2022-12 数学分析 I 阶段练习参考解答 2022-12

班级____________学号____________姓名______________

1. 求极限 (1). $\lim\limits_{x\to0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$; (2). $\lim\limits_{x\to0}\dfrac{e-(1+x)^{\frac{1}{x}}}{x}$; (3). $\lim\limits_{n\to\infty}\left(\dfrac{\left(1+\frac{1}{n}\right)^n}{e}\right)^n$.

(1). $\lim\limits_{x\to0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)=\lim\limits_{x\to0}\dfrac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim\limits_{x\to0}\dfrac{x^2-\sin^2 x}{x^4}=\lim\limits_{x\to0}\dfrac{2x-\sin 2x}{4x^3}\overset{2x=t}{=\!=\!=}2\lim\limits_{t\to0}\dfrac{t-\sin t}{t^3}$

$=2\lim\limits_{t\to0}\dfrac{1-\cos t}{3t^2}=2\lim\limits_{t\to0}\dfrac{\sin t}{6t}=\dfrac{1}{3}$.

(2).解 $(1+x)^{\frac{1}{x}}=e^{\ln(1+x)^{\frac{1}{x}}}=e^{\frac{1}{x}\ln(1+x)}$,

$\therefore$ 原$\overset{\frac{0}{0}}{=\!=}\lim\limits_{x\to0}\dfrac{0-\left[(1+x)^{\frac{1}{x}}\right]'}{1}=-\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}\left(\dfrac{\ln(1+x)}{x}\right)'=-\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}\cdot\dfrac{\frac{x}{1+x}-\ln(1+x)}{x^2}$

其中$\lim\limits_{x\to0}\dfrac{\frac{x}{1+x}-\ln(1+x)}{x^2}=\lim\limits_{x\to0}\dfrac{x-(1+x)\ln(1+x)}{x^2(1+x)}=\lim\limits_{x\to0}\dfrac{1-1-\ln(1+x)}{2x+3x^2}=-\dfrac{1}{2}$, $\therefore$ 原$=\dfrac{1}{2}e$.

解二 原式$=\lim\limits_{x\to0}\dfrac{e-e^{\frac{\ln(1+x)}{x}}}{x}=-e\lim\limits_{x\to0}\dfrac{e^{\frac{\ln(1+x)}{x}-1}-1}{x}=-e\lim\limits_{x\to0}\dfrac{e^{\frac{\ln(1+x)-x}{x}}-1}{x}\overset{t\to0\text{时}}{\underset{e^t-1\sim t}{=\!=\!=\!=}}-e\lim\limits_{x\to0}\dfrac{\frac{\ln(1+x)-x}{x}}{x}$

$=-e\lim\limits_{x\to0}\dfrac{\ln(1+x)-x}{x^2}\overset{\frac{0}{0}}{=\!=}-e\lim\limits_{x\to0}\dfrac{\frac{1}{1+x}-1}{2x}=-e\lim\limits_{x\to0}\dfrac{-x}{2x(1+x)}=\dfrac{1}{2}e$.

(3).令$y=\left(\dfrac{\left(1+\frac{1}{x}\right)^x}{e}\right)^x$, 则 $\lim\limits_{x\to+\infty}\ln y=\lim\limits_{x\to+\infty}x\left[x\ln\left(1+\dfrac{1}{x}\right)-1\right]\overset{\frac{1}{x}=t}{=\!=\!=}\lim\limits_{t\to0}\dfrac{\ln(1+t)-t}{t^2}=\lim\limits_{t\to0}\dfrac{\frac{1}{1+t}-1}{2t}=-\dfrac{1}{2}$,

$\therefore$ 原式$=\lim\limits_{x\to+\infty}e^{\ln y}=e^{-\frac{1}{2}}$.

2. (1). 讨论函数$f(x)=(x-2)\sqrt[3]{x^2}$ 的极值情况.

解 $f'(0)=\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to0}\dfrac{(x-2)\sqrt[3]{x^2}}{x}=\infty$, $f'(0)$不存在.

$f'(x)=\sqrt[3]{x^2}+(x-2)\cdot\dfrac{2}{3}\cdot\dfrac{1}{\sqrt[3]{x}}=\dfrac{5x-4}{3\sqrt[3]{x}}$, $\cdots$, $x=0,\dfrac{4}{5}$都是极值点$\cdots$

2.(2). 讨论函数$f(x)=\dfrac{2x}{1+x^2}$ 的单调性,极值情况,凹凸区间,给出曲线$y=f(x)$的拐点.

解 普通问题正常做… $f''(x)=\dfrac{-4x\left(3-x^2\right)}{\left(1+x^2\right)^3}$,拐点$\left(-\sqrt{3},-\dfrac{\sqrt{3}}{2}\right)$,$(0,0)$,$\left(\sqrt{3},\dfrac{\sqrt{3}}{2}\right)$.

2.(3). 给出曲线$y=\sqrt[3]{x}$ 的拐点.

解 $f''(0)$不存在,$x=0$两侧$f''(x)$变号,$(0,0)$是拐点.

3. 设在$[0,c]$上 $f(x)$二阶可导且为凸函数， $f(0)=0$，证明：当$0\le a\le b\le a+b\le c$时有

$f(a+b)\ge f(a)+f(b)$.

解 $[0,c]$上$f(x)$凸, $f(0)=0, 0\le a\le b\le a+b\le c$, $f(a+b)\ge f(a)+f(b)\Leftrightarrow$
$f(a+b)-f(b)\ge f(a)-f(0)\Leftrightarrow f(a+b)-f(b)\ge f(a)-f(0)\Leftrightarrow f'(\xi)\ge f'(\eta)$,
$0\le a<\eta<b<\xi<a+b$, $f''(x)\ge 0$.

4. 给出函数 $f(x)=xe^{-x^2}$ 的带 Peano 型余项的 Maclaurin 展开式，给出 $f^{(9)}(0), f^{(10)}(0)$.

解 $e^t=1+t+\dfrac{t^2}{2!}+\cdots+\dfrac{t^n}{n!}+o\left(t^n\right), f(x)=xe^{-x^2}=x-x^3+\dfrac{x^5}{2!}+\cdots+\dfrac{(-1)^n}{n!}\cdot x^{2n+1}+o\left(x^{2n+1}\right)$.

又$f(x)=f(0)+f'(0)x+\dfrac{f''(0)}{2!}x^2+\dfrac{f'''(0)}{3!}x^3+\cdots+\dfrac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}+o\left(x^{2n+1}\right)$,

$\therefore f^{(2n)}(0)=0, \dfrac{f^{(2n+1)}(0)}{(2n+1)!}=\dfrac{(-1)^n}{n!}$,即$f^{(2n+1)}(0)=(-1)^n\dfrac{(2n+1)!}{n!}$.

5. 设 $x>0$, 证明: $e^x>1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}$.

解 $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots+\dfrac{x^n}{n!}+\dfrac{e^{\theta x}}{(n+1)!}x^{n+1}, \theta\in(0,1)$.

$x>0, R_n(x)=\dfrac{e^{\theta x}}{(n+1)!}x^{n+1}>0, \Rightarrow \forall n\in\mathbb{N}, e^x>1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots+\dfrac{x^n}{n!}$.

一种小变化: $e^x=1+x+\dfrac{x^2}{2!}+\cdots+\dfrac{x^{n-1}}{(n-1)!}+\dfrac{e^{\theta x}}{n!}x^n, \theta\in(0,1), \because x>0, \therefore e^{\theta x}>1$.

$\therefore e^x=1+x+\dfrac{x^2}{2!}+\cdots+\dfrac{x^{n-1}}{(n-1)!}+\dfrac{e^{\theta x}}{n!}x^n>1+x+\dfrac{x^2}{2!}+\cdots+\dfrac{x^{n-1}}{(n-1)!}+\dfrac{x^n}{n!}$.

注: 用函数e^x的带*Peano*型余项的*Maclaurin*展开式来证明是不充分的,因为*Peano*型余项是一种定性而非定量的表达式.

法二,使用单调性,设$\varphi(x)=e^x-\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}\right), \varphi(0)=0,\cdots$

法二 设$\varphi(x)=e^x-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right),\varphi'(x)=e^x-\left(1+x+\frac{x^2}{2!}\right),$

$\varphi''(x)=e^x-(1+x),\varphi'''(x)=e^x-1.$ $\varphi(x)$各阶导数都连续.

$x>0,\varphi'''(x)>0,\Rightarrow\varphi''(x)$递增,$\therefore x>0,\varphi''(x)(x)>\varphi''(0)=0,\Rightarrow\varphi'(x)$递增,

$\therefore x>0,\varphi'(x)>\varphi'(0)=0,\Rightarrow\therefore x>0,\varphi(x)$递增,$\Rightarrow\therefore x>0,\varphi(x)>\varphi(0)=0.$ 证毕

6. 证明：在$x>0$时函数$f(x)=\left(1+\frac{1}{x}\right)^x$ 严格单调递增 .

解 $f(x)=\left(1+\frac{1}{x}\right)^x=e^{x\ln\left(1+\frac{1}{x}\right)},f'(x)=\left(1+\frac{1}{x}\right)^x\left[x\ln\left(1+\frac{1}{x}\right)\right]'=\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right].$

$x>0,\ln\left(1+\frac{1}{x}\right)=\ln(1+x)-\ln x=\frac{1}{\xi}$, $x<\xi<1+x,$

$\therefore x>0,f'(x)=\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]>0,f(x)\nearrow.$

7. 求证：$x>0,x-\frac{x^2}{2}<\ln(1+x)<x$；

8. 设a为常数，求证：$\lim\limits_{n\to\infty}n^2\left(\arctan\frac{a}{n}-\arctan\frac{a}{n+1}\right)=a$.

解 (1). $a=0$时,原式$=0.$

(2). $a\neq0$时,法1:由*Lagrange*中值定理知$\left(\arctan\frac{a}{n}-\arctan\frac{a}{n+1}\right)=\left(\frac{a}{n}-\frac{a}{n+1}\right)\frac{1}{1+\xi^2}$,$\xi$介于$\frac{a}{n+1},\frac{a}{n}$间,

$$\therefore\lim_{n\to\infty}\left[n^2\left(\arctan\frac{a}{n}-\arctan\frac{a}{n+1}\right)\right]=\lim_{n\to\infty}\left[n^2\left(\frac{a}{n}-\frac{a}{n+1}\right)\frac{1}{1+\xi^2}\right]=\lim_{n\to\infty}\left[\frac{n^2a}{n(n+1)}\cdot\frac{1}{1+\xi^2}\right]=a.$$

法2:记$\arctan\frac{a}{n}=\alpha,\ \arctan\frac{a}{n+1}=\beta,$ 由$\tan(\alpha-\beta)=\dfrac{\frac{a}{n}-\frac{a}{n+1}}{1+\frac{a}{n}\cdot\frac{a}{n+1}}=\dfrac{a}{n^2+n+a^2}$得

$$\therefore\lim_{n\to\infty}\left[n^2\left(\arctan\frac{a}{n}-\arctan\frac{a}{n+1}\right)\right]=\lim_{n\to\infty}\left(n^2\arctan\frac{a}{n^2+n+a^2}\right)=\lim_{n\to\infty}\frac{a\cdot n^2}{n^2+n+a^2}=a.$$

法3：根据归结原则(*Heine*定理)，可用*L'Hopital*法则.

$$\text{原式}=\lim_{x\to+\infty}\left[x^2\left(\arctan\frac{a}{x}-\arctan\frac{a}{x+1}\right)\right]\overset{\frac{1}{x}=t}{=\!=\!=}\lim_{t\to0}\frac{\arctan at-\arctan\frac{at}{1+t}}{t^2}$$

$$=\lim_{t\to0}\frac{\frac{a}{1+a^2t^2}-\frac{1}{1+\left(\frac{at}{1+t}\right)^2}\cdot\frac{a(1+t)-at}{(1+t)^2}}{2t}=\lim_{t\to0}\frac{\frac{a}{1+a^2t^2}-\frac{a}{(1+t)^2+a^2t^2}}{2t}$$

$$=a\lim_{t\to0}\frac{1+2t+t^2+a^2t^2-1-a^2t^2}{2t\cdot(1+a^2t^2)\left[(1+t)^2+a^2t^2\right]}=a\lim_{t\to0}\frac{2t+t^2}{2t}\cdot\frac{1}{(1+a^2t^2)\left[(1+t)^2+a^2t^2\right]}=a.$$

法4：使用无穷小量的等价替换(这里是无穷小量加减运算时用了等价替换!)

$$\lim_{n\to\infty}\left[n^2\left(\arctan\frac{a}{n}-\arctan\frac{a}{n+1}\right)\right]=\lim_{n\to\infty}\left[n^2\left(\frac{a}{n}-\frac{a}{n+1}\right)\right]=\lim_{n\to\infty}\frac{n^2a}{n(n+1)}=a.$$

9. 证明可导的奇函数的导函数是偶函数.又问：连续的偶函数的原函数是奇函数吗?

解 若函数$f(x)$在$(-\infty,+\infty)$上可导且为奇函数，$f(-x)=-f(x)$，

$\therefore\ f'(-x)(-1)=-f'(x)$，即$f'(-x)=f'(x)$，故$f'(x)$为偶函数.

同理，若$f(x)$为$(-\infty,+\infty)$上的偶函数，$f(-x)=f(x)\Rightarrow f'(-x)(-1)=f'(x)$，

则$f'(-x)=-f'(x)$，即$f'(x)$为奇函数.

法二 用导数定义证明：若函数$f(x)$在$(-\infty,+\infty)$上可导且为奇函数，$f(-x)=-f(x)$，

$$\text{则}f'(-x)=\lim_{h\to0}\frac{f(-x+h)-f(-x)}{h}=-\lim_{h\to0}\frac{f(x-h)-f(x)}{h}=\lim_{h\to0}\frac{f(x+(-h))-f(x)}{-h}=f'(x),$$

$\therefore$若$f(x)$是$(-\infty,+\infty)$上的奇函数，则$f'(x)$为偶函数.

$\longrightarrow(-\infty,+\infty)$上连续偶函数的原函数未必是奇函数，如 $1+x^3$ 是$3x^2$ 的一个原函数，不是奇函数.

偶函数的原函数$F(x)$若有$F(0)=0$，则是奇函数.

10. 若$x\ln x$是函数$f(x)$的一个原函数，问$\int xf'(2x)dx=?$

解 考察原函数概念，换元积分法与分部积分法．

$\because\int f(x)dx=x\ln x+C_1$，$\therefore\ f(x)=(x\ln x)'=1+\ln x$.为避免出错，先行换元$\int xf'(2x)dx$

$$\overset{2x=t}{=\!=\!=}\frac{1}{4}\int tf'(t)dt=\frac{1}{4}tf(t)-\frac{1}{4}\int f(t)dt=\frac{1}{4}t(1+\ln t)-\frac{1}{4}t\ln t+C=\frac{1}{4}t+C=\frac{1}{2}x+C\ .$$

11.计算不定积分

(1).$\int\frac{dx}{x+\sqrt{1-x^2}}$；　(2).$\int\sqrt{a^2-x^2}dx$；　(3).$\int\frac{dx}{1+\sqrt{2x}}$；　(4).$\int(x\ln x)^2dx$；　(5).$\int e^{-\sqrt[3]{x}}dx$；

(6).$\int\frac{\arctan x}{\sqrt{(1+x^2)^3}}dx$；(7).$\int\sqrt{e^x-1}\ dx$.

$$11.(1).\int\frac{dx}{x+\sqrt{1-x^2}}\xlongequal[t\in(-\pi/2,\pi/2)]{x=\sin t}\int\frac{\cos t}{\sin t+\cos t}dt=\int\frac{\cos t}{\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)}dt\xlongequal{t+\frac{\pi}{4}=s}\int\frac{\cos\left(s-\frac{\pi}{4}\right)}{\sqrt{2}\sin s}ds$$

$$=\int\frac{\cos s\cos\frac{\pi}{4}+\sin s\sin\frac{\pi}{4}}{\sqrt{2}\sin s}ds=\frac{1}{2}\int\left(\frac{\cos s}{\sin s}+1\right)ds=\frac{1}{2}\left(\ln|\sin s|+s\right)+C_1=\frac{1}{2}\left(\ln\left|\sin\left(t+\frac{\pi}{4}\right)\right|+t+\frac{\pi}{4}\right)+C_1$$

$$=\frac{1}{2}\left(\ln|\sin t+\cos t|+t\right)+C=\frac{1}{2}\left(\ln\left|x+\sqrt{1-x^2}\right|+\arcsin x\right)+C.$$

解二 $\int\frac{dx}{x+\sqrt{1-x^2}}\xlongequal[t\in(-\pi/2,\pi/2)]{x=\sin t}\int\frac{\cos t}{\sin t+\cos t}dt=\frac{1}{2}\int\frac{\sin t+\cos t+(\cos t-\sin t)}{\sin t+\cos t}dt=\frac{1}{2}t+\frac{1}{2}\int\frac{d(\sin t+\cos t)}{\sin t+\cos t}$

$$=\frac{1}{2}\left(\ln|\sin t+\cos t|+t\right)+C=\frac{1}{2}\left(\ln\left|x+\sqrt{1-x^2}\right|+\arcsin x\right)+C.$$

$$(2).\int\sqrt{a^2-x^2}dx\xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=a\sin t}\int|a\cos t|\cdot a\cos tdt=a^2\int\cos^2tdt=\frac{1}{2}a^2\int(1+\cos 2t)dt$$

$$=\frac{1}{2}a^2\left(t+\frac{1}{2}\sin 2t\right)+C=\frac{1}{2}a^2t+\frac{1}{2}a^2\sin t\cos t+C=\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C.$$

$$(3).\int\frac{dx}{1+\sqrt{2x}}\xlongequal[dx=tdt]{\sqrt{2x}=t}\int\frac{t}{1+t}dt=\int\left(1-\frac{1}{1+t}\right)dt=t-\ln(1+t)+C=\sqrt{2x}-\ln\left(1+\sqrt{2x}\right)+C.$$

$$(4).\int(x\ln x)^2dx=\int x^2\ln^2xdx=\int\left(\frac{1}{3}x^3\right)'\ln^2xdx=\frac{1}{3}x^3\ln^2x-\frac{1}{3}\int x^3\cdot 2\ln x\cdot\frac{1}{x}dx=\frac{1}{3}x^3\ln^2x-\frac{2}{3}\int x^2\ln xdx$$

$$=\frac{1}{3}x^3\ln^2x-\frac{2}{3}\left[\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^3\cdot\frac{1}{x}dx\right]=\frac{1}{3}x^3\ln^2x-\frac{2}{9}x^3\ln x+\frac{2}{27}x^3+C.$$

$$(5).\int t^2e^{-t}dt=\int t^2\left(-e^{-t}\right)'dt=-t^2e^{-t}+\int e^{-t}\cdot 2tdt=-t^2e^{-t}-2te^{-t}+2\int e^{-t}dt=-\left(t^2+2t+2\right)e^{-t}+C,$$

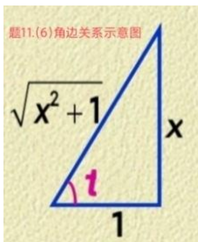
$$\therefore\int e^{-\sqrt[3]{x}}dx\xlongequal[dx=3t^2dt]{\sqrt[3]{x}=t}3\int t^2e^{-t}dt=-3\left(t^2+2t+2\right)e^{-t}+C=-3\left(\sqrt[3]{x^2}+2\cdot\sqrt[3]{x}+2\right)e^{-\sqrt[3]{x}}+C.$$

$$(6).\int\frac{\arctan x}{\sqrt{\left(1+x^2\right)^3}}dx\xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{\arctan x=t}\int\frac{t}{|\sec t|^3}\cdot\sec^2tdt=\int t\cos tdt=t\sin t-\int\sin tdt=t\sin t+\cos t+C$$

$$=\frac{x\arctan x+1}{\sqrt{1+x^2}}+C.$$

$$(7).\int\sqrt{e^x-1}\,dx\xlongequal[x=\ln\left(1+t^2\right)]{\sqrt{e^x-1}=t}\int t\cdot\frac{2t}{1+t^2}dt=2\int\left(1-\frac{1}{1+t^2}\right)dt=2t-2\arctan t+C$$

$$=2\sqrt{e^x-1}-2\arctan\sqrt{e^x-1}+C.$$



12. 不等式证明一箩筐:

(1). $\left|\sin x-\sin y\right|\le\left|x-y\right|$;$\longleftarrow$分别就$x=y,x\ne y$来说明,$Lagrange\ th.$,或用和差化积.

(2). $\forall x\in\mathbb{R},e^{x}\ge 1+x$;$\longleftarrow$可用$Lagrange\ th.$,单调性,极值,凸函数等方法处理.

(3). $x>0,\ln x\le x-1$;$\longleftarrow$可用$Lagrange\ th.$,单调性,极值,函数凹凸性等法处理.

(4). $0<x<\dfrac{\pi}{2},\dfrac{x}{\sin x}<\dfrac{\tan x}{x}$;$\longleftarrow$单调性是容易想到的,需变形.

(5). $x<1$ 时有$e^{x}\le\dfrac{1}{1-x}$;$\longleftarrow$可用极值/最值法处理.变形后与(2).(3)同.用函数图形法亦可.

$(1-x)e^{x}\le 1\leftarrow$ 求$x<1$ 时函数$(1-x)e^{x}$最大值;或变形为$e^{-x}\ge 1-x$,一个熟悉的问题…

(6). $x\in[0,1],p\ge 1,\dfrac{1}{2^{p-1}}\le x^{p}+(1-x)^{p}\le 1$;$\longleftarrow$可用最值法处理.

(7). $a\ne b$,有 $e^{\frac{a+b}{2}}\le\dfrac{e^{b}-e^{a}}{b-a}\le\dfrac{e^{a}+e^{b}}{2}$.$\longleftarrow$可多法处理."降维'打击'"颇有力.

(7).分析:(A).$a\ne b,\dfrac{e^{b}-e^{a}}{b-a}\le\dfrac{e^{a}+e^{b}}{2}\Leftrightarrow\dfrac{e^{b-a}-1}{b-a}\le\dfrac{1+e^{b-a}}{2},b-a\triangleq t,$

$\Leftrightarrow t\ne 0,\dfrac{e^{t}-1}{t}\le\dfrac{1+e^{t}}{2}$ …这种做法形象地称为"降维'打击'".