17-02 复合函数微分法

一. 复合函数微分的链式法则

二. 一阶全微分形式不变性

一.复合函数微分的链式法则

Th.17.5.若函数$u=\varphi(x),v=\psi(x)$在点x处都可导,函数$z=f(u,v)$在点(u,v)处具有连续的偏导数,则$z=f\left(\varphi(x),\psi(x)\right)$在点$x$处可导,且有全导数公式:

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}.$$

证明 设 x 获得增量 Δx,

则 $\Delta u=\varphi(x+\Delta x)-\varphi(x)$,

$\Delta v=\psi(x+\Delta x)-\psi(x)$.

$$\Delta z = f\left(u+\Delta u, v+\Delta v\right) - f(u,v)$$

$$= \left[f\left(u+\Delta u, v+\Delta v\right) - f\left(u, v+\Delta v\right) \right]$$

$$+ \left[f\left(u, v+\Delta v\right) - f(u,v) \right]$$

$$= f_u\left(u+\theta_1\Delta u, v+\Delta v\right)\Delta u + f_v\left(u, v+\theta_2\Delta v\right)\Delta v$$

$\left(\because z = f(u,v)\right.$在点$(u,v)$处有连续的偏导数,$\left.\right)$

$$= f_u\left(u,v\right)\Delta u + f_v\left(u,v\right)\Delta v + \varepsilon_1\Delta u + \varepsilon_2\Delta v$$

$$\varepsilon_i = \varepsilon_i\left(\Delta u, \Delta v\right), \lim_{\substack{\Delta u\to 0\\ \Delta v\to 0}} \varepsilon_i = 0, 0 < \theta_i < 1, i = 1, 2.$$

$\because z = f(u,v)$在点(u,v)处有连续的偏导数,

$$\Delta z = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + \varepsilon_1 \Delta u + \varepsilon_2 \Delta v,$$

当$\Delta u \to 0, \Delta v \to 0$时有$\varepsilon_1 \to 0, \varepsilon_2 \to 0$,

$$\frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta x} + \frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta x} + \varepsilon_1 \frac{\Delta u}{\Delta x} + \varepsilon_2 \frac{\Delta v}{\Delta x},$$

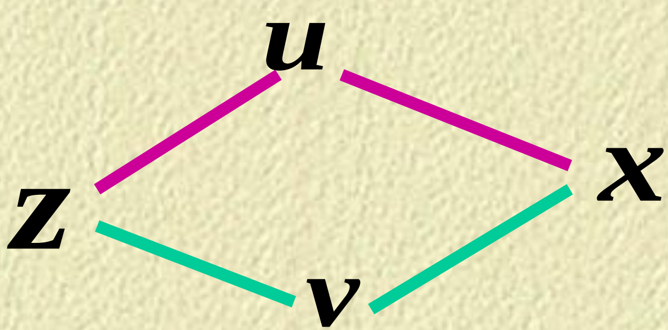
当 $\Delta x \to 0$时有$\Delta u \to 0, \Delta v \to 0$,

$$\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = \frac{du}{dx}, \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} = \frac{dv}{dx},$$

$$\therefore \frac{dz}{dx} = \lim_{\Delta x \to 0}\frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx}.$$

$$z = f(u,v), u = \varphi(x), v = \psi(x)$$

$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx}.$$



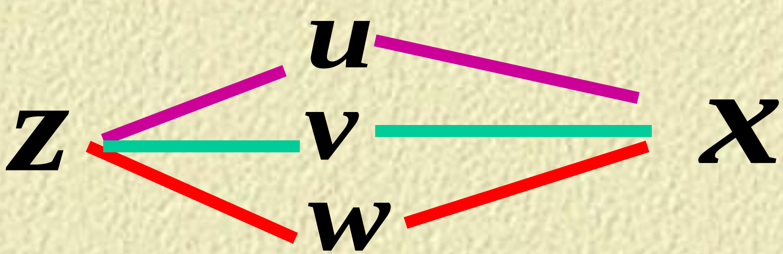
$$z = f(u,v), u = \varphi(x), v = \psi(x)$$

$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx}.$$

该结论可推广至多个中间变量的情形,如

$$z = f(u,v,w), u = \varphi_1(x), v = \varphi_2(x), w = \varphi_3(x),$$

$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dx}.$$



$Th.17.5'$.若函数$u=\varphi(x,y),v=\psi(x,y)$在点$(x,y)$处都可(偏)导,函数$z=f(u,v)$在点$(u,v)$处具有连续的偏导数,则$z=f\left(\varphi(x,y),\psi(x,y)\right)$在点$(x,y)$处可(偏)导,且有全导数公式:

$$\begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}\\[2ex] \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}\end{cases}.$$

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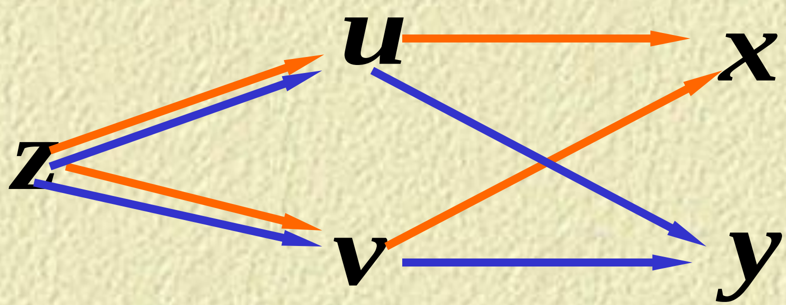
链式法则如图示 (轨道图)



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

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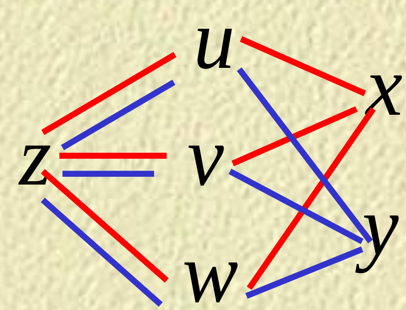
类似地再推广：若函数$u=\varphi_1(x,y)$,$v=\varphi_2(x,y)$,
$w=\varphi_3(x,y)$在点(x,y)处都可(偏)导,函数
$z=f(u,v,w)$在点(u,v,w)处具有连续的偏导数,
则$z=f\big(\varphi_1(x,y),\varphi_2(x,y),\varphi_3(x,y)\big)$在点$(x,y)$处
可(偏)导,且有

$$\begin{cases} \dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}+\dfrac{\partial z}{\partial w}\cdot\dfrac{\partial w}{\partial x} \\ \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}+\dfrac{\partial z}{\partial w}\cdot\dfrac{\partial w}{\partial y} \end{cases}.$$



例1.求幂指函数$y=(\sin x)^{x^2}\ (x\in(0,\pi))$的导数.

解　令$y=u^v$, $\begin{cases} u=\sin x \\ v=x^2 \end{cases}$,用全导数公式:

则 $$\frac{dy}{dx}=\frac{\partial y}{\partial u}\cdot\frac{du}{dx}+\frac{\partial y}{\partial v}\cdot\frac{dv}{dx}$$

$$=v\cdot u^{v-1}\cdot u'_x+u^v\ln u\cdot v'_x$$

$$=\cdots$$

所以为什么这种函数要称为“幂指函数”!

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例2.设$z = e^u \sin v$,而$\begin{cases} u = xy \\ v = x + y \end{cases}$,求:$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1 = e^u \left(y \sin v + \cos v \right),$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1 = e^u \left(x \sin v + \cos v \right).$$

设$z = e^u \sin v$,而$\begin{cases} u = xy \\ v = x + y \end{cases}$,

记$z = e^u \sin v = f(u,v)$,则

$$\frac{\partial z}{\partial u} = e^u \sin v = f_u(u,v),$$

$$\frac{\partial z}{\partial v} = e^u \cos v = f_v(u,v),$$

$f_u(u,v)$, $f_v(u,v)$仍是以u,v为中间变量,以x, y为自变量的两个新的函数.

例3.设f有连续的一阶偏导数,求:$\dfrac{\partial z}{\partial x}$.

(1).$z = f(x+y, xy)$;

(2).$z = f\big(\varphi(x,y), x, y\big)$,$\varphi$可微.

解(1).令$u = x + y, v = xy$,

记$f_1 = \dfrac{\partial f(u,v)}{\partial u}, f_2 = \dfrac{\partial f(u,v)}{\partial v}$,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1 + yf_2.$$

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$$(2).\ z = f\left(\varphi(x,y), x, y\right),\quad \begin{cases} u = \varphi(x,y) \\ v = x \\ w = y \end{cases},$$

$$令\ z = f(u,v,w)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= f_1\cdot\frac{\partial u}{\partial x} + f_2\cdot 1 + f_3\cdot 0,\quad \frac{\partial u}{\partial x} = \varphi_x(x,y),$$

$$\Rightarrow \frac{\partial z}{\partial x} = f_1\cdot\frac{\partial u}{\partial x} + f_2 = f_1\cdot\varphi_x + f_2$$

$$(2).z = f\left(\varphi(x,y),x,y\right) = f(u,v,w),$$

$$u = \varphi(x,y), v = x, w = y,$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}\cdots(1)$$

按理,$v = x, \dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}$,但需注意到,(1)

式左边的$\dfrac{\partial z}{\partial x}$是全部,右边的$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}$

是部分,要注意两者的区别,所以有

的书上记之为$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x}$.

举例说明：

将$z=\sqrt{x^2+y^2}+e^x+y$ 视为

$$z=f(u,x,y)=\sqrt{u}+e^x+y, u=x^2+y^2,$$

$$v=x$$

$$\therefore \frac{\partial z}{\partial x}=\left(\sqrt{x^2+y^2}+e^x+y\right)'_x,$$

而$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}=\left(\sqrt{u}+e^x+y\right)'_x=e^x,$

此处将u,x,y看作地位对等的

都是自变量.

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注1：*Th*.17.5 中外层函数f要求是可微而不仅仅是可偏导,由下面的例子可知f可微的条件是需要的.

例如,$z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$，易知

$f_u(0,0) = f_v(0,0) = 0$,但是f在点(0,0)处不可微.

取$u = v = x$,则 $z = \dfrac{1}{2}x, z'_x = \dfrac{1}{2}$,不过在$x = 0$ 时,

$$\left(\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}\right)\bigg|_{x=0} = 0 \neq \frac{dz}{dx}\bigg|_{x=0}$$

所以,多元函数的情况比一元函数要复杂得多.

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注2：$Th.17.5'$.若函数$u=\varphi(x,y),v=\psi(x,y)$在点$(x,y)$处都可$\left(\text{偏}\right)$导,函数$z=f(u,v)$在点$(u,v)$处具有连续的偏导数,则$z=f\left(\varphi(x,y),\psi(x,y)\right)$在点$(x,y)$处可$\left(\text{偏}\right)$导,且有全导数公式：

$$\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)=\left(\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}\right)\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}.$$

二. 一阶全微分形式不变性

设函数$z = f(u,v)$有连续的偏导数,则

有全微分 $$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

又若$u = \varphi(x,y), v = \psi(x,y)$可微,则有

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

这就是说,无论视函数z是自变量x, y的函数还是中间变量u,v的函数,其全微分的结果形式上是完全一样的.

我们称此为全微分的形式不变性.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy =$$

$$\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

例4.设$z = f(x-y, x^2 y)$, f具有连续的一阶偏导数,求:dz.

解:设$\begin{cases} u = x - y \\ v = x^2 y \end{cases}$,记$f_1 = \dfrac{\partial f(u,v)}{\partial u}$, $f_2 = \dfrac{\partial f(u,v)}{\partial v}$,

$$\frac{\partial z}{\partial x} = f_1 + 2xyf_2, \frac{\partial z}{\partial y} = -f_1 + x^2 f_2,$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= \left(f_1 + 2xyf_2\right)dx + \left(-f_1 + x^2 f_2\right)dy$$

一阶全微分形式不变性

设 $u = x - y, v = x^2 y,$

记 $f_1 = \dfrac{\partial f(u,v)}{\partial u}, f_2 = \dfrac{\partial f(u,v)}{\partial v},$

$$\frac{\partial z}{\partial x} = f_1 + 2xyf_2, \frac{\partial z}{\partial y} = -f_1 + x^2 f_2,$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(f_1 + 2xyf_2\right)dx + \left(-f_1 + x^2 f_2\right)dy$$

$$= f_1\left(dx - dy\right) + f_2\left(2xydx + x^2dy\right)$$

$$= f_1 d\left(x - y\right) + f_2 d\left(x^2 y\right) = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

思考练习

1.设$z = f\left(x^2 - y^2, 2xy\right)$, f有连续的一阶偏导数,求dz.

2*.(P134/总练习3)设 $u = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$

证明:(1).$\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} = 0$; (2).$\sum_{k=1}^{n} x_k \frac{\partial u}{\partial x_k} = \frac{n(n-1)}{2} u$.

1.设$z=f\left(x^2-y^2,2xy\right)$, f有连续的一阶偏导数,求dz.

解　记 $x^2-y^2=u, 2xy=v, z=f\left(u,v\right)$,

$$dz=f_1du+f_2dv=f_1d\left(x^2-y^2\right)+f_2d\left(2xy\right)$$

$$=f_1\left(2xdx-2ydy\right)+f_2\left(2ydx+2xdy\right),$$

$$=2\left(xf_1+yf_2\right)dx+2\left(xf_2-yf_1\right)dy.$$

2.(*P*134/总练习3)设 $u=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$

证明:(1).$\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=0$；(2).$\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}=\frac{n(n-1)}{2}u$.

证一　(1).这是*Van der monde* 行列式,

$$u=\prod_{1\le i<j\le n}\left(x_j-x_i\right)$$

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$$u=\prod_{1\le i<j\le n}\left(x_j-x_i\right)$$

考虑最理想的情况,注意到$\left(\ln(-x)\right)'=\dfrac{1}{x}$,

$$\ln u=\sum_{1\le i<j\le n}\ln\left(x_j-x_i\right)$$

$$u=e^{\ln u}=e^{\sum\limits_{1\le i<j\le n}\ln\left(x_j-x_i\right)},\frac{\partial u}{\partial x_i}=\cdots$$

可以看到,用此最简单的方法来处理问题(2)就显得太拙笨了,但还是值得一试.

简单的方法往往就是好的方法 .

设 $u=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$ 证明：(1). $\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=0$；

(2). $\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}=\frac{n(n-1)}{2}u$.

法二：(1).行列式按第k列展开

$$u=1\cdot A_{1k}+x_kA_{2k}+\cdots+x_k^{n-1}A_{nk}$$

$$\frac{\partial u}{\partial x_k}=\sum_{i=2}^{n}(i-1)x_k^{i-2}A_{ik},$$

A_{ki}中没有x_k的因子

$$\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=\sum_{k=1}^{n}A_{2k}+2\sum_{k=1}^{n}x_k A_{3k}+\cdots+(n-1)\sum_{k=1}^{n}x_k^{n-2}A_{nk}$$

$$=0+0+0+\cdots+0=0.$$

行列式按行计算

$$(2).\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}$$

$$=\sum_{k=1}^{n}x_k A_{2k}+2\sum_{k=1}^{n}x_k^2 A_{3k}+\cdots+(n-1)\sum_{k=1}^{n}x_k^{n-1}A_{nk}$$

$$=u+2u+\cdots+(n-1)u=\frac{n(n-1)}{2}u.$$

法三 (1).注意到 $u = \mathop{\Pi}\limits_{1\le i<j\le n}\left(x_j - x_i\right)$

有一特别巧妙的做法,令

$$g(t) = u\left(x_1 + t, \cdots, x_n + t\right) \equiv u\left(x_1, \cdots, x_n\right) = g(0),$$

$\therefore \forall t, g(t) \equiv C$, $\therefore g'(t) \equiv 0$,

$$g'(t) = \frac{\partial u}{\partial s_1}\cdot\frac{\partial s_1}{\partial t} + \frac{\partial u}{\partial s_2}\cdot\frac{\partial s_2}{\partial t} + \cdots + \frac{\partial u}{\partial s_n}\cdot\frac{\partial s_n}{\partial t}$$

$$= u_1 + u_2 + \cdots + u_n,$$

令 $t = 0$ 就得到(1)的结论.

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(2).利用齐次函数的$Euler$定理$\left(P117/Ex.7\right)$

有 $u\left(tx_1,\cdots,tx_n\right)=t^{1+2+\cdots+n-1}u\left(x_1,\cdots,x_n\right)$,

立即得到结论$\sum_{k=1}^{n}x_k\frac{\partial u}{\partial x_k}=\frac{n\left(n-1\right)}{2}u$.

我们可以看到，比较起来，方法三处理问题的确是简便、快捷，但太不容易想到了，就我而言，首选方法一，你如何?

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