Linear Algebra—Winter/Spring 2026

Instructors: A.S. Arora, S. Das, G. Kurri

Quiz 1

Due: January 30, 2026

Instructions

i. *Studied ahead?* Since we have not yet covered some topics in class (e.g. determinants), if you use them, please be sure to define the appropriate objects and prove any property thereof that you use.

ii. *Duration.* You have a total of 45 minutes.

iii. *Do I need to answer all the questions?* You can score a maximum of $3 \times 5 = 15$ points. We will rescale this to match the allocation for quizzes later. **Your best three answers** will count towards your final score. For instance, suppose you skipped problem (6) and (3), obtained 5 pts for problems (1), (2) but only 2 pts for problem (4) and 1 pt for problem (5). Then, your total would be $(5 \times 2) + 2 = 12$ (your points for problem (3), (5) and (6) will be dropped).

---

1. Let

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

i. [2 pts] For which triples $Y = (y_1, y_2, y_3) \in \mathbb{R}^3$ does the system $AX = Y$ have a solution? Give the most general characterisation.

ii. [3 pts] If $Y = (2, 1, 0)$, describe explicitly all solutions of the system, if any.

2. For each of the statements below, write True or False. ***Note****: Explanations are* ***not*** *required; write only "True" or "False". Make sure your answer to this question fits on a single page of the answer sheet. Answers that extend beyond one page will NOT be evaluated.*

(a) [1 pt] The following system of linear equations always has a non-trivial solution:

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= 0, \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 &= 0, \\ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 &= 0, \end{aligned}$$

where all coefficients are real numbers.

(b) [1 pt] The matrix

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is in row-reduced echelon form.

(c) [1 pt] The set of all $n \times n$ complex Hermitian matrices is a vector space over the field $\mathbb{R}$ of real numbers.

(d) [1 pt] The field $\mathbb{C}$ of complex numbers is a vector space over the filed $\mathbb{R}$ of real numbers.

(e) [1 pt] Let A be an $m \times n$ matrix over a field F and $Y \in F^m$ where Y is non-zero. The solution space of the system of equations $AX = Y$ is *always* a vector space over F.

3. Let A, A' be $m \times n$ matrices and B be an $n \times m'$ matrix (with real entries). Take n, m, m' to be natural numbers.

(a) [2 pt] Which matrix product among AB and BA is well defined for all values of n, m and m'? Are there any constraints on n, m and m' that allow both AB and BA to be well-defined?

(b) [3 pts] Let C be the product that is well defined for all n, m and m'. How many rows and columns does C have?
Now take $m = m'$. For any choice of n, m such that $m > n$, construct A, A' and B that satisfy both of these conditions: (i) $A \neq A'$ and (ii) $BA = BA' = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix (of the appropriate size).

4. Let A be an $n \times n$ matrix (with entries in a subfield of $\mathbb{C}$). Let X and Y be $n \times 1$ column matrices.

(a) [1 pt] Which among the following are equivalent (state without proof)?
(a) A is invertible.
(b) The only solution to $AX = 0$ is $X = 0$.
(c) $AX = Y$ has a solution for every $n \times 1$ column matrix Y.

(b) [2 pts] Suppose A has a left inverse B, i.e. there exists a matrix B such that $BA = \mathbb{I}$. Show that $AX = 0$ has only the trivial solution $X = 0$.

(c) [1 pt] Suppose A has a left inverse B (as in the previous question). Prove that A is invertible using your answers so far.

(d) [1 pt] Suppose A has a right inverse C, i.e. there exists a matrix C such that $AC = \mathbb{I}$. Using your answers so far, show that A is invertible.

5. Using the axioms of vector spaces, prove the following, where c is a scalar and $\vec{\alpha}$ is a vector. Ensure you explicitly write down which axiom is used for each step.

(a) [2 pt] $c\vec{0} = \vec{0}$
(Hint: Use $\vec{0} = \vec{0} + \vec{0}$ and existence of $(-\vec{\alpha})$ given $\vec{\alpha}$ exists)

(b) [1 pt] $0\vec{\alpha} = \vec{0}$
(Hint: Use $0 = 0 + 0$)

(c) [2 pt] For $c \neq 0$ and $c\vec{\alpha} = \vec{0}$ then $\vec{\alpha} = 0$.
(Hint: Start with multiplying c^{-1} and using the first statement, $c\vec{0} = \vec{0}$.)

Note. Here, we explicitly added the vector symbol when writing $\vec{0}$ and $\vec{\alpha}$ to distinguish $\vec{0}$ from 0. In general, the symbol is suppressed.

6. Let V be a vector space and $S \subseteq V$ be some non-empty subset of V. Denote by L the set of all linear combinations of vectors in S and denote by W the intersection of all subspaces of V that contain S.

(a) [1.5 pts] Show that $L \subseteq W$.
(Hint: Start by taking an arbitrary linear combination of vectors in S)

(b) [1.5 pts] Prove that to show $W \subseteq L$, it suffices to show that L is a subspace containing S. (Hint: Look at the definition of W)

(c) [2 pts] Show that L is a subspace (and that it contains S).
(Hint: To show a set W' is a subspace, it suffices to show that $c\alpha + \beta \in W'$ for all vectors α, β in W' and all scalars c in the field.)