Calculation of Maxwell's Stress Tensor with FreeFem++

Computational Electromagnetism Project

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Overview

Outline

Motivation

- Electrostrictive materials - Electroactive polymers (Artificial Muscles)
- Oil and fuel purification

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Goals

- Anisotropic Electrostatics
- Electrostatic (Maxwell) stress coupled with elasticity
- Electrostatic forces moving a ball
- Gaining experience using open source Finite Element software for electromagnetics

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The Maxwell Stress Tensor

The Lorentz Force Law in the presence of an electrostatic field reads:

$$f = \rho \vec{E} \tag{1}$$

and taking into account

$$\nabla \vec{D} = \rho \tag{2}$$

we arrive at

$$\vec{f} = \nabla \cdot \tilde{\sigma} \tag{3}$$

with

$$\sigma_{ij} = -E_i D_j + \frac{1}{2}\delta_{ij}(E_k D_k) \tag{4}$$

The Maxwell Stress Tensor

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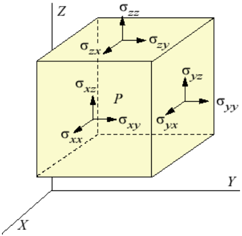


The Maxwell Stress Tensor

- For the general case of a inhomogeneous material $D_i = \epsilon_{ij} E_j$
- The relative permitivity is a tensor which is represented in matrix form as

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} \tag{5}$$

Outline

Strong Form

The strong form of the time independent electrostatics PDE reads

$$\nabla \cdot (\tilde{\epsilon} \nabla \Phi) + \rho = 0 \qquad \text{on} \quad V \tag{6a}$$

$$\Phi = 0, \qquad \text{on} \quad \Gamma_1 \tag{6b}$$

$$(\tilde{\epsilon} \nabla \Phi) \cdot n = g, \qquad \text{on} \quad \Gamma_2 \tag{6c}$$

Where $\Phi$is the electrostatic scalar potential defined by $\vec{E} = -\nabla \Phi$

Weak form of the problem

The weak form, after inserting to the scalar potential the test and trial functions, reads

$$\int_\Omega (\tilde{\epsilon} \nabla u) \nabla v dV = \int_{\Gamma_1} \rho v dV + \int_{\Gamma_2} g v dS \tag{7a}$$

$$v = \sum_i \psi_i \tag{7b}$$

$$u = \sum_i \psi_i w \tag{7c}$$

Formulation in FreeFem++

The variational problem is defined as:

```
problem PoissonVarf(phi,w,solver=CG) =
      int2d(Allh)(
       EPS[0][0] * dx(phi)*dx(w)
     + EPS[0][1] * dx(phi)*dy(w)
     + EPS[1][0] * dy(phi)*dx(w)
     + EPS[1][1] * dy(phi)*dy(w) )
     - int2d(Allh)( source*w )
     + on(Lboxup,phi = phiCup)
     + on(Lboxdown,phi = phiCdown)
    ;
  PoissonVarf;
```

Outline

Elastostatics

Stress - deformation relation:

We define u_i the deformation along direction i,σ_{ij} is the stress tensor and $e_{ij} = \frac{1}{2} u_i, j + u_j, i$is the strain tensor. Then,

$$\sigma_{ij} = C_{ijkl} e_{kl} \tag{8a}$$

$$\tag{8b}$$

where C_{ijkl} is the fourth order stiffness matrix that allows for the couling of any type of stress with any type of deformation (anisotropic materials).

Lamé system

We assumed the materials to be isotropic with respect to elastic response. That allowes them to be modeled with the Lamé system and that allows for a simpler variational formulation. For full anisotropy all the shape function derivatives have to be taken account of with possibly different material coefficients.

Lamé description of the elastic material response reads:

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \tag{9}$$

Formulation in FreeFem++

The elastostatics formulation was based in the lame system (isotropic finite strain elasticity). Example from freefem manual:

```
macro div(u,v) ( dx(u)+dy(v) ) // EOM
macro epsilon(u1,u2) [dx(u1),dy(u2),(dy(u1)+dx(u2))/sqrt2] // EOM
...
Vh f1 = dx(T11)+dy(T21);
Vh f2 = dx(T21)+dy(T22);
...
varf lame([u,v],[uu,vv])= int2d(Th)(
    (reg==regInternal?lambda1:lambda2)*div(u,v)*div(uu,vv)
    +2.*(reg==regInternal?mu1:mu2)* epsilont(u,v) * epsilon(uu,vv))
    + on(3,v=0,u=0);
```

Outline

Element forces

$$f = \nabla\sigma \qquad \Rightarrow \qquad \text{in 2 dim } f = \begin{pmatrix} \partial_x\sigma_{11} + \partial_y\sigma_{21} \\ \partial_x\sigma_{12} + \partial_y\sigma_{22} \end{pmatrix}$$

```
Vh fx = dx(T[0][0])+dy(T[1][0])
Vh fy = dx(T[0][1])+dy(T[1][1])
```

Erratic, inconsistent behaviour (Third order elements, various solvers)!

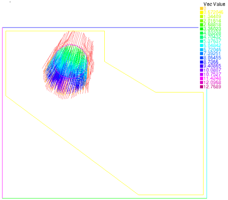
Element forces

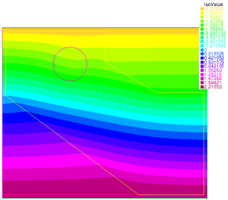
```
Vh fx = - EPS[0][0]*dxx(phi)*dx(phi) - EPS[1][1]*dyx(phi)*dy(phi)
        + 1/2*(EPS[1][0] - EPS[0][1])
            *( dx(phi)*dyx(phi) + dxx(phi)*dy(phi) )
        - EPS[0][0]*(dyy(phi)*dx(phi)+dy(phi)*dyx(phi))
        - EPS[0][1]*(2*dyy(phi)*dy(phi));
Vh fy = - EPS[1][0]*(dxx(phi)*dx(phi)*2)
        - EPS[1][1]*(dxx(phi)*dy(phi)+dx(phi)*dxy(phi))
        + EPS[0][0]*dx(phi)*dyx(phi) - EPS[1][1]*dy(phi)*dyy(phi)
        + 1/2*(EPS[0][1] - EPS[1][0])
            *( dx(phi)*dyy(phi) + dyx(phi)*dy(phi) );
```

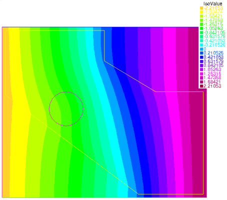
Outline

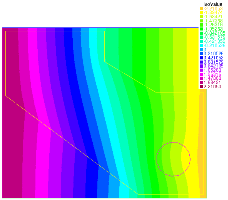
Force on a charged ball

Integrate $\vec{f} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$ over a charged ball $\oint_V \vec{f}\, d\vec{r}$ and apply velocity verlet.

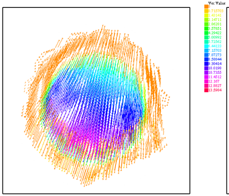


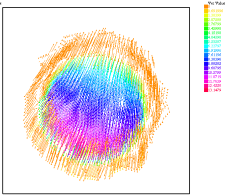


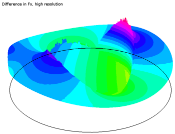


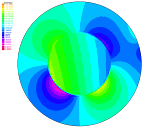


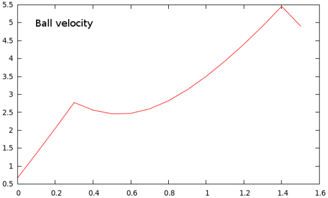
Force on a charged ball

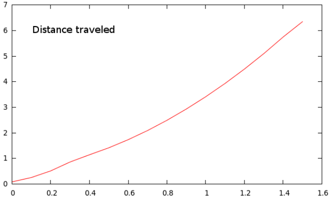












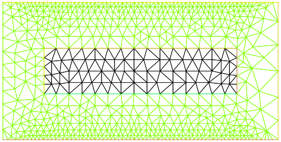
Outline

Electroactive polymer structure

A simple electroactive polymer structure was simulated as a proof of concept example. The structure has fixed displacements at left end and fixed electric potential at top and bottom boundaries. Calculations are done in dimensionless units.

material	e_r	E_{elas}	ν
mat1	11.8	800	0.49
mat2	5.8	40	0.49

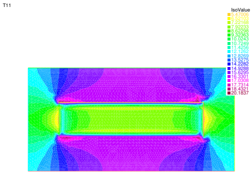
Deformed mesh

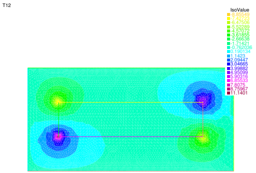


Outline

Electroactive polymer structure

Results (electric potential defines the induced stress and hence the induced body forces)





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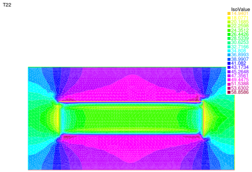

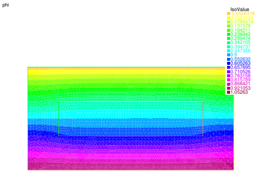

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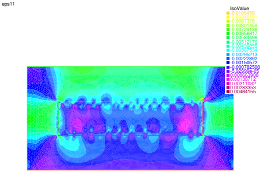

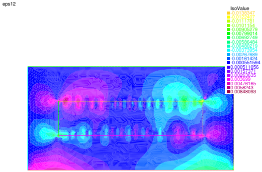

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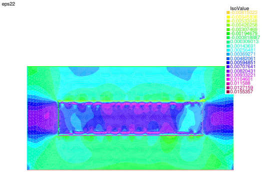

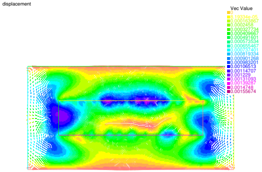

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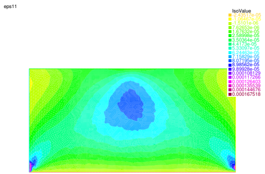
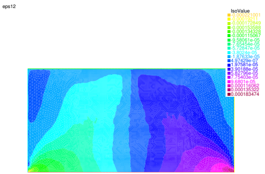
Outline

Homogeneous structure - fixed base





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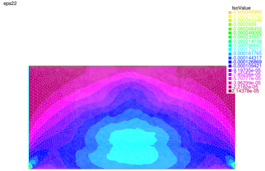
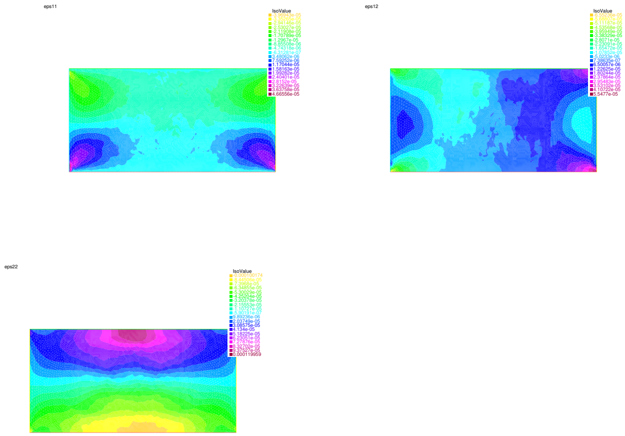


Electroactive polymer structure

Homogeneous structure - fixed base and top



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Outline

Discussion

- Applied knowledge from lecture
- Interesting phenomena
- Working with the FreeFem++ codebase

Questions?

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