Syringe Pump-Driven Flow in an Industry-Facing Microfluidic Channel

David Ruffner[1] and Donald Pierce[2]

[1]Spheryx Inc.
[2]New York University

July 12, 2018

We derive an expression for the average velocity $\bar{v}(t)$ of the xCell fluid flow as a function of the xSight pump volumetric flow rate Q_s and pump cross-sectional area A_s. This expression gives an expectation for the velocity of a fluid in the limit where $Q_s/A_s \approx \bar{v}(t)$ (i.e. where the viscosity of the fluid is not significant enough to cause discrepancies between the pump rate and the flow rate).

Postulates

1. The air in the syringe is approximately an ideal gas:
$$pV_{air} = nRT = const.$$
2. The pressure differential across the xCell channel scales only with the velocity and viscosity of the fluid:
$$\Delta P \propto \bar{v}(t) \cdot \mu.$$
3. Since the air in the syringe is ideal, the V_{air} determined by the syringe flow rate Q_s is approximately equal to the V_{air} determined by the average velocity of the fluid $\bar{v}(t)$:
$$Q_s t - \int_0^t \bar{v}(t) A_c dt \approx 0.$$
where A_c is the cross-sectional area of the xCell channel.

From postulate (2), we are motivated in finding an expression for the pressure differential since we seek an equation of $\bar{v}(t)$. Note that the volume of the air will have an initial volume V_0 which is decreased by fluid flowing toward the syringe, and increased by the syringe pulling away from the fluid:

$$V_{air} = V_0 - \int_0^t \bar{v}(t) A_c dt + Q_s t.$$

Then from postulate (1), the pressure differential is

$$p(t) = \frac{nRT}{V_0 - \int_0^t \bar{v}(t)A)cdt + Q_s t}.$$

Plugging the pressure into the equation of postulate (2), the average velocity can be expressed:

$$\mu\bar{v}(t) = p_0 - \frac{nRT}{V_0 - \int_0^t \bar{v}(t)A)cdt + Q_s t}. \tag{1}$$

This is a differential equation for $\bar{v}(t)$, which can be solved by assuming postulate (3). Since the denominator of (1) is such that $V_0 >> Q_s t - \int_0^t \bar{v}(t)A)cdt$, we may invoke the binomial approximation. Equation (1) then becomes

$$\mu\bar{v}(t) \approx p_0 - \frac{nRT}{V_0}\left(1 - \frac{Q_s t - \int_0^t \bar{v}(t}{A}_c dtV_0\right). \tag{2}$$

Differentiating (2) with respect to time yields

$$\frac{d\bar{v}(t)}{dt} \approx \frac{nRT}{\mu V_0^2}\left(Q_s - \bar{v}(t)A_c\right).$$

This is a first order linear differential equation for $\bar{v}(t)$ and has the solution:

$$\bar{v}(t) = \frac{Q_s}{A_c} + \left(\bar{v}_0 - \frac{Q_s}{A_c}\right) e^{-\frac{nRTA_c}{\mu V_0^2} t}.$$

We see that the average velocity of the fluid under these postulates is exponentially-decaying in time. From this decay, we can extrapolate a time constant

$$\tau = \frac{\mu V_0^2}{nRTA_c},$$

after which the velocity of the fluid will have decayed by a factor of e. We see that in the case where initial $\bar{v}(t)$ is less than the pump rate (i.e. no xSight Kick-start) that the fluid approaches equilibrium logarithmically from below (Figure 1a); and in the case where initial $\bar{v}(t)$ is greater than the pump rate (i.e. an xSight Kick-start) that the fluid approaches equilibrium logarithmically from above (Figure 1b).

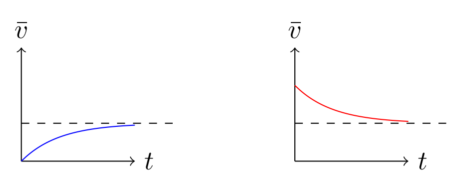

Figure 1: (a) *Blue.* Case where initial velocity of the fluid is less than the pump rate. (b) *Red.* Case where initial velocity of fluid is greater than the pump rate.