# Linear Algebra—Winter/Spring 2026

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## Problem Set 3

**Due:** 8 PM, Tuesday January 27, 2026

### Instructions

a. *Timeline.* The first "half" of a problem set will be released by Tuesday nights (or whenever the first lecture of the week takes place). The full problem set will be released Friday nights (or whenever the second lecture of the week takes place). The full assignment will, *typically*, be due on Tuesday nights, 8 PM.

b. *Punctuality.* Aim to get the assignment done three days prior to the deadline. We will not be able to entertain any requests for extensions. If you seek clarification from us and we do not respond on time, do the following: (a) assume whatever seems most reasonable, and solve the question under that assumption and (b) ensure you have evidence for making the clarification request and report the incident to one of the PhD TAs.

c. *Resources.* Feel free to use any resource you like (including generative AI tools) to get help but please make sure you understand what you finally write and submit. The TAs may ask you to explain the reasoning behind your response if something appears suspicious.

d. *Graded vs Practice Questions.* Problems numbered using Arabic numerals (i.e. $1, 2, 3, \dots$) constitute assignment problems that will be graded. Problems numbered using Roman numerals (i.e. i,ii,iii,…) are practice problems that will not be graded.

*This is as early release draft; please reach out to your TAs in case something is unclear. There is a small chance that we may turn a few of the "graded questions" into "practice questions" in the coming days (or vice versa).*

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i. Let $F$ be any field and $S$ be any non-empty set. Let $V$ be the set of all functions from the set $S$ into $F$. The sum of vectors in $V$ is defined as $(f+g)(s) := f(s) + g(s)$ and the product with a scalar $C$ is defined as $(cf)(s) := cf(s)$. Show that $V$ is a vector space (be sure to establish that scalar multiplication satisfies the required properties as well).

ii. Show that the field of complex numbers $\mathbb{C}$ can be regarded as a vector space over the field $\mathbb{R}$ of real numbers.

iii. Let $\alpha$ be a vector and $c$ be a scalar. (1) Prove that for $c = 0$, it holds that $c\alpha$ is the zero vector. (2) Prove that if $c$ is non-zero, and $c\alpha$ is the zero vector, then $\alpha$ is the zero vector.

**1.** Let $H_n(\mathbb{C})$ denote the set of all $n \times n$ complex Hermitian matrices, i.e.,

$$H_n(\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} : A^{\mathrm{H}} = A\},$$

where $A^{\mathrm{H}}$ denotes the conjugate transpose of $A$.

Determine whether $H_n(\mathbb{C})$ is a vector space over the field $\mathbb{C}$. Also, whether $H_n(\mathbb{C})$ is a vector space over the field $\mathbb{R}$.

**2.** Let $W_1$ and $W_2$ be subspaces of a vector space $V$ such that $W_1 \cup W_2$ is also a subspace. Prove that one of the subspaces is contained in the other, i.e. either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ (or both).

iv. What is the minimum possible cardinality of a field? Also, what is the minimum possible cardinality of a vector space? Justify.

v. Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$ (the last type of subspace is, intuitively, a straight line through the origin). Can you describe the subspaces of $\mathbb{R}^3$?

vi. Consider the set $S$ of $n$-tuples, $(x_1, x_2, \ldots x_n)$ where $x_1 = 1 + x_2$ (take $n \geq 2$). Is $S$ a subspace? Prove your answer.

**3.** Suppose $W = \cap_a W_a$ is the intersection of subspaces $\{W_a\}_a$. Let $\alpha, \beta \in W$ and $c$ be a scalar. Show that $c\alpha + \beta$ is also in $W$.

vii. Suppose $W_1, \ldots W_k$ are subspaces of $V$ and let $W := W_1 + \cdots + W_k$. Show that (1) $W$ is a subspace of $V$ and (2) $W$ is the subspace spanned by $W_1 \cup W_2 \cup \cdots \cup W_k$.

viii. Let $V$ be the vector space of all $2 \times 2$ matrices over $F$. Let $W_1$ be the set of matrices of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix},$$

and let $W_2$ be the set of matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

(1) Are $W_1$ and $W_2$ subspaces of $V$? Give a proof for your answer. (2) Describe the form of matrices that constitute $W_1 \cap W_2$. Do they constitute a subspace? Give a proof.