No Coding Farmer

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Abstract

Some Miscellaneous Summary.

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Contents

1 Expectation Maximization Introduction

1.1 EM Induction

$L(\theta) = \sum_{i=1}^{M} logp(X;\theta) = \sum_{i=1}^{m} log \sum_z p(X,Z;\theta)$

let θ_i be some distribution over z's ($\sum_z \theta_i(z) = 1, \theta_i(z) \geq 0$)

$\sum_i logp(X^{(i)};\theta)$

$= \sum_i log \sum_{Z^{(i)}} \theta_i(Z^{(i)}) \frac{p(X^{(i)},Z^{(i)};\theta)}{\theta_i(Z^{(i)})}$

$\geq \sum_i \sum_{Z^{(i)}} \theta_i(Z^i) log \frac{P(X^{(i)}),Z^{(i)};\theta)}{\theta_i(Z^{(i)})} (f(x) = logx \quad is \quad concave.)$

let $\frac{P(X^{(i)}),Z^{(i)};\theta)}{\theta_i(Z^{(i)})} = C$

the equality can be only reached when $\frac{P(X^{(i)}),Z^{(i)};\theta)}{\theta_i(Z^{(i)})}$ is a constant.

we can get: $\sum_i \frac{p(X^{(i)},Z^{(i)};\theta)}{C} = 1$ namely: $\sum_i p(X^{(i)},Z^{(i)};\theta) = C$

further induction: $\theta_i(Z^{(i)}) = \frac{p(X^{(i)},Z^{(i)};\theta)}{\sum_i p(X^{(i)},Z^{(i)};\theta)} = p(Z^{(i)}|X^{(i)};\theta)$

so the procedure of EM algorithm is:

Repeat Until Convergence:

- E-step: for each i,get $Q_i(Z^{(i)}) = p(Z^{(i)}|X^{(i)};\theta)$
- M-step: $\theta := argmax_\theta \sum_i \sum_{Z^i} i(Z^{(i)}) log \frac{p(X^{(i)},Z^{(i)}),\theta)}{Q_i(Z^{(i)})}$

1.2 EM convergence proof

let $l(\theta^{(t)}) = \sum_i \sum_{Z^{(i)}} Q_i^{(t)}(Z^{(i)}) log \frac{p(X^{(i)},Z^{(i)},\theta)}{Q_i^{(t)}(Z^{(i)})}$

then,we have the following inequality:

$l(\theta^{(t+1)})$

$\geq \sum_i \sum_{Z^{(i)}} Q_i^{(t)}(Z^{(i)}) log \frac{p(X^{(i)},Z^{(i)},\theta^{(t+1)})}{Q_i^{(t)}(Z^{(i)})}$

$\geq \sum_i \sum_{Z^{(i)}} Q_i^{(t)}(Z^{(i)}) log \frac{p(X^{(i)},Z^{(i)},\theta^{(t)})}{Q_i^{(t)}(Z^{(i)})}$

$\geq l(\theta^{(t)})$

the first inequality is because :$l(\theta) \geq \sum_i \sum_{Z^{(i)}} Q_i^{(t)}(Z^{(i)}) log \frac{p(X^{(i)},Z^{(i)},\theta)}{Q_i^{(t)}(Z^{(i)})} \forall \theta, Q_i$

the second inequality is because of the maximum of the M-step.

Hence, EM causes the likelihood to converge monotonically.

1.3 Different Writing Style of EM Algorithm

There are many writing style of EM algorithm. here I just mention the book <Statistics Learning Method> by LiHang who is very famous in China.

EM algorithm from LiHang(Li-version):

Algorithm 1 EM from LIHang

```
Require: observation X,hidden variable Z,joint distribution $P(X,Z|\theta)$,conditional distribution $P(Z|Y,\theta)$
  while Not convergence do
      E-Step: let $\theta^{(i)}$ is the i-th estimate of $\theta$,
  $Q(\theta,\theta^{(i)}) = E_z[logP(X,Z|\theta)|X,\theta^{(i)}] = \sum_Z logP(X,Z|\theta)P(Z|X,\theta^{(i)})$
      M-step: $\theta^{(i+1)} = argmax_\theta Q(\theta,\theta^{(i)})$
  end while
  output model parameter $\theta$
```

it seems that Li-version is different from the above version. however, they are the same. because:

- the above version just consider every data, so that it include subscript i. however Li-version only consider one data.
- the above version can be transformed to Li-version.

  $\sum_Z Q(Z)log\frac{P(X,Z:\theta)}{Q(Z)}$
  $= \sum_Z P(Z|X;\theta^{(t)})log\frac{p(X,Z;\theta)}{p(Z|X;\theta^{(t)})}$
  $= \sum_Z P(Z|X;\theta^t)logP(X,Z;\theta) - \sum_Z P(Z|X;\theta^{(t)})logP(Z|X;\theta^{(t)})$

  as the variable is θ,so $\sum_Z P(Z|X;\theta^{(t)})logP(Z|X;\theta^{(t)}$ can be removed.
- $Q(\theta,\theta^{(i)}) = \sum_Z logP(X,Z|\theta)P(Z|X,\theta^{(i)})$ can be also written as $Q(\theta,\theta^{(i)}) = \sum_Z logP(X,Z|\theta)P(Z,X,\theta^{(i)})$,because X is a observation.

2 EM applications

2.1 Gaussian Mix Model

GMM can be solved by EM. **notice here we use the expectation of EM:**

$Q(\theta,\theta^{(i)})$
$= E_\gamma[logP(y,\gamma|\theta)|y,\theta^{(i)}]$
$= E[\sum_{k=1}^K[n_k log\alpha_k + \sum_{j=1}^N \gamma_{jk}[log\frac{1}{\sqrt{2\pi}} - log\sigma_k - \frac{1}{2\sigma_k^2}(y_j-\mu_k)^2]]]$
$= \sum_{k=1}^K[(E\gamma_{jk})log\alpha_k + \sum_{j=1}^N(E\gamma_{jk})[log\frac{1}{\sqrt{2\pi}} - log\sigma_k - \frac{1}{2\sigma_k^2}(y_j-\mu_k)^2]]$

here $(E\gamma_{jk})$ can be easily calculated.

$\hat{\mu_k}, \hat{\sigma_k^2}$ can be acquired by derivation.

$\hat{\alpha_k}$ can be acquired by the derivation on the Lagrangian($\sum_i^K \alpha_k = 1$).

2.2 Hidden Markov Model

HMM Learning Method is also called Baum-Welch algorithm.the target is learning $\lambda = (A,B,\pi)$.

Q function is:

$Q(\lambda,\bar{\lambda}) = \sum_I logP(O,I|\lambda)P(O,I|\bar{\lambda})$

$P(O,I,\lambda) = \pi_{i1}b_{i1}(o_1)a_{i1i2}b_{i2}(o_2)...a_{i_{T-1}i_T}b_{i_T}(o_T)$

so the Q function can also be written as:

$$Q(\lambda, \bar{\lambda}) =$$
$$\sum_I log\pi_{i1}P(O,I|\bar{\lambda}) + \sum_I(\sum_{t=1}^{T-1} loga_{i,i+1})P(O,I|\bar{\lambda}) + \sum_I(\sum_{t=1}^{T} logb_{it}(o_t))P(O,I|\bar{\lambda})$$

note here: I is not only one state. it includes state length from 1 to T,which all start from i_1

so we can solve the maximum of Q function by derivation on the Lagrangian polynomial (because exists these limitations: $\sum_{i=1}^{N} \pi_i = 1, \sum_{j=1}^{N} a_{ij} = 1, \sum_{i=1}^{M} b_i = 1$)

2.3 Naive Bayesian

2.4 other papers

We can use softmax to model transition probability, normal distribution to model emission probability.

it's a good example in Car that Knows Before You Do: Anticipating Maneuvers via Learning Temporal Driving Models, the AIO-HMM can be more complicated,which can be enriched by the graphic model by M.I Jordon.

3 VAE

here is a complete VAE tutorial [2]

$$max \quad logP(x)$$
lhs = $log \int P(x,z)dz$
$= log \int P(x/z)p(z)dz$
= log $\int \frac{P(x/z)}{q(z/x)} q(z/x)p(z)dz$
$= logE_{q(z/x)}[\frac{p(x/z)}{q(z/x)}p(z)]$
jenson's inequality,we can know: $\geq E_{q(z/x)}[log\frac{p(x/z)}{q(z/x)}p(z)]$
$=E_{q(z/x)}[logp(x/z)] + E_{q(z/x)}[log\frac{p(z)}{q(z/x)}]$
$= E_{q(z/x)}[logp(x/z)] - E_{q(z/x)}[log\frac{q(z/x)}{p(z)}]$
$= E_{q(z/x)}[logp(x/z)] - KL(q(z/x)||p(z))$

4 ADMM

minimize H(u) + G(v)
subject to Au + Bv = b

$max_\lambda min_{u,v} H(u) + G(v) + <\lambda, b - Au - Bv> + \frac{\tau}{2}||b - Au - Bv||^2$

Alternating Direction Method of Multipliers
$u_{k+1} = argmin_u H(u) + <\lambda_k, -Au> + \frac{\tau}{2}||b - Au - Bv_k||^2$
$v_{k+1} = argmin_v G(v) + <\lambda_k, -Bv> + \frac{\tau}{2}||b - Au_{k+1} - Bv||^2$
$\lambda_{k+1} = \lambda_k + \tau(b - Au_{k+1} - Bv_{k+1})$

Distributed Problems
minimize g(x) + $\sum_i f_i(x)$
example: sparse least squares:
minimize $\mu||x||_1 + \frac{1}{2}||Ax - b||^2$

minimize $\mu||x||_1 + \sum_i \frac{1}{2}||A_ix - b_i||^2$

data stored on different servers

Transpose Reduction
minimize $\frac{1}{2}||Ax-b||^2$
$x^* = (A^TA)^{-1}A^Tb$

distributed compution:
$A^Tb = \sum A_i^T b_i$
$A^TA = \sum A_i^T A_i$

Unwrapped ADMM
minimize $g(x) + f(Ax) = g(x) + \sum_i f_i(A_i x)$

Example: SVM
minimize $\frac{1}{2}||x||^2 + h(Ax)$
A = data,h = hinge loss

Unwrapped form
minimize $\frac{1}{2}||x||^2 + h(z)$
subject to z=Ax

Transpose Reduction ADMM
scaled augmented Lagrangian:
minimize $\frac{1}{2}||x||^2 + h(z) + \frac{\tau}{2}||z - Ax - \lambda||^2$

ADMM:
$x^{k+1} = min_x \frac{1}{2}||x||^2 + \frac{\tau}{2}||z^k - Ax + \lambda^k||^2$
$z^{k+1} = min_z h(z) + \frac{\tau}{2}||z - Ax^{k+1} + \lambda^k||^2$
$\lambda^{k+1} = \lambda^k + z^{k+1} - Ax^{k+1}$

Minimization Steps
minimize $l(a_3) + \frac{1}{2}||z_2 - W_1a_1||^2 + \frac{1}{2}||a_2 - \sigma(z_2)||^2 + \frac{1}{2}||z_3 - W_2a_2||^2 + \frac{1}{2}||a_3 - \sigma(z_3)||^2$

Solve for weight: least squares(convex)
Solve for activations: least squares + ridge penalty(convex)
Solve for inputs: coordinate-minimization (non-convex but global)

Lagrange Multipliers
minimize $l(a_3) + \frac{1}{2}||z_2 - W_1a_1||^2 + \frac{1}{2}||a_2 - \sigma(z_2)||^2 + < \lambda_1, z_2 - W_1a_1 > + < \lambda_2, a_2 - \sigma(z_2) > + \frac{1}{2}||z_3 - W_2a_2||^2 + \frac{1}{2}||a_3 - \sigma(z_3)||^2 + < \lambda_3, z_3 - W_2a_2 > + < \lambda_4, a_3 - \sigma(z_3) >$

unstable because of non-linear constraints

Bregman Iteration
minimize $l(a_3) + < \lambda, a_3 > + \frac{1}{2}||z_2 - W_1a_1||^2 + \frac{1}{2}||a_2 - \sigma(z_2)||^2 + \frac{1}{2}|||z_3 - W_2a_2||^2 + \frac{1}{2}||a_3 - \sigma(z_3)||^2$

HOG feature dimension: 648

mid layer 1 num: 100

mid layer 2 num: 50

output layer: 1

Algorithm 2 ADMM_NN

```
**Inputs:**
    data number:n=10000,
    data dimension: m=648,
    hidden layer 1 unit number: a=100
    hidden layer 2 unit number: b=50
    output layer unit number: 1
    $a_0$ m-n dimension,
    $W_1$ : a-m dimension
    $z_1$ : a-n dimension
    $a_1$ : a-n dimension
    $W_2$ : b-a dimension
    $z_2$ : b-n dimension
    $a_2$ : b-n dimension
    $W_3$ : 1-b dimension
    $z_3$ : 1-n dimension
    labels:y 1-n dimension
    $\lambda$ : 1-n dimension
    activation function h is ReLu.
**Initialize:**
    allocate $\{a_l\}_{l=1}^{L}$,$\{z_l\}_{l=1}^{L}$ with i.i.d Gaussian Distribution,and $\lambda$
Cache: $a_0^{\dagger}$
Warm Start:
**for** i=1,..,100 **do**
    **for** l=1,2,...,L-1 **do**
$W_l \leftarrow z_l a_{l-1}^{\dagger}$
$a_l \leftarrow (\beta_{l+1} W_{l+1}^T W_{l+1} + \gamma_l I)^{-1}(\beta_{l+1} W_{l+1}^T z_{l+1} + \gamma_l h_l(z_l))$
$z_l \leftarrow argmin_z \gamma_l ||a_l - h_l(z)||^2 + \beta_l ||z - W_l a_{l-1}||^2$

    **end for**
     $W_L \leftarrow z_L a_{L-1}^{\dagger}$
     $z_L \leftarrow argmin_z l(z, y) + < z, \lambda > + \beta_L ||z - W_L a_{L-1}||^2$
**end for**
Start ADMM:
**while** not converge **do**
    **for** l=1,2,...,L-1
 **do** $W_l \leftarrow z_l a_{l-1}^{\dagger}$
$a_l \leftarrow (\beta_{l+1} W_{l+1}^T W_{l+1} + \gamma_l I)^{-1}(\beta_{l+1} W_{l+1}^T z_{l+1} + \gamma_l h_l(z_l))$
$z_l \leftarrow argmin_z \gamma_l ||a_l - h_l(z)||^2 + \beta_l ||z - W_l a_{l-1}||^2$
    **end for**
     $W_L \leftarrow z_L a_{L-1}^{\dagger}$
     $z_L \leftarrow argmin_z l(z, y) + < z, \lambda > + \beta_L ||z - W_L a_{L-1}||^2$
     $\lambda \leftarrow \lambda + \beta_L (z_L - W_L a_{L-1})$
**end while**
```

z_l argmin procedure:

$$z_l = \begin{cases} max(\frac{a_l \gamma_l + W_l a_{l-1} \beta_l}{\gamma_l + \beta_l}, 0) & \text{z} \geq 0 \\ min(W_l a_{l-1}, 0) & \text{z} \leq 0 \end{cases}$$

choose one minimizer z from two choices.

z_L argmin procedure:

when $y_i = 0$:
$f(z) = \beta z^2 - (2\beta w\_a - \lambda)z + max(z, 0)$

$z^* = max(\frac{2\beta w\_a - \lambda - 1}{2\beta}, 0)$ or
$z^* = min(\frac{2\beta w\_a - \lambda}{2\beta}, 0)$
choose one which make f(z) smaller.

when $y_i = 1$:
$f(z) = \beta z^2 - (2\beta w\_a - \lambda)z + max(1 - z, 0)$
$z^* = max(\frac{2\beta w\_a - \lambda}{2\beta}, 1)$ or
$z^* = min(\frac{2\beta w\_a - \lambda + 1}{2\beta}, 1)$
choose one which make f(z) smaller.

z_L argmin procedure(when l is a standard hinge loss):

when $y_i = -1$: $f(z) = max(1 + z) + \lambda z + \beta(z^2 - 2w\_az)$
$z^* = min(\frac{2\beta w\_a - \lambda}{2\beta}, -1)$ or
$z^* = max(\frac{2\beta w\_a - \lambda - 1}{2\beta}, -1)$ choose one which make f(z) smaller.

when $y_i = 1$:
$f(z) = max(1 - z, 0) + \lambda z + \beta(z^2 - 2w\_az)$
$z^* = min(\frac{2\beta w\_a - \lambda + 1}{2\beta}, 1)$ or
$z^* = max(\frac{2\beta w\_a - a - \lambda}{2\beta}, 1)$
choose one which make f(z) smaller.

5 Key steps you must know when building a DL Framework

5.1 Convolution

convert the convolution to matrix multiplication like the fully-connected network.

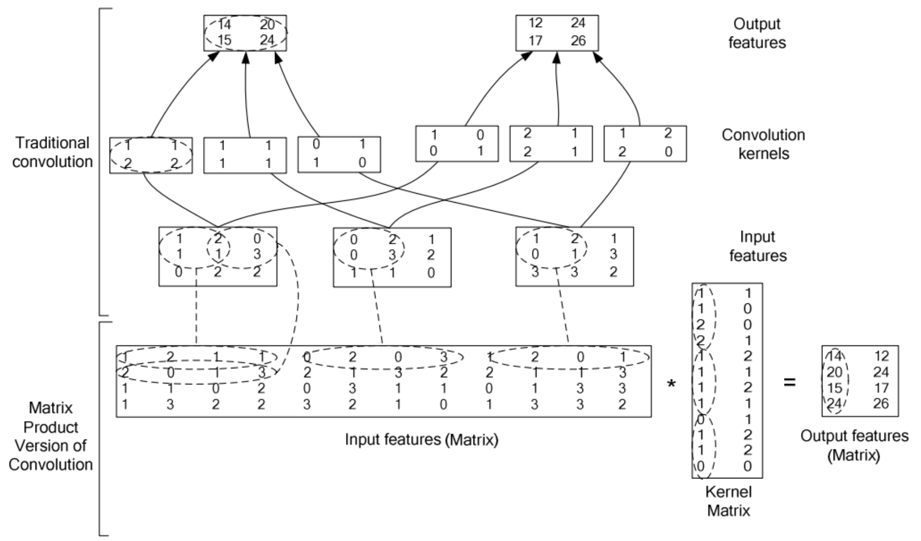


Figure 2. Example convolution operations in a convolutional layer (biases, sub-sampling, and non-linearity omitted). The top figure presents the traditional convolution operations, while the bottom figure presents the matrix version.

figure is from [1].
Let's notate:
H:image height
W:image width
in: input image number
out: output image number

K: convolution kernel size

the matrix product version of convolution,the dimension of two matrix is:
(H*W) *(in*K*K)
(in*K*K) * out

the operation above is like matlab function: img2col
im2col(A,[m n],block_type),where block_type="sliding".

GEMM(GEneral Matrix to Matrix Multiplication) is at the heart of deep learning.`https://petewarden.com/2015/04/20/why-gemm-is-at-the-heart-of-deep-learning/`

6 R-PCA

RPCA problem:
$min_{A,E}\ ||A||_* + \lambda||E||_1$
S.t D=A+E

RPCA dual problem:

Augmented Lagrangian is :
$A_t(A, E; \Lambda) = min\_A, EL(A, E; \Lambda)$
$= min_{A,E}||A||_* + \Lambda||E||_1 + < \Lambda, D - A - E >$
$= min_A||A||_* - < \Lambda, A > + min_E\lambda||E||_1 - < \Lambda, E > + < \Lambda, D >$

both of the sub-problem is conjugate function,according to the property of conjugate function :

$A_t(A, E; \Lambda) =< \Lambda, D >$
$S.t \quad ||\Lambda||_2 \leq 1, ||\Lambda||_\infty \leq \lambda$

so the dual problem is:
$max_\Lambda \quad < \Lambda, D >$
$S.t \quad ||\Lambda||_2 \leq 1, ||\Lambda||_\infty \leq \lambda$

6.1 Solve RPCA by ADMM

the ADMM sub-problem is:
A-sub-problem:
$A_{k+1} = argmin_A||A||_* + \frac{\beta}{2}||D - A - E_k + \Lambda_k/\beta||_F^2$

E-sub-problem:
$E_{k+1} = argmin_E\lambda||E||_1 + \frac{\beta}{2}||D - A_{k+1} - E + \Lambda_k||_F^2$
E-sub-problem has closed-form solution as follows:
$E_{k+1} = S_{\lambda\beta^{-1}}(D - A_k + \Lambda_k/\beta).$

$S_\epsilon = sgn(x)max(|x| - \epsilon, 0)$,which is the same form as shrinkage.

A-sub-problem has a closed-form solution offered by Singular Value Thresholding(SVT):suppose that the SVD of $W = D - E_k + \Lambda_k/\beta_k$ is $W = U\Sigma V^T$,then the optimal solution is $A = US_{\beta^{-1}}(\Sigma)V^T$.

6.2 Adaptive Penalty for ADMM

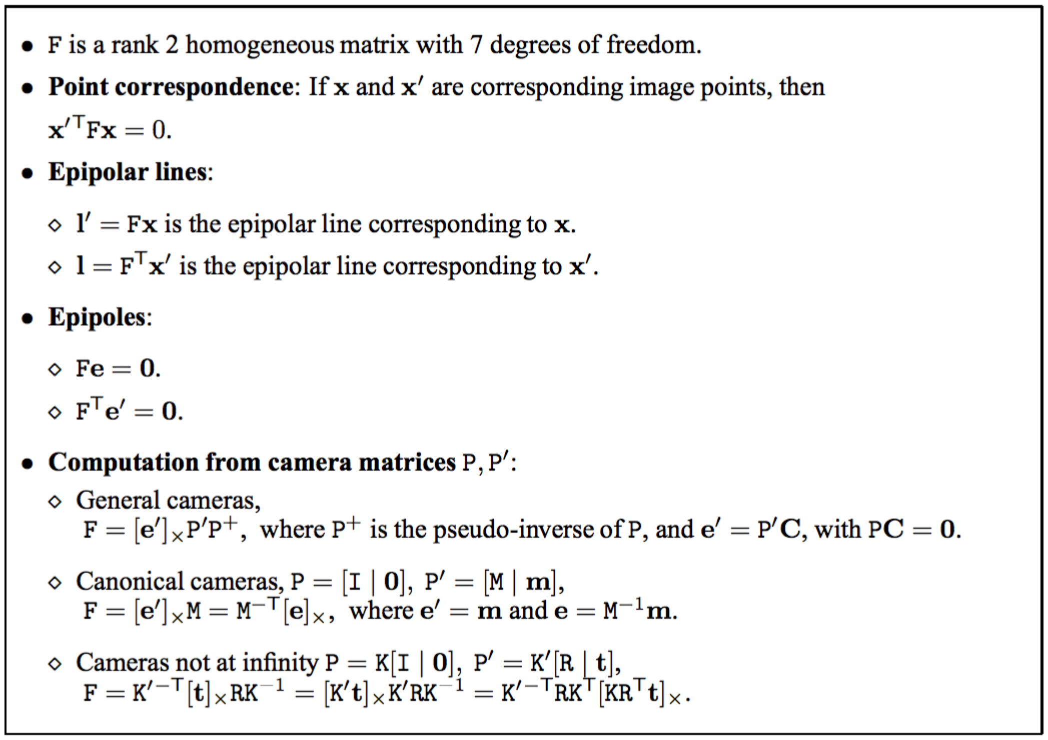
Lin et al.[3] suggest updating the penalty parameter β as follows:
$\beta_{k+1} = min(\beta_{max}, \rho\beta_k)$
where ρ_{max} is an upper bound of $\{\beta_k\}$. the value of ρ is defined as:

$$\rho = \begin{cases} \rho_0 & if \frac{\beta_k max(\sqrt{\eta_A}||x_{k+1} - x_k||_2, \sqrt{\eta_B}||y_{k+1} - y_k||_2)}{||c||_2} < \epsilon_2 \\ 1 & otherwise \end{cases}$$

where η_A, η_B is linearized Taylor second-order factor.

7 SFM

`https://www.robots.ox.ac.uk/~vgg/hzbook/hzbook2/HZepipolar.pdf`



- F is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence**: If $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points, then $\mathbf{x}'^{\mathsf{T}}\mathtt{F}\mathbf{x} = 0$.
- **Epipolar lines**:
  - ◇ $\mathbf{l}' = \mathtt{F}\mathbf{x}$ is the epipolar line corresponding to $\mathbf{x}$.
  - ◇ $\mathbf{l} = \mathtt{F}^{\mathsf{T}}\mathbf{x}'$ is the epipolar line corresponding to $\mathbf{x}'$.
- **Epipoles**:
  - ◇ $\mathtt{F}\mathbf{e} = \mathbf{0}$.
  - ◇ $\mathtt{F}^{\mathsf{T}}\mathbf{e}' = \mathbf{0}$.
- **Computation from camera matrices** $\mathtt{P}, \mathtt{P}'$:
  - ◇ General cameras,
    $\mathtt{F} = [\mathbf{e}']_\times \mathtt{P}'\mathtt{P}^+$, where $\mathtt{P}^+$ is the pseudo-inverse of $\mathtt{P}$, and $\mathbf{e}' = \mathtt{P}'\mathbf{C}$, with $\mathtt{P}\mathbf{C} = \mathbf{0}$.
  - ◇ Canonical cameras, $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$, $\mathtt{P}' = [\mathtt{M} \mid \mathbf{m}]$,
    $\mathtt{F} = [\mathbf{e}']_\times \mathtt{M} = \mathtt{M}^{-\mathsf{T}}[\mathbf{e}]_\times$, where $\mathbf{e}' = \mathbf{m}$ and $\mathbf{e} = \mathtt{M}^{-1}\mathbf{m}$.
  - ◇ Cameras not at infinity $\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]$, $\mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}]$,
    $\mathtt{F} = \mathtt{K}'^{-\mathsf{T}}[\mathbf{t}]_\times \mathtt{R}\mathtt{K}^{-1} = [\mathtt{K}'\mathbf{t}]_\times \mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}\mathtt{R}\mathtt{K}^{\mathsf{T}}[\mathtt{K}\mathtt{R}^{\mathsf{T}}\mathbf{t}]_\times$.

Figure 1: Summary of Fundamental matrix properties

8 Mainfold Learning

`http://www.cad.zju.edu.cn/reports/%C1%F7%D0%CE%D1%A7%CF%B0.pdf`

8.1 Laplace Matrix

8.2 Normalized Cut

8.3 Linear Dimension Reduction

PCA,CDMS,RP

8.4 NonLinear Dimension Reduction

KPCA,ISOMAP,LLE,LSTA,LPCA, LE(Laplacian EigenMaps) , Diffusion Maps, MVU

9 DCT

`http://eeweb.poly.edu/~yao/EE3414/ImageCoding_DCT.pdf`

1D Unitary Transform

Consider the N – point signal $s(n)$ as an N - dimensional vector

$$\mathbf{s} = \begin{bmatrix} s_0 \\ s_1 \\ \\ s_{N-1} \end{bmatrix}$$

The inverse transform says that $\mathbf{s}$ can be represented as the sum of N basis vectors

$\mathbf{s} = t_0\mathbf{u_0} + t_1\mathbf{u}_1 + \ldots + t_{N-1}\mathbf{u}_{N-1}$

where $\mathbf{u}_k$ corresponds to the k - th transform kernel :

$$\mathbf{u}_k = \begin{bmatrix} u_{k,0} \\ u_{k,1} \\ \ldots \\ u_{k,N-1} \end{bmatrix}$$

The forward transform says that the expansion coefficient t_k can be determined by the inner product of $\mathbf{s}$ with $\mathbf{u}_k$:

$$t_k = (\mathbf{u}_k, \mathbf{s}) = \sum_{n=0}^{N-1} u^*_{k;n} s_n$$

1D Discrete Cosine Transform

- Can be considered “real” version of DFT
  - Basis vectors contain only co-sinusoidal patterns

DFT

$$u_{k,n} = \frac{1}{\sqrt{N}} \exp\left(j\frac{2\pi k}{N}n\right) = \frac{1}{\sqrt{N}}\left(\cos\left(\frac{2\pi k}{N}n\right) + j\sin\left(\frac{2\pi k}{N}n\right)\right)$$

$$\mathbf{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} \cos\left(\frac{2\pi k}{N}0\right) \\ \cos\left(\frac{2\pi k}{N}1\right) \\ \ldots \\ \cos\left(\frac{2\pi k}{N}(N-1)\right) \end{bmatrix} + j\frac{1}{\sqrt{N}} \begin{bmatrix} \sin\left(\frac{2\pi k}{N}0\right) \\ \sin\left(\frac{2\pi k}{N}1\right) \\ \ldots \\ \sin\left(\frac{2\pi k}{N}(N-1)\right) \end{bmatrix}$$

DCT

$$u_{k,n} = \alpha(k)\cos\left(\frac{\pi k}{2N}(2n+1)\right)$$

$$\alpha(0) = \sqrt{\frac{1}{N}}, \alpha(k) = \sqrt{\frac{2}{N}}, k = 1,2,\ldots, N-1$$

$$\mathbf{u}_k = \alpha(k) \begin{bmatrix} \cos\left(\frac{\pi k}{2N}1\right) \\ \cos\left(\frac{\pi k}{2N}3\right) \\ \ldots \\ \cos\left(\frac{\pi k}{2N}(2N+1)\right) \end{bmatrix}$$

Example: 4-point DCT

Using $u_{k,n} = \alpha(\mathrm{k})\cos\left(\frac{\mathrm{k}\pi}{2*4}(2n+1)\right), \alpha(0) = \sqrt{\frac{1}{4}} = \frac{1}{2}, \alpha(\mathrm{k}) = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}}, \mathrm{k} \neq 0,$

1D DCT basis are : $\mathbf{u}_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\left(\frac{\pi}{8}\right) \\ \cos\left(\frac{3\pi}{8}\right) \\ \cos\left(\frac{5\pi}{8}\right) \\ \cos\left(\frac{7\pi}{8}\right) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.9239 \\ 0.3827 \\ -0.3827 \\ -0.9239 \end{bmatrix}; \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) \\ \cos\left(\frac{3\pi}{4}\right) \\ \cos\left(\frac{5\pi}{4}\right) \\ \cos\left(\frac{7\pi}{4}\right) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}; \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\left(\frac{3\pi}{8}\right) \\ \cos\left(\frac{9\pi}{8}\right) \\ \cos\left(\frac{15\pi}{8}\right) \\ \cos\left(\frac{21\pi}{8}\right) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.3827 \\ -0.9239 \\ 0.9239 \\ -0.3827 \end{bmatrix}$

For $\mathbf{s} = \begin{bmatrix} 2 \\ 4 \\ 5 \\ 3 \end{bmatrix}$, determine the transform coefficients $t_{k.}$ Also determine the reconstructed vector from all coefficients and two largest coeffcieints.

2D Separable Transform

Consider the $M \times N$ – point image $\mathbf{S}$ as a $M \times N$ - dimensional array (matrix)

$$\mathbf{S} = \begin{bmatrix} S_{0,0} & S_{0,1} & \dots & S_{0,N-1} \\ S_{1,0} & S_{1,1} & \dots & S_{1,N-1} \\ \dots & \dots & \dots & \dots \\ S_{M-1,0} & S_{M-1,1} & \dots & S_{M-1,N-1} \end{bmatrix}$$

The inverse transform says that $\mathbf{s}$ can be represented as the sum of $M \times N$ basis images

$\mathbf{S} = T_{0,0}\mathbf{U}_{\mathbf{0},0} + T_{0,1}\mathbf{U}_{0,1} + \dots + T_{M-1,N-1}\mathbf{U}_{M-1,N-1}$

where $\mathbf{U}_{k,l}$ corresponds to the (k,l) - th transform kernel:

$$\mathbf{U}_{k,l} = \mathbf{u}_k(\mathbf{u}_l)^T = \begin{bmatrix} u_{k,0} \\ u_{k,1} \\ \dots \\ u_{k,N-1} \end{bmatrix} \begin{bmatrix} u_{l,0}^* & u_{l,1}^* & \dots & u_{l,N-1}^* \end{bmatrix} = \begin{bmatrix} u_{k,0}u_{l,0}^* & u_{k,0}u_{l,1}^* & \dots & u_{k,0}u_{l,N-1}^* \\ u_{k,1}u_{l,0}^* & u_{k,1}u_{l,1}^* & \dots & u_{k,1}u_{l,N-1}^* \\ \dots & \dots & \dots & \dots \\ u_{k,N-1}u_{l,0}^* & u_{k,N-1}u_{l,1}^* & \dots & u_{k,N-1}u_{l,N-1}^* \end{bmatrix}$$

The forward transform says that the expansion coefficient S_k can be determined by the inner product of $\mathbf{S}$ and $\mathbf{U}_{k,l}$:

$$T_{k,l} = (\mathbf{U}_{k,l}, \mathbf{S}) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} U_{k,l;m,n}^* S_{m,n}$$

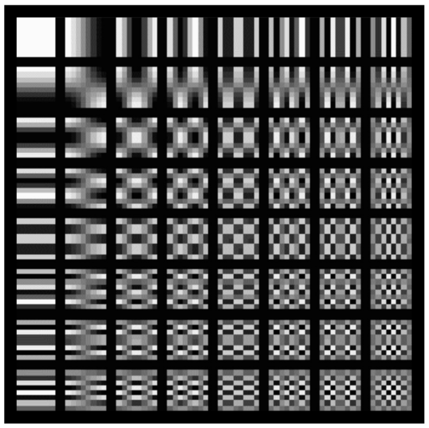
- Basis image = outer product of 1D DCT basis vector

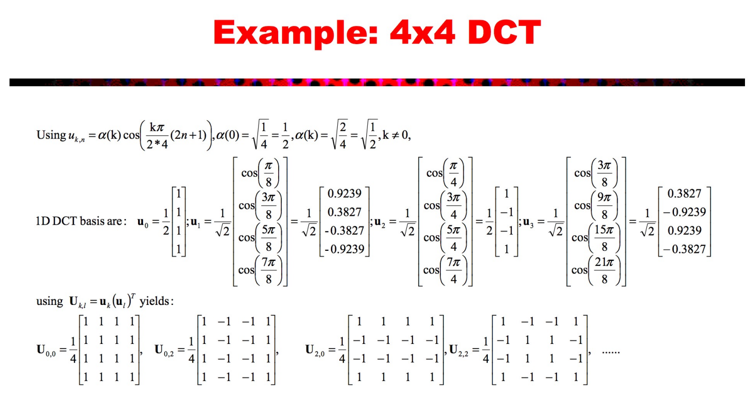
$$\mathbf{u}_{k;N} = \alpha(k)\begin{bmatrix} \cos\left(\frac{\pi k}{2N}1\right) \\ \cos\left(\frac{\pi k}{2N}3\right) \\ \dots \\ \cos\left(\frac{\pi k}{2N}(2N+1)\right) \end{bmatrix}, \quad \alpha(0) = \sqrt{\frac{1}{N}}, \alpha(k) = \sqrt{\frac{2}{N}}, k = 1,2,\dots,N-1$$

$$\mathbf{U}_{k,l;M,N} = \mathbf{u}_{k;M}(\mathbf{u}_{l;N})^T$$

$$= \alpha(k)\alpha(l)\begin{bmatrix} \cos\left(\frac{k\pi}{2M}1\right)\cos\left(\frac{l\pi}{2N}1\right) & \cos\left(\frac{k\pi}{2M}1\right)\cos\left(\frac{l\pi}{2N}3\right) & \dots & \cos\left(\frac{k\pi}{2M}1\right)\cos\left(\frac{l\pi}{2N}(2N+1)\right) \\ \cos\left(\frac{k\pi}{2M}3\right)\cos\left(\frac{l\pi}{2N}1\right) & \cos\left(\frac{k\pi}{2M}3\right)\cos\left(\frac{l\pi}{2N}3\right) & \dots & \cos\left(\frac{k\pi}{2M}3\right)\cos\left(\frac{l\pi}{2N}(2N+1)\right) \\ \dots & \dots & \dots & \\ \cos\left(\frac{k\pi}{2M}(2M+1)\right)\cos\left(\frac{l\pi}{2N}1\right) & \cos\left(\frac{k\pi}{2M}(2M+1)\right)\cos\left(\frac{l\pi}{2N}3\right) & \dots & \cos\left(\frac{k\pi}{2M}(2M+1)\right)\cos\left(\frac{l\pi}{2N}(2N+1)\right) \end{bmatrix}$$

Basis Images of 8x8 DCT





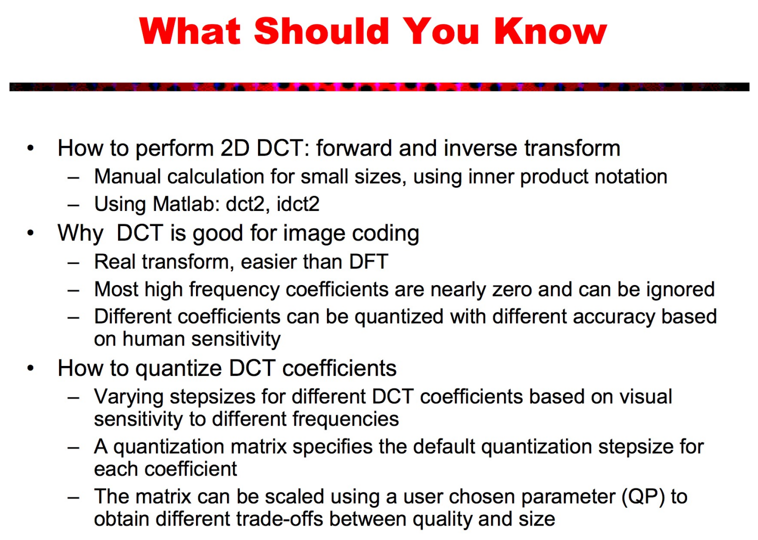
Example: 4x4 DCT

Using $u_{k,n} = \alpha(\mathrm{k}) \cos\left(\frac{\mathrm{k}\pi}{2*4}(2n+1)\right), \alpha(0) = \sqrt{\frac{1}{4}} = \frac{1}{2}, \alpha(\mathrm{k}) = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}}, \mathrm{k} \neq 0,$

$$\text{1D DCT basis are:}\quad \mathbf{u}_0 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}; \mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\left(\frac{\pi}{8}\right)\\ \cos\left(\frac{3\pi}{8}\right)\\ \cos\left(\frac{5\pi}{8}\right)\\ \cos\left(\frac{7\pi}{8}\right)\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0.9239\\0.3827\\-0.3827\\-0.9239\end{bmatrix}; \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\left(\frac{\pi}{4}\right)\\ \cos\left(\frac{3\pi}{4}\right)\\ \cos\left(\frac{5\pi}{4}\right)\\ \cos\left(\frac{7\pi}{4}\right)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}; \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\left(\frac{3\pi}{8}\right)\\ \cos\left(\frac{9\pi}{8}\right)\\ \cos\left(\frac{15\pi}{8}\right)\\ \cos\left(\frac{21\pi}{8}\right)\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0.3827\\-0.9239\\0.9239\\-0.3827\end{bmatrix}$$

using $\mathbf{U}_{k,l} = \mathbf{u}_k(\mathbf{u}_l)^T$ yields:

$$\mathbf{U}_{0,0} = \frac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}, \quad \mathbf{U}_{0,2} = \frac{1}{4}\begin{bmatrix}1&-1&-1&1\\1&-1&-1&1\\1&-1&-1&1\\1&-1&-1&1\end{bmatrix}, \quad \mathbf{U}_{2,0} = \frac{1}{4}\begin{bmatrix}1&1&1&1\\-1&-1&-1&-1\\-1&-1&-1&-1\\1&1&1&1\end{bmatrix}, \mathbf{U}_{2,2} = \frac{1}{4}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}, \quad \ldots\ldots$$



What Should You Know

- How to perform 2D DCT: forward and inverse transform
  - Manual calculation for small sizes, using inner product notation
  - Using Matlab: dct2, idct2
- Why DCT is good for image coding
  - Real transform, easier than DFT
  - Most high frequency coefficients are nearly zero and can be ignored
  - Different coefficients can be quantized with different accuracy based on human sensitivity
- How to quantize DCT coefficients
  - Varying stepsizes for different DCT coefficients based on visual sensitivity to different frequencies
  - A quantization matrix specifies the default quantization stepsize for each coefficient
  - The matrix can be scaled using a user chosen parameter (QP) to obtain different trade-offs between quality and size

10 KL-divergence Application

LINE: Large-scale Information Network Embedding `https://arxiv.org/pdf/1503.03578.pdf`

11 Reinforcement Learning

Markov Decision Process (**MDP**) is tuple$(\mathcal{S}, \mathcal{A}, \mathcal{P}, r, \gamma)$:

r is a reward function,$r(s, a, s^{'})$

γ is a discount factor

$\mathcal{P}$ is the transition probability distribution:
probability from state s with action a to state $s^{'}$: $P(s^{'}|s, a)$

$\mathcal{S}$ is a finite set of states.

$\mathcal{A}$ is a finite set of actions.

Markov Property:
$P(s_{t+1}|s_t) = P(s_{t+1}|s_1, ..., s_t)$

Stochastic Policy:
$\pi(a|s) = P(a_s = a|s_t = s), \pi(a|s)$ here is a probability,we can often see $\pi(s)$, which returns a, means under policy π and state s, you should better take action a.

Value function:
$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$

State-value function:
$V_\pi(s) = E_\pi(G_t|s_t = s)$

Action-value function:
$Q_\pi(s, a) = E_\pi(G_t|s_t = s, a_t = a)$

Bellman Equation:
$V_\pi(s) = E_\pi(r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ...|s_t = s)$
$= E_\pi(r_{t+1} + \gamma(r_{t+2} + \gamma r_{t+3} + ...)|s_t = s)$
$= E_\pi(r_{t+1} + \gamma G_{t+1}|s_t = s)$
$= E_\pi(r_{t+1} + \gamma V_\pi(s_{t+1})|s_t = s)$

For state-value function, Bellman Equation can be written as:
$V_\pi(s) = E_\pi(r_{t+1} + \gamma V_\pi(s_{t+1})|s_t = s)$
$= \sum_{\alpha \in A} \pi(a|s) \sum_{s' \in S} P(s^{'}|s, a)[r(s, a, s^{'}) + \gamma V_\pi(s^{'})]$
$r(s, a, s^{'})$ is same with r_{t+1} to some extent.

For action-value function, Bellman Equation can be written as:
$Q_\pi(s, a) = E_\pi(r(s, a, s^{'}) + \gamma Q_\pi(s^{'}, a^{'})|s, a)$
$= \sum_{s' \in S} P(s^{'}|s, a)[r(s, a, s^{'}) + \gamma \sum_{a' \in A} \pi(a^{'}, s^{'}) Q_\pi(s^{'}, a^{'}))]$

Normally, we just assume the $\pi(a|s), p(s^{'}|s, a), r(s, a, s^{'})$ are known(namely the MDP is known). so we can solve the linear equation above. however, when data become huge, it is not feasible to solve the Bellman Equation directly.

optimal value function:

the optimal state-value function $V_*(s)$ is :
$V_*(s) = max_\pi V_\pi(s)$

the optimal action-value function $Q_*(s, a)$ is:
$Q_*(s, a) = max_\pi Q_\pi(s, a)$

$Q^*(s, a) = max_\pi Q^\pi(s, a)$
following important properties:

$Q^*(s, a) = E[r_{t+1} + \gamma V^*(s_{t+1})|s_t = s, a_t = a]$
$V^*(s) = max_{a \in \mathcal{A}} Q^*(s, a)$
$Q^*(s, a) = \sum_{s' \in \mathcal{S}} p(s^{'}|s, a)[r(s, a, s^{'}) + \gamma V^*(s^{'})]$

The goal for any MDP is finding the optimal value function.

Or equivalent an optimal policy π^* for any policy $\pi, V_{\pi^*}(s) \geq V_\pi(s), \forall s \in S$

how do we take action after we obtain the optimal $V^*(s)$? if we known the $p(s^{'}|s, a), r(s, a, s^{'})$, then we can get: $Q^*(s, a) = \sum_{s' \in S} p(s^{'}|s, a)[R(s, a, s^{'}) + \gamma V^*(s^{'})]$. after $Q^*(s, a)$ is acquired,it's more trivial to take action a when it's at state s.

Relation between Q and V Functions

reference: `http://www.cs.cmu.edu/~mgormley/courses/10601-s17/slides/lecture26-ri.pdf`

Q from V:
$Q^{\pi}(s,a) = \sum_{s^{'} \in S} P(s^{'}|s,a)[R(s,a,s^{'}) + \gamma V^{\pi}(s^{'})]$

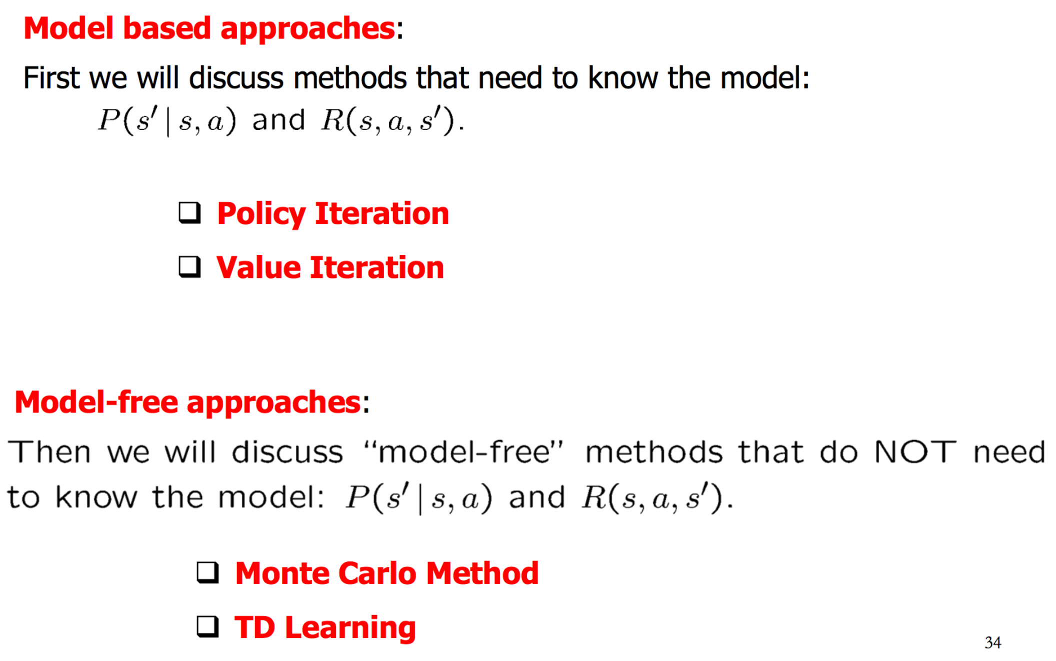
V from Q:
$V^{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) Q^{\pi}(s,a)$

V and Q
$V^{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s^{'} \in S} p(s^{'}|s,a)[r(s,a,s^{'}) + \gamma V^{\pi}(s^{'})]$

Q and V
$Q^{\pi}(s,a) = \sum_{s^{'} \in S} p(s^{'}|s,a)[r(s,a,s^{'}) + \gamma \sum_{a^{'} \in \mathcal{A}} \pi(a^{'}|s^{'}) Q^{\pi}(s^{'},a^{'})]$

more complicated..
$V^{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s^{'} \in S} p(s^{'}|s,a)[r(s,a,s^{'}) + \gamma \sum_{a^{'} \in \mathcal{A}} \pi(a^{'}|s^{'}) Q^{\pi}(s^{'},a^{'})]$

mainly can classified into two main approaches:



Model based approaches:

First we will discuss methods that need to know the model:
$P(s' \mid s,a)$ and $R(s,a,s')$.

- ❑ **Policy Iteration**
- ❑ **Value Iteration**

Model-free approaches:

Then we will discuss "model-free" methods that do NOT need to know the model: $P(s' \mid s,a)$ and $R(s,a,s')$.

- ❑ **Monte Carlo Method**
- ❑ **TD Learning**

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Figure 2: Structure of RL

11.1 Model-based Method

11.1.1 Policy Iteration

One drawback of policy iteration is that each iteration involves policy evaluation.

1. Initialization

$V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ for all $s \in \mathcal{S}$.
$\pi(s)$ is a deterministic policy.
$\delta > 0$ is a small threshold parameter.

2. Policy Evaluation

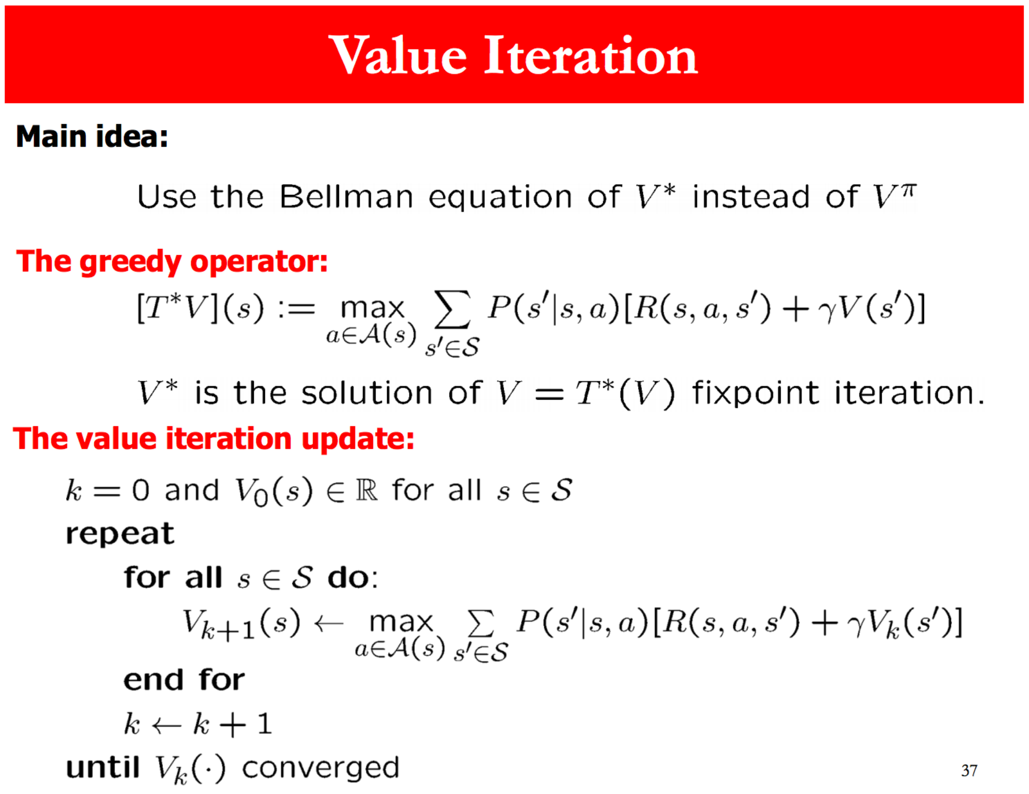
```
repeat
    $\Delta \leftarrow 0$
    for all $s \in \mathcal{S}$ do:
        $v \leftarrow V(s)$
        $a \leftarrow \pi(s)$
        $V(s) \leftarrow \sum_{s' \in \mathcal{S}} P(s'|s,a)[R(s,a,s') + \gamma V(s')]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
    end for
until $\Delta < \delta$
```

3. Policy Improvement

```
$policyStable \leftarrow true$
for all $s \in \mathcal{S}$ do:
    $b \leftarrow \pi(s)$
    $\pi(s) \leftarrow \arg\max_{a \in \mathcal{A}(s)} \sum_{s' \in \mathcal{S}} P(s'|s,a)[R(s,a,s') + \gamma V(s')]$
    if $b \neq \pi(s)$ then
        $policyStable \leftarrow false$
    end if
end for
if $policyStable$ then
    $STOP$
else
    Go to 2 (Policy Evaluation)
end if
```

Policy Improvement just improve the policy $\pi(s)$, then when we back to policy evaluation, the next action a is determined by the policy $\pi(s)$ which was updated in policy improvement. thus the relationship of policy improvement and policy evaluation is founded.

11.1.2 Value Iteration



Value Iteration

Main idea:

Use the Bellman equation of V^* instead of V^π

The greedy operator:

$$[T^*V](s) := \max_{a\in\mathcal{A}(s)} \sum_{s'\in\mathcal{S}} P(s'|s,a)[R(s,a,s') + \gamma V(s')]$$

V^* is the solution of $V = T^*(V)$ fixpoint iteration.

The value iteration update:

```
$k = 0$ and $V_0(s) \in \mathbb{R}$ for all $s \in \mathcal{S}$
repeat
    for all $s \in \mathcal{S}$ do:
        $V_{k+1}(s) \leftarrow \max_{a\in\mathcal{A}(s)} \sum_{s'\in\mathcal{S}} P(s'|s,a)[R(s,a,s') + \gamma V_k(s')]$
    end for
    $k \leftarrow k + 1$
until $V_k(\cdot)$ converged
```

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1, value iteration converges to the true solution of Bellman optimal equations
2, Learn optimal value function directly, unlike policy iteration, there is no explicit policy.

11.2 Model-Free Method

11.2.1 on-policy TD

TD0

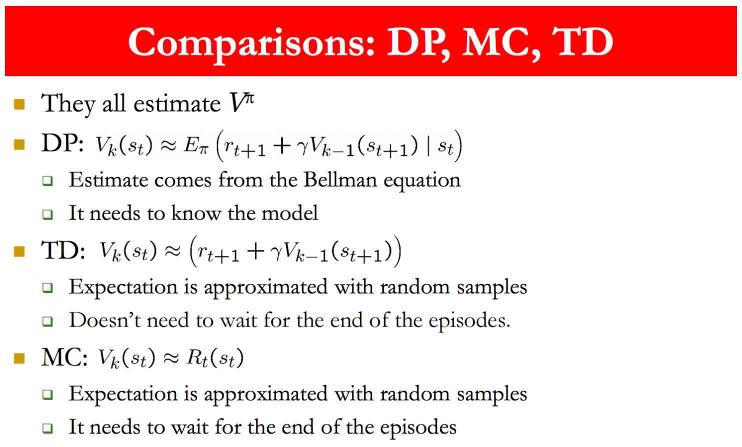
TD(n)

TDλ

11.2.2 on-policy MCMC

11.2.3 off-policy method

11.2.4 Comparisons: DP, MC, TD

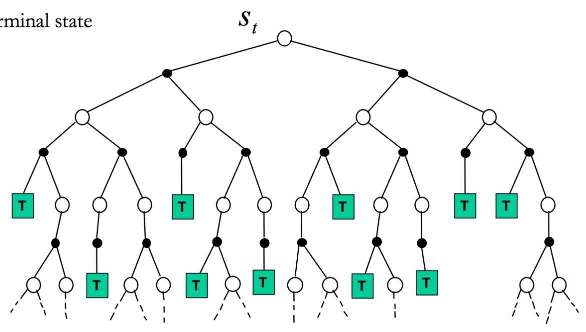


Comparisons: DP, MC, TD

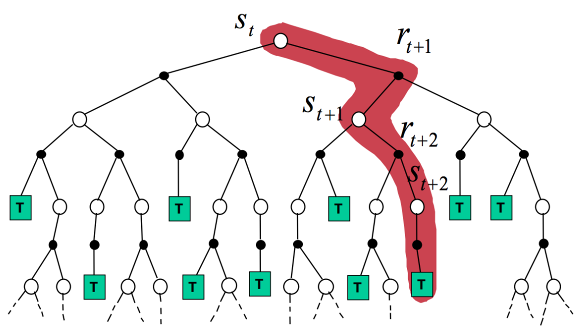
- They all estimate V^{π}
- DP: $V_k(s_t) \approx E_{\pi}\left(r_{t+1} + \gamma V_{k-1}(s_{t+1}) \mid s_t\right)$
  - Estimate comes from the Bellman equation
  - It needs to know the model
- TD: $V_k(s_t) \approx \left(r_{t+1} + \gamma V_{k-1}(s_{t+1})\right)$
  - Expectation is approximated with random samples
  - Doesn't need to wait for the end of the episodes.
- MC: $V_k(s_t) \approx R_t(s_t)$
  - Expectation is approximated with random samples
  - It needs to wait for the end of the episodes

MDP Backup Diagrams

- White circle: state
- Black circle: action
- T: terminal state

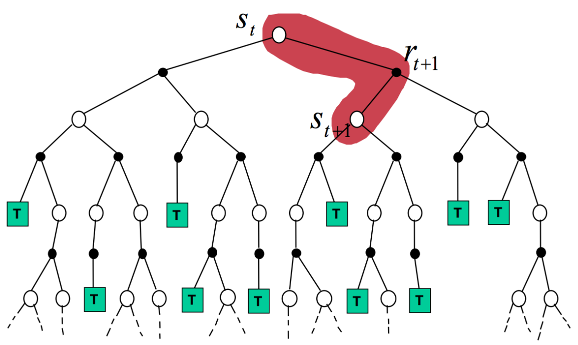


Monte Carlo Backup Diagram



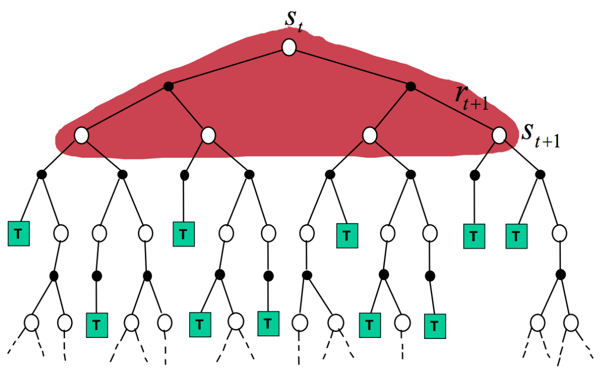
MC estimate: $V_k(s_t) := V_{k-1}(s_t) + \alpha_k \cdot (R_k(s_t) - V_{k-1}(s_t))$

Temporal Differences Backup Diagram



TD estimate: $V_k(s_t) := V_{k-1}(s_t) + \alpha_k \cdot \big((r_{t+1} + \gamma V_{k-1}(s_{t+1})) - V_{k-1}(s_t)\big)$

Dynamic Programming Backup Diagram



$$V^\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s' \in \mathcal{S}} P(s'|s,a)[R(s,a,s') + \gamma V^\pi(s')]$$

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Acknowledgments

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