§ 2.2 不定积分的计算 -分部积分法

前面我们已经看到,利用复合函数的求导公式,推导得到了两类积分换元公式.

下面我们要利用两个函数乘积的 求导公式,推导得到另一样的积分 计算公式——

分部紀分公式







一分部积分法 integration by parts

一问题
$$\int xe^x dx = ?$$

解决思路:利用两个函数乘积的求导公式.

设函数u = u(x)和v = v(x)有连续的导数,则

$$(uv)' = u'v + uv', u'v = (uv)' - uv',$$

$$\int u'vdx = \int (uv)'dx - \int uv'dx,$$

$$\Rightarrow \int u'vdx = uv - \int uv'dx,$$

$$or \int vdu = uv - \int udv$$



分部积分公式



例1.求积分 $J = \int x \cos x dx$.

解(1).令
$$v = \cos x, u' = x$$
,则可取 $u = \frac{1}{2}x^2$,

$$J = \int \left(\frac{1}{2}x^{2}\right)' \cos x dx = \frac{x^{2}}{2} \cos x - \int \frac{x^{2}}{2} (-\sin x) dx,$$

显然 u',v 的选择不当,致使未能解决问题. 60(2) $\Delta v = v \cdot u' = \cos v$ 则可取 $u = \sin v$

解(2).令
$$v = x, u' = \cos x$$
,则可取 $u = \sin x$,

$$\int x \cos x dx = \int x (\sin x)' dx$$

$$= x \sin x - \int \sin x \cdot 1 dx = x \sin x + \cos x + C.$$

在用分部积分法求积分 $\int x \cos x dx$ 时, $\Xi \diamond u' = \cos x,$ 则可取 $u = \sin x,$ 工由u'求u时的常数不用考虑.

中 例1.(2).求积分 $\int x^2 e^{-x} dx$.

工顺便提一下,以下规律性的结果

$$\iint xe^{-x}dx = -(x+1)e^{-x} + C$$

$$\iint x^2e^{-x}dx = -(x^2 + 2x + 2)e^{-x} + C$$

$$\iint x^3e^{-x}dx = -(x^3 + 3x^2 + 6x + 6)e^{-x} + C$$
...



$$\int u'vdx = uv - \int uv'dx$$

$$\int x \cos x dx$$

$$\Leftrightarrow x^{\mu} = v$$

$$\int u' = \cos x$$

$$\mu \in \mathbb{Z}^{+}$$

$$u' = e^{-x}$$

小结 若被积函数是(正整数次幂的)幂函数与正(余)弦函数或指数函数的乘积,就考虑把正(余)弦或指数函数作为u',而设幂函数为v,通过求导使其幂次降低.







$$\ln x dx = \int \left(\frac{x^4}{4}\right) \ln x dx$$

$$\int x^{3} \ln x dx = \int \left(\frac{x^{4}}{4}\right)' \ln x dx$$

$$= \frac{1}{4} x^{4} \ln x - \int \frac{x^{4}}{4} \cdot \frac{1}{x} dx = \frac{1}{4} x^{4} \ln x - \frac{1}{16} x^{4} + C$$





例2.求积分
$$\int x^3 \ln x dx$$
.

第二 令
$$\ln x = t$$
,则 $x = e^t$,
$$\int x^3 \ln x dx = \int e^{3t} \cdot t \cdot e^t dt = \int t e^{4t} dt$$

$$\frac{1}{2} \ln x \, dx = \int e^{3t} \cdot t \cdot e^{t} \, dt = \int t e^{4t} \, dt$$

$$\frac{1}{2} \int u e^{u} \, du = \int (u e^{u} - \int e^{u} \, du)$$

$$\frac{1}{16} = \frac{1}{16} \int ue^{u} du = \frac{1}{16} \left(ue^{u} - \int e^{u} du \right)$$

$$= \frac{1}{16} \left(ue^{u} - e^{u} \right) + C = \frac{1}{16} \left(4te^{4t} - e^{4t} \right)$$

$$\frac{1}{16} = \frac{1}{16} \left(ue^{u} - e^{u} \right) + C = \frac{1}{16} \left(4te^{4t} - e^{4t} \right) + C$$

$$= \frac{1}{4} x^{4} \ln x - \frac{1}{16} x^{4} + C$$



$$\overline{+}$$
 例2.(2). 求积分 $\int x \arctan x dx$.

解 令
$$v = \arctan x$$
, $xdx = d\left(\frac{x}{2}\right) = du$,

解令
$$v = \arctan x$$
, $xdx = d\left(\frac{x^2}{2}\right) = du$,
$$\int x \arctan x dx = \frac{x^2}{2} \arctan x - \int \frac{x^2}{2} (\arctan x)' dx$$

$$\int x \arctan x dx = \frac{1}{2} \arctan x - \int \frac{1}{2} (a) dx$$

$$= \frac{x^2}{2} \arctan x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} (x - \arctan x) + C.$$

$$\frac{c^2}{2}\arctan x - \frac{1}{2}\int \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$\int u'vdx = uv - \int uv'dx$$

$$\int x^3 \ln x dx$$

 $\int u'vdx = uv - \int uv'dx$ $\begin{cases} x^3 \ln x dx \\ \Rightarrow x^{\mu} = u', \\ \mu \neq -1 \end{cases}$ $\begin{cases} v \\ \psi \neq -1 \end{cases}$ 小结 若被积函数是幂函数与对数函数数的乘积,就考虑把幂函数作为u',而设反三角函数为v,通过求导简化被积函 小结若被积函数是幂函数与对数函数或反三角函 数的乘积,就考虑把幂函数作为u',而设对数函数或 反三角函数为 v, 通过求导简化被积函数.







练练手:1.求积分
$$(1).\int x \sin 2x dx ;$$

$$(2).\int \left(xe^{-x}\right)^2 dx ;$$

$$(3).\int \left(\frac{\ln x}{x}\right)^2 dx ;$$

$$(4).\int \arcsin x dx .$$

练练手:1.求积分

$$Ex.1.(1). \int x \sin 2x dx = \int x \left(-\frac{1}{2} \cos 2x \right)' dx$$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx$$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$$

$$Ex.1.(2).\int (xe^{-x})^2 dx = \int x^2 e^{-2x} dx = \int x^2 \left(-\frac{1}{2}e^{-2x}\right)' dx$$

$$= -\frac{1}{2}x^2 e^{-2x} + \frac{1}{2}\int 2xe^{-2x} dx = -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x} dx$$

$$= -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C = -\frac{1}{4}(2x^2 + 2x + 1)e^{-2x} + C$$

或者
$$J = \int x^2 e^{-2x} dx = \frac{1}{8} \int t^2 e^t dt$$
,
$$\int t^2 e^t dt = t^2 e^t - \int 2t e^t dt = t^2 e^t - 2t e^t + \int 2e^t dt$$

$$= t^2 e^t - 2t e^t + 2e^t + C_1,$$

$$\frac{1}{x} Ex.1.(3).\int \left(\frac{\ln x}{x}\right)^2 dx = \int \left(-\frac{1}{x}\right)' \left(\ln x\right)^2 dx$$

$$= -\frac{1}{x} (\ln x)^2 + \int \frac{1}{x} \cdot 2(\ln x) \frac{1}{x} dx$$

$$= -\frac{1}{x}(\ln x)^2 + 2\int \left(-\frac{1}{x}\right)'(\ln x)dx$$

或者,
$$J = \int \left(\frac{\ln x}{x}\right)^2 dx = \int \left(\frac{\ln x}{x}\right)^2 dx = \int t^2 \cdot e^{-2t} \cdot e^t dt = \int t^2 e^{-t} dt$$

$$= \dots = -\left(t^2 + 2t + 2\right)e^{-t} + C = -\frac{1}{x}\left(\ln^2 x + 2\ln x + 2\right) + C$$

$$Ex.1.(4).\int \arcsin x dx = \int (x)' \arcsin x dx$$

$$= x \arcsin x - \int x \frac{1}{\sqrt{1 - x^2}} dx$$

$$= x \arcsin x - \int x \frac{1}{\sqrt{1 - x^2}} dx$$

$$= x \arcsin x - \left(-\frac{1}{2}\right) \int \frac{1}{\sqrt{1 - x^2}} d\left(1 - x^2\right)$$

$$= x \arcsin x + \sqrt{1 - x^2} + C$$

$$= x \arcsin x = t$$

$$= x \sin t - \int \sin t dt$$

首,
$$J = = = \int t(\sin t) dt = t \sin t - \int \sin t dt$$

$$= t \sin t + \cos t + C = x \arcsin x + \sqrt{1 - x^2} + C$$

$$= a \cos x + C = x \arcsin x + \sqrt{1 - x^2} + C$$

$$= a \cos x + C = x \cos x + \sqrt{1 - x^2} + C$$

$$= a \cos x + C = x \cos x + \sqrt{1 - x^2} + C$$

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$$= a \cos x + C = x \cos x + C =$$

$$=$$
 例3.求积分 $\int e^x \sin x dx$.

$$= -e^{x} \cos x + \int \cos x de^{x} = -e^{x} \cos x + \int e^{x} \cos x dx$$
$$= -e^{x} \cos x + \int e^{x} d(\sin x)$$

$$\frac{1}{4} = -e^x \cos x + \int e^x d(\sin x)$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x dx \qquad \text{if } \text{if }$$

在使用分部积分公式时应注意,倘若要接连几次应用分部积分公式,需注意前后几次所选的 u' 应为同类型函数.

例如, $\int e^x \sin x dx$,第一次用分部积分 公式时选择 $u' = \sin x$,

 $\int e^{x} \sin x dx = -e^{x} \cos x + \int e^{x} \cos x dx$ 那么第二次用分部积分公式时应仍 选择 $u' = \cos x$.



或者,

$$\int e^x \sin x dx = \int \sin x d\left(e^x\right)$$

$$\sin x - \int e^x \cos x dx =$$

$$\frac{1}{1+e^x} = e^x \sin x - \int e^x \cos x dx = e^x \sin x - \int \cos x d(e^x)$$

$$\frac{1}{x} = e^x \sin x - \left(e^x \cos x - \int e^x d(\cos x)\right)$$

$$\frac{1}{1} = e^x \left(\sin x - \cos x\right) - \int e^x \sin x dx$$
 注意循环形式





 $\frac{1}{4}$ 求积分 $\int e^x \sin x dx$.

$$\int e^x \sin x dx = \int \sin x de^x$$

$$= e^x \sin x - \int \cos x de^x$$

$$= e^{x} \left(\sin x - \cos x \right) - \int e^{x} \sin x dx$$







将分部积分法改写一下,得到所谓的"联立方程法": $\exists J_1 = \int e^x \sin x dx, J_2 = \int e^x \cos x dx,$ $\begin{cases}
J_1 = \frac{e^x}{2} (\sin x - \cos x) + C_3 \\
J_2 = \frac{e^x}{2} (\sin x + \cos x) + C_4
\end{cases}$

 \uparrow 例3.(2).求积分 $\int e^{ax} \cos bx dx, a^2 + b^2 \neq 0.$ 上解用"联立方程法"较为方便. 工解此方程组得 J₁,J₂…

等等: 2.求积分 $\int e^{-x} \sin 2x dx$ 解 $\int e^{-x} \sin 2x dx$ =

$$\sin 2x dx =$$

合理选择u',v,正确使用分部积分公式

$$\int u'vdx = uv - \int uv'dx$$

(1). $\int x^n e^{ax} dx, \int x^n \cos bx dx, \int x^n \sin bx dx$ $\Re e^{ax}, \cos bx, \sin bx = u', \widehat{\Pi} x^n = v \cdots$

(2). $\int x^{\mu} \ln x dx$, $\int x^{\mu} \arctan x dx$, $\int x^{\mu} \arcsin x dx$ 取 $x^{\mu} = u'$, 而 $\ln x$, $\arctan x$, $\arcsin x = v$... (3). $\int e^{ax} \cos bx dx$, $\int e^{ax} \sin bx dx$ 取 $e^{ax} = u'$, 而 $\cos bx$, $\sin bx = v$, 两次分部积分; 或者取 $\cos bx$, $\sin bx = u'$, 而 $e^{ax} = v$, 两次分部积分.

例4.设 e^{-x^2} 是f(x)的一个原函数,求 $\int xf'(x)dx$.

解 $\int xf'(x)dx = \int xdf(x) = xf(x) - \int f(x)dx$,

 $:: \left(\int f(x)dx\right)' = f(x), \overline{\prod} \int f(x)dx = e^{-x^2} + C,$

两边同时对求x导,得 $f(x) = -2xe^{-x^2}$,

两边同时对求
$$x$$
导,得 $f(x) = -2x$

$$\therefore \int xf'(x)dx = xf(x) - \int f(x)dx$$

$$= -(2x^2 + 1)e^{-x^2} + C.$$

例5.求积分
$$\int e^{-\sqrt{x}}dx$$
.

$$= -2te^{-t} + \int 2e^{-t}dt$$

$$= -2(t+1)e^{-t} + C$$

$$= -2\left(\sqrt{x} + 1\right)e^{-\sqrt{x}} + C$$

下页

之回

$$F$$
例6*.求积分 $J = \int e^{-x} \arctan e^{-x} dx$.

$$+$$
解 考虑作凑微分: $J = -\int \arctan e^{-x} d(e^{-x})$

$$= \int \arctan u du = -\int (u)' \arctan u du$$

$$\frac{1}{1+u^2} = -u \arctan u + \int \frac{u}{1+u^2} du$$

$$\frac{1}{1} = -u \arctan u + \frac{1}{2} \int \frac{1}{1+u^2} d(1+u^2)$$

$$\frac{1}{T} = -u \arctan u + \frac{1}{2} \ln \left(1 + u^2\right) + C$$

$$\frac{1}{4} = \ln \sqrt{1 + e^{-2x}} - e^{-x} \arctan e^{-x} + C \square$$

例6.(2)*.求积分
$$J = \int x \tan^2 x dx$$

解 $J = \int x \tan^2 x dx = \int x (\sec^2 x - 1) dx$

解
$$J = \int x \tan^2 x dx = \int x (\sec^2 x - 1) dx$$

$$= x \tan x - \int \frac{\sin x}{\cos x} dx - \frac{1}{2}x^2$$

$$= x \tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

例7*.设 $\frac{\sin x}{x}$ 是f(x)的一原函数,求 $\int xf'(2x)dx$.

$$\therefore f(x) = \left(\frac{\sin x}{x}\right)' = \frac{x \cos x - \sin x}{x^2},$$

$$\therefore \int xf'(2x)dx \stackrel{2x=u}{==} \frac{1}{4} \int uf'(u)du$$

$$= \frac{1}{4} uf(u) - \frac{1}{4} \int f(u)du$$

$$\therefore \int xf'(2x)dx = \frac{1}{4} \int uf'(u)du$$

$$= \frac{1}{4} u f(u) - \frac{1}{4} \int f(u) du$$

$$=\frac{1}{4}u\cdot\frac{u\cos u-\sin u}{u^2}-\frac{1}{4}\frac{\sin u}{u}+C$$

$$= \frac{u \cos u - 2 \sin u}{4u} + C = \frac{u = 2x}{2} \frac{x \cos 2x - \sin 2x}{4x} + C$$

例7.(2)*.求积分
$$\int \frac{x \arctan x}{\sqrt{1+x^2}} dx.$$

解
$$\because \left(\sqrt{1+x^2}\right)' = \frac{x}{\sqrt{1+x^2}},$$

$$\therefore \int \frac{x \arctan x}{\sqrt{1+x^2}} dx = \int \arctan x d\sqrt{1+x^2}$$

$$= \sqrt{1+x^2} \arctan x - \int \sqrt{1+x^2} d(\arctan x)$$

$$= \sqrt{1 + x^2} \arctan x - \int \sqrt{1 + x^2} \cdot \frac{1}{1 + x^2} dx$$

$$= \sqrt{1+x^2} \arctan x - \int \frac{1}{\sqrt{1+x^2}} dx \Rightarrow x = \tan t$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \int \frac{1}{\sqrt{1+\tan^2 t}} \sec^2 t dt = \int \sec t dt$$

$$= \ln(\sec t + \tan t) + C = \ln(x + \sqrt{1+x^2}) + C$$

$$= \ln(\sec t + \tan t) + C = \ln\left(x + \sqrt{1 + x^2}\right) + C$$

$$\sec t + \tan t + C = \ln(x + \sqrt{1 + x^2}) + C$$

$$x \arctan x$$

$$\frac{x \arctan x}{\sqrt{1 - x^2}} dx$$

$$= \sqrt{1 + x^{2}} \arctan x - \ln(x + \sqrt{1 + x^{2}}) + C.$$

求积分
$$\int \frac{x \arctan x}{\sqrt{1+x^2}} dx$$
.

解二 考虑先作变量代换:
 $\arctan x = t \leftrightarrow x = \tan t, t \in (-\pi/2, \pi/2),$
 $\therefore \int \frac{x \arctan x}{\sqrt{1+x^2}} dx = \int \frac{t \tan t}{|\sec t|} \sec^2 t dt$

$$= \int \frac{t \tan t}{\sec t} \sec^2 t dt = \int t \tan t \sec t dt$$

$$= \int t \sec t - \ln|\tan t + \sec t| + C$$

$$= \sqrt{1+x^2} \cdot \arctan x - \ln(x + \sqrt{1+x^2}) + C$$

求积分
$$\int \frac{x \arctan x}{\sqrt{1+x^2}} dx$$
.
解二 考虑先作变量代换:
arctan $x = t \leftrightarrow x = \tan t, t \in \mathbb{R}$

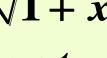


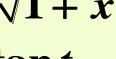


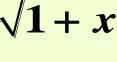


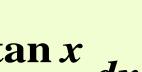


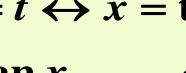


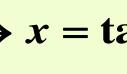


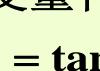




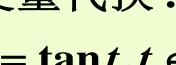


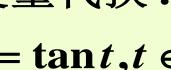




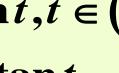


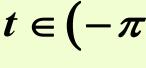






$$\arctan x = t \leftrightarrow x = \tan t, t \in (-\pi/2, \pi/2),$$





例7.(3)*.求积分 $\int \sin(\ln x)dx$.

解 考虑先作变量代换:

$$\ln x = t \to x = e^t \to dx = e^t dt$$

$$\therefore \int \sin(\ln x) dx = \int e^t \sin t dt$$

$$=\frac{1}{2}e^{t}(\sin t - \cos t) + C$$

$$= \frac{x}{2} \left[\sin(\ln x) - \cos(\ln x) \right] + C.$$

例8**.求积分
$$J = \int \sec^3 x dx$$
.

$$= \sec x \tan x - \int \tan x (\sec x)' dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1 \right) dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$\therefore J = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

例8**.(2).求积分
$$J = \int \sqrt{a^2 + x^2} dx, a > 0.$$

解
$$\diamondsuit x = a \tan t, t \in (-\pi/2, \pi/2)$$

$$\int \int J = a^2 \int |\sec t| \sec^2 t dt = a^2 \int \sec^3 t dt$$

$$\frac{1}{2} = \frac{1}{2}a^2 \sec t \tan t + \frac{a^2}{2}\ln|\sec t + \tan t| + C_1$$

$$\frac{1}{2} = \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln(x + \sqrt{a^2 + x^2}) + C$$

练习题

一.求下列不定积分:

$$\frac{1}{4\pi} (1) \cdot \int x^2 \cos^2 \frac{x}{2} dx, \qquad (2) \cdot \int \left(\frac{\ln x}{x}\right)^2 dx,$$

$$(3) \cdot \int e^{3\sqrt{x}} dx, \qquad (4) \cdot \int \cos(\ln x) dx,$$

(5).
$$\int \frac{\arctan x}{\sqrt{\left(1+x^2\right)^3}} dx, \quad (6). \int e^{-x} \cos 2x dx,$$

$$\frac{1}{\sqrt{1+x^2}} dx.$$



$$\frac{1}{2} \int x^2 \cos^2 \frac{x}{2} dx = \frac{1}{2} \int x^2 (1 + \cos x) dx$$

$$\frac{1}{2} \int x^2 dx + \frac{1}{2} \int x^2 \cos x dx$$

$$\int x^2 \cos x dx = \int x^2 (\sin x)' dx$$

$$\frac{1}{4} = x^2 \sin x - \int \sin x \left(x^2\right)' dx = x^2 \sin x - \int 2x \sin x dx$$

$$\frac{1}{1} = x^2 \sin x - \int 2x(-\cos x)' dx$$

$$= x^2 \sin x + 2x \cos x - 2 \int \cos x dx = \cdots$$



$$(5). \Leftrightarrow \arctan x = t,$$

则
$$x = \tan t, t \in (-\pi/2, \pi/2)$$

$$\int \frac{\arctan x}{\sqrt{(1+x^2)^3}} dx$$

$$= \int \frac{t}{|\sec t|^3} \sec^2 t dt = \int t \cos t dt$$

. . .





$$\frac{1}{4\pi} (6) \cdot \int e^{-x} \cos 2x dx = \int (-e^{-x})' \cos 2x dx$$

$$= -e^{-x} \cos 2x + \int e^{-x} (\cos 2x)' dx$$

$$= -e^{-x} \cos 2x + \int e^{-x} (\cos 2x) dx$$

$$= -e^{-x} \cos 2x + \int e^{-x} (-2\sin 2x) dx$$

$$\int e^{-x} \cos 2x dx = \frac{1}{5}e^{-x} \left(2\sin 2x - \cos 2x\right) + C$$

$$\frac{1}{1} (5) \int \frac{\arctan x}{\sqrt{(1+x^2)^3}} dx, (6) \int e^{-x} \cos 2x dx,$$

$$\frac{1}{\sqrt{1+x^2}} dx, (6) \int e^{-x} \cos 2x dx,$$

$$\frac{xe^{\arctan x}}{\sqrt{1+x^2}} dx.$$

$$\frac{1}{\sqrt{1+x^2}} dx.$$

