§2.2 不定积分的计算
——分部积分法

前面我们已经看到,利用复合函数的求导公式,推导得到了**两类积分换元公式.**

下面我们要利用两个函数乘积的求导公式,推导得到另一样的积分计算公式——

分部积分公式

分部积分法 integration by parts

问题 $\int xe^x dx = ?$

解决思路：利用两个函数乘积的求导公式.

设函数$u = u(x)$和$v = v(x)$有连续的导数,则

$$(uv)' = u'v + uv', u'v = (uv)' - uv',$$

$$\int u'vdx = \int (uv)'dx - \int uv'dx,$$

$$\Rightarrow \int u'vdx = uv - \int uv'dx,$$

分部积分公式

$$or \quad \int vdu = uv - \int udv$$

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例1.求积分$J=\int x\cos x dx$.

解(1).令$v=\cos x, u'=x$,则可取$u=\frac{1}{2}x^2$,

$$J=\int\left(\frac{1}{2}x^2\right)'\cos x dx=\frac{x^2}{2}\cos x-\int\frac{x^2}{2}(-\sin x)dx,$$

显然 u',v 的选择不当,致使未能解决问题.

解(2).令$v=x, u'=\cos x$,则可取$u=\sin x$,

$$\int x\cos x dx=\int x(\sin x)' dx$$

自然而然的选择!

$$=x\sin x-\int\sin x\cdot 1 dx=x\sin x+\cos x+C.$$

在用分部积分法求积分$\int x\cos x dx$时，

令$u' = \cos x$,则可取$u = \sin x$,

由u'求u时的常数不用考虑.

$$\because \int x\cos x dx = \int x\left(\sin x + C_1\right)' dx$$

$$= x\left(\sin x + C_1\right) - \int \left(\sin x + C_1\right)\cdot 1 dx$$

$$= x\left(\sin x + C_1\right) + \cos x - C_1 x + C$$

$$= x\sin x + \cos x + C$$

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例1.(2).求积分$\int x^2e^{-x}dx$.

解 令$v=x^2,u'=e^{-x}$,则可取$u=-e^{-x}$,

$$\int x^2e^{-x}dx=-x^2e^{-x}-\left(-\int e^{-x}\cdot 2xdx\right)$$

再次分部积分

$$=-x^2e^{-x}+2\int x\left(-e^{-x}\right)'dx$$

$$=-x^2e^{-x}+2\left(-xe^{-x}+\int e^{-x}\cdot 1dx\right)$$

$$=-x^2e^{-x}+2\left(-xe^{-x}-e^{-x}\right)+C$$

$$=-\left(x^2+2x+2\right)e^{-x}+C$$

顺便提一下,以下规律性的结果

$$\int xe^{-x}dx = -(x+1)e^{-x} + C$$

$$\int x^2e^{-x}dx = -(x^2+2x+2)e^{-x} + C$$

$$\int x^3e^{-x}dx = -(x^3+3x^2+6x+6)e^{-x} + C$$

… …

$$\int u'v\,dx = uv - \int uv'\,dx$$

$$\left.\begin{array}{l}\int x\cos x\,dx \\ \\ \int x^2 e^{-x}\,dx\end{array}\right\} \text{令} x^{\mu} = v,\ \mu \in \mathbb{Z}^{+},\ \left\{\begin{array}{l}u' = \cos x \\ \\ u' = e^{-x}\end{array}\right.$$

小结 若被积函数是(正整数次幂的)幂函数与正(余)弦函数或指数函数的乘积, 就考虑把正(余)弦或指数函数作为u', 而设幂函数为v,通过求导使其幂次降低.

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例2.求积分$\int x^3 \ln x dx$.

解 令$v=\ln x, x^3 dx = d\left(\frac{x^4}{4}\right) = du,$

顺其自然的选择!

$$\int x^3 \ln x dx = \int \left(\frac{x^4}{4}\right)' \ln x dx$$

$$= \frac{1}{4}x^4 \ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

例2.求积分$\int x^3 \ln x dx$.

解二 令$\ln x = t$,则$x = e^t$,

$$\int x^3 \ln x dx = \int e^{3t} \cdot t \cdot e^t dt = \int t e^{4t} dt$$

$$\overset{4t=u}{=\!=\!=} \frac{1}{16}\int u e^u du = \frac{1}{16}\left(u e^u - \int e^u du\right)$$

$$= \frac{1}{16}\left(u e^u - e^u\right) + C = \frac{1}{16}\left(4t e^{4t} - e^{4t}\right) + C$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

顺其自然的选择!

例2.(2). 求积分$\int x\arctan x dx$.

解 令$v=\arctan x$, $xdx=d\left(\dfrac{x^2}{2}\right)=du$,

$$\int x\arctan x dx=\frac{x^2}{2}\arctan x-\int\frac{x^2}{2}(\arctan x)' dx$$

$$=\frac{x^2}{2}\arctan x-\int\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\arctan x-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\arctan x-\frac{1}{2}(x-\arctan x)+C.$$

$$\int u'v\,dx = uv - \int uv'\,dx$$

$$\left.\begin{array}{l}\displaystyle\int x^3 \ln x\,dx \\ \\ \\ \displaystyle\int x \arctan x\,dx\end{array}\right\} \begin{array}{c}\text{令} x^{\mu} = u', \\ \mu \neq -1\end{array} \left\{\begin{array}{l} v = \ln x \\ \\ \\ v = \arctan x\end{array}\right.$$

小结 若被积函数是幂函数与对数函数或反三角函数的乘积,就考虑把幂函数作为u',而设对数函数或反三角函数为 v, 通过求导简化被积函数.

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练练手：1.求积分

(1). $\int x\sin 2x\,dx$ ；

(2). $\int\left(xe^{-x}\right)^2 dx$ ；

(3). $\int\left(\frac{\ln x}{x}\right)^2 dx$ ；

(4). $\int \arcsin x\,dx$.

$$Ex.1.(1).\int x\sin 2x\,dx = \int x\left(-\frac{1}{2}\cos 2x\right)' dx$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

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$$Ex.1.(2).\int\left(xe^{-x}\right)^2dx=\int x^2e^{-2x}dx=\int x^2\left(-\frac{1}{2}e^{-2x}\right)'dx$$

$$=-\frac{1}{2}x^2e^{-2x}+\frac{1}{2}\int 2xe^{-2x}dx=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}+\frac{1}{2}\int e^{-2x}dx$$

$$=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+C=-\frac{1}{4}\left(2x^2+2x+1\right)e^{-2x}+C$$

或者$J=\int x^2e^{-2x}dx\overset{-2x=t}{=\!=\!=\!=}-\frac{1}{8}\int t^2e^tdt,$

$$\int t^2e^tdt=t^2e^t-\int 2te^tdt=t^2e^t-2te^t+\int 2e^tdt$$

$$=t^2e^t-2te^t+2e^t+C_1,$$

$$\therefore J=-\frac{1}{8}\left(4x^2+4x+2\right)e^{-2x}+C$$

$$Ex.1.(3).\int\left(\frac{\ln x}{x}\right)^2 dx=\int\left(-\frac{1}{x}\right)'(\ln x)^2 dx$$

$$=-\frac{1}{x}(\ln x)^2+\int\frac{1}{x}\cdot 2(\ln x)\frac{1}{x}dx$$

$$=-\frac{1}{x}(\ln x)^2+2\int\left(-\frac{1}{x}\right)'(\ln x)dx$$

$$=-\frac{1}{x}(\ln x)^2-\frac{2}{x}\ln x+2\int\frac{1}{x}\cdot\frac{1}{x}dx$$

$$=-\frac{1}{x}(\ln x)^2-\frac{2}{x}\ln x-\frac{2}{x}+C$$

或者, $J=\int\left(\frac{\ln x}{x}\right)^2 dx \overset{\ln x=t}{\underset{x=e^t}{=\!=\!=\!=}} \int t^2\cdot e^{-2t}\cdot e^t dt=\int t^2 e^{-t}dt$

$$=\cdots=-\left(t^2+2t+2\right)e^{-t}+C=-\frac{1}{x}\left(\ln^2 x+2\ln x+2\right)+C$$

$$Ex.1.(4).\int \arcsin x dx = \int (x)' \arcsin x dx$$

$$= x\arcsin x - \int x \frac{1}{\sqrt{1-x^2}} dx$$

$$= x\arcsin x - \left(-\frac{1}{2}\right)\int \frac{1}{\sqrt{1-x^2}} d\left(1-x^2\right)$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

或者, $J \overset{\arcsin x = t}{\underset{x=\sin t}{=\!=\!=\!=\!=\!=\!=}} \int t(\sin t)' dt = t\sin t - \int \sin t dt$

$$= t\sin t + \cos t + C = x\arcsin x + \sqrt{1-x^2} + C$$

$$\left(\arcsin x = t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \cos t = +\sqrt{1-x^2}\right)$$

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例3.求积分$\int e^x \sin x dx$.

解 $\boxed{\int e^x \sin x dx} = \int e^x d(-\cos x)$

$= -e^x \cos x + \int \cos x de^x = -e^x \cos x + \int e^x \cos x dx$

$= -e^x \cos x + \int e^x d(\sin x)$

$= -e^x \cos x + e^x \sin x - \boxed{\int e^x \sin x dx}$ 注意循环形式

$$\therefore \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

在使用分部积分公式时应注意,倘若要接连几次应用分部积分公式,需注意前后几次所选的 u' 应为同类型函数.

例如,$\int e^x \sin x dx$,第一次用分部积分公式时选择 $u' = \sin x$,

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

那么第二次用分部积分公式时应仍选择 $u' = \cos x$.

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或者,

$$\boxed{\int e^{x}\sin x dx}=\int \sin x d\left(e^{x}\right)$$

$$=e^{x}\sin x-\int e^{x}d(\sin x)$$

$$=e^{x}\sin x-\int e^{x}\cos x dx=e^{x}\sin x-\int \cos x d\left(e^{x}\right)$$

$$=e^{x}\sin x-\left(e^{x}\cos x-\int e^{x}d(\cos x)\right)$$

$$=e^{x}\left(\sin x-\cos x\right)-\boxed{\int e^{x}\sin x dx}$$

注意循环形式

$$\therefore \int e^{x}\sin x dx=\frac{e^{x}}{2}\left(\sin x-\cos x\right)+C$$

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求积分$\int e^x \sin x dx$.

解 $\int e^x \sin x dx = \int \sin x de^x$

$= e^x \sin x - \int \cos x de^x$

$= e^x(\sin x - \cos x) - \int e^x \sin x dx$

避免如下错误：

$\therefore \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x)$

注意任意常数

将分部积分法改写一下,得到所谓的“联立方程法”:

记$J_1=\int e^x\sin x dx, J_2=\int e^x\cos x dx,$

$\because \left(e^x\sin x\right)'=e^x\cos x+e^x\sin x,$

$\left(e^x\cos x\right)'=e^x\cos x-e^x\sin x,$

$$\therefore \begin{cases} J_1+J_2=e^x\sin x+C_1 \\ -J_1+J_2=e^x\cos x+C_2 \end{cases}$$

解此方程组得

$$\begin{cases} J_1=\dfrac{e^x}{2}\left(\sin x-\cos x\right)+C_3 \\ J_2=\dfrac{e^x}{2}\left(\sin x+\cos x\right)+C_4 \end{cases}.$$

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例3.(2).求积分$\int e^{ax}\cos bx dx, a^2+b^2\neq 0.$

解 用"联立方程法"较为方便.

记 $J_1=\int e^{ax}\cos bx dx, J_2=\int e^{ax}\sin bx dx,$

$$\because \left(e^{ax}\cos bx\right)'=ae^{ax}\cos bx-be^{ax}\sin bx,$$

$$\left(e^{ax}\sin bx\right)'=be^{ax}\cos bx+ae^{ax}\sin bx,$$

$$\therefore \begin{cases} aJ_1-bJ_2=e^{ax}\cos bx+C_1 \\ bJ_1+aJ_2=e^{ax}\sin bx+C_2 \end{cases}$$

解此方程组得 $J_1, J_2\cdots$

练练手: 2.求积分 $\int e^{-x}\sin 2x dx$

解 $\int e^{-x}\sin 2x dx =$

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小结

合理选择u',v,正确使用分部积分公式

$$\int u'v\,dx = uv - \int uv'\,dx$$

(1).$\int x^n e^{ax}dx, \int x^n \cos bx\,dx, \int x^n \sin bx\,dx$

取 $e^{ax}, \cos bx, \sin bx = u'$,而$x^n = v\cdots$

(2).$\int x^\mu \ln x\,dx, \int x^\mu \arctan x\,dx, \int x^\mu \arcsin x\,dx$

取 $x^\mu = u'$,而$\ln x, \arctan x, \arcsin x = v\cdots$

(3).$\int e^{ax}\cos bx\,dx, \int e^{ax}\sin bx\,dx$

取 $e^{ax} = u'$,而$\cos bx, \sin bx = v$,两次分部积分;

或者取 $\cos bx, \sin bx = u'$,而$e^{ax} = v$,两次分部积分.

例4.设e^{-x^2}是$f(x)$的一个原函数,求$\int xf'(x)dx$.

解 $\int xf'(x)dx = \int xdf(x) = xf(x) - \int f(x)dx$,

$\because \left(\int f(x)dx\right)' = f(x)$,而$\int f(x)dx = e^{-x^2} + C$,

两边同时对求x导,得$f(x) = -2xe^{-x^2}$,

$$\therefore \int xf'(x)dx = xf(x) - \int f(x)dx$$

$$= -\left(2x^2 + 1\right)e^{-x^2} + C.$$

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例5.求积分$\int e^{-\sqrt{x}}dx$.

解 $$\int e^{-\sqrt{x}}dx \xlongequal[dx=2tdt]{t=\sqrt{x}} \int 2te^{-t}dt = \int 2t(-e^{-t})'dt$$

$$= 2t(-e^{-t}) - \int (-e^{-t})(2t)'dt$$

$$= -2te^{-t} + \int 2e^{-t}dt$$

$$= -2(t+1)e^{-t} + C$$

$$= -2\left(\sqrt{x}+1\right)e^{-\sqrt{x}} + C$$

例6*.求积分$J=\int e^{-x}\arctan e^{-x}dx.$

解 考虑作凑微分：$J=-\int \arctan e^{-x}d\left(e^{-x}\right)$

$$\overset{e^{-x}=u}{=\!=\!=}-\int \arctan u\,du=-\int (u)'\arctan u\,du$$

$$=-u\arctan u+\int \frac{u}{1+u^2}du$$

$$=-u\arctan u+\frac{1}{2}\int \frac{1}{1+u^2}d\left(1+u^2\right)$$

$$=-u\arctan u+\frac{1}{2}\ln\left(1+u^2\right)+C$$

$$=\ln\sqrt{1+e^{-2x}}-e^{-x}\arctan e^{-x}+C$$

例6.(2)*.求积分$J=\int x\tan^2 x dx$

解 $J=\int x\tan^2 x dx=\int x\left(\sec^2 x-1\right)dx$

$$=\int x\left(\sec^2 x\right)dx-\int x dx=\int x\left(\tan x\right)' dx-\frac{1}{2}x^2$$

$$=x\tan x-\int \tan x dx-\frac{1}{2}x^2$$

$$=x\tan x-\int \frac{\sin x}{\cos x}dx-\frac{1}{2}x^2$$

$$=x\tan x+\ln\left|\cos x\right|-\frac{1}{2}x^2+C$$

例7*.设$\dfrac{\sin x}{x}$是$f(x)$的一原函数,求$\int xf'(2x)dx$.

解 $\because \dfrac{\sin x}{x}$是$f(x)$的一个原函数,

$$\therefore\ f(x)=\left(\frac{\sin x}{x}\right)'=\frac{x\cos x-\sin x}{x^2},$$

$$\therefore \int xf'(2x)dx \overset{2x=u}{=\!=\!=} \frac{1}{4}\int uf'(u)du$$

$$=\frac{1}{4}uf(u)-\frac{1}{4}\int f(u)du$$

$$=\frac{1}{4}u\cdot\frac{u\cos u-\sin u}{u^2}-\frac{1}{4}\frac{\sin u}{u}+C$$

$$=\frac{u\cos u-2\sin u}{4u}+C \overset{u=2x}{=\!=\!=} \frac{x\cos 2x-\sin 2x}{4x}+C$$

返回

例7.(2)*.求积分$\int \frac{x\arctan x}{\sqrt{1+x^2}}dx.$

解 $\because \left(\sqrt{1+x^2}\right)' = \frac{x}{\sqrt{1+x^2}},$

$$\therefore \int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \arctan x d\sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2} d(\arctan x)$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2} \cdot \frac{1}{1+x^2}dx$$

$$= \sqrt{1+x^2}\arctan x - \boxed{\int \frac{1}{\sqrt{1+x^2}}dx} \quad 令 x = \tan t$$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{1}{\sqrt{1+\tan^2 t}}\sec^2 t dt = \int \sec t dt$$

$$= \ln(\sec t + \tan t) + C = \ln\left(x + \sqrt{1+x^2}\right) + C$$

$$\therefore \quad \int \frac{x \arctan x}{\sqrt{1+x^2}}dx$$

$$= \sqrt{1+x^2}\arctan x - \ln\left(x+\sqrt{1+x^2}\right) + C.$$

求积分$\int \frac{x\arctan x}{\sqrt{1+x^2}}dx.$

解二 考虑先作变量代换：

$$\arctan x = t \leftrightarrow x = \tan t, t \in \left(-\pi/2, \pi/2\right),$$

$$\therefore \int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \frac{t\tan t}{|\sec t|}\sec^2 t dt$$

$$= \int \frac{t\tan t}{\sec t}\sec^2 t dt = \int t\tan t\sec t dt$$

$$= \int t d\left(\sec t\right) = t\sec t - \int \sec t dt$$

$$= t\sec t - \ln\left|\tan t + \sec t\right| + C$$

$$= \sqrt{1+x^2}\cdot\arctan x - \ln\left(x + \sqrt{1+x^2}\right) + C$$

例7.(3)* .求积分$\int \sin(\ln x)dx$.

解 考虑先作变量代换:

$$\ln x = t \to x = e^t \to dx = e^t dt$$

$$\therefore \int \sin(\ln x)dx = \int e^t \sin t dt$$

$$= \frac{1}{2}e^t(\sin t - \cos t) + C$$

$$= \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C.$$

例8**.求积分$J=\int \sec^3 x dx$.

解$J=\int \sec x \sec^2 x dx=\int \sec x(\tan x)' dx$

$=\sec x \tan x-\int \tan x(\sec x)' dx$

$=\sec x \tan x-\int \sec x \tan^2 x dx$

$=\sec x \tan x-\int \sec x(\sec^2 x-1) dx$

$=\sec x \tan x-\int \sec^3 x dx+\int \sec x dx$

$\therefore J=\frac{1}{2}\sec x \tan x+\frac{1}{2}\ln\left|\sec x+\tan x\right|+C$

例8**.(2).求积分$J=\int\sqrt{a^2+x^2}dx, a>0.$

解 令$x=a\tan t, t\in(-\pi/2,\pi/2)$

$$J=a^2\int|\sec t|\sec^2 t\,dt=a^2\int\sec^3 t\,dt$$

$$=\frac{1}{2}a^2\sec t\tan t+\frac{a^2}{2}\ln|\sec t+\tan t|+C_1$$

$$=\frac{1}{2}x\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+C$$

练习题

一.求下列不定积分：

(1). $\int x^2\cos^2\frac{x}{2}dx$,　(2). $\int\left(\frac{\ln x}{x}\right)^2dx$,

(3). $\int e^{3\sqrt{x}}dx$,　(4). $\int\cos(\ln x)dx$,

(5). $\int\frac{\arctan x}{\sqrt{(1+x^2)^3}}dx$,　(6). $\int e^{-x}\cos 2x dx$,

(7). $\int\frac{xe^{\arctan x}}{\sqrt{(1+x^2)^3}}dx$.

$$(1).\int x^2\cos^2\frac{x}{2}dx=\frac{1}{2}\int x^2(1+\cos x)dx$$

$$=\frac{1}{2}\int x^2dx+\frac{1}{2}\int x^2\cos xdx$$

$$\int x^2\cos xdx=\int x^2(\sin x)'dx$$

$$=x^2\sin x-\int\sin x\left(x^2\right)'dx=x^2\sin x-\int 2x\sin xdx$$

$$=x^2\sin x-\int 2x(-\cos x)'dx$$

$$=x^2\sin x+2x\cos x-2\int\cos xdx=\cdots$$

(5).令 $\arctan x = t$,

则 $x = \tan t, t \in (-\pi/2, \pi/2)$

$$\int \frac{\arctan x}{\sqrt{(1+x^2)^3}} dx$$

$$= \int \frac{t}{|\sec t|^3} \sec^2 t dt = \int t \cos t dt$$

$$= \cdots$$

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$$(6).\underline{\int e^{-x}\cos 2x dx} = \int \left(-e^{-x}\right)' \cos 2x dx$$

$$= -e^{-x}\cos 2x + \int e^{-x}\left(\cos 2x\right)' dx$$

$$= -e^{-x}\cos 2x + \int e^{-x}\left(-2\sin 2x\right) dx$$

$$= -e^{-x}\cos 2x - 2\int \left(-e^{-x}\right)' \sin 2x dx$$

$$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x - 2\int e^{-x}\left(\sin 2x\right)' dx$$

$$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x \underline{-4\int e^{-x}\cos 2x dx}$$

$$\therefore \int e^{-x}\cos 2x dx = \frac{1}{5}e^{-x}\left(2\sin 2x - \cos 2x\right) + C$$

$$(5).\int \frac{\arctan x}{\sqrt{\left(1+x^2\right)^3}}dx,\quad (6).\int e^{-x}\cos 2x dx,$$

$$(7).\int \frac{xe^{\arctan x}}{\sqrt{\left(1+x^2\right)^3}}dx.$$