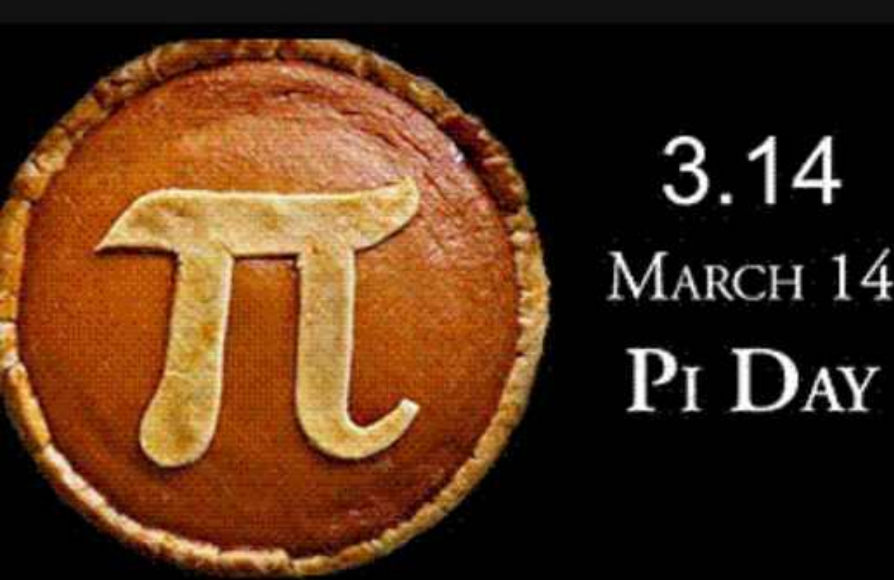

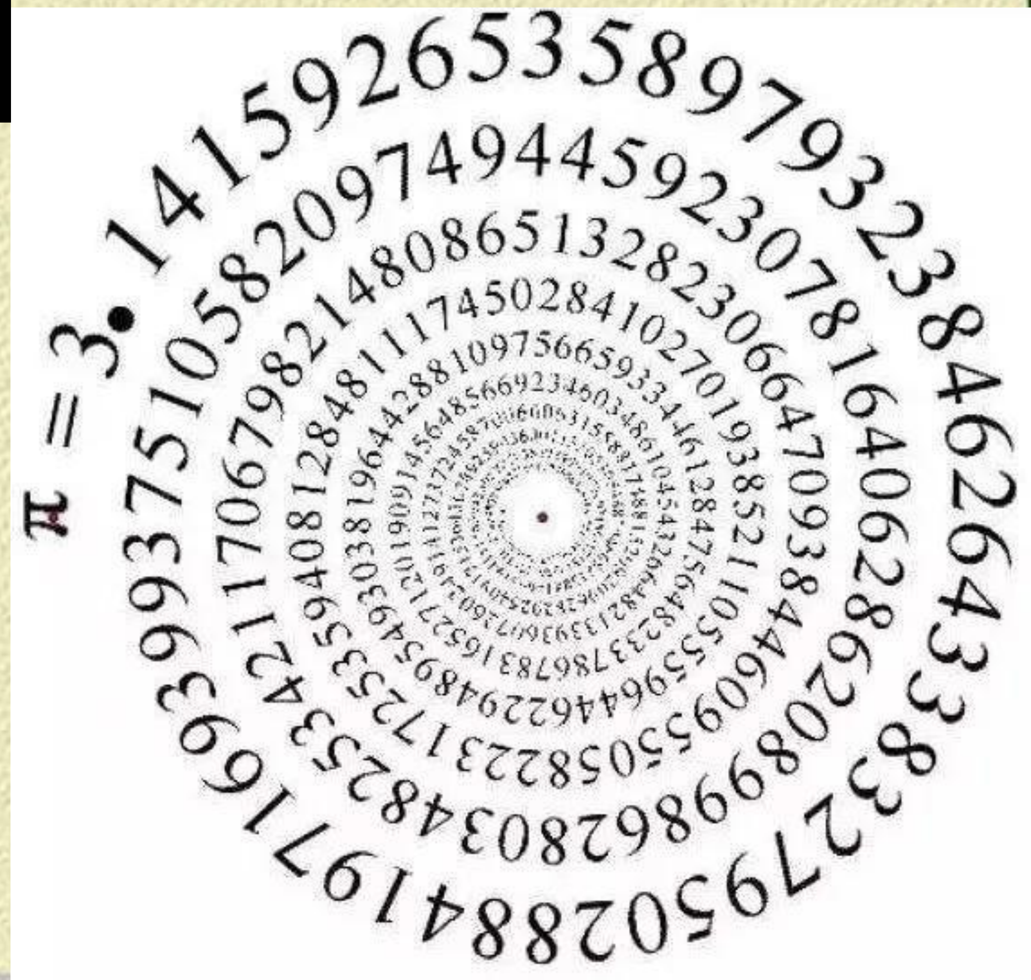

Beauty is truth,
truth beauty.
——John Keats

Chap.9 定积分习题课

2022-03

定积分精选练习 2022 – 03

1.(1).(教材 $P214/Ex.7$)若函数$f(x)$在$[0,1]$上连续.

证明:$\int_0^{\pi} xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

(2).计算(A).$\int_0^{\pi} x\sin^2 xdx$, (B).$\int_0^{\pi} x\sin^4 xdx$.

2.(1).(教材 $P214/Ex.6$)设$f(x)$是$\mathbb{R}$上以$T\,(T>0)$为周期的连续函数,证明:$I=\int_a^{a+T} f(x)dx$的值与a无关.

(1).计算$\int_0^{2022\pi} |\sin x\cos x|dx$.

(2).(教材 $P220/Ex.4$)设$f(x)$是$\mathbb{R}$上以$T\,(T>0)$为周期的连续函数,证明: $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^{x} f(t)dt=\frac{1}{T}\int_0^{T} f(t)dt$.

3.计算积分

$$(1).\int_{-\pi/2}^{\pi/2}\frac{\sin x+\cos x}{1+\sin^2 x}dx.$$

$$(2).\int_{-1}^{1}\left(1+x^{2022}\ln\frac{2+x}{2-x}\right)dx.$$

$$(3).\int_{0}^{2}x(x-1)^{n}(2-x)dx,n\in\mathbb{N}.$$

4.积分计算问题

(1).$\left(P192/Ex.3\right)$若$f(x)$在$\left[a,b\right]$上可积,$F(x)$在$\left[a,b\right]$上连续,且除有限多个点外有$F'(x)=f(x)$,则有

$$\int_a^b f(x)dx=F(b)-F(a).$$

(2).计算$\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

(3).计算$\int_{-1}^{1}\frac{1}{1+x^4}dx$.

5. $f(x), g(x)$在$[-a,a]$上连续, $g(x)$为偶函数,

且$f(x)$满足$f(x)+f(-x)=A$, A为常数.

求证 $\int_{-a}^{a} f(x)g(x)dx = A\int_0^a g(x)dx$,

并由此计算$\int_{-\pi/2}^{\pi/2}\left(\arctan e^x \cdot \cos x\right)dx$.

6.证明: $\int_0^{\pi} e^{\sin^2 x}dx \geq \frac{3}{2}\pi$.

7.证明: $\int_0^{\sqrt{2\pi}} \sin\left(x^2\right)dx > 0$.(与$P214/Ex.12$ 同)

8.设$f \in C[0,1]$, $f(x)>0$.证明:

$$\ln\int_0^1 f(x)dx \geq \int_0^1 \ln f(x)dx.$$

(与$P220/Ex.1,8$ 类同)

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9. 求极限

(1). $$\lim_{x\to 0}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}.$$

(2). $$\lim_{x\to +\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}.$$

(3). $$\lim_{x\to +\infty}\frac{\sqrt{x^2+1}}{\int_0^x (\arctan t)^2 dt}.$$

10.设$f(x)$在$[a,b]$上连续,且$f(x)>0$,

$$\Phi(x)=\int_a^x f(t)dt+\int_b^x \frac{dt}{f(t)},x\in[a,b].$$

证明:(1).$\Phi'(x)\geq 2$；(2).方程$\Phi(x)=0$在(a,b)内有唯一的实根.

11.设函数$f(x)$在$(-\infty,+\infty)$上连续,

证明:$\int_0^x (x-u)f(u)du=\int_0^x\left[\int_0^u f(x)dx\right]du$.

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12.(P214/Ex.9)设函数$f(x)$在$(0,+\infty)$上连续,$\forall a,b\in(0,+\infty)$,$I=\int_a^{ab}f(x)dx$与$a$无关.

证明:$f(x)=\dfrac{C}{x}$,C 为常数.

13.(P214/Ex.13)若$x>0,c>0$,则有

$$\left|\int_x^{x+c}\sin\left(t^2\right)dt\right|\le\frac{1}{x}.$$

14.设函数$f(x)$和$g(x)$在$[a,b]$上均连续,

证明:(1).若在$[a,b]$上$f(x)\geq 0$且$\int_a^b f(x)dx=0$,

则在$[a,b]$上 $f(x)\equiv 0$.

(2).若在$[a,b]$上$f(x)\geq 0$且不恒等于零,

则 $\int_a^b f(x)dx>0$.

同样地有:

(3).若在$[a,b]$上$f(x)\leq g(x)$且$\int_a^b f(x)dx=\int_a^b g(x)dx$,

则在$[a,b]$上 $f(x)\equiv g(x)$.

(4).若在$[a,b]$上$f(x)\leq g(x)$但$f(x),g(x)$不恒相等 .

则 $\int_a^b f(x)dx<\int_a^b g(x)dx$.

15.(1).设$f(x),g(x)$在$[a,b]$上可积,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx \int_a^b g^2(x)dx\ .$$

$\left(Cauchy - Schwarz - Bunijiakovsky\right.$ 不等式$\left.\right)$

15.(2).设$f(x),g(x)$在$[a,b]$上连续,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx \int_a^b g^2(x)dx\ .$$

16.(1).设函数$f(x)$在$[a,b]$上连续,

求证:$\left(\int_a^b f(x)dx\right)^2 \le (b-a)\int_a^b f^2(x)dx\ .$

16.(1).设函数$f(x)$在$[a,b]$上连续,求证:

$$\left(\int_a^b f(x)dx\right)^2 \le (b-a)\int_a^b f^2(x)dx\ .$$

16.(2).设$f(x)$ 在$[a,b]$上连续,且$f(x)>0$.求证:

$$\int_a^b f(x)dx\cdot\int_a^b \frac{dx}{f(x)} \ge (b-a)^2.$$

17.设$f(x)$在$[a,b]$上有连续的导函数,且$f(a)=0$,

$|f'(x)|\le M, x\in[a,b]$.求证:$\left|\int_a^b f(x)dx\right| \le \frac{1}{2}M(b-a)^2$.

18.设$f(x),g(x)$在区间$[a,b]$上连续且同为单调增加或单调减少,则有

$$\int_a^b f(x)dx\int_a^b g(x)dx \le (b-a)\int_a^b f(x)g(x)dx\ .$$

19. 若$f(x)$是$[a,b]$上连续的凸函数,

求证: $\int_a^b f(x)dx \geq (b-a) f\left(\dfrac{a+b}{2}\right)$.

19.$(P205/Ex.11)$若$f(x)$在$[a,b]$上

二阶可导,且$f''(x)>0$.求证:

$$\int_a^b f(x)dx \geq (b-a) f\left(\frac{a+b}{2}\right).$$

20.设函数$f(x)$在$[a,b]$上连续且单调递增,

证明$(a+b)\int_a^b f(x)dx \leq 2\int_a^b xf(x)dx$.

定积分精选练习 2022－03

微积分基本定理：(1).若$f(x)$在区间$[a,b]$上连续，

则$\int_a^x f(t)dt$是$f(x)$在区间$[a,b]$上的一个原函数，即

$$\left(\int_a^x f(t)dt\right)' = f(x).$$

(2).若$F(x)$是连续函数$f(x)$在区间$[a,b]$上的一个

原函数，则 $\int_a^b f(x)dx = F(b) - F(a)$.

命题(调头变换or区间再现)

设$f(x)$是$[a,b]$上的连续函数，则有

$$\int_a^b f(x)dx = \int_a^b f\left(a+b-x\right)dx,$$

$$\Rightarrow \int_a^b f(x)dx = \frac{1}{2}\int_a^b \left[f(x) + f\left(a+b-x\right)\right]dx.$$

返回

1.(1).(教材 $P214/Ex.7$)若函数$f(x)$在$[0,1]$上连续.

证明：$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$

(2).计算(A).$\int_0^\pi x\sin^2 xdx$，(B).$\int_0^\pi x\sin^4 xdx$.

$$(2).(B).\int_0^\pi x\sin^4 xdx = \frac{\pi}{2}\int_0^\pi \sin^4 xdx = \frac{\pi}{2}\int_0^\pi \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{\pi}{8}\int_0^\pi \left(1-2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{\pi}{8}\left(\frac{3}{2} - \sin 2x + \frac{1}{8}\sin 4x\right)\Bigg|_0^\pi = \frac{3\pi^2}{16}.$$

或者由$\int_0^\pi \sin^4 xdx = 2\int_0^{\pi/2} \sin^4 xdx = 2\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2} = \frac{3\pi}{8}\cdots$

2.(1).(教材 $P214/Ex.6$)设$f(x)$是$\mathbb{R}$上以T ($T>0$)为周期的连续函数,

证明: $I=\int_a^{a+T} f(x)dx$的值与a无关.

(1).计算$\int_0^{2022\pi}|\sin x\cos x|dx$.

(1).$\int_0^{2022\pi}|\sin x\cos x|dx = 2022\int_0^{\pi}|\sin x\cos x|dx$

$= 2022\int_{-\pi/2}^{\pi/2}|\sin x\cos x|dx = 4044\int_0^{\pi/2}|\sin x\cos x|dx$

$= 4044\int_0^{\pi/2}\sin x\cos x dx = 2022\sin^2 x\Big|_0^{\pi/2} = 2022.$

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2.(2).(教材 $P220/Ex.4$)设$f(x)$是$\mathbb{R}$上以$T(T>0)$为周期的连续函数,证明: $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(t)dt.$

分析 $\because \lim\limits_{x\to+\infty}\dfrac{\left(\int_0^x f(t)dt\right)'}{(x)'}=\lim\limits_{x\to+\infty} f(x)$未知其存在性,

故不能使用$L'Hopital$法则.

证明 记$x=nT+l, l\in[0,T)$,

$$\int_0^x f(t)dt=\int_0^{nT} f(t)dt+\int_{nT}^{nT+l} f(t)dt\overset{t=nT+s}{=\!=\!=\!=}$$

$$=n\int_0^T f(t)dt+\int_0^l f(nT+s)ds=n\int_0^T f(t)dt+\int_0^l f(s)ds,$$

$$0\le\left|\int_0^l f(s)ds\right|\le\int_0^l |f(s)|ds\le\int_0^T |f(s)|ds,$$

$$\therefore \lim_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\lim_{n\to\infty}\frac{n\int_0^T f(t)dt+\int_0^l f(t)dt}{nT+l}$$

$$=\lim_{n\to\infty}\frac{\int_0^T f(t)dt+\frac{1}{n}\int_0^l f(t)dt}{T+\frac{l}{n}}=\frac{1}{T}\int_0^T f(t)dt.$$

3.计算积分

$$(1).\int_{-\pi/2}^{\pi/2}\frac{\sin x+\cos x}{1+\sin^2 x}dx.$$

$$(2).\int_{-1}^{1}\left(1+x^{2022}\ln\frac{2+x}{2-x}\right)dx.$$

$$(3).\int_0^2 x(x-1)^n(2-x)dx,n\in\mathbb{N}.$$

$$I\overset{x-1=t}{=\!=\!=}\int_0^2 x(x-1)^n(2-x)dx$$

$$=\int_{-1}^{1}(t+1)t^n(1-t)dt=\int_{-1}^{1}t^n\left(1-t^2\right)dt\ ,$$

就n是奇,偶数分别讨论之…

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4.积分计算问题

(1).(P192/Ex.3)若$f(x)$在$[a,b]$上可积,$F(x)$在$[a,b]$上连续,且除有限多个点外有$F'(x)=f(x)$,则有

$$\int_a^b f(x)dx = F(b)-F(a)$$

这叫作拓广的Newton－Leibniz公式.

证明 取$[a,b]$的一个划分$T=\{x_0,x_1,\cdots,x_n\}$,$a=x_0$,$x_n=b$,使得使$F'(x)=f(x)$不成立的点成为划分T的部分分点,在$\Delta_k=[x_{k-1},x_k]$上由Lagrange微分中值定理得

$$F(x_k)-F(x_{k-1})=F'(\xi_k)\Delta x_k=f(\xi_k)\Delta x_k,$$

则$F(b)-F(a)=\sum_{k=1}^{n}[F(x_k)-F(x_{k-1})]=\sum_{k=1}^{n}f(\xi_k)\Delta x_k$,

$\because$ $f(x)$在$[a,b]$上可积,$\therefore$ $\lim_{\|T\|\to 0}\sum_{k=1}^{n}f(\xi_k)\Delta x_k=\int_a^b f(x)dx$.证毕

例如,计算积分:$\int_{-1}^{1}\frac{1}{1+x^2}dx.$

解 在$(-\infty,+\infty)$上,$\arctan x$是$\frac{1}{1+x^2}$的一个原函数,

$$\therefore \int_{-1}^{1}\frac{1}{1+x^2}dx=\left[\arctan x\right]_{-1}^{1}=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}.$$

$x\neq 0$时,$\left(-\arctan\frac{1}{x}\right)'=\frac{1}{1+x^2}$,但是$-\arctan\frac{1}{x}$在$x=0$时没有定义,

所以$-\arctan\frac{1}{x}$不是$\frac{1}{1+x^2}$在$[-1,1]$上的一个原函数.稍作改造,

$$\Phi(x)=\begin{cases}-\arctan\frac{1}{x}, x<0\\ \pi/2\quad, \ x=0\\ \pi-\arctan\frac{1}{x}, x>0\end{cases}$$

,$\Phi(x)$在$[-1,1]$上连续,且$x\neq 0$时$\Phi'(x)=\frac{1}{1+x^2}$,

于是,$\int_{-1}^{1}\frac{1}{1+x^2}dx=\Phi(1)-\Phi(-1)=\pi-\arctan 1-\left(-\arctan\frac{1}{(-1)}\right)=\frac{\pi}{2}.$

4.积分计算(2).$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

$$\int\frac{1}{2+\cos 2x}dx=\int\frac{1}{1+2\cos^2 x}dx=\int\frac{\sec^2 x}{2+\sec^2 x}dx$$

$$=\int\frac{(\tan x)'}{3+\tan^2 x}dx=\int\frac{1}{\left(\sqrt{3}\right)^2+\tan^2 x}d(\tan x)$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+C.$$

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4.积分计算(2).$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

倘由$\int\frac{1}{2+\cos 2x}dx=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+C$

得$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}\Big|_0^{\pi}=0,$

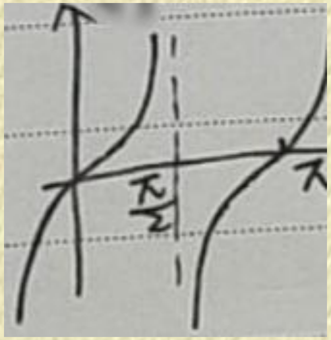
在$[0,\pi]$上$\frac{1}{2+\cos 2x}\geq\frac{1}{3}>0$,我们知道错了!

由$\frac{1}{2+\cos 2x}$在$[0,\pi]$上连续,因而其原函数也

必须是连续的.故$\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}$不是$\frac{1}{2+\cos 2x}$

在$[0,\pi]$上的原函数.

4.积分计算(2).$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.



正解 $\because \frac{1}{2+\cos 2x}$是以$\pi$为周期的连续函数,

$$\therefore I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx=\int_{-\pi/2}^{\pi/2}\frac{1}{2+\cos 2x}dx$$

$$\overset{\text{奇偶性}}{=\!=\!=\!=}2\int_0^{\pi/2}\frac{1}{2+\cos 2x}dx=2\lim_{u\to\frac{\pi}{2}-0}\int_0^{u}\frac{1}{2+\cos 2x}dx$$

$$=\frac{2}{\sqrt{3}}\lim_{u\to\frac{\pi}{2}-0}\arctan\frac{\tan x}{\sqrt{3}}\Big|_0^u=\frac{2}{\sqrt{3}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{3}}.$$

4.积分计算(2). $I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.

$\because \frac{1}{2+\cos 2x}$ 是$(-\infty,+\infty)$上的连续函数,

$\therefore \int_0^{u}\frac{1}{2+\cos 2x}dx=\Phi(u)$ 在$(-\infty,+\infty)$上连续,

$\therefore \Phi\left(\frac{\pi}{2}\right)=\lim\limits_{u\to\frac{\pi}{2}-0}\Phi(u)$,即

$$\int_0^{\pi/2}\frac{1}{2+\cos 2x}dx=\lim_{u\to\frac{\pi}{2}-0}\int_0^{u}\frac{1}{2+\cos 2x}dx.$$

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法二 $\int_0^{\pi}\frac{1}{2+\cos 2x}dx=\int_0^{\pi}\frac{1}{1+2\cos^2 x}dx$

$$=\int_0^{\pi/4}\frac{1}{1+2\cos^2 x}dx+\int_{\pi/4}^{3\pi/4}\frac{1}{1+2\cos^2 x}dx+\int_{3\pi/4}^{\pi}\frac{1}{1+2\cos^2 x}dx$$

$$=\int_0^{\pi/4}\frac{\sec^2 x}{2+\sec^2 x}dx+\int_{\pi/4}^{3\pi/4}\frac{\csc^2 x}{\csc^2 x+2\cot^2 x}dx+\int_{3\pi/4}^{\pi}\frac{\sec^2 x}{2+\sec^2 x}dx$$

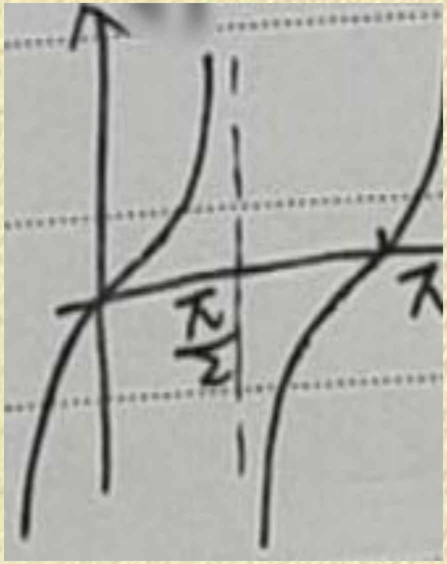
$$=\int_0^{\pi/4}\frac{d\left(\tan x\right)}{(\sqrt{3})^2+\tan^2 x}+\int_{\pi/4}^{3\pi/4}\frac{-d\left(\cot x\right)}{1+\left(\sqrt{3}\cot x\right)^2}dx+\int_{3\pi/4}^{\pi}\frac{d\left(\tan x\right)}{(\sqrt{3})^2+\tan^2 x}$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}\Big|_0^{\pi/4}-\frac{1}{\sqrt{3}}\arctan\left(\sqrt{3}\cot x\right)\Big|_{\pi/4}^{3\pi/4}+\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}\Big|_{3\pi/4}^{\pi}$$

$$=\cdots\cdots=\frac{\pi}{\sqrt{3}}.$$

(注：其中点$\frac{\pi}{4}$的选取是随意的,只要取$\left(0,\frac{\pi}{2}\right)$内的点即可;同样,$\frac{3\pi}{4}$亦如此.)

4.积分计算(2).$I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx$.



法三 由$\int\frac{1}{2+\cos 2x}dx=\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+C$,

据拓广的*Newton – Leibniz*公式,取

$$\Phi(x)=\begin{cases}\frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}},0\le x<\frac{\pi}{2}\\ \frac{\pi}{2\sqrt{3}}\quad,\quad x=\frac{\pi}{2}\\ \frac{1}{\sqrt{3}}\arctan\frac{\tan x}{\sqrt{3}}+\frac{\pi}{\sqrt{3}},\frac{\pi}{2}<x\le\pi\end{cases}\text{，}\Phi(x)\text{在}[0,\pi]\text{上连续,}$$

$$\therefore I=\int_0^{\pi}\frac{1}{2+\cos 2x}dx=\Phi(\pi)-\Phi(0)=\frac{\pi}{\sqrt{3}}\ .$$

4.积分计算(3).$I=\int_{-1}^{1}\frac{1}{1+x^4}dx$.

解 $\int\frac{1}{1+x^4}dx=\frac{1}{2\sqrt{2}}\int\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)dx$

$$=\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\left[\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right]+C.$$

$$\therefore I=\int_{-1}^{1}\frac{1}{1+x^4}dx=\left\{\begin{aligned}&\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|\\&+\frac{1}{2\sqrt{2}}\left[\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right]\end{aligned}\right\}_{-1}^{1}$$

$$=\frac{1}{\sqrt{2}}\left[\ln\left(\sqrt{2}+1\right)+\frac{\pi}{2}\right]=A.$$

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4.积分计算(3).$I=\int_{-1}^{1}\frac{1}{1+x^4}dx$.

解二 $\int\frac{1}{1+x^4}dx=\frac{1}{2}\left(\int\frac{1+x^2}{1+x^4}dx+\int\frac{1-x^2}{1+x^4}dx\right)$

$$=\frac{1}{2}\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx-\frac{1}{2}\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\frac{1}{2}\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}-\frac{1}{2}\int\frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2}$$

$$=\frac{1}{2\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}-\frac{1}{4\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C_1$$

$$=\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}+C_1,$$

4.积分计算(3).$I=\int_{-1}^{1}\frac{1}{1+x^4}dx$.

$$\int\frac{1}{1+x^4}dx=\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}+C_1,$$

作与4.(2)同样的改造,

$$G(x)=\begin{cases}\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}, x<0\\ \frac{\pi}{4\sqrt{2}}\qquad, \qquad x=0\\ \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}+\frac{\pi}{2\sqrt{2}}, x>0\end{cases},$$

则 $\int_{-1}^{1}\frac{1}{1+x^4}dx=G(1)-G(-1)=\frac{1}{\sqrt{2}}\left[\ln\left(\sqrt{2}+1\right)+\frac{\pi}{2}\right]=A.$

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5. $f(x), g(x)$在$[-a,a]$上连续, $g(x)$为偶函数, 且$f(x)$满足

$f(x)+f(-x)=A$, A为常数.求证$\int_{-a}^{a} f(x)g(x)dx = A\int_{0}^{a} g(x)dx$.

并由此计算$\int_{-\pi/2}^{\pi/2}\left(\arctan e^{x}\cdot\cos x\right)dx$.

证明 $L=\int_{-a}^{0} f(x)g(x)dx+\int_{0}^{a} f(x)g(x)dx$

$$\overset{x=-t}{=\!=\!=}-\int_{a}^{0} f(-t)g(-t)dt+\int_{0}^{a} f(x)g(x)dx$$

$$=\int_{0}^{a} f(-x)g(-x)dx+\int_{0}^{a} f(x)g(x)dx$$

$$=\int_{0}^{a} f(-x)g(x)dx+\int_{0}^{a} f(x)g(x)dx$$

$$=\int_{0}^{a}[f(x)+f(-x)]g(x)dx=R\ .$$

而 $\arctan e^{x}+\arctan e^{-x}\equiv\frac{\pi}{2}\cdots$

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6.证明$\int_0^{\pi} e^{\sin^2 x}dx \ge \frac{3}{2}\pi.$

函数$\frac{\sin x}{\sqrt{x}}$在区间$(0,\pi]$上有界,我们取

$$\varphi(x)=\begin{cases}\frac{\sin x}{\sqrt{x}}, & x\ne 0\\ 0, & x=0\end{cases},\varphi(x)\text{在}[0,\pi]\text{上连续},$$

通常我们将$\int_0^{\pi}\varphi(x)dx$表示为$\int_0^{\pi}\frac{\sin x}{\sqrt{x}}dt.$

$Hint: \forall t\in\mathbb{R}, e^t \ge 1+t.$

7.证明:$I=\int_0^{\sqrt{2\pi}}\sin\left(x^2\right)dx>0.$(与$P214/Ex.12$ 同)

$$\text{证明}\quad I \overset{x^2=t}{=\!=\!=} \int_0^{2\pi}\sin t\cdot\frac{1}{2\sqrt{t}}dt=\int_0^{\pi}\frac{\sin t}{2\sqrt{t}}dt+\int_{\pi}^{2\pi}\frac{\sin t}{2\sqrt{t}}dt$$

$$\overset{t=\pi+s}{=\!=\!=}\int_0^{\pi}\frac{\sin t}{2\sqrt{t}}dt+\int_0^{\pi}\frac{\sin(\pi+s)}{2\sqrt{\pi+s}}ds$$

$$=\int_0^{\pi}\frac{\sin t}{2\sqrt{t}}dt-\int_0^{\pi}\frac{\sin s}{2\sqrt{\pi+s}}ds=\frac{1}{2}\int_0^{\pi}\left(\frac{1}{\sqrt{t}}-\frac{1}{\sqrt{\pi+t}}\right)\sin t dt\ge 0,$$

再根据14题结论,$\cdots>0.$

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8.设$f\in C[0,1], f(x)>0.$

证明 $\ln\int_0^1 f(x)dx \ge \int_0^1 \ln f(x)dx.$

(与$P220/Ex.1,8$ 类同)

证明 由$[0,1]$上$f(x)>0$ 知$\int_0^1 f(x)dx = A>0.$

$\because \forall t>-1, \ln(1+t)\le t.$ $\therefore \forall x\in[0,1],$

$$\ln f(x)=\ln A+\ln\left(1+\frac{f(x)}{A}-1\right)\le \ln A+\frac{f(x)}{A}-1,$$

$$\therefore R\le \int_0^1\left[\ln A+\frac{f(x)}{A}-1\right]dx=\ln A+\int_0^1\frac{f(x)}{A}dx-1$$

$=\ln A=L.$

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9. 求极限(1). $\lim\limits_{x\to 0}\dfrac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$.

解 由积分中值定理知, $x\to 0$时$\int_0^x e^{t^2}dt = xe^{\xi^2}\to 0$,

这是$\dfrac{0}{0}$形未定型,用$L'Hopital$法则

$$\lim_{x\to 0}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}=\lim_{x\to 0}\frac{2\int_0^x e^{t^2}dt\cdot e^{x^2}}{e^{2x^2}}=\lim_{x\to 0}\frac{2\int_0^x e^{t^2}dt}{e^{x^2}}=0.$$

9.(2). $\lim\limits_{x\to+\infty}\dfrac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$.

解 $s\geq 0$时, $e^s\geq 1\Rightarrow\int_0^x e^{2t^2}dt\geq\int_0^x 1dt=x$,

$\lim\limits_{x\to+\infty}\int_0^x e^{2t^2}dt\geq\lim\limits_{x\to+\infty}x=+\infty$, $\therefore$ 原问题属于$\dfrac{*}{\infty}$型未定型.

$$\lim_{x\to+\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}=\lim_{x\to+\infty}\frac{2\int_0^x e^{t^2}dt\cdot e^{x^2}}{e^{2x^2}}=2\lim_{x\to+\infty}\frac{\int_0^x e^{t^2}dt}{e^{x^2}}$$

$$\overset{\frac{\infty}{\infty}}{=\!=}2\lim_{x\to+\infty}\frac{e^{x^2}}{e^{x^2}\cdot 2x}=0.$$

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9.(3). $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+1}}{\int_0^x(\arctan t)^2\,dt}$.

解 考虑$x\ge 2$ 时,$\arctan x>\arctan\sqrt{3}=\dfrac{\pi}{3}>1$,

$$\int_0^x(\arctan t)^2dt=\int_0^2(\arctan t)^2dt+\int_2^x(\arctan t)^2dt$$

$$>\int_0^2(\arctan t)^2dt+\int_2^x 1dt=\int_0^2(\arctan t)^2dt+x-2,$$

$\therefore$ 原问题属于$\dfrac{*}{\infty}$型未定型,用$L'Hopital$法则

$$\therefore \lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{\int_0^x(\arctan t)^2dt}=\lim_{x\to+\infty}\frac{x}{\int_0^x(\arctan t)^2dt}$$

$$=\lim_{x\to+\infty}\frac{1}{(\arctan x)^2}=\frac{4}{\pi^2}.$$

$$9.(3).\ \lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{\int_0^x(\arctan t)^2\,dt}.$$

$$法二\ \because\ \lim_{x\to+\infty}\frac{\int_0^x(\arctan t)^2\,dt}{\sqrt{x^2+1}}$$

$$\overset{\frac{*}{\infty}}{=\!=}\lim_{x\to+\infty}\frac{\int_0^x(\arctan t)^2\,dt}{x}$$

$$=\lim_{x\to+\infty}\frac{(\arctan x)^2}{1}=\frac{\pi^2}{4}\cdots$$

10.设$f(x)$在$[a,b]$上连续,且$f(x)>0$,

$$\Phi(x)=\int_a^x f(t)dt+\int_b^x \frac{dt}{f(t)}, x\in[a,b].$$

证明:(1).$\Phi'(x)\geq 2$；(2).方程$\Phi(x)=0$在(a,b)内有唯一的实根.

$Hint$:(2).由$\Phi(a)\Phi(b)<0$据介值定理可得结论

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11.设函数$f(x)$在$(-\infty,+\infty)$上连续,

证明: $\int_0^x (x-u)f(u)du = \int_0^x \left[\int_0^u f(x)dx\right]du$.

证明 $\Phi(u) = \int_0^u f(t)dt$,则

$$R = \int_0^x \Phi(u)du = u\Phi(u)\Big|_0^x - \int_0^x u\Phi'(u)du$$

$$= x\Phi(x) - \int_0^x uf(u)du = x\int_0^x f(u)du - \int_0^x uf(u)du$$

$$= \int_0^x (x-u)f(u)du = L$$

11.设函数$f(x)$在$(-\infty,+\infty)$上连续,

证明:$\int_0^x (x-u)f(u)du=\int_0^x\left[\int_0^u f(x)dx\right]du$.

法二 记$g(x)=\int_0^x (x-u)f(u)du$,$h(x)=\int_0^x\left[\int_0^u f(x)dx\right]du$,

$$g'(x)=\left[\int_0^x (x-u)f(u)du\right]'=\left[x\int_0^x f(u)du-\int_0^x uf(u)du\right]'$$

$$=\int_0^x f(u)du=\Phi(x) ,$$

$$h'(x)=\left[\int_0^x\left[\int_0^u f(x)dx\right]du\right]'=\left[\int_0^x \Phi(u)du\right]'$$

$$=\int_0^x f(u)du=\Phi(x) ,$$

$\therefore g'(x)=h'(x)$,$\therefore g(x)=h(x)+C$

又$\because g(0)=h(0)=0$ $\therefore g(x)\equiv h(x)$.

12.$(P214/Ex.9)$设函数$f(x)$在$(0,+\infty)$上连续,$\forall a,b\in(0,+\infty), I=\int_a^{ab} f(x)dx$与$a$无关.

证明: $f(x)=\dfrac{C}{x}, C$ 常数.

证明 $\int_a^{ab} f(x)dx \equiv Const., \forall a,b\in(0,+\infty),$

$$\Rightarrow \forall a,b\in(0,+\infty), \left[\int_a^{ab} f(x)dx\right]_a' = 0,$$

$\therefore bf(ab)=f(a), \Leftrightarrow abf(ab)=af(a),$

由$a,b\in(0,+\infty)$的任意性,

$\therefore \forall x>0, xf(x)=C$ 常数.

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13.$(P214/Ex.13)$若$x>0,c>0$,则有

$$\left|\int_x^{x+c}\sin\left(t^2\right)dt\right|\le\frac{1}{x}\ .$$

证明　证明方法比较独特.此题缺乏营养.

$$\int_x^{x+c}\sin\left(t^2\right)dt=\int_x^{x+c}\frac{\sin\left(t^2\right)}{2t}dt^2=\frac{-\cos\left(t^2\right)}{2t}\Bigg|_x^{x+c}+\int_x^{x+c}\frac{\cos\left(t^2\right)}{-2t^2}dt,$$

$$=\frac{\cos\left(x^2\right)}{2x}-\frac{\cos\left(x+c\right)^2}{2\left(x+c\right)}-\int_x^{x+c}\frac{\cos\left(t^2\right)}{2t^2}dt,$$

$$\therefore\left|\int_x^{x+c}\sin\left(t^2\right)dt\right|\le\left|\frac{\cos\left(x^2\right)}{2x}\right|+\left|\frac{\cos\left(x+c\right)^2}{2\left(x+c\right)}\right|+\left|\int_x^{x+c}\frac{\cos\left(t^2\right)}{2t^2}dt\right|$$

$$\le\frac{1}{2x}+\frac{1}{2\left(x+c\right)}+\int_x^{x+c}\left|\frac{\cos\left(t^2\right)}{2t^2}\right|dt\le\frac{1}{2x}+\frac{1}{2\left(x+c\right)}+\int_x^{x+c}\frac{dt}{2t^2}=\frac{1}{x}.$$

14.设函数$f(x)$和$g(x)$在$[a,b]$上均连续,

证明:(1).若在$[a,b]$上$f(x)\geq 0$且$\int_a^b f(x)dx=0$,

则在$[a,b]$上 $f(x)\equiv 0$.

(2).若在$[a,b]$上$f(x)\geq 0$且不恒等于零,

则 $\int_a^b f(x)dx>0$.

同样地有:

(3).若在$[a,b]$上$f(x)\leq g(x)$且$\int_a^b f(x)dx=\int_a^b g(x)dx$,

则在$[a,b]$上 $f(x)\equiv g(x)$.

(4).若在$[a,b]$上$f(x)\leq g(x)$但$f(x),g(x)$不恒相等 .

则 $\int_a^b f(x)dx<\int_a^b g(x)dx$.

14.设函数$f(x)$和$g(x)$在$[a,b]$上均连续,

证明:(1).若在$[a,b]$上$f(x)\geq 0$且$\int_a^b f(x)dx=0$,

则在$[a,b]$上 $f(x)\equiv 0$.

证明(1). 反证法　假设结论不成立,

则$\exists x_0\in[a,b]$, $f(x_0)\neq 0$,即$f(x_0)>0$.

$\because$ $f(x)$在$[a,b]$上连续,$\therefore \exists\delta_0>0, \forall |x-x_0|\leq\delta_0$,

有$|f(x)-f(x_0)|\leq\frac{1}{2}f(x_0)$,即$f(x)\geq\frac{1}{2}f(x_0)$,

$$\therefore \int_a^b f(x)dx\geq\int_{x_0-\delta_0}^{x_0+\delta_0}f(x)dx$$

$$\geq\int_{x_0-\delta_0}^{x_0+\delta_0}\frac{1}{2}f(x_0)dx=\delta_0\cdot f(x_0)>0,$$

而这与$\int_a^b f(x)dx=0$矛盾,$\therefore$假设不成立.

上页　下页　返回

14.设函数$f(x)$和$g(x)$在$[a,b]$上均连续,

证明:(1).若在$[a,b]$上$f(x)\geq 0$且$\int_a^b f(x)dx = 0$,

则在$[a,b]$上 $f(x)\equiv 0$.

法二 $\because$ 在$[a,b]$上函数$f(x)$连续、非负,

$$\therefore \forall u \in [a,b], 0 \leq \int_a^u f(x)dx \leq \int_a^b f(x)dx = 0,$$

$$\therefore \forall u \in [a,b], \int_a^u f(x)dx \equiv 0,$$

$$\therefore \forall u \in [a,b], \left(\int_a^u f(x)dx\right)' = f(u) \equiv 0.$$

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14.设函数$f(x)$和$g(x)$在$[a,b]$上均连续,

证明:(1).若在$[a,b]$上$f(x)\geq 0$且$\int_a^b f(x)dx=0$,

则在$[a,b]$上 $f(x)\equiv 0$.

(2).若在$[a,b]$上$f(x)\geq 0$且不恒等于零,则 $\int_a^b f(x)dx>0$.

设函数$f(x)$在$[a,b]$上连续,则

"(1)若在$[a,b]$上$f(x)\geq 0$且$\int_a^b f(x)dx=0$,

则在$[a,b]$上 $f(x)\equiv 0$"与"(2)若在$[a,b]$

上$f(x)\geq 0$且不恒等于零,则$\int_a^b f(x)dx>0$"

是互为逆否命题的两个命题,故而(2)成立.

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15.(1).设$f(x),g(x)$在$[a,b]$上可积,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx \int_a^b g^2(x)dx\ .$$

(Cauchy – Schwarz – Bunijiakovsky 不等式)

证明 据离散形式的Cauchy不等式$\left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$

由积分定义可证得…

法二 $\forall x \in [a,b], \forall \lambda \in \mathbb{R}, (\lambda f(x) + g(x))^2 \ge 0 \Rightarrow$

$\int_a^b [\lambda f(x) + g(x)]^2 dx \ge 0, \therefore \forall \lambda \in \mathbb{R}$,有

$$\lambda^2 \int_a^b f^2(x)dx + 2\lambda \int_a^b f(x)g(x)dx + \int_a^b g^2(x)dx \ge 0,$$

$$\Rightarrow \Delta \le 0 \Leftrightarrow \int_a^b f^2(x)dx \int_a^b g^2(x)dx \ge \left[\int_a^b f(x)g(x)dx\right]^2.$$

15.(2).设$f(x),g(x)$在$[a,b]$上连续,则有:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx \ .$$

证明 设$\varphi(u)=\int_a^u f^2(x)dx\int_a^u g^2(x)dx-\left[\int_a^u f(x)g(x)dx\right]^2$,

$$\varphi'(u)= f^2(u)\int_a^u g^2(x)dx+g^2(u)\int_a^u f^2(x)dx-2f(u)g(u)\int_a^u f(x)g(x)dx$$

$$=\int_a^u f^2(u)g^2(x)dx+\int_a^u g^2(u)f^2(x)dx-2\int_a^u f(x)g(x)f(u)g(u)dx$$

$$=\int_a^u\left[f^2(u)g^2(x)-2f(x)g(x)f(u)g(u)+f^2(x)g^2(u)\right]dx$$

$$=\int_a^u[f(u)g(x)-f(x)g(u)]^2dx\ge 0,$$

即$u\in[a,b]$时$\varphi(u)$单调增加,$\varphi(a)=0$,故$b>a$时$\varphi(b)\ge\varphi(a)=0$.

16.(1).设函数$f(x)$在$[a,b]$上连续,求证:

$$\left(\int_a^b f(x)dx\right)^2 \le (b-a)\int_a^b f^2(x)dx \ .$$

16.(2).设$f(x)$ 在$[a,b]$上连续,且$f(x)>0$.求证:

$$\int_a^b f(x)dx \cdot \int_a^b \frac{dx}{f(x)} \ge (b-a)^2 .$$

17.设$f(x)$在$[a,b]$上有连续的导函数,且$f(a)=0$,

$|f'(x)| \le M, x \in [a,b]$.求证:$\left|\int_a^b f(x)dx\right| \le \dfrac{1}{2}M(b-a)^2$.

18.设$f(x),g(x)$在区间$[a,b]$上连续且同为单调增加或单调减少,则有

$$\int_a^b f(x)dx\int_a^b g(x)dx \le (b-a)\int_a^b f(x)g(x)dx\ .$$

注意,18题中区间$[a,b]$变为$[0,1]$,则结论变为

$$\int_0^1 f(x)dx\int_0^1 g(x)dx \le \int_0^1 f(x)g(x)dx\ .$$

那么证明起来要难得多了.

19. 若$f(x)$是$[a,b]$上连续的凸函数,

求证：$\int_a^b f(x)dx \ge (b-a)f\left(\frac{a+b}{2}\right)$.

由命题(调头变换or区间再现)

设$f(x)$是$[a,b]$上的连续函数,则有

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx, \Rightarrow$$

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b \left[f(x) + f(a+b-x)\right]dx.$$

19. 若$f(x)$是$[a,b]$上连续的凸函数,

求证: $\int_a^b f(x)dx \geq (b-a)f\left(\frac{a+b}{2}\right)$.

证明 $\because$ 函数$f(x)$是$[a,b]$上连续的凸函数,

$$\therefore \frac{1}{2}\left[f(x)+f(a+b-x)\right]$$

$$\geq f\left(\frac{x+(a+b-x)}{2}\right)=f\left(\frac{a+b}{2}\right),$$

$$\therefore \int_a^b f(x)dx=\frac{1}{2}\int_a^b\left[f(x)+f(a+b-x)\right]dx$$

$$\geq \int_a^b f\left(\frac{x+(a+b-x)}{2}\right)dx=\int_a^b f\left(\frac{a+b}{2}\right)dx$$

$=(b-a)f\left(\frac{a+b}{2}\right)$.结论成立.

19.$\left(P205/Ex.11\right)$若$f(x)$在$[a,b]$上二阶可导,且$f''(x)>0$.求证:

$$\int_a^b f(x)dx \ge (b-a)f\left(\frac{a+b}{2}\right).$$

证明 设$\varphi(u)=\int_a^u f(x)dx-(u-a)f\left(\frac{a+u}{2}\right)$,

$u\in[a,b],\varphi(a)=0.$

$$\varphi'(u)=f(u)-f\left(\frac{a+u}{2}\right)-\frac{1}{2}(u-a)f'\left(\frac{a+u}{2}\right)$$

$$=\left(u-\frac{a+u}{2}\right)f'(\xi)-\frac{1}{2}(u-a)f'\left(\frac{a+u}{2}\right)$$

$$=\frac{1}{2}(u-a)\left[f'(\xi)-f'\left(\frac{a+u}{2}\right)\right],a\le\frac{a+u}{2}<\xi<u\le b.\cdots$$

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20.设函数$f(x)$在$[a,b]$上连续且单调递增,

证明$(a+b)\int_a^b f(x)dx \le 2\int_a^b xf(x)dx$.

证明 设$\varphi(u)=2\int_a^u xf(x)dx-(a+u)\int_a^u f(x)dx$

显然,$\varphi(u)$在$[a,b]$上连续,

$$\varphi'(u)=2uf(u)-(a+u)f(u)-\int_a^u f(x)dx$$

$$=(u-a)f(u)-\int_a^u f(x)dx=\int_a^u f(u)dx-\int_a^u f(x)dx$$

$$=\int_a^u [f(u)-f(x)]dx, a\le x\le u\le b,$$

$\because$ 函数$f(x)$在$[a,b]$上连续且单调递增,

$\therefore u\in[a,b], \varphi'(u)\ge 0 \Rightarrow b>a$有$\varphi(b)\ge\varphi(a)=0.$

20.设函数$f(x)$在$[a,b]$上连续且单调递增,

证明$(a+b)\int_a^b f(x)dx \le 2\int_a^b xf(x)dx$.

法二 $\because$函数$f(x)$在$[a,b]$上连续且单调递增,

$\therefore \forall x,t \in [a,b]$有$(x-u)(f(x)-f(u)) \ge 0$,

$\int_a^b (x-u)(f(x)-f(u))dx \ge 0$,得

$\int_a^b [xf(x)+uf(u)-xf(u)-uf(x)]dx \ge 0$,

$\int_a^b xf(x)dx + uf(u)(b-a) - f(u)\cdot\frac{1}{2}(b^2-a^2) - u\int_a^b f(x)dx \ge 0, \cdots\cdots(1)$

再在(1)式两边对变量u积分,得

$$(b-a)\int_a^b xf(x)dx + (b-a)\int_a^b uf(u)du - \frac{1}{2}(b^2-a^2)\int_a^b f(u)du - \frac{1}{2}(b^2-a^2)\int_a^b f(x)dx \ge 0,$$

$\therefore 2(b-a)\int_a^b xf(x)dx \ge (b^2-a^2)\int_a^b f(x)dx,\ a \le b,$

即 $2\int_a^b xf(x)dx \ge (a+b)\int_a^b f(x)dx$.

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20.设函数$f(x)$在$[a,b]$上连续且单调递增,

证明$(a+b)\int_a^b f(x)dx \le 2\int_a^b xf(x)dx$.

定积分的问题处理起来往往会富有技巧性.

法三 利用 $\int_{-a}^{a} f(x)dx = \int_{-a}^{a} f(-x)dx = \frac{1}{2}\int_{-a}^{a}[f(x)+f(-x)]dx$

这一结论, 将积分区间中心化 : 令$x=\frac{a+b}{2}+t$,记$\frac{b-a}{2}=c$.

则 $(a+b)\int_a^b f(x)dx \le 2\int_a^b xf(x)dx \Leftrightarrow \int_a^b\left(x-\frac{a+b}{2}\right)f(x)dx \ge 0$

$$\xleftrightarrow{\frac{a+b}{2}+t=x}\int_a^b\left(x-\frac{a+b}{2}\right)f(x)dx=\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} tf\left(\frac{a+b}{2}+t\right)dt=\int_{-c}^{c} tf\left(\frac{a+b}{2}+t\right)dt \ge 0,$$

而$\int_{-c}^{c} tf\left(\frac{a+b}{2}+t\right)dt=\frac{1}{2}\int_{-c}^{c} t\left[f\left(\frac{a+b}{2}+t\right)-f\left(\frac{a+b}{2}-t\right)\right]dt$

$$=\int_0^c t\left[f\left(\frac{a+b}{2}+t\right)-f\left(\frac{a+b}{2}-t\right)\right]dt$$

而在$[0,c]$上显然有 $f\left(\frac{a+b}{2}+t\right)-f\left(\frac{a+b}{2}-t\right)\ge 0$,

$$\therefore \int_0^c t\left[f\left(\frac{a+b}{2}+t\right)-f\left(\frac{a+b}{2}-t\right)\right]dt \ge 0.\cdots$$

此题做法二三稍麻烦些,亦还有多种其他不同的做法…