Chap17.多元函数微分习题课之一 2022-05

例1.设$\left(axy^3-y^2\cos x\right)dx+\left(1+by\sin x+3x^2y^2\right)dy$是一个二元函数$z=f(x,y)$的全微分.试确定$a,b$的值,并求出函数$z=f(x,y)$.

$$解\because \left(axy^3-y^2\cos x\right)dx+\left(1+by\sin x+3x^2y^2\right)dy=dz,$$

$$\therefore \begin{cases} \dfrac{\partial z}{\partial x}=axy^3-y^2\cos x \\ \dfrac{\partial z}{\partial y}=1+by\sin x+3x^2y^2 \end{cases},$$

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$$dz = (axy^3 - y^2\cos x)dx + (1 + by\sin x + 3x^2y^2)dy$$

$$\therefore \frac{\partial z}{\partial x} = axy^3 - y^2\cos x, \frac{\partial z}{\partial y} = 1 + by\sin x + 3x^2y^2,$$

$$z = \frac{1}{2}ax^2y^3 - y^2\sin x + C(y) \quad \Rightarrow$$

$$\frac{\partial z}{\partial y} = C'(y) - 2y\sin x + \frac{3}{2}ax^2y^2 = 1 + by\sin x + 3x^2y^2,$$

由待定系数法得 $a = 2, b = -2.$

$$\begin{cases} \dfrac{\partial z}{\partial x} = 2xy^3 - y^2\cos x \cdots\cdots(1) \\ \dfrac{\partial z}{\partial y} = 1 - 2y\sin x + 3x^2y^2 \cdots(2) \end{cases},$$

由(1)得$z = x^2y^3 - y^2\sin x + C(y)\cdots(3)$,

对(3)再计算$\dfrac{\partial z}{\partial y} = 3x^2y^2 - 2y\sin x + C'(y)$,

与(2)比较得：$C'(y) = 1, \therefore C(y) = y + C$,

$\therefore z = x^2y^3 - y^2\sin x + y + C$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$(其中$C$为任意常数).

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例2.设$z = f(x-y, x^2y)$, f具有连续的偏导数,求 dz.

解 令设 $u = x-y, v = x^2y, \rightarrow z = f(u,v)$,

则 $dz = \dfrac{\partial z}{\partial u}du + \dfrac{\partial z}{\partial v}dv$

$= f_1 d(x-y) + f_2 d(x^2y)$

$= f_1(dx-dy) + f_2(2xydx + x^2dy)$

$= (f_1 + 2xyf_2)dx + (-f_1 + x^2 f_2)dy$

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设 $u = x - y, v = x^2 y, \rightarrow z = f(x-y, x^2 y) = f(u,v)$,

则 $dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = f_1 d(x-y) + f_2 d(x^2 y)$

$= f_1(dx - dy) + f_2(2xydx + x^2 dy)$

$= (f_1 + 2xyf_2)dx + (-f_1 + x^2 f_2)dy$

记 $f_1 = \frac{\partial f(u,v)}{\partial u}, f_2 = \frac{\partial f(u,v)}{\partial v}$,

$\frac{\partial z}{\partial x} = f_1 + 2xyf_2, \frac{\partial z}{\partial y} = -f_1 + x^2 f_2$,

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (f_1 + 2xyf_2)dx + (-f_1 + x^2 f_2)dy.$

这就是一阶全微分的形式不变性!

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例3. 设$z = f(x+y, xy)$,函数f有连续的二阶偏导数,求$\dfrac{\partial z}{\partial x}$与$\dfrac{\partial^2 z}{\partial x \partial y}$.

解 令$u = x + y, v = xy$.

记$f_1 = \dfrac{\partial f(u,v)}{\partial u}, f_{12} = \dfrac{\partial^2 f(u,v)}{\partial u \partial v}$,

同样有 f_2, f_{11}, f_{22}

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1 + y f_2,$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(f_1 + y f_2) = \frac{\partial f_1}{\partial y} + f_2 + y\frac{\partial f_2}{\partial y},$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} = f_{11} + x f_{12},$$

$$\frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} = f_{21} + x f_{22};$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = f_{11} + x f_{12} + f_2 + y(f_{21} + x f_{22})$$

$$= f_{11} + (x + y) f_{12} + xy f_{22} + f_2$$

例4.已知函数 $z=z(x,y)$具有连续的二阶偏导数,设$\begin{cases} u=x+y \\ v=x-y \end{cases}$,试将方程$\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}=0$化为新坐标系中的形式,并由此求解该偏微分方程.

解　合理的理解是,$z=\varphi(u,v)=f(x,y)$,函数φ, f有连续的二阶偏导数.

$$\begin{cases} \dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v} \\ \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}=\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v} \end{cases},$$

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$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)=\frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}$$

$$+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$-\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial y} - \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u\partial v} - \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0 \Rightarrow \frac{\partial^2 z}{\partial u \partial v}=0,\left(P135/7\right)$$

$$\frac{\partial^2 z}{\partial u \partial v}=0 \Leftrightarrow \frac{\partial z}{\partial u}=g(u),$$

$$\Rightarrow z=\int g(u)du=G(u)+H(v),$$

$$\therefore \text{方程} \frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0 \text{ 的解}$$

为$z=G(x+y)+H(x-y)$

$G(s),H(t)$为任意的有连续导数的函数.

极值判断(充分条件)定理–对比：

二元：$f_x(x_0,y_0)=0,\ f_y(x_0,y_0)=0.$

$f_{xx}(x_0,y_0)=A,\ f_{xy}(x_0,y_0)=B,$

$f_{yy}(x_0,y_0)=C.$

当$AC-B^2>0$时有极值：

$A>0(<0)$时有极小(大)值.

一元：$f'(x_0)=0,\ f''(x_0)=A,$

则当$A\neq 0$时有极值：

$A>0(<0)$时有极小(大)值.

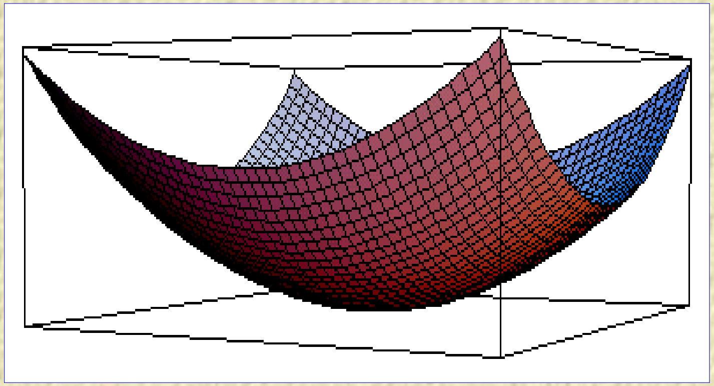
通过具体例子帮助记住定理结论.

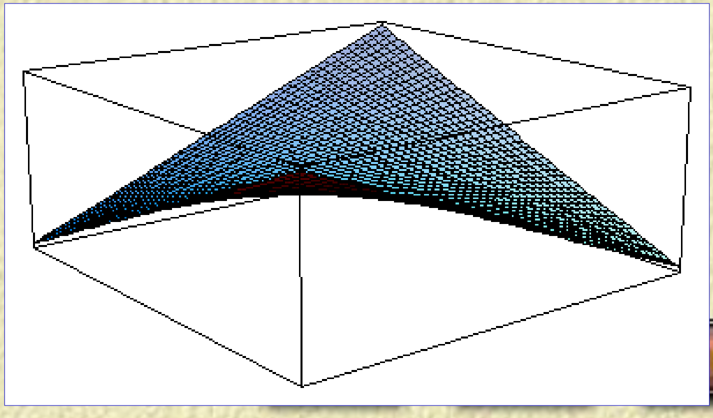
(1).$z = x^2 + 2y^2$在驻点$(0,0)$处取得极小值,且有

$$A = 2, B = 0, C = 4, AC - B^2 = 8 > 0.$$

(2).$z = x^2 - y^2$在驻点$(0,0)$处无极值,

$$A = 2, B = 0, C = -2, AC - B^2 = -4 < 0.$$


(1)


(2)

例5.求$f(x,y)=x^3-y^3+3x^2+3y^2-9x$的极值.

解 第一步 求驻点.

解方程组$\begin{cases} f_x(x,y)=3x^2+6x-9=0 \\ f_y(x,y)=-3y^2+6y=0 \end{cases}$

解得 $x=1$或3 ,$y=0$或2.

$\therefore$ 驻点有$(1,0),(1,2),(3,0),(3,2)$.

第二步 判别.

求二阶偏导数 $f_{xx}=6x+6\cdots A$,

$f_{xy}=0\cdots B$, $f_{yy}=-6y+6\cdots C$

项目 / 驻点	$A=f_{xx}$	$B=f_{xy}$	$C=f_{yy}$	$AC-B^2$	极值情况
(-3,0)	-12	0	6	<0	不取极值
(-3,2)	-12 < 0	0	-6	>0	$\max f(x,y)=f(-3,2)$
(1,0)	12 > 0	0	6	>0	$\min f(x,y)=f(1,0)$
(1,2)	12	0	-6	<0	不取极值

以例说明定理的条件是充分而非必要的.

(不要求掌握)

e.g.讨论函数$z = x^3 + y^3$与$z = \left(x^2 + y^2\right)^2$

在点(0,0)处的极值情况.

解　显然(0,0)都是它们的驻点,且在(0,0)

处都有　$AC - B^2 = 0.$

$z = x^3 + y^3$在点(0,0)的邻域内的取值可

能为: 正 \ 零 \ 负,

$\therefore$ (0,0)处$z = x^3 + y^3$不取极值.

而当 $x^2 + y^2 \neq 0$时, $z = \left(x^2 + y^2\right)^2 > z\big|_{(0,0)} = 0,$

$\therefore z(0,0) = \left(x^2 + y^2\right)^2\Big|_{(0,0)} = 0$为极小值.

例6.设$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$,求 $f_{xy}(0,0)$, $f_{yx}(0,0)$.

解 $$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$\left.\begin{aligned} f_{xy}(0,0)&=\lim_{\Delta y\to 0}\boxed{\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1 \\ f_{yx}(0,0)&=\lim_{\Delta x\to 0}\boxed{\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1 \end{aligned}\right\}$$

象这样的分段函数的问题可以忽略.尽管是有用的,但不是重要的.

返回

多元函数的微分部分,理解概念并掌握结论与方法是重要的,但首先重要的计算,而非利用定义与二重极限等作可微性的讨论.

首要的计算!计算!!计算!!!

例7*.设函数$z = f(x,y)$在$\mathbb{R}^2$上可微,

$\forall ab \neq 0$,若有$b\dfrac{\partial z}{\partial x} = a\dfrac{\partial z}{\partial y}$.

求证:必定有$z = \varphi(ax+by)$的形式.

分析 若$z = g(u,v)$,则$\dfrac{\partial z}{\partial u} \equiv 0 \Leftrightarrow$

z相对于变量u而言是常数,

那么,必定有$z = \varphi(v)$的形式.

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证明 $\because ab \neq 0, \therefore \begin{cases} u = x \\ v = ax + by \end{cases}$ 是一个

可逆的线性变换,且 $\begin{cases} x = \quad u \\ y = -\dfrac{a}{b}u + \dfrac{1}{b}v \end{cases}.$

$\because x, y$ 是函数 $z = f(x, y)$ 的两个独立的自变量,

$\therefore u, v$ 是函数 $z = f(x, y) = g(u, v)$ 的两个独立的自变量.

$\forall ab \neq 0,$

$$b\frac{\partial z}{\partial x} = a\frac{\partial z}{\partial y}.$$

$$\begin{cases} u = x \\ v = ax + by \end{cases}, \quad \begin{cases} x = u \\ y = -\dfrac{a}{b}u + \dfrac{1}{b}v \end{cases},$$

$\because$ f 在 $\mathbb{R}^2$ 上可微, $\therefore \forall (x, y) \in \mathbb{R}^2$,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} - \frac{a}{b} \cdot \frac{\partial z}{\partial y} \equiv 0,$$

$\therefore$ z 作为变量 u,v 的函数, z 相对于变量 u 而言是常数, 故必定有

$z = \varphi(v) = \varphi\big(ax + by\big)$ 的形式.

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例8*.(P134/总练习3)设 $u=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$

证明:(1).$\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=0$;(2).$\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}=\frac{n(n-1)}{2}u$.

证一 (1).这是Van der monde 行列式,

$$u=\prod_{1\le i<j\le n}(x_j-x_i),$$

可以通过硬算 $\frac{\partial u}{\partial x_i}=\cdots$,用此最简单的方法来处理问题(1)应属可行.用此法来处理问题(2)就显得太拙笨了,但还是值得一试.

简单的方法往往就是好的方法.

$$u = \prod_{1 \le i < j \le n} \left(x_j - x_i \right)$$

为计算$\dfrac{\partial u}{\partial x_i}$,如果考虑用对数求导法,如果

$x_j - x_i \neq 0$ 时是可行的$\cdots$ $\left(注意到 (\ln(-x))' = \dfrac{1}{x} \right)$

$$\ln u = \sum_{1 \le i < j \le n} \ln\left(x_j - x_i \right)$$

$$u = e^{\ln u} = e^{\sum\limits_{1 \le i < j \le n} \ln\left(x_j - x_i \right)}, \frac{\partial u}{\partial x_i} = \cdots$$

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设 $u=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$ 证明：(1). $\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=0$；

(2). $\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}=\frac{n(n-1)}{2}u$.

法二：(1).行列式按第k列展开

$$u=1\cdot A_{1k}+x_kA_{2k}+\cdots+x_k^{n-1}A_{nk}=\sum_{i=1}^{n}x_k^{i-1}A_{ik},$$

$$\frac{\partial u}{\partial x_k}=\sum_{i=2}^{n}(i-1)x_k^{i-2}A_{ik},$$

A_{ik}中没有x_k的因子

返回

$$\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=\sum_{k=1}^{n}A_{2k}+2\sum_{k=1}^{n}x_kA_{3k}+\cdots+(n-1)\sum_{k=1}^{n}x_k^{n-2}A_{nk}$$

$$=0+0+0+\cdots+0=0.$$

行列式按行计算

$$(2).\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}$$

$$=\sum_{k=1}^{n}x_kA_{2k}+2\sum_{k=1}^{n}x_k^2A_{3k}+\cdots+(n-1)\sum_{k=1}^{n}x_k^{n-1}A_{nk}$$

$$=u+2u+\cdots+(n-1)u=\frac{n(n-1)}{2}u.$$

法三 (1).注意到 $u=\prod_{1\le i<j\le n}\left(x_j-x_i\right)$

有一特别巧妙的做法,令

$$g(t)=u\left(x_1+t,\cdots,x_n+t\right)\equiv u\left(x_1,\cdots,x_n\right)=g(0),$$

$\therefore \forall t, g(t)\equiv C$, $\therefore g'(t)\equiv 0$,

$$g'(t)=\frac{\partial u}{\partial s_1}\cdot\frac{\partial s_1}{\partial t}+\frac{\partial u}{\partial s_2}\cdot\frac{\partial s_2}{\partial t}+\cdots+\frac{\partial u}{\partial s_n}\cdot\frac{\partial s_n}{\partial t}$$

$$=u_1+u_2+\cdots+u_n,$$

令 $t=0$ 就得到(1)的结论.

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(2).利用齐次函数的Euler定理$\left(P117/Ex.7\right)$

有 $u\left(tx_1,\cdots,tx_n\right)=t^{1+2+\cdots+n-1}u\left(x_1,\cdots,x_n\right)$,

立即得到结论$\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}=\frac{n\left(n-1\right)}{2}u$.

我们可以看到,比较起来,方法三处理问题的确是简便、快捷,但太不容易想到了.就我而言,首选方法一,历尽艰辛而不得,然后再去寻求所谓巧妙的做法.你如何?

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例9*.(P117/7)若函数$u=F(x,y,z)$满足恒等式

$F(tx,ty,tz)=t^k F(x,y,z)\ (t>0)$,则称$F(x,y,z)$

为k次齐次函数.

试证明下述关于齐次函数的Euler定理:

可微函数$F(x,y,z)$为 k 次齐次函数 $\Leftrightarrow$

$$xF_x(x,y,z)+yF_y(x,y,z)+zF_z(x,y,z)=kF(x,y,z).$$

证明思路分析

回顾:可微函数$f(x),x\in(0,+\infty)$满足

$xf'(x)=kf(x)$,问$f(x)=?\longleftarrow$联想

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证明思路分析

回顾：可微函数$f(x), x\in(0,+\infty)$满足

$xf'(x)=kf(x)$,问$f(x)=?$

解决途径：据微分形式不变性,由$xf'(x)=kf(x)$,

变化为 $\dfrac{df(x)}{f(x)}=k\dfrac{dx}{x}$,即$d\left(\ln f(x)-k\ln x\right)=0$,

得 $\ln f(x)-k\ln x=\ln C$,即$\dfrac{f(x)}{x^k}=C\cdots$

故设辅助函数$\varphi(x)=\dfrac{f(x)}{x^k}$,即证$\varphi(x)=C$.

$Q: f(x)$可微, $x \in (0,+\infty)$满足$xf'(x)=kf(x)$,问$f(x)=?$

$A:$设辅助函数$\varphi(x)=\dfrac{f(x)}{x^k}, x \in (0,+\infty)$,往证$\varphi(x)=C$.

联想,类比$\longrightarrow$

$Q:$试证明:可微函数$F(x,y,z)$为 k 次齐次函数$\Leftrightarrow$

$xF_x(x,y,z)+yF_y(x,y,z)+zF_z(x,y,z)=kF(x,y,z)$.

$A:$设辅助函数$\varphi(t)=\dfrac{F(tx,ty,tz)}{t^k}$,即往证 $t \in (0,+\infty)$

时$\varphi(t)=C$.

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试证明下述关于齐次函数的*Euler*定理：

可微函数$F(x,y,z)$为 k 次齐次函数 $\Leftrightarrow$

$$xF_x(x,y,z)+yF_y(x,y,z)+zF_z(x,y,z)=kF(x,y,z).$$

证明 "$\Leftarrow$" 的证明：设$\varphi(t)=\dfrac{F(tx,ty,tz)}{t^k}, t\in(0,+\infty)$,

$$\varphi'(t)=\frac{t^k\cdot\dfrac{d}{dt}F(tx,ty,tz)-F(tx,ty,tz)\cdot kt^{k-1}}{t^{2k}}$$

$$=\frac{txF_1(tx,ty,tz)+tyF_2(tx,ty,tz)+tzF_3(tx,ty,tz)-kF(tx,ty,tz)}{t^{k+1}}$$

$$\equiv 0,$$

$\therefore t\in(0,+\infty)$时$\varphi(t)\equiv C$,于是$\varphi(t)=\varphi(1)$,结论得证.

"$\Rightarrow$" 的证明是容易的.