	371.62
12	394.06	394.07	394.06	394.06	394.06
13	417.56	417.57	417.56	417.57	418.22
14	441.94	442.61	443.28	442.61	443.29
15	469.36	469.36	469.34	469.36	469.36
16	497.62	497.22	497.23	497.82	497.62

Table 1: Raw collected data

The frequencies are taken up to 2d.p because this is the smallest the software can measure.

Table of processed data

Fret number	Fret distance (m)	Avg frequency (Hz)	Note	Standard frequency (Hz)	Δf (Hz)	Abs. unc. Δf (Hz)
0	0.648	196.22	G3	196.00	0.22	
7	0.432	294.76	D4	293.66	1.10	0.005
8	0.408	312.42	D#4	311.13	1.29	0.7
9	0.385	331.01	E4	329.63	1.39	0.3
10	0.364	350.90	F4	349.23	1.68	0.3
11	0.343	371.89	F#4	369.99	1.90	0.3
12	0.324	394.06	G4	392.00	2.06	0.005
13	0.306	417.70	G#4	415.30	2.39	0.3
14	0.289	442.74	A4	440.00	2.74	0.7
15	0.272	469.35	A#4	466.16	3.19	0.01
16	0.257	497.51	B4	493.88	3.63	0.3

Table 2: Table of processed data

Sample calculations

All example calculations are taken from values for fret 7.

- Fret distance is l_n, calculated from the luthier equation (6)
  $l_7 = \frac{.648}{\sqrt[12]{2^7}} = 0.432\,\mathrm{m}$

- Average frequency is simply average value of the 5 trials.
  avg.$f = \frac{294.76+294.76+294.77+294.76+294.77}{5} = 294.76\,\mathrm{Hz}$

- Note names correspond to standard note positions on G string on a standard guitar

- Standard frequency corresponds to notes, as taken from this source []

- Δf is the intonation deviation at each fret, calculated as the difference between the average frequency and the standard frequency.
  $\Delta f = 294.76 - 293.66 = 1.10\,\mathrm{Hz}$

- Absolute uncertainty of Δf is half the range of trial values up to 1 s.f
  Abs. unc. of $\Delta f = (294.77 - 294.76)/2 = 0.005\,\mathrm{Hz}$

2.5 Analysis of results

Graphs

After collecting the data, I can take the average value of $f_0 = 196.22\,\mathrm{Hz}$ as the constant value of f_0 for Equation (31). Now I can plot the graph relating Δf and l_n. The range of the graph is $\{0 \leq x \leq 0.648\}$, because this is the scale length of the guitar. Below is the general form of the graph with the constants.
From the graph, we can see it matches with my expectation. Overall Δf is lower for larger l_n (further from the bridge, lower frets), and increases when l_n is smaller (higher frets, closer to the bridge).
Observing the graph, we can see a few abnormal behaviors at the bounds. There are 2 vertical asymptotes, at 0 and at $l_n = l$. Mathematically the vertical asymptotes at 0 and at l come from Equation (31) when the denominator of the terms inside brackets is equal to 0 ($l_n = 0$ & $l - l_n = 0$). At $\lim_{l_n \to 0^+}$ the function approaches $-\infty$, while $\lim_{l_n \to l^-} = +\infty$. This is physically impossible. My explanation of this is because of the error terms in our first order approximation. When l_n or $l - l_n$ is very small and on a comparable order to a or b, the approximation at (18) and (26) no longer holds because the higher order error terms will

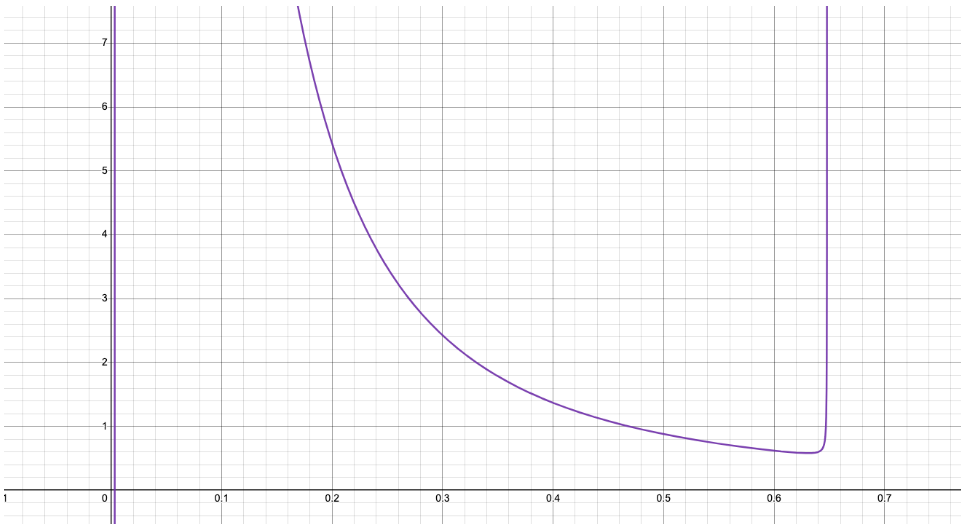

Figure 8: General form of the graph without data points

take over. This means the function will no longer accurately model the behavior of the string at that point. There is one point where $\Delta f = 0$ at $l_n = 0.0032$, meaning at this only point the intonation will be spot on. This is very close to the bridge (3.2mm) and does not match the position of any fret or any harmonic node of the string. I can't make a feasible intepretation for this zero, the only guess I can think of is this is also due to the irregular behaviors of error terms.
To analyze the graph further I can take the first derivative of this with respect to l_n. The graph of the first derivative has a zero at approx. 0.0048, indicating there is an extremum of the initial function at this point that is not shown on the graph. Plugging $l_n = 0.0048$ into the initial function, it returns a value of approx. 3193 Hz, so this is a very large maximum. A possible physical interpretation of this is the breaking point of the string. Notice how the value of $l_n = 0.0048$ is very small, this is very close to the bridge, just 4.3 mm away. If you try to fret very close to the nut or the bridge, the "breaking angle" (angle between the fret and the string) θ of the string in Figure 9 gets very large, therefore the amount of force needed to counteract the tension to depress the string to touch the fret increases dramatically, which also increases the tension of the string accordingly, which then in turn requires an even larger force to fret down. This explains the steep increase in Δf approaching $l_n = l$ (tending to $+\infty$ as a vertical asymptote) and the maxima. At the maxima, the tension in the

string gets too large and the string break, explaining the abnormal peak.

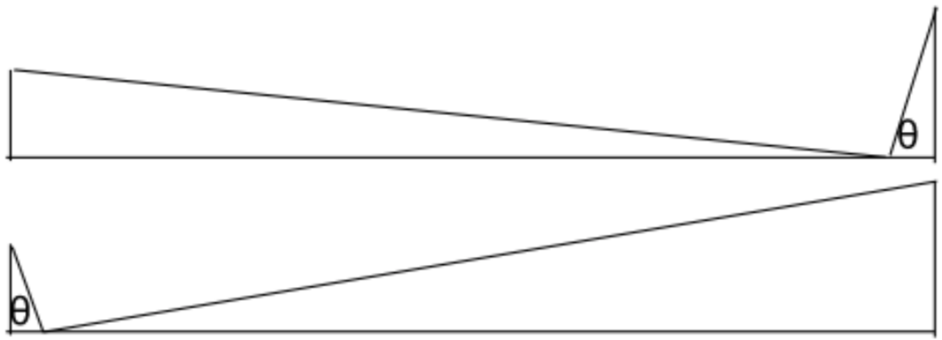

Figure 9: Breaking angle θ of string when fretting near bridge and nut