09. 积分部分习题课 2022-12

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1.积分计算问题

(1).$\left(P192/Ex.3\right)$若$f(x)$在$[a,b]$上可积，

$F(x)$在$[a,b]$上连续，且除有限多个点

外有$F'(x)=f(x)$，则有

$$\int_a^b f(x)dx = F(b)-F(a).$$

(2).计算$\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

(3).计算$\int_{-1}^{1}\frac{1}{1+x^4}dx$.

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2.证明：$\int_0^{2\pi} e^{\sin^2 x} dx \geq 3\pi.$

3.设$f(x)$在$[a,b]$上有连续的导函数,且$f(a)=0$,

$|f'(x)| \leq M, x \in [a,b]$.求证：$\left|\int_a^b f(x)dx\right| \leq \frac{1}{2}M(b-a)^2.$

4.设$f \in C[0,1], f(x)>0$.证明

$$\ln\int_0^1 f(x)dx \geq \int_0^1 \ln f(x)dx.$$(与$P220/Ex.1,8$ 类同)

5.设$f(x),g(x)$在区间$[a,b]$上连续且同为单调增加或单调减少,则有

$$\int_a^b f(x)dx\int_a^b g(x)dx \leq (b-a)\int_a^b f(x)g(x)dx\ .$$

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6. 设$f(x),g(x)$在$[a,b]$上可积,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx\ .$$

(Cauchy – Schwarz – Bunijiakovsky 不等式)

7.设函数$f(x)$在$[a,b]$上连续,求证:

$$\left(\int_a^b f(x)dx\right)^2 \le (b-a)\int_a^b f^2(x)dx\ .$$

8.(P205/Ex.11)若$f(x)$在$[a,b]$上二阶可导,且$f''(x)>0$.求证:

$$\int_a^b f(x)dx \ge (b-a)\, f\left(\frac{a+b}{2}\right).$$

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9.计算不定积分

$$(1).\int \frac{dx}{x+\sqrt{1-x^2}}\ ;\qquad (2).\int \sqrt{a^2-x^2}\,dx\ ;\qquad (3).\int \frac{dx}{1+\sqrt{2x}}\ ;$$

$$(4).\int (x\ln x)^2\,dx\ ;\qquad (5).\int e^{-\sqrt[3]{x}}\,dx\ ;$$

$$(6).\int \frac{\arctan x}{\sqrt{\left(1+x^2\right)^3}}\,dx\ ;\quad (7).\int \sqrt{e^x-1}\,dx\ ;$$

$$(8).\int \frac{1}{\left(1+x^2\right)^2}\,dx\ ;\qquad (9).\int \frac{x+2x^3}{1+x+x^2}\,dx\ .$$

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1.积分计算问题

(1).$(P192/Ex.3)$若$f(x)$在$[a,b]$上可积,$F(x)$在$[a,b]$上连续,且除有限多个点外有$F'(x)=f(x)$,则有

$$\int_a^b f(x)dx = F(b)-F(a)$$

这叫作拓广的$Newton-Leibniz$公式.

证明 取$[a,b]$的一个划分$T=\{x_0,x_1,\cdots,x_n\}$,$a=x_0,x_n=b$,使得使$F'(x)=f(x)$不成立的点成为划分T的部分分点,

在$\Delta_k=[x_{k-1},x_k]$上由$Lagrange$微分中值定理得

$$F(x_k)-F(x_{k-1})=F'(\xi_k)\Delta x_k = f(\xi_k)\Delta x_k,$$

则$F(b)-F(a)=\sum_{k=1}^{n}[F(x_k)-F(x_{k-1})]=\sum_{k=1}^{n} f(\xi_k)\Delta x_k$,

$\because f(x)$在$[a,b]$上可积,$\therefore \lim_{\|T\|\to 0}\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \int_a^b f(x)dx$.证毕

例如,计算积分:$\int_{-1}^{1}\frac{1}{1+x^2}dx$.

解 在$(-\infty,+\infty)$上,$\arctan x$是$\frac{1}{1+x^2}$的一个原函数,

$$\therefore \int_{-1}^{1}\frac{1}{1+x^2}dx=\left[\arctan x\right]_{-1}^{1}=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}.$$

$x\neq 0$时,$\left(-\arctan\frac{1}{x}\right)'=\frac{1}{1+x^2}$,但是$-\arctan\frac{1}{x}$在$x=0$时没有定义,

所以$-\arctan\frac{1}{x}$不是$\frac{1}{1+x^2}$在$[-1,1]$上的一个原函数.稍作改造,

$$\Phi(x)=\begin{cases}-\arctan\frac{1}{x}, x<0\\ \pi/2 \quad , \quad x=0\\ \pi-\arctan\frac{1}{x}, x>0\end{cases}$$

,$\Phi(x)$在$[-1,1]$上连续,且$x\neq 0$时$\Phi'(x)=\frac{1}{1+x^2}$,

于是,$\int_{-1}^{1}\frac{1}{1+x^2}dx=\Phi(1)-\Phi(-1)=\pi-\arctan 1-\left(-\arctan\frac{1}{(-1)}\right)=\frac{\pi}{2}$.

1.积分计算(2). $I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

$$\int\frac{1}{2+\cos 2x}dx=\int\frac{1}{1+2\cos^2 x}dx=\int\frac{\sec^2 x}{2+\sec^2 x}dx$$

$$=\int\frac{(\tan x)'}{3+\tan^2 x}dx=\int\frac{1}{\left(\sqrt{3}\right)^2+\tan^2 x}d(\tan x)$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+C.$$

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1.积分计算(2).$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

倘由$\int\frac{1}{2+\cos 2x}dx=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+C$

得$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}\Big|_0^{\pi}=0,$

在$[0,\pi]$上$\frac{1}{2+\cos 2x}\geq\frac{1}{3}>0,$我们知道错了!

由$\frac{1}{2+\cos 2x}$在$[0,\pi]$上连续,因而其原函数也

必须是连续的.故$\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}$不是$\frac{1}{2+\cos 2x}$

在$[0,\pi]$上的原函数.

解 $\int_0^{\pi}\frac{1}{2+\cos 2x}dx=\int_0^{\pi}\frac{1}{1+2\cos^2 x}dx$

$$=\int_0^{\pi/4}\frac{1}{1+2\cos^2 x}dx+\int_{\pi/4}^{3\pi/4}\frac{1}{1+2\cos^2 x}dx+\int_{3\pi/4}^{\pi}\frac{1}{1+2\cos^2 x}dx$$

$$=\int_0^{\pi/4}\frac{\sec^2 x}{2+\sec^2 x}dx+\int_{\pi/4}^{3\pi/4}\frac{\csc^2 x}{\csc^2 x+2\cot^2 x}dx+\int_{3\pi/4}^{\pi}\frac{\sec^2 x}{2+\sec^2 x}dx$$

$$=\int_0^{\pi/4}\frac{d(\tan x)}{(\sqrt{3})^2+\tan^2 x}+\int_{\pi/4}^{3\pi/4}\frac{-d(\cot x)}{1+\left(\sqrt{3}\cot x\right)^2}dx+\int_{3\pi/4}^{\pi}\frac{d(\tan x)}{(\sqrt{3})^2+\tan^2 x}$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}\Bigg|_0^{\pi/4}-\frac{1}{\sqrt{3}}\arctan\left(\sqrt{3}\cot x\right)\Bigg|_{\pi/4}^{3\pi/4}+\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}\Bigg|_{3\pi/4}^{\pi}$$

$$=\cdots\cdots=\frac{\pi}{\sqrt{3}}.$$

(注：其中点$\frac{\pi}{4}$的选取是随意的,只要取$\left(0,\frac{\pi}{2}\right)$内的点即可;同样,$\frac{3\pi}{4}$亦如此.)

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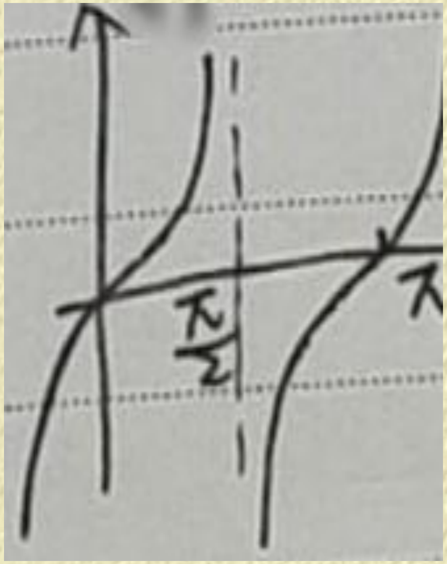

1.积分计算(2).$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

法二 由$\int\frac{1}{2+\cos 2x}dx=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+C$,

据拓广的$Newton-Leibniz$公式,取

$$\Phi(x)=\begin{cases}\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}},0\le x<\frac{\pi}{2}\\ \frac{\pi}{2\sqrt{3}}\quad,\quad x=\frac{\pi}{2}\\ \frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+\frac{\pi}{\sqrt{3}},\frac{\pi}{2}<x\le\pi\end{cases}$$, $\Phi(x)$在$[0,\pi]$上连续,

在$x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$时$\Phi'(x)=\frac{1}{2+\cos 2x}$.

$$\therefore I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx=\Phi(\pi)-\Phi(0)=\frac{\pi}{\sqrt{3}}\ .$$

1.积分计算(3). $I=\int_{-1}^{1}\frac{1}{1+x^4}dx$.

解 $\int\frac{1}{1+x^4}dx=\frac{1}{2\sqrt{2}}\int\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)dx$

$=\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\left[\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right]+C.$

$$\therefore I=\int_{-1}^{1}\frac{1}{1+x^4}dx=\left\{\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\left[\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right]\right\}_{-1}^{1}$$

$=\frac{1}{\sqrt{2}}\left[\ln\left(\sqrt{2}+1\right)+\frac{\pi}{2}\right]=A.$

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1.积分计算(3). $I=\int_{-1}^{1}\frac{1}{1+x^4}dx$.

解二 $\int\frac{1}{1+x^4}dx=\frac{1}{2}\left(\int\frac{1+x^2}{1+x^4}dx+\int\frac{1-x^2}{1+x^4}dx\right)$

$$=\frac{1}{2}\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx-\frac{1}{2}\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\frac{1}{2}\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}-\frac{1}{2}\int\frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2}$$

$$=\frac{1}{2\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}-\frac{1}{4\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C_1$$

$$=\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}+C_1,$$

1.积分计算(3).$I=\int_{-1}^{1}\frac{1}{1+x^4}dx$.

$$\int\frac{1}{1+x^4}dx=\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}+C_1,$$

作与 1.(2)同样的改造,

$$G(x)=\begin{cases}\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}, & x<0\\ \frac{\pi}{4\sqrt{2}}, & x=0\\ \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}+\frac{\pi}{2\sqrt{2}}, & x>0\end{cases}$$ 连续,

在$x\neq 0$ 时$G'(x)=\frac{1}{1+x^4}$.

则 $\int_{-1}^{1}\frac{1}{1+x^4}dx=G(1)-G(-1)=\frac{1}{\sqrt{2}}\left[\ln\left(\sqrt{2}+1\right)+\frac{\pi}{2}\right]=A.$

2.证明：$\int_0^{2\pi} e^{\sin^2 x} dx \geq 3\pi$.

Hint. $\forall t \in \mathbb{R}$有$e^t \geq 1+t \ \cdots$

3.设$f(x)$在$[a,b]$上有连续的导函数,且$f(a)=0$,

$|f'(x)| \leq M, x \in [a,b]$.求证：$\left|\int_a^b f(x)dx\right| \leq \frac{1}{2}M(b-a)^2$.

Hint. $x \in [a,b]$,

$$f(x) = f(a) + f'(\xi)(x-a) = f'(\xi)(x-a), \xi \in (a,x)$$

$$|f'(x)| \leq M \Rightarrow -M(x-a) \leq f(x) \leq M(x-a), x \in [a,b]$$

$$\therefore -\frac{1}{2}M(b-a)^2 = -M\int_a^b (x-a)dx \leq \int_a^b f(x)dx$$

$$\leq M\int_a^b (x-a)dx = \frac{1}{2}M(b-a)^2.$$

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4.设$f \in C[0,1]$, $f(x)>0$.证明

$$\ln\int_0^1 f(x)dx \geq \int_0^1 \ln f(x)dx.$$（与$P220/Ex.1,8$ 类同）

证明 由$[0,1]$上$f(x)>0$ 知$\int_0^1 f(x)dx = A > 0.$

$\because \forall t>-1, \ln(1+t)\leq t.$ $\therefore \forall x\in[0,1]$,

$$\ln f(x) = \ln A + \ln\left(1+\frac{f(x)}{A}-1\right) \leq \ln A + \frac{f(x)}{A}-1,$$

$$\therefore R \leq \int_0^1\left[\ln A + \frac{f(x)}{A}-1\right]dx = \ln A + \int_0^1 \frac{f(x)}{A}dx - 1$$

$$= \ln A = L.$$

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5.设$f(x),g(x)$在区间$[a,b]$上连续且同为单调增加或单调减少,则有

$$\int_a^b f(x)dx\int_a^b g(x)dx \le (b-a)\int_a^b f(x)g(x)dx \ .$$

证明

6. 设$f(x),g(x)$在$[a,b]$上可积,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2\le\int_a^b f^2(x)dx\int_a^b g^2(x)dx\ .$$

$\left(Cauchy-Schwarz-Bunijiakovsky\right.$ 不等式$\left.\right)$

证明 据离散形式的$Cauchy$不等式$\left(\sum_{i=1}^n a_ib_i\right)^2\le\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$,

$f(x),g(x)$在$[a,b]$上可积$\Rightarrow f(x)g(x)$在$[a,b]$上可积,

对区间$[a,b]$作划分$T=\{\Delta_1,\Delta_2,\cdots,\Delta_n\}$,则相应的$Riemann$积分和

有$\left(\sum_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i\right)^2\le\left(\sum_{i=1}^n f^2(\xi_i)\Delta x_i\right)\left(\sum_{i=1}^n g^2(\xi_i)\Delta x_i\right)$,于是,

$$\lim_{\|T\|\to 0}\left(\sum_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i\right)^2\le\lim_{\|T\|\to 0}\left(\sum_{i=1}^n f^2(\xi_i)\Delta x_i\right)\cdot\lim_{\|T\|\to 0}\left(\sum_{i=1}^n g^2(\xi_i)\Delta x_i\right),$$

即得$\left[\int_a^b f(x)g(x)dx\right]^2\le\int_a^b f^2(x)dx\int_a^b g^2(x)dx\ .$

6. 设$f(x),g(x)$在$[a,b]$上可积,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx\ .$$

(*Cauchy – Schwarz – Bunijiakovsky* 不等式)

证二 $\forall x \in [a,b], \forall \lambda \in \mathbb{R}, (\lambda f(x)+g(x))^2 \ge 0 \Rightarrow$

$\int_a^b [\lambda f(x)+g(x)]^2 dx \ge 0, \therefore \forall \lambda \in \mathbb{R}$,有

$\lambda^2\int_a^b f^2(x)dx + 2\lambda\int_a^b f(x)g(x)dx + \int_a^b g^2(x)dx \ge 0,$

$\int_a^b f^2(x)dx > 0, \Delta \le 0 \Leftrightarrow$

$$\int_a^b f^2(x)dx\int_a^b g^2(x)dx \ge \left[\int_a^b f(x)g(x)dx\right]^2.$$

(2).$\int_a^b f^2(x)dx = 0, \cdots$(回到证法一)

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6. 设$f(x),g(x)$在$[a,b]$上可积,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx \ .$$

(*Cauchy* – *Schwarz* – *Bunijiakovsky* 不等式)

7.设函数$f(x)$在$[a,b]$上连续,求证:

$$\left(\int_a^b f(x)dx\right)^2 \le (b-a)\int_a^b f^2(x)dx \ .$$

证明…

8. 若$f(x)$是$[a,b]$上二阶可导的凸函数,

求证: $\int_a^b f(x)dx \geq (b-a) f\left(\frac{a+b}{2}\right)$.

证明 $\because$ $f(x)$是$[a,b]$上可导的凸函数,

$\therefore$ $x \in [a,b]$时有$f(x) \geq f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)$,

$$\therefore \int_a^b f(x)dx \geq \int_a^b \left[f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)\right]dx$$

$$= (b-a) f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\int_a^b \left(x-\frac{a+b}{2}\right)dx$$

$$= (b-a) f\left(\frac{a+b}{2}\right) + 0 = (b-a) f\left(\frac{a+b}{2}\right).$$

8. 若$f(x)$是$[a,b]$上二阶可导的凸函数,

求证: $\int_a^b f(x)dx \geq (b-a)f\left(\frac{a+b}{2}\right)$.

证二　$\because f(x)$是$[a,b]$上二阶可导函数,

$\forall x \in [a,b], f(x) =$

$$= f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right) + \frac{f''(\xi)}{2}\left(x-\frac{a+b}{2}\right)^2,$$

$\because f(x)$是$[a,b]$上二阶可导的凸函数,故$f''(x) \geq 0$,

$$\therefore f(x) \geq f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)\cdots$$