

Algorithms for Big Data Analysis : Homework #2

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Problem 1

5.11 Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^N \|A_i x + b_i\|_2 + (1/2) \|x - x_0\|_2^2.$$

The problem data are $A_i \in \mathbf{R}^{m_i \times n}$, $b_i \in \mathbf{R}^{m_i}$, and $x_0 \in \mathbf{R}^n$. First introduce new variables $y_i \in \mathbf{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.

5.11

Solution

The Lagrangian is

$$L(x, y, \lambda) = \sum_{i=1}^N \|y_i\|_2 + \frac{1}{2} \|x - x_0\|_2^2 + \sum_{i=1}^N \lambda_i^T (y_i - A_i x - b_i)$$

firstly, minimize x by deviation on x ,this yield:

$$x = x_0 + \sum_{i=1}^N A_i^T \lambda_i$$

which we use the following derivation principle:

$$\frac{\partial a^T x}{\partial x} = a$$

secondly,minimize over y_i :

$$\begin{aligned} \inf_{y_i} (\|y_i\|_2 + \lambda_i^T y_i) &= 0, \text{ if } \|\lambda_i\|_2 \leq 1 \\ \inf_{y_i} (\|y_i\|_2 + \lambda_i^T y_i) &= -\infty, \text{ o.w} \end{aligned}$$

because when $\|\lambda_i\|_2 \geq 1$,where always exist a y_i ,let the inf unbounded to $-\infty$

so totally,we can get :

$$\begin{aligned} y_i &= 0, x = x_0 + \sum_{i=1}^N A_i^T \lambda_i \\ L(x, \lambda) &= \frac{1}{2} \|\sum_{i=1}^N A_i^T \lambda_i\|_2^2 + \sum_{i=1}^N \lambda_i^T [-A_i(x_0 + \sum_{i=1}^N A_i^T \lambda_i) - b_i] \\ &= \frac{1}{2} \|\sum_{i=1}^N A_i^T \lambda_i\|_2^2 + \sum_{i=1}^N \lambda_i^T A_i (\sum_{i=1}^N A_i^T \lambda_i) - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i) \\ &= \frac{1}{2} \|\sum_{i=1}^N A_i^T \lambda_i\|_2^2 + (\sum_{i=1}^N A_i^T \lambda_i)^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i) \\ &= \frac{1}{2} \|\sum_{i=1}^N A_i^T \lambda_i\|_2^2 + \|\sum_{i=1}^N A_i^T \lambda_i\|_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i) \\ &= -\frac{1}{2} \|\sum_{i=1}^N A_i^T \lambda_i\|_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i) \end{aligned}$$

so,the dual form is :

$$\begin{aligned} \max_{\lambda_i} \quad & -\frac{1}{2} \|\sum_{i=1}^N A_i^T \lambda_i\|_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x_0 + b_i) \\ \text{S.t} \quad & \|\lambda_i\|_2 \leq 1, i = 1, 2, \dots, n \end{aligned}$$

Problem 2

5.17 Robust linear programming with polyhedral uncertainty. Consider the robust LP

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && \sup_{a \in \mathcal{P}_i} a^T x \leq b_i, \quad i = 1, \dots, m, \end{aligned}$$

with variable $x \in \mathbf{R}^n$, where $\mathcal{P}_i = \{a \mid C_i a \preceq d_i\}$. The problem data are $c \in \mathbf{R}^n$, $C_i \in \mathbf{R}^{m_i \times n}$, $d_i \in \mathbf{R}^{m_i}$, and $b \in \mathbf{R}^m$. We assume the polyhedra \mathcal{P}_i are nonempty. Show that this problem is equivalent to the LP

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && d_i^T z_i \leq b_i, \quad i = 1, \dots, m \\ & && C_i^T z_i = x, \quad i = 1, \dots, m \\ & && z_i \succeq 0, \quad i = 1, \dots, m \end{aligned}$$

with variables $x \in \mathbf{R}^n$ and $z_i \in \mathbf{R}^{m_i}$, $i = 1, \dots, m$. *Hint.* Find the dual of the problem of maximizing $a_i^T x$ over $a_i \in \mathcal{P}_i$ (with variable a_i).

5.19

Solution

the original problem can be fomulated as :

$$\begin{aligned} & \min_x c^T x \\ & \text{S.t. } f_i(x) \leq b_i, i = 1, \dots, m \end{aligned}$$

we can define $f_i(x)$ as the optimal value of the following LP:

$$\begin{aligned} & \max_a x^T a \\ & \text{S.t. } c_i a \leq d_i \end{aligned}$$

Lagrangian of $f_i(x)$ is:

$$L(x, \lambda) = x^T a + \lambda^T (c_i a - d_i) = (x^T + \lambda^T c_i) a - \lambda^T d_i$$

$$\inf_a L(x, \lambda) = -\lambda^T d_i, \text{ when } x^T + \lambda^T c_i = 0$$

$$\inf_a L(x, \lambda) = -\infty, \text{ o.w}$$

the dual problem is:

$$\begin{aligned} & \max_{\lambda} -\lambda^T d_i \\ & \text{S.t. } x^T + \lambda^T c_i = 0 \\ & \lambda \geq 0 \end{aligned}$$

→

$$\begin{aligned} & \min_{\lambda} \lambda^T d_i \\ & \text{S.t. } x^T + \lambda^T c_i = 0 \\ & \lambda \geq 0 \end{aligned}$$

the optimal value of this LP also equals to $f_i(x)$, so the final LP can be written as:

$$\begin{aligned}
& \min_x \quad c^T x \\
& \text{S.t.} \quad d^T \lambda_i \leq b_i \\
& \quad \quad c^T \lambda_i = x \\
& \quad \quad \lambda_i \geq 0, i = 1, \dots, m
\end{aligned}$$

Problem 3

5.19 *The sum of the largest elements of a vector.* Define $f : \mathbf{R}^n \rightarrow \mathbf{R}$ as

$$f(x) = \sum_{i=1}^r x_{[i]},$$

where r is an integer between 1 and n , and $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[r]}$ are the components of x sorted in decreasing order. In other words, $f(x)$ is the sum of the r largest elements of x . In this problem we study the constraint

$$f(x) \leq \alpha.$$

As we have seen in chapter 3, page 80, this is a convex constraint, and equivalent to a set of $n!/(r!(n-r)!)$ linear inequalities

$$x_{i_1} + \dots + x_{i_r} \leq \alpha, \quad 1 \leq i_1 < i_2 < \dots < i_r \leq n.$$

The purpose of this problem is to derive a more compact representation.

- (a) Given a vector $x \in \mathbf{R}^n$, show that $f(x)$ is equal to the optimal value of the LP

$$\begin{aligned} & \text{maximize} && x^T y \\ & \text{subject to} && 0 \preceq y \preceq \mathbf{1} \\ & && \mathbf{1}^T y = r \end{aligned}$$

with $y \in \mathbf{R}^n$ as variable.

- (b) Derive the dual of the LP in part (a). Show that it can be written as

$$\begin{aligned} & \text{minimize} && rt + \mathbf{1}^T u \\ & \text{subject to} && t\mathbf{1} + u \succeq x \\ & && u \succeq 0, \end{aligned}$$

where the variables are $t \in \mathbf{R}$, $u \in \mathbf{R}^n$. By duality this LP has the same optimal value as the LP in (a), *i.e.*, $f(x)$. We therefore have the following result: x satisfies $f(x) \leq \alpha$ if and only if there exist $t \in \mathbf{R}$, $u \in \mathbf{R}^n$ such that

$$rt + \mathbf{1}^T u \leq \alpha, \quad t\mathbf{1} + u \succeq x, \quad u \succeq 0.$$

These conditions form a set of $2n+1$ linear inequalities in the $2n+1$ variables x, u, t .

- (c) As an application, we consider an extension of the classical Markowitz portfolio optimization problem

$$\begin{aligned} & \text{minimize} && x^T \Sigma x \\ & \text{subject to} && \bar{p}^T x \geq r_{\min} \\ & && \mathbf{1}^T x = 1, \quad x \succeq 0 \end{aligned}$$

discussed in chapter 4, page 155. The variable is the portfolio $x \in \mathbf{R}^n$; \bar{p} and Σ are the mean and covariance matrix of the price change vector p .

Suppose we add a *diversification constraint*, requiring that no more than 80% of the total budget can be invested in any 10% of the assets. This constraint can be expressed as

$$\sum_{i=1}^{\lfloor 0.1n \rfloor} x_{[i]} \leq 0.8.$$

Formulate the portfolio optimization problem with diversification constraint as a QP.

Solution

(a).

intuitively, $0 \leq y \leq 1$, a number is either positive or negative, we want a maximum and need $\mathbf{1}^T y = r$, so for every large positive, the weight y_i mostly reach 1. finally we can choose r number which is constrained by $\mathbf{1}^T y = r$

(b).

$$\begin{aligned}
L(y, m, n, p) &= -x^T y = m^T y - n^T(1 - y) + p(1^T y - r) \\
&= (-x^T - m^T + n^T + p1^T)y - n^T - pr \\
&= (-x - m + n - p1)^T y - n^T - pr
\end{aligned}$$

therefore, dual function is:

$$\begin{aligned}
\inf_{y, m, n, p} L &= -n^T - pr, \text{ when } -x - m + n + p1 = 0 \\
\inf_{y, m, n, p} L &= -\infty, \text{ o.w}
\end{aligned}$$

dual problem is:

$$\begin{aligned}
\max \quad & -n^T 1 - pr \\
\text{s.t.} \quad & -x - m + n + p1 = 0 \\
& m \succeq 0 \\
& n \succeq 0 \\
& p \succeq 0
\end{aligned}$$

→

$$\begin{aligned}
\min \quad & n^T 1 + pr \\
\text{S.t.} \quad & n + p1 \succeq x \\
& n \succeq 0 \\
& p \succeq 0
\end{aligned}$$

we eliminate m by noting that it acts as a slack variable in the first constraint.

$$\begin{aligned}
\min \quad & 1^T u + rt \\
& u + t1 \succeq x \\
& u \succeq 0
\end{aligned}$$

(c).

from problem (b), we can know :

$$r = \lfloor \frac{n}{10} \rfloor$$

so we have the following:

$$\begin{aligned}
\min \quad & x^T \Sigma x \\
\text{S.t.} \quad & \bar{p}^T \geq r_{\min} \\
& 1^T x = 1, x \succeq 0 \\
& \lfloor \frac{n}{10} \rfloor t + 1^T u \leq 0.8 \\
& \lambda 1 + u \succeq 0 \\
& u \succeq 0
\end{aligned}$$

Problem 4



13.9 Consider the following linear program:

$$\begin{aligned} \min \quad & -5x_1 - x_2 \quad \text{subject to} \\ & x_1 + x_2 \leq 5, \\ & 2x_1 + (1/2)x_2 \leq 8, \\ & x \geq 0. \end{aligned}$$

- (a) Add slack variables x_3 and x_4 to convert this problem to standard form.
- (b) Using Procedure 13.1, solve this problem using the simplex method, showing at each step the basis and the vectors λ , s_N , and x_B , and the value of the objective function. (The initial choice of \mathcal{B} for which $x_B \geq 0$ should be obvious once you have added the slacks in part (a).)

Procedure 13.1 (One Step of Simplex).

Given $\mathcal{B}, \mathcal{N}, x_B = B^{-1}b \geq 0, x_N = 0$;

Solve $B^T \lambda = c_B$ for λ ,

Compute $s_N = c_N - N^T \lambda$; (* pricing *)

if $s_N \geq 0$

stop; (* optimal point found *)

Select $q \in \mathcal{N}$ with $s_q < 0$ as the entering index;

Solve $Bd = A_q$ for d ;

if $d \leq 0$

stop; (* problem is unbounded *)

Calculate $x_q^+ = \min_{i \mid d_i > 0} (x_B)_i / d_i$, and use p to denote the minimizing i ;

Update $x_B^+ = x_B - dx_q^+, x_N^+ = (0, \dots, 0, x_q^+, 0, \dots, 0)^T$;

Change \mathcal{B} by adding q and removing the basic variable corresponding to column p of B .

Solution

(a).

by add slack variant, the standard fomulation is as follows:

$$\begin{aligned}
& \min_x -5x_1 - x_2 \\
& \text{s.t. } x_1 + x_2 + x_3 = 5 \\
& 2x_1 + \frac{1}{2}x_2 + x_4 = 8 \\
& x \geq 0
\end{aligned}$$

(b).

I programmed a matlab program named "simplex.demo.m", from the script:


$$\lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, S_n = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, X_B = \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \text{optimal value} = 0$$

after a iteration, we choose $q=1$:

$$\lambda = \begin{pmatrix} 0 \\ -2.5 \end{pmatrix}, S_n = \begin{pmatrix} 2.5 \\ 0.25 \end{pmatrix}, X_B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \text{optimal value} = -20$$

and $S_n \geq 0$, so we stop and return optimal value : -20

Problem 5

 **14.6** In the long-step path-following method (Algorithm 14.2), give a procedure for calculating the maximum value of α such that (14.20) is satisfied.

Algorithm 14.2 (Long-Step Path-Following).

Given $\gamma, \sigma_{\min}, \sigma_{\max}$ with $\gamma \in (0, 1), 0 < \sigma_{\min} \leq \sigma_{\max} < 1$,
and $(x^0, \lambda^0, s^0) \in \mathcal{N}_{-\infty}(\gamma)$;

for $k = 0, 1, 2, \dots$

 Choose $\sigma_k \in [\sigma_{\min}, \sigma_{\max}]$;

 Solve (14.10) to obtain $(\Delta x^k, \Delta \lambda^k, \Delta s^k)$;

 Choose α_k as the largest value of α in $[0, 1]$ such that

$$(x^k(\alpha), \lambda^k(\alpha), s^k(\alpha)) \in \mathcal{N}_{-\infty}(\gamma); \tag{14.20}$$

 Set $(x^{k+1}, \lambda^{k+1}, s^{k+1}) = (x^k(\alpha_k), \lambda^k(\alpha_k), s^k(\alpha_k))$;

end (for).

Solution

TODO