Linear Algebra—Winter/Spring 2026

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Problem Set 5

Due: 8 PM, Tuesday February 24, 2026

Instructions

a. *Timeline.* In general, expect the first "half" of a problem set to be released by Tuesday nights (or whenever the first lecture of the week takes place). Expect the full problem set to be released by Friday nights (or whenever the second lecture of the week takes place). The full assignment will, *typically*, be due on Tuesday nights, 8 PM.

b. *Punctuality.* Aim to get the assignment done three days prior to the deadline. We will not be able to entertain any requests for extensions. If you seek clarification from us and we do not respond on time, do the following: (a) assume whatever seems most reasonable, and solve the question under that assumption and (b) ensure you have evidence for making the clarification request and report the incident to one of the PhD TAs.

c. *Resources.* Feel free to use any resource you like (including generative AI tools) to get help but please make sure you understand what you finally write and submit. The TAs may ask you to explain the reasoning behind your response if something appears suspicious.

d. *Graded vs Practice Questions.* Problems numbered using Arabic numerals (i.e. $1, 2, 3, \dots$) constitute assignment problems that will be graded. Problems numbered using Roman numerals (i.e. i,ii,iii,...) are practice problems that will not be graded.

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1. (a) Prove that the dimension of the vector space $\mathbb{R}$ over the field $\mathbb{R}$ is one.

(b) Prove that the dimension of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ is infinite.
Hint. You need not explicitly construct a basis. It suffices to exhibit an infinite set of real numbers that is linearly independent over $\mathbb{Q}$. For example, if $p_1, p_2, \dots, p_n$ are distinct prime numbers, consider whether the real numbers $\log p_1, \log p_2, \dots, \log p_n$ are linearly independent.

2. Let V be an n-dimensional vector space over a field F, and let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered bases of V. Suppose that $[\alpha]_{\mathcal{B}} = [\alpha]_{\mathcal{B}'}$ for all $\alpha \in V$. Show that $\mathcal{B} = \mathcal{B}'$.

3. Let V be an n-dimensional vector space over a field F. Suppose that vectors

$$\alpha_1, \alpha_2, \dots, \alpha_n \in V$$

are linearly independent.

Using Corollary 1 of Chapter 2 of the textbook, which states that if W is a proper subspace of V, then $\dim W < \dim V$, show that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of V.

4. Let V be an n-dimensional vector space over a field F, and let

$$\alpha_1, \alpha_2, \ldots, \alpha_n \in V$$

be vectors that span V. Show that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of V.

5. Let $\mathcal{P}_2$ denote the vector space of all polynomials of degree less than or equal to 2 over the field $\mathbb{R}$. Let

$$B = \{1, x, x^2\} \quad \text{and} \quad B' = \{1 + x,\ 2 + 3x,\ 5 + x + x^2\}.$$

Show that B and B' are ordered bases of $\mathcal{P}_2$. Find a matrix P such that

$$[p(x)]_B = P\,[p(x)]_{B'} \quad \text{for all } p(x) \in \mathcal{P}_2.$$

6. Consider the vectors in $\mathbb{R}^4$ defined by $\alpha_1 = (-1, 0, 1, 2)$, $\alpha_2 = (3, 4, -2, 5)$, $\alpha_3 = (1, 4, 0, 9)$. Find a homogeneous system of linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the three given vectors.

7. Let m, n be positive integers. Suppose R is an $m \times n$ row-reduced echelon matrix over some field F and denote its non-zero row vectors by $\rho_1, \rho_2 \ldots \rho_r$. Let W be its row space and suppose the vector $\beta = (b_1, b_2 \ldots b_n)$ is in W. Below, we use k_i to denote the index of the first non-zero column in the row vector ρ_i.

i. Find the coefficients $c_1, \ldots c_r$ (in terms of $b_1, \ldots b_n$) when β is written as $\beta = c_1\rho_1 + \ldots c_r\rho_r$.

ii. Let $1 \leq s \leq r$. Prove that there is exactly one vector σ_s in W such that the k_sth coordinate of $\sigma_s = 1$ and the remaining k_ith coordinates (for $i \in \{1 \ldots r\} \setminus s$) of σ_s are zero.

iii. Is $\sigma_s = \rho_s$? Prove your answer.

8. Let m, n be positive integers. Suppose W is a subspace of F^n with $\dim W \leq m$. Let $\beta = (b_1, \ldots b_n)$ be a vector in W. Let $T := \{k_1 \ldots k_r\}$ denote a set of integers such that for any $t \in T$, there is some $\beta \neq 0$ in W such that its first non-zero coordinate of β occurs in column t, i.e. $\beta = (0, \ldots, 0, b_t, \ldots b_n)$ where $b_t \neq 0$.

i. Explain briefly why one can always assume $k_1 < k_2 < \cdots < k_r$ without loss of generality.

ii. Let k_i, k_j be distinct indices. Show that $\beta_1 = (0, \ldots 0, b_{k_i}, b_{k_i+1}, \ldots, b_r)$ and $\beta_2 = (0, \ldots 0, b'_{k_j}, b'_{k_j+1}, \ldots, b'_r)$ are linearly independent (assume $b_{k_i} \neq 0$ and $b'_{k_j} \neq 0$).

iii. Prove that for each integer k_s there is exactly one vector γ_s in W such that the k_sth coordinate of γ_s is 1 and the k_ith coordinate of γ_s is 0, for $i \neq s$.

9. Which of the following functions T from $\mathbb{R}^2$ to $\mathbb{R}^2$ are linear transformations? Explain.

(a) $T(x_1, x_2) = (x_1, 1 + x_2)$;

(b) $T(x_1, x_2) = (x_1^2, x_2)$;

(c) $T(x_1, x_2) = (0, x_1 - x_2)$.

i. Let V be a finite-dimensional vector space over a field F, and let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered bases of V. In Theorem 7 of Chapter 2 of the textbook, it is shown that there exists a matrix P such that

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'} \quad \text{for all } \alpha \in V.$$

Explain why this matrix P is unique. In particular, indicate how uniqueness follows implicitly from the construction used in the proof.

ii. Let V be a finite-dimensional vector space over a field F, and let $\mathcal{B}$ be an ordered basis of V. In Theorem 8 of Chapter 2 of the textbook, it is shown that for any $n \times n$ invertible matrix P there exists an ordered basis $\mathcal{B}'$ of V such that

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'} \quad \text{for all } \alpha \in V.$$

Explain why the ordered basis $\mathcal{B}'$ is unique.

iii. *This is a variant of Example 21 that focuses on using the "second method" in the textbook.* Let W be the subspace spanned by $\alpha_1 = (2, 4, 4, 2), \alpha_2 = (2, 8, 4, 4), \alpha_3 = (-2, 0, -4, 3)$. Let B be the 4×3 matrix formed by taking $\alpha_1, \alpha_2, \alpha_3$ as its columns.

   i. Using the augmented matrix method, find the condition on Y such that $BX = Y$ has a solution.

   ii. Using your answer, how would you characterise vectors $\beta = (b_1, b_2, b_3, b_4) \in W$ (i.e. can you give an explicit condition on the coordinates of β to ensure β is in W)?

   iii. Using the answer to the first part, given that $\beta = (b_1, \ldots, b_4) \in W$, find the coordinates (x_1, x_2, x_3) for β in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$.

iv. Solve Example 22 from the textbook.