Operations of MDDs and matrix diagrams [1]

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In this note, we show the details of the operations about MDDs and matrix diagrams. In particular, those operations include the union of two MDDs, the union of two matrix diagrams, and the relational product of an MDD and a matrix diagram, which are in accordance with Definitions 8, 9, and 10 of the paper "Symbolic Observer of Timed Labeled Petri Nets With Application to Current-state Opacity", respectively. For the sake of brevity, the listed examples are displayed under a relatively simple Petri net that is shown in Fig. 1.

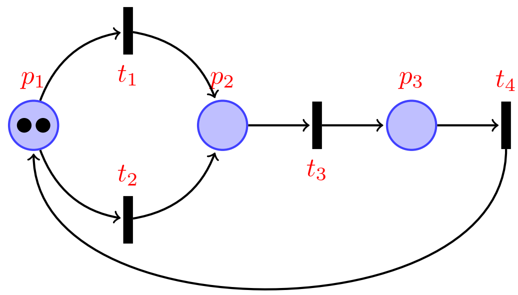


Figure 1: A Petri net.

1 Union of two MDDs

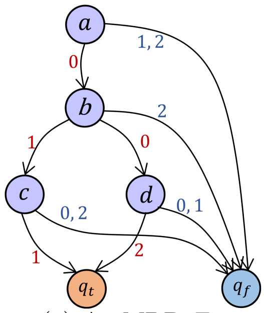
Let $\mathcal{M}_1 = \{[1,1,0]^T, [2,0,0]^T\}$ and $\mathcal{M}_2 = \{[0,2,0]^T, [1,1,0]^T\}$ be two sets of markings. The two marking sets $\mathcal{M}_1$ and $\mathcal{M}_2$ can be represented by two MDDs $F_1 = (Q', D, q'_0, q'_t, q'_f, \delta'_t)$ and $F_2 = (Q'', D, q''_0, q''_t, q''_f, \delta''_t)$ that are shown in Figs. 2(a) and 2(b), respectively. The MDD $F = (Q, D, (q'_0, q''_0), (q'_t, q''_t), (q'_f, q''_f), \delta_t)$ that represents the marking set $\mathcal{M}_1 \cup \mathcal{M}_2$ is the union of F_1 and F_2 (portrayed in Fig. 2(c)). We now show the detail of the union of these two MDDs (Please see Definition 8 of the paper).

Initially, by Definition 8, the root vertex of F is $(q'_0, q''_0) = (a, A)$. Starting from (a, A), we now recursively define the other non-terminal vertexes in F. For all labels $d \in D = \{0, 1, 2\}$, since both vertexes $a \in Q'$ and $A \in Q''$ are extensible with respect to true child vertexes, i.e., b and B (when $d = 0$), the vertex (b, B) is added to Q, and $\delta_t((a, A), 0) = (b, B)$ is defined. For labels 1 and 2, we have $\delta'_t(a, 1) = q'_f$, $\delta'_t(a, 2) = q'_f$, $\delta''_t(A, 1) = q''_f$, and $\delta''_t(A, 2) = q''_f$, i.e., $a[d]_f$ and $A[d]_f$ for $d = 1, 2$[2]. Then we define $\delta_t((a, A), 1) = (q'_f, q''_f)$ and $\delta_t((a, A), 2) = (q'_f, q''_f)$.

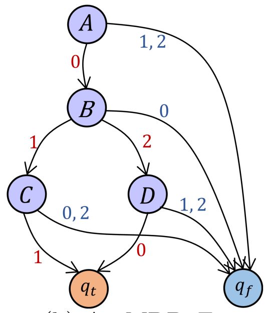
Now, we come to the vertex (b, B). When $d \in D = 0$, we have $\delta'_t(b, 0) = d$ (in this equation, d is a vertex where $d \in Q'$) and $\delta''_t(B, 0) = q''_f$. By Definition 8, we add the vertex (d, q''_f) to the vertex

[1]To cite this note, please refer to Y. F. Dong, N. Q. Wu, and Z. W. Li, "Operations of MDDs and matrix diagrams," Available: https://github.com/dongyifan199/Opacity-on-TLPN, Nov. 2021.

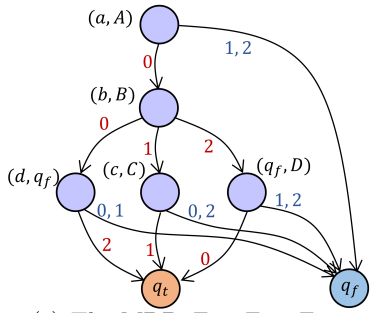
[2]For the sake of the unified representation of MDDs, in Figs. 2(a), 2(b), and 2(c), the terminal vertex valued with **True** (resp., **False**) is unified represented as q_t (resp., q_f). However, please note that in this text, the two terminal vertexes are represented by q'_t (q'_f), q''_t (q''_f), and (q'_t, q''_t) ((q'_f, q''_f)) in MDDs F_1, F_2, and F, respectively.



(a) An MDD F_1.



(b) An MDD F_2.



(c) The MDD $F = F_1 \cup F_2$.

Figure 2: Union of two MDDs.

set Q, since the vertex satisfies $\neg(b[d]_f \ \& \ B[d]_f)$, and we define $\delta_t((b,B),0) = (d, q''_f)$. Analogously, we add vertexes (c,C) and (q'_f, D) to Q, and define $\delta_t((b,B),1) = (c,C)$ and $\delta_t((b,B),2) = (q'_f, D)$.

Then, we analyze the vertexes in level 1, i.e., (d, q''_f), (c,C), and (q'_f, D). For the vertex (d, q''_f), only when $d \in D = 2$, the vertex d is extensible with respect to the true terminal vertex, i.e., $d[2]_t$ (note that here $d \in Q'$ is a vertex while in "$d \in D = 2$", d is a label). Then we have $\delta_t((d, q''_f), 2) = (q'_t, q''_f) = (q'_t, q''_t)$ (here $(q'_t, q''_f) = (q'_t, q''_t)$ is obtained from Definition 8). The vertexes (c,C) and (q'_f, D) can be analogously analyzed.

2 Union of two matrix diagrams

Indeed, the union of two matrix diagrams is similar to the union of two MDDs. Here, we just briefly describe it. Figs. 3(a) and 3(b) show two matrix diagrams $H_1 = (Q', \mathcal{D}, q'_0, q'_t, q'_f, \delta'_t)$ and $H_2 = (Q'', \mathcal{D}, q''_0, q''_t, q''_f, \delta''_t)$ that are decided by transitions t_1 and t_3 of the Petri net depicted in Fig. 1, respectively. The union of H_1 and H_2, i.e., $H = H_1 \cup H_2 = (Q, \mathcal{D}, (q'_0, q''_0), (q'_t, q''_t), (q'_f, q''_f), \delta_t)$ is shown in Fig. 3(c).

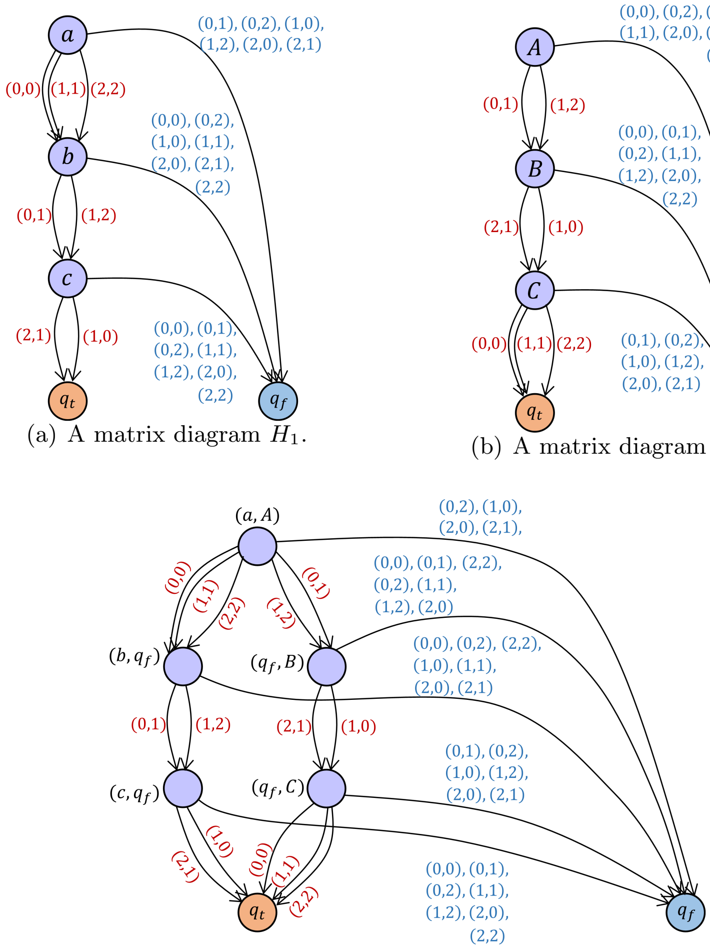
The root vertex (q'_0, q''_0) of H is initialized by (a, A). At (a, A), for all label pairs $(d, d') \in \mathcal{D} = \{0,1,2\} \times \{0,1,2\}$, we have $a[(0,0)]_t$, $a[(1,1)]_t$, and $a[(2,2)]_t$. However, for these three label pairs, the vertex $A \in Q''$ is extensible with respect to the false child vertex, i.e., $A[(0,0)]_f$, $A[(1,1)]_f$, and $A[(2,2)]_f$. Then we add the vertex (b, q''_f) to Q and define $\delta_t((a,A),(0,0)) = (b, q''_f)$, $\delta_t((a,A),(1,1)) = (b, q''_f)$, and $\delta_t((a,A),(2,2)) = (b, q''_f)$. As for label pairs $(0,1)$ and $(1,2)$, we have $A[(0,1)]_t$, $A[(1,2)]_t$, $a[(0,1)]_f$, and $a[(1,2)]_f$. Then, the vertex (q'_f, B) is added to Q, and the labeling function can be defined by $\delta_t((a,A),(0,1)) = (q'_f, B)$ and $\delta_t((a,A),(1,2)) = (q'_f, B)$. Similarly, we can define the remainder of vertexes and the labeling function associated with them.

3 Relational product of an MDD and a matrix diagram

In this part, we touch upon the relational product of an MDD and a matrix diagram by showing two examples (Please see Definition 10 of the paper).

(1) The first instance is shown in Fig. 4. The relational product represents the computation of the next-state function $\mathcal{M}'_t = \mathcal{N}(\mathcal{M}_t, \mathcal{T}_t)$, where $\mathcal{M}_t = \{[1,1,0]^T, [2,0,0]^T\}$, $\mathcal{T}_t = \{t_1, t_3\}$, and $\mathcal{M}'_t = \{[1,1,0]^T, [0,2,0]^T, [1,0,1]^T\}$. The set $\mathcal{M}_t$ is represented by the MDD $F_1 = (Q', D, q'_0, q'_t, q'_f, \delta'_t)$ (depicted in Fig. 4(a)) and the transition relation decided by transitions t_1 and t_3 is represented by the matrix diagram $H_2 = (Q'', \mathcal{D}, q''_0, q''_t, q''_f, \delta''_t)$ (portrayed in Fig. 4(b)). We have $F = F_1 \otimes H_2 = (Q, D, (q'_0, q''_0), (q'_t, q''_t), (q'_f, q''_f), \delta_t)$ that is shown in Fig. 4(c).

We now show the operation of the relational product. At the root vertex $(q'_0, q''_0) \in Q$, for all $d \in D = \{0,1,2\}$ and for all $(d, d') \in \mathcal{D} = \{0,1,2\} \times \{0,1,2\}$, we have $a[0]_t$, $A[(0,0)]_t$, $A[(1,1)]_t$,



(a) A matrix diagram H_1.

(b) A matrix diagram H_2.

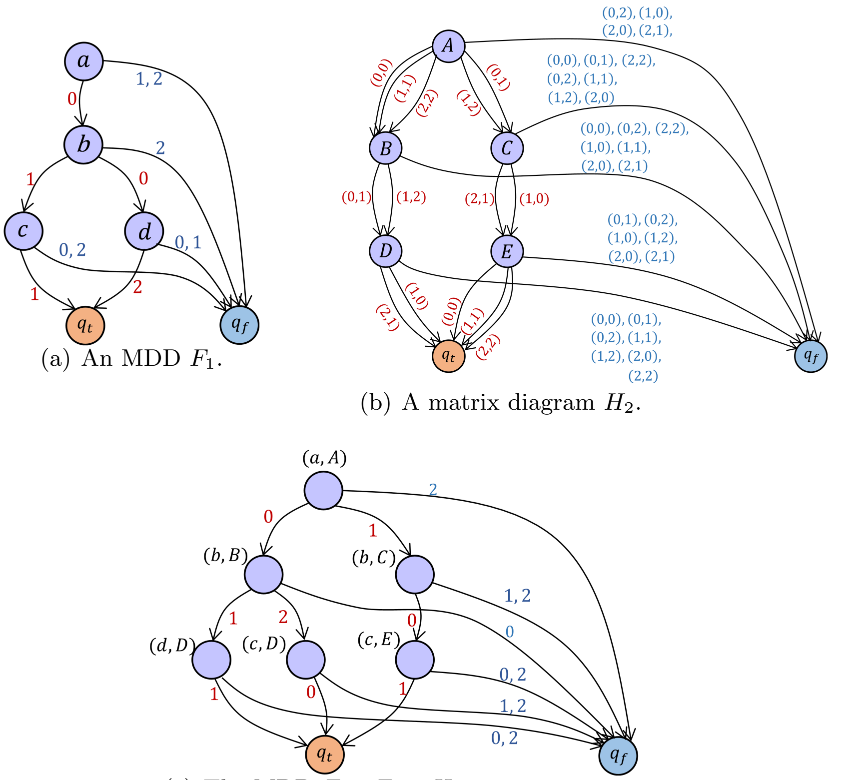
(c) The matrix diagram $H = H_1 \cup H_2$.

Figure 3: Union of two matrix diagrams.

$A[(2,2)]_t$, $A[(0,1)]_t$, and $A[(1,2)]_t$. By Definition 10, only when the condition $q'_i[d]_t$ & $q''_i[(d,d')]_t$ holds, the vertex $(q'_i[d], q''_i[(d,d')])$ can be added to Q, and the labeling function can be defined by $\delta_t((q'_i, q''_i), d') = (q'_i[d], q''_i[(d,d')])$. Then the vertex (b, B) is defined, since it satisfies that $a[0]_t$ and $A[(0,0)]_t$, where $a[0] = b$ and $A[(0,0)] = B$. The labeling function is defined by $\delta_t((a, A), 0) = (b, B)$. Besides, we have $A[(0,1)]_t$ with $A[(0,1)] = C$. Therefore, the vertex (b, C) is defined and $\delta_t((a, A), 1) = (b, C)$ is defined. Otherwise, when $a[d]_f$ or $(a[d]_t$ & $A[(d,d')]_f)$, $\delta_t((a, A), d') = (q'_f, q''_f)$ is defined. At (b, B), we have $b[0]_t$, $b[1]_t$, $B[(0,1)]_t$, and $B[(1,2)]_t$. Therefore, the vertexes (d, D) and (c, D) are defined, since both $b[0]_t$ & $B[(0,1)]_t$ and $b[1]_t$ & $B[(1,2)]_t$ are satisfied. The labeling function is defined by $\delta_t((b, B), 1) = (d, D)$ and $\delta_t((b, B), 2) = (c, D)$. As for the labels in set $\{2\} = D\backslash\{0, 1\}$, we define $\delta_t((b, B), 2) = (q'_f, q''_f)$. At the vertex (b, C), we have $b[0]_t$, $b[1]_t$, $C[(2,1)]_t$, and $C[(1,0)]_t$. However, as for $C[(2,1)]_t$, when $d \in D = 2$, we have $b[2]_f$ and when $d \in D = 0$, we have $C[(d,d')]_f$. Therefore, we can only add the vertex (c, E) to Q and define $\delta_t((b, C), 0) = (c, E)$.

Then, at the vertex (d, D), only when $d \in D = 2$, the condition $d[2]_t$ & $D[(2,1)]_t$ holds. The labeling function is defined as $\delta_t((d, D), 1) = (q'_t, q''_t)$. Otherwise, we define $\delta_t((d, D), 0) = (q'_f, q''_f)$ and $\delta_t((d, D), 2) = (q'_f, q''_f)$. The labeling function for vertexes (c, D) and (c, E) can be analogously defined (note that by Definition 10, we have $(q'_t, q''_f) = (q'_f, q''_t) = (q'_f, q''_f)$).

(2) We consider the following procedure in Fig. 5 as another example. In this case, the input set of markings is replaced by $\mathcal{M}_{tn} = \{[0, 2, 0]^T, [0, 0, 2]^T\}$ that is shown in Fig. 5(a). The matrix



(a) An MDD F_1.

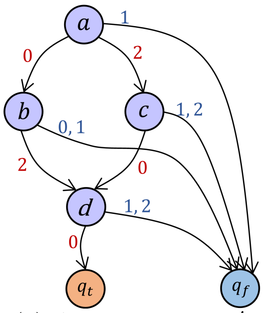
(b) A matrix diagram H_2.

(c) The MDD $F = F_1 \otimes H_2$.

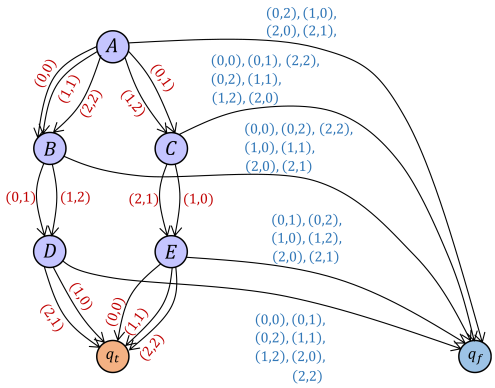
Figure 4: Relational product of an MDD and a matrix diagram.

diagram remains the same as the previous example. Similarly, we initialize the root vertex (a, A) and define the remainder of non-terminal vertexes (please see Fig. 5(c)).

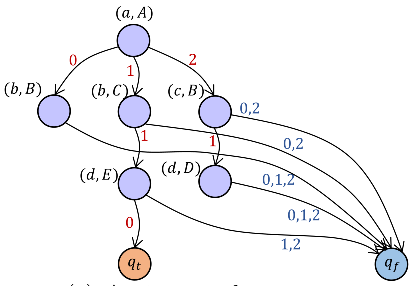
At the vertex (b, B), for all $d \in D$ and for all $(d, d') \in D \times D$, we cannot find a case that satisfies $b[d]_t$ & $B[(d, d')]_t$. Then by Definition 10, the set of the child vertexes of (b, B) in level 1 is an empty set and we define $\delta_t((b, B), 0) = (q'_f, q''_f)$, $\delta_t((b, B), 1) = (q'_f, q''_f)$, and $\delta_t((b, B), 2) = (q'_f, q''_f)$. In this case, by Definition 10, since for all $d \in D = \{0, 1, 2\}$, the condition $\delta_t((b, B), d) = (q'_f, q''_f)$ holds, we delete the vertex (b, B), and modify the labeling function from $\delta_t((a, A), 0) = (b, B)$ to $\delta_t((a, A), 0) = (q'_f, q''_f)$. As for the vertex (d, D), since for all $d \in D$ such that $\delta_t((d, D), d) = (q'_f, q''_f)$, we delete (d, D), and modify the labeling function to $\delta_t((c, B), 1) = (q'_f, q''_f)$. At (c, B), we now have $\delta_t((c, B), d) = (q'_f, q''_f)$ for all $d \in D$. Then we delete the vertex (c, B) and define $\delta_t((a, A), 2) = (q'_f, q''_f)$. By removing vertexes (b, B), (d, D), and (c, B), and changing the labeling function associated with these vertexes, we can obtain the MDD $F' = F'_1 \otimes H'_2$ that is shown in Fig. 5(d). We can see that $F' = F'_1 \otimes H'_2$ is corresponding to the computation of the next-state function $\{[0, 1, 1]^T\} = \mathcal{N}(\{[0, 2, 0]^T, [0, 0, 2]^T\}, \{t_1, t_3\})$.



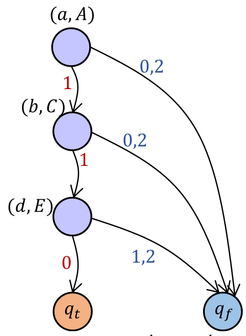
(a) Another MDD F_1'.



(b) Another matrix diagram H_2'.



(c) An intermediate state.



(d) The MDD $F' = F_1' \otimes H_2'$.

Figure 5: Another example for the operation of relational product.