RBF Interpolation for FV Schemes

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1 Introduction

Radial basis function(RBF) interpolation is a useful tool for multivariate scattered data approximation. We especially choose the shape parameters for RBFs to obtain highly accurate approximation.

2 RBF interpolation

Suppose that a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is known only at a set of discrete points $X := \{x_1, ..., x_N\}$ in $\Omega \subset \mathbb{R}^d$. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radial in the sense that $\phi(x) = \phi(|x|)$, where $|\cdot|$ is the usual Euclidean norm. Radial basis function interpolation to f on X starts with choosing a basis function ϕ, and then it defines an interpolant by

$$Af_X(x) := \sum_{k=1}^{m} \beta_k p_k(x) + \sum_{j=1}^{N} \alpha_j \phi(x - x_j) \tag{1}$$

where $\{p_1, \ldots, p_m\}$ is a basis for Π_m and the coefficients α_j and β_i are chosen to satisfy the linear system

$$\begin{aligned} Af_X(x_i) &= f(x_i), \quad i = 1, \ldots, N, \\ \sum_{j=1}^{N} \alpha_j p_k(x_j) &= 0, \quad k = 1, \ldots, m. \end{aligned} \tag{2}$$

For a wide choice of functions ϕ and polynomial orders m, the existence and uniqueness of the solution of the linear system (2) is ensured when ϕ is a conditionally positive definite function.

Definition 1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We say that ϕ is conditionally positive definite of order $m \in \mathbb{N}$ if for every finite set of pairwise distinct points $X = \{x_1, \ldots x_N\} \subset \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N \backslash \{0\}$ satisfying $\sum_{j=1}^{N} \alpha_j p(x_j) = 0$ for $\forall p \in \Pi_m$, the quadric form*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i - x_j)$$

is positive definite.

This leads to the linear system (2) on the matrix form

$$\begin{bmatrix} \Phi & P^T \\ P & O \end{bmatrix} \begin{bmatrix} \alpha^T \\ \beta^T \end{bmatrix} = \begin{bmatrix} \mathbf{f}^T \\ O \end{bmatrix} \tag{3}$$

to be solvable where $\Phi = \{\phi(x_i - x_j) : i, j\}$, $P = \{p_k(x_j) : k, j\}$ and $\mathbf{f} = \{f(x_i) : i\}$. If we assume $m = 0$ in (1) then from the equations (1) and (3) the interpolation could be represented by

$$Af_X(x) = \phi\Phi^{-1}\mathbf{f}^T \tag{4}$$

where $\phi = \{\phi(x - x_j) : j\}$. We note that $\phi\Phi^{-1}$ in the representation (4) is the kernel completely independent of the function values $\mathbf{f}$. (This kernel will be compared to the GP approximation later.)

3 Optimal choice for the shape parameter

In [1], the authors introduced a WENO scheme based on the exponential polynomial space, and improved the order of accuracy by using the optimal control parameter $\lambda \in \mathbb{R}$ or $i\mathbb{R}$ for the exponential basis functions of the form $e^{\lambda x}$ in [2]. We follow this idea but use the infinitely smooth RBFs $\phi := \phi_\lambda$ with a shape parameter λ, as basis functions instead of exponential polynomials. There are commonly used RBFs: a Gaussian function $\phi(x) = e^{-(\lambda x)^2}$ and a multiquadric $\phi(x) = \sqrt{1 + (\lambda x)^2}$, etc.

We consider a system of conservation laws $\partial_t u + \nabla \cdot f(u) = 0$ where u is a vector of conserved quantities and f is a vector of vector valued of flux functions. Here we focus on the one-dimensional spatial case:

$$\partial_t u + \partial_x f(u) = 0. \tag{5}$$

We restrict our considerations to uniformly distributed spatial grids with grid cells $I_i := [x_{i-1/2}, x_{i+1/2}]$ and grid spacing $\Delta x = x_{i+1/2} - x_{i-1/2}$ for all i. A finite volume method for the equation (5) can be written in the semi-discrete form

$$\frac{d}{dt}u_i = -\frac{1}{\Delta x}(\hat{F}_{i+1/2} - \hat{F}_{i-1/2}). \tag{6}$$

Given the cell averages $\bar{u}_i(x)$ of a function $u(x)$ for each cell I_i, we construct the RBF reconstruction from cell averages.

We now present here the detailed analysis for the case of $N=2$. To reconstruct the approximation at the cell boundary $x=x_{1/2}$ from the cell averages $\{\bar{u}_0,\bar{u}_1\}$, we define a primitive function

$$U(x)=\int_{x_{-1/2}}^{x} u(\xi)d\xi$$

so that $U(x_{i-1/2})=\Delta x\sum_{k=0}^{i}\bar{u}_k$ for $i=0,\cdots,N$. Then we find an approximant

$$AU(x):=\sum_{j=0}^{N}\alpha_j\phi(x-x_{j-1/2})$$

satisfying

$$AU(x_{i-1/2})=U(x_{i-1/2}),\quad i=0,\cdots,N,$$

and then the derivative of approximant for U, $(AU)'(x)=\sum_{j=0}^{N}\alpha_j\phi'(x-x_{j-1/2})$, approximates $U'(x)=u(x)$. For example, if we choose a basis with Gaussian function $\phi(x)=e^{-(\lambda x)^2}$ with a shape parameter $\lambda\in\mathbb{R}$ or $i\mathbb{R}$, we can have the final approximation

$$AU'(x_{1/2})=\sum_{i=0}^{2}c_iU_{i-1/2}=2\Delta x^2\lambda^2\frac{e^{3\Delta x^2\lambda^2}}{e^{4\Delta x^2\lambda^2}-1}(\bar{u}_0+\bar{u}_1)$$

with kernel coefficients $\{c_i:i=0,1,2\}$ and Taylor expansion for the right-hand side gives

$$AU'(x_{1/2})=u(x_{1/2})+(\frac{1}{6}u'(x_{1/2})+\lambda^2u(x_{1/2}))\Delta x^2+\mathscr{O}(\Delta x^4).$$

So we have the second order approximation for the cell boundary point and we can obtain the fourth order approximation if we choose the shape parameter with

$$\lambda^2=-\frac{u''(x_{1/2})}{6u(x_{1/2})}+\mathscr{O}(\Delta x^2).$$

A. Connection with GP

In [3], the authors use kernel-based Gaussian process(GP) prediction methods to interpolate/reconstruct high-order approximation for solving hyperbolic PDEs. Given function values $f = [f(x_1), \cdots, f(x_N)]$, GP predictions aim to make a probabilistic statement about the value $f_* = f(x_*)$ of the unknown function $f \sim GP(\bar{f}, K)$ at a new spatial point x_*. By utilizing the conditioning property of GP from the theory of Bayesian inference, an updated posterior mean function is obtained by

$$\tilde{f}_* = \bar{f}(x_*) + k_*^T K^{-1} \cdot (f - \bar{f}) \tag{7}$$

where $k_* = \{K(x_*, x_i) : i\}$ and $K = \{K(x_i, x_j) : i, j\}$. The mean $\tilde{f}_*$ of the distribution is introduced as the interpolation of the function f at the point x_*, where f is any given fluid variable.

For reconstruction in FV, the pointwise function value $f(x_*)$ at the point x_*, reconstructed from the volume-averaged data $G = \{G_k : k\}$ with

$$G_k = \int f(x) dg_k(x),$$

is given by

$$\tilde{f}_* = \bar{f}(x_*) + T_*^T C^{-1} (G - \bar{G}) \tag{8}$$

where $T_* = \{T_{*,k} : k\}$ and $C_* = \{C_{k,h} : k, h\}$ with

$$T_{*,k} = \int K(x, x_*) dg_k(x) \quad \text{and} \quad C_{k,h} = \int K(x, y) dg_k(x) dg_h(y).$$

Questions

1. How did you find $\bar{f}(x_*)$ in (7) and (8) and how accurate are they?
2. If the covariance function K is chosen as just a Gaussian function and a zero mean is assumed, i.e., $\bar{f}(x) = 0$, in (7), what is the difference between this approximation and the RBF interpolation?

References

[1] Y. Ha, C. H. Kim, H. Yang, and J. Yoon. Sixth-order weighted essentially nonoscillatory schemes based on exponential polynomials. *SIAM Journal on Scientific Computing*, 38:1987–2017, 2016.

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[3] A. Reyes, D. Lee, C. Graziani, and P. Tzeferacos. A new class of high-order methods for fluid dynamics simulations using gaussian process modeling. *J Sci Comput*, 76:443–480, 2018.