§8-02. 不定积分的计算

———换元积分法&分部积分法

1. 第一类换元法——凑微分法
2. 第二类换元法——变量代换法
3. 分部积分法

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1.第一类换元法——凑微分法

问题 $\int \cos 2x dx \overset{?}{=} \sin 2x + C,$

解决方法 利用复合函数,设置中间变量

处理过程 令 $t = 2x \Rightarrow dx = \frac{1}{2} dt,$

$$\int \cos 2x dx = \frac{1}{2}\int \cos(2x) 2dx = \frac{1}{2}\int \cos(2x) d(2x)$$

$$= \frac{1}{2}\int \cos t dt = \frac{1}{2}\sin t + C = \frac{1}{2}\sin 2x + C.$$

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在一般情形下：

设$F'(u)=f(u)$,则$\int f(u)du=F(u)+C$.

如果$u=\varphi(x)$可微,

$\because dF[\varphi(x)]=f[\varphi(x)]\varphi'(x)dx$,

$$\therefore \int f[\varphi(x)]\varphi'(x)dx=F[\varphi(x)]+C$$

$$=\left[\int f(u)du\right]_{u=\varphi(x)}$$

由此可得换元积分法定理

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定理1.设$f(u)$有原函数,$u=\varphi(x)$有连续的导数,则有积分换元公式

$$\int f[\varphi(x)]\varphi'(x)dx=\left[\int f(u)du\right]_{u=\varphi(x)}$$

通常称此法为第一类换元积分法(凑微分法).

说明:使用此公式的关键在于将

$\int g(x)dx$ 变化为$\int f[\varphi(x)]\varphi'(x)dx$.

所以,凑微分法难就难在这第一步.

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$$F'(u) = f(u),$$

$$\int f(u)du = F(u) + C.$$

$\varphi'(x)dx = du,$凑微分

$$\therefore \int f[\varphi(x)]\varphi'(x)dx = \int f(u)du,$$

$$= F(u) + C = F[\varphi(x)] + C$$

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例1.求积分$\int \sin 2x dx$.

解1.$\int \sin(2x)dx = \frac{1}{2}\int \sin(2x)d(2x)$

$= -\frac{1}{2}\cos 2x + C$;

解2.$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C_1$;

解3.$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C_2$.

求积分$\int \sin 2x dx$.

解1. $\int \sin(2x)dx = \frac{1}{2}\int \sin(2x)d(2x)$

$= -\frac{1}{2}\cos 2x + C;$

解1适用的面更广,因而解1比解2解3更有价值.如

$\int \sin(\pi x)dx = \frac{1}{\pi}\int \sin(\pi x)d(\pi x)$

$= -\frac{1}{\pi}\cos(\pi x) + C$

例2.求积分$\int \frac{1}{3+2x}dx$.

解 $\because (3+2x)' = 2, \therefore 2dx = d(3+2x)$

$$\therefore \int \frac{1}{3+2x}dx = \frac{1}{2}\int \frac{1}{3+2x}\cdot(3+2x)'dx$$

$$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|3+2x| + C.$$

一般地$\int f(ax+b)dx = \frac{1}{a}\left[\int f(u)du\right]_{u=ax+b}$

$u = ax+b, (a \neq 0)$

例3.求积分$\int \frac{1}{x(1+2\ln x)}dx$.

解 $\int \frac{1}{x(1+2\ln x)}dx = \int \frac{1}{1+2\ln x}d(\ln x)$

$= \frac{1}{2}\int \frac{1}{1+2\ln x}d(1+2\ln x)$

$\downarrow u = 1+2\ln x$

$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+2\ln x| + C.$

例4.求积分 $\int \frac{x}{(1+x)^3}dx.$

解 $\int \frac{x}{(1+x)^3}dx = \int \frac{x+1-1}{(1+x)^3}dx$

$$= \int \left[\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}\right] d(1+x)$$

$$= \frac{1}{2(1+x)^2} - \frac{1}{1+x} + C$$

例5.求积分$\int \frac{1}{a^2+x^2}dx, a>0.$

解 $\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$

$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a}+C.$

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例5.(2).求积分$\int \frac{1}{\sqrt{a^2-x^2}}dx, a>0.$

解 $\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} d\left(\frac{x}{a}\right)$

$= \arcsin\frac{x}{a} + C.$

例6.求积分$\int \frac{1}{x^2-6x+9}dx$.

解$\int \frac{1}{x^2-6x+9}dx = \int \frac{1}{(x-3)^2}dx$

$= \int \frac{1}{(x-3)^2}d(x-3)$

$= -\frac{1}{x-3} + C = \frac{1}{3-x} + C.$

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例6.(2).求积分$\int \frac{1}{x^2-8x-9}dx = ?$

$$\int \frac{1}{x^2-8x-9}dx = \int \frac{1}{(x-9)(x+1)}dx$$

$$= \frac{1}{10}\int\left(\frac{1}{x-9}-\frac{1}{x+1}\right)dx$$

$$= \frac{1}{10}\int\frac{d(x-9)}{x-9}-\frac{1}{10}\int\frac{d(x+1)}{x+1}$$

$$= \frac{1}{10}\ln\left|\frac{x-9}{x+1}\right|+C.$$

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例6.(3).求积分$\int \frac{1}{x^2-8x+25}dx$.

解$\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$

$$= \frac{1}{3^2}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}dx = \frac{1}{3}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}d\left(\frac{x-4}{3}\right)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

例6.(4).求积分$\int\frac{x^3-5x^2+3}{x^2-8x+25}dx.$

解 $\frac{x^3-5x^2+3}{x^2-8x+25}=\frac{x^3-8x^2+25x+3x^2-24x+75-x-72}{x^2-8x+25}$

$=x+3-\frac{x+72}{x^2-8x+25}$ $\qquad\left(x^2-8x+25\right)'=2x-8,$

$\therefore\int\frac{x+72}{x^2-8x+25}dx=\frac{1}{2}\int\frac{2x-8+152}{x^2-8x+25}dx$

$=\frac{1}{2}\int\frac{\left(x^2-8x+25\right)'}{x^2-8x+25}dx+\int\frac{76}{x^2-8x+25}dx$

$\int\frac{\left(x^2-8x+25\right)'}{x^2-8x+25}dx=\int\frac{d\left(x^2-8x+25\right)}{x^2-8x+25}=\ln\left(x^2-8x+25\right)+C$

… …

凑微分法

$$\int f[\varphi(x)]\varphi'(x)dx$$

$$=\int f[\varphi(x)]d\varphi(x)dx=\left[\int f(u)du\right]_{u=\varphi(x)}$$

难就难在这第一步

$$\int g(x)dx=\int f[\varphi(x)]\varphi'(x)dx.$$

用凑微分法进行不定积分的计算,一般并没有什么普遍的规律.我们需要通过一定的训练,在无规律中寻找规律.

$$\int g(x)dx = \int f[\varphi(x)]\varphi'(x)dx$$

$$= \int f[\varphi(x)]d\varphi(x)dx = \left[\int f(u)du\right]_{u=\varphi(x)}$$

凑微分法难在第一步——如何把被积分函数$g(x)$分拆成 $f[\varphi(x)]\varphi'(x)$.

其基础是基于我们对函数的复合过程了然于胸,基于对复合函数的求导过程十分熟悉,基于我们对不定积分的第一批基本公式的熟稔.

用凑微分法进行不定积分的计算,
是在无规律中寻找规律.

若$\int f(u)du = F(u) + C$,则

$$\int \frac{1}{\sqrt{x}} f\left(\sqrt{x}\right)dx = 2\int f\left(\sqrt{x}\right)d\left(\sqrt{x}\right),$$

$$\int x^{\mu-1} f\left(x^{\mu}\right)dx = \frac{1}{\mu}\int f\left(x^{\mu}\right)d\left(x^{\mu}\right), \mu \neq 0;$$

$$\int \frac{1}{x} f(\ln x)dx = \int f(\ln x)d(\ln x);$$

$$\int f(ax+b)dx \overset{a\neq 0}{=\!=} \frac{1}{a}\int f(ax+b)d(ax+b);$$

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用凑微分法进行不定积分的计算,

在无规律中寻找规律.

若$\int f(u)du = F(u)+C$,则

$$\int f(e^x)e^x dx = \int f\left(e^x\right)d\left(e^x\right)\ ;$$

$$\int \frac{f(\arctan x)}{1+x^2}dx = \int f(\arctan x)d(\arctan x)\ ;$$

$$\int f(\sin x)\cos x dx = \int f(\sin x)d(\sin x)\ ;$$

$$\cdots \quad \cdots$$

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例7.求积分$\int \frac{1}{1+e^x}dx.$

解 $\int \frac{1}{1+e^x}dx = \int \frac{e^x}{e^x\left(1+e^x\right)}dx$

$$\overset{e^x=u}{=\!=} \int \frac{du}{u(1+u)} = \int \left(\frac{1}{u} - \frac{1}{1+u}\right)du$$

$$= \ln\frac{u}{1+u} + C = \ln\frac{e^x}{1+e^x} + C$$

$$= x - \ln\left(1+e^x\right) + C$$

解二 $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$

$$= \int \left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d\left(1+e^x\right)$$

$$= x - \ln\left(1+e^x\right) + C.$$

解三

$$\int \frac{1}{1+e^{x}}dx = \int \frac{e^{-x}}{1+e^{-x}}dx$$

$$= -\int \frac{\left(1+e^{-x}\right)'}{1+e^{-x}}dx = -\ln\left(1+e^{-x}\right)+C$$

经过分析,可以知道两种表面上不同的结果其实是完全一样的。

例7.(2).求积分$J=\int \frac{1}{1+e^{2x}}dx.$

解 $J=\frac{1}{2}\int \frac{1}{1+e^{2x}}d(2x) \overset{2x=u}{=\!=\!=} \frac{1}{2}\int \frac{1}{1+e^{u}}du$

$$=\frac{1}{2}\int\left(1-\frac{e^{u}}{1+e^{u}}\right)du=\frac{1}{2}u-\frac{1}{2}\int\frac{d\left(1+e^{u}\right)}{1+e^{u}}$$

$$=\frac{1}{2}u-\frac{1}{2}\ln\left(1+e^{u}\right)+C$$

$$\overset{u=2x}{=\!=\!=} x-\ln\sqrt{1+e^{2x}}+C.$$

课堂练习1.计算不定积分

$$(1).\int \frac{e^{2x}-2e^{x}}{4e^{2x}+1}dx\ ;$$

$$(2)^{*}.\int \frac{1-2e^{x}}{7e^{2x}+4e^{x}+1}dx\ .$$

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例8.求积分$\int \frac{1}{1+\cos x}dx$.

解$\int \frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx$

$$= \int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}dx$$

$$= \int \frac{1}{\sin^2 x}dx - \int \frac{1}{\sin^2 x}d(\sin x)$$

$$= -\cot x + \csc x + C$$

解二 $\displaystyle\int \frac{1}{1+\cos x}dx = \int \frac{1}{2\cos^2\frac{x}{2}}dx$

$$= \int \sec^2\left(\frac{x}{2}\right)d\left(\frac{x}{2}\right) = \tan\frac{x}{2} + C,$$

比较 $\displaystyle\int \frac{1}{1+\cos x}dx = \csc x - \cot x + C.$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1-\cos x}{\sin x}$$

此解法二比解法一要更有价值.

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例6.求积分$\int \frac{1}{1+\cos 2x}dx.$

解$\int \frac{1}{1+\cos 2x}dx = \int \frac{1}{2\cos^2 x}dx$

$= \frac{1}{2}\int \sec^2 x dx = \frac{1}{2}\tan x + C$

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例8.(2).求积分$\int \frac{1}{2+3\cos x}dx$.

解$\int \frac{1}{2+3\cos x}dx = \int \frac{1}{6\cos^2\frac{x}{2}-1}dx$

$$\overset{\frac{x}{2}=t}{=\!=}2\int \frac{1}{6\cos^2 t-1}dt = 2\int \frac{\sec^2 t}{6-\sec^2 t}dt$$

$$= 2\int \frac{1}{5-\tan^2 t}d(\tan t) \overset{\tan t=u}{=\!=\!=} 2\int \frac{1}{5-u^2}du$$

$$= \frac{1}{\sqrt{5}}\int\left(\frac{1}{\sqrt{5}-u}+\frac{1}{\sqrt{5}+u}\right)du$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+u}{\sqrt{5}-u}\right|+C = \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+\tan\frac{x}{2}}{\sqrt{5}-\tan\frac{x}{2}}\right|+C$$

例8.(2).求积分$\int \frac{1}{2+3\cos x}dx$.

解二$\int \frac{1}{2+3\cos x}dx = \int \frac{2-3\cos x}{(2+3\cos x)(2-3\cos x)}dx$

$$= \int \frac{2}{4-9\cos^2 x}dx - \int \frac{3\cos x}{4-9\cos^2 x}dx$$

$$= 2\int \frac{\sec^2 x}{4\sec^2 x - 9}dx - 3\int \frac{(\sin x)'}{9\sin^2 x - 5}dx$$

$$= 2\int \frac{1}{4\tan^2 x - 5}d(\tan x) - 3\int \frac{1}{9\sin^2 x - 5}d(\sin x)$$

例9.求积分$\int \sin^2 xdx$.

解 $\because \cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1 = 1 - 2\sin^2 x$,

$$\therefore \sin^2 x = \frac{1-\cos 2x}{2},$$

$$\int \sin^2 xdx = \int \frac{1-\cos 2x}{2} dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

例9.(2).求积分$\int \cos^4 x dx$.

解 $\int \cos^4 x dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)dx$

$= \frac{1}{4}\int \left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$

$= \frac{1}{4}\left(\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right)+C$

例9.(3).求积分$\int \cos^3 x dx$.

解

$$\int \cos^3 x dx = \int \cos^2 x \cdot \underline{\cos x dx}$$

$$= \int \left(1 - \sin^2 x\right) d\left(\sin x\right)$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

例9.(4).求积分$\int \sin^2 x\cos^5 x dx$

解$\int \sin^2 x\cos^5 x dx = \int \sin^2 x\cos^4 x \cdot \underline{\cos x dx}$

$= \int \sin^2 x\cos^4 x d(\sin x)$

$= \int \sin^2 x(1-\sin^2 x)^2 d(\sin x)$

$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$

总结经验 当被积函数是正\余弦函数相乘时, 拆开奇次项去凑微分.

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例9.(5).求积分$J=\int \cos 3x\cos 2x dx$

解 由$\cos 2\alpha = 2\cos^2\alpha - 1 \Rightarrow$

$$\cos 3\alpha = \cos\left(2\alpha+\alpha\right) = \cdots$$

$$= 4\cos^3\alpha - 3\cos\alpha$$

$$J = \int\left(8\cos^5 x + \cdots\right)dx = \cdots$$

倘若这样做虽能进行到底,但比较麻烦.

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例9.(5).求积分$\int \cos 3x \cos 2x dx$.

解 $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

$$\Rightarrow \cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\therefore \int \cos 3x \cos 2x dx = \frac{1}{2}\int(\cos x + \cos 5x)dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

课堂练习2.计算不定积分

(1). $\int \cos^6 x dx$;

(2). $\int \sin^3 x \cos^3 x dx$;

(3). $\int \cos^2 x \sin^4 x dx$;

(4). $\int \sin^3 x \cos^4 x dx$.

一般地,我们会更多地根据以下三组公式来处理三角函数的有理式的不定积分问题:

(1) $\cos^2\alpha + \sin^2\alpha = 1, (\sin x)' = \cos x,$

$(\cos x)' = -\sin x, \int \cos x dx = \sin x + C;$

(2) $\sec^2 x = 1 + \tan^2 x, (\tan x)' = \sec^2 x,$

$(\sec x)' = \sec x \tan x, \int \sec^2 x dx = \tan x + C;$

(3) $\csc^2 x = 1 + \cot^2 x, (\cot x)' = -\csc^2 x,$

$(\csc x)' = -\csc x \cot x.$

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例10.求积分$\int \frac{1}{\sin^4 x} dx$.

解 $\int \frac{1}{\sin^4 x} dx = \int \csc^4 x dx$

$= \int \csc^2 x \cdot \underline{\csc^2 x dx}$

$= -\int (1 + \cot^2 x) d(\cot x)$,

$\xlongequal{\cot x = u} -\int (1 + u^2) du = -u - \frac{1}{3} u^3 + C$

$\xlongequal{u = \cot x} -\cot x - \frac{1}{3} \cot^3 x + C$

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解二 $\displaystyle\int \frac{1}{\sin^4 x}dx = \int \frac{\sec^4 x}{\tan^4 x}dx$

$$= \int \frac{\sec^2 x}{\tan^4 x}\sec^2 x dx = \int \frac{1+\tan^2 x}{\tan^4 x}d(\tan x)$$

$$= -\frac{1}{3}\frac{1}{\tan^3 x} - \frac{1}{\tan x} + C$$

解一 $\displaystyle\int \frac{1}{\sin^4 x}dx = -\cot x - \frac{1}{3}\cot^3 x + C$

两个做法结果一模一样.

例10.(2).求积分$\int \frac{1}{\sin^2 x \cos^4 x} dx.$

解 $\int \frac{1}{\sin^2 x \cos^4 x} dx = \int \frac{\sec^6 x}{\tan^2 x} dx$

$$= \int \frac{\left(\sec^2 x\right)^2}{\tan^2 x} \sec^2 x dx = \int \frac{\left(1+\tan^2 x\right)^2}{\tan^2 x} d\left(\tan x\right)$$

$d\left(\tan x\right) = (\tan x)' dx = \sec^2 x dx$

$$\overset{\tan x = u}{=\!=\!=} \int \frac{\left(1+u^2\right)^2}{u^2} du = \int \left(\frac{1}{u^2} + 2 + u^2\right) du = \cdots$$

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$$J=\int\frac{1}{\sin^2 x\cos^4 x}dx=\int\frac{\sec^6 x}{\tan^2 x}dx$$

$$=\int\frac{\left(\sec^2 x\right)^2}{\tan^2 x}\sec^2 x dx=\int\frac{\left(1+\tan^2 x\right)^2}{\tan^2 x}d\left(\tan x\right)$$

$$\overset{\tan x=u}{=\!=\!=}\int\frac{\left(1+u^2\right)^2}{u^2}du=\int\left(\frac{1}{u^2}+2+u^2\right)du=\cdots$$

或者$J=\int\frac{1}{\sin^2 x\cos^4 x}dx=\int\frac{\csc^6 x}{\cot^4 x}dx$

$$=\int\frac{\csc^4 x}{\cot^4 x}\csc^2 x dx=-\int\frac{\left(1+\cot^2 x\right)^2}{\cot^4 x}d\left(\cot x\right)$$

两个做法完全等效.

例10.(3).求积分$\int \frac{1}{\sin^3 x \cos^5 x} dx.$

解 $\int \sin^{-3} x \cdot \cos^{-5} x dx = \int \sec^5 x \csc^3 x dx = ??$

$$\int \sin^{-3} x \cdot \cos^{-5} x dx = \int \frac{1}{\sin^3 x \cdot \cos^5 x} dx$$

$$= \int \frac{\cos x}{\sin^3 x \cdot \cos^6 x} dx = \int \frac{d(\sin x)}{\sin^3 x \cdot (1-\sin^2 x)^3}$$

$$\overset{\sin x = u}{=\!=\!=} \int \frac{du}{u^3(1-u^2)^3} = ? = \int \frac{u du}{u^4(1-u^2)^3}$$

$$= \frac{1}{2} \int \frac{d\left(u^2\right)}{u^4\left(1-u^2\right)^3} \overset{t=u^2}{=\!=\!=} \frac{1}{2} \int \frac{dt}{t^2(1-t)^3} = ?$$

返回

$$\int \sin^{-3} x \cdot \cos^{-5} x dx$$

$$= \int \frac{1}{\tan^3 x \cdot \cos^8 x} dx = \int \frac{\sec^6 x}{\tan^3 x} \sec^2 x dx$$

$$= \int \frac{\left(\sec^2 x\right)^3}{\tan^3 x} \sec^2 x dx$$

$$= \int \frac{\left(1 + \tan^2 x\right)^3}{\tan^3 x} d \tan x,$$

$$d \tan x = (\tan x)' dx = \sec^2 x dx$$

$$= \int \frac{\left(1 + u^2\right)^3}{u^3} du = \cdots$$

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例10.(4).求积分$J=\int \frac{\sin^5 x}{\cos^3 x}dx$.

解$J=\int \frac{\sin^5 x}{\cos^3 x}dx=\int \frac{\sin^4 x}{\cos^3 x}(\sin x)dx$

$$=-\int \frac{\left(\sin^2 x\right)^2}{\cos^3 x}d(\cos x)$$

$$=-\int \frac{\left(1-\cos^2 x\right)^2}{\cos^3 x}d(\cos x)=\cdots$$

例10.(4).求积分$J=\int\frac{\sin^5 x}{\cos^3 x}dx.$

解二$J=\int\frac{\sin^5 x}{\cos^4 x}(\cos x)dx$

$$=\int\frac{\sin^5 x}{\left(1-\sin^2 x\right)^2}d(\sin x)\overset{\sin x=u}{=\!=\!=}\int\frac{u^5}{\left(1-u^2\right)^2}du=\cdots$$

解三$J=\int\frac{\tan^5 x}{\sec^2 x}dx=\int\frac{\tan^5 x}{\sec^4 x}\left(\sec^2 x\right)dx$

$$=\int\frac{\tan^5 x}{\left(1+\tan^2 x\right)^2}d(\tan x)\overset{\tan x=u}{=\!=\!=}\int\frac{u^5}{\left(1+u^2\right)^2}du=\cdots$$

例10.(5).求积分$J=\int\frac{1}{\sin^3 x\cos^2 x}dx.$

解 $\int\frac{1}{\sin^3 x\cos^2 x}dx=\int\frac{\sec^5 x}{\tan^3 x}dx=???$

$$J=\int\frac{\sec^5 x}{\tan^3 x}dx=\int\frac{\sec^4 x}{\tan^4 x}(\sec x\tan x)dx$$

$$=\int\frac{\sec^4 x}{\tan^4 x}d(\sec x)=\int\frac{\sec^4 x}{\left(\sec^2 x-1\right)^2}d(\sec x)$$

$$\overset{\sec x=u}{=\!=\!=}\int\frac{u^4}{\left(u^2-1\right)^2}du=?$$

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一般地,我们会更多地根据以下三组公式来处理三角函数的有理式的不定积分问题:

$(1)\cos^2\alpha+\sin^2\alpha=1,(\sin x)'=\cos x,$

$(\cos x)'=-\sin x,\int\cos xdx=\sin x+C;$

$(2)\sec^2 x=1+\tan^2 x,(\tan x)'=\sec^2 x,$

$(\sec x)'=\sec x\tan x,\int\sec^2 xdx=\tan x+C;$

$(3)\csc^2 x=1+\cot^2 x,(\cot x)'=-\csc^2 x,$

$(\csc x)'=-\csc x\cot x.$

课堂练习3.计算不定积分

(1).$\int \sec^4 x dx$;

(2).$\int \sec x \cdot \tan^3 x dx$;

(3).$\int \sin^{-3} x \cdot \cos^{-3} x dx$.

例11.求积分$\int \sec x dx$.

解 $\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx$

$$= \int \frac{1}{1-\sin^2 x} d(\sin x) \overset{u=\sin x}{=\!=\!=}$$

$$= \int \frac{1}{1-u^2} du = \frac{1}{2} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$$

$$= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C = \frac{1}{2} \ln \left| \frac{1+\sin x}{1-\sin x} \right| + C$$

$$= \frac{1}{2} \ln \frac{(1+\sin x)^2}{1-\sin^2 x} + C = \ln \left| \frac{1+\sin x}{\cos x} \right| + C$$

$$= \ln \left| \sec x + \tan x \right| + C.$$

解二 $\int \sec x dx = \int \frac{1}{\cos x} dx$

$$= \int \frac{1}{2\cos^2\frac{x}{2} - 1} dx = 2\int \frac{\sec^2\frac{x}{2}}{2 - \sec^2\frac{x}{2}} d\left(\frac{x}{2}\right)$$

$$= 2\int \frac{1}{1 - \tan^2\frac{x}{2}} d\left(\tan\frac{x}{2}\right) = \ln\left|\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right| + C$$

$$= \ln\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right| + C = \ln\left|\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right| + C$$

$$= \ln\left|\frac{1 + \sin x}{\cos x}\right| + C = \ln\left|\sec x + \tan x\right| + C.$$

法三 $\int \sec x dx = \int \frac{(\sec x + \tan x)\sec x}{\sec x + \tan x} dx$

$$= \int \frac{(\sec x + \tan x)'}{\sec x + \tan x} dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln|\sec x + \tan x| + C.$$

此法妙则妙矣,但过于巧妙,不容易想到,就方法的角度而言,于初学者价值不大.

例11.(2).求积分$\int \csc x dx$.

解 $\int \csc x dx = \int \frac{1}{\sin x} dx$

$$= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right)$$

$$= \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right) = \ln\left|\tan\frac{x}{2}\right| + C$$

(使用了三角函数恒等变形)

$$= \ln\left|\csc x - \cot x\right| + C.$$

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解二 $\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \xlongequal{u=\cos x}$$

$$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$$

$$= \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$$

$$= \frac{1}{2}\ln\frac{(1-\cos x)^2}{1-\cos^2 x} + C = \ln\left|\frac{1-\cos x}{\sin x}\right| + C$$

$$= \ln\left|\csc x - \cot x\right| + C.$$

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例12.设$f'(\sin^2 x)=\cos^2 x$,求$f(x)$.

解　令$u=\sin^2 x \Rightarrow \cos^2 x = 1-u$,

$$f'(u)=1-u,$$

$$f(u)=\int(1-u)du = u-\frac{1}{2}u^2+C,$$

$$f(x)=x-\frac{1}{2}x^2+C, 0\le x\le 1.$$

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Addendum. 三角函数公式系列

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

先考虑特殊情形$0<\alpha,\beta,\alpha+\beta<\dfrac{\pi}{2}$

利用三角形面积计算的方法:

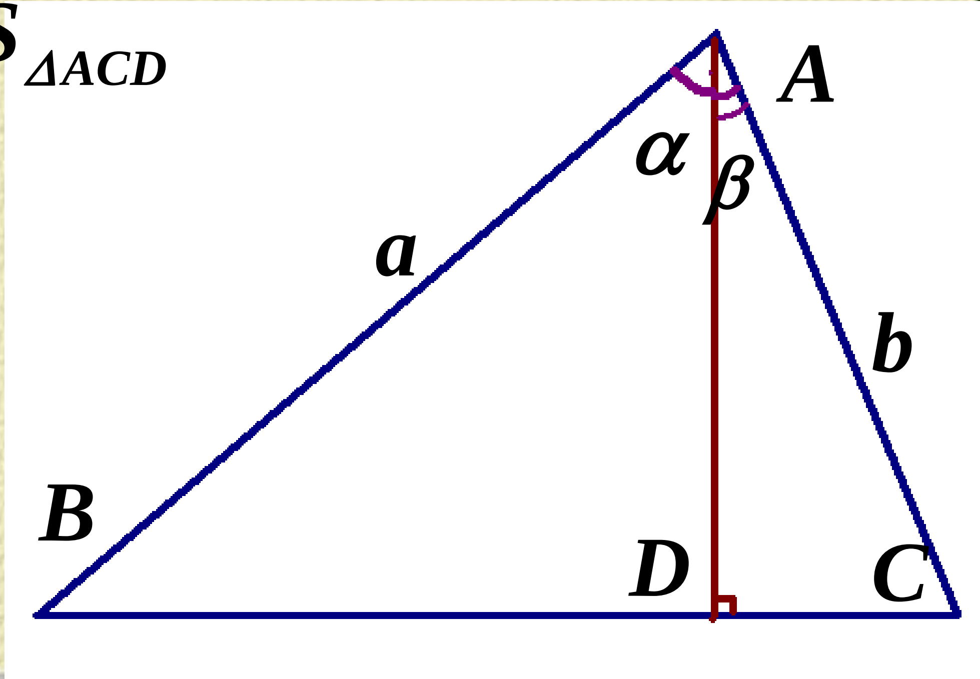
$$2S_{\Delta ABC}=2S_{\Delta ABD}+2S_{\Delta ACD}$$

$$ab\sin(\alpha+\beta)=$$

$$AD\cdot BD+AD\cdot CD$$

$$=b\cos\beta\cdot a\sin\alpha$$

$$+a\cos\alpha\cdot b\sin\beta$$



$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

先考虑特殊情形$0<\alpha,\beta,\alpha+\beta<\dfrac{\pi}{2}$

利用三角形面积计算的方法:

$$2S_{\Delta ABC}=2S_{\Delta ABD}+2S_{\Delta ACD}$$

$$ab\sin(\alpha+\beta)=AD\cdot BD+AD\cdot CD$$

$$=b\cos\beta\cdot a\sin\alpha+a\cos\alpha\cdot b\sin\beta\Rightarrow$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

再考虑角一般的情形,得到

普遍适用的公式.

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$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\Rightarrow \sin(\alpha-\beta)=\sin[\alpha+(-\beta)]$$

$$=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$

$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\Rightarrow \cos(\alpha+\beta)=\sin\left[\frac{\pi}{2}-(\alpha+\beta)\right]$$

$$=\sin\left[\left(\frac{\pi}{2}-\alpha\right)-\beta\right]$$ **和角公式**

$$=\sin\left(\frac{\pi}{2}-\alpha\right)\cos\beta-\cos\left(\frac{\pi}{2}-\alpha\right)\sin\beta$$

$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\begin{cases}\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta\end{cases}\Rightarrow$$

$$\begin{cases}\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta\cdots(1)\\ \sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta\cdots(2)\end{cases}$$

记$\alpha+\beta=A,\alpha-\beta=B$,则(1),(2)式变为

和差化积公式

$$\begin{cases}\sin A+\sin B=2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}\\ \sin A-\sin B=2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}\end{cases}$$

$$\begin{cases}\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta\end{cases}\Rightarrow$$

$$\begin{cases}\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta\cdots(1)\\ \sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta\cdots(2)\end{cases}$$

得积化和差公式:

$$\begin{cases}\sin\alpha\cos\beta=\dfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\\ \cos\alpha\sin\beta=\dfrac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]\end{cases}$$

积化和差公式：

$$\begin{cases} \cos\alpha\cos\beta = \dfrac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \\ \sin\alpha\sin\beta = -\dfrac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)] \end{cases}$$

令$\alpha=\beta$代入上述两式得

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\begin{cases} \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \end{cases}$$

$$\Rightarrow \begin{cases} \sin 2\alpha = 2\sin\alpha\cos\beta \\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \\ \cos^2\alpha + \sin^2\alpha = 1 \end{cases}$$

$$\begin{cases} \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \end{cases}$$

$$\Rightarrow \tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\begin{cases} \sin 2\alpha = 2\sin\alpha\cos\alpha \\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \\ \cos^2\alpha + \sin^2\alpha = 1 \end{cases} \Rightarrow$$

$$\tan 2\alpha = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\cos^2\alpha + \sin^2\alpha = 1 \Leftrightarrow$$

$$\sec^2 x = 1 + \tan^2 x \Leftrightarrow$$

$$\csc^2 x = 1 + \cot^2 x$$

Pythagoras Theorem=勾股定理

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2. 第二类换元法——变量代换法

问题 $\int x^5\sqrt{1-x^2}\,dx = ?$

解决方法 改变中间变量的设置方式.

解决过程 令 $x=\sin t \Rightarrow dx=\cos t\,dt$,

$$\int x^5\sqrt{1-x^2}\,dx=\int(\sin t)^5\sqrt{1-\sin^2 t}\cos t\,dt$$

$$=\int \sin^5 t\cos^2 t\,dt=\cdots\cdots$$

(应用“凑微分”即可求出结果)

暂时有些不严格

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定理2.设$t\in I$时$x=\varphi(t)$严格单调且有连续的导数,$t=\varphi^{-1}(x)$为$x=\varphi(t)$的反函数.若$f[\varphi(t)]\varphi'(t)$有原函数$\Phi(t)$.则

$$\int f(x)dx \xlongequal{x=\varphi(t)} \int f[\varphi(t)]\varphi'(t)dt$$

$$=\Phi(t)+C \xlongequal{t=\varphi^{-1}(x)} \Phi\left[\varphi^{-1}(x)\right]+C.$$

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例13.求积分$\int \frac{1}{\sqrt{x^2+a^2}}dx,(a>0)$.

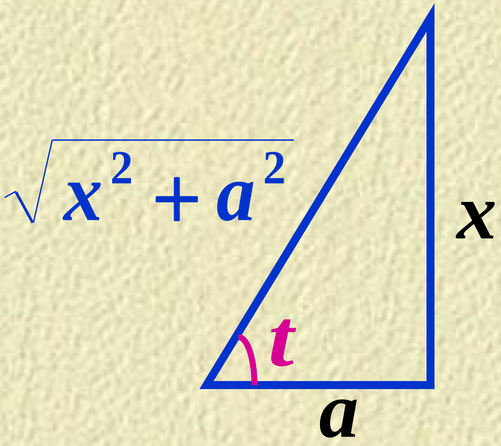
解　令$x = a\tan t \Rightarrow dx = a\sec^2 t dt, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a|\sec t|} \cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C_1$$



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例14.求积分$\int\sqrt{4-x^2}dx.$

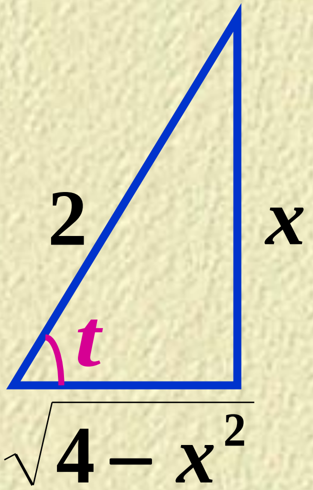
解　令$x=2\sin t$, $dx=2\cos t dt, t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\int\sqrt{4-x^2}dx=\int\sqrt{4-4\sin^2 t}\cdot 2\cos t dt$$

$$=4\int\cos^2 t dt=2\int(1+\cos 2t)dt$$

$$=2t+\sin 2t+C=2t+2\sin t\cos t+C$$

$$=2\arcsin\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+C,$$



一般地,$\int\sqrt{a^2-x^2}dx=\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$

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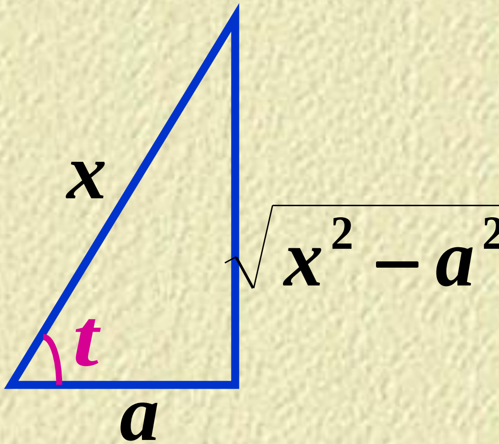
例15.求积分$\int \frac{1}{\sqrt{x^2-a^2}}dx, (a>0)$.

解　令$x = a\sec t, dx = a\sec t\tan t dt,$

$$t \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$J = \int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t \cdot \tan t}{a|\tan t|}dt$$



(1).$t \in \left(0, \frac{\pi}{2}\right)$时$J = \int \frac{a\sec t \cdot \tan t}{a|\tan t|}dt$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C_1$$

$$(C_1 = C - \ln a)$$

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$$J=\int\frac{1}{\sqrt{x^2-a^2}}dx=\int\frac{a\sec t\cdot\tan t}{a|\tan t|}dt$$

(2). $t\in\left(\frac{\pi}{2},\pi\right)$ 时 $J=\int\frac{a\sec t\cdot\tan t}{a|\tan t|}dt$

$$=-\int\sec t\,dt=-\ln|\sec t+\tan t|+C$$

$\sec t=\frac{x}{a}$, $\tan t=-\frac{\sqrt{x^2-a^2}}{a}$,

$$J=-\ln|\sec t+\tan t|+C$$

$$=-\ln\left|\frac{x}{a}-\frac{\sqrt{x^2-a^2}}{a}\right|+C=\ln\left|x+\sqrt{x^2-a^2}\right|+C-\ln a$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C_1$$

关于第二类换元积分法的说明:

(1).换元积分法中作变量代换的目的是简化被积表达式. 以上几例所作的均为三角代换.

三角代换的一般做法是:

当被积函数中含有 $(a > 0)$

(A). $\sqrt{a^2 - x^2}$, 可令 $x = a\sin t$ 或 $x = a\cos t$;

(B). $\sqrt{a^2 + x^2}$, 可令 $x = a\tan t$ 或 $x = a\cot t$;

(C). $\sqrt{x^2 - a^2}$, 可令 $x = a\sec t$ 或 $x = a\csc t$.

(2).当被积函数中含有根式$\sqrt[k]{x}, \sqrt[l]{x}, \cdots$时,可令$\sqrt[n]{x}=t$,其中$n$为各个根式开根次数的最小公倍数.

例16.求$\int \frac{1}{\sqrt{x}\left(1+\sqrt[3]{x}\right)}dx$.

解　令$\sqrt[6]{x}=t \Rightarrow dx=6t^5dt$,

$$\int \frac{1}{\sqrt{x}\left(1+\sqrt[3]{x}\right)}dx=\int \frac{6t^5}{t^3(1+t^2)}dt=\int \frac{6t^2}{1+t^2}dt$$

$$=6\int \frac{t^2+1-1}{1+t^2}dt=6\int\left(1-\frac{1}{1+t^2}\right)dt$$

$$=6\left(t-\arctan t\right)+C=6\left(\sqrt[6]{x}-\arctan\sqrt[6]{x}\right)+C$$

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(3).在某些问题中偶尔也可考虑用倒代换$x=\frac{1}{t}$.

例17.求$\int\frac{1}{x\left(x^7+2\right)}dx$.

解 令$x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}dt$,

$$\int\frac{1}{x\left(x^7+2\right)}dx=\int\frac{t}{\left(\frac{1}{t}\right)^7+2}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$=-\int\frac{t^6}{1+2t^7}dt=-\frac{1}{14}\ln\left|1+2t^7\right|+C$$

$$=-\frac{1}{14}\ln\left|2+x^7\right|+\frac{1}{2}\ln|x|+C$$

法二 $\int \frac{1}{x\left(x^{7}+2\right)} dx$

$$= \int \frac{x^{6} dx}{x^{7}\left(x^{7}+2\right)} = \int \frac{d\left(x^{7}\right)}{7x^{7}\left(x^{7}+2\right)}$$

$$= \frac{1}{7} \cdot \frac{1}{2} \int \left(\frac{1}{x^{7}} - \frac{1}{x^{7}+2} \right) d\left(x^{7}\right)$$

这是凑微分法.

(4).变量代换可以不拘一格.

例18. $\int \frac{1}{\sqrt{1+e^x}} dx.$

解 令$t=\sqrt{1+e^x} \Rightarrow e^x = t^2-1,$

$$x=\ln\left(t^2-1\right), dx=\frac{2t}{t^2-1}dt,$$

$$\int \frac{1}{\sqrt{1+e^x}} dx = \int \frac{2}{t^2-1} dt = \int \left(\frac{1}{t-1}-\frac{1}{t+1}\right) dt$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C = 2\ln\left(\sqrt{1+e^x}-1\right)-x+C$$

求$\int \frac{1}{\sqrt{1+e^x}}dx.$

法二 令$e^x = \tan^2 t, t \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right),$

$$x = \ln\left(\tan^2 t\right), dx = \frac{2\sec^2 t}{\tan t}dt = \frac{2}{\sin t \cos t}dt,$$

原式$= 2\int \csc t dt = 2\ln\left|\csc t - \cot t\right| + C$

$$= 2\ln\left|\sqrt{1+e^{-x}} - e^{-\frac{1}{2}x}\right| + C$$

$$= 2\ln\left(\sqrt{1+e^x} - 1\right) - x + C$$

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继续研究例7.求$I=\int\frac{1}{1+e^{x}}dx.$

解法四　令$e^{x}=t$, $x=\ln t, dx=\frac{1}{t}dt,$

$$I=\int\frac{1}{t(t+1)}dt=\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t}{t+1}\right|+C=\ln\frac{e^{x}}{1+e^{x}}+C$$

$$=x-\ln\left(1+e^{x}\right)+C.$$

回顾例7求$\int \frac{1}{1+e^x}dx$ 以前的做法.

$$\int \frac{1}{1+e^x}dx = \int \frac{e^x}{e^x(1+e^x)}dx$$

$$\overset{e^x=u}{=\!=} \int \frac{du}{u(1+u)} = \int \left(\frac{1}{u} - \frac{1}{1+u} \right) du$$

$$= \ln \frac{u}{1+u} + C = \ln \frac{e^x}{1+e^x} + C$$

$$= x - \ln(1+e^x) + C$$

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法二 $\int \frac{1}{1+e^{x}}dx = \int \frac{1+e^{x}-e^{x}}{1+e^{x}}dx$

$$= \int \left(1 - \frac{e^{x}}{1+e^{x}}\right)dx = \int dx - \int \frac{e^{x}}{1+e^{x}}dx$$

$$= \int dx - \int \frac{1}{1+e^{x}}d(1+e^{x})$$

$$= x - \ln(1+e^{x}) + C.$$

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又 解法三

$$\int \frac{1}{1+e^{x}}dx = \int \frac{e^{-x}}{1+e^{-x}}dx$$

$$= -\int \frac{(1+e^{-x})'}{1+e^{-x}}dx = -\ln(1+e^{-x}) + C$$

经过分析,可以知道两种表面上不同的结果其实是完全一样的。

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例19.计算$\int \frac{1}{\sqrt{x-x^2}}dx = ?$

解$\int \frac{1}{\sqrt{x-x^2}}dx = \int \frac{1}{\sqrt{\frac{1}{4}-\left(\frac{1}{4}-x+x^2\right)}}dx$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}dx = \int \frac{2dx}{\sqrt{1-(2x-1)^2}}$$

$$= \int \frac{d(2x-1)}{\sqrt{1-(2x-1)^2}} = \arcsin(2x-1)+C$$

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法二 此法针对此特殊问题,有局限性.

$$\int \frac{1}{\sqrt{x-x^2}}dx = \int \frac{1}{\sqrt{1-x}} \cdot \frac{1}{\sqrt{x}}dx$$

$x - x^2 > 0$

$\Leftrightarrow 0 < x < 1$

$$= 2\int \frac{1}{\sqrt{1-x}}d\left(\sqrt{x}\right) = 2\int \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}d\left(\sqrt{x}\right)$$

$$= 2\arcsin\sqrt{x} + C$$

$$\int \frac{1}{\sqrt{x-x^2}}dx = \arcsin(2x-1) + C$$

法三 变量代换可以不拘一格.

$\because 0 < x < 1$,故令$\sqrt{x} = \sin t, t \in \left(0, \frac{\pi}{2}\right)$,

$$J = \int \frac{1}{\sqrt{x}\sqrt{1-x}} dx = \int \frac{1}{\sqrt{1-\sin^2 t}} \cdot \frac{1}{\sin t} d\left(\sin^2 t\right)$$

$$= \int \frac{1}{\sin t \cos t} \cdot 2\sin t \cos t \cdot dt = \int 2dt$$

$$= 2t + C = 2\arcsin\sqrt{x} + C$$

此法与法二本质上是相同的.

基本积分表(II)

$$(10).\int \tan x dx = -\ln|\cos x| + C\ ;$$

$$\int \cot x dx = \ln|\sin x| + C\ ;$$

$$(11).\int \sec x dx = \ln|\sec x + \tan x| + C\ ;$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C\ ;$$

$$(12).\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\frac{x}{a} + C, (a > 0)\ ;$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\frac{x}{a} + C, (a > 0)\ ;$$

$$(13).\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C,(a>0)\ ;$$

$$(14).\int \frac{1}{\sqrt{x^2\pm a^2}}dx=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C,$$

$$(a>0);$$

$$(15).\int \sqrt{a^2-x^2}dx=\frac{1}{2}x\sqrt{a^2-x^2}$$

$$+\frac{1}{2}a^2\arcsin\frac{x}{a}+C,(a>0)\ .$$

小结

两类积分换元法：

$$\begin{cases}\text{（一）凑微分}\\ \text{（二）三角代换、根式代换}\end{cases}$$

基本积分表 (2) 要能够

独自推导，熟练运用.

第一类换元积分法(凑微分法)

$$F'(u)=f(u),\int f(u)du=F(u)+C.$$

$\varphi'(x)dx=du$,凑微分

$$\therefore \int f[\varphi(x)]\varphi'(x)dx=\int f(u)du$$

$$=F(u)+C=F[\varphi(x)]+C$$

第二类换元积分法

$$\int f(x)dx \xlongequal{x=\varphi(t)} \int f[\varphi(t)]\varphi'(t)dt$$

$$= \Phi(t) + C \xlongequal{t=\varphi^{-1}(x)} \Phi\left[\varphi^{-1}(x)\right] + C$$

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自我检测题：计算下列不定积分.

1.(1). $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx$; (2). $\int \frac{1}{4 + 9x^2} dx$;

(3). $\int \frac{x}{\sqrt{1 - x^2}} dx$; (4). $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$;

(5). $\int \frac{dx}{x \ln x \ln(\ln x)}$; (6). $\int \frac{dx}{e^x + e^{-x}}$.

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自我检测题 .

2.(1). $\int\sqrt{\frac{a+x}{a-x}}dx,(a>0)$; (2). $\int\frac{\sin x\cos x}{1+\sin^4 x}dx$;

(3). $\int\tan\sqrt{1+x^2}\cdot\frac{xdx}{\sqrt{1+x^2}}$; (4). $\int x^2\sqrt{1+x^3}dx$;

(5). $\int\frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx$; (6). $\int\frac{1-x}{\sqrt{9-4x^2}}dx$;

(7). $\int\frac{x^3}{9+x^2}dx$; (8). $\int\frac{dx}{x(x^6+4)}$.

$$(9).\int \frac{10^{2\arccos x}}{\sqrt{1-x^2}}dx; \qquad (10).\int \frac{\ln\tan x}{\cos x\sin x}dx;$$

$$(11).\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx; \qquad (12).\int \frac{x+1}{x(1+xe^x)}dx.$$

$$3.(1).\int \frac{dx}{x+\sqrt{1-x^2}}; \qquad (2).\int \frac{dx}{\sqrt{(x^2+1)^3}};$$

$$(3).\int \frac{dx}{1+\sqrt{2x}}; \qquad (4).\int x\sqrt{\frac{x}{2a-x}}dx, a>0.$$

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解答

$$1.(6).\int \frac{dx}{e^{x}+e^{-x}}=\int \frac{e^{-x}dx}{e^{2x}+1}=\int \frac{de^{x}}{1+\left(e^{x}\right)^{2}};$$

$$2.(1).a>0,\int\sqrt{\frac{a+x}{a-x}}dx=\int\frac{a+x}{\sqrt{a^2-x^2}}dx$$

$$=\int\frac{a}{\sqrt{1-\left(\frac{x}{a}\right)^2}}d\left(\frac{x}{a}\right)-\frac{1}{2}\int\frac{\left(a^2-x^2\right)'}{\sqrt{a^2-x^2}}dx$$

$$=a\arcsin\frac{x}{a}-\frac{1}{2}\int\frac{d\left(a^2-x^2\right)}{\sqrt{a^2-x^2}}$$

$$=a\arcsin\frac{x}{a}-\sqrt{a^2-x^2}+C$$

2.(1).法二$a>0,$

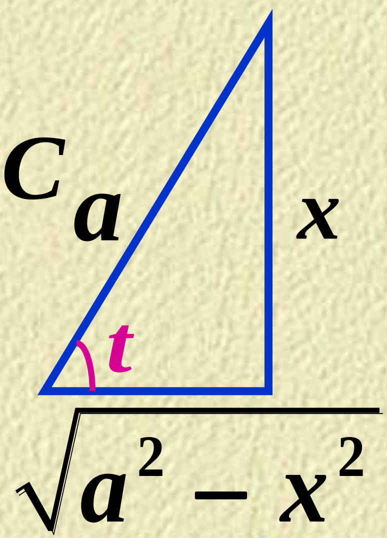
$$\int\sqrt{\frac{a+x}{a-x}}dx=\int\frac{a+x}{\sqrt{a^2-x^2}}dx=I$$

设$x=a\sin t,$则

$$I=\int\frac{a+a\sin t}{a\cos t}a\cos t dt$$

$$=\int(a+a\sin t)dt=at-a\cos t+C$$

$$=a\arcsin\frac{x}{a}-\sqrt{a^2-x^2}+C$$



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2.(1).法三 $a>0$,令$\sqrt{\dfrac{a+x}{a-x}}=t$,则

$$x=\frac{a(t^2-1)}{t^2+1},dx=\frac{4at}{\left(t^2+1\right)^2}dt$$

$$\therefore\int\sqrt{\frac{a+x}{a-x}}dx=\int t\cdot\left[\frac{a(t^2-1)}{t^2+1}\right]'dt$$

$$=\int t\cdot\frac{4at}{\left(t^2+1\right)^2}dt$$ 用分部积分法

$$\int t\cdot\frac{4at}{\left(t^2+1\right)^2}dt=\int t\cdot\left[\frac{a(t^2-1)}{t^2+1}\right]'dt$$

用分部积分法

$$I=t\cdot\frac{a(t^2-1)}{t^2+1}-\int\frac{a(t^2-1)}{t^2+1}dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-a\int\frac{t^2+1-2}{t^2+1}dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-at+2a\arctan t+C$$

$$\int\sqrt{\frac{a+x}{a-x}}dx=\int t\cdot\left[\frac{a(t^2-1)}{t^2+1}\right]'dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-\int\frac{a(t^2-1)}{t^2+1}dt$$

$$=t\cdot\frac{a(t^2-1)}{t^2+1}-at+2a\arctan t+C$$

将$t=\sqrt{\frac{a+x}{a-x}}$ 代入上式得

$$\int\sqrt{\frac{a+x}{a-x}}dx=a\arcsin\frac{x}{a}-\sqrt{a^2-x^2}+C$$

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$$2.(2).\ \because \left(\sin^2 x\right)' = 2\sin x\cos x,$$

$$\therefore \int \frac{\sin x\cos x}{1+\sin^4 x}dx$$

$$= \frac{1}{2}\int \frac{1}{1+\left(\sin^2 x\right)^2}d\left(\sin^2 x\right);$$

$$2.(3).\int \tan\sqrt{1+x^2}\cdot\frac{x\,dx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}\int \tan\sqrt{1+x^2}\cdot\frac{d(x^2+1)}{\sqrt{1+x^2}}$$

$$=\int \tan\sqrt{1+x^2}\cdot d\sqrt{1+x^2}$$

$$=-\ln\left|\cos\sqrt{1+x^2}\right|+C$$

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$$2.(3).\int \tan\sqrt{1+x^2}\cdot\frac{xdx}{\sqrt{1+x^2}}$$

$$\xlongequal[x=\tan t]{\text{法二}}\int \tan(\sec t)\cdot\frac{\tan t\sec^2 tdt}{\sec t}$$

$$=\int \tan(\sec t)\cdot\tan t\sec tdt$$

$$=\int \tan(\sec t)\cdot(\sec t)' dt$$

$$=\int \tan(\sec t)d(\sec t)$$

$$=-\ln\left|\cos(\sec t)\right|+C=-\ln\left|\cos\sqrt{1+x^2}\right|+C$$

$$2.(6).\int \frac{1-x}{\sqrt{9-4x^2}}dx$$

$$=\frac{1}{2}\int \frac{1}{\sqrt{3^2-(2x)^2}}d(2x)+\frac{1}{8}\int \frac{(9-4x^2)'}{\sqrt{9-4x^2}}dx$$

$$=\frac{1}{2}\arcsin\frac{2x}{3}+\frac{1}{8}\cdot 2\sqrt{9-4x^2}+C$$

$$2.(7).\int \frac{x^3}{9+x^2}dx=\frac{1}{2}\int \frac{x^2+9-9}{9+x^2}d(x^2)=\cdots$$

$$2.(8).\int \frac{dx}{x(x^6+4)}=\int \frac{x^5dx}{x^6(x^6+4)}=\frac{1}{6}\int \frac{d(x^6)}{x^6(x^6+4)}$$

$$2.(9).\int\frac{10^{2\arccos x}}{\sqrt{1-x^2}}dx=-\int 100^{\arccos x}d\left(\arccos x\right)$$

$$=-\frac{1}{\ln 100}100^{\arccos x}+C$$

$$2.(10).\int\frac{\ln\tan x}{\cos x\sin x}dx=\int\frac{\ln\tan x}{\cos^2 x\tan x}dx$$

$$=\int\frac{\ln\tan x}{\tan x}\sec^2 x dx=\int\frac{\ln\tan x}{\tan x}d\tan x$$

$$=\int\ln\tan x d\left(\ln\tan x\right)=\frac{1}{2}\left(\ln\tan x\right)^2+C$$

$$2.(11).\int \frac{\arctan\sqrt{x}}{\sqrt{x}\left(1+x\right)}dx = 2\int \frac{\arctan\sqrt{x}}{1+x}d\left(\sqrt{x}\right)$$

$$= 2\int \arctan\sqrt{x}\cdot\frac{1}{1+\left(\sqrt{x}\right)^2}d\left(\sqrt{x}\right)$$

$$= 2\int \arctan\sqrt{x}d\left(\arctan\sqrt{x}\right)$$

$$= \left(\arctan\sqrt{x}\right)^2 + C$$

$$2.(11).\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx$$

法二 $t=\arctan\sqrt{x}, x=\tan^2 t,$

$$I=\int \frac{t}{\tan t\sec^2 t}\cdot 2\tan t\cdot\sec^2 t dt$$

$$=\int 2tdt=t^2+C=\left(\arctan\sqrt{x}\right)^2+C$$

$$2.(12).\int \frac{x+1}{x\left(1+xe^{x}\right)}dx = I$$

$$\because \left(xe^{x}\right)' = (1+x)e^{x},$$

$$\therefore I = \int \frac{(1+x)e^{x}}{xe^{x}\left(1+xe^{x}\right)}dx \xlongequal{xe^{x}=t} \int \frac{1}{t(1+t)}dt$$

$$= \int \left(\frac{1}{t} - \frac{1}{1+t}\right)dt = \ln\left|\frac{t}{1+t}\right| + C$$

$$= \ln\left|\frac{xe^{x}}{1+xe^{x}}\right| + C$$

$$3.(1).\int \frac{dx}{x+\sqrt{1-x^2}} \xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=\sin t} = \int \frac{\cos t}{\sin t+\cos t}dt = \cdots$$

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$$3.(2).\int\frac{dx}{\sqrt{(x^2+1)^3}}\overset{x=\tan t}{\underset{t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}{=\!=\!=\!=\!=\!=\!=}}$$

$$=\int\frac{1}{\left|\sec^3 t\right|}\cdot\sec^2 t\,dt=\int\cos t\,dt$$

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$$3.(3).\int \frac{dx}{1+\sqrt{2x}};$$

$$3.(4).\int x\sqrt{\frac{x}{2a-x}}dx.$$

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3. 分部积分法

前面我们已经看到,利用复合函数的求导公式,推导得到了**两类积分换元公式.**

下面我们要利用两个函数乘积的求导公式,推导得到另一样的积分计算公式——

分部积分公式

分部积分法 integration by parts

问题 $\int xe^x dx = ?$

解决思路：利用两个函数乘积的求导公式.

设函数$u = u(x)$和$v = v(x)$有连续的导数,则

$$(uv)' = u'v + uv', u'v = (uv)' - uv',$$

$$\int u'vdx = \int (uv)'dx - \int uv'dx,$$

$$\Rightarrow \int u'vdx = uv - \int uv'dx,$$

分部积分公式

$$or \quad \int vdu = uv - \int udv$$

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基本初等函数乘积的分部积分:

x^{μ};e^{ax};$\ln x$; $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$;

$\arcsin x, \arccos x, \arctan x, \operatorname{arccot} x$.

(1).$\int x^{\mu} e^{ax} dx, \int x^{\mu} \sin bx dx, \int x^{\mu} \cos bx dx$仅$\mu \in \mathbb{N}$时可做.

(2).$\int x^{\mu} \ln dx, \int x^{\mu} \arcsin x dx, \int x^{\mu} \arctan x dx$ 可做.

(3).$\int e^{ax} \sin x dx, \int e^{ax} \cos x dx$ 可做.

(4).$\int x \tan^2 x dx, \int x \sec^4 x dx, \int x \csc^2 x dx \cdots$可做.

(5).$\int x^{\mu} \tan x dx, \int x^{\mu} \sec x dx, \int x^{\mu} \cot x dx$ 不可做.

(6).$\int e^{ax} \tan x dx, \int e^{ax} \sec x dx, \int e^{ax} \ln x dx$,

$\int e^{ax} \arcsin x dx, \int \sec x \ln x dx, \int \cos x \ln x dx \cdots$皆不可做.

(7)*.$\int \ln x \arcsin x dx$可做,$\int \ln x \arctan x dx \cdots$不可做.

返回

例20.求积分$J=\int x\cos x dx$.

解(1).令$v=\cos x, u'=x$,则可取$u=\frac{1}{2}x^2$,

$$J=\int\left(\frac{1}{2}x^2\right)'\cos x dx=\frac{x^2}{2}\cos x-\int\frac{x^2}{2}(-\sin x)dx,$$

显然 u',v 的选择不当,致使未能解决问题.

解(2).令$v=x, u'=\cos x$,则可取$u=\sin x$,

$$\int x\cos x dx=\int x(\sin x)' dx$$

自然而然的选择!

$$=x\sin x-\int\sin x\cdot 1dx=x\sin x+\cos x+C.$$

在用分部积分法求积分$\int x\cos x dx$时，

令$u' = \cos x$,则可取$u = \sin x$,

由u'求u时的常数不用考虑.

$$\because \int x\cos x dx = \int x\left(\sin x + C_1\right)' dx$$

$$= x\left(\sin x + C_1\right) - \int\left(\sin x + C_1\right)\cdot 1dx$$

$$= x\left(\sin x + C_1\right) + \cos x - C_1 x + C$$

$$= x\sin x + \cos x + C$$

例21.求积分$\int x^2e^{-x}dx$.

解　令$v=x^2, u'=e^{-x}$,则可取$u=-e^{-x}$,

$$\int x^2e^{-x}dx=-x^2e^{-x}-\left(-\int e^{-x}\cdot 2xdx\right)$$

$\boxed{\text{再次分部积分}}$

$$=-x^2e^{-x}+2\int x\left(-e^{-x}\right)'dx$$

$$=-x^2e^{-x}+2\left(-xe^{-x}+\int e^{-x}\cdot 1dx\right)$$

$$=-x^2e^{-x}+2\left(-xe^{-x}-e^{-x}\right)+C$$

$$=-\left(x^2+2x+2\right)e^{-x}+C$$

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顺便提一下,以下规律性的结果

$$\int xe^{-x}dx = -(x+1)e^{-x} + C$$

$$\int x^2e^{-x}dx = -(x^2+2x+2)e^{-x} + C$$

$$\int x^3e^{-x}dx = -(x^3+3x^2+6x+6)e^{-x} + C$$

$\cdots \quad \cdots$

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$$\int u'v\,dx = uv - \int uv'\,dx$$

$$\left.\begin{array}{l}\int x\cos x\,dx \\ \\ \int x^2 e^{-x}\,dx\end{array}\right\} \begin{array}{c}\text{令} x^{\mu} = v \\ \mu \in \mathbb{Z}^+\end{array},\ \begin{cases} u' = \cos x \\ \\ u' = e^{-x}\end{cases}$$

小结 若被积函数是(正整数次幂的)幂函数与正(余)弦函数或指数函数的乘积, 就考虑把正(余)弦或指数函数作为u', 而设幂函数为v,通过求导使其幂次降低.

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例22.求积分$\int x^3 \ln x dx$.

解 令$v=\ln x, x^3 dx = d\left(\dfrac{x^4}{4}\right) = du$,

顺其自然的选择!

$$\int x^3 \ln x dx = \int \left(\frac{x^4}{4}\right)' \ln x dx$$

$$= \frac{1}{4}x^4 \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

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例22.求积分$\int x^3 \ln x dx$.

解二 令$\ln x = t$,则$x = e^t$,

$$\int x^3 \ln x dx = \int e^{3t} \cdot t \cdot e^t dt = \int te^{4t} dt$$

$$\overset{4t=u}{=\!=\!=} \frac{1}{16}\int ue^u du = \frac{1}{16}\left(ue^u - \int e^u du\right)$$

$$= \frac{1}{16}\left(ue^u - e^u\right) + C = \frac{1}{16}\left(4te^{4t} - e^{4t}\right) + C$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

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顺其自然的选择!

例23.求积分$\int x\arctan x dx$.

解 令$v=\arctan x$, $xdx=d\left(\frac{x^2}{2}\right)=du$,

$$\int x\arctan x dx=\frac{x^2}{2}\arctan x-\int\frac{x^2}{2}(\arctan x)'dx$$

$$=\frac{x^2}{2}\arctan x-\int\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\arctan x-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\arctan x-\frac{1}{2}(x-\arctan x)+C.$$

例23.求积分$\int x\arctan x dx$

解二　可令$\arctan x = t$,则$x=\tan t$,

$$\int x\arctan x dx = \int \tan t \cdot t \cdot \sec^2 t dt$$

$$= \int t\left(\frac{1}{2}\tan^2 t\right)' dt = t\cdot\frac{1}{2}\tan^2 t - \int \frac{1}{2}\tan^2 t dt$$

$$= \frac{1}{2}t\tan^2 t - \frac{1}{2}\int\left(\sec^2 t - 1\right)dt$$

$$= \frac{1}{2}t\tan^2 t - \frac{1}{2}\tan t + \frac{1}{2}t + C$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C$$

此时换元法并不是必要的选择!

例23.(2).求积分$\int \arctan x dx$

解 $\int \arctan x dx = \int (x)' \arctan x dx$

$$= x\arctan x - \int x \cdot \frac{1}{1+x^2} dx$$

$$= x\arctan x - \frac{1}{2}\int \left(\ln\left(1+x^2\right)\right)' dx$$

$$= x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) + C$$

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$$\int u'v\,dx = uv - \int uv'\,dx$$

$$\left.\begin{array}{l}\displaystyle\int x^3 \ln x\,dx \\ \\ \\ \displaystyle\int x \arctan x\,dx\end{array}\right\} \begin{array}{c}\text{令}x^{\mu} = u', \\ \mu \neq -1\end{array} \begin{cases} v = \ln x \\ \\ v = \arctan x \end{cases}$$

小结 若被积函数是幂函数与对数函数或反三角函数的乘积,就考虑把幂函数作为u',而设对数函数或反三角函数为 v, 通过求导简化被积函数.

课堂练习4.计算不定积分

(1).$\int x\sin 2x dx$;

(2).$\int \left(xe^{-x}\right)^2 dx$;

(3).$\int \left(\dfrac{\ln x}{x}\right)^2 dx$;

(4).$\int \arcsin x dx$.

$$Ex.1.(1).\int x\sin 2x\,dx=\int x\left(-\frac{1}{2}\cos 2x\right)'dx$$

$$=-\frac{1}{2}x\cos 2x+\frac{1}{2}\int \cos 2x\,dx$$

$$=-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+C$$

$$Ex.1.(2).\int\left(xe^{-x}\right)^2dx=\int x^2e^{-2x}dx=\int x^2\left(-\frac{1}{2}e^{-2x}\right)'dx$$

$$=-\frac{1}{2}x^2e^{-2x}+\frac{1}{2}\int 2xe^{-2x}dx=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}+\frac{1}{2}\int e^{-2x}dx$$

$$=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+C=-\frac{1}{4}\left(2x^2+2x+1\right)e^{-2x}+C$$

或者$J=\int x^2e^{-2x}dx\overset{-2x=t}{=\!=\!=\!=}-\frac{1}{8}\int t^2e^tdt,$

$$\int t^2e^tdt=t^2e^t-\int 2te^tdt=t^2e^t-2te^t+\int 2e^tdt$$

$$=t^2e^t-2te^t+2e^t+C_1,$$

$$\therefore J=-\frac{1}{8}\left(4x^2+4x+2\right)e^{-2x}+C$$

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$$Ex.1.(3).\int\left(\frac{\ln x}{x}\right)^2 dx=\int\left(-\frac{1}{x}\right)'(\ln x)^2 dx$$

$$=-\frac{1}{x}(\ln x)^2+\int\frac{1}{x}\cdot 2(\ln x)\frac{1}{x}dx$$

$$=-\frac{1}{x}(\ln x)^2+2\int\left(-\frac{1}{x}\right)'(\ln x)dx$$

$$=-\frac{1}{x}(\ln x)^2-\frac{2}{x}\ln x+2\int\frac{1}{x}\cdot\frac{1}{x}dx$$

$$=-\frac{1}{x}(\ln x)^2-\frac{2}{x}\ln x-\frac{2}{x}+C$$

或者, $J=\int\left(\frac{\ln x}{x}\right)^2 dx \overset{\ln x=t}{\underset{x=e^t}{=\!=\!=\!=}} \int t^2\cdot e^{-2t}\cdot e^t dt=\int t^2 e^{-t}dt$

$$=\cdots=-\left(t^2+2t+2\right)e^{-t}+C=-\frac{1}{x}\left(\ln^2 x+2\ln x+2\right)+C$$

$$Ex.1.(4).\int \arcsin x dx = \int (x)' \arcsin x dx$$

$$= x\arcsin x - \int x \frac{1}{\sqrt{1-x^2}} dx$$

$$= x\arcsin x - \left(-\frac{1}{2}\right)\int \frac{1}{\sqrt{1-x^2}} d\left(1-x^2\right)$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

或者, $J \overset{\arcsin x = t}{\underset{x=\sin t}{=\!=\!=\!=\!=\!=\!=}} \int t(\sin t)' dt = t\sin t - \int \sin t dt$

$$= t\sin t + \cos t + C = x\arcsin x + \sqrt{1-x^2} + C$$

$$\left(\arcsin x = t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \cos t = +\sqrt{1-x^2}\right)$$

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例24.求积分$\int e^x \sin x dx$.

解 $\boxed{\int e^x \sin x dx} = \int e^x d(-\cos x)$

$= -e^x \cos x + \int \cos x de^x = -e^x \cos x + \int e^x \cos x dx$

$= -e^x \cos x + \int e^x d(\sin x)$

$= -e^x \cos x + e^x \sin x - \boxed{\int e^x \sin x dx}$

注意循环形式

$$\therefore \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

在使用分部积分公式时应注意,倘若要接连几次应用分部积分公式,需注意前后几次所选的 u' 应为同类型函数.

例如, $\int e^x \sin x dx$, 第一次用分部积分公式时选择 $u' = \sin x$,

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

那么第二次用分部积分公式时应仍选择 $u' = \cos x$.

或者,

$$\boxed{\int e^x \sin x dx} = \int \sin x d\left(e^x\right)$$

$$= e^x \sin x - \int e^x d(\sin x)$$

$$= e^x \sin x - \int e^x \cos x dx = e^x \sin x - \int \cos x d\left(e^x\right)$$

$$= e^x \sin x - \left(e^x \cos x - \int e^x d(\cos x)\right)$$

$$= e^x \left(\sin x - \cos x\right) - \boxed{\int e^x \sin x dx}$$

注意循环形式

$$\therefore \int e^x \sin x dx = \frac{e^x}{2}\left(\sin x - \cos x\right) + C$$

求积分$\int e^x \sin x dx$.

解 $\int e^x \sin x dx = \int \sin x de^x$

$= e^x \sin x - \int \cos x de^x$

$= e^x(\sin x - \cos x) - \int e^x \sin x dx$

避免如下错误：

$\therefore \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x)$

注意任意常数

将分部积分法改写一下,得到所谓的“联立方程法”:

记$J_1 = \int e^x \sin x dx, J_2 = \int e^x \cos x dx,$

$$\because \left(e^x \sin x\right)' = e^x \cos x + e^x \sin x,$$

$$\left(e^x \cos x\right)' = e^x \cos x - e^x \sin x,$$

$$\therefore \begin{cases} J_1 + J_2 = e^x \sin x + C_1 \\ -J_1 + J_2 = e^x \cos x + C_2 \end{cases}$$

解此方程组得

$$\begin{cases} J_1 = \dfrac{e^x}{2}\left(\sin x - \cos x\right) + C_3 \\ J_2 = \dfrac{e^x}{2}\left(\sin x + \cos x\right) + C_4 \end{cases}.$$

例24.(2).求积分$\int e^{ax}\cos bx dx, a^2+b^2\neq 0.$

解 用"联立方程法"较为方便.

记 $J_1=\int e^{ax}\cos bx dx, J_2=\int e^{ax}\sin bx dx,$

$$\because \left(e^{ax}\cos bx\right)'=ae^{ax}\cos bx-be^{ax}\sin bx,$$

$$\left(e^{ax}\sin bx\right)'=be^{ax}\cos bx+ae^{ax}\sin bx,$$

$$\therefore \begin{cases} aJ_1-bJ_2=e^{ax}\cos bx+C_1 \\ bJ_1+aJ_2=e^{ax}\sin bx+C_2 \end{cases}$$

解此方程组得 $J_1, J_2\cdots$

课堂练习5.计算不定积分

$$\int e^{-x}\sin 2x dx$$

解 $\int e^{-x}\sin 2x dx =$

小结

合理选择u', v,正确使用分部积分公式

$$\int u'v\,dx = uv - \int uv'\,dx$$

(1).$\int x^n e^{ax}dx, \int x^n \cos bx\,dx, \int x^n \sin bx\,dx$

取 $e^{ax}, \cos bx, \sin bx = u'$,而$x^n = v \cdots$

(2).$\int x^\mu \ln x\,dx, \int x^\mu \arctan x\,dx, \int x^\mu \arcsin x\,dx$

取 $x^\mu = u'$,而$\ln x, \arctan x, \arcsin x = v \cdots$

(3).$\int e^{ax}\cos bx\,dx, \int e^{ax}\sin bx\,dx$

取 $e^{ax} = u'$,而$\cos bx, \sin bx = v$,两次分部积分;

或者取 $\cos bx, \sin bx = u'$,而$e^{ax} = v$,两次分部积分.

例25.设e^{-x^2}是$f(x)$的一个原函数,求$\int xf'(x)dx$.

解 $\int xf'(x)dx = \int xdf(x) = xf(x) - \int f(x)dx$,

$\because \left(\int f(x)dx\right)' = f(x)$,而$\int f(x)dx = e^{-x^2} + C$,

两边同时对求x导,得$f(x) = -2xe^{-x^2}$,

$$\therefore \int xf'(x)dx = xf(x) - \int f(x)dx$$

$$= -\left(2x^2 + 1\right)e^{-x^2} + C.$$

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例26.求积分$\int e^{-\sqrt{x}}dx$.

解 $\int e^{-\sqrt{x}}dx \xlongequal[dx=2tdt]{t=\sqrt{x}} \int 2te^{-t}dt = \int 2t(-e^{-t})'dt$

$= 2t(-e^{-t}) - \int (-e^{-t})(2t)'dt$

$= -2te^{-t} + \int 2e^{-t}dt$

$= -2(t+1)e^{-t} + C$

$= -2\left(\sqrt{x}+1\right)e^{-\sqrt{x}} + C$

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例27*.求积分$J=\int e^{-x}\arctan e^{-x}dx$.

解 考虑作凑微分：$J=-\int \arctan e^{-x}d\left(e^{-x}\right)$

$$\xlongequal{e^{-x}=u}-\int \arctan u\,du=-\int (u)'\arctan u\,du$$

$$=-u\arctan u+\int \frac{u}{1+u^2}du$$

$$=-u\arctan u+\frac{1}{2}\int \frac{1}{1+u^2}d\left(1+u^2\right)$$

$$=-u\arctan u+\frac{1}{2}\ln\left(1+u^2\right)+C$$

$$=\ln\sqrt{1+e^{-2x}}-e^{-x}\arctan e^{-x}+C$$

例27.(2)*.求积分$J=\int x\tan^2 x dx$

解 $J=\int x\tan^2 x dx=\int x\left(\sec^2 x-1\right)dx$

$$=\int x\left(\sec^2 x\right)dx-\int x dx=\int x\left(\tan x\right)' dx-\frac{1}{2}x^2$$

$$=x\tan x-\int \tan x dx-\frac{1}{2}x^2$$

$$=x\tan x-\int \frac{\sin x}{\cos x}dx-\frac{1}{2}x^2$$

$$=x\tan x+\ln\left|\cos x\right|-\frac{1}{2}x^2+C$$

例28*.设$\dfrac{\sin x}{x}$是$f(x)$的一原函数,求$\int xf'(2x)dx$.

解 $\because \dfrac{\sin x}{x}$是$f(x)$的一个原函数,

$$\therefore\ f(x)=\left(\frac{\sin x}{x}\right)'=\frac{x\cos x-\sin x}{x^2},$$

$$\therefore \int xf'(2x)dx \overset{2x=u}{=\!=\!=} \frac{1}{4}\int uf'(u)du$$

$$=\frac{1}{4}uf(u)-\frac{1}{4}\int f(u)du$$

$$=\frac{1}{4}u\cdot\frac{u\cos u-\sin u}{u^2}-\frac{1}{4}\frac{\sin u}{u}+C$$

$$=\frac{u\cos u-2\sin u}{4u}+C \overset{u=2x}{=\!=\!=} \frac{x\cos 2x-\sin 2x}{4x}+C$$

返回

例28.(2)*.求积分$\int \frac{x\arctan x}{\sqrt{1+x^2}}dx$.

解 $\because \left(\sqrt{1+x^2}\right)' = \frac{x}{\sqrt{1+x^2}}$,

$$\therefore \int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \arctan x d\sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2}\, d(\arctan x)$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2} \cdot \frac{1}{1+x^2}dx$$

$$= \sqrt{1+x^2}\arctan x - \boxed{\int \frac{1}{\sqrt{1+x^2}}dx}$$ 令$x = \tan t$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{1}{\sqrt{1+\tan^2 t}}\sec^2 t\,dt = \int \sec t\,dt$$

$$= \ln(\sec t + \tan t) + C = \ln\left(x + \sqrt{1+x^2}\right) + C$$

$$\therefore \quad \int \frac{x\arctan x}{\sqrt{1+x^2}}dx$$

$$= \sqrt{1+x^2}\arctan x - \ln\left(x + \sqrt{1+x^2}\right) + C.$$

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求积分$\int \frac{x\arctan x}{\sqrt{1+x^2}}dx.$

解二 考虑先作变量代换：

$\arctan x = t \leftrightarrow x = \tan t, t \in (-\pi/2, \pi/2),$

$$\therefore \int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \frac{t\tan t}{|\sec t|}\sec^2 t dt$$

$$= \int \frac{t\tan t}{\sec t}\sec^2 t dt = \int t \tan t \sec t dt$$

$$= \int t d(\sec t) = t\sec t - \int \sec t dt$$

$$= t\sec t - \ln|\tan t + \sec t| + C$$

$$= \sqrt{1+x^2}\cdot \arctan x - \ln\left(x + \sqrt{1+x^2}\right) + C$$

返回

例28.(3)*.求积分$\int \sin(\ln x)dx$.

解　考虑先作变量代换：

$$\ln x = t \to x = e^t \to dx = e^t dt$$

$$\therefore \int \sin(\ln x)dx = \int e^t \sin t dt$$

$$= \frac{1}{2}e^t(\sin t - \cos t) + C$$

$$= \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

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例29**.求积分$J=\int \sec^3 x dx$.

解$J=\int \sec x \sec^2 x dx=\int \sec x(\tan x)' dx$

$=\sec x \tan x-\int \tan x(\sec x)' dx$

$=\sec x \tan x-\int \sec x \tan^2 x dx$

$=\sec x \tan x-\int \sec x(\sec^2 x-1) dx$

$=\sec x \tan x-\int \sec^3 x dx+\int \sec x dx$

$\therefore J=\frac{1}{2}\sec x \tan x+\frac{1}{2}\ln|\sec x+\tan x|+C$

例29**.(2).求积分$J=\int\sqrt{a^2+x^2}dx, a>0.$

解 令$x=a\tan t, t\in(-\pi/2,\pi/2)$

$$J=a^2\int|\sec t|\sec^2 t dt=a^2\int\sec^3 t dt$$

$$=\frac{1}{2}a^2\sec t\tan t+\frac{a^2}{2}\ln|\sec t+\tan t|+C_1$$

$$=\frac{1}{2}x\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+C$$

例30*.求积分$I=\int \frac{1}{\left(1+x^2\right)^2}dx.$

解$I=\int \frac{1+x^2-x^2}{\left(1+x^2\right)^2}dx=\int \frac{1}{1+x^2}dx-\int \frac{x^2}{\left(1+x^2\right)^2}dx$

$$=\arctan x-\int x\cdot\left(\frac{-1}{2}\cdot\frac{1}{1+x^2}\right)'dx$$

$$=\arctan x+\frac{x}{2\left(1+x^2\right)}-\frac{1}{2}\int \frac{1}{1+x^2}dx$$

$$=\frac{1}{2}\arctan x+\frac{x}{2\left(1+x^2\right)}+C$$

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例30.(2)**.求积分$I(n)=\int\frac{1}{\left(1+x^2\right)^n}dx.\left(n\geq 2\right)$

$$\because\left(\frac{1}{\left(1+x^2\right)^{n-1}}\right)'=\frac{-2\left(n-1\right)x}{\left(1+x^2\right)^n},$$

$$\therefore I(n)=\int\frac{1+x^2-x^2}{\left(1+x^2\right)^n}dx=I\left(n-1\right)-\int\frac{x^2}{\left(1+x^2\right)^n}dx$$

$$=I\left(n-1\right)-\int x\cdot\left(\frac{-1}{2(n-1)}\cdot\frac{1}{\left(1+x^2\right)^{n-1}}\right)'dx$$

$$=I\left(n-1\right)+\frac{1}{2\left(n-1\right)}\cdot\frac{x}{\left(1+x^2\right)^{n-1}}-\frac{1}{2\left(n-1\right)}\int\frac{1}{\left(1+x^2\right)^{n-1}}dx$$

$$=\frac{2n-3}{2n-2}I\left(n-1\right)+\frac{1}{2\left(n-1\right)}\cdot\frac{x}{\left(1+x^2\right)^{n-1}}.$$

进一步地,求积分

$$J(a,n)=\int \frac{1}{\left(a^2+x^2\right)^n}dx.\left(a>0,\begin{matrix} n\in\mathbb{Z}^+ \\ n\geq 2\end{matrix}\right)$$

$$J(a,n)=\int \frac{1}{\left(a^2+x^2\right)^n}dx$$

$$=\frac{1}{a^{2n-1}}\int \frac{1}{\left[1+\left(\frac{x}{a}\right)^2\right]^n}d\left(\frac{x}{a}\right).$$

我们可由$I(n)=\int \frac{1}{\left(1+x^2\right)^n}dx$ 计算得到结论.

$$a>0, n\in\mathbb{Z}^+, n\ge 2$$

$$J(a,n)=\int\frac{1}{\left(a^2+x^2\right)^n}dx$$

$$=\frac{1}{a^{2n-1}}\int\frac{1}{\left[1+\left(\frac{x}{a}\right)^2\right]^n}d\left(\frac{x}{a}\right).$$

$$I(n)=\int\frac{1}{\left(1+x^2\right)^n}dx=$$

$$\begin{cases}\dfrac{2n-3}{2n-2}I(n-1)+\dfrac{1}{2(n-1)}\cdot\dfrac{x}{\left(1+x^2\right)^{n-1}}, & n\ge 2\\ \arctan x+C, & n=1\end{cases}$$

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求积分$J(a,n)=\int\frac{1}{\left(a^2+x^2\right)^n}dx$ 时,也可

用变量代换:$x=a\tan t, t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

$a>0, n\in\mathbb{Z}^+, n\geq 2$

$$J(a,n)=\int\frac{1}{a^n\sec^{2n}t}a\sec^2 t dt$$

$$=\frac{1}{a^{n-1}}\int\left(\cos^2 t\right)^{n-1}dt=\cdots$$

例31.求不定积分$\int \frac{x+\sin 2x}{1+\cos 2x}dx$.

解　原$=\int \frac{x+\sin 2x}{1+\cos 2x}dx$

$=\int \frac{x+2\sin x\cos x}{2\cos^2 x}dx=\frac{1}{2}\int\left(x\sec^2 x+\tan x\right)dx$

其中$\int x\sec^2 x dx=\int x(\tan x)' dx$

$=x\tan x-\int \tan x dx,$

$\therefore$ 原$=\frac{1}{2}x\tan x+C.$

例31.(2). 求不定积分

$$(A).\ \int \frac{1+\cos x}{x+\sin x}\,dx \quad ;$$

$$(B).\ \int \frac{x+\sin x}{1+\cos x}\,dx \quad .$$

自我检测题：计算下列不定积分.

3.(1). $\int x^2\cos^2\frac{x}{2}dx$,　(2). $\int\left(\frac{\ln x}{x}\right)^2 dx$,

(3). $\int e^{3\sqrt{x}}dx$,　(4). $\int\cos(\ln x)dx$,

(5). $\int\frac{\arctan x}{\sqrt{(1+x^2)^3}}dx$,　(6). $\int e^{-x}\cos 2x dx$,

(7). $\int\frac{xe^{\arctan x}}{\sqrt{(1+x^2)^3}}dx$.

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$$(1).\int x^2\cos^2\frac{x}{2}dx=\frac{1}{2}\int x^2(1+\cos x)dx$$

$$=\frac{1}{2}\int x^2dx+\frac{1}{2}\int x^2\cos xdx$$

$$\int x^2\cos xdx=\int x^2(\sin x)'dx$$

$$=x^2\sin x-\int\sin x\left(x^2\right)'dx=x^2\sin x-\int 2x\sin xdx$$

$$=x^2\sin x-\int 2x(-\cos x)'dx$$

$$=x^2\sin x+2x\cos x-2\int\cos xdx=\cdots$$

(5).令 $\arctan x = t$,

则 $x = \tan t, t \in \left(-\pi/2, \pi/2\right)$

$$\int \frac{\arctan x}{\sqrt{\left(1+x^2\right)^3}}dx$$

$$= \int \frac{t}{\left|\sec t\right|^3}\sec^2 t dt = \int t \cos t dt$$

$= \cdots$

$$(6).\ \underline{\int e^{-x}\cos 2x\,dx} = \int \left(-e^{-x}\right)'\cos 2x\,dx$$

$$= -e^{-x}\cos 2x + \int e^{-x}\left(\cos 2x\right)' dx$$

$$= -e^{-x}\cos 2x + \int e^{-x}\left(-2\sin 2x\right) dx$$

$$= -e^{-x}\cos 2x - 2\int \left(-e^{-x}\right)'\sin 2x\,dx$$

$$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x - 2\int e^{-x}\left(\sin 2x\right)' dx$$

$$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x \underline{-4\int e^{-x}\cos 2x\,dx}$$

$$\therefore \int e^{-x}\cos 2x\,dx = \frac{1}{5}e^{-x}\left(2\sin 2x - \cos 2x\right) + C$$

$$(5).\int \frac{\arctan x}{\sqrt{\left(1+x^2\right)^3}}dx, \quad (6).\int e^{-x}\cos 2x dx,$$

$$(7).\int \frac{xe^{\arctan x}}{\sqrt{\left(1+x^2\right)^3}}dx.$$