Gaussian processes for high order finite volume methods

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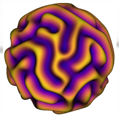

Introduction

Solve the compressible ideal MHD equations (2D)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}) + \frac{\partial}{\partial y}\mathbf{G}(\mathbf{U}) = 0$$

$$\mathbf{U} = \begin{pmatrix} \rho & \rho u & \rho v & \rho w & B_x & B_y & B_z & E \end{pmatrix}^T$$

$$\mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p_{tot} - B_x^2 \\ \rho u v - B_x B_y \\ \rho u w - B_x B_z \\ 0 \\ u B_y - v B_x (= -E_z) \\ u B_z - w B_x (= E_y) \\ u(E + p_{tot}) - B_x\left(\mathbf{u} \cdot \mathbf{B}\right) \end{pmatrix} \quad \mathbf{G}(\mathbf{U}) = \begin{pmatrix} \rho v \\ \rho u v - B_x B_y \\ \rho v^2 + p_{tot} - B_y^2 \\ \rho v w - B_y B_z \\ v B_x - u B_y (= E_z) \\ 0 \\ v B_z - w B_y (= -E_x) \\ v(E + p_{tot}) - B_y\left(\mathbf{u} \cdot \mathbf{B}\right) \end{pmatrix}$$

while satisfying

$$\nabla \cdot \mathbf{B} = 0$$

Solver overview

Hydrodynamic subsystem

- Hydrodynamic variables reconstructed as before
- Fluxes in subsystem corrected as before

Magnetic field

- Field is only stored as cell average values
- Face centered field reconstructed through divergence free GP
- Magnetic fluxes corrected to obey something like constrained transport

No staggered grid!

Divergence free kernel function

Prediction of vector valued function requires a matrix valued kernel. We can construct an inherently divergence free kernel from a scalar kernel via

$$\mathbf{K}_{div}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} (\partial_{2,2}+\partial_{3,3}) & -\partial_{1,2} & -\partial_{1,3} \\ -\partial_{2,1} & (\partial_{1,1}+\partial_{3,3}) & -\partial_{2,3} \\ -\partial_{3,1} & -\partial_{3,2} & (\partial_{1,1}+\partial_{2,2}) \end{pmatrix} K_{scl}(\mathbf{x},\mathbf{y})$$

where $\partial_{k,l} = \frac{\partial^2}{\partial x_k \partial y_l}$.

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In 2D with squared-exponential

With $x_3 = y_3 = 0$, the above becomes

$$\mathbf{K}^{div}(\mathbf{x},\mathbf{y}) = \frac{1}{\ell^2} \begin{pmatrix} \left(2 - \frac{(x_2-y_2)^2}{\ell^2}\right) & \frac{(x_1-y_1)(x_2-y_2)}{\ell^2} & 0 \\ \frac{(x_1-y_1)(x_2-y_2)}{\ell^2} & \left(2 - \frac{(x_1-y_1)^2}{\ell^2}\right) & 0 \\ 0 & 0 & \ell^2 \end{pmatrix} e^{-\frac{||\mathbf{x}-\mathbf{y}||}{2\ell^2}}$$

Integrated kernel

Let K and C be the 1D point and integrated kernel functions respectively. The integrated divergence free kernel is then

$$
\begin{aligned}
\mathbf{C}_{xx}^{div}(\mathbf{x},\mathbf{y}) &= \frac{2}{\ell^2} C(x_1,y_1)\left(2C(x_2,y_2) - K(x_2,y_2)\right) \\
\mathbf{C}_{yy}^{div}(\mathbf{x},\mathbf{y}) &= \frac{2}{\ell^2} C(x_2,y_2)\left(2C(x_1,y_1) - K(x_1,y_1)\right) \\
\mathbf{C}_{xy}^{div}(\mathbf{x},\mathbf{y}) &= \frac{1}{\ell^3}\sqrt{\frac{\pi}{2}}\left(erf\left(\frac{(x_1-y_1)+1}{\ell\sqrt{2}}\right) + erf\left(\frac{(x_1-y_1)-1}{\ell\sqrt{2}}\right)\right) \\
&\quad \left(erf\left(\frac{(x_2-y_2)+1}{\ell\sqrt{2}}\right) + erf\left(\frac{(x_2-y_2)-1}{\ell\sqrt{2}}\right)\right)
\end{aligned}
$$

where ℓ and the cell center locations x_i, y_i are given in units of the grid spacing.

Covariance matrix

Each entry

$$\mathbf{C}_{ij} = \mathbf{C}^{div}(\mathbf{x}_i, \mathbf{x}_j)$$

is now a 3×3 block. In total, $\mathbf{C}$ is still symmetric positive definite.

Prediction matrix

Sample vector becomes a sample matrix with 3 columns. As for $\mathbf{C}$, $\mathbf{T}$ can be written using scalar GP pieces. Prediction follows exactly as before,

$$\mathbf{B}_* = \mathbf{T}^T\mathbf{C}^{-1}\mathbf{q}$$

WENO Revisited

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WENO Revisited

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Temporary method for $\mathbf{B}$

- Scatter field magnitude onto 2D scalar stencil
- Calculate smoothness indicators, effective weights
- Apply those effective weights here
- Weakly justified by considering a field with 1 nonzero component

Toth's Flux-interpolated CD scheme

- Use shock-capturing code to generate fluxes at all cell interfaces
- Correct the magnetic fluxes prior to update to maintain $\nabla \cdot \mathbf{B} = 0$
- Discretize $\nabla \cdot$ with cell-centered finite differences

Consider $\Omega = E_z$

$$\Omega_{i,j} = \frac{1}{4}\left(-f^{B_y}_{i-1/2,j} - f^{B_y}_{i+1/2,j} + g^{B_x}_{i,j-1/2} + g^{B_x}_{i,j+1/2}\right),$$

differencing this to obtain B_x and B_y fluxes satisfies $\nabla \cdot \mathbf{B} = 0$ discretely to ϵ_{mach}. (see board)

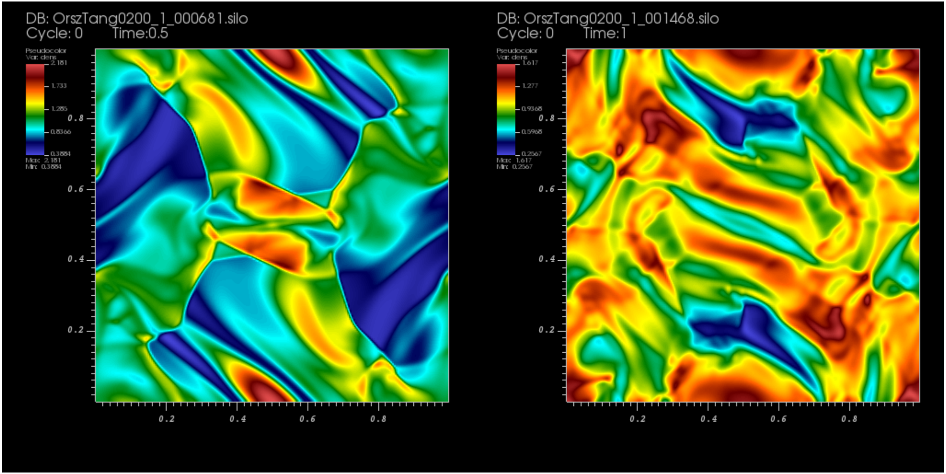
Distinguishing between average and point data

Toth's correction at an edge is similar point$\rightarrow$average correction from earlier

- Grow stencil to more edges, larger radii
- Prune stencil to match symmetry of discrete divergence operator
- Solve for stencil weights satisfying:
  - Divergence free constraint
  - Consistency
  - Integrate transverse polynomials exactly

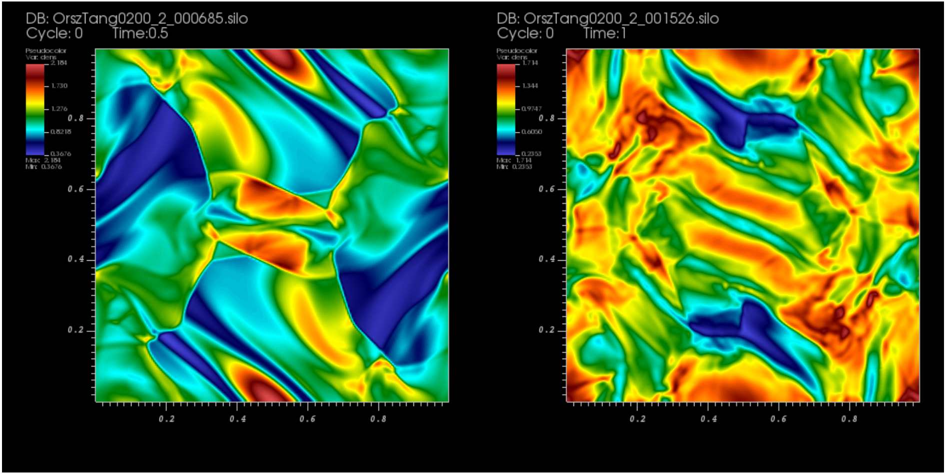
Orszag-Tang vortex

200×200, Radius 1, $\ell = 12\Delta$, $\sigma = 3\Delta$



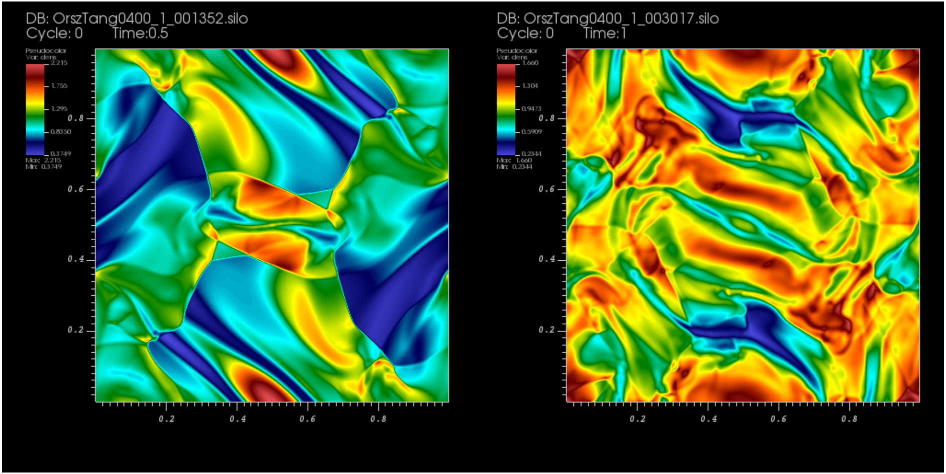
Orszag-Tang vortex

200×200, Radius 2, $\ell = 12\Delta$, $\sigma = 3\Delta$



Orszag-Tang vortex

400×400, Radius 1, $\ell = 12\Delta$, $\sigma = 3\Delta$



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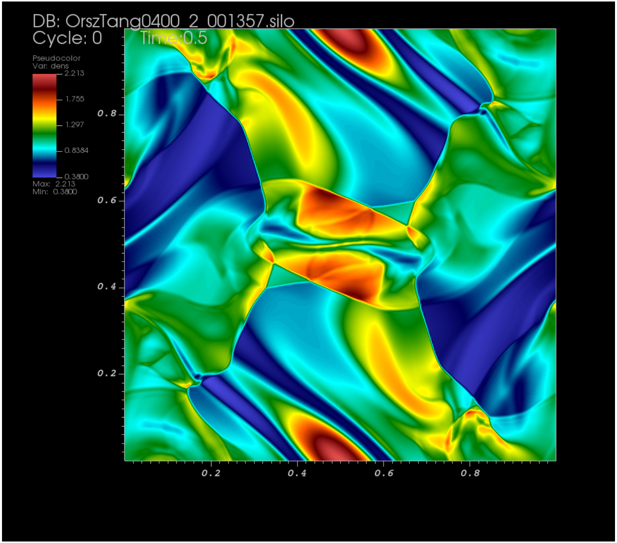

Final thoughts

Conclusion

- An unstaggered, unsplit method is possible
- Toth's 2^{nd} order scheme can nominally be extended to higher order
- Measuring $\nabla \cdot \mathbf{B}$ to lower accuracy than remaining method seems fine

Next steps

- Perform convergence study (CPAW)
- Extend to rectangular cells
- Implement MHD characteristic variables
  - Is there a convenient way to transform $\mathbf{B}$ too?
- Generalize flux formulation to arbitrary radius
- . . .
- Extend to 3D and AMR