A summary of 'Statistical Calibration and Validation of Mathematical Model to Predict Motion of Paper Helicopter' with further details of statistics

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Abstract

To understand physical phenomena, mathematical models are broadly used to reduce experiment expenses. However, due to simplified assumptions and uncertain parameters, there are bound to be differences from the actual phenomena. In this study, statistical calibration and validation are used to compare mathematical results and experimental measurements. Paper helicopter experiments are conducted as an actual phenomenon example. Fall time is measured with three nominally identical paper helicopters. The drag coefficient, the single unknown parameter of the two mathematical models, is quantified using Bayesian Calibration. Predicted fall time data is presented in the form of probability distributions. The mathematical models are validated by comparing predicted distribution and experimental data distribution. In addition, Analysis of Variance test is used to compare manufacturing and experimental error. All three paper helicopters are regarded as one identical model.

Keywords: Bayesian Calibration, Predictive Validation, Analysis of Variance, Markov Chain Monte-Carlo Simulation, Uncertainty Quantification

1 Introduction

There have been various studies that utilize paper helicopters because of its cost effectiveness and relatively easy experiments. Most studies are focused on optimization [1]. However, in this project, the validity of mathematical models is investigated with statistical methods. [2]. Drag coefficient (C_D) is the single unknown parameter of the mathematical models, so it is suitable to quantify the uncertainty with Markov chain Monte Carlo (MCMC) [3].

Basic concept of paper helicopter and analytical models

Newton's second law

$$\sum \mathrm{F} = \mathrm{m}\frac{\mathrm{dV}}{\mathrm{dt}} = \mathrm{W} - \mathrm{F_D}$$

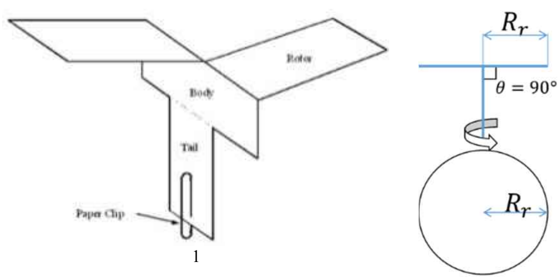

Figure. 1

Linear model

$$m\frac{dV}{dt} = mg - kV$$

$$V_{ss} = \frac{mg}{k} \qquad k = \frac{1}{2}\rho_{\mathrm{air}}\mathrm{V_0 A C_D}$$

$$\mathrm{V}(t) = V_{ss}(1 - \mathrm{e}^{-\frac{kt}{m}})$$

$$h(t) = V_{ss}\left(t + \frac{e^{-ct}}{c}\right) - \frac{\mathrm{V_{ss}}}{c} \qquad \text{(Equation 1)}$$

Quadratic model

$$m\frac{dV}{dt} = mg - kV^2$$

$$V_{ss} = \sqrt{\frac{mg}{k}} \qquad k = \frac{1}{2}\rho_{\mathrm{air}}\mathrm{A C_D}$$

$$\mathrm{V}(t) = V_{ss}\tanh\left(\frac{g}{V_{ss}}\right)t$$

$$h(t) = \frac{V_{ss}^2}{g}\ln\cosh\frac{g}{V_{ss}}t \qquad \text{(Equation 2)}$$

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[1] Image reference: http://www.papertoys.com/spinning-helicopter.htm
The original contents are translated from Korean and modified by Dongyoung Kim.
Statistical details are added including graphs and tables.

The experiments are conducted in four different conditions at two different heights (10.67m, 6.82m) with one or two clips as a weight factor (Appendix. A). In each experiment, three nominally identical paper helicopters are used and fall time is measured 7 times. With the Analysis of Variance (ANOVA) test, manufacturing and experimental errors are compared. Furthermore, the uncertainty of a single parameter, C_D, is quantified with Bayesian calibration for predictive validation.

2 ANOVA test result

To compare manufacturing and experimental errors, ANOVA test is conducted with fall time data of helicopter 1, helicopter 2, and helicopter 3 (Appendix A) as follows:

Table 1 One-way ANOVA (10.67m / 1 clip)

Source	DF	SS	MS	F	P
Factor	2	0.036	0.018	0.95	0.404
Error	18	0.336	0.019		
Total	20	0.372			

S = 0.1366 R-Sq = 9.57% R-Sq(adj) = 0.00%

Table 2 Levene's test

Source	DF	SS	MS	F	P
Factor	2	0.0106	0.0053	0.92	0.415
Error	18	0.1027	0.0057		

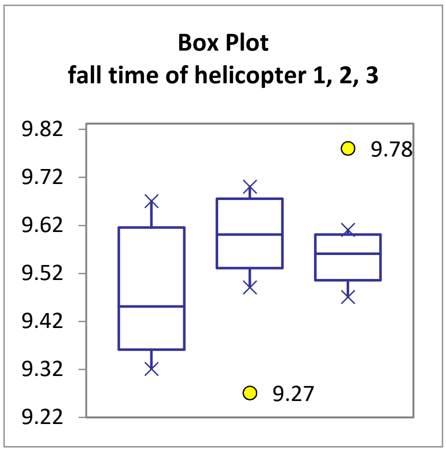

Figure 2 Boxplot of fall time (Condition 1)

The assumption of equal variances is verified with Levene's test (Table 2). Since the p value is higher than 0.05, the null hypothesis (equal variances) is retained. The Factor (MSbetween) is considered as manufacturing error between three helicopters, and the Error (MSwithin) is considered as experimental error from repetitive measuring of fall time. One-way ANOVA test result (Table 1) shows that three helicopters (H1, H2, H3) can be treated as the same helicopter model so that 21 fall time data can be treated as one helicopter.

3 Statistical calibration and predictive validation

Statistical calibration and validation is a method to validate mathematical models by quantifying the uncertainty with procedures of Calibration, Validation, and Prediction (Figure 3).

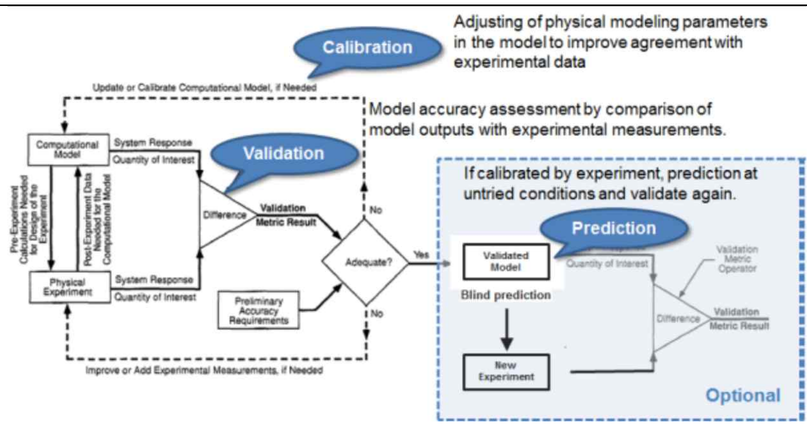

Figure 3 Overall procedures of calibration, validation, and prediction (Oberkampf and Barone, 2006) [4]

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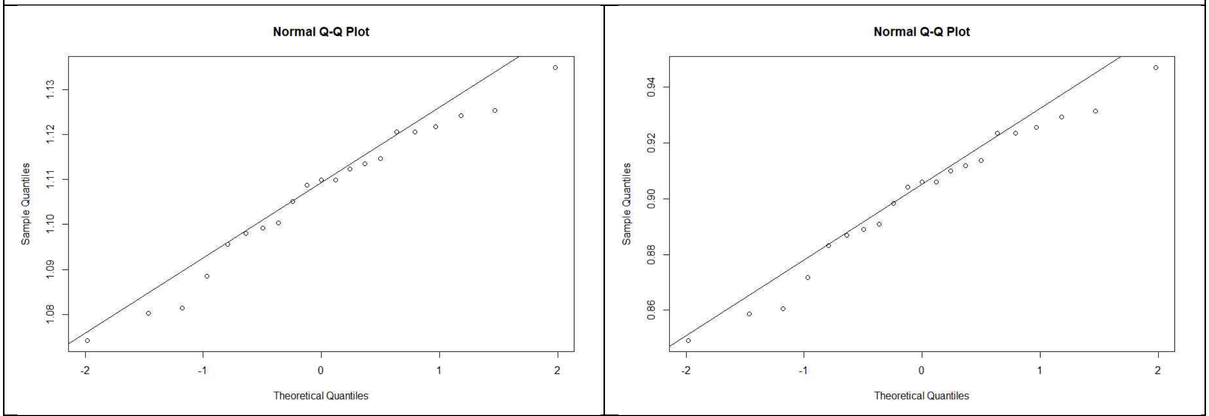
3.1 Normal distribution assumption

With the analytical model, fall time data can be converted into C_D. The distribution of C_D is assumed to follow a normal distribution with μ_{C_D}, σ_{C_D}.

$$C_D \sim \mathrm{N}(\mu_{C_D}, \sigma_{C_D})$$

The assumption of normal distribution is verified with the quantile-quantile plot as follows:

Figure 4 QQ plot of the linear model and quadratic model



Linear model	Quadratic model

3.2 Bayesian calibration

The likelihood of C_D, which is calculated from fall time of 21 times (Condition 1) is expressed as

$$\mathrm{p}\left(C_{D,test}^{i} \,\middle|\, \mu_{C_D}\sigma_{C_D}\right)$$

Non-informative prior is assumed for the mean and standard deviation and the posterior distribution is expressed as

$$\mathrm{p}\left(\mu_{C_D}, \sigma_{C_D} \middle| C_{D,test}^{1}\ , \dots, C_{D,test}^{N}\ \right) \propto \prod_{i=1}^{N} \mathrm{p}\left(C_{D,test}^{i} \,\middle|\, \mu_{C_D}\sigma_{C_D}\right)\frac{1}{\sigma_{C_D}^{2}}$$

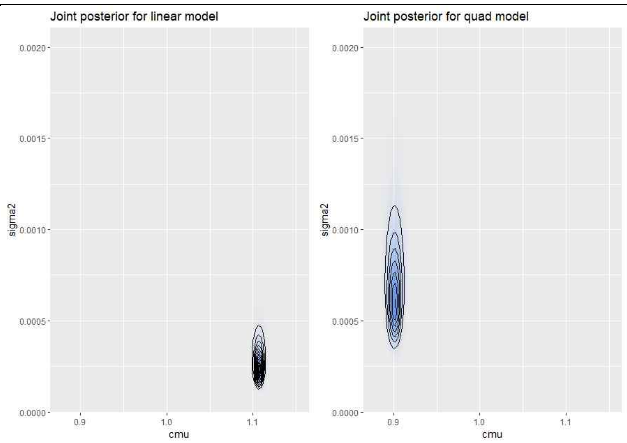
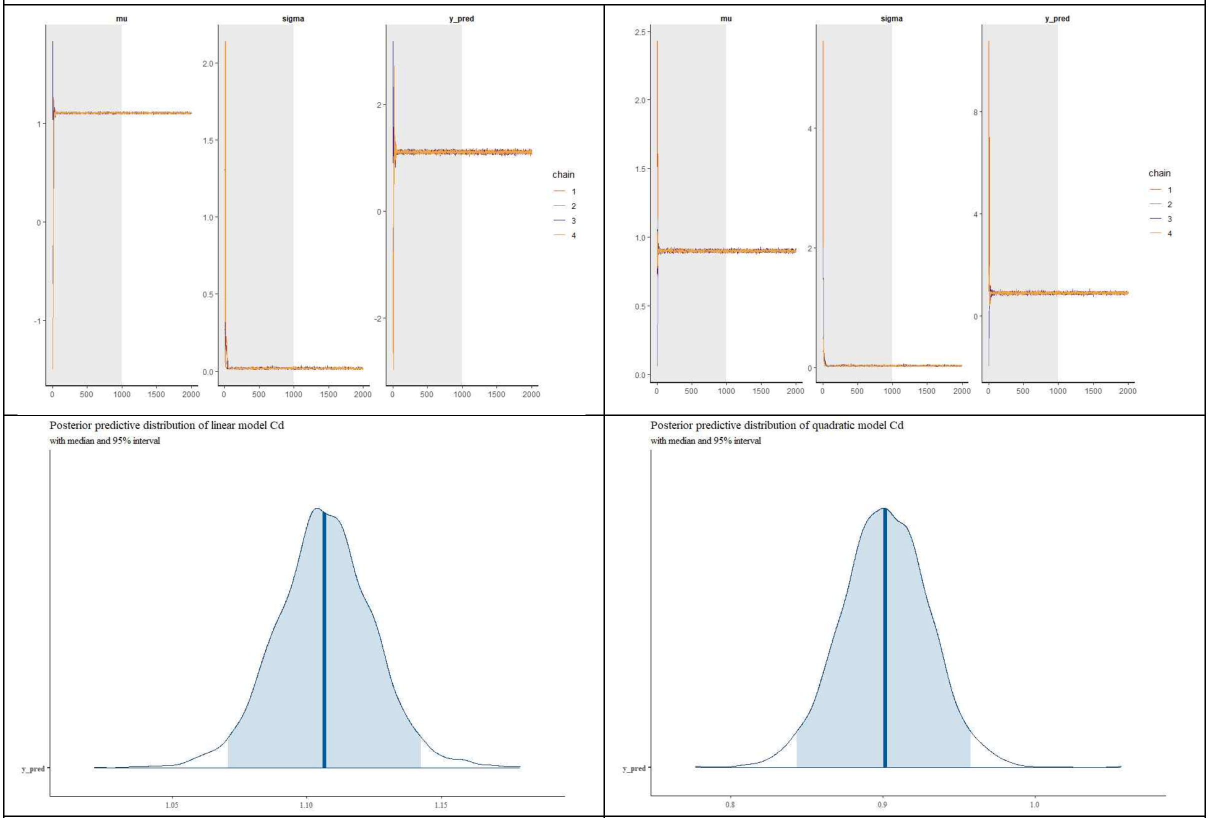


Figure 5 The posterior distribution of C_D with Non-informative two parameter normal distribution assumption (linear model on the left, quadratic model on the right)

Markov Chain Monte Carlo (Hamiltonian Montel Carlo algorithm) is applied to obtain the distribution in the form of samples and Posterior predictive distribution of C_D is computed [5].

Figure 6 The Posterior predictive distribution of linear model and quadratic model



Linear model	Quadratic model

3.3 Predictive Validation area metric

The validity of the two models, the linear model and the quadratic model, is compared through predictive validation. With the calibrated drag coefficient (C_D) distribution of the experiment condition 1 (10.67m / 1clip), the fall time in the other conditions (Test2, Test3, Test4) is predicted with Eq (1) and (2). The predicted fall time distributions (probability density function, PDF) (Figure 7) are converted to cumulative distribution functions (CDFs) (Figure 8).

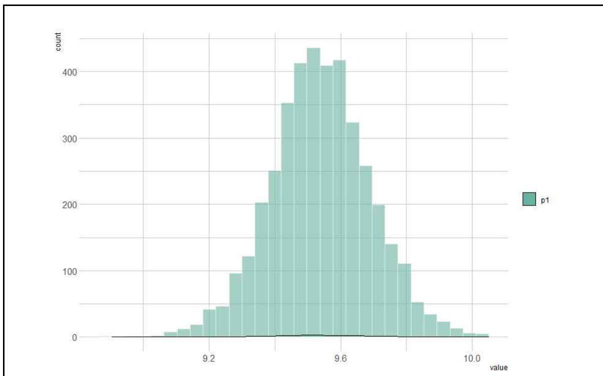


Figure 7 Predicted fall time distribution

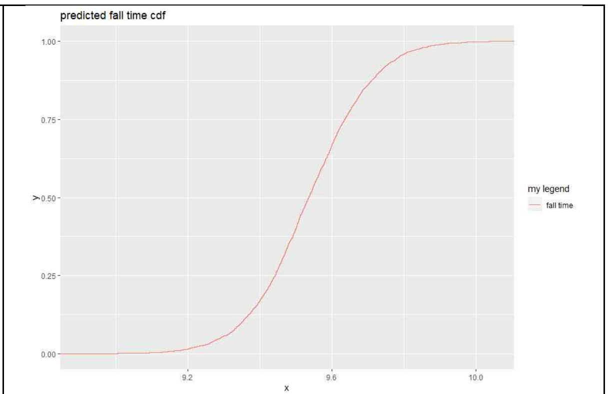


Figure 8 CDF of predicted fall time

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The actual fall time data are converted to CDFs which we define as empirical CDFs (EDFs). Area metric between predicted CDF(F) and EDF(S_n) is expressed as

$$d(F, S_n) = \int_{-\infty}^{\infty} |F(x) - S_n| dx$$

The epistemic uncertainty is quantified with area metrics $d(F, S_n)$ (Figure 9) [6].

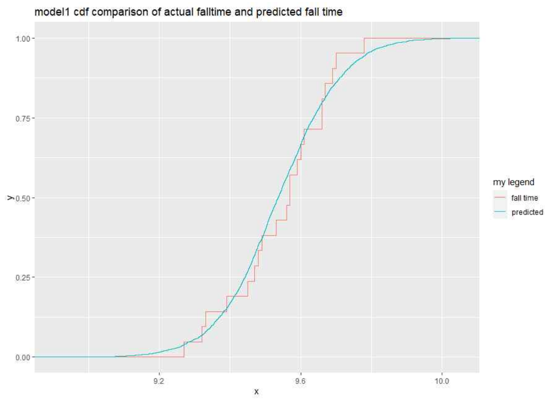

Figure 9 Predictive Validation area metric

Table 3 Predictive distribution area metrics of one helicopter for four conditions under the quadratic and linear model

Conditions	Quadratic model	Linear model
10.67m / 1clip (for calibrations)	0.0221 (1.00)	0.0229 (1.00)
6.82m / 1clip	0.0561 (2.53)	0.0681 (2.97)
6.82m / 2clips	0.0422 (1.91)	0.8157 (35.58)
10.67m / 2clips	0.0227 (1.02)	1.3632 (59.46)

The Area metric of condition 1 (10.67m / 1clip) is calibrated into 1.00 (Table 3) so that the other area metrics are compared based on condition 1. For example, the area metric of condition 2 (6.82m / 1clip), for the quadratic model 0.0561, is divided by 0.0221 (condition 1 area metric) which is equal to 2.53 times. By comparing calibrated numbers based on the condition 1, the quadratic model clearly shows better performance than the linear model. For the calibrated area metrics of the quadratic model, the numbers are below 3 times, whereas the linear model can go over 59 times (10.67m / 2clips). The Linear model could predict fall time for height changes but for the weight changes it showed larger area metrics.

4 Conclusion

In this study, statistical methods were used to validate mathematical models predicting motion of paper helicopters. It was beneficial to use paper helicopters for its economical nature and easy production. Also, their analytical model is relatively simple, which means statistical techniques could be applied effectively. Fall time is measured in four different conditions, and one of the conditions is used for calibration to predict fall time of the other three conditions. The three conditions were used to validate the performance of the prediction.

Three nominally identical helicopters are considered as one identical model with Analysis of Variance (ANOVA) test result. The uncertainty of a single parameter is quantified through Bayesian calibration so that estimated fall time of helicopters was predicted in the form of distribution. Markov chain Monte Carlo (MCMC) technique is applied to generate samples from posterior distributions. Predictive validation was conducted by comparing distributions of actual fall time and predicted fall time. The result showed the validity of two mathematical models. The quadratic model demonstrated better performance for weight and height changes. On the other hand, the linear model showed poor prediction with weight change.

In this paper, statistical calibration and validation are applied to mitigate the difference between actual phenomena and mathematical prediction model. Future work will extend the present single parameter to more parameters through additional experiments for optimization.

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Appendix A Fall time data for four conditions

Condition 1				Condition 2			
Height:10.67m and clips:1				Height:10.67m and clips:2			
M.N	H1	H2	H3	M.N	H1	H2	H3
1	9.67	9.60	9.48	1	7.77	7.62	7.55
2	9.57	9.70	9.78	2	7.86	7.83	7.88
3	9.45	9.27	9.53	3	7.73	7.73	7.90
4	9.32	9.69	9.47	4	7.71	7.88	8.00
5	9.33	9.66	9.56	5	7.93	7.68	7.84
6	9.39	9.49	9.61	6	7.69	7.71	7.95
7	9.66	9.57	9.59	7	7.72	7.73	7.68
AVG	9.48	9.57	9.57	AVG	7.77	7.74	7.83
STV	0.15	0.15	0.11	STV	0.09	0.09	0.16
COV	0.02	0.02	0.01	COV	0.01	0.01	0.02

Condition 3				Condition 4			
Height:6.82m and clips:1				Height:6.82m and clips:2			
M.N	H1	H2	H3	M.N	H1	H2	H3
1	6.24	6.01	6.05	1	5.06	5.01	5.05
2	6.03	6.16	5.99	2	4.94	5.05	4.79
3	6.16	6.12	6.12	3	4.85	4.95	4.93
4	6.09	6.14	6.10	4	4.94	5.05	5.00
5	6.00	5.84	6.15	5	5.02	5.01	4.99
6	5.85	6.12	5.96	6	4.95	5.01	4.96
7	6.06	6.23	6.06	7	5.04	4.91	4.99
AVG	6.06	6.09	6.06	AVG	4.97	5.00	4.96
STV	0.12	0.13	0.07	STV	0.07	0.05	0.08
COV	0.02	0.02	0.01	COV	0.01	0.01	0.02

Appendix B Predictive Validation area metric

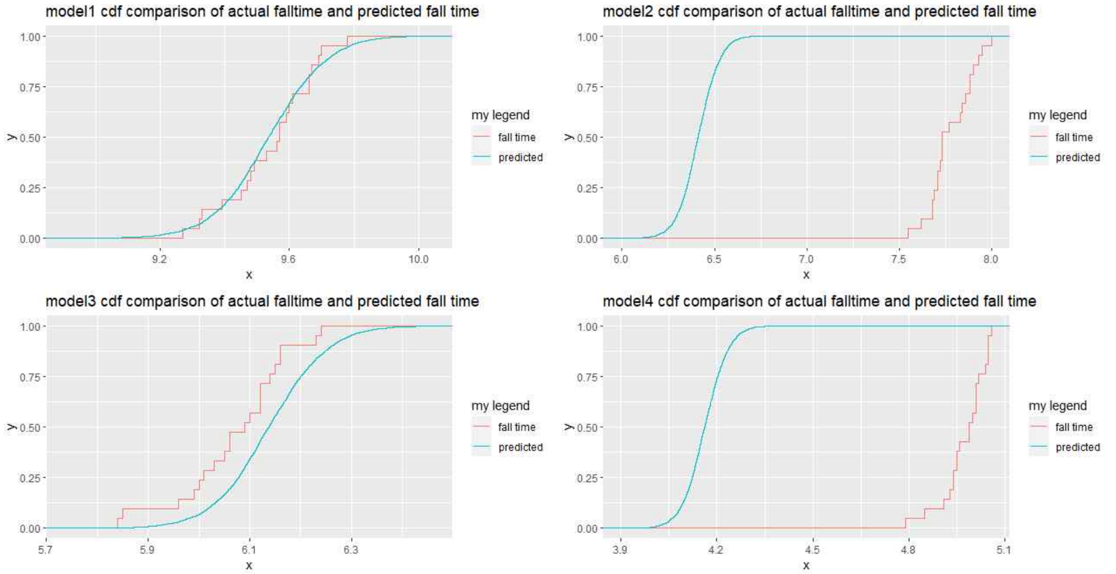

Figure 10 Predictive Validation area metric for the linear model

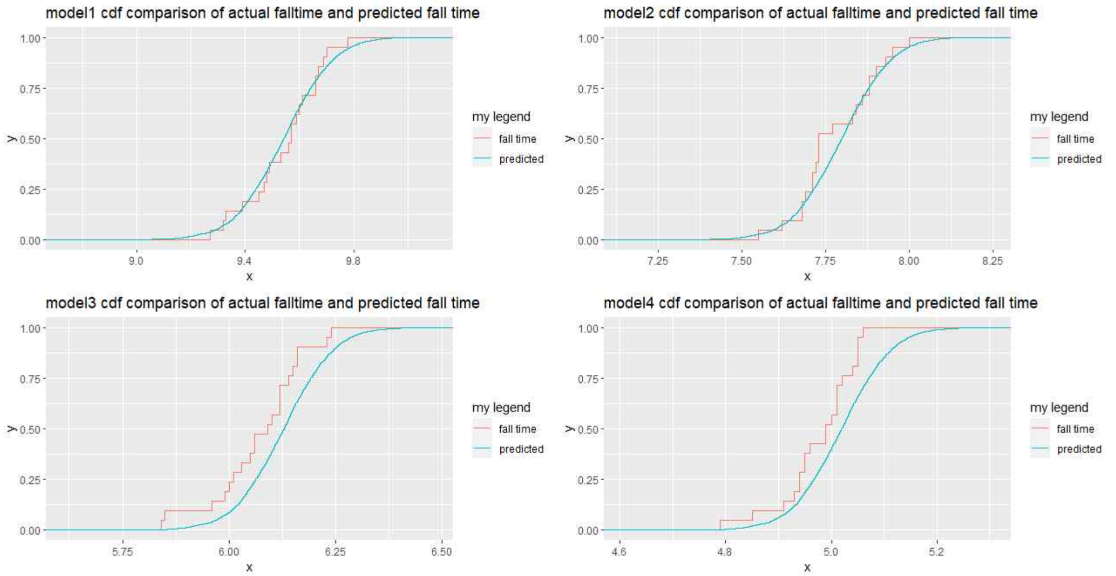

Figure 11 Predictive Validation area metric for the quadratic model