Quantum Aspects of Cryptography

End-sem Examination
VERSION: $\beta.2$

Tuesday, April 29, 2025

Important Instructions.

1. Ground rules.
   (a) *Permitted.* As announced, you are allowed to carry one A4 sheet with handwritten notes (you may write on both sides) but nothing beyond that, to explicitly help you with your exam.
   (b) *Forbidden.* In particular, devices such as a tablet, phone etc. must not be used and no interaction with your peers is allowed. Please call me instead, should there be any confusion.
   (c) *Penalty.* You will immediately lose 50% of your points should any violation of these ground rules be observed. The second violation will cause you to lose all points for this exam.
   (d) *Duration.* Please try to finish your exam within *three hours.* I will do my best to offer as much extra time as logistically possible.
2. Grading/points.
   (a) 5 points for Assignment Exercises (these are taken verbatim from your assignments with additional explanations) are split as follows:
      i. 1 point for Haar
      ii. 1 point for PRS => Commitments
      iii. 1 point for QMA=BQP and yet PRUs exist
      iv. 1 point for QFHE
      v. 1 point for Self-Testing
   (b) 15 points for 'new' questions (these may overlap with your assignments, but won't be exactly the same) are split as follows:
      i. 3 points for Uncloneable Encryption (security in the ROM)
      ii. 3 points for Haar measure
      iii. 2 points for Pseudorandom States => Commitments
      iv. 3 points for QMA=BQP and yet PRUs exist
      v. 1 point for QFHE
      vi. 1 point for Self-Testing
      vii. 2 points for Verification
3. Optional remarks
   (a) Unlike the mid-sem, there are no options this time.
   (b) While the question paper is long, this is primarily because it explains all the material needed to answer the questions. It does not expect you to remember much and is designed to primarily test comprehension of key concepts.
   (c) The difficulty of the questions varies quite a bit. The length of the question does not always correlate with the points or hardness. Proceed accordingly.

Please let me know if you spot a mistake or if something is unclear or feels suspicious. Good luck.

1 Review | Exercises from the Assignment

1.1 Haar Measure

Reminder:

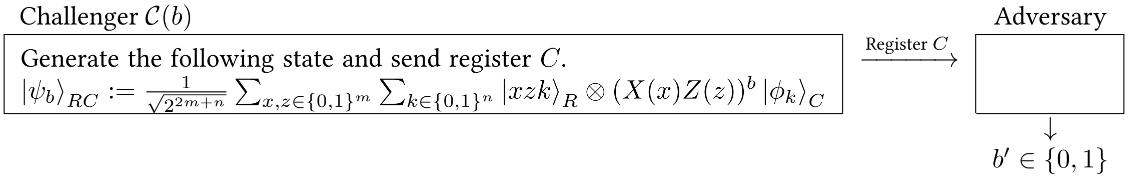
- The moment operator is defined as $\mathcal{M}^{(k)}(\cdot) := \mathbb{E}_{U \leftarrow \mu_H}[U^{\otimes k}(\cdot)U^{\dagger\otimes k}]$ where μ_H is the Haar measure.
- The commutant
$$\mathrm{Comm}(\mathrm{U}(d), k) := \{A \in \mathcal{L}((\mathbb{C}^d)^{\otimes k}) : [A, U^{\otimes k}] = 0 \,\forall\, U \in \mathrm{U}(d)\}$$
is the set of linear operators A that commute with all$U^{\otimes k}$. Also, it holds that

Exercise 1 (1 points; Haar)**.** Show that $\mathcal{M}^{(k)} \in \mathrm{Comm}(\mathrm{U}(d), k)$ and that if $A \in \mathrm{Comm}(\mathrm{U}(d), k)$, then $\mathcal{M}^{(k)}(A) = A$.

1.2 PRSG $\implies$ Commitments

Reminder: Recall that for proving computational hiding of their commitment scheme, Morimae and Yamakawa [4] consider the following hybrids. Below, $|\phi_k\rangle$ is an m-qubit state produced by the state generator function of PRSG (with security parameter λ) and μ_m is the Haar measure over m-qubit states.

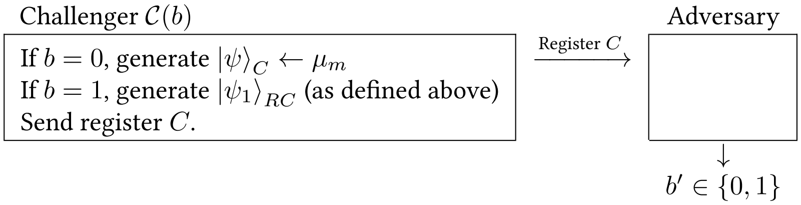
- $\mathrm{Hyb}_0(b)$



Challenger $\mathcal{C}(b)$

Generate the following state and send register C.
$|\psi_b\rangle_{RC} := \frac{1}{\sqrt{2^{2m+n}}} \sum_{x,z\in\{0,1\}^m} \sum_{k\in\{0,1\}^n} |xzk\rangle_R \otimes (X(x)Z(z))^b |\phi_k\rangle_C$

- $\mathrm{Hyb}_1(b)$



Challenger $\mathcal{C}(b)$

If $b = 0$, generate $|\psi\rangle_C \leftarrow \mu_m$
If $b = 1$, generate $|\psi_1\rangle_{RC}$ (as defined above)
Send register C.

Exercise 2 (1 point; PRSG $\implies$ Commitments)**.** Show that $|\Pr[1 \leftarrow \mathrm{Hyb}_0(b)] - \Pr[1 \leftarrow \mathrm{Hyb}_1(b)| \leq \mathrm{negl}$ for $b = 1$ (note, $b = 0$ is not asked, only $b = 1$).

1.3 BQP=QMA yet PRUs exist

Reminder:

- Semi-formally, a pseduorandom unitary is a keyed family of unitaries $\{U_k\}_{k\in\{0,1\}^\kappa}$ acting on $n(\kappa)$-qubits (also referred to as a PRU ensemble), where κ is a security parameter with the following two properties:
  - It can be applied efficiently (i.e. given k and a quantum state $|\psi\rangle$, one can efficiently compute $U_k |\psi\rangle$).
  - No efficient quantum algorithm $\mathcal{A}$ can distinguish whether it is given oracle access to a Haar random unitary, i.e. $U \leftarrow \mu_{2^n}$, or a pseudorandom unitary, i.e. U_k where $k \leftarrow \{0,1\}^\kappa$.
- To show that relative to an oracle, PRUs can exist, even when BQP=QMA, Kretschmer [3] uses two oracles: $\mathcal{U}$ (a quantum oracle) and $\mathcal{C}$ (a classical oracle) which is independent of $\mathcal{U}$. The quantum oracle $\mathcal{U}$ is defined as the set $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ where for each n, $\mathcal{U}_n$ denotes a direct sum 2^n independent Haar random unitaries, acting on n qubits, i.e. $\mathcal{U}_n \leftarrow \mu_{2^n}^{2^n}$. Description of $\mathcal{C}$ is not relevant here.

Exercise 3 (1 point; BQP=QMA yet PRUs exist)**.** State the construction for the PRU ensemble used by Kretschmer.

1.4 QFHE

Reminder:

- Recall that Mahadev's QFHE scheme encrypts a quantum state $|\psi\rangle$ as $(X^x Z^z |\psi\rangle, (\hat{x}, \hat{z}))$ where x, z are sampled uniformly and $\hat{x}$ and $\hat{z}$ denote classical FHE encryptions of x and z respectively.

Exercise 4 (1 point; QFHE)**.** Explain how the Eval function is implemented for Clifford gates, in Mahadev's construction.

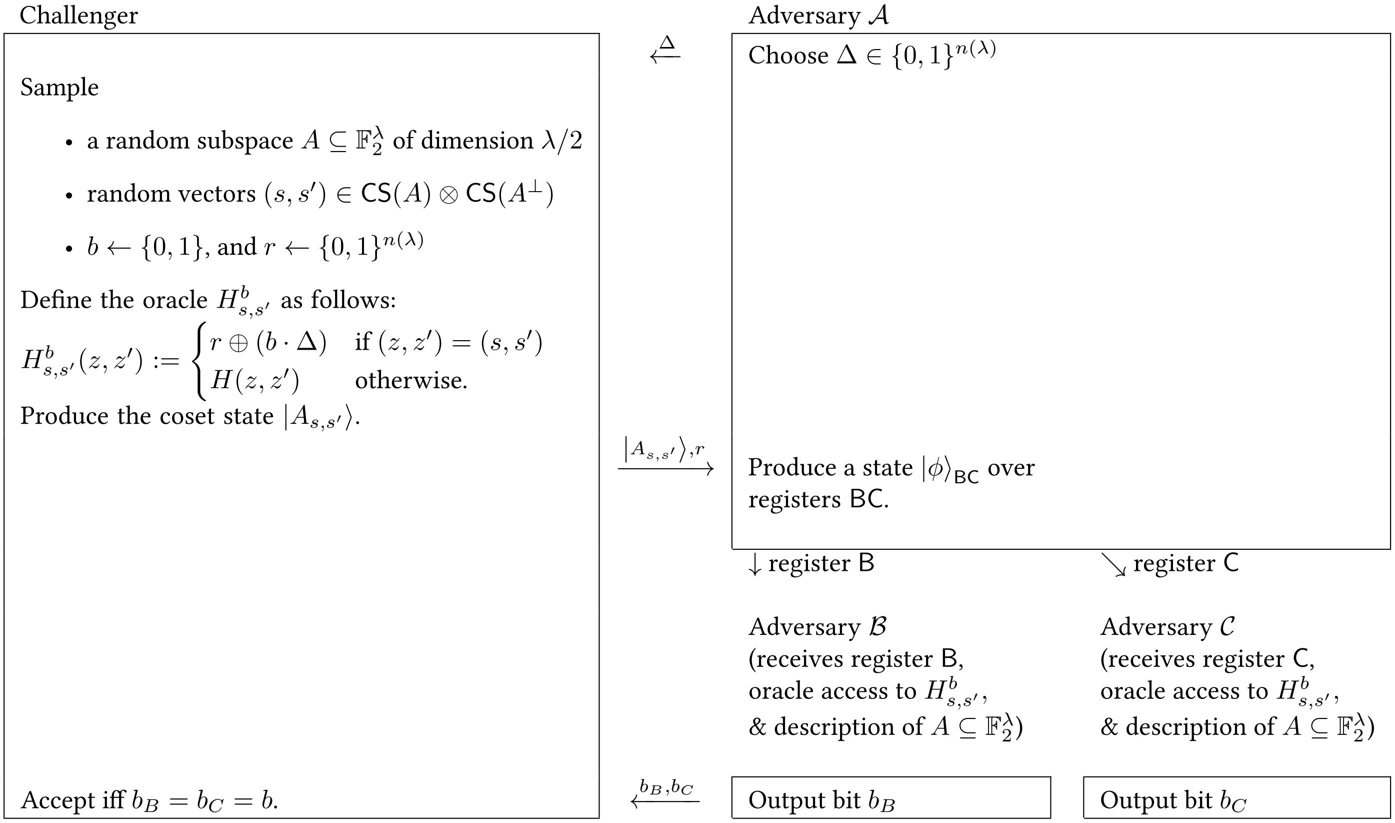
1.5 Self-testing

Reminder: In class, we only considered self-testing of the two-qubit maximally entangled state $|\phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We considered the CHSH game where Alice and Bob are asked questions $x, y \in \{0,1\}$ and respond with answers $a, b \in \{\pm 1\}$ respectively. To describe the most general quantum strategy, denote the quantum state shared by Alice and Bob by $|\psi\rangle_{AB}$ where Alice holds register A and Bob register B. Denote the observable used by Alice (resp. Bob) to answer question x, by A_x (resp. B_y). We observed that A_x and B_y are Hermitian and unitary.

Exercise 5 (1 point; Self-testing)**.** Suppose $\frac{A_0 \pm A_1}{\sqrt{2}} |\psi\rangle = B_{0/1} |\psi\rangle$ where $\{A_x\}_{x\in\{0,1\}}$ and $\{B_y\}_{y\in\{0,1\}}$ denote Alice's and Bob's operators in a general strategy they follow to play the CHSH game. Prove that B_0 and B_1 anti-commute on $|\psi\rangle$, i.e. $\{B_0, B_1\}|\psi\rangle = 0$.

2 Uncloneable Encryption

A word of advice: Do this one after you've attempted all other questions. It ended up being quite long for a 3 point question. Reminder: Let us recall Hybrid 2 in the security proof of the uncloneable encryption scheme introduced by [1].



Challenger

Sample

- a random subspace $A \subseteq \mathbb{F}_2^\lambda$ of dimension $\lambda/2$
- random vectors $(s, s') \in \mathsf{CS}(A) \otimes \mathsf{CS}(A^\perp)$
- $b \leftarrow \{0,1\}$, and $r \leftarrow \{0,1\}^{n(\lambda)}$

Define the oracle $H^b_{s,s'}$ as follows:

$$H^b_{s,s'}(z,z') := \begin{cases} r \oplus (b \cdot \Delta) & \text{if } (z,z') = (s,s') \\ H(z,z') & \text{otherwise.} \end{cases}$$

Produce the coset state $|A_{s,s'}\rangle$.

Accept iff $b_B = b_C = b$.

Adversary $\mathcal{A}$

Choose $\Delta \in \{0,1\}^{n(\lambda)}$

$\xleftarrow{\Delta}$

$\xrightarrow{|A_{s,s'}\rangle, r}$

Produce a state $|\phi\rangle_{\mathsf{BC}}$ over registers BC.

$\downarrow$ register B

$\searrow$ register C

Adversary $\mathcal{B}$
(receives register B,
oracle access to $H^b_{s,s'}$,
& description of $A \subseteq \mathbb{F}_2^\lambda$)

Adversary $\mathcal{C}$
(receives register C,
oracle access to $H^b_{s,s'}$,
& description of $A \subseteq \mathbb{F}_2^\lambda$)

Output bit b_B

Output bit b_C

$\xleftarrow{b_B, b_C}$

where

- H is a random oracle $H : \mathbb{F}_2^\lambda \times \mathbb{F}_2^\lambda \to \{0,1\}^{n(\lambda)}$,
- $\mathcal{A}, \mathcal{B}, \mathcal{C}$ all get access to P_{A+s} and $P_{A^\perp+s'}$ after the first message is sent by the challenger, and

- $\mathsf{CS}(A)$ is the set of canonical representatives for A, i.e. $\mathsf{CS}(A) = \{\mathsf{Can}_A(s) : s \in \mathbb{F}_2^n\}$.
  - $\mathsf{Can}_A(s)$ is the (lexicographically) smallest vector in $A + s := \{a + s : a \in A\}$.
- $|A_{s,s'}\rangle := \frac{1}{\sqrt{|A|}} \sum_{a \in A} (-1)^{s' \cdot a} |a + s\rangle$

For the proof, we will need to define projectors Π_0^B, Π_1^B and Π_0^C, Π_1^C. For a fixed A, r, Δ, s, s' they are defined as follows:

- Π_0^B: runs $\mathcal{B}$ with oracle access to $H_{s,s'}^0$ (B and A are the same), projects on outcome 0, and undoes the computation.
- Π_1^B : runs $\mathcal{B}$ with oracle access to $H_{s,s'}^1$ (B and A are the same), projects on outcome 1, and undoes the computation.
- Π_0^C and Π_1^C are defined similarly, with $(\mathcal{B}, \mathsf{B})$ replaced with $(\mathcal{C}, \mathsf{C})$.

We will need some more notation.

- Denote by $\{(|\phi_i\rangle, \lambda_i)\}_i$ the (eigenvectors, eigenvalues) of $(\Pi_0^B + \Pi_1^B)/2$. Similarly, let $\{(|\psi_j\rangle, \mu_j)\}_j$ denote the corresponding quantities for $(\Pi_0^C + \Pi_1^C)/2$.
- NB: $|\phi\rangle_{BC} = \sum_{ij} \alpha_{ij} |\phi_i\rangle_B \otimes |\psi_j\rangle_C$ without loss of generality.

Finally, we will assume the following holds about the state $|\phi\rangle_{BC}$ produced by $\mathcal{A}$: for every polynomial p, we have

$$\sum_{\substack{i:|\lambda_i - 1/2| > 1/p \\ j:|\mu_j - 1/2| > 1/p}} |\alpha_{ij}|^2 \leq \text{negl}(n).$$

Question 1 (3 points). *Denote by p_2 the probability that Hybrid 2 above outputs accept. We will show some of the main steps in establishing that $p_2 \leq \frac{1}{2} + \text{negl}(\lambda)$.*

1. *Show that the state $|\phi\rangle_{BC}$ is negligibly close to*
$$|\phi'\rangle_{BC} := \sum_{i:|\lambda_i - 1/2| \leq 1/p} \alpha_{ij} |\phi_i\rangle_B \otimes |\psi_j\rangle_C + \sum_{\substack{i:|\lambda_i - 1/2| > 1/p \\ j:|\mu_j - 1/2| \leq 1/p}} \alpha_{ij} |\phi_i\rangle_B \otimes |\psi_j\rangle_C .$$

2. *Show that*
$$\frac{\left(\left| (\Pi_0^B \otimes \Pi_0^C) |\phi'\rangle_{BC} \right|^2 + \left| (\Pi_1^B \otimes \Pi_1^C) |\phi'\rangle_{BC} \right|^2 \right)}{2} \tag{1}$$
is negligibly close to the probability that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is accepted by the Challenger.

3. *Show that the expression above (Eq 1) can be upper bounded by*
$$\frac{1}{2} \cdot \Bigg(\underbrace{\langle\phi'_B| (\Pi_0^B \otimes I) |\phi'_B\rangle + \langle\phi'_B| (\Pi_1^B \otimes I) |\phi'_B\rangle}_{\text{I}} + \underbrace{\langle\phi'_C| (I \otimes \Pi_0^C) |\phi'_C\rangle + \langle\phi'_C| (I \otimes \Pi_1^C) |\phi'_C\rangle}_{\text{II}} \Bigg)$$
$$+ \Re \Bigg(\underbrace{\langle\phi'_C| (\Pi_0^B \otimes \Pi_0^C) |\phi'_C\rangle}_{\text{III}} + \underbrace{\langle\phi'_B| (\Pi_1^B \otimes \Pi_1^C) |\phi'_C\rangle}_{\text{IV}} \Bigg).$$

4. *Establish that* I *(resp.* II*) is at most $\left(\frac{1}{2} + \frac{1}{p}\right) ||\phi'_B\rangle|^2$ (resp. $\left(\frac{1}{2} + \frac{1}{p}\right) ||\phi'_C\rangle|^2$).*

5. *Suppose it holds that for indices i, i' satisfying $\lambda_i + \lambda_{i'} \neq 1$, we have $\langle\phi_i| \Pi_0^B |\phi_{i'}\rangle = 0$ (we proved this in class). Using this, show that both* III *and* IV *are zero.*

6. *Quick conceptual questions:*
   (a) *In your answers above, where did you (most directly) use properties of the coset state $|A_{s,s'}\rangle$?*
   (b) *Assertion: $\Pi_0^B + \Pi_1^B = \mathbb{I}$. Either prove the assertion or justify why you don't expect the projectors to sum to identity.*

(NB. Combining these, you should be able to deduce that $p_2 \leq \left(\frac{1}{2} + \frac{1}{p}\right) \left(||\phi'_B\rangle|^2 + ||\phi'_C\rangle|^2 \right) + \text{negl} \leq \frac{1}{2} + \frac{1}{p} + \text{negl}$ and since this is true for all polynomials p, we have proved $p_2 \leq \frac{1}{2} + \text{negl}$.)

3 Haar Measure

Reminder:

- $P_{\text{sym}} := \frac{1}{k!}\sum_{\pi\in S_k} V_d(\pi)$ where S_k is the symmetric group (group of permutations over k indices), and $V_d(\pi) = \sum_{i_1,\dots i_k\in[d]} \left|i_{\pi^{-1}(1)}\dots i_{\pi^{-1}(k)}\right\rangle \langle i_1 \dots i_k|$ affects (the inverse of) this permutation on k many d-dimensional quantum systems.
- Schur-Weyl duality says that $\mathcal{M}^k(O) = \mathbb{E}_{U\leftarrow\mu_H}[U^{\otimes k} O U^{\otimes k}]$ can be expressed as a linear combination of $V_d(\pi)$, where O is a linear operator on $(\mathbb{C}^d)^{\otimes k}$.

Question 2 (3 points). *Prove that*

$$\mathcal{M}^k(|\phi\rangle\langle\phi|^{\otimes k}) = \mathbb{E}_{U\leftarrow\mu_H}[U^{\otimes k}|\phi\rangle\langle\phi|^{\otimes k} U^{\dagger\otimes k}] = \frac{P_{\text{sym}}}{\mathsf{tr}(P_{\text{sym}})}$$

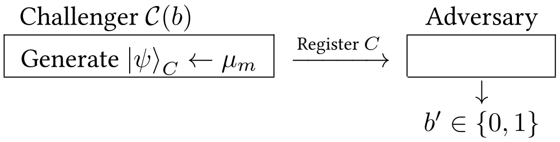
where $|\phi\rangle \in \mathbb{C}^d$. You could either do this directly yourself, or do it by establishing the following sub-steps.

1. *Show that $V_d(\sigma^{-1})\mathcal{M}(\phi) = \mathcal{M}(\phi)$ where we use $\mathcal{M}$ to denote $\mathcal{M}^k$ and ϕ to denote $|\phi\rangle\langle\phi|$.*
2. *Show that $\sum_{\pi\in S_k} c_\pi V_d(\sigma^{-1})V_d(\pi) = \sum_{\pi\in S_k} c_{\sigma\pi} V_d(\pi)$.*
3. *Use these two observations to prove that $\mathcal{M}(\phi) = \sum_{k\in S_k} c_{\sigma\pi}V_d(\pi)$.*
4. *Using your result above, reason that $c_\sigma = c_I$ for all $\sigma\in\pi$ (here I is the identity permutation).*
5. *Now, show that $\mathcal{M}^k(\phi) \propto P_{\text{sym}}$*
6. *Finally, establish the normalisation to complete the proof.*

4 PRS $\implies$ Commitments

Reminder (PRSG $\implies$ Commitments). In addition to $\text{Hyb}_0, \text{Hyb}_1$ as defined above 2, Morimae and Yamakawa [4] also define Hyb_2 as follows.

- $\text{Hyb}_2(b)$


Challenger $\mathcal{C}(b)$
Generate $|\psi\rangle_C \leftarrow \mu_m$
Register C
Adversary
$b' \in \{0,1\}$

Question 3 (2 points). *We want to establish computational hiding of [4]'s commitment scheme by showing*

$$|\Pr[1\leftarrow\text{Hyb}_0(0)] - \Pr[1\leftarrow\text{Hyb}_0(1)]| \le \text{negl}(\lambda). \tag{2}$$

To this end, prove the following sub-claims (where we use $a\approx b$ to denote $|a-b|\le\text{negl}$).

1. $\Pr[1\leftarrow\text{Hyb}_0(b)] \approx \Pr[1\leftarrow\text{Hyb}_1(b)]$ *for $b=0$ as well. Show that if this were not the case, then one can construct an adversary $\mathcal{A}'$ that breaks the security of the underlying PRSG (pseudorandom state generator) by distinguishing a pseudorandom state from a Haar random state.*
2. $\Pr[1\leftarrow\text{Hyb}_1(b)] \approx \Pr[1\leftarrow\text{Hyb}_2(b)]$ *for $b=0$ should be immediate (write one line justifying it). How will you modify $\mathcal{A}'$ above to establish the $b=1$ case? Using these observations, establish (2).*

5 QMA=BQP and yet PRUs exist

Question 4 (3 points). *The answers to the following, should be enough for you to prove that relative to $(\mathcal{U},\mathcal{C})$ (as defined above Exercise 3), PRUs exist.*

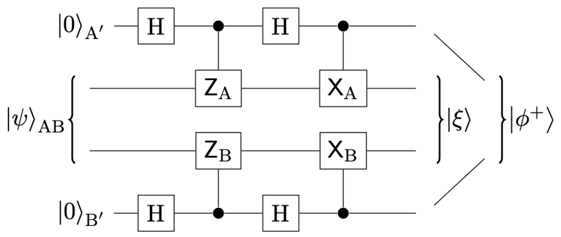


Figure 1: Local Isometry Φ taken form [5].

1. *Fix the system size (on which the unitaries act) to be n qubits. Denote the PRU ensemble you defined in Exercise 3 as $\tilde{U} := \{\tilde{U}_k\}_{k\in\{0,1\}^n}$. To establish that $\tilde{U}$ is a PRU, does it suffice to show that for all efficient quantum oracle-algorithms $\mathcal{A}$,*

$$\mathsf{adv}(\mathcal{A}) := \Pr_{k\leftarrow\{0,1\}^n}[1 \leftarrow \mathcal{A}^{\tilde{U}_k}] - \Pr_{U\leftarrow\mu_{2^n}}[1 \leftarrow \mathcal{A}^{U}] \leq \text{negl}(n)?$$

   If so, justify your answer (Hint: Do you need $\mathcal{U}$ to efficiently generate $\tilde{U}$? If so, should that appear somewhere in your definition of a secure PRU, relative to $\mathcal{U}$).
   If not, how will you modify adv*? (Hint: Note that $\mathcal{C}$ is uncorrelated to $\mathcal{U}$)*
   (NB. You can neglect non-uniformity considerations i.e. it suffices, in the definition of PRUs to be secure against uniform adversaries—no need to consider advice, for the purposes of this exam).

2. *Suppose $\mathcal{A}^{O,U}$ is an oracle algorithm that makes queries to $U = (U_1, \ldots U_N) \in \mathbb{U}(D)^N$. Prove that*

$$\mathbb{E}_{U\leftarrow\mu_D^N}\left[\Pr_{k\leftarrow[N]}[1 \leftarrow \mathcal{A}^{U_k,U}]\right] = \Pr_{k\leftarrow[N]}[1 \leftarrow \mathcal{B}^{e_k}] \tag{3}$$

   where $[N] = \{1, \ldots, N\}$, $e_k = 00\ldots010\ldots0$ is an N-bit all-zero string except it has 1 at the kth position, and $\mathcal{B}$ is defined as follows:

   - *$\mathcal{B}$ samples $(V_0, V_1 \ldots V_N) \leftarrow \mu_D^{N+1}$ (which are unitaries in $\mathbb{U}(D)^{N+1}$).*
   - *$\mathcal{B}$ runs $\mathcal{A}$, replacing queries to O by queries to V_0 and queries to $U_k \in U$ by V_0 if $x_k = 1$, and by V_k if $x_k = 0$.*

   Prove also that

$$\mathbb{E}_{U\leftarrow\mu_D^N}\left[\Pr_{O\leftarrow\mu_D}[1 \leftarrow \mathcal{A}^{O,U}]\right] = \Pr[1 \leftarrow \mathcal{B}^{0^N}]. \tag{4}$$

(NB. Using a bound on unstructured search, one can now argue that the difference between Eq. (3) and (4) must be at most cT^2/N, where T is the number of queries made by $\mathcal{A}$ and c is a universal constant.).

5.1 Self-testing

Reminder: Continuing with the discussion preceding Exercise 5, recall that we explicitly defined the local isometry Φ as shown in Figure 1. For $Z_A = \frac{1}{\sqrt{2}}(A_0 + A_1), Z_B = B_0, X_A = \frac{1}{\sqrt{2}}(A_0 - A_1), X_B = B_1$, we proved in class that

$$\Phi[|\psi\rangle] = \sum_{i,j\in\{0,1\}} |ij\rangle_{A'B'} \otimes \underbrace{\left(\frac{1}{4}X_A^i(\mathbb{I} + (-1)^i Z_A)X_B^j(\mathbb{I} - (-1)^j Z_B)\right)}_{=:\hat{f}_{ij}} |\psi\rangle_{AB}\,.$$

It may help to also recall that we proved the following:

- $\{Z_A, X_A\} = 0, \{Z_B, X_B\}|\psi\rangle = 0$
- $Z_A|\psi\rangle = Z_B|\psi\rangle$ and $X_A|\psi\rangle = X_B|\psi\rangle$

Question 5 (1 point; self-testing). *Using the aforementioned, show that that $\hat{f}_{01}|\psi\rangle = \hat{f}_{10}|\psi\rangle = 0$ and $\hat{f}_{11}|\psi\rangle = \hat{f}_{00}|\psi\rangle$.*

5.2 QFHE

Reminder: We look at some of the steps that go into applying the encrypted CNOT operation in Mahadev's QFHE scheme. The question will ask you to fill in a detail or two. Let $|\psi\rangle = \sum_{a,b\in\{0,1\}} \alpha_{ab} |ab\rangle$ be an arbitrary pure two qubit state. The goal is to apply CNOT^s where $\hat{s}$ is given (recall $\hat{s}$ is a classical FHE encryption of s).

[BEGIN SNIPPET]

1. Clasically, compute a description of the claw-free pair (f_0, f_1) corresponding to $\hat{s}$ (a reminder about the claw-free pair is appended, just in case).
2. Prepare the state
$$|\psi_1\rangle = \sum_{\substack{a,b,\mu\in\{0,1\}\\ r\in\mathcal{R}}} \alpha_{ab} |ab\rangle |\mu r\rangle |f_a(\mu, r)\rangle$$
and measure the last register to obtain a y. Denote by $|\psi_{1'}\rangle$ the resulting state.
3. XOR μ_a into the second register. This should result in
$$|\psi_3\rangle := \sum_{ab\in\{0,1\}} \alpha_{ab}(\mathbb{I}\otimes X^{\mu_0})\mathsf{CNOT}^s_{12} |a,b\rangle |\mu_a r_a\rangle$$
where $f_0(\mu_0, r_0) = f_1(\mu_1, r_1) = y$.
4. Measure the last register in the Hadamard basis...

[END SNIPPET]

Question 6 (1 point; QFHE). *Write down $|\psi_{1'}\rangle$ and prove that following step 3, one indeed gets the state $|\psi_3\rangle$ (up to a change in variable names, depending on the choice you make in specifying $|\psi_{1'}\rangle$).*
(Hint: it may help to observe that $|b\oplus\mu_a\rangle = X^{\mu_0}|b + a\cdot s\rangle$; if you use this, prove that it is true first)

Reminder: Additional properties we need from (N)TCFs (Noisy Trapdoor Claw-free Functions)—Given an encryption $\hat{s}$ of the bit s, the following must hold:

1. One can efficiently compute trapdoor claw-free functions $f_0, f_1 : \{0,1\}\times\mathcal{R}\to\mathcal{Y}$ s.t. $\forall\,(\mu_0, r_0),(\mu_1, r_1)\in\{0,1\}\times\mathcal{R}$ which is a claw (i.e. $f_0(\mu_0, r_0) = f_1(\mu_1, r_1)$), it holds that $\mu_0\oplus\mu_1 = s$.
2. One can also compute a classical FHE encryption of the trapdoor (which allows one to invert the function) corresponding to the pair (f_0, f_1).

5.3 Verification

Reminder:

- An XZ Local Hamiltonian acting on n qubits is of the form $H = \sum_{\ell\in[m]}\gamma_\ell H_\ell$ where H_ℓ consists of tensor products of σ_x and σ_z (and σ_I) matrices, acting non-trivially (i.e. non-identity) on at most k qubits, i.e. for each ℓ, $H_\ell = \otimes_{j\in n}\sigma_{W_{j,\ell}} \in \{\sigma_x, \sigma_z, \sigma_I\}^{\otimes n}$ with $\left|\{j | j\in[n]\wedge\sigma_{W_{j,\ell}}\neq\sigma_I\}\right| \le k$.
- The Hamiltonian Test $G(H)$ for such Hamiltonians, as constructed by [2], is as shown in Figure 2.
- In class, we proved the following (as Lemma 7). Let $\omega_h(H) := 1 - p\left(\frac{1}{2m}\sum_{\ell\in[m]}|\gamma_\ell| - \frac{1}{2}\lambda_0(H)\right)$ where $\lambda_0(H)$ denotes the smallest eigenvalue of H. If the provers use the honest strategy in the Pauli-Braiding test, then the maximum acceptance probability in $G(H)$ is $\omega_h(H)$. We now write down some of the main steps that go into this proof and ask you to fill in some of the gaps in the argument.

Question 7 (2 points, Verification). *Given the premise above, prove the following assertions.*

1. *The acceptance probability in $G(H)$ depends uniquely on the strategy of the first prover in the energy test (in particular, it does not depend on the strategy of the second prover in the energy test).*
2. *For a fixed H_ℓ, the verifier rejects with probability*
$$\frac{|\gamma_\ell| + \gamma_\ell\mathbb{E}[\Pi_{i\in[n]}d_i]}{2}. \tag{5}$$

(a) Pauli-Braiding Test: (We don't need the details here, except that the view of Prover 2 is exactly the same as that in the Energy Test).

(b) Energy Test

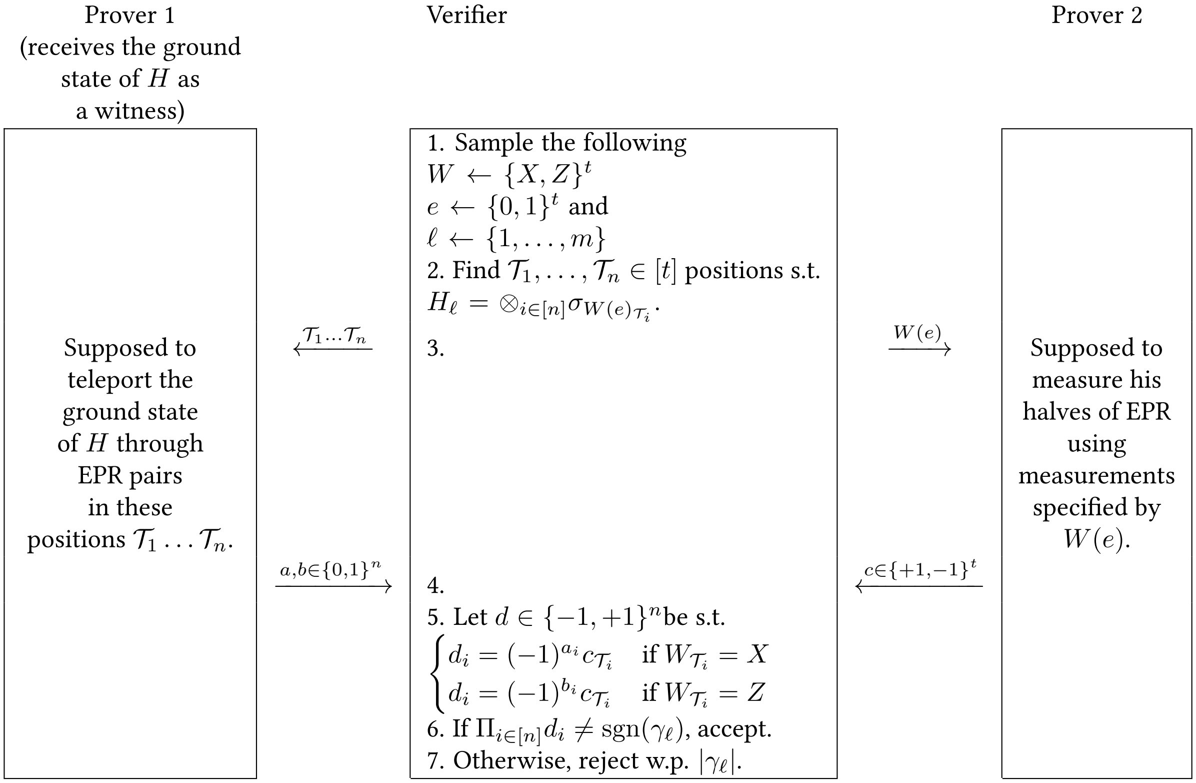

Prover 1
(receives the ground state of H as a witness)
Verifier
Prover 2
Supposed to teleport the ground state of H through EPR pairs in these positions $\mathcal{T}_1 \ldots \mathcal{T}_n$.
1. Sample the following
$W \leftarrow \{X, Z\}^t$
$e \leftarrow \{0,1\}^t$ and
$\ell \leftarrow \{1, \ldots, m\}$
2. Find $\mathcal{T}_1, \ldots, \mathcal{T}_n \in [t]$ positions s.t.
$H_\ell = \otimes_{i\in[n]} \sigma_{W(e)_{\mathcal{T}_i}}$.
$\xleftarrow{\mathcal{T}_1 \ldots \mathcal{T}_n}$
3.
$\xrightarrow{W(e)}$
$\xrightarrow{a,b \in \{0,1\}^n}$
4.
$\xleftarrow{c \in \{+1,-1\}^t}$
5. Let $d \in \{-1,+1\}^n$be s.t.
$\begin{cases} d_i = (-1)^{a_i} c_{\mathcal{T}_i} & \text{if } W_{\mathcal{T}_i} = X \\ d_i = (-1)^{b_i} c_{\mathcal{T}_i} & \text{if } W_{\mathcal{T}_i} = Z \end{cases}$
6. If $\Pi_{i\in[n]} d_i \neq \text{sgn}(\gamma_\ell)$, accept.
7. Otherwise, reject w.p. $|\gamma_\ell|$.
Supposed to measure his halves of EPR using measurements specified by $W(e)$.

Figure 2: The verifier performs the Pauli-Braiding Test with probability $1-p$ and the Energy Test with probability p. The provers are assumed to share t EPR pairs with t scaling roughly as $n \log n$.

Denote by τ the reduced state held by the second prover on qubits $\mathcal{T} := (\mathcal{T}_1, \ldots \mathcal{T}_n)$ of his EPR halves (after 'teleportation').

3. *Given the answers (a, b) of the first prover, the following behaviours of the second prover are equivalent:*

   (a) *Measure τ using H_ℓ and obtain outcome $\Pi_{i\in\mathcal{T}} c_i$.*

   (b) *Measure $\rho = Z^b X^a \tau X^a Z^b$ using H_ℓ and obtain outcome $\Pi_{i\in\mathcal{T}} d_i$.*

   Hint: Measuring a qubit $|\phi\rangle$ in the Z-basis w/ outcome $f \in \{\pm 1\}$ is equivalent to getting outcome $(-1)^g f$ when measuring $Z^h X^g |\phi\rangle$ in the Z-basis. Similarly, measuring $|\phi\rangle$ in the X-basis w/ outcome $f \in \{\pm 1\}$ is equivalent to getting outcome $(-1)^h f$ when measuring $Z^h X^g |\phi\rangle$ in the X-basis. (Equivalence as in the output distribution is identical).

4. *The rejection probability in Eq (5), averaged over ℓ, is*

$$\frac{1}{2m} \sum_{\ell\in[m]} |\gamma_\ell| + \frac{1}{2}\mathsf{tr}(\rho H).$$

   The acceptance *probability in $G(H)$ is at most*

$$1 - p\left(\frac{1}{2m} \sum_{\ell\in[m]} |\gamma_\ell| - \frac{1}{2}\lambda_0(H)\right) = w_h(H).$$

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