# EXERCISE 1 BASIC MATHEMATICAL KNOWLEDGE REQUIRED FOR CONDUCTING THE GRAPHIC SYSTEMS COURSE

The basic mathematical knowledge required for a computer graphics course usually includes the following topics:

**1. Algebra**

- **Systems of linear equations** – for understanding transformations and calculations in 3D space**.**
- **Scalars and vectors** – for working with directions, distances, and motion.
- **Matrices and determinants:**
  - Matrix addition, multiplication, and transformations.
  - Inverse matrix – for applying inverse transformations.

**2. Geometry**

- **Plane geometry**:
  - Working with points, lines, angles, polygons, and circles.
- **Solid geometry**:
  - Working with 3D objects such as spheres, cubes, cones, and pyramids.
- **Vector geometry**:
  - Dot product – for calculating angles between vectors.
  - Cross product – for calculating surface normal vectors.

**3. Trigonometry**

- **Sine, cosine, tangent** – for rotations, projections, and angle calculations.
- **Pythagorean theorem** – for distances between points.
- **Periodic functions** – for animations and working with waves.

**4. Linear algebra**

- **Spaces and bases**:
  - Coordinate systems (Cartesian, normalized, and homogeneous).
- **Transformations**:
  - Transformations such as scaling, rotation, translation, and shearing.

- **Homogeneous coordinates** – for representing perspective transformations.

**5. Analytic geometry**

- **Equations of lines and planes** – for describing objects in space.
- **Distance between points, lines, and planes**.
- **Parametric equations** – for animations and describing curves.

**6. Differential and integral calculus**

- **Differentiation** – for determining slopes, velocity, and optimization.
- **Integration** – for calculating areas and volumes.

**7. Mathematical logic**

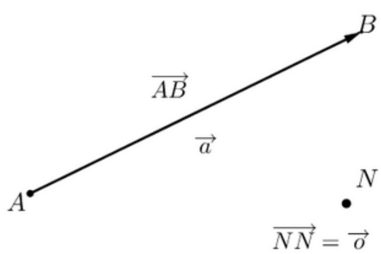
- Basic understanding of Boolean logic and bitwise operations – often used in graphics algorithms.

All this knowledge has been acquired in the Mathematics – Part 1 and Part 2 courses at TU-Varna. In the present exercise, we will review some fundamental concepts that will be necessary throughout the entire course.

## VECTORS

A directed line segment in space defined by two points:

- Initial (base) point – this is the point from which the vector originates;
- Terminal point – the point at which the vector ends.




It is represented as an arrow, where:

- The length of the arrow represents the magnitude (length) of the vector;
- The direction of the arrow indicates the orientation of the vector.

Main characteristics of a vector:

- Magnitude (length) of a vector – the length of the directed line segment. If the vector is $\vec{v}$, it is calculated as follows (in 2D and 3D respectively):

$$|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Geometrically, the magnitude is the distance between the initial and the terminal point.

- Direction of a vector – this is the orientation of the vector relative to the coordinate system. It can be expressed by an angle with respect to the positive direction of the x-axis (in 2D) or by a direction in space (in 3D).

**Equal (identical) vectors** – when they have the same magnitude and direction, regardless of their initial point.

**Opposite vectors** – the vector $-\vec{v}$ is opposite to $\vec{v}$, if $\vec{v} \neq \vec{0}$, the two vectors have the same magnitude and opposite directions.

**Unit vector** - $|\vec{v}| = 1$

**Zero vector** - $\vec{0}$ – has length 0 and no defined direction.

## Vector in a coordinate system

Vectors play a key role in graphics systems, as they allow quantities with both magnitude and direction to be described. In a coordinate system, vectors are represented by numbers that define their position and orientation in space.

The vector $\vec{v}$ in the Cartesian coordinate system is defined as:

- A directed line segment with a specified initial point and terminal point.
- A point in space (in its final form).

Let us consider:

- The initial point of the vector is $A(x_1, y_1, z_1)$
- The terminal point is $B(x_2, y_2, z_2)$

Then the coordinates of the vector are the difference between the coordinates of the terminal point and the initial point:

$$\vec{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Example: if the initial point is A(1, 2, 3) and the terminal point is B(4, 6, 8), the vector is:

$$\vec{v} = (4 - 1, 6 - 2, 8 - 3) = (3, 4, 5)$$

### Vector normalization

Vector normalization means transforming a vector into a vector of length 1 (a unit vector) while preserving its direction. This is important for applications such as determining directions and lighting calculations in graphics systems.

The formula is:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

Example:

If $\vec{v} = (3,4)$, then:

- The length of the vector is 5;
- The normalized vector is:

$$\vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right) = (0.6, 0.8)$$

## Vector operations

### Vector addition

Vector addition is a fundamental operation in linear algebra and has important applications in graphics systems, physics, and computational geometry. It can be represented both algebraically and geometrically:

If we have two vectors $\vec{u}$ and $\vec{v}$:

- In two-dimensional space:
  $\vec{u} = (u_x, u_y), \qquad \vec{v} = (v_x, v_y)$
- In three-dimensional space:
  $\vec{u} = (u_x, u_y, u_z), \qquad \vec{v} = (v_x, v_y, v_z)$

The addition of vectors is performed by adding their corresponding components:

- In two-dimensional space:
  $\vec{u} + \vec{v} = (u_x + v_x, u_y + v_y)$
- In three-dimensional space:
  $\vec{u} + \vec{v} = (u_x + v_x, u_y + v_y, u_z + v_z)$

  Example:
- In two-dimensional space:
  $\vec{u} = (2, 3), \qquad \vec{v} = (4, 1)$
  $\vec{u} + \vec{v} = (2 + 4, 3 + 1) = (6,4)$
- In three-dimensional space:
  $\vec{u} = (1, 2, 3), \qquad \vec{v} = (4, 5, 6)$

$$\vec{u} + \vec{v} = (1 + 4, 2 + 5, 3 + 6) = (5, 7, 9)$$

## Vector subtraction

- Algebraic vector subtraction

If we have two vectors $\vec{u}$ and $\vec{v}$:

- In two-dimensional space:
  $\vec{u} = (u_x, u_y), \qquad \vec{v} = (v_x, v_y)$
- In three-dimensional space:
  $\vec{u} = (u_x, u_y, u_z), \qquad \vec{v} = (v_x, v_y, v_z)$

Vector subtraction is performed by subtracting their corresponding components:

- In two-dimensional space:
  $\vec{u} - \vec{v} = (u_x - v_x, u_y - v_y)$
- In three-dimensional space:
  $\vec{u} - \vec{v} = (u_x - v_x, u_y - v_y, u_z - v_z)$

Example:

- In two-dimensional space:
  $\vec{u} = (4, 6), \qquad \vec{v} = (1, 3)$
  $\vec{u} - \vec{v} = (4 - 1, 6 - 3) = (3, 3)$

- In three-dimensional space:
  $\vec{u} = (5, 8, 2), \qquad \vec{v} = (3, 4, 1)$
  $\vec{u} - \vec{v} = (5 - 3, 8 - 4, 2 - 1) = (2, 4, 1)$

## Vector multiplication

- Multiplication of a vector by a scalar

The multiplication of a vector by a number (scalar) is an operation in which the vector is scaled by multiplying each of its components by that number. This is a fundamental operation in linear algebra that preserves the direction of the vector but changes its magnitude.

If we have a vector $\vec{v}$:

- In two-dimensional space:
  $\vec{v} = (v_x, v_y)$
- In three-dimensional space:
  $\vec{v} = (v_x, v_y, v_z)$

and a scalar λ (a number), then the product of $\vec{v}$ and λ is a new vector

- In two-dimensional space:
  $\lambda.\vec{v} = (\lambda.v_x, \lambda.v_y)$
- In three-dimensional space:

$$\overrightarrow{\lambda.v} = (\lambda.v_x, \lambda.v_y, \lambda.v_z)$$

Multiplying a vector by a scalar scale its magnitude. If the scalar is negative, the direction of the vector is reversed. If the scalar is 0, the result is zero vector.

Vectors can participate in two main types of products:

- **Dot product** (the result is a scalar).
- **Cross product** (the result is a new vector) – only in three-dimensional space.

Both types of products have different geometric and physical meanings and are used in different contexts.

## Dot product

Two vectors are given: $\vec{u}(u_1,\ u_2,\ u_3)$ и $\vec{v}(v_1,\ v_2,\ v_3)$

- Geometric formula:

The dot product of two vectors $\vec{u}$ and $\vec{v}$ is a scalar value calculated as the product of the magnitudes of the two vectors and the cosine of the angle between them:

$$\vec{u}.\vec{v} = |\vec{u}|.\,|\vec{v}|.\cos(\theta)$$

Where:

$|\vec{u}|$ and $|\vec{v}|$ are the magnitudes of the vectors;

$\theta$ is the angle between them $(0^o \le \theta \le 180^o)$

The dot product measures the projection of one vector onto the other, multiplied by the magnitude of the other vector.

- Analytical formula:

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Example: The following vectors are given $\vec{u} = (5, 8, 2)$, $\vec{v} = (3, 4, 1)$

$$|\vec{u}| = \sqrt{5^2 + 8^2 + 2^2} = \sqrt{93}$$
$$|\vec{v}| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$$
$$\vec{u}.\vec{v} = 5 * 3 + 8 * 4 + 2 * 1 = 49$$
$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|\vec{u}|.\,|\vec{v}|} = \frac{49}{\sqrt{93} * \sqrt{26}} = 0.99648$$

$$\theta = arccos(0.99648) = 4.81^o$$

## Cross product

The following vectors are given: $\vec{u}(u_1,\ u_2,\ u_3)$ и $\vec{v}(v_1,\ v_2,\ v_3)$

- Geometric expression:

The cross product of two vectors $\overrightarrow{u}$ and $\vec{v}$ is a vector that is perpendicular to the plane formed by $\overrightarrow{u}$ and $\vec{v}$. Its magnitude is determined by the formula:

$$\overrightarrow{u} \times \vec{v} = |\overrightarrow{u}|.\,|\overrightarrow{v}|.\sin(\theta)$$

Where:

$|\overrightarrow{u}|$ and $|\overrightarrow{v}|$ are the magnitudes of the vectors;

$\theta$ is the angle between them $(0^o \leq \theta \leq 180^o)$

- Analytical expression – determinant form:

$$\overrightarrow{u} \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where $\vec{\imath}, \vec{\jmath}$, and $\vec{k}$ are the unit basis vectors along the axes of an arbitrary Cartesian coordinate system.

$$\overrightarrow{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{\imath} + (u_3v_1 - u_1v_3)\vec{\jmath} + (u_1v_2 - u_2v_1)\vec{k}$$

Example: The following vectors are given $\vec{u} = (5, 8, 2)$, $\vec{v} = (3, 4, 1)$

$$\overrightarrow{u} \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 5 & 8 & 2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$\overrightarrow{u} \times \vec{v} = (8 * 1 - 2 * 4)\vec{\imath} + (2 * 3 - 5 * 1)\vec{\jmath} + (5 * 4 - 8 * 3)\vec{k} = 0\vec{\imath} + 1\vec{\jmath} - 4\vec{k} \quad => (0, 1, -4)$$

# MATRICES

Matrices are a fundamental tool in mathematics and are widely used in computer graphics. They represent ordered tables of numbers arranged in rows and columns, and they allow efficient handling of multiple equations and transformations.

## Main characteristics

1. Rows and columns
    - Matrix A has m rows and n columns
    - Its size is written as m × n
2. Elements of the matrix
    - Each element $a_{ij}$ is located in the i-th row and the j-th column.

    Example – a matrix of size 2 × 3:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

## Types of matrices

Square matrix – a matrix with an equal number of rows and columns (m × n)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Diagonal matrix – a square matrix in which only the elements on the main diagonal are different from 0

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Identity matrix – a square matrix in which all elements on the main diagonal are 1 and all other elements are 0.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Zero matrix – a matrix in which all elements are 0.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row matrix (row vector) – a matrix that has only one row

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Column matrix (column vector) – a matrix that has only one column

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

## Matrix operations

1. Addition and subtraction – two matrices can be added/subtracted if they have the same dimensions (m × n)

   $A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$

   Example:

   $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

   $A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

2. Multiplication of a matrix by a scalar – each element of the matrix is multiplied by the number.

   $\lambda.A = \begin{bmatrix} \lambda.a_{11} & \lambda.a_{12} \\ \lambda.a_{21} & \lambda.a_{22} \end{bmatrix}$

3. Matrix multiplication – two matrices A (m × n) and B (n × p) can be multiplied if the number of columns of the first matrix equals the number of rows of the second matrix:

   $$C_{ij} = \sum_{k=1}^{n} A_{ik} * B_{kj}$$

   Example:

   $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

   $A.B = \begin{bmatrix} (1.5 + 2.7) & (1.6 + 2.8) \\ (3.5 + 4.7) & (3.6 + 4.8) \end{bmatrix}$

   Unlike ordinary numbers, in matrix multiplication **the order matters**.

If A and B are matrices, in the general case: **A · B ≠ B · A**. Even if the dimensions allow multiplication in both orders, the results may be different.

4. Matrix transposition – transforming rows into columns and columns into rows

   $A^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

5. Inverse matrix ($A^{-1}$) – a matrix for which:

   $A.\ A^{-1} = A^{-1}.\ A = I$

   where **I** is the identity matrix

Only square matrices can have an inverse matrix. In computer graphics, it is used to return objects to their original position.

If we have a matrix A:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The inverse matrix $A^{-1}$ is calculated as:

$A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Where $|A| = a.d - b.c$ (the determinant of $A$)

Example:

$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$

$|A| = 4.6 - 7.2 = 10$

$A^{-1} = \frac{1}{10} \cdot \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$

## Multiplication of a matrix by a column vector

The multiplication of a column vector by a matrix is a fundamental operation in linear algebra. This operation is of great importance in computer graphics. The result of the multiplication is a new column vector that represents a transformation of the original vector.

To multiply a matrix **A** by a column vector $\vec{v}$, the following conditions must be satisfied:

- The matrix must have dimensions **m × n** (m rows and n columns)
- he column vector must have **n rows** (n × 1)

The size of the result will be **m × 1**, i.e., a new column vector with m elements.

Let us consider:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The multiplication A . $\vec{v}$ is calculated as:

$$A \cdot \vec{v} = \begin{bmatrix} a_{11} \cdot v_1 + a_{12} \cdot v_2 + \cdots + a_{1n} \cdot v_n \\ a_{21} \cdot v_1 + a_{22} \cdot v_2 + \cdots + a_{2n} \cdot v_n \\ \vdots \\ a_{m1} \cdot v_1 + a_{m2} \cdot v_2 + \cdots + a_{mn} \cdot v_n \end{bmatrix}$$

Example:

$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$A.\vec{v} = \begin{bmatrix} 2.5 + 3.2 \\ 1.5 + 4.2 \end{bmatrix} = \begin{bmatrix} 16 \\ 13 \end{bmatrix}$

Geometrically, the multiplication of a column vector by a matrix represents a transformation of the vector into a new space. Examples of such transformations include:

- **Rotations:** Rotating a vector in 2D or 3D.
- **Scaling:** Increasing or decreasing the magnitude of the vector.

## Determinant

**The determinant** is a scalar value associated with a square matrix (a matrix with an equal number of rows and columns). It is useful for analyzing the properties of the matrix, such as whether it is invertible, for calculating areas or volumes, and for solving systems of linear equations. It is denoted as |A| or det(A).

Formulas for the determinant:

- For a 2 × 2 matrix:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = a.d - b.c$

- For a 3 × 3 matrix:

$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$|A| = a.(e.i - f.h) - b.(d.i - f.g) + c.(d.h - e.g)$

## EXERCISES

**Exercise 1**. Find:

а) A.P

Where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.1 \\ 1 \end{bmatrix}$$

б) A.P

Where:

$$A = \begin{bmatrix} \cos(45^o) & -\sin(45^o) & 0 & 0 \\ \sin(45^o) & \cos(45^o) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.2 \\ 1 \end{bmatrix}$$

в) $C = R_z * S$

Where:

$$S = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_z = \begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ) & 0 & 0 \\ \sin(90^\circ) & \cos(90^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us consider:

$$P = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.1 \\ 1 \end{bmatrix}$$

Find $P' = C.P$

г) $C = S * R_z$

Where:

$$S = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_z = \begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ) & 0 & 0 \\ \sin(90^\circ) & \cos(90^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us consider:

$$P = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.1 \\ 1 \end{bmatrix}$$

Find $P' = C.P$

**Exercise 2.** The following vectors are given:

1) $\vec{a}(2, -1, 2)$ and $\vec{b}(4, 4, -2)$
2) $\vec{a}(3, 4, -1)$ and $\vec{b}(5, -3, 2)$

Find:

a) their magnitudes
b) their dot product
c) the angle between them
d) their cross product