§17-04 中值定理与多元函数的极值

一. 高阶偏导数

二. 中值定理

三. 二元函数的泰勒公式

四. 多元函数的无条件极值

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一. 高阶偏导数

1. 高阶偏导数

函数$z = f(x, y)$的二阶偏导数

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) \end{cases},$$

二阶混合偏导

$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y) \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y) \end{cases}.$$

类似地可以定义更高阶的偏导数.

例如,$z = f(x, y)$关于x的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3},$$

$z = f(x, y)$关于x的$n-1$阶偏导数,再关于y的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = \frac{\partial^n z}{\partial x^{n-1}\partial y},$$

习惯上,二阶及二阶以上的偏导数统称为高阶偏导数.

例1.设$z = x^3y^2 - 3xy^3 - xy$,求二阶偏导.

解 $\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$,

$$\frac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x \ ;$$

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2, \frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy \ ;$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(3x^2y^2 - 3y^3 - y\right)'_y = 6x^2y - 9y^2 - 1,$$

$$\frac{\partial^2 z}{\partial y \partial x} = \left(2x^3y - 9xy^2 - x\right)'_x = 6x^2y - 9y^2 - 1.$$

例2.设$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0\quad, & x^2+y^2=0 \end{cases}$,求 $f_{xy}(0,0)$, $f_{yx}(0,0)$.

解 $$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{\left(x^2+y^2\right)^2}, & x^2+y^2\neq 0 \\ 0\quad, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4-4x^2y^2-y^4}{\left(x^2+y^2\right)^2}, & x^2+y^2\neq 0 \\ 0\quad, & x^2+y^2=0 \end{cases}$$

$$\left.\begin{aligned} f_{xy}(0,0)&=\lim_{\Delta y\to 0}\frac{\boxed{f_x(0,\Delta y)-f_x(0,0)}}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1 \\ f_{yx}(0,0)&=\lim_{\Delta x\to 0}\frac{\boxed{f_y(\Delta x,0)-f_y(0,0)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1 \end{aligned}\right\}$$

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Q.构成相同,次序不同的混合偏导数何时能够相等?

Th.17.6.若函数$z = f(x, y)$的二阶混合偏导数 $f_{xy}(x, y), f_{yx}(x, y)$在点$(x, y)$处连续,则

$$f_{xy}(x, y) = f_{yx}(x, y).$$

该定理结论可以推广到更高阶的偏导数或多个自变量的情形.在许多问题中,混合偏导数是连续的,因而不用考虑求导的次序,这给我们解题带来了便利.

例3.已知函数$z=f(x,y)$满足$\dfrac{\partial^2 z}{\partial x\partial y}=4xy$,

且$f_x(x,0)=3x^2$, $f(0,y)=y$,求函数表达式.

解 $\dfrac{\partial^2 z}{\partial x\partial y}=4xy \Rightarrow \dfrac{\partial z}{\partial x}=2xy^2+C_1(x)=f_x(x,y)$,

$\because f_x(x,0)=3x^2 \Rightarrow C_1(x)=3x^2$,

$\dfrac{\partial z}{\partial x}=2xy^2+3x^2 \Rightarrow z=x^2y^2+x^3+C_2(y)$,

由$f(0,y)=y \Rightarrow C_2(y)=y$,

$\therefore f(x,y)=x^2y^2+x^3+y.$

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(续) *Sec*.1例4.试问$\left(1+x^2y\right)dx+\left(e^x-\sin y\right)dy$

是否是一个二元函数的全微分?

分析:若$\left(1+x^2y\right)dx+\left(e^x-\sin y\right)dy$

是函数$z=f(x,y)$的全微分,由*Th*.2

——函数可微的必要条件知

$$\frac{\partial z}{\partial x}=1+x^2y,\ \frac{\partial z}{\partial y}=e^x-\sin y.$$

而一个函数的两个偏导数$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$

之间是有着紧密的关系的…

解 若$\left(1+x^2y\right)dx+\left(e^x-\sin y\right)dy$

是函数$z=f(x,y)$的全微分,由$Th.2$

——函数可微的必要条件知

$$\frac{\partial z}{\partial x}=1+x^2y,\ \frac{\partial z}{\partial y}=e^x-\sin y\cdots\cdots(1)$$

由$\dfrac{\partial z}{\partial x}=1+x^2y\Rightarrow z=x+\dfrac{1}{3}x^3y+C(y),$

$\Rightarrow\dfrac{\partial z}{\partial y}=\dfrac{1}{3}x^3+C'(y),$而这与(1)式相矛盾.

$\therefore$ 题设不可能是某函数的全微分.

解二 若$\left(1+x^2y\right)dx+\left(e^x-\sin y\right)dy$是函数

$z=f(x,y)$的全微分,由可微的必要条件知

$$\frac{\partial z}{\partial x}=1+x^2y,\ \ \frac{\partial z}{\partial y}=e^x-\sin y$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=x^2,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=e^x,$$

而初等函数x^2,e^x在$\mathbb{R}^2$上有定义因而连续,

由$Th.17.6$知应有$\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$,由此矛盾知

题设不可能是某函数的全微分.

例4*.已知函数$z=f\left(\sqrt{x^2+y^2}\right)$满足$\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$,其中$f$有连续的二阶导数.求 $z=f\left(\sqrt{x^2+y^2}\right)$表达式.

解 记$z=f(r), r=\sqrt{x^2+y^2}$, $\therefore \dfrac{\partial r}{\partial x}=\dfrac{x}{r}$,

$$\frac{\partial^2 r}{\partial x^2}=\frac{r-x\cdot r'_x}{r^2}=\frac{r^2-x^2}{r^3},$$

$$\frac{\partial z}{\partial x}=f'(r)\cdot r'_x=f'(r)\cdot\frac{x}{r},$$

$$\frac{\partial^2 z}{\partial x^2}=f''(r)\cdot\left(\frac{x}{r}\right)^2+f'(r)\cdot\frac{r^2-x^2}{r^3},$$

$$\frac{\partial^2 z}{\partial x^2} = f''(r)\cdot\left(\frac{x}{r}\right)^2 + f'(r)\cdot\frac{r^2 - x^2}{r^3},$$

同理：$$\frac{\partial^2 z}{\partial y^2} = f''(r)\cdot\left(\frac{y}{r}\right)^2 + f'(r)\cdot\frac{r^2 - y^2}{r^3}.$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f''(r)\cdot\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) + f'(r)\cdot\frac{2r^2 - x^2 - y^2}{r^3}$$

$$= f''(r) + \frac{1}{r} f'(r) = 0,$$

$\therefore$ 函数$z = f(r)$满足一个二阶可降阶的微分方程

$$rf''(r) + f'(r) = 0$$

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对于特殊的二阶微分方程

$$rf''(r)+f'(r)=0$$

即 $\left(rf'(r)\right)'_r=0\Rightarrow rf'(r)=C_1,$

$$f'(r)=\frac{C_1}{r},\therefore\ f(r)=C_1\ln r+C_2\ ,$$

C_1,C_2 是任意常数,于是

得到函数$z=f\left(\sqrt{x^2+y^2}\right)$的表达式 .

例5. 已知函数 $z = z(x, y)$具有连续的二阶偏导数,设$\begin{cases} u = x + y \\ v = x - y \end{cases}$,试将方程$\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$化为新坐标系中的形式,并由此求解该偏微分方程.

解 $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v},\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial y}-\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial y}-\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}$$

$$=\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}.$$

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0 \Rightarrow \frac{\partial^2 z}{\partial u \partial v}=0 \ \left(P135/7\right)$$

$$\begin{cases} u = x + y \\ v = x - y \end{cases} \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$,记为$U = AX$,

矩阵A可逆,故自变量x, y 相互独立 $\Leftrightarrow$ u,v 相互独立.

$$\therefore \frac{\partial^2 z}{\partial u \partial v}=0 \Leftrightarrow \frac{\partial z}{\partial u}=g(u), z=\int g(u)du = G(u)+H(v),$$

$\therefore$ 满足方程$\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}=0$ 的解为

$$z = G\left(x+y\right)+H\left(x-y\right),$$

$G(s), H(t)$为任意的可导函数.

例6.设$z = f(x+y, xy)$, f具有连续的二阶偏导数,求:$\dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial x \partial y}$.

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至为重要

解 令$u = x + y, v = xy$,

记$f_1 = \dfrac{\partial f(u,v)}{\partial u}$,同理$f_2 = \dfrac{\partial f(u,v)}{\partial v}$,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1 + y f_2;$$

$$\frac{\partial^2 z}{\partial x \partial y} = (f_1 + y f_2)'_y = \frac{\partial f_1}{\partial y} + f_2 + y \frac{\partial f_2}{\partial y}$$

$$u = x + y, v = xy, \frac{\partial z}{\partial x} = f_1 + yf_2.$$

$$\frac{\partial^2 z}{\partial x \partial y} = (f_1 + yf_2)'_y = \frac{\partial f_1}{\partial y} + f_2 + y\frac{\partial f_2}{\partial y},$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v} \cdot \frac{\partial v}{\partial y} = f_{11} + xf_{12},$$

记$f_1 = \dfrac{\partial f(u,v)}{\partial u}$, f_1, f_2仍是以u,v为中间变量,以x, y为自变量的两个新的函数.

$$\frac{\partial f_1}{\partial u} = f_{11}, \quad \frac{\partial f_1}{\partial v} = f_{12}, \quad \frac{\partial f_2}{\partial v} = f_{22}.$$

返回

$$z = f(u,v), u = x + y, v = xy,$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1 + yf_2;$$

$$\frac{\partial^2 z}{\partial x\partial y} = (f_1 + yf_2)'_y = \frac{\partial f_1}{\partial y} + f_2 + y\frac{\partial f_2}{\partial y},$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} = f_{11} + xf_{12},$$

$$\frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} = f_{21} + xf_{22},$$

f有连续的二阶偏导数 $\Rightarrow f_{12} = f_{21} \cdots$

返回

思考题 1.

1.(1). 验证函数$u(x,y)=\ln\sqrt{x^2+y^2}$ 满足

*Laplace*方程 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

1.(2). 记 $r=\sqrt{x^2+y^2+z^2}$,证明函数$u=\dfrac{1}{r}$满足

*Laplace*方程 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0.$

2.设$\left(axy^3-y^2\cos x\right)dx+\left(1+by\sin x+3x^2y^2\right)dy$

是一个二元函数$z=f(x,y)$的全微分.试确定a,b

的值,并求出函数$z=f(x,y)$.

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思考题 1.参考解答

1.(1).验证函数$u(x,y)=\ln\sqrt{x^2+y^2}$ 满足

*Laplace*方程$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

解 $\dfrac{\partial u}{\partial x}=\dfrac{x}{x^2+y^2},\dfrac{\partial u}{\partial y}=\dfrac{y}{x^2+y^2},$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\left(x^2+y^2\right)-x\cdot 2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\left(x^2+y^2\right)-y\cdot 2y}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}.$$

注意对称性

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}+\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=0.$$

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1.(2).记 $r=\sqrt{x^2+y^2+z^2}$,证明函数$u=\dfrac{1}{r}$满足

*Laplace*方程$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0.$

解 $$\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{r},\frac{\partial u}{\partial x}=-\frac{r_x}{r^2}=-\frac{x}{r^3},$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{r^3-x\cdot 3r^2\cdot\frac{x}{r}}{r^6}=\frac{3x^2-r^2}{r^5}.$$

由形式对称性得

$$\frac{\partial^2 u}{\partial y^2}=\frac{3y^2-r^2}{r^5},\frac{\partial^2 u}{\partial z^2}=\frac{3z^2-r^2}{r^5},$$

代入*Laplace*方程,结论成立.

注:这里的函数$u=\dfrac{1}{r}$常称为“位势函数”.

2.设$\left(axy^3 - y^2\cos x\right)dx + \left(1 + by\sin x + 3x^2y^2\right)dy$ 是一个二元函数$z = f(x, y)$的全微分.试确定a,b 的值,并求出函数$z = f(x, y)$.

解 $\because \left(axy^3 - y^2\cos x\right)dx$

$$+\left(1 + by\sin x + 3x^2y^2\right)dy = dz,$$

$$\therefore \begin{cases} \dfrac{\partial z}{\partial x} = axy^3 - y^2\cos x \\ \dfrac{\partial z}{\partial y} = 1 + by\sin x + 3x^2y^2 \end{cases},$$

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$$\left(axy^3 - y^2\cos x\right)dx + \left(1 + by\sin x + 3x^2y^2\right)dy = dz,$$

$$\therefore \frac{\partial z}{\partial x} = axy^3 - y^2\cos x, \frac{\partial z}{\partial y} = 1 + by\sin x + 3x^2y^2,$$

$$z = \frac{1}{2}ax^2y^3 - y^2\sin x + C(y) \qquad \Rightarrow$$

$$\frac{\partial z}{\partial y} = C'(y) - 2y\sin x + \frac{3}{2}ax^2y^2 = 1 + by\sin x + 3x^2y^2,$$

$$\therefore a = 2, b = -2, C'(y) = 1.$$

$$\left(2xy^3 - y^2\cos x\right)dx + \left(1 - 2y\sin x + 3x^2y^2\right)dy = dz,$$

$$\therefore \begin{cases} \dfrac{\partial z}{\partial x} = 2xy^3 - y^2\cos x \cdots\cdots\cdots(1) \\ \dfrac{\partial z}{\partial y} = 1 - 2y\sin x + 3x^2y^2 \cdots(2) \end{cases},$$

由(1)得$z = x^2y^3 - y^2\sin x + C(y)\cdots(3)$,

对(3)计算$\dfrac{\partial z}{\partial y} = 3x^2y^2 - 2y\sin x + C'(y)$,

与(2)比较得:$C'(y) = 1, \therefore C(y) = y + C$,

$\therefore z = x^2y^3 - y^2\sin x + y + C$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$(其中$C$为任意常数).

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或者,

$$\left(axy^3 - y^2\cos x\right)dx + \left(1 + by\sin x + 3x^2y^2\right)dy = dz,$$

$$\therefore \frac{\partial z}{\partial x} = axy^3 - y^2\cos x, \frac{\partial z}{\partial y} = 1 + by\sin x + 3x^2y^2,$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3axy^2 - 2y\cos x, \frac{\partial^2 z}{\partial y \partial x} = by\cos x + 6xy^2,$$

由$Th.1$:“二阶混合偏导函数若连续则相等”知

$$3axy^2 - 2y\cos x = by\cos x + 6xy^2,$$

$$\therefore a = 2, b = -2.$$

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二. 二元函数的中值定理

2. 二元函数的微分中值定理

定理17.7(中值定理) 若函数$z=f(x,y)$在凸开域$D\subset\mathbb{R}^2$上连续,在D内可微,则 $\forall P(a,b)$, $Q(a+h,b+k)\in D$,$\exists\,\theta\in(0,1)$,使得

$$f(a+h,b+k)-f(a,b)=$$

$$f_x(a+\theta h,b+\theta k)h+f_y(a+\theta h,b+\theta k)k.$$

证明:设$\Phi(t)=f(a+th,b+tk)$,$t\in[0,1]$,

则$\Phi(t)$在$[0,1]$上满足*Lagrange* 微分中值定理的条件,

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证明 设$\Phi(t)=f\left(a+th,b+tk\right),t\in\left[0,1\right]$,

则$\Phi(t)$在$\left[0,1\right]$上满足*Lagrange* 微分中值

定理的条件,所以$\exists\,\theta\in\left(0,1\right)$,使得

$$\Phi(1)-\Phi(0)=\Phi'(\theta),$$ 而

$$\Phi'(\theta)=f_x\left(a+\theta h,b+\theta k\right)h+f_y\left(a+\theta h,b+\theta k\right)k.$$

由此可得推论:

若函数$z=f(x,y)$在凸开域D内可微,

且$f_x=f_y\equiv 0$,则在区域D内$f(x,y)\equiv C$.

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对比.

定理17.3 设函数$f(x,y)$在点(x_0,y_0)的某邻域内有偏导数,若点(x,y)属于该邻域,则$\exists\xi = x_0+\theta_1(x-x_0),\eta = y_0+\theta_2(y-y_0)$,$0<\theta_1<1$, $0<\theta_2<1$,使得

$$f(x,y)-f(x_0,y_0)=$$

$$f_x(\xi,y)(x-x_0)+f_y(x_0,\eta)(y-y_0),$$

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三. 二元函数的泰勒公式

3. 二元函数的*Taylor*定理

一元函数的*Taylor*公式

$$f(x)=f(x_0)+f'(x_0)(x-x_0)$$

$$+\frac{f''(x_0)}{2}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$+\frac{f^{(n+1)}\left(x_0+\theta(x-x_0)\right)}{(n+1)!}(x-x_0)^{n+1}\qquad(0<\theta<1).$$

意义：可用n次多项式来近似表达函数$f(x)$，且误差是当$x\to x_0$时比$(x-x_0)^n$高阶的无穷小.

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$Th.17.8$（二元函数的带$Lagrange$型余项的$Taylor$定理）

设$z=f(x,y)$在点(x_0,y_0)的某一邻域$U(x_0,y_0)$内有$n+1$阶连续的偏导数，$\forall (x_0+h,y_0+k)\in U(x_0,y_0)$，则有

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k),\ (0<\theta<1)$$

$$\cdots\cdots\cdots\cdots(1)$$

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其中,记号$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$表示

$$hf_x(x_0,y_0)+kf_y(x_0,y_0),$$

$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$表示

$$h^2 f_{xx}(x_0,y_0)+2hkf_{xy}(x_0,y_0)+k^2 f_{yy}(x_0,y_0),$$

一般地,$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f(x_0,y_0)$表示

$$\sum_{i=0}^{m} C_m^i h^i k^{m-i} \frac{\partial^m f}{\partial x^i \partial y^{m-i}}\bigg|_{(x_0,y_0)}.$$

$n=0$ 时(1)即为

$$
\begin{aligned}
f\left(x_0+h, y_0+k\right) = {} & f\left(x_0, y_0\right) \\
& + hf_x\left(x_0+\theta h, y_0+\theta k\right) \\
& + kf_y\left(x_0+\theta h, y_0+\theta k\right),\ \left(0<\theta<1\right)
\end{aligned}
$$

上式称为二元函数的*Lagrange*微分中值公式,即前面的定理17.7.

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$Th.17.8$的证明 设$\Phi(t)=f\left(x_0+th,y_0+tk\right)$,

由$Th.17.5$(全导数公式)知$\Phi(t)$在$[0,1]$上满足Taylor th.条件,

于是有

$$\Phi(1)=\Phi(0)+\Phi'(0)+\frac{\Phi''(0)}{2!}+\cdots+\frac{\Phi^{(n)}(0)}{n!}+\frac{\Phi^{(n+1)}(\theta)}{(n+1)!},\theta\in(0,1),$$

由全导数公式得$\Phi^{(m)}(t)=\dfrac{1}{m!}\left(h\dfrac{\partial}{\partial x}+k\dfrac{\partial}{\partial y}\right)^m f\left(x_0+th,y_0+tk\right)$,

$$m\in\{1,2,\cdots,n,n+1\}$$

于是 $\Phi^{(m)}(0)=\dfrac{1}{m!}\left(h\dfrac{\partial}{\partial x}+k\dfrac{\partial}{\partial y}\right)^m f\left(x_0,y_0\right),m\in\{1,2,\cdots,n\}$

$$\Phi^{(n+1)}(t)=\frac{1}{m!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f\left(x_0+\theta h,y_0+\theta k\right),$$

将此结果代入上式即得Taylor th.结论.

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$Th.17.9$ (二元函数的带$Peano$型余项的$Taylor$定理)

设$z=f(x,y)$在点(x_0,y_0)的某一邻域$U(x_0,y_0)$内有n阶连续的偏导数,$\forall(x_0+h,y_0+k)\in U(x_0,y_0)$,则当$\rho=\sqrt{h^2+k^2}\to 0$时有

$$f(x_0+h,y_0+k)=$$

$$=f(x_0,y_0)+\sum_{i=1}^{n}\frac{1}{i!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^i f(x_0,y_0)+o(\rho^n).$$

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下面是*Th*.17.9（二元函数的带*Peano*型余项的*Taylor*定理）

我们常用到的形式：设$z=f(x,y)$在点(x_0,y_0)的某一邻域$U(x_0,y_0)$内有二阶连续的偏导数，

$\forall (x_0+h,y_0+k)\in U(x_0,y_0)$，则当$\rho=\sqrt{h^2+k^2}\to 0$时有

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left[f_x(x_0,y_0)h+f_y(x_0,y_0)k\right]$$

$$+\frac{1}{2!}\left[f_{xx}(x_0,y_0)h^2+2f_{xy}(x_0,y_0)hk+f_{yy}(x_0,y_0)k^2\right]+o(\rho^2)$$

或表示为

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(f_x(x_0,y_0),f_y(x_0,y_0)\right)\begin{pmatrix}h\\k\end{pmatrix}$$

$$+\frac{1}{2!}(h,k)\begin{pmatrix}f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0)\\ f_{xy}(x_0,y_0) & f_{yy}(x_0,y_0)\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix}+o(\rho^2)$$

下面是*Th.*17.9（二元函数的带*Peano*型余项的*Taylor*定理）

我们常用到的形式：设$z=f(x,y)$在点(x_0,y_0)的某一邻域

$U(x_0,y_0)$ 内有二阶连续的偏导数，记$X_0=(x_0,y_0)$，

$X=(x_0+h,y_0+k),\forall X\in U(X_0),\Delta X=(h,k)$，

则当$\rho=\sqrt{h^2+k^2}=\|\Delta X\|\to 0$ 时有

$$f(X)=f(X_0)+\nabla f(X_0)\cdot\Delta X+\frac{1}{2!}\Delta X\cdot H_{f(X_0)}\cdot\Delta X^T+o\left(\|\Delta X\|^2\right)$$

$$\nabla f(X_0)=\left(f_x,f_y\right)_{X_0}=\operatorname{grad}f\big|_{X_0},\leftarrow \text{梯度}$$

$$H_{f(X_0)}=\begin{pmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}_{X_0}\leftarrow \textit{Hesse}\ \text{矩阵}$$

(*Hessian Matrix*)

$\Delta X\cdot H_{f(X_0)}\cdot\Delta X^T$ 是一个实的二次型

四.多元函数的(无条件)极值

设函数$z=f(x,y)$在点$P_0(x_0,y_0)$的某邻域$U(P_0)$内有定义,$\forall(x,y)\in U(P_0)$有

$$f(x,y)\le(\ge)f(x_0,y_0),$$

则称函数在点P_0处取得极大(小)值.

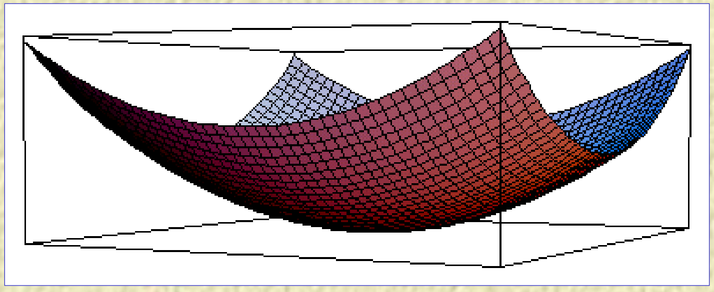
极大值,极小值统称为极值.

使函数取得极值的点称为函数的极值点.

观察函数在(0,0)处的极值情况.

(1). $z = x^2 + 2y^2$

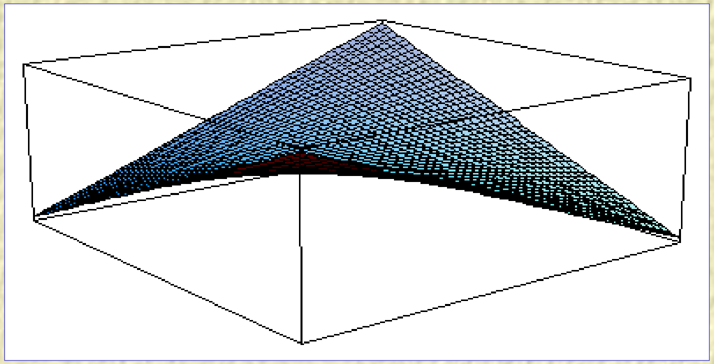
在 (0,0) 处有极小值.



(1)

(2). $z = x^2 - y^2$

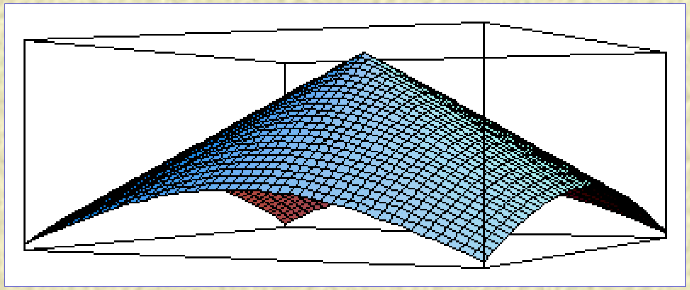
在 (0,0) 处无极值.



(2)

(3). $z = -\sqrt{x^2 + y^2}$

在 (0,0) 处有极大值.



(3)

4.多元函数取得极值的条件

Th.17.10.（必要条件）设函数$z=f(x,y)$在点(x_0,y_0)处有偏导数,且在该点处取得极值,则有$f_x(x_0,y_0)=0, f_y(x_0,y_0)=0.$

证明　不妨设$z=f(x,y)$在点(x_0,y_0)处有极大值,则对于(x_0,y_0)的某邻域内的任意一点(x,y)都有$f(x,y)\le f(x_0,y_0)$.

故当$y=y_0, x\neq x_0$时有$f(x,y_0)\leq f(x_0,y_0)$,

即$f(x,y_0)$在$x=x_0$处取得极大值,

$$\therefore\ f_x(x_0,y_0)=0.$$

同样地,有$f_y(x_0,y_0)=0.$

Th.17.10.(必要条件)的推广:

函数$u=f(x,y,z)$在点(x_0,y_0,z_0)处有偏导数,

且在该点处取得极值,则有$f_x(x_0,y_0,z_0)=0$,

$f_y(x_0,y_0,z_0)=0,\ f_z(x_0,y_0,z_0)=0.$

思考题:

若函数$f(x, y_0)$在点$x = x_0$处取得极值,函数$f(x_0, y)$在点$y = y_0$处也取得极值.

问:函数$f(x, y)$在点(x_0, y_0)处一定取得极值么?

A: *No!*

直觉上就是不能“以偏概全”.

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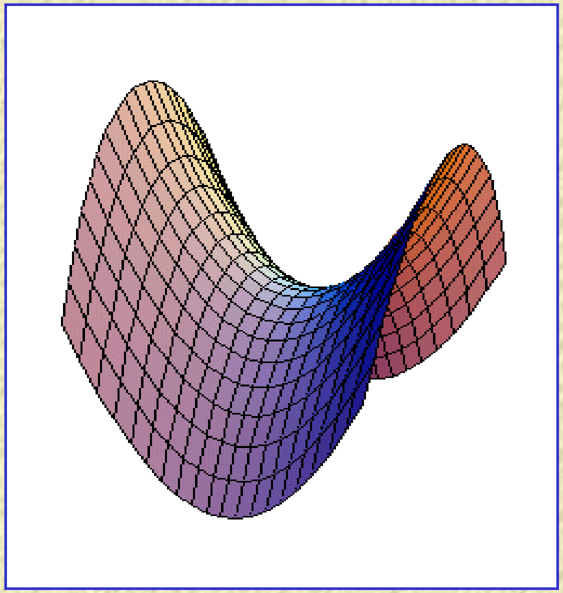
思考题解答：

No! 如$f(x,y)=x^2-y^2$,

当$x=0$时$f(0,y)=-y^2$

在(0,0)处取得极大值;



当$y=0$时$f(x,0)=x^2$在(0,0)处取得极小值;

但$f(x,y)=x^2-y^2$在(0,0)处取不到极值.

点(0,0,0)是马鞍面$z=x^2-y^2$的鞍点,是曲面的不稳定的平衡点.

凡使得函数的一阶偏导数同时为零的点称为函数的驻点.

需注意,驻点$\neq$极值点.

如$(0,0)$点是函数$z = x^2 - y^2$的驻点但不是极值点,而点$(0,0)$是函数$z = -\sqrt{x^2 + y^2}$的极值点但不是驻点.

Q:如何判定一个驻点是函数的极值点?

定理17.11.(充分条件)设函数$f(x,y)$在驻点(x_0,y_0)的某邻域内有连续的一阶和二阶偏导数.

令 $f_{xx}(x_0,y_0)=A, f_{xy}(x_0,y_0)=B, f_{yy}(x_0,y_0)=C.$

则函数$f(x,y)$在点(x_0,y_0)处取得极值的情况如下:

(1).$AC-B^2>0$ 时有极值:$A>0$ 时有极小值,

$A<0$ 时有极大值.

(2).$AC-B^2<0$ 时一定没有极值.

(3).$AC-B^2=0$ 时极值情况不确定.

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设$P_0(x_0,y_0)$为函数f的驻点,

$$H_{f(P_0)}=\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{P_0}=\begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

Hessian matrix

Hesse 矩阵

记$|H|=\begin{vmatrix} A & B \\ B & C \end{vmatrix}=AC-B^2$.则

(1).当$|H|>0$ 时,函数f在(x_0,y_0)处取极值,

且$A>0$,函数取极小值,$A<0$,函数取极大值.

(2).$|H|<0$ 时,函数f在(x_0,y_0)处不取极值.

(3).$|H|=0$ 时,函数f在(x_0,y_0)处的极值情况无法确定.

极值充分条件的定理要利用二元函数的泰勒公式来证明.

极值判断(充分条件)定理–对比:

二元: $f_x(x_0,y_0)=0,\ f_y(x_0,y_0)=0.$

$f_{xx}(x_0,y_0)=A,\ f_{xy}(x_0,y_0)=B,$

$f_{yy}(x_0,y_0)=C.$

当$AC-B^2>0$时有极值:

$A>0(<0)$时有极小(大)值.

一元: $f'(x_0)=0,\ f''(x_0)=A,$

则当$A\neq 0$时有极值:

$A>0(<0)$时有极小(大)值.

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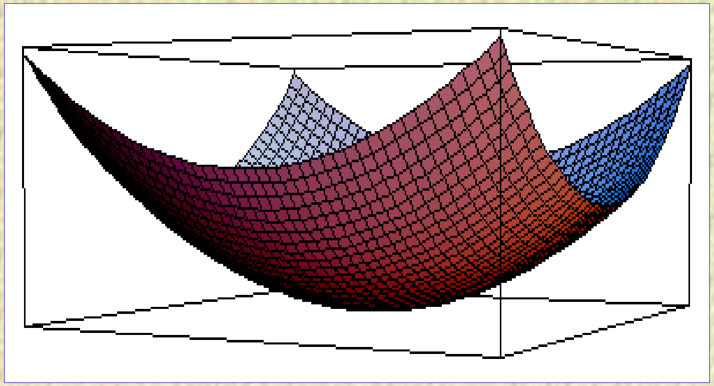
通过具体例子帮助记住定理结论.

(1). $z = x^2 + 2y^2$ 在驻点 $(0,0)$ 处取得极小值,且有

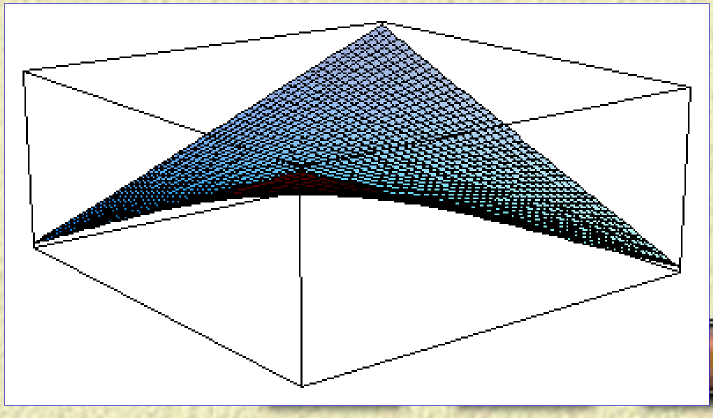
$$A = 2, B = 0, C = 4, AC - B^2 = 8 > 0.$$

(2). $z = x^2 - y^2$ 在驻点 $(0,0)$ 处无极值,

$$A = 2, B = 0, C = -2, AC - B^2 = -4 < 0.$$



(1)



(2)

例7.求$f(x,y)=x^3-y^3+3x^2+3y^2-9x$的极值.

解 第一步 求驻点.

解方程组$\begin{cases} f_x(x,y)=3x^2+6x-9=0 \\ f_y(x,y)=-3y^2+6y \quad =0 \end{cases}$

解得 $x=1$ 或 3 ,$y=0$ 或 2.

$\therefore$ 驻点有$(1,0),(1,2),(3,0),(3,2)$.

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第二步 判别.

求二阶偏导数 $f_{xx}=6x+6\cdots A,$

$f_{xy}=0\cdots B,\ f_{yy}=-6y+6\cdots C$

项目 / 驻点	$A=f_{xx}$	$B=f_{xy}$	$C=f_{yy}$	$AC-B^2$	极值情况
(-3,0)	-12	0	6	<0	不取极值
(-3,2)	$-12<0$	0	-6	>0	$\max f(x,y)=f(-3,2)$
(1,0)	$12>0$	0	6	>0	$\min f(x,y)=f(1,0)$
(1,2)	12	0	-6	<0	不取极值

以例说明定理17.10的条件是充分而非必要的.

(不要求掌握)

$e.g.$讨论函数$z=x^3+y^3$与$z=\left(x^2+y^2\right)^2$

在点(0,0)处的极值情况.

解　显然(0,0)都是它们的驻点,且在(0,0)

处都有　$AC-B^2=0.$

$z=x^3+y^3$在点(0,0)的邻域内的取值可

能为:正\零\负,

$\therefore$ (0,0)处$z=x^3+y^3$不取极值.

而当 $x^2+y^2\neq 0$ 时,$z=\left(x^2+y^2\right)^2>z\big|_{(0,0)}=0,$

$\therefore z(0,0)=\left(x^2+y^2\right)^2\big|_{(0,0)}=0$为极小值.

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一个多元函数极值判断的结论,在实际问题中的方法简单、但极其有用的应用:

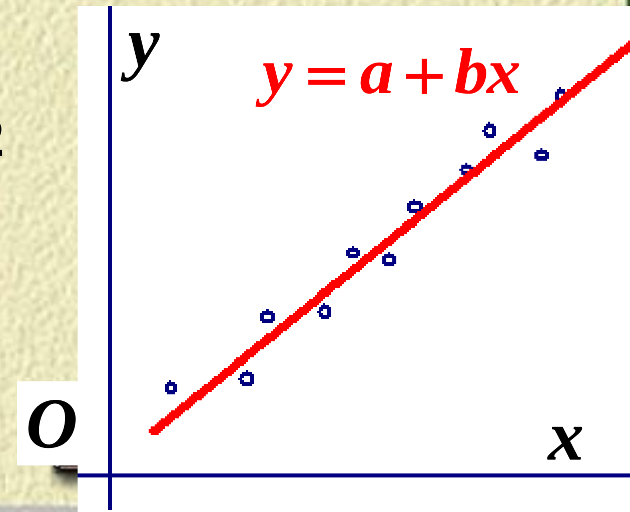
……·最小二乘法

…寻找经验公式.

例8.计量经济学中的线性回归就是要找经验公式.设某一问题中,我们由抽样调查得到的数据$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$可以认为,内生变量$x$与变量$y$是一个线性函数关系, $y = a + bx$,那么由最小二乘法知,系数a,b是根据

$$\min L(a,b) = \sum_{1}^{n} (a + bx_i - y_i)^2$$

来确定的. (不要求掌握)



最小二乘法,就是确定系数a,b,使得

$$L(a,b)=\sum_{1}^{n}\left(a+bx_i-y_i\right)^2$$取值最小.

$$\therefore\begin{cases}\dfrac{\partial L}{\partial a}=0\\[2mm] \dfrac{\partial L}{\partial b}=0\end{cases},\text{即}\begin{cases}\displaystyle\sum_{1}^{n}\left(a+bx_i-y_i\right)=0\\ \displaystyle\sum_{1}^{n}\left(a+bx_i-y_i\right)x_i=0\end{cases}.$$

$$\begin{cases}na+\left(\displaystyle\sum_{1}^{n}x_i\right)b=\displaystyle\sum_{1}^{n}y_i\\ \left(\displaystyle\sum_{1}^{n}x_i\right)a+\left(\displaystyle\sum_{1}^{n}x_i^2\right)b=\displaystyle\sum_{1}^{n}x_iy_i\end{cases},\cdots\ \cdots$$

5.多元连续函数的最值

与一元函数相类似，我们可以利用函数的极值来求函数的最大值和最小值.

求最值的一般方法：

将函数在D内的所有驻点处的函数值及在D的边界上的最大值和最小值相互比较，其中最大者即为最大值，最小者即为最小值.

最值应用问题

依据

函数 f 在闭域上连续

$\Downarrow$

函数 f 在闭域上可达到最值

最值可疑点 $\begin{cases} \text{驻点} \\ \text{边界上的最值点} \end{cases}$

例9 某厂要用铁板做一个体积为2 m^3的有盖长方体水问当长、宽、高各取怎样的尺寸时，才能使用料最省?

解: 设水箱长, 宽分别为 x , y m ,则高为 $\frac{2}{xy}$ m,

则水箱所用材料的面积为

$$A=2\left(xy+y\cdot\frac{2}{xy}+x\cdot\frac{2}{xy}\right)=2\left(xy+\frac{2}{x}+\frac{2}{y}\right),\begin{pmatrix}x>0\\ y>0\end{pmatrix}$$

令$\begin{cases}A_x=0\\ A_y=0\end{cases}$,即$\left(y-\frac{2}{x^2}\right)=\left(x-\frac{2}{y^2}\right)=0$得驻点$\left(\sqrt[3]{2},\sqrt[3]{2}\right)$.

根据实际问题可知最小值在定义域内应存在，因此可断定此唯一驻点就是最小值点. 即当长、宽均为 $\sqrt[3]{2}$

高为 $\frac{2}{\sqrt[3]{2}\cdot\sqrt[3]{2}}=\sqrt[3]{2}$ 时，水箱所用材料最省.

但是要特别注意，当二元函数在某区域内部只有一个极值点P 并且是极小值点时，该点函数值未必是函数在区域内的最小值。这是二元函数与一元函数的极值、最值问题中的一大差别，尤需注意！

例 10. 求二元函数 $z = f(x, y) = x^3 - 4x^2 + 2xy - y^2$ 在闭区域 $D = [-5,5] \times [-1,1]$ 上的最大值与最小值.

解 $z = f(x, y) = x^3 - 4x^2 + 2xy - y^2$

(1). 先求得f的驻点$P_1(0,0) \in D, P_2(2,2) \notin D$

(2). $P_1(0,0), H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} < 0, f(0,0) = \max f(x).$

(3). ∂D上， $x=5, y\in[-1,1], f(5,y)=25+10y-y^2\uparrow$

$x=-5, y\in[-1,1], f(-5,y)=-225-10y-y^2\downarrow$

$y=1, x\in[-5,5], f(x,1)=x^3-4x^2+2x-1$,驻点$x=\dfrac{4\pm\sqrt{10}}{3}$,

$y=-1, x\in[-5,5], f(x,1)=x^3-4x^2-2x-1$,驻点$x=\dfrac{4\pm\sqrt{22}}{3}$,

(4). $f(0,0)=0, f(5,-1)=14, f(5,1)=34,$

$f(-5,-1)=-216, f(-5,1)=-236,$

$$f\left(\frac{4+\sqrt{10}}{3},1\right)\approx-5.42,\ f\left(\frac{4-\sqrt{10}}{3},1\right)\approx-0.73,$$

$$f\left(\frac{4+\sqrt{22}}{3},-1\right)\approx-16.05,\ f\left(\frac{4-\sqrt{22}}{3},-1\right)\approx-0.76.$$

$$(4).\ f(0,0)=0,\ f(5,-1)=14,\ f(5,1)=34,$$

$$f(-5,-1)=-216,\ f(-5,1)=-236,$$

$$f\left(\frac{4+\sqrt{10}}{3},1\right)\approx -5.42,\ f\left(\frac{4-\sqrt{10}}{3},1\right)\approx -0.73,$$

$$f\left(\frac{4+\sqrt{22}}{3},-1\right)\approx -16.05,\ f\left(\frac{4-\sqrt{22}}{3},-1\right)\approx -0.76.$$

所以,函数f在区域D上的最大值为$f(5,1)=34$,

最小值为$f(-5,1)=-236$.

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