2021-11-数学分析 I 阶段练习(一)　2021-11

1. 用“$\varepsilon-N$”定义证明（两选一）：(1). $\lim\limits_{n\to\infty}\dfrac{n-\sin n}{n^3+6}=0$.　　(2). $\lim\limits_{n\to\infty}\dfrac{n^2+(-1)^n}{n^2-n}=1$.

2. 若 $x\to 0$ 时, $e^{x\cos(x^2)}-e^x$ 与 ax^n 为等价无穷小量，问 $n=?$

3. 求极限 $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{\sqrt[4]{1+x}-\sqrt[4]{1-x}}$.

4. 求极限 (1). $\lim\limits_{n\to\infty}\left(\dfrac{3^n+4^n}{2}\right)^{\frac{1}{n}}$;(2). $\lim\limits_{n\to\infty}\left(\dfrac{3^{\frac{1}{n}}+4^{\frac{1}{n}}}{2}\right)^n$; (3). $\lim\limits_{n\to\infty}\left(\dfrac{n+5}{3n-2}\right)^n$. (4). $\lim\limits_{n\to\infty}\left(\dfrac{n^2-5}{n^2+5}\right)^n$.

5. 设 $a_n\le a\le b_n\ (n=1,2,\cdots)$, 且 $\lim\limits_{n\to\infty}(a_n-b_n)=0$. 求证：$\lim\limits_{n\to\infty}a_n=a$, $\lim\limits_{n\to\infty}b_n=a$.

6. 叙述关于数列极限的 Cauchy 收敛准则.试依此证明数列 $a_n=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{n^2}$ 收敛.

7. 如果我们已经证明, $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ 为递增数列, $\left\{\left(1+\dfrac{1}{n}\right)^{n+1}\right\}$ 为递减数列且均有界，因而它们都收敛.

(I).记 $A=\left\{\left(1+\dfrac{1}{n}\right)^{n+1}, n\in\mathbb{N}^*\right\}$, 问 $\sup A=?$　$\inf A=?$

(II).证明 : (1). $n\in\mathbb{N}^*$, $\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}$;　　(2). $\left\{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n\right\}$ 收敛 ;

(3). $\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}\right)=\ln 2$;　　(4). $\left\{\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{2^2}\right)\cdots\left(1+\dfrac{1}{2^n}\right)\right\}$ 收敛.

1. 用“$\varepsilon-N$”定义证明（两选一）：(1). $\lim\limits_{n\to\infty}\dfrac{n-\sin n}{n^3+6}=0$.　　(2). $\lim\limits_{n\to\infty}\dfrac{n^2+(-1)^n}{n^2-n}=1$.

(1). $\lim\limits_{n\to\infty}\dfrac{n-\sin n}{n^3+6}=0$.　　分析 $\forall\varepsilon>0$, 欲找到 N, 使在 $n>N$ 时有 $\left|\dfrac{n-\sin n}{n^3+6}-0\right|<\varepsilon$.

而 $\left|\dfrac{n-\sin n}{n^3+6}-0\right|\le\dfrac{n+1}{n^3+6}<\dfrac{2n}{n^3}=\dfrac{2}{n^2}\le\dfrac{2}{n}$, 当 $\dfrac{2}{n}<\varepsilon$ 时就有 $\left|\dfrac{n-\sin n}{n^3+6}-0\right|<\varepsilon$. $\dfrac{2}{n}<\varepsilon\Leftrightarrow n>\dfrac{2}{\varepsilon}$.

证明 $\forall\varepsilon>0, \exists N\ge\dfrac{2}{\varepsilon}, \forall n>N, s.t. \left|\dfrac{n-\sin n}{n^3+6}-0\right|\le\dfrac{n+1}{n^3+6}<\dfrac{2n}{n^3}=\dfrac{2}{n^2}\le\dfrac{2}{n}<\dfrac{2}{N}\le\dfrac{2}{2/\varepsilon}=\varepsilon$.证毕

(2). $\lim\limits_{n\to\infty}\dfrac{n^2+(-1)^n}{n^2-n}=1.$

分析 $\forall\varepsilon>0$,欲找到N,使在$n>N$时有$\left|\dfrac{n^2+(-1)^n}{n^2-n}-1\right|<\varepsilon.$

而$\left|\dfrac{n^2+(-1)^n}{n^2-n}-1\right|=\dfrac{n+(-1)^n}{n^2-n}\le\dfrac{2n}{n^2-n}$,当$n\ge 2$时,$\dfrac{2n}{n^2-n}\le\dfrac{2n}{n^2-\frac{1}{2}n^2}=\dfrac{4}{n}$,

当$\dfrac{4}{n}<\varepsilon$时就有$\left|\dfrac{n^2+(-1)^n}{n^2-n}-1\right|<\varepsilon.$ $\dfrac{4}{n}<\varepsilon\Leftrightarrow n>\dfrac{4}{\varepsilon}.$

证明 $\forall\varepsilon>0,\exists N\ge\max\left(\dfrac{4}{\varepsilon},2\right),\forall n>N,s.t.\left|\dfrac{n^2+(-1)^n}{n^2-n}-1\right|=\dfrac{n+(-1)^n}{n^2-n}\le\dfrac{2n}{n^2-n}\le\dfrac{2n}{n^2-\frac{1}{2}n^2}=\dfrac{4}{n}$

$<\dfrac{4}{N}\le\dfrac{4}{4/\varepsilon}=\varepsilon$.证毕

2. 若 $x\to 0$ 时,$e^{x\cos\left(x^2\right)}-e^x$ 与 ax^n 为等价无穷小量，问 $n=?$

解 $x\to 0$时,$e^x-1\sim x,1-\cos x\sim\dfrac{1}{2}x^2.$

$\therefore x\to 0$时,$e^{x\cos\left(x^2\right)}-e^x=e^x\left[e^{x\cos\left(x^2\right)-x}-1\right]\sim e^{x\cos\left(x^2\right)-x}-1\sim x\cos\left(x^2\right)-x$

$\sim x\left[\cos\left(x^2\right)-1\right]\sim x\left(-\dfrac{1}{2}x^4\right)=-\dfrac{1}{2}x^5,\ \therefore n=5.$

3. 求极限 $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{\sqrt[4]{1+x}-\sqrt[4]{1-x}}$.

解 若分式分子分母直接作无理式有理化,则书写过于麻烦,故作变量代换:

令$\sqrt[12]{1+x}=a,\sqrt[12]{1-x}=b,\lim\limits_{x\to 0}a=\lim\limits_{x\to 0}b=1.$

$\therefore\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{\sqrt[4]{1+x}-\sqrt[4]{1-x}}=\lim\limits_{x\to 0}\dfrac{a^4-b^4}{a^3-b^3}=\lim\limits_{x\to 0}\dfrac{(a-b)(a+b)\left(a^2+b^2\right)}{(a-b)\left(a^2+ab+b^2\right)}=\lim\limits_{x\to 0}\dfrac{(a+b)\left(a^2+b^2\right)}{\left(a^2+ab+b^2\right)}=\dfrac{4}{3}.$

我比较喜欢作变量代换,再加等价无穷小量的等价替换:

$t\to 0,\mu\ne 0$,则$(1+t)^\mu-1\sim\mu t.$

原式$=\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x}}{\sqrt[4]{1-x}}\cdot\dfrac{\sqrt[3]{\dfrac{1+x}{1-x}}-1}{\sqrt[4]{\dfrac{1+x}{1-x}}-1}=\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x}}{\sqrt[4]{1-x}}\cdot\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\left(\dfrac{1+x}{1-x}-1\right)}-1}{\sqrt[4]{1+\left(\dfrac{1+x}{1-x}-1\right)}-1}=\lim\limits_{x\to 0}\dfrac{\frac{1}{3}\left(\dfrac{1+x}{1-x}-1\right)}{\frac{1}{4}\left(\dfrac{1+x}{1-x}-1\right)}=\dfrac{4}{3}.$

4. 求极限(1). $\lim\limits_{n\to\infty}\left(\dfrac{3^n+4^n}{2}\right)^{\frac{1}{n}}$;(2). $\lim\limits_{n\to\infty}\left(\dfrac{3^{\frac{1}{n}}+4^{\frac{1}{n}}}{2}\right)^n$;

解 (1),(2)两题虽都是幂指函数的形式,但属于不同类型的问题.(1)为∞^0,(2)为1^∞.

$$\lim_{n\to\infty}\left(\frac{3^n+4^n}{2}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left[4^n\left(\left(\frac{3}{4}\right)^n+1\right)\cdot\frac{1}{2}\right]^{\frac{1}{n}}=4\lim_{n\to\infty}\left[\left(\left(\frac{3}{4}\right)^n+1\right)^{\frac{1}{n}}\cdot\frac{1}{\sqrt[n]{2}}\right]=4\cdot1^0\cdot1=4.$$

或者用*Squeeze th.*,由$\dfrac{4^n}{2}<\dfrac{3^n+4^n}{2}<4^n$ 得 $4\cdot\dfrac{1}{\sqrt[n]{2}}<\left(\dfrac{3^n+4^n}{2}\right)^{\frac{1}{n}}<4,\lim\limits_{n\to\infty}\sqrt[n]{2}=1$ $\cdots$

(2).由$t\to0$,有$e^t-1\sim t$,得$\lim\limits_{n\to\infty}\dfrac{3^{\frac{1}{n}}-1}{\frac{1}{n}}=\lim\limits_{n\to\infty}\dfrac{e^{\frac{1}{n}\ln3}-1}{\frac{1}{n}}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n}\ln3}{\frac{1}{n}}=\ln3$,同样,$\lim\limits_{n\to\infty}\dfrac{4^{\frac{1}{n}}-1}{\frac{1}{n}}=\ln4$.

$$\therefore\lim_{n\to\infty}\frac{n\left(3^{\frac{1}{n}}+4^{\frac{1}{n}}-2\right)}{2}=\frac{1}{2}\lim_{n\to\infty}\frac{3^{\frac{1}{n}}-1+4^{\frac{1}{n}}-1}{\frac{1}{n}}=\frac{\ln3+\ln4}{2}=\frac{1}{2}\ln12=\ln\sqrt{12},$$

$$\text{则}\lim_{n\to\infty}\left(\frac{3^{\frac{1}{n}}+4^{\frac{1}{n}}}{2}\right)^n=\lim_{n\to\infty}\left[\left(1+\frac{3^{\frac{1}{n}}+4^{\frac{1}{n}}-2}{2}\right)^{\frac{2}{3^{\frac{1}{n}}+4^{\frac{1}{n}}-2}}\right]^{\frac{3^{\frac{1}{n}}+4^{\frac{1}{n}}-2}{2}\cdot n}=e^{\ln\sqrt{12}}=\sqrt{12}.$$

(1),(2)亦可用*L'Hopital*法则解之:

$$(1).\lim_{n\to\infty}\left(\frac{3^n+4^n}{2}\right)^{\frac{1}{n}}=\lim_{x\to+\infty}\left(\frac{3^x+4^x}{2}\right)^{\frac{1}{x}}\overset{\infty^0}{=\!=\!=}\lim_{x\to+\infty}e^{\frac{\ln(3^x+4^x)-\ln2}{x}}=e^{\lim\limits_{x\to+\infty}\frac{\ln(3^x+4^x)-\ln2}{x}},$$

$$\lim_{x\to+\infty}\frac{\ln(3^x+4^x)-\ln2}{x}\overset{\frac{\infty}{\infty}}{=\!=}\lim_{x\to+\infty}\frac{\frac{3^x\ln3+4^x\ln4}{3^x+4^x}}{1}=\lim_{x\to+\infty}\frac{\left(\frac{3}{4}\right)^x\ln3+\ln4}{1+\left(\frac{3}{4}\right)^x}=\ln4,\therefore\text{ 原}=4.$$

$$(2).\lim_{n\to\infty}\left(\frac{3^{\frac{1}{n}}+4^{\frac{1}{n}}}{2}\right)^n=\lim_{x\to0}\left(\frac{3^x+4^x}{2}\right)^{\frac{1}{x}}\overset{1^\infty}{=\!=}\lim_{x\to0}e^{\frac{\ln(3^x+4^x)-\ln2}{x}}=e^{\lim\limits_{x\to0}\frac{\ln(3^x+4^x)-\ln2}{x}},$$

$$\lim_{x\to0}\frac{\ln(3^x+4^x)-\ln2}{x}\overset{\frac{0}{0}}{=\!=}\lim_{x\to0}\frac{\frac{3^x\ln3+4^x\ln4}{3^x+4^x}}{1}=\lim_{x\to0}\frac{\ln3+\ln4}{1+1}=\ln\sqrt{12},\therefore\text{ 原}=\sqrt{12}.$$

4. 求极限(3). $\lim\limits_{n\to\infty}\left(\dfrac{n+5}{3n-2}\right)^n$. (4). $\lim\limits_{n\to\infty}\left(\dfrac{n^2-5}{n^2+5}\right)^n$.

(3). $\lim\limits_{n\to\infty}\left(\frac{n+5}{3n-2}\right)^n$: $\lim\limits_{n\to\infty}\frac{n+5}{3n-2}=\frac{1}{3}$, 且 $\frac{n+5}{3n-2}>\frac{1}{3}$. 可以发现, $n>12$ 时有 $\frac{n+5}{3n-2}<\frac{1}{2}$,

$\therefore n>12$ 时有 $\frac{1}{3}<\frac{n+5}{3n-2}<\frac{1}{2}$.

$\therefore n>12$ 时有 $\left(\frac{1}{3}\right)^n<\left(\frac{n+5}{3n-2}\right)^n<\left(\frac{1}{2}\right)^n$, $\lim\limits_{n\to\infty}\left(\frac{1}{3}\right)^n=\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n=0$, 由 *Squeeze th.*, 原 $=0$.

或者, $\lim\limits_{n\to\infty}\left(\frac{n+5}{3n-2}\right)^n=\lim\limits_{n\to\infty}\left[\left(\frac{1}{3}\right)^n\left(\frac{1+\frac{5}{n}}{1-\frac{2}{3n}}\right)^n\right]=\lim\limits_{n\to\infty}\left[\left(\frac{1}{3}\right)^n\frac{\left(1+\frac{5}{n}\right)^{\frac{n}{5}\cdot 5}}{\left(1-\frac{2}{3n}\right)^{-\frac{3n}{2}\cdot\left(-\frac{2}{3}\right)}}\right]=0\cdot\frac{e^5}{e^{-\frac{2}{3}}}=0.$

(4). $\lim\limits_{n\to\infty}\left(\frac{n^2-5}{n^2+5}\right)^n=\lim\limits_{n\to\infty}\left[\left(1+\frac{-10}{n^2+5}\right)^{\frac{n^2+5}{-10}}\right]^{-\frac{10n}{n^2+5}}=e^0=1.$

5. 设 $a_n\le a\le b_n\ (n=1,2,\cdots)$, 且 $\lim\limits_{n\to\infty}(a_n-b_n)=0$. 求证: $\lim\limits_{n\to\infty}a_n=a$, $\lim\limits_{n\to\infty}b_n=a$.

证明 $\because a_n\le a\le b_n\ (n=1,2,\cdots)$, $\therefore 0\le a-a_n\le b_n-a_n$, 而 $\lim\limits_{n\to\infty}(b_n-a_n)=0$,

由 *Squeeze th.* 知 $\lim\limits_{n\to\infty}(a-a_n)=0$, 得 $\lim\limits_{n\to\infty}a_n=a$.

$\therefore \lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}(b_n-a_n+a_n)=\lim\limits_{n\to\infty}(b_n-a_n)+\lim\limits_{n\to\infty}a_n=a$.

6. 叙述关于数列极限的 Cauchy 收敛准则.试依此证明数列 $a_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}$ 收敛.

Cauchy criterion: $\{a_n\}$ 收敛 $\Leftrightarrow \forall\varepsilon>0,\exists N>0,\forall n>N,\forall p\in\mathbb{N}^*, s.t.\left|a_n-a_{n+p}\right|<\varepsilon.$

对于 $a_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}$, $\forall\varepsilon>0,\exists N\ge\frac{1}{\varepsilon},\forall n>N,\forall p\in\mathbb{N}^*$,

$$s.t.\left|a_n-a_{n+p}\right|=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(n+p)^2}$$

$$<\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\cdots+\frac{1}{(n+p-1)(n+p)}$$

$$=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}+\cdots+\frac{1}{n+p-1}-\frac{1}{n+p}=\frac{1}{n}-\frac{1}{n+p}<\frac{1}{n}<\frac{1}{N}\le\frac{1}{1/\varepsilon}=\varepsilon.$$

题 7.或许难度稍大了些，诸位可以将其暂时放到一边，暇时琢磨琢磨。

7. 我们可以证明，$\left\{\left(1+\frac{1}{n}\right)^n\right\}$ 为递增数列，$\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ 为递减数列且均有界，因而它们都收敛.

(I).记$A=\left\{\left(1+\dfrac{1}{n}\right)^{n+1},n\in\mathbb{N}^*\right\}$,问$\sup A=?\ \inf A=?$

(II).证明:(1).$n\in\mathbb{N}^*,\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}$; (2).$\left\{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n\right\}$收敛;

(3).$\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}\right)=\ln 2$; (4).$\left\{\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{2^2}\right)\cdots\left(1+\dfrac{1}{2^n}\right)\right\}$收敛.

解 (I).$\because$数列$\left\{\left(1+\dfrac{1}{n}\right)^{n+1}\right\}$单调递减有下界,对于$A=\left\{\left(1+\dfrac{1}{n}\right)^{n+1},n\in\mathbb{N}^*\right\}$,

有$\sup A=\max A=4,\inf A=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+1}=e.\therefore\left(1+\dfrac{1}{n}\right)^{n+1}\geq e.$

(II).(1).$\because\left\{\left(1+\dfrac{1}{n}\right)^{n}\right\}$单调递增有上界,$\therefore\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}=e=\sup\left\{\left(1+\dfrac{1}{n}\right)^{n}\right\}$,

$\therefore\left(1+\dfrac{1}{n}\right)^{n}\leq e$.由于数$e$是无理数,而$\left(1+\dfrac{1}{n}\right)^{n},\left(1+\dfrac{1}{n}\right)^{n+1}$都是有理数,$\therefore\left(1+\dfrac{1}{n}\right)^{n}<e<\left(1+\dfrac{1}{n}\right)^{n+1}$.

$\Rightarrow n\ln\left(1+\dfrac{1}{n}\right)<1,(n+1)\ln\left(1+\dfrac{1}{n}\right)>1.\therefore n\in\mathbb{N}^*$时有$\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}$.

(2).$\because n\in\mathbb{N}^*,\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}$,$\Rightarrow\dfrac{1}{2}<\ln 2<1,\dfrac{1}{3}<\ln\dfrac{3}{2}<\dfrac{1}{2},\cdots,\dfrac{1}{n+1}<\ln\dfrac{n+1}{n}<\dfrac{1}{n}$,

这n个不等式相加,得$\dfrac{1}{2}+\cdots+\dfrac{1}{n}+\dfrac{1}{n+1}<\ln(n+1)<1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}$,

记$a_n=1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n,\therefore a_n>0$,且$a_{n+1}-a_n=\dfrac{1}{n+1}-\ln(n+1)+\ln n=\dfrac{1}{n+1}-\ln\left(1+\dfrac{1}{n}\right)<0$,

$\therefore$数列$\{a_n\}$单调递减且有下界$(a_n>0)$,$\Rightarrow\left\{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n\right\}$收敛.

$\longrightarrow Add.\lim\limits_{n\to\infty}\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n\right)=\gamma$,这是著名的$Euler$常数,$\gamma=0.5772156649015328\cdots$

(3).$\because\lim\limits_{n\to\infty}a_n=\gamma$,则$\lim\limits_{n\to\infty}a_{2n}=\gamma$,于是$\lim\limits_{n\to\infty}(a_{2n}-a_n)=0$,

即$\lim\limits_{n\to\infty}\left[1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}+\dfrac{1}{n+1}+\cdots+\dfrac{1}{n+n}-\ln(2n)-\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n\right)\right]=0.$

$\Rightarrow\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}-\ln 2\right)=0.\therefore\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}\right)=\ln 2$.

(4).$\because n\in\mathbb{N}^*,\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n},\therefore\left(1+\dfrac{1}{n}\right)<e^{\frac{1}{n}},\therefore b_n=\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{2^2}\right)\cdots\left(1+\dfrac{1}{2^n}\right)<e^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}$,

记$\dfrac{1}{2}+\dfrac{1}{2^2}+\cdots+\dfrac{1}{2^n}=H$,于是$\dfrac{1}{2^2}+\cdots+\dfrac{1}{2^n}+\dfrac{1}{2^{n+1}}=\dfrac{1}{2}H$,错位相减得$H=\dfrac{\frac{1}{2}-\frac{1}{2^{n+1}}}{1-\frac{1}{2}}=1-\dfrac{1}{2^n}$.

$\therefore b_n=\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{2^2}\right)\cdots\left(1+\dfrac{1}{2^n}\right)<e^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}=e^{1-\frac{1}{2^n}}<e$,$\therefore$数列$\{b_n\}$单调递增有上界,

$\therefore$数列$\{b_n\}=\left\{\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{2^2}\right)\cdots\left(1+\dfrac{1}{2^n}\right)\right\}$收敛.(当然,该收敛数列的极限求不出来!)