Power

Statistical Inference

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Power

- Power is the probability of rejecting the null hypothesis when it is false
- Ergo, power (as it's name would suggest) is a good thing; you want more power
- A type II error (a bad thing, as its name would suggest) is failing to reject the null hypothesis when it's false; the probability of a type II error is usually called β
- Note Power $= 1 - \beta$

Notes

- Consider our previous example involving RDI
- $H_0 : \mu = 30$ versus $H_a : \mu > 30$
- Then power is

$$P\left(\frac{\bar{X} - 30}{s/\sqrt{n}} > t_{1-\alpha,n-1} \mid \mu = \mu_a\right)$$

- Note that this is a function that depends on the specific value of μ_a!
- Notice as μ_a approaches 30 the power approaches α

Calculating power for Gaussian data

Assume that n is large and that we know σ

$$
\begin{aligned}
1-\beta &= P\left(\frac{\bar{X}-30}{\sigma/\sqrt{n}} > z_{1-\alpha} \mid \mu=\mu_a\right) \\
&= P\left(\frac{\bar{X}-\mu_a+\mu_a-30}{\sigma/\sqrt{n}} > z_{1-\alpha} \mid \mu=\mu_a\right) \\
&= P\left(\frac{\bar{X}-\mu_a}{\sigma/\sqrt{n}} > z_{1-\alpha} - \frac{\mu_a-30}{\sigma/\sqrt{n}} \mid \mu=\mu_a\right) \\
&= P\left(Z > z_{1-\alpha} - \frac{\mu_a-30}{\sigma/\sqrt{n}} \mid \mu=\mu_a\right)
\end{aligned}
$$

Example continued

- Suppose that we wanted to detect a increase in mean RDI of at least 2 events / hour (above 30).
- Assume normality and that the sample in question will have a standard deviation of 4;
- What would be the power if we took a sample size of 16?
  - $Z_{1-\alpha} = 1.645$
  - $\frac{\mu_a - 30}{\sigma / \sqrt{n}} = 2 / (4 / \sqrt{16}) = 2$
  - $P(Z > 1.645 - 2) = P(Z > -0.355) = 64\%$

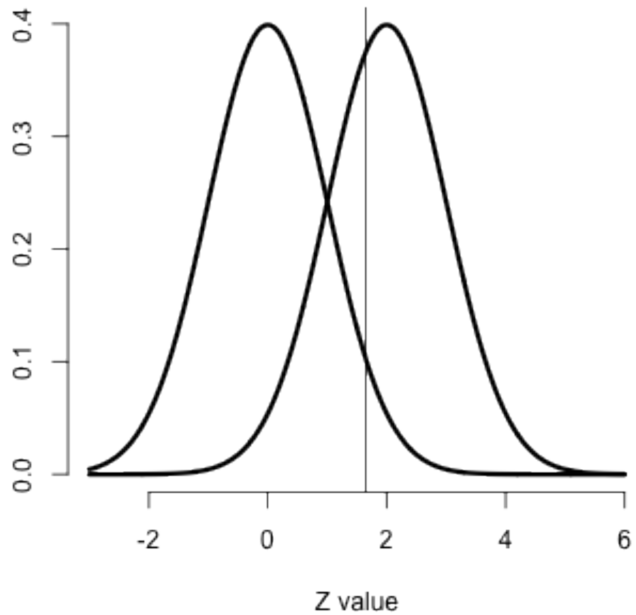
```
pnorm(-0.355, lower.tail = FALSE)
```

```
## [1] 0.6387
```

Note

- Consider $H_0 : \mu = \mu_0$ and $H_a : \mu > \mu_0$ with $\mu = \mu_a$ under H_a.
- Under H_0 the statistic $Z = \frac{\sqrt{n}(\bar X - \mu_0)}{\sigma}$ is $N(0, 1)$
- Under H_a Z is $N\left(\frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma}, 1\right)$
- We reject if $Z > Z_{1-\alpha}$

```r
sigma <- 10; mu_0 = 0; mu_a = 2; n <- 100; alpha = .05
plot(c(-3, 6),c(0, dnorm(0)), type = "n", frame = FALSE, xlab = "Z value", ylab = "")
xvals <- seq(-3, 6, length = 1000)
lines(xvals, dnorm(xvals), type = "l", lwd = 3)
lines(xvals, dnorm(xvals, mean = sqrt(n) * (mu_a - mu_0) / sigma), lwd =3)
abline(v = qnorm(1 - alpha))
```


0.0
0.1
0.2
0.3
0.4
-2
0
2
4
6
Z value

Question

- When testing $H_a : \mu > \mu_0$, notice if power is $1 - \beta$, then

$$1 - \beta = P\left(Z > z_{1-\alpha} - \frac{\mu_a - \mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_a\right) = P(Z > z_\beta)$$

- This yields the equation

$$z_{1-\alpha} - \frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma} = z_\beta$$

- Unknowns: μ_a, σ, n, β
- Knowns: μ_0, α
- Specify any 3 of the unknowns and you can solve for the remainder

Notes

- The calculation for $H_a : \mu < \mu_0$ is similar
- For $H_a : \mu \neq \mu_0$ calculate the one sided power using $\alpha / 2$ (this is only approximately right, it excludes the probability of getting a large TS in the opposite direction of the truth)
- Power goes up as α gets larger
- Power of a one sided test is greater than the power of the associated two sided test
- Power goes up as μ_1 gets further away from μ_0
- Power goes up as n goes up
- Power doesn't need μ_a, σ and n, instead only $\frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma}$
  - The quantity $\frac{\mu_a - \mu_0}{\sigma}$ is called the effect size, the difference in the means in standard deviation units.
  - Being unit free, it has some hope of interpretability across settings

T-test power

- Consider calculating power for a Gossett's T test for our example
- The power is

$$P\left(\frac{\bar{X} - \mu_0}{S/\sqrt{n}} > t_{1-\alpha,n-1} ~|~ \mu = \mu_a\right)$$

- Calcuting this requires the non-central t distribution.
- `power.t.test` does this very well
  - Omit one of the arguments and it solves for it

Example

```
power.t.test(n = 16, delta = 2/4, sd = 1, type = "one.sample", alt = "one.sided")$power
```

```
## [1] 0.604
```

```
power.t.test(n = 16, delta = 2, sd = 4, type = "one.sample", alt = "one.sided")$power
```

```
## [1] 0.604
```

```
power.t.test(n = 16, delta = 100, sd = 200, type = "one.sample", alt = "one.sided")$power
```

```
## [1] 0.604
```

Example

```
power.t.test(power = 0.8, delta = 2/4, sd = 1, type = "one.sample", alt = "one.sided")$n
```

```
## [1] 26.14
```

```
power.t.test(power = 0.8, delta = 2, sd = 4, type = "one.sample", alt = "one.sided")$n
```

```
## [1] 26.14
```

```
power.t.test(power = 0.8, delta = 100, sd = 200, type = "one.sample", alt = "one.sided")$n
```

```
## [1] 26.14
```