LINEAR ALGEBRA | winter/spring 2025

Hoffman & Kunze

Ch. 1 | Line Equations

§1.1 Fields

story: Abstract out relevant properties of Real/complex numbers.

Def^n: Let F be a set &
suppose two operations
"addition" &
"multiplication"
are given by
$+ : F \times F \to F$
$\cdot : F \times F \to F$

We say that $(F, +, \cdot)$ is a field if the following are satisfied:

1. Addition is commutative: $x + y = y + x \quad \forall x, y \in F$

2. Addition is associative
$x + (y + z) = (x + y) + z$
$\forall x, y, z \in F$.

3. There exists $(\exists)$ a unique element "$\underline{0}$" (zero) $\in F$ s.t.
$x + \underline{0} = x \quad \forall x$

4. For all $x \in F$,
$\exists$ a unique element $(-x) \in F$
s.t. $x + (-x) = \underline{0}$

5. Multiplic^n is commutative
$xy = yx \quad \forall x, y \in F$

6. Multiplic^n is associative
$x(yz) = (xy)z \quad \forall x, y, z \in F$

7. $\exists$ a unique nonzero element 1 (one) in F s.t.
$x \cdot 1 = x \quad \forall x$

8. To each nonzero x in F,
$\exists$ a corresponding unique element $\underline{x^{-1}}$ (or $\underline{1/x}$) in F s.t.
$x \cdot x^{-1} = 1$

9. Multiplication distributes over addition
i.e. $x(y+z) = xy + xz$
$\forall x, y, z \in F$.

story: • Intuitively, fields are defined to mimic properties of real/complex #s.

• elements of a field — "scalars"

NB. The set of real/complex #s form a field wrt std $+$ & $\times$.

story: We now define "subfields"

Def^n: Let $(F, +, \cdot)$ be a field
Then $(F', +, \cdot)$ is a subfield of $(F, +, \cdot)$
if $F' \subseteq F$ &
$(F', +, \cdot)$ is also a field.

Tanisha
Masood
Bharath
Tej
Anusha

NB. The set of reals is a subfield of complex #s.

Notation: Sometimes we will use F instead of $(F, +, \cdot)$.

Story: We focus on subfields of $\mathbb{C}$
$\mathbb{C}$ — complex #s.

e.g. 1 The set of positive int $(1, 2, 3, \dots)$
Is this a subfield of $\mathbb{C}$?
not even a field
Ambika

e.g. 2. The set of integers
$\dots, -2, -1, 0, 1, 2, \dots$
Is this a subfield of $\mathbb{C}$?
No multiplicative inverse
Ashwin

e.g. 3 The set of rationals
Is this a subfield?
Anusha

NB. Every subfield of $\mathbb{C}$ must contain the set of rationals.

False [$\mathbb{C}$ has only one subfield which is the set of rationals]
Manan

e.g. 4 The set of all complex numbers of the form
$x + y\sqrt{2}$ $\quad x, y$ are rationals.
is a subfield of $\mathbb{C}$.

$3 \equiv 2$

Assumption: We restrict to subfields of $\mathbb{C}$.
Why?
"fields of characteristic n"
It can happen that in some field F
$\underbrace{1 + 1 \dots + 1}_{n\text{-many}} = 0$

§1.2 Systems of Linear Equations

Def^n. Suppose F is a field
A system of m linear equations in n unknowns
is any problem of the following form
find n scalars (elements of F)
$x_1, \dots, x_n$
s.t.

$$(1\text{-}1) \quad \begin{cases} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = y_1 \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n = y_m \end{cases}$$

where $y_1, \dots, y_m$ & $\{A_{ij}\}_{\substack{i \in \{1, \dots, m\} \\ j \in \{1, \dots, n\}}}$ are elements of F.

Any n-tuple $(x_1, \dots, x_n)$ of elements in F satisfying each equation is called a solution (of the system).

If $y_1 = y_2 = \dots = y_m = 0$ the system is called homogeneous
Masood

story: Can one determine whether two different systems have the same set of solutions?

We start w/ an illustr^n.

Illustration: Suppose we want to solve

$$\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0 \end{aligned}\right] \text{ I}$$

: add (-2) times 2nd eq^n to 1st

$-7x_2 - 7x_3 = 0$
$\iff x_2 = -x_3$

: add (3) times 1st eq^n to 2nd

$7x_1 + 7x_2 = 0 \iff x_1 = -x_3$

II

Conclusion: Suppose $x_1\ x_2\ x_3$ solve I.
Then, they must also satisfy
$x_1 = x_2 = -x_3$.

Conversely, one can check that
$(a, a, -a)$ is a sol^n to I.

story: To formalize this method of showing two systems have the same sol^n.

We define "linear combination".

Def^n: Recall the previous def^n & (1-1) in particular (1-1)

Suppose: one selects scalars $c_1 \dots c_m$
[multiplies the jth eq^n w/ c_j &
sums them to obtain

$$\left.\begin{aligned} (c_1 A_{11} + \dots + c_m A_{m1})x_1 + \dots \\ (c_1 A_{1n} + \dots + c_m A_{mn})x_n = \\ c_1 y_1 + c_2 y_2 \dots c_m y_m \end{aligned}\right]$$

Such an equation, is called a linear combination of equations in (1-1)

NB.
Question: Is a sol^n to (1-1) also a solution to the equation above?
Yes.

Story: We will use this to define a notion of "equivalent systems".

Def^n: Recall (1-1) & consider another sys of eq^{n}s

$$(1\text{-}2) \quad \begin{cases} B_{11}x_1 + \dots + B_{1n}x_n = z_1 \\ \vdots \\ B_{k1}x_1 + \dots + B_{kn}x_n = z_k \end{cases}$$

linear combin^n of eq^{n}s in (1-1)

Those two systems are equivalent if each equation in each system is a linear combination of equations in the other system.

Story: Clearly, if the systems are equivalent, & $x_1 \dots x_n$ is a sol^n to (1-1) then it is also a sol^n to each equation in (1-2).

Conversely, as well,
If $x_1 \dots x_n$ is a sol^n to (1-2)
then it is also a sol^n to (1-1)

Theorem 1: Equivalent systems (of linear eq^{n}s) have exactly the same set of sol^{n}s.

§1.3 Matrices & Elementary Row Operations

Make the notation more brief

story:

Not^n: We abbreviate