§2.1不定积分的计算
——换元积分法

一.第一类换元法——凑微分法

二.第二类换元法——变量代换法

一.第一类换元法——凑微分法

问题 $\int \cos 2x dx \overset{?}{=} \sin 2x + C,$

解决方法 利用复合函数,设置中间变量

处理过程 令 $t = 2x \Rightarrow dx = \frac{1}{2}dt,$

$$\int \cos 2x dx = \frac{1}{2}\int \cos(2x)2dx = \frac{1}{2}\int \cos(2x)d(2x)$$

$$= \frac{1}{2}\int \cos t dt = \frac{1}{2}\sin t + C = \frac{1}{2}\sin 2x + C.$$

在一般情形下：

设$F'(u)=f(u)$,则$\int f(u)du=F(u)+C.$

如果$u=\varphi(x)$可微,

$\because dF[\varphi(x)]=f[\varphi(x)]\varphi'(x)dx,$

$\therefore \int f[\varphi(x)]\varphi'(x)dx=F[\varphi(x)]+C$

$=\left[\int f(u)du\right]_{u=\varphi(x)}$

由此可得换元积分法定理

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定理1.设$f(u)$有原函数,$u=\varphi(x)$有连续的导数,则有积分换元公式

$$\int f[\varphi(x)]\varphi'(x)dx=\left[\int f(u)du\right]_{u=\varphi(x)}$$

通常称此法为第一类换元积分法(凑微分法).

说明:使用此公式的关键在于将$\int g(x)dx$ 变化为$\int f[\varphi(x)]\varphi'(x)dx$.

所以,凑微分法难就难在这第一步.

$$F'(u) = f(u),$$

$$\int f(u)du = F(u) + C.$$

$\varphi'(x)dx = du$,凑微分

$$\therefore \int f[\varphi(x)]\varphi'(x)dx = \int f(u)du,$$

$$= F(u) + C = F[\varphi(x)] + C$$

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例1.求积分$\int \sin 2x dx$.

解1.$\int \sin(2x)dx = \frac{1}{2}\int \sin(2x)d(2x)$

$= -\frac{1}{2}\cos 2x + C$;

解2.$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C_1$;

解3.$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C_2$.

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求积分$\int \sin 2x dx$.

解1.$\int \sin(2x)dx = \frac{1}{2}\int \sin(2x)d(2x)$

$= -\frac{1}{2}\cos 2x + C$;

解1适用的面更广,因而解1比解2解3更有价值.如

$$\int \sin(\pi x)dx = \frac{1}{\pi}\int \sin(\pi x)d(\pi x)$$

$$= -\frac{1}{\pi}\cos(\pi x) + C$$

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例2.求积分$\int \frac{1}{3+2x}dx.$

解　$\because (3+2x)'=2, \therefore 2dx = d(3+2x)$

$$\therefore \int \frac{1}{3+2x}dx = \frac{1}{2}\int \frac{1}{3+2x}\cdot(3+2x)'dx$$

$$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|3+2x| + C.$$

一般地$\int f(ax+b)dx = \frac{1}{a}\left[\int f(u)du\right]$

$$u = ax+b, (a \neq 0)$$

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例2.(2).求积分$\int \frac{1}{x(1+2\ln x)}dx.$

解 $\int \frac{1}{x(1+2\ln x)}dx = \int \frac{1}{1+2\ln x} d(\ln x)$

$= \frac{1}{2}\int \frac{1}{1+2\ln x} d(1+2\ln x)$

$\downarrow u = 1+2\ln x$

$= \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+2\ln x| + C.$

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例2.(3).求积分$\int \frac{x}{(1+x)^3}dx$.

解 $\int \frac{x}{(1+x)^3}dx = \int \frac{x+1-1}{(1+x)^3}dx$

$$= \int \left[\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3} \right] d(1+x)$$

$$= \frac{1}{2(1+x)^2} - \frac{1}{1+x} + C$$

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例3.求积分$\int \frac{1}{a^2+x^2}dx, a>0.$

解 $\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$

$$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a} + C.$$

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例3.(2).求积分$\int \frac{1}{\sqrt{a^2-x^2}}dx, a>0.$

解 $\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}d\left(\frac{x}{a}\right)$

$= \arcsin\frac{x}{a} + C.$

例4.求积分$\int \frac{1}{x^2-6x+9}dx$.

解$\int \frac{1}{x^2-6x+9}dx = \int \frac{1}{(x-3)^2}dx$

$= \int \frac{1}{(x-3)^2}d(x-3)$

$= -\frac{1}{x-3}+C = \frac{1}{3-x}+C.$

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例4.(2).求积分$\int \frac{1}{x^2-8x-9}dx=?$

$$\int \frac{1}{x^2-8x-9}dx=\int \frac{1}{(x-9)(x+1)}dx$$

$$=\frac{1}{10}\int\left(\frac{1}{x-9}-\frac{1}{x+1}\right)dx$$

$$=\frac{1}{10}\int\frac{d(x-9)}{x-9}-\frac{1}{10}\int\frac{d(x+1)}{x+1}$$

$$=\frac{1}{10}\ln\left|\frac{x-9}{x+1}\right|+C.$$

例4.(3).求积分$\int \frac{1}{x^2-8x+25}dx$.

解$\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$

$$= \frac{1}{3^2}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}dx = \frac{1}{3}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}d\left(\frac{x-4}{3}\right)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

例4.(4).求积分$\int \frac{x^3-5x^2+3}{x^2-8x+25}dx.$

解 $\frac{x^3-5x^2+3}{x^2-8x+25}=\frac{x^3-8x^2+25x+3x^2-24x+75-x-72}{x^2-8x+25}$

$=x+3-\frac{x+72}{x^2-8x+25}$ $\quad (x^2-8x+25)'=2x-8,$

$\therefore \int \frac{x+72}{x^2-8x+25}dx=\frac{1}{2}\int \frac{2x-8+152}{x^2-8x+25}dx$

$=\frac{1}{2}\int \frac{(x^2-8x+25)'}{x^2-8x+25}dx+\int \frac{76}{x^2-8x+25}dx$

$\int \frac{(x^2-8x+25)'}{x^2-8x+25}dx=\int \frac{d(x^2-8x+25)}{x^2-8x+25}=\ln(x^2-8x+25)+C$

… …

凑微分法

$$\int f[\varphi(x)]\varphi'(x)dx$$

$$=\int f[\varphi(x)]d\varphi(x)dx=\left[\int f(u)du\right]_{u=\varphi(x)}$$

难就难在这第一步

$$\int g(x)dx=\int f[\varphi(x)]\varphi'(x)dx.$$

用凑微分法进行不定积分的计算,一般并没有什么普遍的规律.我们需要通过一定的训练,在无规律中寻找规律.

$$\int g(x)dx = \int f[\varphi(x)]\varphi'(x)dx$$

$$= \int f[\varphi(x)]d\varphi(x)dx = \left[\int f(u)du\right]_{u=\varphi(x)}$$

凑微分法难在第一步——如何把被积分函数$g(x)$分拆成 $f[\varphi(x)]\varphi'(x)$.

其基础是基于我们对函数的复合过程了然于胸,基于对复合函数的求导过程十分熟悉,基于我们对不定积分的第一批基本公式的熟稔.

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用凑微分法进行不定积分的计算,

是在无规律中寻找规律.

若$\int f(u)du = F(u) + C$,则

$$\int \frac{1}{\sqrt{x}} f\left(\sqrt{x}\right)dx = 2\int f\left(\sqrt{x}\right)d\left(\sqrt{x}\right) ,$$

$$\int x^{\mu-1} f\left(x^{\mu}\right)dx = \frac{1}{\mu}\int f\left(x^{\mu}\right)d\left(x^{\mu}\right), \mu \neq 0 ;$$

$$\int \frac{1}{x} f(\ln x)dx = \int f(\ln x)d(\ln x) ;$$

$$\int f(ax+b)dx \overset{a\neq 0}{=\!=} \frac{1}{a}\int f(ax+b)d(ax+b) ;$$

用凑微分法进行不定积分的计算,

在无规律中寻找规律.

若 $\int f(u)du = F(u)+C$,则

$$\int f(e^x)e^x dx = \int f\left(e^x\right)d\left(e^x\right) \;;$$

$$\int \frac{f(\arctan x)}{1+x^2}dx = \int f(\arctan x)d(\arctan x) \;;$$

$$\int f(\sin x)\cos x dx = \int f(\sin x)d(\sin x) \;;$$

$\cdots \quad \cdots$

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例5.求积分$\int \frac{1}{1+e^x}dx$.

解 $\int \frac{1}{1+e^x}dx = \int \frac{e^x}{e^x\left(1+e^x\right)}dx$

$$\overset{e^x=u}{=\!=} \int \frac{du}{u(1+u)} = \int\left(\frac{1}{u} - \frac{1}{1+u}\right)du$$

$$= \ln\frac{u}{1+u} + C = \ln\frac{e^x}{1+e^x} + C$$

$$= x - \ln\left(1+e^x\right) + C$$

解二 $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$

$$= \int \left(1 - \frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d\left(1+e^x\right)$$

$$= x - \ln\left(1+e^x\right) + C.$$

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解三

$$\int \frac{1}{1+e^{x}}dx = \int \frac{e^{-x}}{1+e^{-x}}dx$$

$$= -\int \frac{\left(1+e^{-x}\right)'}{1+e^{-x}}dx = -\ln\left(1+e^{-x}\right)+C$$

经过分析,可以知道两种表面上不同的结果其实是完全一样的。

例5.(2).求积分$J=\int \frac{1}{1+e^{2x}}dx.$

解 $J=\frac{1}{2}\int \frac{1}{1+e^{2x}}d(2x) \overset{2x=u}{=\!=\!=} \frac{1}{2}\int \frac{1}{1+e^{u}}du$

$$=\frac{1}{2}\int\left(1-\frac{e^{u}}{1+e^{u}}\right)du=\frac{1}{2}u-\frac{1}{2}\int\frac{d\left(1+e^{u}\right)}{1+e^{u}}$$

$$=\frac{1}{2}u-\frac{1}{2}\ln\left(1+e^{u}\right)+C$$

$$\overset{u=2x}{=\!=\!=} x-\ln\sqrt{1+e^{2x}}+C.$$

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练习1.计算不定积分

(1).$\int \frac{e^{2x}-2e^{x}}{4e^{2x}+1}dx$;

(2)*.$\int \frac{1-2e^{x}}{7e^{2x}+4e^{x}+1}dx$.

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例6.求积分$\int \frac{1}{1+\cos x}dx$.

解$\int \frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx$

$= \int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}dx$

$= \int \frac{1}{\sin^2 x}dx - \int \frac{1}{\sin^2 x}d(\sin x)$

$= -\cot x + \csc x + C$

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例6.求积分$\int \frac{1}{1+\cos x}dx$.

解 $$\int \frac{1}{1+\cos x}dx = \int \frac{1}{2\cos^2 \frac{x}{2}}dx$$

$$= \int \sec^2\left(\frac{x}{2}\right)d\left(\frac{x}{2}\right) = \tan\frac{x}{2} + C$$

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例6.求积分$\int \frac{1}{1+\cos 2x} dx$.

解 $\int \frac{1}{1+\cos 2x} dx = \int \frac{1}{2\cos^2 x} dx$

$= \frac{1}{2}\int \sec^2 x dx = \frac{1}{2}\tan x + C$

例6.(2).求积分$\int \frac{1}{2+\cos 2x}dx$.

解$\int \frac{1}{2+\cos 2x}dx = \int \frac{1}{1+2\cos^2 x}dx$

$$= \int \frac{\sec^2 x}{2+\sec^2 x}dx = \int \frac{1}{3+\tan^2 x}d(\tan x)$$

$$= \frac{1}{\sqrt{3}}\arctan \frac{\tan x}{\sqrt{3}} + C$$

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例6.(3).求积分$\int \frac{1}{2+3\cos x}dx$.

解$\int \frac{1}{2+3\cos x}dx = \int \frac{1}{6\cos^2\frac{x}{2}-1}dx$

$$\overset{\frac{x}{2}=t}{=\!=} 2\int \frac{1}{6\cos^2 t-1}dt = 2\int \frac{\sec^2 t}{6-\sec^2 t}dt$$

$$= 2\int \frac{1}{5-\tan^2 t}d(\tan t) \overset{\tan t=u}{=\!=\!=} 2\int \frac{1}{5-u^2}du$$

$$= \frac{1}{\sqrt{5}}\int\left(\frac{1}{\sqrt{5}-u}+\frac{1}{\sqrt{5}+u}\right)du$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+u}{\sqrt{5}-u}\right|+C = \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+\tan\frac{x}{2}}{\sqrt{5}-\tan\frac{x}{2}}\right|+C$$

例7.求积分$\int \sin^2 x dx$.

解 $\because \cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1 = 1 - 2\sin^2 x$,

$$\therefore \sin^2 x = \frac{1-\cos 2x}{2},$$

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

例7.(2).求积分$\int \cos^4 x dx$.

解 $\int \cos^4 x dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x) dx$

$= \frac{1}{4}\int \left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right) dx$

$= \frac{1}{4}\left(\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right)+C$

例7.(3).求积分$\int \cos^3 x dx$.

解

$$\int \cos^3 x dx = \int \cos^2 x \cdot \underline{\cos x dx}$$

$$= \int \left(1 - \sin^2 x\right) d\left(\sin x\right)$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

例7.(4).求积分$\int \sin^2 x\cos^5 x dx$

解$\int \sin^2 x\cos^5 x dx = \int \sin^2 x\cos^4 x \cdot \underline{\cos x dx}$

$= \int \sin^2 x\cos^4 x d(\sin x)$

$= \int \sin^2 x(1-\sin^2 x)^2 d(\sin x)$

$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$

总结经验 当被积函数是正\余弦函数相乘时, 拆开奇次项去凑微分.

例7.(5).求积分$J=\int\cos 3x\cos 2x dx$

解　由$\cos 2\alpha = 2\cos^2\alpha - 1 \Rightarrow$

$$\cos 3\alpha = \cos\left(2\alpha + \alpha\right) = \cdots$$

$$= 4\cos^3\alpha - 3\cos\alpha$$

$$J = \int\left(8\cos^5 x + \cdots\right)dx = \cdots$$

倘若这样做虽能进行到底,但比较麻烦.

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例7.(5).求积分$\int \cos 3x \cos 2x dx$.

解 $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

$$\Rightarrow \cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\therefore \int \cos 3x \cos 2x dx = \frac{1}{2}\int(\cos x + \cos 5x)dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

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练习2.计算不定积分

(1).$\int \cos^6 x dx$;

(2).$\int \sin^3 x \cos^3 x dx$;

(3).$\int \cos^2 x \sin^4 x dx$;

(4).$\int \sin^3 x \cos^4 x dx$.

一般地,我们会更多地根据以下三组公式来处理三角函数的有理式的不定积分问题:

$(1)\cos^2\alpha+\sin^2\alpha=1,(\sin x)'=\cos x,$

$(\cos x)'=-\sin x,\int\cos x dx=\sin x+C;$

$(2)\sec^2 x=1+\tan^2 x,(\tan x)'=\sec^2 x,$

$(\sec x)'=\sec x\tan x,\int\sec^2 x dx=\tan x+C;$

$(3)\csc^2 x=1+\cot^2 x,(\cot x)'=-\csc^2 x,$

$(\csc x)'=-\csc x\cot x.$

例8.求积分$\int \frac{1}{\sin^4 x} dx.$

解 $\int \frac{1}{\sin^4 x} dx = \int \csc^4 x dx$

$$= \int \csc^2 x \cdot \underline{\csc^2 x dx}$$

$$= -\int \left(1+\cot^2 x\right) d\left(\cot x\right),$$

$$\overset{\cot x = u}{=\!=\!=} -\int \left(1+u^2\right) du = -u - \frac{1}{3}u^3 + C$$

$$\overset{u = \cot x}{=\!=\!=} -\cot x - \frac{1}{3}\cot^3 x + C$$

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解二 $\displaystyle\int \frac{1}{\sin^4 x}dx = \int \frac{\sec^4 x}{\tan^4 x}dx$

$$= \int \frac{\sec^2 x}{\tan^4 x}\sec^2 x dx = \int \frac{1+\tan^2 x}{\tan^4 x}d(\tan x)$$

$$= -\frac{1}{3}\frac{1}{\tan^3 x} - \frac{1}{\tan x} + C$$

解一 $\displaystyle\int \frac{1}{\sin^4 x}dx = -\cot x - \frac{1}{3}\cot^3 x + C$

两个做法结果一模一样.

例8.(2).求积分$\int \frac{1}{\sin^2 x \cos^4 x} dx.$

解 $\int \frac{1}{\sin^2 x \cos^4 x} dx = \int \frac{\sec^6 x}{\tan^2 x} dx$

$$= \int \frac{\left(\sec^2 x\right)^2}{\tan^2 x} \sec^2 x dx = \int \frac{\left(1 + \tan^2 x\right)^2}{\tan^2 x} d\left(\tan x\right)$$

$$d\left(\tan x\right) = (\tan x)' dx = \sec^2 x dx$$

$$\overset{\tan x = u}{=\!=\!=} \int \frac{\left(1 + u^2\right)^2}{u^2} du = \int \left(\frac{1}{u^2} + 2 + u^2\right) du = \cdots$$

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$$J=\int\frac{1}{\sin^2 x\cos^4 x}dx=\int\frac{\sec^6 x}{\tan^2 x}dx$$

$$=\int\frac{\left(\sec^2 x\right)^2}{\tan^2 x}\sec^2 x dx=\int\frac{\left(1+\tan^2 x\right)^2}{\tan^2 x}d\left(\tan x\right)$$

$$\overset{\tan x=u}{=\!=\!=}\int\frac{\left(1+u^2\right)^2}{u^2}du=\int\left(\frac{1}{u^2}+2+u^2\right)du=\cdots$$

或者$J=\int\frac{1}{\sin^2 x\cos^4 x}dx=\int\frac{\csc^6 x}{\cot^4 x}dx$

$$=\int\frac{\csc^4 x}{\cot^4 x}\csc^2 x dx=-\int\frac{\left(1+\cot^2 x\right)^2}{\cot^4 x}d\left(\cot x\right)$$

两个做法完全等效.

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例8.(3).求积分$\int \frac{1}{\sin^3 x \cos^5 x} dx.$

解 原式$= \int \frac{1}{\tan^3 x \cdot \cos^8 x} dx = \int \frac{\sec^6 x}{\tan^3 x} \sec^2 x dx$

$= \int \frac{\left(\sec^2 x\right)^3}{\tan^3 x} \sec^2 x dx = \int \frac{\left(1+\tan^2 x\right)^3}{\tan^3 x} d\tan x$,

$d\tan x = (\tan x)' dx = \sec^2 x dx$

$= \int \frac{\left(1+u^2\right)^3}{u^3} du = \cdots$

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一般地,我们会更多地根据以下三组公式来处理三角函数的有理式的不定积分问题:

$$(1)\cos^2\alpha+\sin^2\alpha=1,(\sin x)'=\cos x,$$

$$(\cos x)'=-\sin x,\int\cos x dx=\sin x+C;$$

$$(2)\sec^2 x=1+\tan^2 x,(\tan x)'=\sec^2 x,$$

$$(\sec x)'=\sec x\tan x,\int\sec^2 x dx=\tan x+C;$$

$$(3)\csc^2 x=1+\cot^2 x,(\cot x)'=-\csc^2 x,$$

$$(\csc x)'=-\csc x\cot x.$$

练习3.计算不定积分

(1).$\int \sec^4 x dx$ ；

(2).$\int \sec x \cdot \tan^3 x dx$ ；

(3).$\int \sin^{-3} x \cdot \cos^{-3} x dx$.

例9.求积分$\int \sec x dx$.

解 $\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx$

$= \int \frac{1}{1-\sin^2 x} d(\sin x) \overset{u=\sin x}{=\!=\!=}$

$= \int \frac{1}{1-u^2} du = \frac{1}{2} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$

$= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C = \frac{1}{2} \ln \left| \frac{1+\sin x}{1-\sin x} \right| + C$

$= \frac{1}{2} \ln \frac{(1+\sin x)^2}{1-\sin^2 x} + C = \ln \left| \frac{1+\sin x}{\cos x} \right| + C$

$= \ln |\sec x + \tan x| + C.$

解二 $\int \sec x dx = \int \frac{1}{\cos x} dx$

$$= \int \frac{1}{2\cos^2 \frac{x}{2} - 1} dx = 2\int \frac{\sec^2 \frac{x}{2}}{2 - \sec^2 \frac{x}{2}} d\left(\frac{x}{2}\right)$$

$$= 2\int \frac{1}{1 - \tan^2 \frac{x}{2}} d\left(\tan \frac{x}{2}\right) = \ln\left|\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right| + C$$

$$= \ln\left|\frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}}\right| + C = \ln\left|\frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}\right| + C$$

$$= \ln\left|\frac{1 + \sin x}{\cos x}\right| + C = \ln\left|\sec x + \tan x\right| + C.$$

法三 $\int \sec x dx = \int \frac{(\sec x + \tan x)\sec x}{\sec x + \tan x} dx$

$$= \int \frac{(\sec x + \tan x)'}{\sec x + \tan x} dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln|\sec x + \tan x| + C.$$

此解法妙则妙矣,但过于巧妙,非常人易想到,就方法的角度而言,于我们价值不大.

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例9.(2).求积分$\int \csc x dx$.

解 $\int \csc x dx = \int \frac{1}{\sin x} dx$

$$= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right)$$

$$= \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right) = \ln\left|\tan\frac{x}{2}\right| + C$$

(使用了三角函数恒等变形)

$$= \ln\left|\csc x - \cot x\right| + C.$$

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解二 $\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \overset{u=\cos x}{=\!=\!=}$

$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$

$= \frac{1}{2} \ln \left| \frac{1-u}{1+u} \right| + C = \frac{1}{2} \ln \left| \frac{1-\cos x}{1+\cos x} \right| + C$

$= \frac{1}{2} \ln \frac{(1-\cos x)^2}{1-\cos^2 x} + C = \ln \left| \frac{1-\cos x}{\sin x} \right| + C$

$= \ln \left| \csc x - \cot x \right| + C.$

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Addendum. 三角函数公式系列

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

先考虑特殊情形$0<\alpha,\beta,\alpha+\beta<\dfrac{\pi}{2}$

利用三角形面积计算的方法：

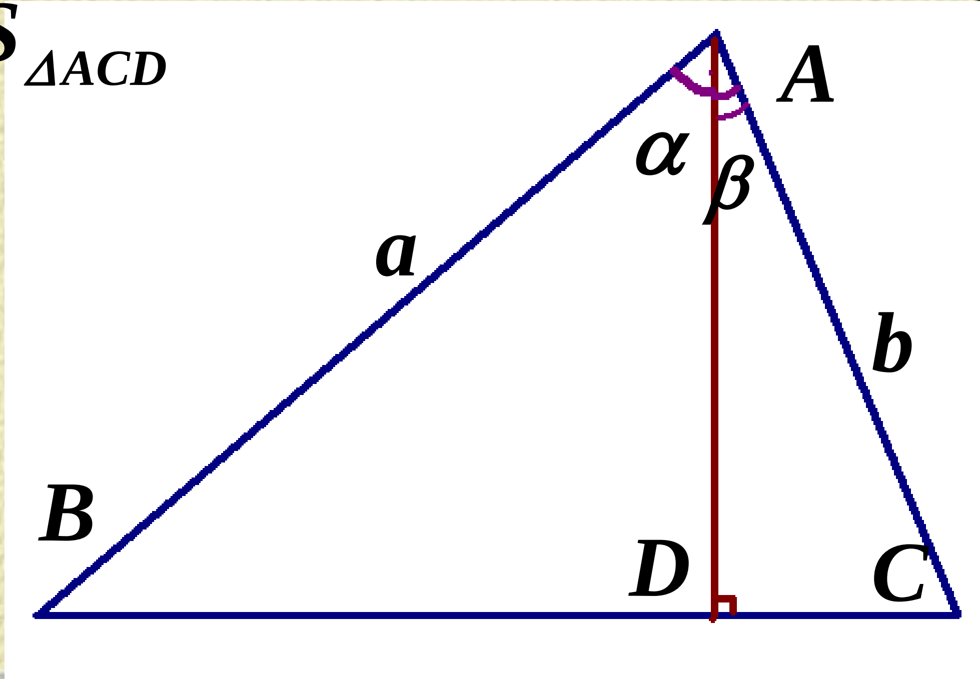
$$2S_{\Delta ABC}=2S_{\Delta ABD}+2S_{\Delta ACD}$$

$$ab\sin(\alpha+\beta)=$$

$$AD\cdot BD+AD\cdot CD$$

$$=b\cos\beta\cdot a\sin\alpha$$

$$+a\cos\alpha\cdot b\sin\beta$$



$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

先考虑特殊情形$0<\alpha,\beta,\alpha+\beta<\dfrac{\pi}{2}$

利用三角形面积计算的方法：

$$2S_{\Delta ABC}=2S_{\Delta ABD}+2S_{\Delta ACD}$$

$$ab\sin(\alpha+\beta)=AD\cdot BD+AD\cdot CD$$

$$=b\cos\beta\cdot a\sin\alpha+a\cos\alpha\cdot b\sin\beta\Rightarrow$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

再考虑角一般的情形,得到

普遍适用的公式.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\Rightarrow \sin(\alpha-\beta)=\sin[\alpha+(-\beta)]$$

$$=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$

$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\Rightarrow \cos(\alpha+\beta)=\sin\left[\frac{\pi}{2}-(\alpha+\beta)\right]$$

$$=\sin\left[\left(\frac{\pi}{2}-\alpha\right)-\beta\right]$$ **和角公式**

$$=\sin\left(\frac{\pi}{2}-\alpha\right)\cos\beta-\cos\left(\frac{\pi}{2}-\alpha\right)\sin\beta$$

$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\begin{cases}\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta\end{cases}\Rightarrow$$

$$\begin{cases}\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta\cdots(1)\\ \sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta\cdots(2)\end{cases}$$

记$\alpha+\beta=A,\alpha-\beta=B$,则(1),(2)式变为

和差化积公式

$$\begin{cases}\sin A+\sin B=2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}\\ \sin A-\sin B=2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}\end{cases}$$

$$\begin{cases} \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \end{cases} \Rightarrow$$

$$\begin{cases} \sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta \cdots (1) \\ \sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta \cdots (2) \end{cases}$$

得积化和差公式:

$$\begin{cases} \sin\alpha\cos\beta=\dfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right] \\ \cos\alpha\sin\beta=\dfrac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right] \end{cases}$$

积化和差公式:

$$\begin{cases} \cos\alpha\cos\beta = \dfrac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \\ \sin\alpha\sin\beta = -\dfrac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)] \end{cases}$$

令$\alpha=\beta$代入上述两式得

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

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$$\begin{cases} \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \end{cases}$$

$$\Rightarrow \begin{cases} \sin 2\alpha = 2\sin\alpha\cos\beta \\ \cos 2\alpha = \cos^2\alpha-\sin^2\alpha \\ \cos^2\alpha+\sin^2\alpha = 1 \end{cases}$$

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$$\begin{cases}\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta\\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta\end{cases}$$

$$\Rightarrow \tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

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$$\begin{cases} \sin 2\alpha = 2\sin\alpha\cos\alpha \\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \\ \cos^2\alpha + \sin^2\alpha = 1 \end{cases} \Rightarrow$$

$$\tan 2\alpha = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\cos^2\alpha + \sin^2\alpha = 1 \Leftrightarrow$$

$$\sec^2 x = 1 + \tan^2 x \Leftrightarrow$$

$$\csc^2 x = 1 + \cot^2 x$$

Pythagoras Theorem=勾股定理

二.第二类换元法——变量代换法

问题 $\int x^5\sqrt{1-x^2}dx = ?$

解决方法 改变中间变量的设置方式.

解决过程 令$x=\sin t \Rightarrow dx=\cos t dt$,

$$\int x^5\sqrt{1-x^2}dx=\int(\sin t)^5\sqrt{1-\sin^2 t}\cos t dt$$

$$=\int \sin^5 t\cos^2 t dt = \cdots\cdots$$

(应用"凑微分"即可求出结果)

暂时有些不严格

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定理2.设$t\in I$时$x=\varphi(t)$严格单调且有连续的导数,$t=\varphi^{-1}(x)$为$x=\varphi(t)$的反函数.若$f\left[\varphi(t)\right]\varphi'(t)$有原函数$\Phi(t)$.则

$$\int f(x)dx \overset{x=\varphi(t)}{=\!=\!=\!=\!=} \int f\left[\varphi(t)\right]\varphi'(t)dt$$

$$=\Phi(t)+C \overset{t=\varphi^{-1}(x)}{=\!=\!=\!=\!=} \Phi\left[\varphi^{-1}(x)\right]+C.$$

例10.求积分$\int \frac{1}{\sqrt{x^2+a^2}}dx, (a>0)$.

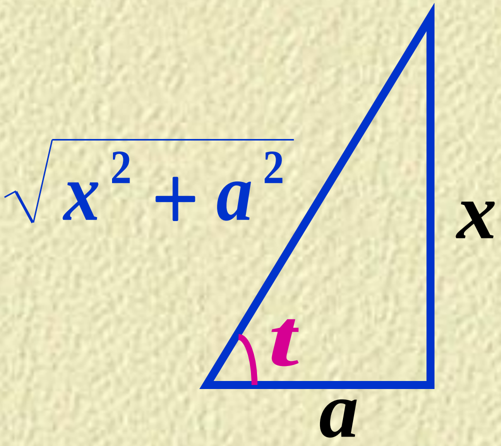
解　令$x = a\tan t \Rightarrow dx = a\sec^2 t dt, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a|\sec t|} \cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C_1$$



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例10.(2). 求积分$\int\sqrt{4-x^2}dx$.

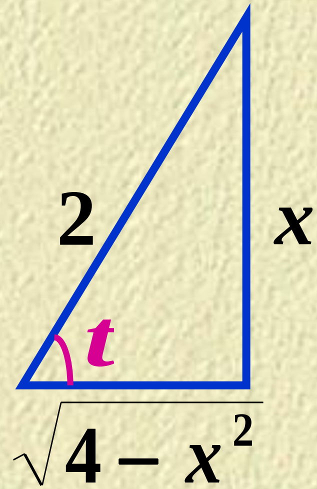
解　令$x=2\sin t$, $dx=2\cos t dt$, $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\int\sqrt{4-x^2}dx=\int\sqrt{4-4\sin^2 t}\cdot 2\cos t dt$$

$$=4\int\cos^2 t dt=2\int(1+\cos 2t)dt$$

$$=2t+\sin 2t+C=2t+2\sin t\cos t+C$$

$$=2\arcsin\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+C,$$



一般地, $\int\sqrt{a^2-x^2}dx=\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$

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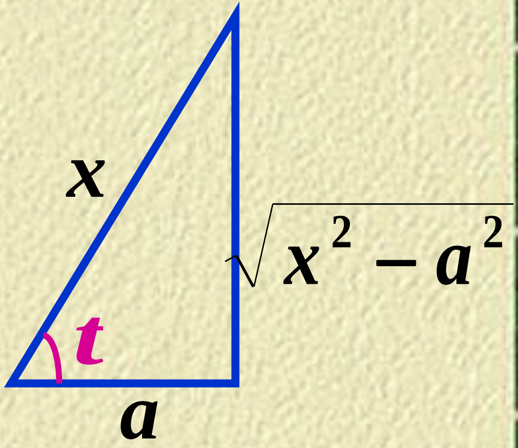
例10.(3)*. 求积分$\int \frac{1}{\sqrt{x^2-a^2}}dx, (a>0)$.

解 令$x = a\sec t, dx = a\sec t\tan t dt$,

$$t \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$J = \int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t \cdot \tan t}{a|\tan t|}dt$$



$$(1).t \in \left(0, \frac{\pi}{2}\right) \text{时} J = \int \frac{a\sec t \cdot \tan t}{a|\tan t|}dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C_1$$

$$\left(C_1 = C - \ln a\right)$$

$$J=\int\frac{1}{\sqrt{x^2-a^2}}dx=\int\frac{a\sec t\cdot\tan t}{a|\tan t|}dt$$

$$(2).t\in\left(\frac{\pi}{2},\pi\right)\text{时}J=\int\frac{a\sec t\cdot\tan t}{a|\tan t|}dt$$

$$=-\int\sec t dt=-\ln|\sec t+\tan t|+C$$

$$\sec t=\frac{x}{a},\tan t=-\frac{\sqrt{x^2-a^2}}{a},$$

$$J=-\ln|\sec t+\tan t|+C$$

$$=-\ln\left|\frac{x}{a}-\frac{\sqrt{x^2-a^2}}{a}\right|+C=\ln\left|x+\sqrt{x^2-a^2}\right|+C-\ln a$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C_1$$

关于第二类换元积分法的说明：

(1).换元积分法中作变量代换的目的是简化被积表达式. 以上几例所作的均为三角代换.

三角代换的一般做法是：

当被积函数中含有 $(a>0)$

(A).$\sqrt{a^2-x^2}$,可令$x=a\sin t$或$x=a\cos t$;

(B).$\sqrt{a^2+x^2}$,可令$x=a\tan t$或$x=a\cot t$;

(C).$\sqrt{x^2-a^2}$,可令$x=a\sec t$或$x=a\csc t$.

(2).当被积函数中含有根式$\sqrt[k]{x},\sqrt[l]{x},\cdots$时,可令$\sqrt[n]{x}=t$,其中$n$为各个根式开根次数的最小公倍数.

例11.求$\displaystyle\int\frac{1}{\sqrt{x}\left(1+\sqrt[3]{x}\right)}dx$.

解　令$\sqrt[6]{x}=t\Rightarrow dx=6t^5dt$,

$$\int\frac{1}{\sqrt{x}\left(1+\sqrt[3]{x}\right)}dx=\int\frac{6t^5}{t^3(1+t^2)}dt=\int\frac{6t^2}{1+t^2}dt$$

$$=6\int\frac{t^2+1-1}{1+t^2}dt=6\int\left(1-\frac{1}{1+t^2}\right)dt$$

$$=6\left(t-\arctan t\right)+C=6\left(\sqrt[6]{x}-\arctan\sqrt[6]{x}\right)+C$$

回想：求极限 $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$,

解 令 $\sqrt[6]{x}=t$, 则 $x\to 1$ 时有 $t\to 1$,

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}=\lim_{t\to 1}\frac{t^3-1}{t^2-1}=\lim_{t\to 1}\frac{(t-1)(t^2+t+1)}{(t-1)(t+1)}$$

$$=\lim_{t\to 1}\frac{t^2+t+1}{t+1}=\frac{3}{2}$$

使用变量代换，使得极限的解题过程表达更简洁.

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(3).变量代换可以不拘一格.

例12. $\int \frac{1}{\sqrt{1+e^x}}dx.$

解 令$t=\sqrt{1+e^x} \Rightarrow e^x = t^2-1,$

$$x=\ln\left(t^2-1\right), dx=\frac{2t}{t^2-1}dt,$$

$$\int \frac{1}{\sqrt{1+e^x}}dx = \int \frac{2}{t^2-1}dt = \int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C = 2\ln\left(\sqrt{1+e^x}-1\right)-x+C$$

求$\int \frac{1}{\sqrt{1+e^x}}dx.$

运用之妙，存乎一心！

法二 令$e^x = \tan^2 t, t \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right),$

$$x = \ln\left(\tan^2 t\right), dx = \frac{2\sec^2 t}{\tan t}dt = \frac{2}{\sin t \cos t}dt,$$

$$原式 = 2\int \csc t dt = 2\ln\left|\csc t - \cot t\right| + C$$

$$= 2\ln\left|\sqrt{1+e^{-x}} - e^{-\frac{1}{2}x}\right| + C$$

$$= 2\ln\left(\sqrt{1+e^x} - 1\right) - x + C$$

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继续研究例5.求$I=\int\frac{1}{1+e^x}dx.$

解法四 令$e^x=t$, $x=\ln t, dx=\frac{1}{t}dt,$

$$I=\int\frac{1}{t(t+1)}dt=\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t}{t+1}\right|+C=\ln\frac{e^x}{1+e^x}+C$$

$$=x-\ln\left(1+e^x\right)+C.$$

回顾例5求$\int \frac{1}{1+e^x}dx$ 以前的做法.

$$\int \frac{1}{1+e^x}dx = \int \frac{e^x}{e^x(1+e^x)}dx$$

$$\overset{e^x=u}{=\!=} \int \frac{du}{u(1+u)} = \int \left(\frac{1}{u} - \frac{1}{1+u}\right)du$$

$$= \ln\frac{u}{1+u} + C = \ln\frac{e^x}{1+e^x} + C$$

$$= x - \ln(1+e^x) + C$$

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法二 $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$

$$= \int\left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$$

$$= x - \ln(1+e^x) + C.$$

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又 解法三

$$\int \frac{1}{1+e^{x}}dx = \int \frac{e^{-x}}{1+e^{-x}}dx$$

$$= -\int \frac{(1+e^{-x})'}{1+e^{-x}}dx = -\ln(1+e^{-x}) + C$$

经过分析,可以知道两种表面上不同的结果其实是完全一样的。

例12.(2). 计算$\int \frac{1}{\sqrt{x-x^2}}dx = ?$

解$\int \frac{1}{\sqrt{x-x^2}}dx = \int \frac{1}{\sqrt{\frac{1}{4}-\left(\frac{1}{4}-x+x^2\right)}}dx$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}dx = \int \frac{2dx}{\sqrt{1-(2x-1)^2}}$$

$$= \int \frac{d(2x-1)}{\sqrt{1-(2x-1)^2}} = \arcsin(2x-1)+C$$

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法二 此法针对此特殊问题,有局限性.

$$\int \frac{1}{\sqrt{x-x^2}}dx = \int \frac{1}{\sqrt{1-x}} \cdot \frac{1}{\sqrt{x}}dx$$

$$x - x^2 > 0 \Leftrightarrow 0 < x < 1$$

$$= 2\int \frac{1}{\sqrt{1-x}}d\left(\sqrt{x}\right) = 2\int \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}d\left(\sqrt{x}\right)$$

$$= 2\arcsin\sqrt{x} + C$$

$$\int \frac{1}{\sqrt{x-x^2}}dx = \arcsin(2x-1) + C$$

法三 变量代换可以不拘一格.

$\because 0 < x < 1$, 故令 $\sqrt{x} = \sin t, t \in \left(0, \frac{\pi}{2}\right)$,

$$J = \int \frac{1}{\sqrt{x}\sqrt{1-x}} dx = \int \frac{1}{\sqrt{1-\sin^2 t}} \cdot \frac{1}{\sin t} d\left(\sin^2 t\right)$$

$$= \int \frac{1}{\sin t \cos t} \cdot 2\sin t \cos t \cdot dt = \int 2dt$$

$$= 2t + C = 2\arcsin\sqrt{x} + C$$

此法与法二本质上是相同的.

基本积分表(II)

$$(10).\int \tan x dx = -\ln\left|\cos x\right| + C \;;$$

$$\int \cot x dx = \ln\left|\sin x\right| + C \;;$$

$$(11).\int \sec x dx = \ln\left|\sec x + \tan x\right| + C \;;$$

$$\int \csc x dx = \ln\left|\csc x - \cot x\right| + C \;;$$

$$(12).\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\frac{x}{a} + C, (a > 0) \;;$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\frac{x}{a} + C, (a > 0) \;;$$

$$(13).\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C,(a>0)\ ;$$

$$(14).\int \frac{1}{\sqrt{x^2\pm a^2}}dx=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C,$$

$$(a>0);$$

$$(15).\int \sqrt{a^2-x^2}dx=\frac{1}{2}x\sqrt{a^2-x^2}$$

$$+\frac{1}{2}a^2\arcsin\frac{x}{a}+C,(a>0)\ .$$

小结

第一类换元积分法(凑微分法)

$$F'(u) = f(u), \int f(u)du = F(u) + C.$$

$\varphi'(x)dx = du$,凑微分

$$\therefore \int f[\varphi(x)]\varphi'(x)dx = \int f(u)du$$

$$= F(u) + C = F[\varphi(x)] + C$$

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第二类换元积分法

$$\int f(x)dx \overset{x=\varphi(t)}{=\!=\!=\!=\!=} \int f[\varphi(t)]\varphi'(t)dt$$

$$= \Phi(t) + C \overset{t=\varphi^{-1}(x)}{=\!=\!=\!=\!=} \Phi\left[\varphi^{-1}(x)\right] + C$$

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练习题

1.计算下列不定积分.

(1). $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx$;　　(2). $\int \frac{1}{4 + 9x^2} dx$;

(3). $\int \frac{x}{\sqrt{1 - x^2}} dx$;　　(4). $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$;

(5). $\int \frac{dx}{x \ln x \ln(\ln x)}$;　　(6). $\int \frac{dx}{e^x + e^{-x}}$.

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2.计算下列不定积分.

$$(1).\int\sqrt{\frac{a+x}{a-x}}dx,\left(a>0\right);\qquad (2).\int\frac{\sin x\cos x}{1+\sin^4 x}dx;$$

$$(3).\int\tan\sqrt{1+x^2}\cdot\frac{xdx}{\sqrt{1+x^2}};\qquad (4).\int x^2\sqrt{1+x^3}dx;$$

$$(5).\int\frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx;\qquad (6).\int\frac{1-x}{\sqrt{9-4x^2}}dx;$$

$$(7).\int\frac{x^3}{9+x^2}dx;\qquad (8).\int\frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx\ .$$

3. 计算下列不定积分.

$$(1).\int \frac{dx}{x+\sqrt{1-x^2}};\qquad (2).\int \frac{dx}{\sqrt{(x^2+1)^3}};$$

$$(3).\int \frac{dx}{1+\sqrt{2x}};\qquad (4).\int x\sqrt{\frac{x}{2a-x}}\,dx, a>0.$$

解答

$$1.(6).\int\frac{dx}{e^x+e^{-x}}=\int\frac{e^{-x}dx}{e^{2x}+1}=\int\frac{de^x}{1+\left(e^x\right)^2};$$

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$$2.(1).a>0,\int\sqrt{\frac{a+x}{a-x}}dx=\int\frac{a+x}{\sqrt{a^2-x^2}}dx$$

$$=\int\frac{a}{\sqrt{1-\left(\frac{x}{a}\right)^2}}d\left(\frac{x}{a}\right)-\frac{1}{2}\int\frac{\left(a^2-x^2\right)'}{\sqrt{a^2-x^2}}dx$$

$$=a\arcsin\frac{x}{a}-\frac{1}{2}\int\frac{d\left(a^2-x^2\right)}{\sqrt{a^2-x^2}}$$

$$=a\arcsin\frac{x}{a}-\sqrt{a^2-x^2}+C$$

2.(1).法二 $a > 0$,

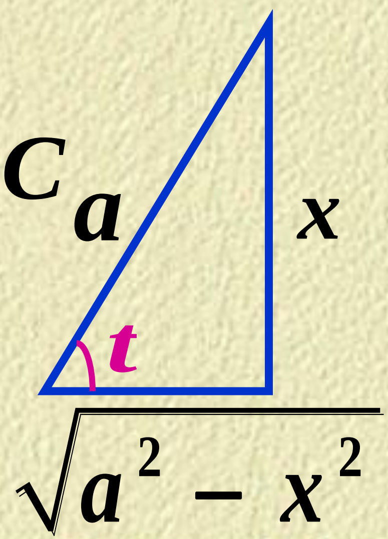
$$\int \sqrt{\frac{a+x}{a-x}}\,dx = \int \frac{a+x}{\sqrt{a^2-x^2}}\,dx = I$$

设 $x = a\sin t$, 则

$$I = \int \frac{a + a\sin t}{a\cos t}\, a\cos t\,dt$$

$$= \int \left(a + a\sin t\right)dt = at - a\cos t + C$$

$$= a\arcsin\frac{x}{a} - \sqrt{a^2 - x^2} + C$$



2.(1).法三 $a>0$, 令 $\sqrt{\dfrac{a+x}{a-x}}=t$, 则

$$x=\frac{a(t^2-1)}{t^2+1},\ dx=\frac{4at}{\left(t^2+1\right)^2}dt$$

$$\therefore \int\sqrt{\frac{a+x}{a-x}}dx=\int t\cdot\left[\frac{a(t^2-1)}{t^2+1}\right]'dt$$

$$=\int t\cdot\frac{4at}{\left(t^2+1\right)^2}dt$$ 用分部积分法

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$$\int t\cdot\frac{4at}{\left(t^2+1\right)^2}dt=\int t\cdot\left[\frac{a(t^2-1)}{t^2+1}\right]'dt$$

用分部积分法

$$I=t\cdot\frac{a(t^2-1)}{t^2+1}-\int\frac{a(t^2-1)}{t^2+1}dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-a\int\frac{t^2+1-2}{t^2+1}dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-at+2a\arctan t+C$$

$$\int\sqrt{\frac{a+x}{a-x}}dx=\int t\cdot\left[\frac{a(t^2-1)}{t^2+1}\right]'dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-\int\frac{a(t^2-1)}{t^2+1}dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-at+2a\arctan t+C$$

将$t=\sqrt{\dfrac{a+x}{a-x}}$ 代入上式得

$$\int\sqrt{\frac{a+x}{a-x}}dx=a\arcsin\frac{x}{a}-\sqrt{a^2-x^2}+C$$

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$$2.(2).\ \because \left(\sin^2 x\right)' = 2\sin x\cos x,$$

$$\therefore \int \frac{\sin x\cos x}{1+\sin^4 x}dx$$

$$= \frac{1}{2}\int \frac{1}{1+\left(\sin^2 x\right)^2}d\left(\sin^2 x\right);$$

$$2.(3).\int \tan\sqrt{1+x^2}\cdot\frac{xdx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}\int \tan\sqrt{1+x^2}\cdot\frac{d(x^2+1)}{\sqrt{1+x^2}}$$

$$=\int \tan\sqrt{1+x^2}\cdot d\sqrt{1+x^2}$$

$$=-\ln\left|\cos\sqrt{1+x^2}\right|+C$$

$$2.(3).\int \tan\sqrt{1+x^2}\cdot\frac{xdx}{\sqrt{1+x^2}}$$

$$\xlongequal[x=\tan t]{\text{法二}}\int \tan(\sec t)\cdot\frac{\tan t\sec^2 tdt}{\sec t}$$

$$=\int \tan(\sec t)\cdot \tan t\sec tdt$$

$$=\int \tan(\sec t)\cdot(\sec t)' dt$$

$$=\int \tan(\sec t)d(\sec t)$$

$$=-\ln\left|\cos(\sec t)\right|+C=-\ln\left|\cos\sqrt{1+x^2}\right|+C$$

$$2.(6).\int \frac{1-x}{\sqrt{9-4x^2}}dx$$

$$=\frac{1}{2}\int \frac{1}{\sqrt{3^2-(2x)^2}}d(2x)+\frac{1}{8}\int \frac{(9-4x^2)'}{\sqrt{9-4x^2}}dx$$

$$=\frac{1}{2}\arcsin\frac{2x}{3}+\frac{1}{8}\cdot 2\sqrt{9-4x^2}+C$$

$$2.(7).\int \frac{x^3}{9+x^2}dx=\frac{1}{2}\int \frac{x^2+9-9}{9+x^2}d(x^2)=\cdots$$

$$2.(8).\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx = 2\int \frac{\arctan\sqrt{x}}{1+x}d\left(\sqrt{x}\right)$$

$$= 2\int \arctan\sqrt{x} \cdot \frac{1}{1+\left(\sqrt{x}\right)^2}d\left(\sqrt{x}\right)$$

$$= 2\int \arctan\sqrt{x}d\left(\arctan\sqrt{x}\right)$$

$$= \left(\arctan\sqrt{x}\right)^2 + C$$

2.(8). $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx$

法二 $t=\arctan\sqrt{x},\ x=\tan^2 t,$

$$I=\int\frac{t}{\tan t\sec^2 t}\cdot 2\tan t\cdot\sec^2 t\,dt$$

$$=\int 2t\,dt=t^2+C=\left(\arctan\sqrt{x}\right)^2+C$$

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$$3.(1).\int \frac{dx}{x+\sqrt{1-x^{2}}} \xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=\sin t}$$

$$=\int \frac{\cos t}{\sin t+\cos t}dt=\cdots$$

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$$3.(2).\int \frac{dx}{\sqrt{(x^2+1)^3}} \xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=\tan t}$$

$$=\int \frac{1}{\left|\sec^3 t\right|}\cdot \sec^2 t dt=\int \cos t dt$$

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$$3.(3).\int \frac{dx}{1+\sqrt{2x}};$$

$$3.(4).\int x\sqrt{\frac{x}{2a-x}}dx.$$

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