MDP : discrete time

Time epochs : $t = 1, 2, 3, \dots, T$

State : $x_t \in \mathcal{X}$

Action : $a_t \in \mathcal{U}_t(x_t)$, feasible action

Transition Kernel : $\mathbb{P}(x_{t+1} \mid x_t, a_t)$

Reward Function : $R_t(x_t, a_t)$

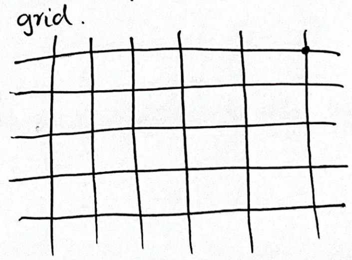
Final Reward : $R_T(x_T, a_T) = R_T(x_T)$

Objective : To get optimal policy, $\Pi = \{\pi_1, \pi_2, \dots, \pi_{T-1}\}$.

$$V_t^{\Pi}(x_t) = \mathbb{E}\left[\left(\sum_{\tau=t}^{T} \beta^{\tau-1} R_\tau(x_\tau, a_\tau)\right) \Big| \, x_t\right]$$

$$\begin{cases} V_t^{*}(x_t) = \max\limits_{a \in \mathcal{U}_t(x_t)} \left[ R_t(x_t, a) + \beta \mathbb{E}\left[ V_{t+1}^{*}(X_{t+1}) \,\middle|\, x_t, a_t \right] \right] \\ V_T^{*}(x_T) = R_T(x_T). \end{cases}$$

Discretization of $\mathcal{X}$ : Calculate every value on the grid.



$x_{i,j}$ $\quad x_t \in \{x_{i,j}\}$.

Backward Induction Solution.

Time : $t = 1, 2, 3, \cdots, T$

State: $x_t \in \mathcal{X}$ , $x_t = [\underline{W_t, n_{t-1}}, e_t, S_t, A_t, \underline{M_t, g_{t-1}}]$ $(H, r_h, m, O_{t-1} = 1)$

Action: $a_t = [c_t, b_t, k_t, i_t, q_t]$

↳ owning share of the house $[0, 1]$.

Reward: $R_t(x_t, a_t)$

If $A_t = 0$ :

$R_t = uB(TB_t)$ , $uB(0) = 0$ should be satisfied.

$u(c) = 1 - \frac{1}{c+1}$ with gamma $= 2$, then $uB(TB_t) = 2(1 - \frac{1}{TB_t + 1})$

where $TB_t = (W_t + n_{t-1}) + (H + (1-\chi)(1-\delta) g_{t-1}) P_t - M_t$

If $A_t = 1$ :

If $q_t = 1$ : $V_h = (1+\kappa) \cdot h_t$ , $h_t = H + [(1-\delta) g_{t-1} + i_t]$

$R_t = u(c_t^{\alpha} \cdot V_h^{1-\alpha})$

If $q_t \neq 1$ : $V_h = (1-\kappa)[h_t - (1-q_t) H]$. $h_t = H + [(1-\delta) g_{t-1}]$

$R_t = u(c_t^{\alpha} \cdot V_h^{1-\alpha})$

Terminal Reward: $R_T(x_T)$

$$R_T(x_T) = uB(TB_T) = (W_T + n_{T-1}) + (H + (1-\chi)(1-\delta) g_{T-1}) P_t - M_t$$

Transition:

$x_t = [w_t, n_{t-1}, e_t, S_t, A_t, M_t, g_{t-1}]$ ( H, $r_n$, $m$, $O_t = 1$ )

$a_t = [c_t, b_t, k_t, i_t, q_t]$

$\mathbb{P}(x_{t+1} \mid x_t, a_t)$

If $A_t = 0$: then $a_t =$ None. or $[0, 0, 0, 0, 0]$

$$x_{t+1} = \begin{bmatrix} w_{t+1} = 0 \\ n_t = 0 \\ e_{t+1} = 0 \\ S_{t+1} \\ A_{t+1} = 0 \\ M_{t+1} = 0 \\ g_{t+1} = 0 \end{bmatrix}$$

with $P_s(S_t, S_{t+1} = 1)$, $S_{t+1} = 1$

with $P_s(S_t, S_{t+1} = 0)$, $S_{t+1} = 0$

If $A_t = 1$: Income: $y_t^{AT} = \begin{cases} (1-\tau_L)(y_t - i_t^n), & t < T_R \\ (1-\tau_R)y_t - i_t^n, & t \geq T_R. \end{cases}$

① If $q_t = 1$:

...

② If $q_t \neq 1$:

...

① If $q_t = 1$:

If $t \geq 45$:

$n_t = g_n(n_{t-1}, t, y_t)$ → small fix here.

$e_{t+1} = 0$

for $s_{t+1}$ in [0, 1]:

for $A_{t+1}$ in [0, 1]:

$w_{t+1} = (1 + r_b(s_t) b_t) + (1 + r_k(s_t, s_{t+1})) k_t$.

$g_t = (1-\delta) g_{t-1} + i_t$ - - - - - (*)

$M_{t+1} = M_t (1 + r_h) - m$

If $t < 45$:

$n_t = g_n(n_{t-1}, t, y_t)$

for $s_{t+1}$ in [0, 1]:

for $e_{t+1}$ in [0, 1]:

for $A_{t+1}$ in [0, 1]:

$\boxed{\text{Same here}}$ · · · · · (*)

② If $q_t \neq 1$:

If $t \geq 45$:

$n_t = g_n(n_{t-1}, t, y_t)$

$e_{t+1} = 0$

for $S_{t+1}$ in $[0, 1]$:

for $A_{t+1}$ in $[0, 1]$:

$w_{t+1} = (1 + r_b(S_t))b_t + (1 + r_k(S_t, S_{t+1}))k_t$.

$g_t = (1 - \delta)g_{t-1}$ ---- $(*)$

$M_{t+1} = M_t(1 + r_h) - m$.

If $t < 45$:

$n_t = g_n(n_{t-1}, t, y_t)$

for $S_{t+1}$ in $[0, 1]$:

for $e_{t+1}$ in $[0, 1]$:

for $A_{t+1}$ in $[0, 1]$:

$\boxed{\text{Same}}$ ---- $(*)$.

Value iteration.

$$V_T(X_T) = R_T(X_T).$$

$$V_t(x_t) = \max_{a \in \mathcal{U}_t(x_t)} \underbrace{\left[ R_t(x_t, a) + \beta \mathbb{E}[ \underbrace{V_{t+1}(x_{t+1})}_{\text{from approximation}} \mid x_t, a_t ] \right]}_{\text{objective function.}}$$

① what is the shape of the obj(a).

$$\max_{a \in \mathcal{U}_t(x_t)} obj(a), \quad a = [c_t, b_t, k_t, i_t, q_t]$$

high dimension approximation

high dimension optimization.

Linear constain: with $q_t = 1$, budget constain: $\mathcal{U}_t(x_t) \Rightarrow$

$$\left\{ \begin{aligned} & y_t^{AT} + W_t = c_t + b_t + k_t + m + (1+\chi)P_t \cdot i_t + C_h \cdot \mathbb{I}(i_t > 0) \\ & q_t = 1 \\ & c_t \geq 0 \\ & b_t \geq 0 \\ & k_t \geq 0 \\ & i_t \geq 0 \end{aligned} \right.$$

with $q_t \neq 1$, budget constain : $u_t(x_t) \Rightarrow$ (

$$\begin{cases} y_t^{AT} + W_t + P_t^r (1-q_t) H = C_t + b_t + k_t + m \\ 0 \le q_t \le 1 \\ C_t \geq 0 \\ b_t \geq 0 \\ k_t \geq 0 \\ i_t = 0 \end{cases}$$