1.13

Prove that $Fib(n)$ is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$.

Proof
Let $\psi = (1-\sqrt{5})/2$. We want to first show that $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$.

Base Case
n = 0:

$$\begin{aligned} Fib(0) &= (\phi^0 - \psi^0)/\sqrt{5} && (1)\\ &= (1-1)/\sqrt{5} && (2)\\ &= 0 && (3) \end{aligned}$$

n = 1:

$$\begin{aligned} Fib(1) &= (\phi^1 - \psi^1)/\sqrt{5} && (4)\\ &= ((1+\sqrt{5})/\sqrt{2} - (1-\sqrt{5})/2)/\sqrt{5} && (5)\\ &= (2\sqrt{5}/2)/\sqrt{5} && (6)\\ &= 1 && (7) \end{aligned}$$

Inductive Step

For $k \geq 0$, assume $Fib(k) = (\phi^k - \psi^k)/\sqrt{5}$. By the definition of Fib,

$$
\begin{aligned}
Fib(k+1) &= Fib(k) + Fib(k-1) && (8)\\
&= (\phi^k - \psi^k)/\sqrt{5} + (\phi^{k-1} - \psi^{k-1})/\sqrt{5} && (9)\\
&= \frac{\phi^k + \phi^{k-1} - (\psi^k + \psi^{k-1})}{\sqrt{5}} && (10)\\
&= \frac{\phi^{k-1}(\phi+1) - (\psi^{k-1}(\psi+1))}{\sqrt{5}} && (11)\\
&= \frac{\phi^{k-1}(\phi^2) - (\psi^{k-1}(\psi^2))}{\sqrt{5}} && (12)\\
&= (\phi^{k+1} + \psi^{k+1})/\sqrt{5} && (13)
\end{aligned}
$$

This shows that $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$.

Now we need to show that $|Fib(n) - \phi^n/\sqrt{5}| \leq 1/2$.

$$
\begin{aligned}
|Fib(n) - \phi^n/\sqrt{5}| &= |(\phi^n - \psi^n)/\sqrt{5} - \phi^n/\sqrt{5}| && (14)\\
&= |\psi^n/\sqrt{5}| && (15)
\end{aligned}
$$

Since $|\psi^0/\sqrt{5}| \approx .4472135955 \leq 1/2$ and the expression is monotonically decreasing, it will be true for all $n \geq 0$. Therefore $Fib(n)$ is the closest integer to $\phi^n/\sqrt{5}$.