Algorithms for Big Data Analysis : Homework #2

Due on March 14, 2017 at 23:59pm

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Problem 1

5.11 Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^N \|A_i x + b_i\|_2 + (1/2)\|x - x_0\|_2^2.$$

The problem data are $A_i \in \mathbf{R}^{m_i \times n}$, $b_i \in \mathbf{R}^{m_i}$, and $x_0 \in \mathbf{R}^n$. First introduce new variables $y_i \in \mathbf{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.

5.11

Solution
The Lagranian is

$$L(x, y, \lambda) = \sum_{i=1}^N ||y_i||_2 + \frac{1}{2}||x - x_0||_2^2 + \sum_{i=1}^N \lambda_i^T (y_i - A_i x_i - b_i)$$

firstly, minimize x by deviation on x ,this yield:

$$x = x_0 + \sum_{i=1}^N A_i^T \lambda_i$$

which we use the following deriviation principle:

$$\frac{\partial a^T x}{\partial x} = a$$

secondly,minimize over y_i:

$$inf_{y_i}(||y_i||_2 + \lambda_i^T y_i) = 0, if \ ||\lambda_i||_2 \leq 1$$
$$inf_{y_i}(||y_i||_2 + \lambda_i^T y_i) = -\infty, o.w$$

because when $||\lambda_i||_2 \geq 1$,where always exist a y_i,let the inf unbounded to $-\infty$

so totally,we can get :

$$y_i = 0, x = x_0 + \sum_{i=1}^N A_i^T \lambda_i$$

$$L(x, \lambda) = \frac{1}{2}|| \sum_{i=1}^N A_i^T \lambda_i||_2^2 + \sum_{i=1}^N \lambda_i^T [-A_i(x_0 + \sum_{i=1}^N A_i^T \lambda_i) - b_i]$$

$$= \frac{1}{2}|| \sum_{i=1}^N A_i^T \lambda_i||_2^2 + \sum_{i=1}^N \lambda_i^T A_i (\sum_{i=1}^N A_i^T \lambda_i) - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i)$$

$$= \frac{1}{2}|| \sum_{i=1}^N A_i^T \lambda_i||_2^2 + (\sum_{i=1}^N A_i^T \lambda_i)^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i)$$

$$= \frac{1}{2}|| \sum_{i=1}^N A_i^T \lambda_i||_2^2 + || \sum_{i=1}^N A_i^T \lambda_i||_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i)$$

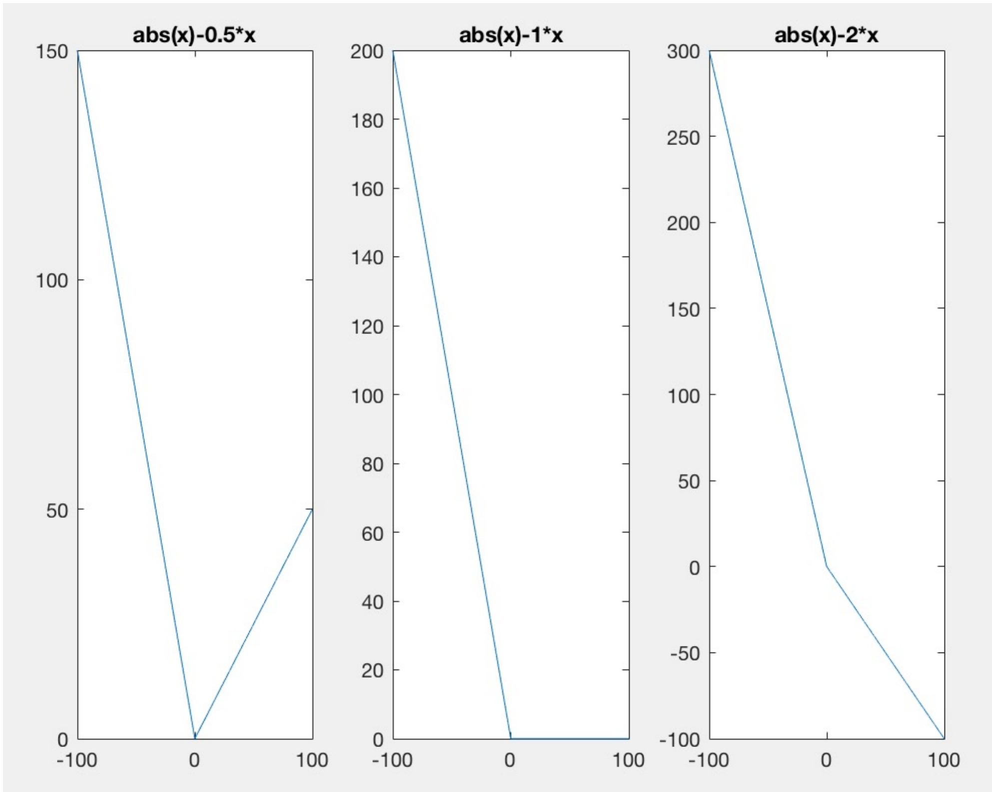
$$= -\frac{1}{2}|| \sum_{i=1}^N A_i^T \lambda_i||_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i)$$

so,the dual form is :

$$max_{\lambda_i} \ -\frac{1}{2}|| \sum_{i=1}^N A_i^T \lambda_i||_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i)$$
$$S.t \ ||\lambda_i||_2 \leq 1, i = 1, 2, ..., n$$

Notice: why $||\lambda_i||_2 \leq 1$?,we can think about the 1-dimension problem,only when $|a| \leq 1$ can it have infimal.:

Problem 1 continued on next page. . .



Problem 2

5.17 *Robust linear programming with polyhedral uncertainty.* Consider the robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \sup_{a \in \mathcal{P}_i} a^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

with variable $x \in \mathbf{R}^n$, where $\mathcal{P}_i = \{a \mid C_i a \preceq d_i\}$. The problem data are $c \in \mathbf{R}^n$, $C_i \in \mathbf{R}^{m_i \times n}$, $d_i \in \mathbf{R}^{m_i}$, and $b \in \mathbf{R}^m$. We assume the polyhedra $\mathcal{P}_i$ are nonempty. Show that this problem is equivalent to the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & d_i^T z_i \leq b_i, \quad i = 1, \ldots, m \\ & C_i^T z_i = x, \quad i = 1, \ldots, m \\ & z_i \succeq 0, \quad i = 1, \ldots, m \end{array}$$

with variables $x \in \mathbf{R}^n$ and $z_i \in \mathbf{R}^{m_i}$, $i = 1, \ldots, m$. *Hint.* Find the dual of the problem of maximizing $a_i^T x$ over $a_i \in \mathcal{P}_i$ (with variable a_i).

5.19

Solution
the original problem can be fomulated as :

$$min_x \; c^T x$$
$$S.t \; f_i(x) \leq b_i, i = 1, ..., m$$

we can define $f_i(x)$ as the optimal value of the following LP:

$$max_a \ x^T a$$
$$S.t \ c_i a \leq d_i$$

Lagrangian of $f_i(x)$ is:

$$L(x, \lambda) = x^T a + \lambda^T (c_i a - d_i) = (x^T + \lambda^T c_i)a - \lambda^T d_i$$

$$inf_a \ L(x, \lambda) = -\lambda^T d_i, when \ x^T + \lambda^T c_i = 0$$

$$inf_a \ L(x, \lambda) = -\infty, o.w$$

the dual problem is:

$$\begin{gathered} max_\lambda - \lambda^T d_i \\ S.t \ x^T + \lambda^T c_i = 0 \\ \lambda \geq 0 \end{gathered}$$

$\rightarrow$

$$\begin{gathered} min_\lambda \ \lambda^T d_i \\ S.t \ x^T + \lambda^T c_i = 0 \\ \lambda \geq 0 \end{gathered}$$

the optimal value of this LP also equals to $f_i(x)$,so the final LP can be written as:

$$\begin{gathered} min_x \ \ c^T x \\ S.t \ d^T \lambda_i \leq b_i \\ c^T \lambda_i = x \\ \lambda_i \geq 0, i = 1.., m \end{gathered}$$

Problem 3

5.19 *The sum of the largest elements of a vector.* Define $f : \mathbf{R}^n \rightarrow \mathbf{R}$ as

$$f(x) = \sum_{i=1}^{r} x_{[i]},$$

where r is an integer between 1 and n, and $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[r]}$ are the components of x sorted in decreasing order. In other words, $f(x)$ is the sum of the r largest elements of x. In this problem we study the constraint

$$f(x) \leq \alpha.$$

As we have seen in chapter 3, page 80, this is a convex constraint, and equivalent to a set of $n!/(r!(n-r)!)$ linear inequalities

$$x_{i_1} + \cdots + x_{i_r} \leq \alpha, \quad 1 \leq i_1 < i_2 < \cdots < i_r \leq n.$$

The purpose of this problem is to derive a more compact representation.

Problem 3 continued on next page. . .

(a) Given a vector $x \in \mathbf{R}^n$, show that $f(x)$ is equal to the optimal value of the LP

$$\begin{array}{ll} \text{maximize} & x^T y \\ \text{subject to} & 0 \preceq y \preceq \mathbf{1} \\ & \mathbf{1}^T y = r \end{array}$$

with $y \in \mathbf{R}^n$ as variable.

(b) Derive the dual of the LP in part (a). Show that it can be written as

$$\begin{array}{ll} \text{minimize} & rt + \mathbf{1}^T u \\ \text{subject to} & t\mathbf{1} + u \succeq x \\ & u \succeq 0, \end{array}$$

where the variables are $t \in \mathbf{R}$, $u \in \mathbf{R}^n$. By duality this LP has the same optimal value as the LP in (a), *i.e.*, $f(x)$. We therefore have the following result: x satisfies $f(x) \leq \alpha$ if and only if there exist $t \in \mathbf{R}$, $u \in \mathbf{R}^n$ such that

$$rt + \mathbf{1}^T u \leq \alpha, \qquad t\mathbf{1} + u \succeq x, \qquad u \succeq 0.$$

These conditions form a set of $2n+1$ linear inequalities in the $2n+1$ variables x, u, t.

(c) As an application, we consider an extension of the classical Markowitz portfolio optimization problem

$$\begin{array}{ll} \text{minimize} & x^T \Sigma x \\ \text{subject to} & \bar{p}^T x \geq r_{\min} \\ & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

discussed in chapter 4, page 155. The variable is the portfolio $x \in \mathbf{R}^n$; $\bar{p}$ and Σ are the mean and covariance matrix of the price change vector p.

Suppose we add a *diversification constraint*, requiring that no more than 80% of the total budget can be invested in any 10% of the assets. This constraint can be expressed as

$$\sum_{i=1}^{\lfloor 0.1n \rfloor} x_{[i]} \leq 0.8.$$

Formulate the portfolio optimization problem with diversification constraint as a QP.

Solution
(a).
intuitionally,$0 \leq y \leq 1$,a number is either positive or negative,we want a maximum and need $1^T y = r$,so for every large positive,the weight y_i mostly reach 1.finally we can choose r number which is constrained by $1^T y = r$

(b).

$$\begin{aligned} L(y,m,n,p) &= -x^T y = m^T y - n^T(1-y) + p(1^T y - r) \\ &= (-x^T - m^T + n^T + p1^T)y - n^T - pr \\ &= (-x - m + n - p1)^T y - n^T - pr \end{aligned}$$

therefore,dual function is:

$$\begin{aligned} inf_{y,m,n,p}\ L &= -n^T - pr, when\ -x - m + n + p1 = 0 \\ inf_{y,m,n,p}\ L &= -\infty, o.w \end{aligned}$$

dual problem is:

$$\begin{aligned} max\ & -n^T 1 - pr \\ s.t\ & -x - m + n + p1 = 0 \\ & m \succeq 0 \\ & n \succeq 0 \\ & p \succeq 0 \end{aligned}$$

$\rightarrow$

$$\begin{aligned} & min\ n^T 1 + pr \\ & S.t\ n + p1 \succeq x \\ & n \succeq 0 \\ & p \succeq 0 \end{aligned}$$

we eliminate m by noting that it acts as a slack variable in the first constraint.

$$\begin{aligned} & min 1^T u + rt \\ & u + t1 \succeq x \\ & u \succeq 0 \end{aligned}$$

(c).
from problem (b),we can know :

$$r = \lfloor \tfrac{n}{10} \rfloor$$

so we have the following:

$$\begin{aligned} & min x^T \Sigma x \\ & S.t\ \bar{p}^T \geq r_{min} \\ & 1^T x = 1, x \succeq 0 \\ & \lfloor \tfrac{n}{10} \rfloor t + 1^T u \preceq 0.8 \\ & \lambda 1 + u \succeq 0 \\ & u \succeq 0 \end{aligned}$$

Problem 4

13.9 Consider the following linear program:

$$\min -5x_1 - x_2 \quad \text{subject to}$$
$$x_1 + x_2 \le 5,$$
$$2x_1 + (1/2)x_2 \le 8,$$
$$x \ge 0.$$

(a) Add slack variables x_3 and x_4 to convert this problem to standard form.

(b) Using Procedure 13.1, solve this problem using the simplex method, showing at each step the basis and the vectors λ, s_N, and x_B, and the value of the objective function. (The initial choice of $\mathcal{B}$ for which $x_B \ge 0$ should be obvious once you have added the slacks in part (a).)

Procedure 13.1 (One Step of Simplex).

Given $\mathcal{B}, \mathcal{N}, x_B = B^{-1}b \ge 0, x_N = 0$;
Solve $B^T\lambda = c_B$ for λ,
Compute $s_N = c_N - N^T\lambda$; (* pricing *)
if $s_N \ge 0$
  stop; (* optimal point found *)
Select $q \in \mathcal{N}$ with $s_q < 0$ as the entering index;
Solve $Bd = A_q$ for d;
if $d \le 0$
  stop; (* problem is unbounded *)
Calculate $x_q^+ = \min_{i \mid d_i > 0} (x_B)_i / d_i$, and use p to denote the minimizing i;
Update $x_B^+ = x_B - dx_q^+$, $x_N^+ = (0, \ldots, 0, x_q^+, 0, \ldots, 0)^T$;
Change $\mathcal{B}$ by adding q and removing the basic variable corresponding to column p of B.

Solution
(a).
by add slack variant,the standard fomulation is as follows:

$$\begin{aligned} &min_x \ -5x_1 - x_2 \\ &S.t\ x_1 + x_2 + x_3 = 5 \\ &2x_1 + \tfrac{1}{2}x_2 + x_4 = 8 \\ &x \geq 0 \end{aligned}$$

(b).
I programmed a matlab program named "simplex.demo.m" ,from the script:

$$\lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, S_n = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, X_B = \begin{pmatrix} 5 \\ 8 \end{pmatrix}, optimal\ value\ = 0$$

after a iteration,we choose q=1:

$$\lambda = \begin{pmatrix} 0 \\ -2.5 \end{pmatrix}, S_n = \begin{pmatrix} 2.5 \\ 0.25 \end{pmatrix}, X_B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, optimal\ value\ = -20$$

and $S_n \geq 0$,so we stop and return optimal value : -20

Problem 5

14.6 In the long-step path-following method (Algorithm 14.2), give a procedure for calculating the maximum value of α such that (14.20) is satisfied.

Algorithm 14.2 (Long-Step Path-Following).
Given $\gamma, \sigma_{\min}, \sigma_{\max}$ with $\gamma \in (0, 1)$, $0 < \sigma_{\min} \leq \sigma_{\max} < 1$,
and $(x^0, \lambda^0, s^0) \in \mathcal{N}_{-\infty}(\gamma)$;
for $k = 0, 1, 2, \ldots$
Choose $\sigma_k \in [\sigma_{\min}, \sigma_{\max}]$;
Solve (14.10) to obtain $(\Delta x^k, \Delta \lambda^k, \Delta s^k)$;
Choose α_k as the largest value of α in $[0, 1]$ such that

$$(x^k(\alpha), \lambda^k(\alpha), s^k(\alpha)) \in \mathcal{N}_{-\infty}(\gamma); \tag{14.20}$$

Set $(x^{k+1}, \lambda^{k+1}, s^{k+1}) = (x^k(\alpha_k), \lambda^k(\alpha_k), s^k(\alpha_k))$;
end (for).

Solution
TODO