Chapter 2—Vector spaces

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§ 2.1 Vector spaces

Story:

- One often encounters sets
  where it makes sense to deal with
  "linear combinations" of the elements in the set.

  E.g. We saw while studying linear equations
  that it is quite natural to consider
  linear combination of rows of a matrix

  Some of us, may also have seen
  linear combinations of functions
  (e.g. when looking at differential equations)

  Many would also have some experience with
  vectors (in say 3d Euclidean space)
  and must have considered linear combinations of these vectors.

- Loosely speaking
  linear algebra is a branch of mathematics
  that treats the "common properties" of
  algebraic systems which
  consist of a set,
  together with a reasonable notion of a "linear combination"
  of elements in the set.

  In this section, we define the mathematical object
  which (from experience)
  turns out to be a very useful abstraction
  of this kind of an *algebraic system*.

Definition.

A **vector space** (or **linear** space)
consists of the following:

(1) a field F of *scalars*
(2) a set V of objects, called *vectors*
(3) a rule (or operation),
called vector addition which
associates with each pair of vectors
α, β in V
a vector
$\alpha + \beta$ in V
(called the sum of α and β)
such that
(a) addition is commutative, i.e. $\alpha + \beta = \beta + \alpha$;
(b) addition is associative, i.e.. $\big(\alpha + (\beta + \gamma)\big) = (\alpha + \beta) + \gamma$;
(c) there is a unique vector 0 in V
called the zero vector,
such that $\alpha + 0 = \alpha$ for all α in V
(d) for each vector α in V

there is a unique vector $-\alpha$ in V
such that
$\alpha + (-\alpha) = 0$

(4) a rule (or operation)
called scalar multiplication
which associates with
each scalar c in F and
each vector α in V
a vector $c\alpha$ in V
called the product of c and α in a way that
(a) $1\alpha = \alpha$ for every α in V
(b) $(c_1 c_2)\alpha = c_1(c_2\alpha)$
(c) $c(\alpha + \beta) = c\alpha + c\beta$
(d) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$

Story:

- Note that the definition says that
  a vector space is a "composite object"
  conisting of a field and a set of "vectors"
  together with two operations—that have special properties.

  It is possible that the same set of "vectors" is
  a part of a number of different vector spaces
  (we will see this in Example 5 below).

- (remark about notation)
  When there is no chance of confusion
  we may simply refer to the vector space as V
  or
  when it is desirable to specify the field, we will say
  V is a vector space over the fnield F.

- Let us look at a few examples.

Example 1.
The n-tuple space F^n.

Let F be any field.
Let V be the set of all n-tuples
$\alpha = (x_1 \dots x_n)$ of scalars $x_i \in F$.
For $\beta = (y_1 \dots y_n) \in F$,
(2-1) $\alpha + \beta := (x_1 + y_1, \dots, y_n + x_n)$.

The product of a scalar $c \in F$ & a vector $\alpha \in V$
is defined as
(2-2) $c\alpha = (cx_1, cx_2, \dots cx_n)$.

Example 2. The space of $m \times n$ matrices. $F^{m \times n}$. Let F be any field and let m and n be positive integers. Let $F^{m\times n}$ be the set of all $m \times n$ matrices over the field F. The sum of two vectors A and B in $F^{m\times n}$ is defined by

$$(A + B)_{ij} = A_{ij} + B_{ij}. \tag{2-3}$$

The product of a scalar c and the matrix A is defined by

$$(cA)_{ij} = cA_{ij}. \tag{2-4}$$

Note that $F^{1\times n} = F^n$.

Example 3

The space of functions from a set to a field.

Let
F be any field and
S be any non-empty set

Let V be the set of all functions
from the set S into F.

The sum of two vectors f and g in V
is defined to be the vector $f + g$
i.e. the function from S to F defined by
$(f + g)(s) \coloneqq f(s) + g(s)$.

The product of the scalar c and the function f is
the function cf defined by
$(cf)(s) \coloneqq cf(s)$.

NB. The preceding examples
are a special case of Example 3.

Justification
(Example 1 as a special case of Example 3):
To see this, note that
an n-tuple of elements of F
can be seen as a function
from the set S of integers $\{1 \dots n\}$
into F.
(basically, you input the index, it tells you the value)

(Example 2 as a special case):
Similarly, an $m\times n$ matrix over the field F
can be seen as a function
from the set S of pairs of integers
(i, j), where $1 \le i \le m, 1 \le j \le n$
into the field F.
(again, you basically input the row and column index
and it (i.e. the function) tells you the value (assigned to that matrix element)

Story

- We will now look at, a bit more carefully, how
  the operations in example 3
  do, in fact, satisfy the conditions (3) and (4) (in the definition of a vector space).

Justification: Example 3 is indeed a vector space

(a) Since addition in F is commutative
it holds that
$f(s) + g(s) = g(s) + f(s)$
for each s in S
so the functions $f + g$ and $g + f$
are identical.

(b) Since addition in F is associative,
$f(s) + [g(s) + h(s)] = [g(s) + f(s)] + h(s)$

for each s so
$f + (g + h)$
is exactly the same function as
$(f + g) + h$.

(c) The unique zero vector is the zero function
that assigns to each element of S
the scalar 0 in F.

(d) For each f in V,
$(-f)$ is the function that is given by
$(-f)(g) \coloneqq -f(s)$.

One can similarly check that
scalar multiplication satisfies the conditions of (4). HW

Story:

- Let us continue with a few more examples.

Example 4
The space of polynomial functions over a field F.

Defn (Polynomial function).
Let
F be a field and
V be the set of all functions f
from F into F that have a rule of the form
$f(x) = c_0 + c_1 x + \cdots + c_n x^n$
where $c_0, \ldots c_n$ are fixed scalars in F (independent of x).
A function of this type is called
a **polynomial function** on F.

Let addition and scalar multiplication be as in Example 3.

NB. If f and g are polynomials,
and scalars c are in F
(there would a be a set of scalars
for each f and g)
then $f + g$ and cf are again polynomial functions.

Recall

The sum of two vectors f and g in V
is defined to be the vector $f + g$
i.e. the function from S to F defined by
$(f + g)(s) \coloneqq f(s) + g(s)$.

The product of the scalar c and the function f is
the function cf defined by
$(cf)(s) \coloneqq cf(s)$.

Example 5. ★ HW

(i)

The field C of complex numbers
may be regarded as a
vector space over the field R of real numbers.

(ii)

More generally,
let F be the field of real numbers and
let V be the set of n-tuples $\alpha = (x_1 \dots x_n)$
where $x_1 \dots x_n$ are *complex* numbers.

Recall

For $\beta = (y_1 \cdots y_n) \in F$,

&V $\alpha + \beta := (x_1 + y_1, \cdots, y_n + x_n)$.

The product of a scalar $c \in F$ & a vector $\alpha \in V$
is defined as

(2-2) $c\alpha = (cx_1, cx_2, \cdots cx_n)$.

Define addition of vectors and
scalar multiplication as in Example 1.

In this way
one obtains a vector space over the field R
which is different from both
C^n and R^n.

Story:

- We now look at some simple facts
that follow almost immediately
from the definition of vector spaces

Recall

(3) a rule (or operation),
called vector addition which
associates with each pair of vectors
α, β in V
a vector
$\alpha + \beta$ in V
(called the sum of α and β)
such that
(a) addition is commutative, i.e. $\alpha + \beta = \beta + \alpha$;
(b) addition is associative, i.e.. $(\alpha + (\beta + \gamma)) = (\alpha + \beta) + \gamma$;
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there is a unique vector $-\alpha$ in V
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a vector $c\alpha$ in V
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(a) $1\alpha = \alpha$ for every α in V
(b) $(c_1c_2)\alpha = c_1(c_2\alpha)$
(c) $c(\alpha + \beta) = c\alpha + c\beta$
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NB1. If c is a scalar and 0 is the zero vector,
then it holds that
$c0 = 0$.

Proof.

We have
$c0 = c(0 + 0)$, using 3(c)
$= c0 + c0$ using 4(c)

Adding $-(c0)$ and using 3(d) we have
$c0 = 0$.

■

★ HW (verify)

NB2. Similarly, for the scalar 0 and any vector α
it holds that $0\alpha = 0$.

Proof.

$0\alpha = (0 + 0)\alpha$ since 0 is a scalar,
$= 0\alpha + 0\alpha$ using $4(c)$

Adding $-(0\alpha)$ and using 3(d)
$0\alpha = 0$

■

NB3. If c is a non-zero scalar, and α is a vector s.t.
$c\alpha = 0$ then $\alpha = 0$.

Proof.
From NB1, it holds that $c^{-1}(c\alpha) = 0$.
However,
$$c^{-1}(c\alpha) = (c^{-1}c)\alpha = 1\alpha = \alpha$$

thus it means $\alpha = 0$.

Summary: If $c\alpha = 0$ where c is a scalar and α is a vector
then either
c is the zero scalar or
α is the zero vector
(or both)

NB4. If α is any vector in V
then $(-1)\alpha = -\alpha$.

Proof.
$0 = 0\alpha = (1-1)\alpha$
$= 1\alpha + (-1)\alpha = \alpha + (-1)\alpha$

and that in turn shows $(-1)\alpha = -\alpha$
as asserted.
■

Remark. From the associative and commutative properties
of vector addition, it follows that
a sum involving a number of vectors
is independent of the way
in which these vectors are combined
and associated.
i.e. one can unambigiously write
$\alpha_1 + \alpha_2 \ldots + \alpha_n$
for vectors α_i in V.

Definition (linear combination).
A vector β in V is said to be a linear combination of the vectors
$\alpha_1 \ldots \alpha_n$ in V if
there are scalars $c_1 \ldots c_n$ in F such that
$$\beta = c_1\alpha_1 + \cdots + c_n\alpha_n = \sum_{i=1}^{n} c_i\alpha_i$$

NB. The following follow from the associative and distributive properties of the two operations

$n \quad n \quad n$

$$\sum_{i=1}^{n} c_i\alpha_i + \sum_{i=1}^{n} d_i\alpha_i = \sum_{i=1}^{n} (c_i + d_i)\alpha_i$$

$$c \sum_{i=1}^{n} c_i\alpha_i = \sum_{i=1}^{n} (cc_i)\alpha_i.$$

Story:

- Certain parts of linear algebra are
  intimitaly related to geometry.
  - E.g. the very word "space" suggests omething geometrical
    (as does the word "vector", from physics for instance).
  - A lot of the terminology will carry geometric connotation.
  - We therefore spend some time explaining this connection
    and indicating the origin of the term "vector space".

- Consider the vector space R^3.
  - One often identifies triples (x_1, x_2, x_3)
    of real numbers with
    points in the three-dimensional Euclidean space.
  - Recall from physics (e.g.) that
    one usually defined a vector as a directed line-segment
    PQ from the point P in space to another point Q.
  - This is essentially a careful formulation
    of the idea of an "arrow" from P to Q.
  - Intuitively, we have used vectors to convey length and direction,
    and so accordingly
    we must identify two directed line segments
    if they have the same length and direction.

- Ok, so with this in mind, observe that
  a directed line segment PQ from $P = (x_1, x_2, x_3)$ to the point $Q = (y_1, y_2, y_3)$
  has the same length and direction as
  the directed line segment from
  the origin $O = (0,0,0)$ to the point $(y_1 - x_1, y_2 - x_2, y_3 - x_3)$.

  Furthermore, this is the only segment emanting from the origin
  which has the same length and direction as PQ.

  Thus, if one restricts to vectors emanating from teh orogin
  there is exactly one vector
  associated with each given length and direction.

- The vector OP
  from the origin to $P = (x_1, x_2, x_3)$
  is completely determined by P.

  In our definition of the vector space R^3
  the vectors are simply defined to be
  the triples (x_1, x_2, x_3).

- What about vector addition?

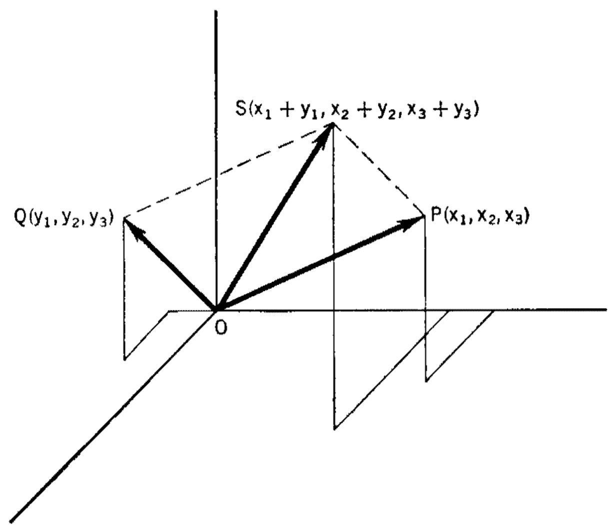

FIGURE 1

- And what about scalar multiplication?

  Scalar c multiplied to a vector, geometrically interpreted, does the following

  Changes the length to $|c|$ times the original length

  And flips the direction of the vector if $c < 0$