Ch 2—§2.2 onwards

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2.2 Subspaces

Story

- We now look at some of the basic concepts
  of vector spaces.

Definition (subspace).

Let V be a vector space over the field F.
A **subspace** of V is
a subset W of V which is
itself a vector space over F with
the operations of vector addition and scalar multiplication
on V

NB

By inspection (of the definition of a vector space)
it is easy to see that
the subset W of V is a subspace if
for each α and β in W
the vector $\alpha+\beta$ is also in W
the 0 vector is in W
for each α in W the vector $(-\alpha)$ in W
for each α in W and each scalar c
the vector $c\alpha$ is in W.

The commutativity and associativity properties
of vector addition &
properties 4a,b,c and d of scalar multiplications
do not need to be checked
(since these are properties of operations on V).

Recall:

(4) a rule (or operation)
called scalar multiplication
which associates with
each scalar c in F and
each vector α in V
a vector $c\alpha$ in V
called the product of c and α in a way that
(a) $1\alpha=\alpha$ for every α in V
(b) $(c_1c_2)\alpha=c_1(c_2\alpha)$
(c) $c(\alpha+\beta)=c\alpha+c\beta$
(d) $(c_1+c_2)\alpha=c_1\alpha+c_2\alpha$

Story.

One can simplify things further.

Theorem 1.

A non-empty subset W of V is a subspace of V
iff
for
each pair of vectors α,β in W and
each scalar c in F
the vector $c\alpha+\beta$
is again in W.

Proof.

Suppose that W is a non-empty subset of V such that
$c\alpha+\beta$ belongs to W
for all vectors α,β in W
and
all scalars c in F.

each α and β in W
the vector $\alpha+\beta$ is also in W
the 0 vector is in W
for each α in W the vector $(-\alpha)$ in W
for each α in W and each scalar c
the vector $c\alpha$ is in W.

Since W is a non-empty,
there is a vector ρ in W,
hence $(-1)\rho+\rho=0$ is also in W. <-- Zero vector in W

Now, if α is any vector in W
and $c$is any scalar
$c\alpha=c\alpha+0$ is in W. <-- $c\alpha$ is in W

Specifically, $(-1)\alpha=-\alpha$ is in W. <--- $(-\alpha)$ is in W

Finally, if α and β are in W

Finally, if α and β are in W
then $\alpha + \beta = 1\alpha + \beta$ is in W. <--- $\alpha + \beta$ is in W

Thus W is a subspace of V.

Conversely,
if W is a subspace of V
α and β are in W
and
c is a scalar
then $c\alpha + \beta$ is in W.

■

Story

- Some authors use the $c\alpha + \beta$ property in Theorem 1
  as the definition of a subspace.

  But as Theorem 1 showed,
  these two notions are equivalent and
  thus it makes no difference which is taken to be the definition

- The crucial point is that
  if W is a non-empty subset of V
  such that $c\alpha + \beta$ is in V (for all α, β, c)
  then
  (with the operations inherited from V)
  W is a vector space.

- This allows us to consider many new examples of vector spaces

Example 6

(a) Trivial examples
Let V be a vector space.
Then,
V is a subspace of V
the subset of the zero vector alone
is a subspace of V
(called the **zero subspace** of V).

(b) In F^n
the set of n-tuples $(x_1 \ldots x_n)$ with $x_1 = 0$
is a subspace
however
the set of n-tuples
with $x_1 = 1 + x_2$ is
not a subspace ($n \geq 2$). ★ HW

$(1+x_2, x_2)$
$c\alpha+\beta \in V$
$2 + \underbrace{2x_2}_{x_2'}, 2x_2$

$1+x_2, x_2, x_3$
$2+2x_2, 2x_2, 2x_3$
$1+c+x_2+cx_2', \quad x_2+cx_2', \quad x_3+cx_3'$
not of the form

(c) The space of polynomial functions
over the field F is
a subspace of the space of all functions
from F into F.

(d) An $n \times n$ (square) matrix A over the field F <-- Defn
is **symmetric** if $A_{ij} = A_{ji}$ for all i, j.

Symmetric matrices form a subspace of
the space of all $n \times n$ matrices over F.

(e) An $n \times n$ (square) matrix A over the field C
of complex numbers is **Hermitian** (or self-adjoint) if <-- Defn
$A_{jk} = \overline{A}_{kj}$ (where the bar is complex conjugation)
for all j, k.

A 2×2 matrix is Hermitian iff
it has the form

$$\begin{bmatrix} z & x + iy \\ x - iy & w \end{bmatrix}$$

where x, y, z and w are real numbers.

The set of all Hermitian matrices is *not*
a subspace of the space of all $n \times n$ matrices over $\mathbb{C}$.

Why?
For any Hermitian matrix A, the diagonal entries
A_{11}, A_{22} ... must be real
However, iA will not have real entries (in general).

The set of Hermitian matrices (allowing comlpex entries)
is a vector space of the field $\mathbb{R}$ of real numbers
(with usual operations).

Example 7. The solution space of a system of homogeneous linear equations.

Let A be an $m \times n$ matrix over F.
The set of all column matrices (i.e. ($n \times 1$) matrices) X over F
such that $AX = 0$
is a subspace of all $n \times 1$ matrices over F.

Why?
It suffices to show that if X, Y are such that $AX = AY = 0$
then for all scalars c it holds that
$A(cX + Y) = 0$
but this is immediate from the linearity of matrix multiplication
(more precisely stated as a lemma below).

Lemma. *If* A *is an* m $\times$ n *matrix over* F *and* B, C *are* n $\times$ p *matrices over* F *then*

(2-11) $$A(dB + C) = d(AB) + AC$$

for each scalar d *in* F.

Proof.
$$\begin{aligned}[A(dB + C)]_{ij} &= \sum_k A_{ik}(dB + C)_{kj} \\ &= \sum_k (dA_{ik}B_{kj} + A_{ik}C_{kj}) \\ &= d\sum_k A_{ik}B_{kj} + \sum_k A_{ik}C_{kj} \\ &= d(AB)_{ij} + (AC)_{ij} \\ &= [d(AB) + AC]_{ij}. \quad \blacksquare\end{aligned}$$

Similarly one can show that $(dB + C)A = d(BA) + CA$, if the matrix sums and products are defined.

Story:

- Here's a useful property of subspaces.

Theorem 2.

Let V be a vector spcae over the field F.
The intersection of any collection of subspaces of V
is also a subspace of V.

Proof.

Let $\{W_a\}$ be a collection of subspaces of V
and
let $W = \cap_a W_a$ be their intersection.

Recall
W is the set of elements such that each elemnt belongs to every W_a.

Since each W_a is a subspace
each contains the zero vector.

Thus,
the zero vector is in the intersection of W and
W is non-empty.

Let $\alpha, \beta \in W$ and c bea scalar
Then,
by definition of W
both α and β belong to each W_a
and because each W_a is a subspace
$(c\alpha + \beta)$ is in every W_a.

Thus,
$(c\alpha + \beta)$ is again in W.

By Theorem 1, W is a subspace of V.

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Theorem 1.
A non-empty subset W of V is a subspace of V
iff
for
each pair of vectors α, β in W and
each scalar c in F
the vector $c\alpha + \beta$
is again in W.

Story:

- From Theorem 2
  it follows that if S is any collection of vectors in V
  then there is a smallest subspace of V
  which contains S
  i.e. a subspace which contains S and
  which is contained in
  every other subspace
  containing S.

  Why?
  [intuition]
  Well, simply take all the subspaces that contain S
  and take their intersection
  This is a well-defined unique subspace

  And it is also the smallest
  (because otherwise it would shrink upon intersection)

Definition.

Let S be a set of vectors in a vector space V.
The subspace spanned by S is defined to be
the intersection W of all subspaces of V
which contain S.

When S is a finite set of vectors

$S = \{\alpha_1 \dots \alpha_n\}$

we simply call W the

subspace spanned by the vectors $\alpha_1 \dots \alpha_n$.

Story

- This definition may not feel all that "constructive" but
  the following theorem makes it very concrete

Theorem 3.

The subspace spanned by
a non-empty subset S of a vector space V is
the set of all linear combinations of vectors in S.

Proof.

Let L be the set of all linear combinations of vectors in S.
Let W be the subspace spanned by S.

We first show $L \subseteq W$

Let α_1
Then each linear combination

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$$

of vectors α_i in S
is clearly in W.

[Why? Because W is a subspace that contains S]

We conclude that
L is in W, i.e. $L \subseteq W$.

We now show $W \subseteq L$.

It suffices to show that L is a subspace containing S.
(because W is the smallest subspace containing S).

NB. L contains S and is non-empty.
Claim: L is a subspace.
If $\alpha, \beta \in L$ then
α is a linear combination

$$\alpha = x_1\alpha_1 + \cdots x_m\alpha_m$$

and
β is a linear combination

$$\beta = y_1\beta_1 + \cdots + y_n\beta_n.$$

Clearly, for each scalar

$$c\alpha + \beta = \sum_i (cx_i)\,\alpha_i + \sum_j y_j\beta_j$$

is also in L.

Thus, L is a subspace.

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Story:

- In view of this, one can equivalently define a subspace spanned by
  $\alpha_1 \dots \alpha_n$ as the set of all linear combinations of these vectors,
  i.e. $\{c_1\alpha_1 + \cdots c_n\alpha_n\}_{c_1 \dots c_n}$ where c_is are scalars.

- We now introduce some notation to
  easily speak of linear combination of vectors

easily speak of linear combination of vectors
in different sets.

Definition.

If $S_1 \dots S_k$ are subsets of a vector space V
the set of all sums
$\alpha_1 + \cdots + \alpha_k$
of vectors α_i in S_i is called the
sum of the subsets $S_1, S_2 \dots S_k$ and
is denoted by
$S_1 + \cdots + S_k$
or by
$\sum_i S_i$.

Story:

- This notation comes in handy when
one considers

NB. If $W_1 \dots W_k$ are subspaces of V then
the sum $W = W_1 + \cdots + W_k$
is easily seen to be a subspace of V which
contains each of the subspaces W_i.

Claim: W is the subspace
spanned by the union of $W_1 \dots W_k$.

HW

Proof idea.
Follows from NB and Theorem 3.

Example 8

Let F be a subfield of the field C of complex numbers.

Suppose
$$\begin{aligned} \alpha_1 &= (1, 2, 0, 3, 0) \\ \alpha_2 &= (0, 0, 1, 4, 0) \\ \alpha_3 &= (0, 0, 0, 0, 1). \end{aligned}$$

Let W be the subspace spanned by $\alpha_1, \alpha_2, \alpha_3$

By Theorem 3
a vector $\alpha \in W$
iff
there exist scalars such that
$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$.

Thus, W consistes of all vectors of the form
$\alpha = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$

where c_1, c_2, c_3 are arbitrary scalars in F.

Alternatively, W can be described as the set of all vectors
$\alpha = (x_1, \dots x_5)$

with $x_i \in F$ such that
$x_2 = 2x_1$ and
$x_4 = 3x_1 + 4x_3$

(just a rewriting of α as $(x_1 \dots x_5)$
then equating it to $(c_1, 2c_1, c_2 \dots)$
and writing the constraints, eliminating the cs)

One can check that
$(-3,-6,1,-5,2)$ is in W but
$(2,4,6,7,8)$ is not.

Example 9

Let
F be a subfield of the field C of complex numbers and
V be the vector space of all 2×2 matrices over F.

Let
W_1 be the subset of V consisiting of all matrices
of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

where x, y, z are scalars in F.

Let
W_2 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

★ HW

Then, W_1 and W_2 are subspaces of V.

Also, $V = W_1 + W_2$ ★ HW

Finally, the subspace $W_1 \cap W_2$ consists of
all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

Example 10

Let A be an $m\times n$ matrix over a field F.

The **row vectors** of A
are the vectors in F^n given by
$\alpha_i = (A_{i1}, \dots, A_{in})$
for $i = 1 \dots m$.

The subspace of F^n spanned by
the row vectors of A is called
the **row space** of A.

The subspace in Example 8
is the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(baby) Claim: It is also the row space of the matrix ★ HW

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -4 & -8 & 1 & -8 & 0 \end{bmatrix}.$$

Example 11

Let V be the space of all polynomial functions over F.

Let S be the subset of V consisting of the polynomials (monomials really)
$f_0, f_1 \dots$
defined by
$f_n(x) = x^n, n = 0,1,2 \dots$

HW

Then V is the subspace spanned by the set S.

§ 2.3 Bases and Dimensions

Story:

- We now look at how to generalise the intuitive notion of dimension
  to vector spaces.
- To this end, we consider
  an algebraic definition of dimension
  for vector spaces
- This is done through
  the concept of a "basis" for the vector space.

Definition

Let V be a vector space over F.
A subset S of V is said to be
linearly dependent (or simply dependent)
if there exist
distinct vectors
$\alpha_1 \ldots \alpha_n$ in S and
scalars
$c_1 \ldots c_n$ in F that are not all zeros
such that
$c_1\alpha_1 + \cdots c_n\alpha_n = 0$

A set which is not linearly dependent is called **linearly independent**.

If the set S contains only *finitely* many vectors
$\alpha_1 \ldots \alpha_n$
we sometimes say that
$\alpha_1 \ldots \alpha_n$ are dependent (or indepnedent)
instead of saying S is dependent (or independent).

Story:

- Here are some easy consequences of this definition

Observations.

1. Any set that contains a linearly dependent set
   is linearly dependent.
2. Any subset of a linearly independent set
   is linearly independent.
3. Any set which contains the 0 vector
   is linearly dependent
   (because $1 \cdot \vec{0} = \vec{0}$)
4. A set S of vectors is linearly independent
   iff
   each finite subset of S is linearly independent
   (i.e.
   for any distinct vectors
   $\alpha_1 \ldots \alpha_n$ of S
   $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$
   implies each $c_i = 0$)

Definition.

Let V be a vector space.
A **basis** for V is

a linearly independent set of vectors in V
which spans the space V.

The space V is **finite-dimensional**
if it has a finite basis.

EXAMPLE 12. Let F be a subfield of the complex numbers. In F^3 the vectors

$$\begin{aligned} \alpha_1 &= (\ \ 3, 0, -3) \\ \alpha_2 &= (-1, 1, \ \ 2) \\ \alpha_3 &= (\ \ 4, 2, -2) \\ \alpha_4 &= (\ \ 2, 1, \ \ 1) \end{aligned}$$

are linearly dependent, since

$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0 \cdot \alpha_4 = 0.$$

The vectors

$$\begin{aligned} \epsilon_1 &= (1, 0, 0) \\ \epsilon_2 &= (0, 1, 0) \\ \epsilon_3 &= (0, 0, 1) \end{aligned}$$

are linearly independent

EXAMPLE 13. Let F be a field and in F^n let S be the subset consisting of the vectors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ defined by

$$\begin{aligned} \epsilon_1 &= (1, 0, 0, \ldots, 0) \\ \epsilon_2 &= (0, 1, 0, \ldots, 0) \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \epsilon_n &= (0, 0, 0, \ldots, 1). \end{aligned}$$

Let $x_1, x_2, \ldots, x_n$ be scalars in F and put $\alpha = x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_n\epsilon_n$. Then

$$\alpha = (x_1, x_2, \ldots, x_n). \tag{2-12}$$

This shows that $\epsilon_1, \ldots, \epsilon_n$ span F^n. Since $\alpha = 0$ if and only if $x_1 = x_2 = \cdots = x_n = 0$, the vectors $\epsilon_1, \ldots, \epsilon_n$ are linearly independent. The set $S = \{\epsilon_1, \ldots, \epsilon_n\}$ is accordingly a basis for F^n. We shall call this particular basis the **standard basis** of F^n.

EXAMPLE 14. Let P be an invertible $n \times n$ matrix with entries in the field F. Then $P_1, \ldots, P_n$, the columns of P, form a basis for the space of column matrices, $F^{n\times 1}$. We see that as follows. If X is a column matrix, then

$$PX = x_1P_1 + \cdots + x_nP_n.$$

Since $PX = 0$ has only the trivial solution $X = 0$, it follows that $\{P_1, \ldots, P_n\}$ is a linearly independent set. Why does it span $F^{n\times 1}$? Let Y be any column matrix. If $X = P^{-1}Y$, then $Y = PX$, that is,

$$Y = x_1P_1 + \cdots + x_nP_n.$$

So $\{P_1, \ldots, P_n\}$ is a basis for $F^{n\times 1}$.

Example 15 and 16 ★ HW
(I do hope to cover this in the next class
but please take a look on your own)

Theorem 4.
Let V be a vector space that is
spanned by a finite set of vectors
$\beta_1 \ldots \beta_m$.

Then
any independent set of vectors in V
is finite and contains
no more than m elements.

Proof.

Strategy: To prove the theorem
it is enough to show that every subset S of V
that contains more than m vectors
is linearly dependent.

Let S be a subset of V
containing $n > m$ vectors
$\alpha_1 \ldots \alpha_n$

Since $\beta_1 \ldots \beta_m$ span V
there exist scalars A_{ij} in F such that

$$\alpha_j = \sum_{i \in \{1 \ldots m\}} A_{ij} \beta_i$$

For any n scalars $x_1 \ldots x_n$, we have

$$x_1\alpha_1 + \cdots + x_n\alpha_n = \sum_{j \in \{1 \ldots n\}} x_j \alpha_j$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} (A_{ij}x_j)\beta_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}x_j \right) \beta_i.$$

Recall Theorem 6 (from Chapter 1):

Theorem 6.
If A is an $m \times n$ matrix and $m < n$
then the homogenous system of linear equations
$AX = 0$ has a non-trivial solution.

Now, since $n > m$, Theorem 6 says that
there exist scalars $x_1 \ldots x_n$ (not all 0) such that

$$\sum_{j \in \{1 \ldots n\}} A_{ij} x_j = 0 \text{ for } 1 \le i \le m$$

Hence,
$x_1\alpha_1 + \cdots + x_n\alpha_n = 0.$

This shows that S is a linearly dependent set.

∎