Superconducting Quantum Circuit and Quantum Error Correction Code

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Abstract

This talk is divided into two parts, superconducting quantum circuit and quantum error correction code.

In the first part, we first introduce some physical platforms to realize quantum computing and to demonstrate the importance of superconducting quantum circuit and quantum error correction code. Then, to fill the gap, we briefly summarize how to quantize a nondissipative LC circuit. With the second quantization, we endow a bosonic interpretation to nondissipative LC circuits. In order to realize control of qubits, we need to introduce some nonlinear elements to the quantum circuit. Thus, the third part is the second quantization of circuit with Josephson junction, one of the popular nonlinear elements used in quantum circuits, and we talk about the coupling between different qubits and microwave control. To simplify the Hamiltonian and think about the physical meaning behind it, we introduce the interaction picture and rotating wave approximation and talk about the leakage in quantum circuits. The fourth part is physics behind superconductivity. We will intuitively introduce carrier in superconductivity, Cooper pair, and the macroscopic quantum phenomenon. After that, we will derive the Josephson relation.

In the second part, we introduce stabilizer group by an atomic physics example, i.e., quantum numbers of Hydrogen atom. Then, we give the formalism of stabilizer code and talk about how to realize it in quantum circuit with examples. To encode topological quantum information, we realize stabilizer code on lattice to get surface code. By introducing periodic boundary condition, surface code is embeded on topologically nontrivial manifold which is illustrated by toric code.

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Part I

Quantum Circuit

1 Introduction

1.1 Physical platforms of quantum computing

Actually, there are several methods to realize quantum computing nowadays (and within my knowledge) and they can be classified into three groups, liquid nuclear magnetic resonance (NMR), quantum optics based and solid state based.

1. The first one, liquid NMR which is a quite mature technique, uses organic molecules as qubits whose resonance with strong magenetic field is used to proceed qubits computing. It was successfully used to perform Shor's algorithm to decomposite $15 = 3 \times 5$ in 2001 by IBM. However, the NMR system is only used to demonstrate how quantum computing works.

2. The second group is based on quantum optics, such as photons, Ion trap, microcavity trapped atom. For instance, we can use the polarization states of photons to represent the state of qubits. The wonderful advantage of quantum optics is that those systems have a satisfactory coherence, which means it is relatively stable against the noise. However, the relatively huge experimental device of such systems greatly limit their application to quantum computer. (The huge here is compared with electronic chips)

3. The third group, which is thought to be the most promising method that can be used in real quantum computer, is based on solid state, such as superconductivity, quantum dots and semiconductor. The solid state based systems have a satisfactory small size which can be integrated in electronic elements. Therefore, we can expect that the large-scale quantum computing is possible to perform on such systems as what electronic computer can do. However, the sensitivity of solid state qubit result in a fragile quantum computing due to decoherence.

In order to stablize the solid state qubits, we need to develop efficient error detection and correction techniques to protect qubits against decoherence. That is why we are intersted in superconducting quantum circuit and quantum error correction code.

2 Physics behind circuit

2.1 How to quantize a nondissipative circuit

Let's assume we are working on a simple circuit consisting of an inductor L (transfering magnetic energy to electronic energy) and a capacitor C (storing and releasing electronic energy), which is socalled LC circuit. There is no dissipation caused by resistance because we want to introduce this quantization in superconducting quantum circuit. According to the physical law of those two circuit elements, the relation between charge Q and magnetic flux Φ is given by the following equation describing the evolution of a LC circuit.

$$
\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \frac{Q}{C} = \frac{\partial \mathcal{H}}{\partial Q} \\
\frac{\mathrm{d}Q}{\mathrm{d}t} &= -\frac{\Phi}{L} = -\frac{\partial \mathcal{H}}{\partial \Phi}
\end{aligned} \tag{1}
$$

where the Hamiltonian is $\mathcal{H}(\Phi, Q) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$. Thus, we map the LC circuit to a Hamiltonian system where Φ, Q behave like coordinate, momentum. It's not hard to tell the behavior of the LC circuit is quite similar to an oscillator which can be obeserved in experiments. Thus, following this harmonic osicillator like Hamiltonian, we can do the second quantization to rewrite the Hamiltonian w.r.t. the creation and annihilation operator of bosons with commuation relation $[\Phi, Q] = i\hbar$.

$$
\begin{cases}
a = \dfrac{\Phi + iZQ}{\sqrt{2\hbar Z}} \\
a^\dagger = \dfrac{\Phi - iZQ}{\sqrt{2\hbar Z}}
\end{cases} \Rightarrow \mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{2}
$$

where the characteristic impedance is $Z = \sqrt{L/C}$ and $\omega = 1/\sqrt{LC}$. It's not surperising that the energy spectrum is equally spaced due to harmonic oscillator.

2.2 Why we can't use simple LC circuit for quantum computing

In a quantum circuit, we always use bounded states to represent qubits state, such as the number of bosons here $|n\rangle$. If we want to control a quantum circuit, we give some energy to change the state. Let's imagine we deliver $\hbar\omega$ energy to the circuit, which kinds of transition can occur? The answer is infinitely many because all $|n\rangle \to |n+1\rangle$ transitions happen with energy $\hbar\omega$. Thus, we should introduce some anharmonic terms in the Hamiltonian, i.e., nonlinear elements in circuit. Similarly, the circuit will behave like a nonlinear oscillator. One of the most important and promising elements in quantum circuit is Josephson junction.

3 Quantum circuit with Josephson junction

3.1 What is junction

In our classical commonsense, we know that there is no electronic current if we place an insulator between two conductor in an circuit. The physical intuition behins it is the insulator will create a extremely high energy barrier to prevent electrons from crossing it. However, let's think more quantum mechanically. Energy barrier can be crossed in quantum mechanics by tunneling because there is only kets (wave functions if you choose a basis) playing the role. The current (supercurrent) passing the conductor-insulator-conductor junction (tunneling junction) is socalled tunnel current.

3.2 What is Josephson junction

Josephson junction is a specific junction with supersonductor-insulator-superconductor structure. The relation between the Josephson phase ϕ and voltage, current follows the Josephson relation. Let's first directly show the Josephson relation and it will be proved in next section. In the proof, the meaning of Josephson phase will be revealed, which can be controlled by external magnetic flux.

$$\begin{cases} U = \dfrac{\Phi_0}{2\pi}\dfrac{\mathrm{d}\phi}{\mathrm{d}t} \\ I = I_c \sin\phi \end{cases} \tag{3}$$

where $\Phi_0 = h/2e$ is magnetic flux quantum and I_c is the critical current, a parameter, of Josephson junction.

3.3 Anharmonicity due to Josephson junction

Let's replace the inductor L in LC circuit, which is a linear element, by Josephson junction and do the second quantization to such circuit. The energy of capacitor, kinetic energy in Hamiltonian, is still $Q^2/2C$. However, the contribution of inductor is replaced by that of Josephson junction. As we know, the power of an element is $P = \dfrac{\mathrm{d}V}{\mathrm{d}t} = UI$. Thus, the energy of Josephson junction is

$$V = \int_0^t UI\mathrm{d}t' = \frac{\Phi_0 I_c}{2\pi}\int_0^{\phi(t)} \sin\phi\mathrm{d}\phi = \frac{\Phi_0 I_c}{2\pi}(1-\cos\phi) \approx E_J\left(\frac{\phi^2}{2} - \frac{\phi^4}{24}\right) \tag{4}$$

where $E_J = \dfrac{\Phi_0 I_c}{2\pi}$ and the time dependence is dropped for convinience. The Hamiltonian is given by

$$\mathcal{H} = \frac{Q^2}{2C} + V = 4\frac{e^2}{2C}\left(\frac{Q}{2e}\right)^2 + \frac{E_J}{2}\phi^2 - \frac{E_J}{24}\phi^4 = 4E_c n^2 + \frac{E_J}{2}\phi^2 - \frac{E_J}{24}\phi^4 \tag{5}$$

where the charging energy $E_c = e^2/2C$ is the energy of capcitor storing an electron, $n = Q/2e$ is the Cooper pair, which is consisted of two electrons, number. Thus, using the similar technique, we can second quantize the harmonic part, dirst two terms, of the Hamiltonian w.r.t. $[\phi, n] = i\hbar$ and treat the anharmonicity with perturbation theory.

$$\begin{cases} a = \dfrac{1}{\sqrt{2\hbar}}\left(\dfrac{E_J}{8E_c}\right)^{1/4}\left(\phi + i\left(\dfrac{8E_c}{E_J}\right)^{1/2} n\right) \\ a^\dagger = \dfrac{1}{\sqrt{2\hbar}}\left(\dfrac{E_J}{8E_c}\right)^{1/4}\left(\phi - i\left(\dfrac{8E_c}{E_J}\right)^{1/2} n\right) \end{cases} \Rightarrow \mathcal{H}_0 = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{6}$$

where $\omega = \sqrt{8E_cE_J}$. Thus, the anharmonicity is

$$\mathcal{H}_1 = -\frac{\hbar^2 E_c}{12}(a + a^\dagger)^4 \tag{7}$$

The first order pertubation gives

$$\langle n|\mathcal{H}_1|n\rangle = -\frac{\hbar^2 E_c}{12}\langle n|(a + a^\dagger)^4|n\rangle = -\frac{\hbar^2 E_c}{4}(2n^2 + 2n + 1) \tag{8}$$

Thus, the energy gap between two energy levels of Josephson junction is

$$\Delta_n = E_{n+1} - E_n = \hbar\sqrt{8E_cE_J} - \hbar^2 E_c(n + 1) \tag{9}$$

which uniquely depends on energy level. Thus, we can use different energy pulsion to realize transition control. Moreover, we can find the bigger the anharmonicity (charging energy) E_c, the greater the difference between energy gaps. Thus, a small capacity C in circuit should be beneficial for us to control a quantum circuit.

3.4 Coupling between qubits

As we derived above, each Josephson LC circuit can be quantized as a qubit whose state is represented by the oscillation energy level. Thus, a natural question for us to ask is how do two qubits interact in a complicated quantum computing device. Actually, the coupling between two Josephson LC circuits can be turned on by capacitive coupling in real world experiments. The capacitive coupling leads to a coupling energy contribution between Josephson junctions.

$$\mathcal{H}_2 = -2C_X U_1 U_2 \propto \frac{\mathrm{d}\phi_1}{\mathrm{d}t}\frac{\mathrm{d}\phi_2}{\mathrm{d}t} \propto (a_1^\dagger - a_1)(a_2^\dagger - a_2) \tag{10}$$

where C_X is the coup;ing constant.

3.5 Control a quantum circuit with microwave pulse

In order to control qubits in quantum computers, we need to introduce some outside resources in Hamiltonian. One of the promising methods is using microwave pulse to couple qubits (oscillation modes). Similar to capacitive coupling, the coupling Hamiltonian takes the form as

$$\mathcal{H}_3 = \frac{1}{2}C'_X U_{control} U_{qubit} \tag{11}$$

where U stands for the voltage. From Josephson relation and second quantization, the qubit's voltage is propotional to the time derivative of phase, i.e., $U_{qubit} \propto \frac{\mathrm{d}\phi}{\mathrm{d}t} \propto i(a - a^\dagger)$. We assume the microwave takes the following cosine wave form.

$$U_{control} = f(t) \left(e^{i(\omega t+\alpha)} + e^{-(i\omega t+\alpha)} \right) \propto f(t) \cos(\omega t + \alpha) \tag{12}$$

where α is the initial phase, ω and $f(t)$ are frequency and amplitude. Since we can perform different control microwaves to different Josephson junctions, the control coupling Hamiltonian is

$$\mathcal{H}_3 \propto \sum_j i(a_j - a_j^\dagger) f_j(t) \cos(\omega_j t + \alpha_j) \tag{13}$$

Part II
Quantum Error Correction Code

4 Introduction

4.1 Why we need quantum error correction code

After the proposal of Shor's algorithm, scientists were excited to expect the huge success of quantum computing. There are at least three difficulties awaiting us to overcome if we want to realize quantum computing, hardware, information and software. In past few years, physicists had various breakthroughs on hardwares but one thing confused them was how to read the results generating from quantum computing. You can imagine that the measurement of states after computing (evolution) gives a probability distribution which doen't give computational results explicitly. However, Shor said you can follow my algorithm to capture what is given by quantum states' evolution which made quantum computing community more confident. Do you think it's enough? Actually, we have another question to answer, how to protect quantum information against decoherence. Given that the comuting power of quantum computer is exponentially greater than that of electronic computer, qubits are so fragile that they can't afford large-scale and long-time calculation. You may argue that we can put our quantum computer in vacuum to stabilize it. In classical world, it is true. Nevertheless, quantum mechanics tells us that vacuum can also interact with matters resulting in spontaneous emission. That means we have no choice but to consider how to protect quantum information.

Remark 4.1 *The interaction with vacuum also answers what Prof. Lin asked in the talk. Because we need excited states in quantum circuit to do calculation, it's necessary for us to develop efficient code to protect qubits.*

There are several methods proposed to protect qubits against decoherence.

1. We can measure the target state all the time to confine it that is socalled quantum Zeno effect. As we know, measurement is projection. Thus, we can projectour state to a given subspace by measurement. However, you can tell this method is not useful and practical for integrated qubits' system.

2. Second method is called decoherence free subspaces in which qubits can maintain a high coherence.

3. Another method is based on information theory. We spreadly encode quantum information to more qubits and do the error detection and correction. However, at least until now, qubits are scarce resources which requires us to propose some efficient ways to encode quantum information. That is our topic, quantum error correction code.

5 Stabilizer Code

5.1 Atomic physics as introduction

As we know in atomic physics, e.g., Hydrogen atom, different energy levels are labeled by different quantum numbers, i.e., $|nlm_l m_s\rangle$, which corresponds to a complete set of commuting operators (CSCO), $\{\mathcal{H}, L^2, L_z, S_z\}$. Moreover, CSCO maintain the state (it doesn't change the state up to a $\mathbb{C}$ number). Thus, CSCO gives us a compact description of states, we can use a set of operators instead of a whole function space to describe states.

Remark 5.1 *More precisely, we use CSCO to decompose state space (Hilbert space) to several invariant subspaces. CSCO can be constructed by considering symmetries of the physical system. To illustrate, L^2 represents isotropy, L_z stands for rotational symmetry around z-axis.*

Assume the spin of the electron in Hydrogen atom is flipped by some noise occasionally which results in a new state $|\psi\rangle = S_x|nlm_l m_s\rangle$. Recalling the anticommutation of spin operators, if we measure the spin of the new state, what will happen?

$$S_z|\psi\rangle = S_z S_x|nlm_l m_s\rangle = -S_x S_z|nlm_l m_s\rangle = -m_l\hbar|\psi\rangle \tag{14}$$

where other operators in CSCO have the same measurement results as before. Thus, all the evidences from CSCO imply the possible error is spin flip. To fix it, we can flip the spin again to recover the original state, $|nlm_l m_s\rangle = S_x|\psi\rangle$.

The idea of stabilizer code is quite similar to CSCO. Once we find a set of commuting operators, the state space can be decomposed into several subspaces which can be used to encode quantum information, detect and correct errors.

5.2 What is stabilizer code

As we pointed out in last subsection, the stabilizer code is trying to find a set of commuting operators to decompose Hilbert space. Rather than search in the sea of operators, we use Pauli group alternatively because of its wonderful properties.

Definition 5.1 (Pauli Group) *Pauli group is defined as the group generated by all three Pauli operators including the phase $\pm 1, \pm i$.*

$$G_1 := \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

The n-fold Pauli group is defined as all possible tensor product in n Pauli groups.

$$G_n := \{P_1 \otimes P_2 \otimes \cdots \otimes P_n : P_j \in G_1, \forall j\}$$

Remark 5.2 *1. The notation here is different in what was used in talk, $\Pi \to G$, to avoid the confusion with product.*

2. *Arbitarily picking two elements $P_1, P_2 \in G_n$, they are either commuting or anticommuting which is a trivial corollary of the property of G_1.*

3. *All elements in G_n have only two eigenvalues.*

Imagine we want to encode k logical qubits into $n > k$ physical qubits. It's not hard to tell we need $n-k$ labels to mark our encoded states, which means we need $n-k$ independent commuting operators $\{g_1, g_2, \cdots, g_{n-k}\} \subset G_n$. Those operators should maintain the state or, equivalently speaking, have trivial action $g_j |\psi\rangle = |\psi\rangle, \forall j$, which are therefore called **stabilizer**s. Thus, we define a encoded state, or **codeword**, as the simultaneous $+1$-eigenstate corresponding to stabiliers. And all codewords form a subspace of $\mathscr{H}^{2n}$, which is socalled **codespace**. Since all stabilizers are independent and commuting, they generate an Abelian subgroup of n-fold Pauli group, which is usually called stabilizer as well, under multiplication. We denote stabilizer as $\mathcal{S} = \langle g_1, g_2, \cdots, g_{n-k}\rangle$, codespace as $\mathcal{C}(\mathcal{S})$.

Remark 5.3 *To have the simultaneous +1-eigenspace, we require $-I \notin \mathcal{S}$ which implies that $\pm iI \notin \mathcal{S}$ as well.*

Distict from CSCO in introduction, the stabilizer is not complete. Otherwise, there is no extra degree of freedom for us to encode states. Thus, there exists other operators in G_n commuting with $\mathcal{S}$. We define the **centralizer** $\mathcal{Z}(\mathcal{S})$ as the set of all operators in G_n that commute with stabilizer. Actually, $\mathcal{Z}(\mathcal{S})$ is a group but not necessary to be Abelian. (In the talk, I said it's equivalent to the largest Abelian subgroup. This claim is not correct.) For element $L \in \mathcal{Z}(\mathcal{S}) - \mathcal{S}$, the encoded state is not even necessary to be eigenstate of them. So, it may change the encoded state although it commutes with stabilizer. Thus, those operators can corrupt the quantum information in logical qubits. We therefore denote the elements in $\mathcal{Z}(\mathcal{S}) - \mathcal{S}$ as **logical operator**s. However, because of the commuting relation, $L|\psi\rangle$ is still stabilized by $\mathcal{S}$, which means we can't detect error even if error occurs.

All operators $E \in G_n$ is possible to be errors. For instance, $Z : |a\rangle \mapsto (-1)^a |a\rangle$ is phase flip error, $X : |a\rangle \mapsto |a \oplus 1\rangle$ is bit flip error. Using the beautiful (anti)commuting relation of G_n, the error can be classified into three groups.

1. $E \in \mathcal{S}$. Therefore, $E = \prod_j g_j^{x_j}$ where $x_j = 0, 1$ and acts trivially on the encoded state.

2. There exists an operator $P \in \mathcal{S}$ with which E anticommutes. Thus, given an encoded state $|\psi\rangle$, the error can be detected by the action of stabilizer. $P(E|\psi\rangle) = -EP|\psi\rangle = -E|\psi\rangle$ implies that error flips the eigenvalue of stabilizer. The conjugate operator $E^\dagger$ can be applied to recover encoded qubits. Thus, those errors are **correctable**.

3. The most intractable situation is that $E \in \mathcal{Z}(\mathcal{S}) - \mathcal{S}$ which we discussed before. The error may corrupt the quantum information but it can't be detected by stabilizer.

5.3 Hamiltonian of stabilizer code

Similarly, the Hamiltonian of encoded state is

$$\mathcal{H} = -\sum_j g_j \tag{15}$$

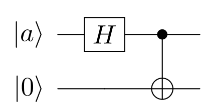
whose ground state is codeword. By definition, codewords are simultaneous +1-eigenstates, which means they can be represented by projection operators.

$$|\text{codeword}\rangle = \prod_j \frac{I + g_j}{2} |\psi\rangle, \ \forall |\psi\rangle \in \mathscr{H}^n \tag{16}$$

The symmetry of stabilizers are natually obtained by codewords. For instance, in the following example, $|\beta_{a0}\rangle$ is protected from two bits phase flip error because of the $Z \otimes Z$ symmetry of stabilizer. The Hamiltonian is even useful in next section, surface code, in which the Hamiltonian (stabilizer) captures all local symmetry to protect quantum information. Hamiltonian also provides us another viewpoint. Since codeword sits on the ground state and correctable errors bring codeword out to other orthogonal subspaces, resulting in a higher eigenvalue of Hamiltonian. We can therefore imagine codespace as vacuum and correctable errors create particles (anyons) from vacuum. It is more interesting for anyons on nontrivial topology but we don't plan to talk about it in this talk.

5.4 Our first stabilizer code

Suppose we are working with the following encoding quantum circuit.

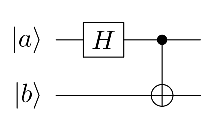


The output state is

$$CNOT(1,2)(H \otimes I)|a0\rangle = CNOT(1,2)\frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}} |0\rangle = \frac{|00\rangle + (-1)^a |11\rangle}{\sqrt{2}} := |\beta_{a0}\rangle \tag{17}$$

where a is the phase ($\pm$ sign of combination) of the encoded state. Thus, we can use that quantum circuit to encode a single logical qubit as two physical qubits $|a\rangle \mapsto |\beta_{a0}\rangle$, which implies that we need to find one stabilizer.

Remark 5.4 *The usual notation for EPR (Bell) state is* $|\beta_{ab}\rangle = \dfrac{|0\rangle |b\rangle + (-1)^a |1\rangle |b \oplus 1\rangle}{\sqrt{2}}$. *The parameter a stands for phase ($\pm$ sign of superposition) and b stands for parity (even or odd). They are actually four maximally entangled states that measurement of one qubit reveals information of the other. In quantum circuit, EPR states can be prepared by*



It is obvious that the parity of encoded states is even despite what initial $|a\rangle$ is. Therefore, it's easy to verify the encoded state is invariant under $Z \otimes Z$ for all inputs $|a\rangle$. This observation motivates us to choose $Z \otimes Z$ as stabilizer. Thus, the stabilizer is a Z_2 group $\mathcal{S} = \{I \otimes I, Z \otimes Z\}$.

Let's consider errors. For single bit flip error, say, $I \otimes X : |\beta_{a0}\rangle \mapsto |\beta_{a1}\rangle$. The result state is therefore with odd parity and can be detected by parity checking operator $Z \otimes Z$. However, for single bit phase flip error, say, $I \otimes Z : |\beta_{a0}\rangle \mapsto |\beta_{a\oplus 1,0}\rangle$. Stabilizer can't notice us the state has been changed. Once we decode physical qubits, the output is $|a \oplus 1\rangle$ which is totally different from what we encode. Using the stabilizer formalism, the corrupted error $I \otimes Z \in \mathcal{Z}(\mathcal{S}) - \mathcal{S}$ as we expecting.

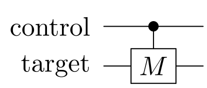
5.5 Measure stabilizer in quantum circuit

As we know, measurement of an operator is equivalent to the projection onto their eigenspaces. In quantum circuit, we can use the combination of Hadamard gate and controlled-M gate to measure multi-qubits operator M.

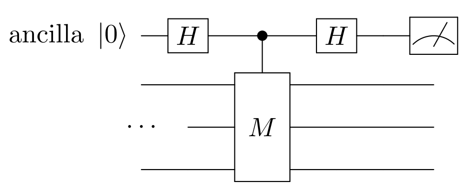
Definition 5.2 (Controlled-M gate) *Controlled-M (CM) gate has two kinds of inputs, control and target, and it is triggered only if the controlled state is* $|1\rangle \cdots |1\rangle$.

$$\begin{aligned} &CM : (|1\rangle, |b\rangle) \mapsto |1\rangle \otimes M |b\rangle \\ &\text{otherwise, } CM = I \end{aligned}$$

The notation of CM is



Specifically, $CNOT = CX$. Measurements can be realized by the following circuit.



The output state before measuring ancilla is

$$\begin{aligned} H_1 CM(1,\cdot) H_1 |0\rangle |a\rangle &= H_1 CM(1,\cdot) \frac{|0\rangle + |1\rangle}{\sqrt{2}} |a\rangle = H_1 \frac{|0\rangle |a\rangle + |1\rangle M |a\rangle}{\sqrt{2}} \\ &= |0\rangle \frac{I+M}{2} |a\rangle + |1\rangle \frac{I-M}{2} |a\rangle \end{aligned} \tag{18}$$

Thus, if the measurement of ancilla gives $|0\rangle$, the target qubits are in +1 eigenspace of M, $|1\rangle$ for -1 eigenspace respectively. Suppose we encode one

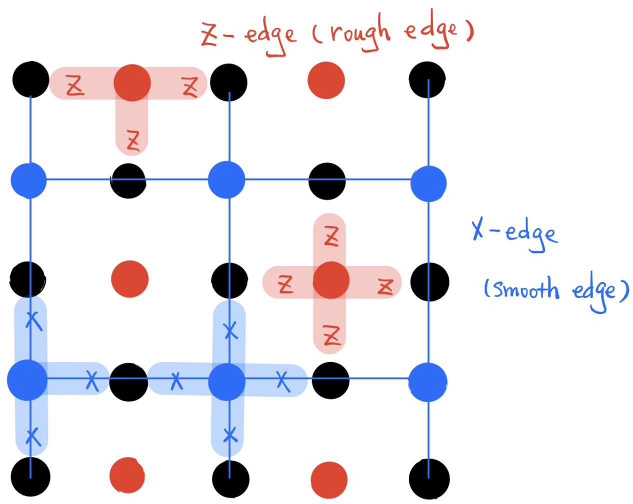
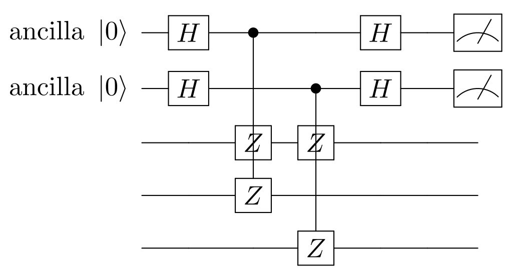


Figure 1: Illustration of surface code. Blue lines: lattice. Black dots: physical qubits. Blue dots: vertex ancilla with X operations. Red dots: plaquette ancilla with Z operations.

logical qubit to three physical qubits with stabilizer $\mathcal{S} = \langle Z \otimes Z \otimes I, Z \otimes I \otimes Z \rangle$. The error detection quantum circuit is



6 Surface Code

6.1 Stabilizer code on lattice

A natural question is how to proceed with stabilizer code formalism. As we said in introduction, one of the promising ways to realize quantum computing is solid state based system. For a solid system, the structure is always described by lattice packed with particles. Can we encode quantum inofrmation on lattice? Surface code is actually based on this idea that quantum inofrmation is encoded on sites sitting on the middle of each edge. As illustrated in 1, let's work with $L \times L$ lattice whose distance between rough edges and smooth edges are both L. It's not hard to calculate number of physical qubits is $L^2 + (L-1)^2$, the number of vertexes and plaquettes are both $L(L-1)$. The smart idea comes now. We define stabilizers on each vertex and plaquette.

$$
\begin{aligned}
A_v &= \prod_{i \in nn(v)} X_i \\
B_p &= \prod_{i \in nn(p)} Z_i
\end{aligned}
\tag{19}
$$

Since each vertex and plaquette share 0 or 2 nearst neighbors, $[A_v, B_p] = 0, \forall v, p$. Thus, those $2L(L-1)$ operators generate an Abelian subgroup $\mathcal{S}$ of $G_{L^2+(L-1)^2}$ to encode $L^2 + (L-1)^2 - 2L(L-1) = 1$ logical qubit.

How to modify logical information? If we draw a path Γ_x along lattice connecting two physical qubits on different smooth edges (Γ_z for rough edges respectively), we can define the following two operators.

$$
\begin{aligned}
\bar{X} &= \prod_{i \in \Gamma_x} X_i \\
\bar{Z} &= \prod_{i \in \Gamma_z} Z_i
\end{aligned}
\tag{20}
$$

It can be shown that $\bar{X}, \bar{Y}$ commute with $\mathcal{S}$ but they can't be represented by the multiplication of A_v, B_p. Thus, $\bar{X}, \bar{Z}$ are logical operators corrupting quantum information and represent logically bit flip and phase flip errors. You may ask whether logical operators are unique or, equivalently, how about choosing other two pathes to define logical operators. The answer is the following claim.

Claim 6.1 *There are only two indepedent logical operators. For given $\bar{X}, \bar{Z}$ and $\bar{X}', \bar{Z}'$, there exist $P_1, P_2 \in \mathcal{S}$ such that $\bar{X}' = P_1\bar{X}, \bar{Z}' = P_2\bar{Z}$.*

Remark 6.1 *Equivalently speaking, the path defining logical operators can be deformed by stabilizer. We can therefore define an quivalence relation between deformed pathes. With such equivalence, the path, defining logical operator, is just an arbitary representative element.*

In this talk, this claim is not planned to be proved rigorously. We give the following example to show how it works heuristically.

Example 6.1 *Suppose we define logical operators corresponding to two pathes Γ_x, Γ_x' in 2.*

$$
\bar{X}' = X_4X_5X_2X_3 = X_4X_5X_1X_1X_2X_3 = A_{v1}\bar{X} \tag{21}
$$

6.2 Topological quantum information, Toric code

The quantum inofrmation is encoded on a torus if we endow periodic boundary condition to the lattice above. Because of the periodic boundary condition, $\prod_v A_v = \prod_p B_p = I$ which means one of A_v's and one of B_p's can

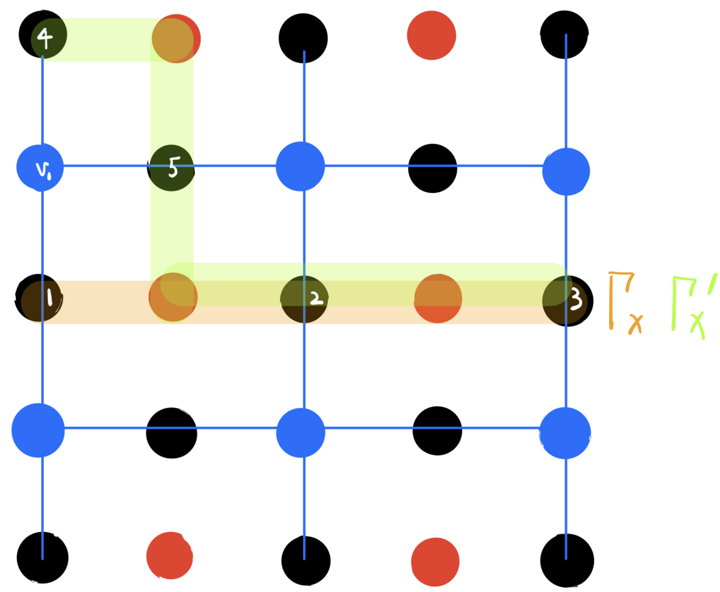


Figure 2: Path deformation.

be represented as the multiplication of others. Thus, the generator of stabilizer decreases by 2. The number of logical qubits that can be encoded is $N_L = 2 + \#(edge) - \#(vertex) - \#(plaquette) = 2 - \chi$ where $\chi = 0$ is Euler characteristic of torus. The logical operator, or path, is required to be close due to the periodic boundary condition. Since there is no bounday at all, we should classify loops on torus. With the topologically nontrivial structure, uncontractable loops on torus are used to define logical operators and there are two indepedent $\bar{X}, \bar{Z}$ corresponding to two encoded degree of freedom. Also because of the absence of rough/smooth boundary, there is no vertical/horizontal direction requirement of loops defining logical operators. Different loops can be deformed by stabilizers with similar reason in surface code. For contractable loops, it's obvious their action is similar to identical operator since those loops can be contracted by stabilizers inside.

Remark 6.2 *Although contractable and uncontractable loops are homologically different on torus, they can't be distinguished by looking only at a local region. Due to the locality of stabilizer, logical errors cannot be detected.*