Linear Algebra—Winter/Spring 2026

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Problem Set 4

Due: 8 PM, Monday February 9, 2026

Instructions

a. *Timeline.* Note for this week: The full assignment will contain only 3 graded problems. It is due on Monday at 8:00 PM (you are encouraged to submit by this Saturday). There will be no assignment next week.

b. *Punctuality.* Aim to get the assignment done three days prior to the deadline. We will not be able to entertain any requests for extensions. If you seek clarification from us and we do not respond on time, do the following: (a) assume whatever seems most reasonable, and solve the question under that assumption and (b) ensure you have evidence for making the clarification request and report the incident to one of the PhD TAs.

c. *Resources.* Feel free to use any resource you like (including generative AI tools) to get help but please make sure you understand what you finally write and submit. The TAs may ask you to explain the reasoning behind your response if something appears suspicious.

d. *Graded vs Practice Questions.* Problems numbered using Arabic numerals (i.e. $1, 2, 3, \dots$) constitute assignment problems that will be graded. Problems numbered using Roman numerals (i.e. i,ii,iii,...) are practice problems that will not be graded.

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1. Let V be the space of polynomial functions over the field F (where F is a subfield of the complex numbers). Construct a basis for V and prove that it is indeed a basis.

2. Suppose V is a vector space that is spanned by a finite set of vectors $\beta_1 \dots \beta_m$. Show that any independent set of vectors in V must be finite and cannot contain more than m elements. *You can use Theorem 6 in Chapter 1 from the text that essentially says, if A is a matrix with more columns than rows, then $AX = 0$ has a non-trivial solution.*

3. Suppose $W \subsetneq V$ where W and V are vector spaces (and V is finite dimensional). Show that $\dim W < \dim V$.

i. How is the dimension of a vector space defined? Using this definition, how would you argue that the dimension of the zero subspace (i.e. $\{\vec{0}\}$) is 0?

ii. Understand and write down Example 15 in your own words (from Hoffman Kunze).

iii. Show that (a) any set containing the zero vector $\vec{0}$ is linearly dependent, and (b) any subset of a linearly independent set, is linearly independent.

iv. Suppose P is an $n \times n$ invertible matrix. Show that the columns of P form a basis for $\mathbb{F}^n$.

v. Suppose V is a finite-dimensional vector space and let $n = \dim V$. Show that any subset of V that contains more than n vectors, is linearly dependent.

vi. Show that for any finite dimensional vector space V, if S and S' are both bases for V, then $|S| = |S'|$, i.e. S and S' have the same number of vectors.

vii. Suppose A is an $n \times n$ matrix (over some field $\mathbb{F}$) such that the row vectors of A form a linearly independent set of vectors in $\mathbb{F}^n$. Prove that A is invertible.