Linear Algebra—Winter/Spring 2026

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Problem Set 2

Due: 8 PM, Tuesday January 20, 2026

Instructions

a. *Timeline.* The first "half" of a problem set will be released by Tuesday nights (or whenever the first lecture of the week takes place). The full problem set will be released Friday nights (or whenever the second lecture of the week takes place). The full assignment will, *typically*, be due on Tuesday nights, 8 PM.

b. *Punctuality.* Aim to get the assignment done three days prior to the deadline. We will not be able to entertain any requests for extensions. If you seek clarification from us and we do not respond on time, do the following: (a) assume whatever seems most reasonable, and solve the question under that assumption and (b) ensure you have evidence for making the clarification request and report the incident to one of the PhD TAs.

c. *Resources.* Feel free to use any resource you like (including generative AI tools) to get help but please make sure you understand what you finally write and submit. The TAs may ask you to explain the reasoning behind your response if something appears suspicious.

d. *Graded vs Practice Questions.* Problems numbered using Arabic numerals (i.e. $1, 2, 3, \dots$) constitute assignment problems that will be graded. Problems numbered using Roman numerals (i.e. i,ii,iii,...) are practice problems that will not be graded.

In the following, assume the field (when unspecified) is any subfield of the complex field.

1. Let A be an $n \times n$ matrix. Can it be the case that there exist two distinct matrices B and B' such that both B and B' are left inverses of A, i.e. $BA = B'A = \mathbb{I}$? If so, give an example. Otherwise, prove that this is impossible.

i. How was an elementary matrix defined? Is it related to an elementary row operation?

ii. Show that every elementary matrix can be inverted.

iii. Show that every matrix A can be written as $A = PR$ where R is in the row-reduced echelon form and P is an invertible matrix.

iv. Let R be the row-reduced echelon matrix, row-equivalent to a square matrix A. Is the following true (and if so, give a proof; otherwise give a counter-example): If the last row of R is not all-zeros, then R is the identity matrix.

2. What is the augmented matrix A' corresponding to the system $AX = Y$? Use Examples 15 and 16 in the textbook to explain how using A' one might obtain A^{-1} (when A is invertible) via two different routes.

v. Let P be the product of elementary matrices that maps A to the row-reduced echelon matrix R. If A is invertible, show that $A^{-1} = P$.

vi. Let A be an $n \times n$ matrix. Show that the following two statements are equivalent: (i) A is row equivalent to the $n \times n$ identity matrix and (ii) $AX = 0$ has only the trivial solution (i.e. the only solution is $X = 0$).

3. Describe a general procedure (at a high level) for solving a system of non-homogeneous linear equations $AX = Y$ and illustrate this procedure by finding an expression for X (in terms of A and Y) where

$$A := \begin{bmatrix} 0 & 5 & -1 \\ 1 & 8 & -1 \\ 2 & 1 & 1 \end{bmatrix}.$$

4. Let A be an $n \times n$ matrix. Show that the following statements are equivalent: (i) A is invertible, and (ii) the system of equations $AX = Y$ has a solution for each $n \times 1$ column matrix Y.

5. Consider $n \times n$ matrices, $A, A_1 \ldots A_k$ satisfying $A = A_1 A_2 \ldots A_k$. Show that if A is invertible, then for each j, A_j is also invertible.

6. Prove the following corollary after Theorem 12 in the textbook: If A is an invertible $n \times n$ matrix and if a sequence of elementary row operations reduces A to the identity, then that same sequence of operations when applied to I yields A^{-1}.

7. Suppose A has a left inverse. Show that $AX = 0$ has only the trivial solution.

8. Let A be an $m \times n$ matrix over a field F, and let $u_1, u_2, \ldots, u_r$ denote the non-zero row vectors in the row-reduced echelon form of A. Determine whether the following statement is true or false.

For scalars $c_1, \ldots, c_r \in F$, *if* $\sum_{i=1}^{r} c_i u_i = 0$, *then* $c_i = 0$ *for all* $i \in \{1, 2, \ldots, r\}$.