Remarks on Birkhoff's Theorem

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I provide an explanation of Birkhoff's theorem which explains how the constancy of 'mass' is not a result of spherical symmetry, but of the vacuum case.

The constancy of mass ($\dot{m} = 0$) is a property of the unique solution (Schwarzschild) to the spherically symmetric vacuum Einstein equations. For a symmetric spacetime,

$$ds^2 = -e^{\nu(r,t)}dt^2 + e^{\lambda(r,t)}dr^2 + d\Omega^2, \tag{1}$$

one of these equations yields

$$G_{\hat{r}\hat{t}} \propto \dot{\lambda}, \tag{2}$$

where $d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Since the vacuum case involves a null stress-energy tensor, equation (2) implies

$$\dot{\lambda} = 0 \Longrightarrow \lambda(t) = const. \tag{3}$$

This implies that Einstein's equations may be solved while neglecting the time-dependence of λ without loss of generality. Further, the combination $G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = 0$, which can be used to solve for $\nu(r,t)$ in equation (1), reveals ν to be a function only of r and λ. Since λ does not depend on t by equation (3), ν must be solely a function of r and not of t. Thus for a spherically-symmetric spacetime such as (1), λ and ν are constant in time.

One can approach the same problem with a more general, but still spherically-symmetric, spacetime

$$ds^2 = -e^{\nu(r,t)}dt^2 + e^{\lambda(r,t)}dr^2 + e^{\kappa(r,t)}drdt + d\Omega^2. \tag{4}$$

In which case (2) takes the form

$$G_{\hat{r}\hat{t}} \propto \kappa(r,t)\dot{\lambda}, \tag{5}$$

where κ is an arbitrary function. Since the $G_{\hat{r}\hat{t}}$ equation describes the time-evolution of the radial component, κ and λ both parameterize the 'mass'. In the null stress-energy case, (5) requires that when $\kappa(r,t) \neq 0$, $\dot{\lambda} = 0$; conversely, if $\dot{\lambda} \neq 0$, $\kappa(r,t)$ must be 0. This is a physical contradiction, because $\dot{\lambda}$ and κ both parameterize 'mass'. Thus $\kappa = \dot{\lambda} = 0$. Therefore 'mass' must be static.

This phenomenon, which is proven to be true for any spherically-symmetric spacetime, can be explained by an assumption of the equations of curvature $R_{\alpha\beta}$, that the acceleration of a separation vector S^α between two nearby geodesics does not explicitly depend on time. The effect of this assumption can be visualized either by considering two points on nearby geodesics which approach a center of symmetry in vacuum space (Fig 1a), or by considering two points on nearby geodesics on a sphere (Fig. 1b).

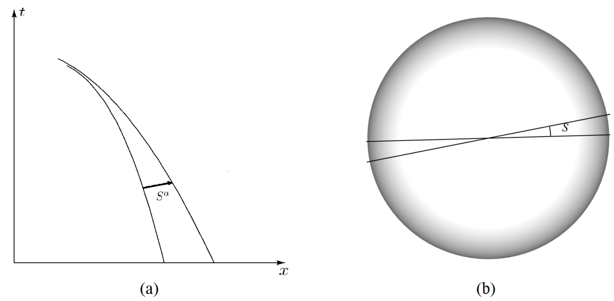

Figure 1: (a) Two geodesics separated by a four-vector S^α. (b) Two geodesics on the surface of a sphere separated by a three-vector S.

As the points in Fig. 1a move in unison along their respective geodesics toward the center of symmetry, the vector S^α between them accelerates constantly. However, if the 'mass' were growing during this descent, then the separation vector S^α would accelerate at a slower rate. Equivalently, if the sphere in Fig. 1b were growing during the uniform motion of the points along their geodesics, the three-vector S would accelerate slower than if the sphere were static. There is a metaphor for this postulate: if two ants bound to intersect were walking along geodesics on the surface of a balloon, it would take longer for them to intersect if the balloon were inflating as they walked. This means that for a spacetime metric to be a function of time, the distance between geodesics must also be a function of time. This last statement requires no assumption of symmetry.

I. Proof of Static Geodesics

I will now show that the vacuum equations of curvature ($R_{\alpha\beta} = 0$) only consider geodesics with constant separation, and thus can only predict spacetimes with constant masses, regardless of symmetry. I will begin by recalling the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma, \tag{6}$$

where x^α is a point on the fiducial geodesic, and u^δ is the coordinate velocity $dx^\delta/d\tau$. Since neighboring geodesics are considered to be very close, the geodesic equation for the nearby geodesics located at $x^\alpha + S^\alpha$, where S^α is the separation vector, may be approximated to first-order using a Taylor expansion around x^α. This is

$$\frac{d^2(x^\alpha + S^\alpha)}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\epsilon}u^\beta u^\gamma S^\epsilon - 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{dS^\gamma}{d\tau}. \tag{7}$$

Subtracting (7) from (6) reveals

$$\frac{d^2S^\alpha}{d\tau^2} = \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\epsilon}u^\beta u^\gamma S^\epsilon + 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{dS^\gamma}{d\tau}. \tag{8}$$

This is the equation for the proper time acceleration of the separation vector between two nearby geodesics. However, arriving at it required a fundamental assumption: that the separation vector has no explicit time-dependence, that 'mass' is static. For a time-dependent S^α, the full first-order Taylor expansion is

$$\begin{aligned}\frac{d^2(x^\alpha + S^\alpha(t))}{d\tau^2} &= -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\epsilon}u^\beta u^\gamma S^\epsilon(t)\\ -2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{dS^\gamma(t)}{d\tau} &- \frac{d\Gamma^\alpha_{\beta\gamma}}{dt}u^\beta u^\gamma\Delta t - 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{du^\gamma}{dt}\Delta t,\end{aligned} \tag{9}$$

where $\Delta t = 0$, restoring equation (7), only in the case that S^α does not change with time. Note that Δt must be a small dt for the first order approximation to be accurate. Subtracting (9) from (6) to obtain the new proper time acceleration yields

$$\begin{aligned}\frac{d^2S^\alpha(t)}{d\tau^2} =&\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\epsilon}u^\beta u^\gamma S^\epsilon(t) + 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{dS^\gamma(t)}{d\tau}\\ &+ \frac{d\Gamma^\alpha_{\beta\gamma}}{dt}u^\beta u^\gamma dt + 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{du^\gamma}{dt}dt.\end{aligned} \tag{10}$$

The difference between this equation and (8) are the two dt terms. These terms are not accounted for in (8) because $S^0 \neq dt$. This is the most important mathematical statement of this paper: $S^0 = \epsilon t$ measures the infinitesimal time separation between each spacetime coordinate of the nearby geodesics, while dt measures the explicit change in time. Because they are independent of each other and not the same variable, (10) is necessary to allow for the distance between geodesics to depend on time explicitly, and consequently for mass to be time-dependent.

To show how equation (10) changes the Einstein equations of curvature, I will recall that the acceleration of the (time-independent) separation vector, taken with the complete derivative ∇_u to account for changing basis vectors, gives forth the Riemann tensor

$$(\nabla_u\nabla_u S)^\alpha = -R^\alpha_{\beta\gamma\delta}u^\beta S^\gamma u^\delta. \tag{11}$$

The covariant derivative ∇_u of a vector is always the change in the vector components throughout coordinate space plus the change in the basis vectors throughout coordinate space, all in the direction of the coordinate velocity u^δ. The expression for the second covariant derivative of the separation vector S^α is

$$\begin{aligned}(\nabla_u\nabla_u S)^\alpha = &\frac{d^2S^\alpha}{d\tau^2} + \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\epsilon}u^\epsilon u^\beta S^\gamma + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\tau}S^\gamma\\ &+ 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{dS^\gamma}{d\tau} + \Gamma^\alpha_{\epsilon\sigma}\Gamma^\sigma_{\beta\gamma}u^\epsilon u^\beta S^\gamma.\end{aligned} \tag{12}$$

To obtain (11), one must substitute (8) into (12). This would show that

$$\begin{aligned}(\nabla_u\nabla_u S)^\alpha = &-\left(\frac{\partial\Gamma^\alpha_{\beta\delta}}{x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{x^\delta} + \Gamma^\alpha_{\gamma\epsilon}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma}\right)u^\beta S^\gamma u^\delta\\ &\equiv -R^\alpha_{\beta\gamma\delta}u^\beta S^\gamma u^\delta.\end{aligned} \tag{13}$$

However, if instead equation (10) were inserted into (12) to account for time-dependent mass, the equations of curvature would not simplify to the Riemann tensor of equation (13).

Before demonstrating this, I would like to resolve any discomforts one may have with whether or not the same covariant derivative used to calculate (12) may also be used for time-dependent vectors. The equation for the covariant derivative ∇_u of a static vector v^α

$$(\nabla_u v)^\alpha = u^\beta\nabla_\beta v^\alpha = u^\beta\left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma}v^\gamma\right), \tag{14}$$

is the same as for a time-dependent vector $v^\alpha(t)$. The justification for this is that the time-derivatives in equation (14) will measure the change in each component of $v^\alpha(t)$ and $\hat{e}_\beta(t)$ by the chain rule, i.e. for

$$\frac{\partial v^\alpha(t)}{x^0} \quad\text{and}\quad \Gamma^\alpha_{\beta 0} = \left(\frac{\partial\hat{e}_\beta(t)}{\partial x^0}\right)^\alpha. \tag{15}$$

Thus the covariant derivative needs no alterations to measure the correct change in time-dependent vectors.

Finally, the new equation for the acceleration of the separation vector is

$$\begin{aligned}(\nabla_u\nabla_u S)^\alpha = &-R^\alpha_{\beta\gamma\delta}u^\beta S^\gamma u^\delta\\ &+ \left(\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial t}u^\beta + 2\Gamma^\alpha_{\beta\gamma}\frac{\partial u^\beta}{\partial t}\right)dt u^\gamma.\end{aligned} \tag{16}$$

The terms in the parenthesis in (16) describe the change in spacetime curvature over time, and suggest two things: the rate at which basis vectors change throughout coordinate space (i.e. $\Gamma^\alpha_{\beta\gamma}$) changes with time, and the proper time velocity (i.e. u^β) of a point changes with time. This is expected. The next step is to calculate the object which is analogous to the Ricci curvature. Since $R_{\alpha\beta}$ is obtained with a sum taken over the components of the separation vector $R^\gamma_{\beta\gamma\delta} \equiv R_{\beta\delta}$, with the motivation that these components describe geodesic deviation, the analogous sum that must be taken to describe time-dependent geodesic deviation is over time,

$$C^\alpha_\gamma = \int\left(\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial t}u^\beta + 2\Gamma^\alpha_{\beta\gamma}\frac{\partial u^\beta}{\partial t}\right)dt. \tag{17}$$

This gives rise to a new requirement for vacuum equations of curvature, namely

$$R_{\alpha\beta} = 0 \tag{18a}$$

$$C^\alpha_\beta = 0. \tag{18b}$$

These requirements imply that the vacuum case is satisfied only when geodesics do not deviate in spacetime (18a) *and* in explicit time (18b). Although obvious, this fact is somewhat obscured by the popular discourse on Birkhoff's theorem; because the constancy of mass has little to do with spherical symmetry. The unique Schwarzschild solution is a result of spherical symmetry;

the constant and stationary property of mass is a result of the inability of the Riemann tensor to describe time-dependent geodesics in the vacuum case. The Riemann and Ricci tensors, considered alone, are static.

II. Proof of Static Field

It is important, and perhaps clarifying, to consider the Newtonian analogy. I will begin with the force law

$$m\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial\Phi(\vec{x},t)}{\partial x^j}. \tag{19}$$

This is the Newtonian equation of motion for a particle x^i subject to a field Φ, where m can be set to 1 for our purposes. By employing the full Taylor expansion around the point (x^i, t), then subtracting this from (19), the equation for the acceleration of a small separation vector S^i between two nearby particles is

$$\frac{d^2S^i}{dt^2} = -\delta^{ij}\left(\frac{\partial^2\Phi(\vec{x},t)}{\partial x^k\partial x^j}S^k + \frac{\partial^2\Phi(\vec{x},t)}{\partial t\partial x^j}\Delta t\right). \tag{20}$$

This is almost the same as the traditional *Newtonian Deviation Equation*

$$\frac{d^2S^i}{dt^2} = -\delta^{ij}\frac{\partial^2\Phi(\vec{x})}{\partial x^k\partial x^j}S^k, \tag{21}$$

except where the second term in equation (20) has been added to account for the explicit time-dependence of the separation between the paths of particles, caused by a time-dependent field. (20) gives rise to an altered Newtonian field equation

$$\nabla^2\Phi = \delta^{ij}\left(\frac{\partial^2\Phi(\vec{x},t)}{\partial x^i\partial x^j} + \frac{\partial^2\Phi(\vec{x},t)}{\partial t\partial x^j}\right) = 4\pi G\mu(\vec{x},t), \tag{22}$$

where the mass density μ must be time-dependent.

III. Discussion

It is important to notice that freely-falling particles are not easily confined to their geodesics during the expansion of a spherical mass. That is to say, it would take energy to keep free-falling points on the same geodesics if the mass were changing with time. A thought experiment which makes this extremely clear is to consider two nearby particles which are so far from the mass that they don't *seem* to be falling at all. In this case, a rapidly-changing mass would most certainly not increase the separation vector S^α between them. Instead, the particles would energetically favor entering onto new geodesics. Equivalently, but not as obviously, a free particle moving on a geodesic on the surface of a sphere is not easily confined to that geodesic during the sphere's expansion. This is evidence of the fact that the Hamiltonian is not conserved for spacetime metrics with time-dependent masses. Due to the conclusions of this paper, Birkhoff's theorem should correctly be stated as "the unique solution to a spherically-symmetric, vacuum spacetime is the Schwarzschild solution."