Ch 2—§2.5 Summary of Row-Equivalence & §2.6 Computatinos Concerning Subspaces

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Theorem 11.

Let
m and n be positive integers and
F be a field.

Suppose
W is a subspace of F^n and
$\dim W \leq m$.

Then
there is precisely one $m\times n$ row-reduced echelon matrix over F
which has W as its row space.

Proof.

There is at least one $m\times n$ row reduced echelon matrix
with row space W.

Since $\dim W \leq m$
we can select some m vectors
$\alpha_1 \dots \alpha_m$ in W
that span W.

Let A be the
$m\times n$ matrix with row vectors
$\alpha_1, \dots \alpha_m$.
Let R be a row-reduced echelon matrix which is
row-equivalent to A.

NB: The row space of R is W (using Theorem 10;
because row space of R = row space of A
and row space of A is by constrution W)

Now
let R be any row-reduced echelon matrix
that has W as its row space.

Let $\rho_1 \dots \rho_r$ be the non-zero row vectors of R
and
suppose that
the leading non-zero entry of ρ_i occurs
in column k_i
(for $i = 1 \dots r$).

NB: The vectors $\rho_1, \dots, \rho_r$ form a basis for W.

Recall (from the proof of Theorem 10):
If $\beta = (b_1, \dots, \beta_n)$ is in W then

one can write $\beta = c_1\rho_1 + \dots + c_r\rho_r$
where
$c_i = b_{k_i}$

i.e. the unique expression for β as
a linear combination of $\rho_1 \dots \rho_r$ is
$\beta = \sum_{i\in\{1\dots r\}} b_{k_i}\rho_i$.

Consequently, any vector β
is determined if one knows the coordinates b_{k_i}
for $i \in \{1, \dots r\}$.

(e.g. ρ_s is the unique vector in W that has its
k_sth coordinate 1 and
k_ith coordinate 0 for all $i \neq s$
(here coordinate is relative to
the standard basis)

Suppose β is in W and $\beta \neq 0$.

Claim. The first non-zero coordinate (standard basis) of β

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occurs in one of the columns k_s.

Proof.
Since $\beta = \sum_{i=\{1\ldots r\}} b_{k_i}\rho_i$ and $\beta \neq 0$
one can write

$\beta = \sum_{i=s}^{r} b_{k_i}\rho_i$ where $b_{k_s} \neq 0$

(there has to be at least one non-zero b_j)

From the conditions
(a) $R(i, j) = 0$ if $j < k_i$
(b) $R(i, k_j) = \delta_{ij}$
(c) $k_1 < \cdots < k_r$.

it follows that $R_{ij} = 0$
if $i > s$ and $j \leq k_s$.
(i.e. for rows beyond the sth row
the first k_s columns are zero)

Thus,
$\beta = (0, \ldots 0, b_{k_s}, \ldots, b_n)$ for $b_{k_s} \neq 0$.

(by basically writing out the linear combination over ρ_is and
using the form of R above)

Clearly, the first non-zero coordinate of β occurs in column k_s
as claimed.

□

NB: For each k_s (for $s \in \{1 \ldots r\}$)
there is a vector in W that has a non-zero k_sth coordinate
i.e. ρ_s.

Claim. R is uniquely determined by W.

Proof.
Let us describe R in terms of W explicitly.

Consider all vectors $\beta = (b_1 \ldots b_n)$ in W.

If $\beta \neq 0$
then the first non-zero entry occurs in some column t
i.e.
$\beta = (0, \ldots, 0, \quad b_t, \ldots b_n)$ where $b_t \neq 0$.

Let $T \coloneqq \{k_1, \ldots k_r\}$ denote the set of integers t such that
for any $t \in T$ the following holds:
there is some $\beta \neq 0$ in W such that
the first non-zero coordinate occurs in column t.

Without loss of generality
assume $k_1 < k_2 < \cdots < k_r$.

NB: For each of the positive integers k_s
there will be one and only one vector ρ_s in W
such that
the k_sth coordinate of ρ_s is 1
and
the k_ith coordinate of ρ_s is 0 for $i \neq s$.

<---- How?
One can take
linear combinations
and remove
all the non-zero entries in k_is for $i \neq s$
(just as one does for row-reduced echelon forms)

But why should it be unique? Couldn't the remaining entries allow multiple such vectors to exist?
No, because if there was more than one vector that has non-zero entries
beyond say k_r
then one could take their linear combinations and find a vector
whose k_rth entry is zero
Ok, but uniqueness still needs a proof that I am missing—or maybe it is obvious somehow.

Now, R is the $m \times n$ matrix
that has $\rho_1, \ldots \rho_r, 0, \ldots 0$ as row vectors.

■

Corollary.
Each $m \times n$ matrix A is row-equivalent to one and only one row-reduced echelon matrix.

Proof.
We have that A is row-equivalent to
at least one row reduced echelon matrix R.

Suppose A is row-equivalent to
another row reduced echelon matrix R'.

Using the Theorem 10,
we conclude that both R and R'
have the same row span.
Therefore
R and R' must be identical.

Corollary.
Let A and B be $m \times n$ matrices over the field F.
Then A and B are row-equivalent
iff
they have the same row space.

Proof.
$\Rightarrow$
From Theorem 9, it follows that
if A and B are row-equivalent
then they have the same row space

$\Leftarrow$
Suppose that A and B have the same row space.
Now A is row-equivalent to a row-reduced echelon matrix R
And B is row-equivalent to a row-reduced echelon matrix R'.

Since A and B have the same row space
R and R'also have the same row space.

From Theorem 11,
it follows that $R = R'$
and that in turn means that
A is row-equivalent to B.

Story:
To summarise
if A and B are $m \times n$ matrices over the field F
the following statements are equivalent:

(1) A and B are row-equivalent
(2) A and B have the same row space
(3) $B = PA$ where P is an invertible $m \times m$ matrix.

Remark:
A fourth equivalent statement is that
(4) the homogeneous systems $AX = 0$ and $BX = 0$
have the same solutions;

While we know row equivalence of A and B
implies that both systems have the same solution,
we skipe the proof of the converse for now.

§ 2.6 Computations Concerning Subspaces

Story:
- We now show
  how elementary row operations provide
  a standarised method of answering
  certain concrete questions
  concerning subpsaces of F^n.
  - We already derived the facts we are going to need
    but we gather them here for convenience.
  - This discussion applies to
    any n-dimensional vector space

over the field F
if one selects a fixed orderd basis $\mathcal{B}$ and
describes each vector α in V by the n-tuple
$(x_1 \dots x_n)$ that gives th ecoordinate of α
in the ordered basis $\mathcal{B}$.

- Suppose we are given m vectors $\alpha_1 \dots \alpha_m$ in F^n.
  Here, we look at the following questions.

  a. How does one determine if
     the vectors $\alpha_1 \dots \alpha_m$ are linearly independent?

     More generally
     how does one find the dimension of the subspace W
     spanned by these vectors?

  b. Given β in F^n
     how does one determine whether β
     is a linear combination of $\alpha_1 \dots \alpha_m$
     (i.e. whether β is in the subspace W)?

  c. How can one give an explicit description of
     the subspace W?

  (The third point is a bit vague—what does "explicit description" mean—
  but it will be clarified by an example shotrly).

- Let us start with (a) and (b)

  - Let A be the $m \times n$ matrix with
    row vectors $\alpha_i = (A_{i1}, \dots, A_{in})$.

    Perform a sequence of elementary row operations
    starting with A and
    terminating with a row-reduced echelon matrix R.

    We have previously described how to do this.

    At this point
    the dimension of W (the row space of A)
    is apparent
    (it is simply the number of non-zero row vectors of R).

    If $\rho_1 \dots \rho_r$ are the nonzero row vectors of R
    then
    $\mathcal{B} = \{\rho_1 \dots \rho_r\}$
    is a basis for W.

    If the first non-zero coordinate of ρ_i is
    the k_ith one
    then, we have the following for $i < r$

    $R(i,j) = 0 \quad \text{if } j < k_i$
    $R(i,k_j) = \delta_{ij}$
    $k_1 < \cdots < k_r$

    The subspace W consists of all vectors
    $\beta = c_1\rho_1 + \cdots + c_r\rho_r = \sum_{i \in \{1 \dots r\}} c_i(R_{i1}, \dots R_{in})$.

    The coordinates $b_1 \dots b_n$ of such a vector β are then

    $b_j = \sum_{i \in \{1 \dots r\}} c_i R_{ij}$.

    In particular, $b_{k_j} = c_j$ and so
    if $\beta = (b_1 \dots b_n)$ is a linear combination of the ρ_is
    it must be the particular linear combination

    $\beta = \sum_{i \in \{1 \dots r\}} b_{k_i}\rho_i$. [2-24]

    The conditions on β that [2-24] holds are
    $b_j = \sum_{i \in \{1 \dots r\}} b_{k_i} R_{ij}$ for $j \in \{1 \dots n\}$. [2-25]

    Now [2-25] is the explicit description of the subspace W
    (Recall: W the row space of A)

(Recall: W the row space of A)
(recall: W is the subspace spanned by $\alpha_1 \dots \alpha_m$;
(recall: α_i were the row vectors of the initial matrix A)
What is this explicit description?
It is the subspace consisting of vectors β in F^n
whose coordinates
satisfy the conditions in [2-25].

- What kind of description is [2-25]?

First, it describes W as
all solutions to the homologeous linear equations [2-25].

The system of equations has a very special nature
it expresses $(n - r)$ of the coordinates as
linear combinations of the
r distinguished coordinates
$b_{k_1} \dots b_{k_r}$.

One has complete freedom of choice
in the coordinates b_{k_i},
i.e. if $c_1, \dots c_r$ are any r scalars
there is
one and only one vector β in W
that has c_i as its k_ith coordinate
(me: $c_i = b_{k_i}$ for each $i \in \{1 \dots r\}$
as the remaining coordinates
get fixed).

- The main point is the following:
Given the vectors α_is (row vectors of A)
row-reduction is a straightforward method
of determininng
the integers $r, k_1 \dots k_r$
and
the scalars R_{ij} that are used in [2-25]
(to describe the subspace spanned by α_is)

One should observe (as we did in Theorem 11)
that every subspace W of F^n
has a description of the type [2-25].

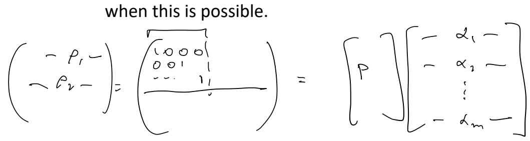
We also point out a few things about question (b)
[recall: given $\beta \in F^n$, how do we test
if $\beta \in W$?]

Recall from Section 1.4
we already discussed how one can find
an $m \times m$ invertible matrix P such that
$R = PA$.

The knowledge of P enables one to find the scalars
$x_1 \dots x_m$ such that

$$\beta = x_1\alpha_1 + \cdots + x_m\alpha_m$$

when this is possible.



β would specify y_is s.t. $\beta = \sum y_i \rho_i$

one can write using P in terms of α_is

and find x_js.

To see this, note that

$$\rho_i = \sum P_{ij}\alpha_j$$

$j \in \{1 \ldots m\}$

(i.e. the ith row of the row-echelon matrix
can be written as above)

so that if $\beta = (b_1 \ldots b_n)$ is a linear combination of α_js, we have

$$\beta = \sum_{i \in \{1 \ldots r\}} b_{k_i} \rho_i$$

$$= \sum_i b_{k_i} \sum_{j=1}^{m} P_{ij} \alpha_j = \sum_{j,i} b_{k_i} P_{ij} \alpha_j$$

and that means

$$x_j = \sum_{i \in \{1 \ldots r\}} b_{k_i} P_{ij}$$

is one possible choice for x_i (there may be many—depending on P).

"SECOND METHOD"

The question of whether $\beta = (b_1 \ldots b_n)$ is
a linear combination of α_is and if so
what the scalars x_i are
can also be looked at by asking
whether the system of equations

$$\sum_{i=1}^{m} A_{ij} x_i = b_j \text{ for } j \in \{1 \ldots n\}$$

has a solution and what those solutions are.

The coefficient matrix of this system of equations
is the $n \times m$ matrix B with
column vectors $\alpha_1 \ldots \alpha_m$.

In Chapter 1
we discussed the use of elementary row operations in
solving a system of equations $BX = Y$.

Let us consider one exmalpe in which
we adopt both points of view in
answering questions about subspaces of F^n.

Example 21.

Let W be the subspace of R^4 spanned by the vectors

$$\alpha_1 = (1, 2, 2, 1)$$
$$\alpha_2 = (0, 2, 0, 1)$$
$$\alpha_3 = (-2, 0, -4, 3).$$

(a) Prove that $\alpha_1, \alpha_2, \alpha_3$ form a basis for W
(i.e. these vectors obviously span W by definition
show that they are also linearly independent)

(b) Let $\beta = (b_1, b_2, b_3, b_4)$ be a vector in W.
What are the coordinates of β
relative to the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$?

(c) Let

$$\alpha_1' = (1, 0, 2, 0)$$
$$\alpha_2' = (0, 2, 0, 1)$$
$$\alpha_3' = (0, 0, 0, 3).$$

Show that $\alpha_1', \alpha_2', {\alpha_3}'$ form a basis for W.

(d) If β is in W,
let X denote the coordinate matrix of β
relative to the α-basis, and
let X' denote the coordinate matrix of β
relative to the α'-basis.

Find the 3×3 matrix P such that $X = PX'$ for every such β.

To answer these questions using the first method
(introduced in this chapter)
we form the matrix A with row vectors $\alpha_1, \alpha_2, \alpha_3$
find the row-reduced echelon matrix R that is

find the row reduced echelon matrix R that is
row-equivalent to A
and simultaneously perform the same operations on identity
to obtain the invertible matrix Q such that
$R = QA$

(more explicitly, we have the following:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to Q = \frac{1}{6}\begin{bmatrix} 6 & -6 & 0 \\ -2 & 5 & -1 \\ 4 & -4 & 2 \end{bmatrix}$$

(a) Clearly, R has rank 3
so $\alpha_1, \alpha_2, \alpha_3$ are independent.

(b) Which vectors $\beta = (b_1, b_2, b_3, b_4)$ are in W?

We have the basis for W given by ρ_1, ρ_2, ρ_3
the row vectors of R.

It is immediate that
the span of ρ_1, ρ_2, ρ_3 consists of vectors β
such that $b_3 = 2b_1$.

For such a β
we have

$$\begin{aligned} \beta &= b_1\rho_1 + b_2\rho_2 + b_4\rho_3 \\ &= [b_1, b_2, b_4]R \\ &= [b_1 \quad b_2 \quad b_4]QA \\ &= x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 \end{aligned}$$

where $x_i = [b_1 \; b_2 \; b_4]Q_i$, i.e.

$$\begin{aligned} x_1 &= \quad b_1 - \tfrac{1}{3}b_2 + \tfrac{2}{3}b_4 \\ x_2 &= -b_1 + \tfrac{5}{6}b_2 - \tfrac{2}{3}b_4 \\ x_3 &= \qquad\; - \tfrac{1}{6}b_2 + \tfrac{1}{3}b_4. \end{aligned} \qquad [2\text{-}26]$$

(c) Clearly, the vectors $\alpha_1', \alpha_2', \alpha_3'$
are all of the form (y_1, y_2, y_3, y_4) with
$y_3 = 2y_1$
and thus
they are in W.

Independence is easy to verify by inspection.
(simply multiply each vector by a scalar coefficient
and sum them and equate to the zero vector
and show each scalar coefficient must be zero)

(d) The matrix P has for its columns
$P_j = [\alpha_j']_{\mathcal{B}}$

where $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$.

The equations [2-26]
tell us how to find the coordinate matrices
$\alpha_1', \alpha_2', \alpha_3'$.

For instance, with $\beta = \alpha_1'$
we have $b_1 = 1, b_2 = 0, b_3 = 2, b_4 = 0$ and

$$\begin{aligned} x_1 &= \quad 1 - \tfrac{1}{3}(0) + \tfrac{2}{3}(0) = \quad 1 \\ x_2 &= -1 + \tfrac{5}{6}(0) - \tfrac{2}{3}(0) = -1 \\ x_3 &= \qquad - \tfrac{1}{6}(0) + \tfrac{1}{3}(0) = \quad 0. \end{aligned}$$

Thus,
$\alpha_1' = \alpha_1 - \alpha_2$.

Similarly,
we obtain
$\alpha_2' = \alpha_2$ and
$\alpha_3' = 2\alpha_1 - 2\alpha_2 + \alpha_3$.

Hence, the matrix P becomes

$$P = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

We now answere these same questions, using the second method.

We start by forming the 4×3 matrix B
with column vectors $\alpha_1, \alpha_2, \alpha_3$:

$$B = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 2 & 0 & -4 \\ 1 & 1 & 3 \end{bmatrix}$$

We now ask
for which y_1, y_2, y_3, y_4
the system $BX = Y$ has a solution.

Using what we learnt in Chapter 1, we have

$$\begin{bmatrix} 1 & 0 & -2 & y_1 \\ 2 & 2 & 0 & y_2 \\ 2 & 0 & -4 & y_3 \\ 1 & 1 & 3 & y_4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & y_1 \\ 0 & 2 & 4 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_3 - 2y_1 \\ 0 & 1 & 5 & y_4 - y_1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & -2 & y_1 \\ 0 & 0 & -6 & y_2 - 2y_4 \\ 0 & 1 & 5 & y_4 - y_1 \\ 0 & 0 & 0 & y_3 - 2y_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_4 \\ 0 & 0 & 1 & \frac{1}{6}(2y_4 - y_2) \\ 0 & 1 & 0 & -y_1 + \frac{5}{6}y_2 - \frac{2}{3}y_4 \\ 0 & 0 & 0 & y_3 - 2y_1 \end{bmatrix}$$

The condition that
system $BX = Y$ has a solution is
$y_3 = 2y_1$.

So $\beta = (b_1, b_2, b_3, b_4)$ is in W
iff
$b_3 = 2b_2$.

If β is in W
then the coordinates (x_1, x_2, x_3)
in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$
can be read off from
the last matrix above.

(i.e. $x_1 = y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_4$, $x_3 = \frac{1}{6}(2y_4 - y_2)$, $x_2 = \cdots$)

These should (and do) match the formulae we obtained in [2-26]

These already answer parts (a) and (b).

For parts (c) and (d), we proceed as before.

Example 22.
Left as a HW assignment

Recall:

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(a) Prove that $\alpha_1, \alpha_2, \alpha_3$ form a basis for W
(i.e. these vectors obviously span W by definition
show that they are also linearly independent)

(b) Let $\beta = (b_1, b_2, b_3, b_4)$ be a vector in W.
What are the coordinates of β
relative to the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$?

(c) Let

$$\alpha_1' = (1, 0, 2, 0)$$
$$\alpha_2' = (0, 2, 0, 1)$$
$$\alpha_3' = (0, 0, 0, 3).$$

Show that $\alpha_1', \alpha_2', \alpha_3{}'$ form a basis for W.

(d) If β is in W,
let X denote the coordinate matrix of β
relative to the α-basis, and
let X' denote the coordinate matrix of β
relative to the α'-basis.

Find the 3×3 matrix P such that $X = PX'$ for every such β.