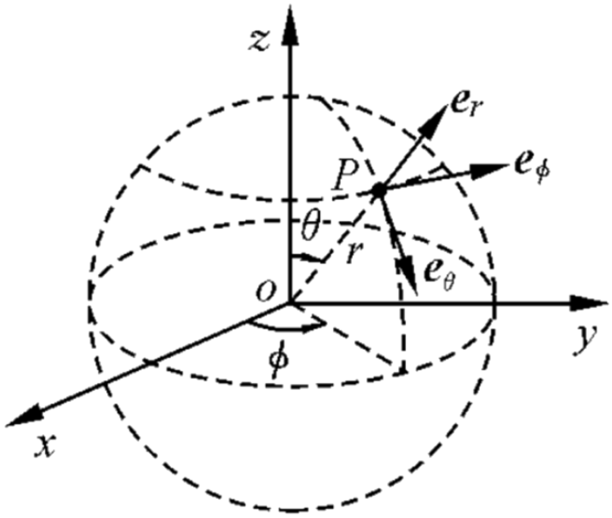
球坐标下的梯度公式



坐标变量之间的转换

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \cos^{-1}\dfrac{z}{\sqrt{x^2+y^2+z^2}} \\ \phi = \mathrm{tg}^{-1}\dfrac{y}{x} \end{cases}$$

单位矢量之间的转换

$$\begin{cases} \vec{e}_x = \sin\theta\cos\phi\,\vec{e}_r + \cos\theta\cos\phi\,\vec{e}_\theta - \sin\phi\,\vec{e}_\phi \\ \vec{e}_y = \sin\theta\sin\phi\,\vec{e}_r + \cos\theta\sin\phi\,\vec{e}_\theta + \cos\phi\,\vec{e}_\phi \\ \vec{e}_z = \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta \end{cases}$$

$$\begin{cases} \vec{e}_r = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z \\ \vec{e}_\theta = \cos\theta\cos\phi\,\vec{e}_x + \cos\theta\sin\phi\,\vec{e}_y - \sin\theta\,\vec{e}_z \\ \vec{e}_\phi = -\sin\phi\,\vec{e}_x + \cos\phi\,\vec{e}_y \end{cases}$$

$$\nabla f(r,\theta,\phi) = \vec{e}_x\frac{\partial f}{\partial x} + \vec{e}_y\frac{\partial f}{\partial y} + \vec{e}_z\frac{\partial f}{\partial y}$$

替换单位矢量

$$\nabla f(r,\theta,\phi) = \left(\sin\theta\cos\phi\,\vec{e}_r + \cos\theta\cos\phi\,\vec{e}_\theta - \sin\phi\,\vec{e}_\phi\right)\frac{\partial f}{\partial x}$$

$$+\left(\sin\theta\sin\phi\,\vec{e}_r + \cos\theta\sin\phi\,\vec{e}_\theta + \cos\phi\,\vec{e}_\phi\right)\frac{\partial f}{\partial y} + (\cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta)\frac{\partial f}{\partial y}$$

按单位矢量整理得

$$
\begin{aligned}
\nabla f(r,\theta,\phi) = \vec{e}_r &\left(\sin\theta\cos\phi\frac{\partial f}{\partial x} + \sin\theta\sin\phi\frac{\partial f}{\partial y} + \cos\theta\frac{\partial f}{\partial z}\right) \\
&+ \vec{e}_\theta\left(\cos\theta\cos\phi\frac{\partial f}{\partial x} + \cos\theta\sin\phi\frac{\partial f}{\partial y} - \sin\theta\frac{\partial f}{\partial z}\right) \\
&+ \vec{e}_\phi\left(-\sin\phi\frac{\partial f}{\partial x} + \cos\phi\frac{\partial f}{\partial y}\right) \qquad (1)
\end{aligned}
$$

转换 3 个偏导数 $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$

$$
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}
$$

$$
\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}
$$

$$
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}
$$

$$
\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta\cos\phi}{r} = \sin\theta\cos\phi
$$

$$
\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} = \frac{r^2\sin\theta\cos\theta\cos\phi}{r^3\sin\theta} = \frac{\cos\theta\cos\phi}{r}
$$

$$
\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\mathrm{tg}^{-1}\frac{y}{x} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta} = -\frac{\sin\phi}{r\sin\theta}
$$

$$
\text{整理得}\begin{cases}
\frac{\partial r}{\partial x} = \sin\theta\cos\phi \\
\frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\phi}{r} \\
\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}
\end{cases} \qquad (2)
$$

略去中间步骤，同理得

$$
\begin{cases}
\frac{\partial r}{\partial y} = \sin\theta\sin\phi \\
\frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\phi}{r} \\
\frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}
\end{cases} \qquad (3) \qquad\qquad
\begin{cases}
\frac{\partial r}{\partial z} = \cos\theta \\
\frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r} \\
\frac{\partial \phi}{\partial z} = 0
\end{cases} \qquad (4)
$$

将（2）、（3）、（4）代入（1），得

$\vec{e}_r$ 方向的分量

$$
\begin{aligned}
&\sin\theta\cos\phi\frac{\partial f}{\partial x}+\sin\theta\sin\phi\frac{\partial f}{\partial y}+\cos\theta\frac{\partial f}{\partial z}\\
&=\sin\theta\cos\phi\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}\right)+\sin\theta\sin\phi\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}\right)\\
&+\cos\theta\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}\right)\\
&=\sin\theta\cos\phi\left(\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial \theta}\frac{\cos\theta\cos\phi}{r}-\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}\right)\\
&+\sin\theta\sin\phi\left(\frac{\partial f}{\partial r}\sin\theta\sin\phi+\frac{\partial f}{\partial \theta}\frac{\cos\theta\sin\phi}{r}+\frac{\partial f}{\partial \phi}\frac{\cos\phi}{r\sin\theta}\right)+\cos\theta\left(\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\right)\\
&=(\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta)\frac{\partial f}{\partial r}\\
&+\left(\frac{\sin\theta\cos\theta\cos^2\phi}{r}+\frac{\sin\theta\cos\theta\sin^2\phi}{r}-\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial f}{\partial \theta}+\left(\frac{\sin\phi\cos\phi}{r}-\frac{\sin\phi\cos\phi}{r}\right)\frac{\partial f}{\partial \phi}\\
&=\frac{\partial f}{\partial r}
\end{aligned}
$$

略去中间步骤，同理得

$\vec{e}_\theta$ 方向的分量

$$\cos\theta\cos\phi\frac{\partial f}{\partial x}+\cos\theta\sin\phi\frac{\partial f}{\partial y}-\sin\theta\frac{\partial f}{\partial z}=\frac{1}{r}\frac{\partial f}{\partial \theta}$$

$\vec{e}_\phi$ 方向的分量

$$-\sin\phi\frac{\partial f}{\partial x}+\cos\phi\frac{\partial f}{\partial y}=\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

所以，球坐标下的梯度计算公式为

$$\nabla f=\vec{e}_r\frac{\partial f}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$