Self-Testing | Quick Introduction by simple example.

Focus: $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Setup: Alice & Bob get two questions $x, y \in \{0,1\}$

Modelling the most general strategy: & produce two answers $a, b \in \{+1,-1\}$.

measurement operators

Let $A_x := M_{+1|x} - M_{-1|x}$

taken to be projective

$B_y := N_{+1|y} - N_{-1|y}$

NB1: $A_x^\dagger = A_x$; $A_x^2 = \mathbb{I}$ similarly, $B_x^\dagger = B_x$; $B_x^2 = \mathbb{I}$ (20)

Let: $|\psi\rangle_{ABP}$ be the state shared b/w Alice & Bob &

suppose P is the purification register.

Not^n: $p(a,b|x,y)$ denote the prob. of getting outcomes $a\,b$

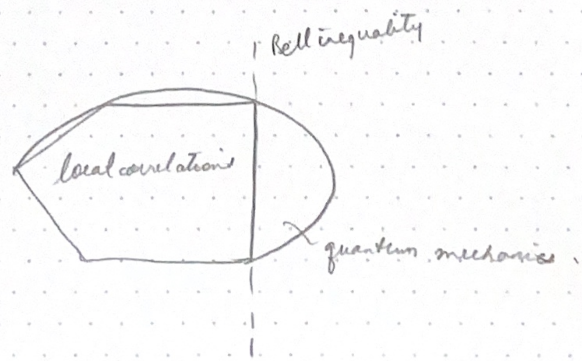
given questions x, y

local if $p(ab|xy) = \sum_\lambda \pi(\lambda)\, p_A(a|x\lambda)\, p_B(b|y\lambda)$

(non-local otherwise).

: "correlation vector" $\vec{p} := (p(00|00), p(01|00), \dots)$

: Set of $\vec{p}$'s form a convex set as illustrated (for both local & non-local correlations)



$\beta_{CHSH} = \langle A_0 B_0\rangle + \langle A_1 B_0\rangle + \langle A_0 B_1\rangle - \langle A_1 B_1\rangle \overset{\text{local}}{\leq} 2$

$\langle A_x B_y\rangle = \sum_{ab} a \cdot b\; p(ab|xy)$ $\quad \overset{\text{q.m.}}{\leq} 2\sqrt{2}$

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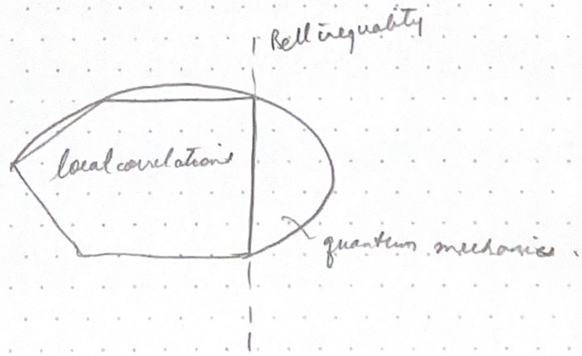

: $\beta_{CHSH} = \langle A_0 B_0\rangle + \langle A_1 B_0\rangle + \langle A_0 B_1\rangle - \langle A_1 B_1\rangle \overset{\text{local}}{\leq} 2$

$\rangle\ \langle A_x B_y\rangle = \sum_{ab} a \cdot b\ p(ab|xy) \qquad \overset{\text{q.m.}}{\leq} 2\sqrt{2}$

Recall: B_{CHSH} can be saturated by measuring $|\Phi^+\rangle$ w/

$$A_0 := \sigma_x \qquad B_0 := \frac{\sigma_z + \sigma_x}{\sqrt{2}} \quad \&$$
$$A_1 := \sigma_z \qquad B_1 := \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

Goal: Show that any state & measurements that max. violate Bell, are the same as (up to local isometries) to $|\Phi^+\rangle$, $A_{0/1}$, $B_{0/1}$

Story: Here, we show this only for $|\Phi^+\rangle$.

§ 4.2

The key idea is to establish $\{A_0, A_1\}|\psi\rangle = \{B_0, B_1\}|\psi\rangle = 0$.

Once this is done, one can build the required isometry.

The general idea here is to use a Sum of Squares decomposition

(i.e. Let: $\mathcal{B} = \sum_{abxy} w_{abxy} M_{ax} \otimes M_{by}$

be the "Bell operator"

β_Q be the max. achievable violation.

NB1: $\beta_Q \mathbb{I} - \mathcal{B} \geq 0$ (by defn of max. violation)

NB2: $\beta_Q \mathbb{I} - \mathcal{B} = \sum_\lambda P_\lambda^\dagger P_\lambda$ (lin algebra)

NB3: For max violation, say be $|\psi\rangle$,

$$\langle\psi|(\beta_Q \mathbb{I} - \mathcal{B})|\psi\rangle = 0 \Leftrightarrow \sum_\lambda \langle\psi| P_\lambda^\dagger P_\lambda |\psi\rangle = 0$$
$$\Leftrightarrow \sum_\lambda \| P_\lambda |\psi\rangle \|^2 = 0 \Rightarrow \forall \lambda, P_\lambda |\psi\rangle = 0.$$

Recall B_{CHSH} can be saturated by measuring $|\Phi^+\rangle$ w/

$$A_0 := \sigma_x \qquad B_0 := \frac{\sigma_z + \sigma_x}{\sqrt{2}} \quad \&$$

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$$\Leftrightarrow \sum_\lambda \| P_\lambda|\psi\rangle\|^2 = 0 \Rightarrow \forall\lambda, P_\lambda|\psi\rangle = 0.$$

Story: The last step, allows one to deduce statements about the underlying state & measurements.

Story: Back to showing $\{A_0, A_1\}$ & $\{B_0, B_1\}$ on $|\psi\rangle$ are zero.

NB: $$2\sqrt{2}\,\mathbb{1} - \mathcal{B}_{CHSH} = \frac{1}{\sqrt{2}}\left[\left(\frac{A_0 + A_1}{\sqrt{2}} - B_0\right)^2 + \left(\frac{A_0 - A_1}{\sqrt{2}} - B_1\right)^2\right] \quad (30)$$

(one can verify using Eq. (20))

NB2: For any state $|\psi\rangle$ s.t. $\langle\psi|\mathcal{B}_{CHSH}|\psi\rangle = \beta_{CHSH} = 2\sqrt{2}$, we have that

$$\frac{A_0 \pm A_1}{\sqrt{2}}|\psi\rangle = B_{0/1}|\psi\rangle. \quad (31)$$

$\Rightarrow \{B_0, B_1\}|\psi\rangle = 0$

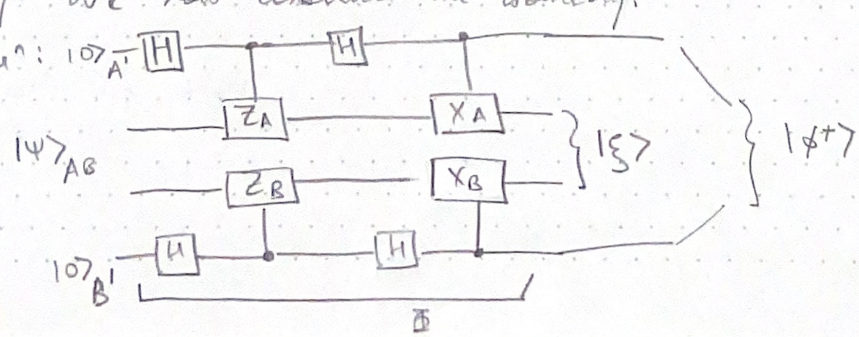
$$\begin{aligned}\{B_0, B_1\}|\psi\rangle &= (B_0 B_1 + B_1 B_0)|\psi\rangle \quad (25)\\ &= (A_0 - A_1)B_0 + (A_0 + A_1)B_1 |\psi\rangle\\ &= (A_0 - A_1)(A_0 + A_1) + (A_0 + A_1)(A_0 - A_1)|\psi\rangle\\ &= 0\end{aligned}$$

One can similarly show $\{A_0, A_1\}|\psi\rangle = 0$

§ 4.3

Story: We now construct the isometry.

Constrⁿ:



Suppose: The physical state $|\psi\rangle$ is a two-qubit state, &

X_A X_B Z_A Z_B are Pauli matrices,

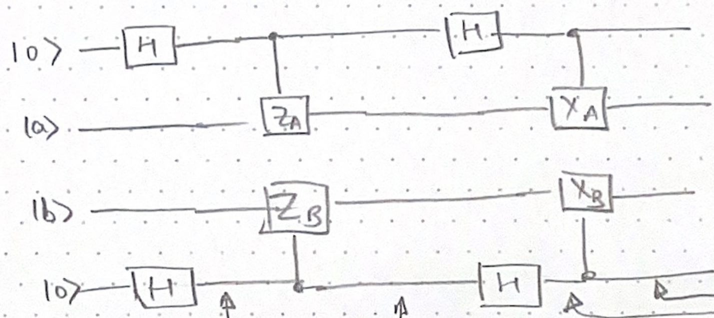
$(\sigma_x)_A$, $(\sigma_x)_B$, $(\sigma_z)_A$, $(\sigma_z)_B$ resp.

: Input

Claim: The action of this circuit, is to swap the physical state $|\psi\rangle$ w/ $|00\rangle$.

Proof

Suppose $|\psi\rangle = |ab\rangle$ for $a, b \in \{0,1\}$



$|+\rangle|ab\rangle|+\rangle$ $\quad (|0\rangle+(-1)^a|1\rangle)|a\rangle|b\rangle(|0\rangle+(-1)^b|1\rangle)$

$|a\rangle|a\rangle|b\rangle|b\rangle$

$|a\rangle|0\rangle|0\rangle|b\rangle$

important to do this when entangled states are considered.

Story: The hope is that this action is preserved, even when the state $|\psi\rangle$ is arbitrary (as in not just a 2-qubit state).

(and the measurement operators are arbitrary),

but maximally violate CHSH.

This indeed turns out to be the case.

Define: $Z_A = \frac{1}{\sqrt{2}}(A_0 + A_1)$ $\left(\frac{\sigma_x+\sigma_z}{\sqrt{2}}\right)$ $X_A = \frac{1}{\sqrt{2}}(A_0 - A_1)$ $\frac{\sigma_x - \sigma_z}{\sqrt{2}}$

$Z_B = B_0$ $\left(\frac{\sigma_x+\sigma_z}{\sqrt{2}}\right)$ $X_B = B_1$ $\left(\frac{\sigma_x-\sigma_z}{\sqrt{2}}\right)$

NB1: $\{Z_A, X_A\} = 0$ by construction

$\left((A_0+A_1)(A_0-A_1) + (A_0-A_1)(A_0+A_1) \right.$

$\left(= \cancel{A_0^2} + \cancel{A_1 A_0} - \cancel{A_0 A_1} - \cancel{A_1^2} + \cancel{A_0^2} - \cancel{A_1 A_0} + \cancel{A_0 A_1} - \cancel{A_1^2} \right)$

$\because A^2 = \mathbb{1}$

NB2: $\{Z_B, X_B\}|\psi\rangle = 0$ from (25)

$\left(\{B_0, B_1\}|\psi\rangle = 0 \right)$

NB3: $Z_A|\psi\rangle = Z_B|\psi\rangle$,

$X_A|\psi\rangle = X_B|\psi\rangle$.

$\because$ Recall: $\frac{A_0 \pm A_1}{\sqrt{2}}|\psi\rangle = B_{0/1}|\psi\rangle$. (31)

Subtlety: To ensure Φ is an isometry, it suffices to have Z_A X_A Z_B X_B be unitary.

This is the case for Z_B & X_B.

For Z_A & X_A, this may not be the case.

(sol^n: "Regularise"

$Z \mapsto \tilde{Z} :=$ Z but 0 eigenvalues are replaced w/ 1.

$\hat{Z} := |\tilde{Z}|^{-1}\tilde{Z}$ ~ normalise eigenvalues of $\tilde{Z}$

Now $\hat{Z}$ is unitary by construction

In our case, it also holds that

$\hat{Z}|\psi\rangle = Z|\psi\rangle$

(proof is in appendix A.2; skipped here).

Convention: Assume X_A Z_A are also unitary.

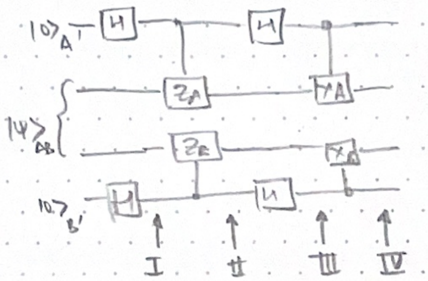
NB: $\Phi[|\psi\rangle] = \frac{1}{4}\Big[|00\rangle \otimes (\mathbb{1}+Z_A)(\mathbb{1}+Z_B)|\psi\rangle$

(by calculation) $+ |01\rangle \otimes (\mathbb{1}+Z_A)X_B(\mathbb{1}-Z_B)|\psi\rangle$

$+ |10\rangle \otimes X_A(\mathbb{1}-Z_A)(\mathbb{1}+Z_B)|\psi\rangle$

$+ |11\rangle \otimes X_A(\mathbb{1}-Z_A)X_B(\mathbb{1}-Z_B)|\psi\rangle\Big]$

Γ



$I = (|0\rangle+|1\rangle)_{A'}(|0\rangle+|1\rangle)_{B'}|\psi\rangle_{AB}$

$II = |00\rangle|\psi\rangle + |01\rangle\, \mathbb{1}\otimes Z|\psi\rangle + |10\rangle\, Z\otimes\mathbb{1}|\psi\rangle + |11\rangle\, Z\otimes Z|\psi\rangle$

$III = (|0\rangle+|1\rangle)(|0\rangle+|1\rangle)|\psi\rangle$

$(|0\rangle+|1\rangle)(|0\rangle-|1\rangle)\,\mathbb{1}\otimes Z|\psi\rangle$

$(|0\rangle-|1\rangle)(|0\rangle+|1\rangle)\,Z\otimes\mathbb{1}|\psi\rangle$

$(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)\,Z\otimes Z|\psi\rangle$

$IV = |00\rangle|\psi\rangle + |01\rangle\,\mathbb{1}\otimes X|\psi\rangle + |10\rangle\,X\otimes\mathbb{1}|\psi\rangle + |11\rangle\,X\otimes X|\psi\rangle$

$+ |00\rangle\,\mathbb{1}\otimes Z|\psi\rangle - |01\rangle\,\mathbb{1}\otimes XZ|\psi\rangle + |10\rangle\,X\otimes Z|\psi\rangle + |11\rangle\,X\otimes XZ|\psi\rangle$

$+$

$+$

$\square$

NB2: $\Phi[|\psi\rangle] = \sum_{ij\in\{0,1\}} |ij\rangle_{A'B'} \otimes \hat{f}_{ij}|\psi\rangle_{AB}$

Convention: Assume X_A, Z_A are also unitary.

NB: $\Phi[|\psi\rangle] = \frac{1}{4}\Big[\, |00\rangle \otimes (\mathbb{1} + Z_A)(\mathbb{1} + Z_B)\, |\psi\rangle$

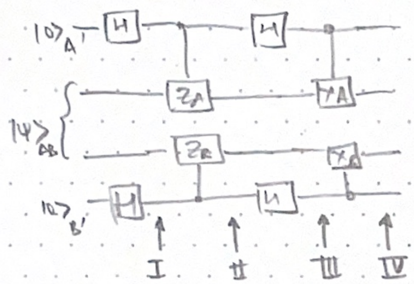
(by calculation)

$+ \; |01\rangle \otimes (\mathbb{1} + Z_A)\, X_B\, (\mathbb{1} - Z_B)\, |\psi\rangle$

$+ \; |10\rangle \otimes X_A\, (\mathbb{1} - Z_A)(\mathbb{1} + Z_B)\, |\psi\rangle$

$+ \; |11\rangle \otimes X_A\, (\mathbb{1} - Z_A)\, X_B\, (\mathbb{1} - Z_B)\, |\psi\rangle \Big]$

⌜



$\mathrm{I} = (|0\rangle + |1\rangle)_{A'} (|0\rangle + |1\rangle)_{B'}\, |\psi\rangle_{AB}$

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$\mathrm{III} = (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)\, |\psi\rangle$

$(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)\, \mathbb{1}\otimes Z\, |\psi\rangle$

$(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)\, Z\otimes\mathbb{1}\, |\psi\rangle$

$(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)\, Z\otimes Z\, |\psi\rangle$

$\mathrm{IV} = |00\rangle|\psi\rangle + |01\rangle\, \mathbb{1}\otimes X\, |\psi\rangle + |10\rangle\, X\otimes\mathbb{1}\, |\psi\rangle + |11\rangle\, X\otimes X\, |\psi\rangle$

$+ \; |00\rangle\, \mathbb{1}\otimes Z|\psi\rangle - |01\rangle\, \mathbb{1}\otimes XZ\, |\psi\rangle + |10\rangle\, X\otimes Z\, |\psi\rangle + |11\rangle\, X\otimes XZ\, |\psi\rangle$

$+$

$+$

$\square$

NB2: $\Phi[|\psi\rangle] = \sum_{ij \in \{0,1\}} |ij\rangle_{A'B'} \otimes \hat{f}_{ij}\, |\psi\rangle_{AB}$

where $\hat{f}_{ij} = \frac{1}{4} X_A^i (\mathbb{1} + (-1)^i Z_A) X_B^j (\mathbb{1} + (-1)^j Z_B)$.

NB3: Using (35) – recall: $Z_A|\psi\rangle = Z_B|\psi\rangle$,
$X_A|\psi\rangle = X_B|\psi\rangle$,

it holds that expressions of the form
$$(\mathbb{1} \pm Z_A)(\mathbb{1} \mp Z_B)|\psi\rangle = 0$$

$$\left[\ \mathbb{1} \mp Z_B \pm Z_A + \underbrace{Z_A Z_B}\ \right.$$

$Z_A^\dagger = Z_A$ so $Z_A|\psi\rangle = Z_B|\psi\rangle$
&
$|\psi\rangle = Z_A Z_B|\psi\rangle$

$$\hat{f}_{01}|\psi\rangle = \hat{f}_{10}|\psi\rangle = 0$$

NB4: $\hat{f}_{11}|\psi\rangle$ can be simplified as follows:

$$\hat{f}_{11}|\psi\rangle = \frac{1}{4} X_A(\mathbb{1} - Z_A) X_B(\mathbb{1} - Z_B)|\psi\rangle$$

(using anti-commutⁿ) $= \frac{1}{4} (\mathbb{1} + Z_A) X_A (\mathbb{1} + Z_B) X_B |\psi\rangle$

(from (35) & unitarity of X_B) $= \frac{1}{4} (\mathbb{1} + Z_A)(\mathbb{1} + Z_B)|\psi\rangle$

$= \hat{f}_{00}|\psi\rangle$.

NB5: $$\Phi\left[|\psi\rangle_{AB}\right] = \underbrace{|\phi^+\rangle_{A'B'}}_{\frac{|00\rangle + |11\rangle}{\sqrt{2}}} \otimes \underbrace{|\xi\rangle_{AB}}_{= \sqrt{2}\,\hat{f}_{00}|\psi\rangle}$$

Conclusion: If $|\psi\rangle$ yields a maximal violation,
then $|\psi\rangle$ is $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ up to local isometries.

We have "self-tested" the state.