2022 级数学分析 II 阶段练习 2　　2023-05

一．填空题或选择题（ 选择题正确选项唯一）

1. 下列无穷级数中条件收敛的是 ___(A)___ .

$(A).\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ ；　$(B).\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ ；　$(C).\sum_{n=1}^{\infty}(-1)^n$ ；　$(D).\sum_{n=1}^{\infty}\left(\frac{\sin n}{n^2}-\frac{1}{n}\right)$.

答 $(C),(D)$均发散,(B)绝对收敛,(A)是.

2. 级数 $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=$ ________.

答 $\sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ ，著名的结果直接推导出来并非易事,记住 .

3. 级数$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ 在p ___>0___ 时收敛 .

答 $p>0$ 时$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ 为$Leibniz$ 级数,收敛 .

4. 记级数$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\frac{1}{13}-\frac{1}{16}+\cdots$的和为 A，则刻画级数和 A 大小的选项“$-1<A<0$”、“$0<A<1$”、“$1<A<2$”中正确的结果为 ________________.

答 $0<A<1$,基本结论,应熟知 .

5. 幂级数$\sum_{n=1}^{\infty}\frac{x^n}{3^n\cdot n^3}$ 的收敛域为 ___$[-3,3]$___ .

解 $\lim_{n\to\infty}\sqrt[n]{\left|\frac{x^n}{3^n\cdot n^3}\right|}=\frac{|x|}{3}\lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{n}\right)^3}=\frac{|x|}{3}$,故幂级数收敛半径$R=3.x=3$ 时,$\sum_{n=1}^{\infty}\frac{1}{n^3}$ 收敛,

$x=-3$ 时,$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ 绝对收敛.$\therefore\sum_{n=1}^{\infty}\frac{x^n}{3^n\cdot n^3}$ 的收敛域为$[-3,3]$.

6. 试问以下论断是否正确 ？ 你的回答是___正确___.

对数项级数$\sum a_n$ 而言，如果$\lim_{n\to\infty}\sqrt[n]{|a_n|}=r<1$，则级数$\sum a_n$ 收敛 .

解 正确! $\lim_{n\to\infty}\sqrt[n]{|a_n|}=r<1\Rightarrow\sum_{n=1}^{\infty}|a_n|$ 收敛即$\sum_{n=1}^{\infty}a_n$ 绝对收敛$\Rightarrow\sum_{n=1}^{\infty}a_n$ 收敛.

二．解答题

7. 试判断级数敛散性.(1).$\sum_{1}^{\infty}\frac{1}{n^2-\ln n}$ ；　(2).$\sum_{1}^{\infty}\frac{1}{3^{\ln n}}$.

解 (1). $\because\lim_{n\to\infty}\frac{\frac{1}{n^2-\ln n}}{\frac{1}{n^2}}=1$,或由$\lim_{n\to\infty}\frac{\ln n}{n^2}=0$,知$\exists n_0,n\ge n_0$时$\ln n<\frac{1}{2}n^2$,于是$n\ge n_0$时有

$0<\frac{1}{n^2-\ln n}<\frac{1}{n^2-\frac{1}{2}n^2}=\frac{2}{n^2}$,据比较判别法,由$\sum_{1}^{\infty}\frac{1}{n^2}$ 收敛$\Rightarrow\sum_{1}^{\infty}\frac{1}{n^2-\ln n}$ 收敛 .

(2). $3^{\ln n}=e^{\ln n\ln 3}=\left(e^{\ln n}\right)^{\ln 3}=n^{\ln 3}$,即$\dfrac{1}{3^{\ln n}}=\dfrac{1}{n^{\ln 3}}$,$\ln 3>1$,据$p-$级数结论知,$\sum_{1}^{\infty}\dfrac{1}{3^{\ln n}}$ 收敛 .

注1:这2个问题都是与$p-$级数相关,(广义的)$p-$级数问题用比值/根值法皆失效,
是因为用比值/根值法时极限为1,故方法失效.

注2:对于正项级数$\sum u_n$,若存在某$n_0\in\mathbb{Z}^+$,(1).存在常数$r<1$,使对$\forall n>n_0$,有$\dfrac{u_{n+1}}{u_n}\le r$
或者 $\sqrt[n]{u_n}\le r$,则级数$\sum u_n$收敛 . 这里的"存在常数$r<1$"这一条件不可少,仅有条件
$\dfrac{u_{n+1}}{u_n}<1$ 或 $\sqrt[n]{u_n}<1$ 是不够的,$p>0$时的$p-$级数$\sum_{1}^{\infty}\dfrac{1}{n^p}$ 就是一个典型的例子.
(2).$\forall n>n_0$,$\dfrac{u_{n+1}}{u_n}\ge 1$ 或者$\sqrt[n]{u_n}\ge 1$,则$\sum u_n$发散.

8. 试问级数$\sum_{n=1}^{\infty}\dfrac{2^n\cos n+(-3)^n}{n\cdot 3^n}$ 是否收敛?给出结论，说明理由 .

解 $\left|\dfrac{2^n\cos n}{n\cdot 3^n}\right|<\dfrac{2^n}{n\cdot 3^n}$,$\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{2^n}{n\cdot 3^n}}=\dfrac{2}{3}\lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{n}}=\dfrac{2}{3}<1$,$\sum_{n=1}^{\infty}\dfrac{2^n\cos n}{n\cdot 3^n}$绝对收敛,故$\sum_{n=1}^{\infty}\dfrac{2^n\cos n}{n\cdot 3^n}$收敛;
$\sum_{n=1}^{\infty}\dfrac{(-3)^n}{n\cdot 3^n}=\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$是交错级数,满足*Leibniz*定理条件,故$\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$收敛.
由收敛级数加法性质知原级数收敛.

注1:由$\dfrac{2^n\cos n}{n\cdot 3^n}\le\dfrac{2^n}{n\cdot 3^n}$ 而据$\sum_{n=1}^{\infty}\dfrac{2^n}{n\cdot 3^n}$收敛 $\Rightarrow\sum_{n=1}^{\infty}\dfrac{2^n\cos n}{n\cdot 3^n}$收敛是不对的.须注意比较判别法
只适用于正项级数敛散性的判断. 级数绝对收敛 $\Rightarrow$ 收敛.

注2:关于$\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$敛散性的判断是点到为止即可.

9. 试给出$\sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{(2n+1)!}$的收敛域.在该收敛域内记$S(x)=\sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{(2n+1)!}$.验证$S(x)$满足$S''(x)=S(x)$,
$S(0)=0$,$S'(0)=1$.试求出$S(x)$初等函数形式的表达式.

解 $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{\dfrac{x^{2n+3}}{(2n+3)!}}{\dfrac{x^{2n+1}}{(2n+1)!}}\right|=\lim\limits_{n\to\infty}\dfrac{x^2}{(2n+2)(2n+3)}=0,$

$\therefore$ 级数对任意的$x\in\mathbb{R}$都绝对收敛,幂级数的收敛域为$(-\infty,+\infty)$.

$$S(x)=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}+\cdots,$$

$$S'(x)=\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}+\cdots\right)'=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{2n}}{(2n)!}+\cdots$$

$$S''(x)=\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{2n}}{(2n)!}+\cdots\right)'=0+x+\frac{x^3}{3!}+\frac{x^5}{5!}\cdots+\frac{x^{2n-1}}{(2n-1)!}+\cdots$$

$\therefore S''(x)=S(x),S(0)=0,S'(0)=1.$

由 $\sum\limits_0^\infty\dfrac{x^n}{n!}=e^x$ 得$\sum\limits_0^\infty\dfrac{(-x)^n}{n!}=e^{-x}$,于是$\left(e^x-e^{-x}\right)=2\sum\limits_0^\infty\dfrac{x^{2n+1}}{(2n+1)!}$,$\therefore S(x)=\dfrac{1}{2}\left(e^x-e^{-x}\right).$

注:用“+”表示比较直观: $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dfrac{x^5}{5!}+\dfrac{x^6}{6!}+\cdots+\dfrac{x^{2n-1}}{(2n-1)!}+\dfrac{x^{2n}}{(2n)!}+\cdots=e^x,\cdots(A)$

$1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\dfrac{x^4}{4!}-\dfrac{x^5}{5!}+\dfrac{x^6}{6!}+\cdots-\dfrac{x^{2n-1}}{(2n-1)!}+\dfrac{x^{2n}}{(2n)!}+\cdots=e^{-x},\cdots(B),$

$(A),(B)$两式相减,得: $2\left(x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\cdots+\dfrac{x^{2n-1}}{(2n-1)!}+\cdots\right)=e^x-e^{-x}.$

10. 求级数的和:(1).$\sum\limits_0^\infty\dfrac{(-1)^n}{(2n+1)3^n}$;　　(2).$\sum\limits_0^\infty\dfrac{(-1)^n(2n+1)}{3^n}$.

解 (1).$\sum\limits_{n=0}^\infty\dfrac{(-1)^n}{(2n+1)3^n}=\sqrt{3}\sum\limits_{n=0}^\infty\dfrac{(-1)^n}{2n+1}\left(\dfrac{1}{\sqrt{3}}\right)^{2n+1}$,考察幂级数$\sum\limits_{n=0}^\infty\dfrac{(-1)^n}{2n+1}x^{2n+1}=S(x)$,

$$S(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots,x\in[-1,1],S'(x)=1-x^2+x^4-x^6+\cdots=\frac{1}{1+x^2},x\in(-1,1).$$

$$S(0)=0,\Rightarrow S(x)-S(0)=\int_0^x S'(t)dt=\int_0^x\frac{1}{1+t^2}dt=\arctan x.$$

$$\therefore\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)3^n}=\sqrt{3}S\left(\frac{1}{\sqrt{3}}\right)=\sqrt{3}\arctan\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{6}\pi.$$

解 (2).$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{3^n}=\sum_{n=0}^{\infty}(-1)^n(2n+1)\left(\frac{1}{\sqrt{3}}\right)^{2n}$,记$\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n}=S(x),x\in(-1,1)$,

$S(x)=1-3x^2+5x^4-7x^6+\cdots=\left(x-x^3+x^5-x^7+\cdots\right)'=\left(\frac{x}{1+x^2}\right)'=\frac{1+x^2-x\cdot 2x}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$,

$\therefore S=S\left(1/\sqrt{3}\right)=\frac{3}{8}$.

注:$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{3^n}=2\sum_{n=0}^{\infty}n\cdot\left(-\frac{1}{3}\right)^n+\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n$,而 $|q|<1$ 时求$\sum_{n=0}^{\infty}n\cdot q^n$用错位相减法是基本题.

11. 对于级数$\sum_{n=1}^{\infty}a_n$,(1). 举例说明:$\sum_{n=1}^{\infty}\left(a_{2n-1}+a_{2n}\right)$收敛,$\sum_{n=1}^{\infty}a_n$未必收敛;

(2). 证明:若$\sum_{n=1}^{\infty}a_n$是正项级数,$\sum_{n=1}^{\infty}\left(a_{2n-1}+a_{2n}\right)$收敛,则$\sum_{n=1}^{\infty}a_n$收敛 .

解 (1).对于级数$1-1+1-1+1-1+\cdots$,$S_{2n-1}=1,S_{2n}=0$,故$\lim\limits_{n\to\infty}S_n$不存在,原级数发散,

但级数$(1-1)+(1-1)+(1-1)+\cdots$收敛.

(2).对于正项级数$\sum_{n=1}^{\infty}a_n$,若$\sum_{n=1}^{\infty}\left(a_{2n-1}+a_{2n}\right)$收敛于$A$,则$S_{2n}=\sum_{k=1}^{n}\left(a_{2k-1}+a_{2k}\right)\le A$,

于是有 $S_n\le S_{2n}\le A,\forall n=1,2,3,\cdots$. 又因为数列$\{S_n\}$单调递增,所以数列$\{S_n\}$收敛.

$\therefore$ 该级数收敛,且$\sum_{n=1}^{\infty}a_n=A$.

12. 曲面$z=x^2+y^2$在点$(1,1,2)$处的与z轴正向夹角为锐角的单位化的法向量$\overrightarrow{n^o}=$?

解 $z=f(x,y)$上点(x,y,z)处法向量$\vec{n}=\pm\left(f_x,f_x,-1\right)$,

对于$z=x^2+y^2$在点$(1,1,2)$处有$\vec{n}=\pm(2,2,-1)$,$\therefore$ 所求为 $\overrightarrow{n^o}=\frac{1}{3}(-2,-2,1)$.

13. 设 $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, x^2+y^2\neq 0\\ 0 \quad , x^2+y^2=0\end{cases}$,试问在$O(0,0)$处函数$f(x,y)$是否连续?是否可微?

解 由 $x^2y^2\le\frac{1}{4}(x^2+y^2)^2$ 得$0\le\dfrac{x^2y^2}{x^2+y^2}\le\frac{1}{4}(x^2+y^2)$,$\lim\limits_{\substack{x\to0\\y\to0}}f(x,y)=\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2y^2}{x^2+y^2}=0$,

$\therefore$ $f(x,y)$在点$(0,0)$处连续.

$f_x(0,0)=\lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x}=\lim\limits_{x\to0}\dfrac{0}{x}=0$, $f_y(0,0)=\lim\limits_{y\to0}\dfrac{f(0,y)-f(0,0)}{y}=\lim\limits_{y\to0}\dfrac{0}{y}=0$,

由 $0\le\dfrac{x^2y^2}{\sqrt{(x^2+y^2)^3}}\le\frac{1}{4}\sqrt{x^2+y^2}$,得

$$\lim_{\sqrt{x^2+y^2}\to0}\frac{f(x,y)-f(0,0)-\left[f_x(0,0)x+f_y(0,0)y\right]}{\sqrt{x^2+y^2}}=\lim_{\sqrt{x^2+y^2}\to0}\frac{x^2y^2}{\sqrt{(x^2+y^2)^3}}=0,$$

$\therefore$ $f(x,y)$在点$(0,0)$处可微.

14. 求曲面$\dfrac{x^2}{2}+y^2+\dfrac{z^2}{4}=1$上点与平面$2x+2y+z+5=0$上点之间的最短距离.

解 根据几何意义知,距离最小时椭球面在点(x_0,y_0,z_0)处的切平面平行于给定平面,

椭球面法向量 $\vec{n_0}=\left(x_0,2y_0,\frac{1}{2}z_0\right)$,平面法向量 $\vec{n_1}=(2,2,1)$,

于是可设 $x_0=2t,2y_0=2t,\frac{1}{2}z_0=t$,代入$\dfrac{x^2}{2}+y^2+\dfrac{z^2}{4}=1$ 得 $t=\pm\frac{1}{2}$,

于是,$(x_0,y_0,z_0)=\pm\left(1,\frac{1}{2},1\right)$,$\left(1,\frac{1}{2},1\right)$到平面$2x+2y+z+5=0$的距离为$d_1=\dfrac{\left|2\times1+2\times\frac{1}{2}+1+5\right|}{\sqrt{2^2+2^2+1^2}}=3$,

$\left(-1,-\frac{1}{2},-1\right)$到平面$2x+2y+z+5=0$的距离为$d_2=\dfrac{\left|2\times(-1)+2\times\left(-\frac{1}{2}\right)-1+5\right|}{\sqrt{2^2+2^2+1^2}}=\dfrac{1}{3}$,

$\therefore$所求平面与椭球面的最小距离为$\dfrac{1}{3}$.

解二 椭球面上点(x,y,z)到给定平面的距离为$d=\dfrac{|2x+2y+z+5|}{\sqrt{2^2+2^2+1^2}}$,

考虑条件极值问题$\begin{cases}\min G=(2x+2y+z+5)^2\\ s.t.\quad \dfrac{x^2}{2}+y^2+\dfrac{z^2}{4}=1\end{cases}$,

取*Lagrange* 乘子函数 $L=(2x+2y+z+5)^2-\lambda\left(\dfrac{x^2}{2}+y^2+\dfrac{z^2}{4}-1\right)$

$$\begin{cases}L_x=0\\L_y=0\\L_z=0\\L_\lambda=0\end{cases},\Rightarrow\begin{cases}4(2x+2y+z+5)-\lambda x=0\\4(2x+2y+z+5)-2\lambda y=0\\2(2x+2y+z+5)-\dfrac{1}{2}\lambda z=0\\\dfrac{x^2}{2}+y^2+\dfrac{z^2}{4}=1\end{cases}$$,求得驻点$\pm\left(1,\dfrac{1}{2},1\right)$,

根据问题的实际意义知所求距离有最小值与最大值,

点$\left(1,\dfrac{1}{2},1\right)$到平面的距离为$d_1=3$, 点$\left(-1,-\dfrac{1}{2},-1\right)$到平面的距离为$d_2=\dfrac{1}{3}$,

$\therefore$ 所求平面与椭球面的最小距离为$\dfrac{1}{3}$.

15. 设 f'' 存在, 且 $x^2+y^2+z^2=xyf\left(z^2\right)$, 求$\dfrac{\partial^2 z}{\partial x\partial y}$.

解 对 $x^2+y^2+z^2=xyf\left(z^2\right)$两边对$x$求导,$2x+2z\dfrac{\partial z}{\partial x}=y\left[f\left(z^2\right)+xf'\left(z^2\right)\cdot 2z\dfrac{\partial z}{\partial x}\right]$解得

$\dfrac{\partial z}{\partial x}=\dfrac{2x-yf\left(z^2\right)}{2xyzf'\left(z^2\right)-2z}$,由变量$x,y$的对称性可得$\dfrac{\partial z}{\partial y}=\dfrac{2y-xf\left(z^2\right)}{2xyzf'\left(z^2\right)-2z}$.

$$\frac{\partial^2 z}{\partial x\partial y}=\left(\frac{2x-yf\left(z^2\right)}{2xyzf'\left(z^2\right)-2z}\right)'_y=\frac{-\left[f\left(z^2\right)+yf'\left(z^2\right)\cdot 2z\dfrac{\partial z}{\partial y}\right]\left[2xyzf'\left(z^2\right)-2z\right]-\left[2x-yf\left(z^2\right)\right]\left[2xyzf'\left(z^2\right)-2z\right]'_y}{\left[2xyzf'\left(z^2\right)-2z\right]^2},\cdots\cdots(1)$$

$$\left[2xyzf'\left(z^2\right)-2z\right]'_y=2x\left[zf'\left(z^2\right)+yf'\left(z^2\right)\frac{\partial z}{\partial y}+yzf''\left(z^2\right)\cdot 2z\frac{\partial z}{\partial y}\right]-2\frac{\partial z}{\partial y}\cdots\cdots\cdots\cdots(2)$$

将$\dfrac{\partial z}{\partial y}$代入上述(2)式,再代入上述(1)式,即得结果,结果从略.

解二 设 $F(x,y,z)=x^2+y^2+z^2-xyf\left(z^2\right)$,$\dfrac{\partial F}{\partial x}=F_x=2x-yf\left(z^2\right)$,$F_y=2y-xf\left(z^2\right)$,

$F_z=2z-xyf'\left(z^2\right)\cdot 2z$,$\therefore\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$,$\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}\cdots$

法三 在等式 $x^2+y^2+z^2=xyf\left(z^2\right)$两边作微分,$d\left(x^2+y^2+z^2\right)=d\left(xyf\left(z^2\right)\right)$,

即 $2xdx+2ydy+2zdz=yf\left(z^2\right)dx+xf\left(z^2\right)dy+xyf'\left(z^2\right)\cdot 2zdz$,整理得$dz=Adx+Bdy$,

即$A=\dfrac{\partial z}{\partial x}$,$B=\dfrac{\partial z}{\partial y}\cdots$

16. 证明：曲面$\Phi\left(x-az,y-bz\right)=0$上任一点处的切平面均与一定直线平行 .

解 设$F\left(x,y,z\right)=\Phi\left(x-az,y-bz\right)=\Phi\left(u,v\right),F_x=\frac{\partial\Phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\Phi}{\partial v}\cdot\frac{\partial v}{\partial x}=\Phi_1\cdot 1+\Phi_2\cdot 0,$

$F_y=\frac{\partial\Phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\Phi}{\partial v}\cdot\frac{\partial v}{\partial y}=\Phi_1\cdot 0+\Phi_2\cdot 1,F_z=\frac{\partial\Phi}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial\Phi}{\partial v}\cdot\frac{\partial v}{\partial z}=\Phi_1\cdot\left(-a\right)+\Phi_2\cdot\left(-b\right),$

曲面$\Phi\left(x-az,y-bz\right)=0$ 的法向量$\vec{n}=\left(F_x,F_y,F_z\right)=\left(\Phi_1,\Phi_2,-a\Phi_1-b\Phi_2\right)$与确定的向量$\left(a,b,1\right)$正交,

$\therefore$曲面$\Phi\left(x-az,y-bz\right)=0$ 上任意一点处的切平面与一条方向向量为$\left(a,b,1\right)$的直线平行.

17. 在椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$上求一点 $P\left(x_0,y_0,z_0\right)$，使该点处曲面的切平面与三个坐标平面围成的四面体体积最小，并求该最小体积 .

解 由几何对称性,不妨只考虑$P\left(x_0,y_0,z_0\right)$在第一卦限的情形.

设$P\left(x_0,y_0,z_0\right)$是椭球面上的点,令$F\left(x,y,z\right)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1,$

则曲面在P点处法向量$\vec{n}=\left(F_x,F_y,F_z\right)=\left(\frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2}\right),$

过P点的切平面方程为 $\frac{x_0}{a^2}\left(x-x_0\right)+\frac{y_0}{b^2}\left(y-y_0\right)+\frac{z_0}{c^2}\left(z-z_0\right)=0,$整理得$\frac{x\cdot x_0}{a^2}+\frac{y\cdot y_0}{b^2}+\frac{z\cdot z_0}{c^2}=1.$

该切平面在三坐标轴上的截距为$x=\frac{a^2}{x_0},y=\frac{b^2}{y_0},z=\frac{c^2}{z_0},$则所求四面体体积为$V=\frac{1}{6}xyz=\frac{a^2b^2c^2}{6x_0y_0z_0}.$

在条件$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1$ 下求$V=\frac{a^2b^2c^2}{6x_0y_0z_0}=\frac{abc}{6}\cdot\frac{1}{\frac{x_0}{a}\cdot\frac{y_0}{b}\cdot\frac{z_0}{c}}$的最小值.

记$\frac{x_0}{a}=u,\frac{y_0}{b}=v,\frac{z_0}{c}=w,$则考虑在$u^2+v^2+w^2=1$ 条件下求$G=uvw$ 的最大值.

用$Lagrange$ 乘子法：设$L=\ln u+\ln v+\ln w-\lambda\left(u^2+v^2+w^2-1\right),\longleftarrow$（取对数只是为了求导简单些）

$\begin{cases}L_u=0,L_v=0\\L_w=0,L_\lambda=0\end{cases}$,解得驻点$\left(u,v,w\right)=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$,由问题的实际意义知$G=uvw$ 必有最大值.

$\therefore G=uvw$ 在点$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$处取得最大值$\frac{1}{3\sqrt{3}}.$

于是,在切点$\left(x_0,y_0,z_0\right)=\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$处,四面体体积的最小值为$V_{\min}=\frac{\sqrt{3}}{2}abc.$

18. 若函数$u=u\left(x,y,z\right),v=v\left(x,y,z\right)$都可微，证明：$\frac{\partial u}{\partial x}\cdot\frac{\partial\left(u,v\right)}{\partial\left(y,z\right)}+\frac{\partial u}{\partial y}\cdot\frac{\partial\left(u,v\right)}{\partial\left(z,x\right)}+\frac{\partial u}{\partial z}\cdot\frac{\partial\left(u,v\right)}{\partial\left(x,y\right)}=0.$

解 对于$u=u(x,y,z),v=v(x,y,z)$,显然有$\dfrac{\partial(u,u,v)}{\partial(x,y,z)}=\begin{vmatrix} u_x & u_y & u_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}=0$,

对该行列式按第一行展开,得 $u_x\cdot\begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}-u_y\cdot\begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}+u_z\cdot\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=0$,

即为 $\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial(u,v)}{\partial(y,z)}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial(u,v)}{\partial(z,x)}+\dfrac{\partial u}{\partial z}\cdot\dfrac{\partial(u,v)}{\partial(x,y)}=0$.

19. 设函数$z=f(x,y)$在$\mathbb{R}^2$上可微,$\forall ab\neq 0$,若有$b\dfrac{\partial z}{\partial x}=a\dfrac{\partial z}{\partial y}$.

求证:必定有$z=\varphi(ax+by)$的形式.

证明$\because ab\neq 0,\therefore\begin{cases} u=x \\ v=ax+by \end{cases}$是一个可逆的线性变换,且$\begin{cases} x= u \\ y=-\dfrac{a}{b}u+\dfrac{1}{b}v \end{cases}$.

$\because x,y$是函数$z=f(x,y)$的两个独立的自变量,

$\therefore u,v$是函数$z=f(x,y)=g(u,v)$ 的两个独立的自变量.

$$\begin{cases} u=x \\ v=ax+by \end{cases},\begin{cases} x= u \\ y=-\dfrac{a}{b}u+\dfrac{1}{b}v \end{cases},$$

$\because f$在$\mathbb{R}^2$上可微,$\therefore\forall(x,y)\in\mathbb{R}^2,\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u}=\dfrac{\partial z}{\partial x}-\dfrac{a}{b}\cdot\dfrac{\partial z}{\partial y}\equiv 0$,

$\therefore z$作为变量u,v的函数,z相对于变量 u 而言是常数,故必定有$z=\varphi(v)=\varphi(ax+by)$的形式.