Convex Optimization Note

Tao Hu
Department of Computer Science
Peking University
No.5 Yiheyuan Road Haidian District, Beijing, P.R.China
taohu@pku.edu.cn

Abstract

convex optimization note based on the slides in SEEM5350: Numerical Optimization conducted by Shiqian Ma.

1 Proximal Gradient

Indicator function and norm

indicator of convex set C: conjugate is support function of C

$$f(x) = \begin{cases} 0 & \text{x} \in \text{C} \\ +\infty & \text{otherwise} \end{cases}$$

$f^*(y) = sup_x < x, y > -f(y)$, only when f(y) ∈ C can the conjugate function reach maximum: $sup_y < x, y >$

so this the end of the proof.

inversely,conjugate of support function of closed convex set is indicator function

$f(x) = S_C(x) = sup_{y \in C} x^T y \quad f^*(y) = I_C(y)$

proximal mapping of indicator function I_C is Euclidean projection on C

$$Prox_{I_c}(x) = argmin_{u \in C} ||u - x||_2^2 = P_C(X)$$

1.1 Conjugate function

1.2 Norms

conjugate of norm is indicator function of dual norm ball:

$$f(x) = ||x||, f^*(y) = I_B(y), (B = \{y|||y||_* \leq 1\})$$

Author Info: Taohu ,Peking University

proof: recall the definition of dual norm:
$||y||_* = sup_{||x||\leq 1} x^T y$
to evaluate $f^*(y) = sup_x(y^T x - ||x||)$,we distinguish two cases:

- if $||y||_* \leq 1$,then (by Cauchy-Schwarz inequality)
  $y^T x \leq ||x||, \forall x$
  and equality holds if x=0;therefore $sup_x(y^T x - ||x||) = 0$
- if $||y||_* > 1$,therefore exists an x with $||x|| \leq 1, x^T y > 1$;then
  $f^*(y) \geq y^T(tx) - ||tx|| = t(y^T x - ||x||)$
  and rhs goes to infinity if $t- > \infty$

prox-operator of norm:apply Moreau decomposition:

$$prox_{tf}(x) = x - tprox_{t^{-1}f^*}(\frac{x}{t}) = x - tP_B(\frac{x}{t}) = x - P_{tB}(x)$$

useful formula for $prox_{t||.||}$ when projection on $tB = \{x|||x|| \leq t\}$ is cheap.

for example, we can quickly calculate the $||.||_1, ||.||_2$'s prox-operator.

recall the projection of 2-norm:
Euclidian ball: $C = \{x|||x||_2 \leq t\}$
$P_C(x) = \frac{t}{||x||2} x \quad if \quad ||x||_2 > t$
$P_C(x) = x, \quad if \quad ||x||_2 \leq t$
therefore the proximal-operator of 2-norm is :

$$prox_{tf}(x) = \begin{cases} x - \frac{tx}{||x||_2} & ||x||_2 \leq t \\ 0 & \text{otherwise} \end{cases}$$

Acknowledgments

References