Ch2—§2.4 Coordinates

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Story:

- One of the useful features of a basis $\mathcal{B}$ in an
  n-dimensional space V
  is that it
  essentially enables one
  to introduce *coordinates* in V
  analogous to the "natural coordinates"
  x_i of a vector $\alpha = (x_1 \dots x_n)$
  in the space F^n.
  - In this scheme,
    the coordinates of a vector α in V
    (relative to the basis $\mathcal{B}$)
    will be scalars that serve to express
    α as a linear combination of the
    vectors in the basis.
  - Thus,
    we choose the natural coordinates
    of a vector α in F^n as
    being defined by α and
    the standard basis for F^n.
  - However, must be slightly careful—
    If
    $\alpha = (x_1 \dots x_n) = \sum x_i \epsilon_i$

    and $\mathcal{B}$ is teh standard basis for F^n

    how are the coordinates of α
    determined by $\mathcal{B}$ and α?
    - Here's one answer:
      A given vector α has a unique expression
      as a linear combination of
      the standard basis vectors
      and
      the ith coordinate (x_i) of α

is the coefficient ϵ_i in this expression.

- From this point of view
  we are able to say
  which is the ith coordinate
  because we have a
  "natural ordering" of the vectors
  in the standard basis
  (i.e. a rule for which is the "first" vector,
  which is the "second" vector etc.)

- If $\mathcal{B}$ is an arbitrary basis of
  the n-dim space V
  we won't have a "natural ordering" (in general)
  and would therefore have to impose some order.

- Stated differently,
  coordinates will be defined relative to
  a sequence of vectors
  (instead of being defined relative to
  a set of vectors)

Definition.

If V is a finite-dimensional vector space
an **ordered basis** for V is a finite sequence of vectors
which is linearly independent and spans V.

Story:

- If the sequence $\alpha_1 \dots \alpha_n$ is an ordered basis for V
  then the set
  $\{\alpha_1 \dots \alpha_n\}$ is a basis for V.

  The ordered basis is the set
  together with the specified ordering.

  We will abuse the notation slightly
  and describe all that by saying
  $\mathcal{B} = \{\alpha_1 \dots \alpha_n\}$
  is an ordered basis for V.

[Me: the book spent a lot of time on this trivial point; but ok]

Premise: Suppose V is a finite-dimensional vector space over the field F
and that
$\mathcal{B} = \{\alpha_1 \dots \alpha_n\}$ is an ordered basis for V.

Claim. Given α in V
there is a unique n-tuple $(x_1 \dots x_n)$ of scalars such that

$$\alpha = \sum_{i \in \{1 \dots n\}} x_i \alpha_i$$

Proof.

Suppose it were the case that

$$\alpha = \sum_{i \in \{1 \dots n\}} z_i \alpha_i$$

then

$$\sum_{i \in \{1 \dots n\}} (x_i - z_i)\alpha_i = 0$$

and then the linear independence of α_is tells that
for each i, it must be that $x_i - z_i = 0$.

■

Notation.

We call x_i the ith **coordinate of α relative to the orderd basis**
$\mathcal{B} = \{\alpha_1 \dots \alpha_n\}$.

NB1.

If $\beta = \sum_{i \in \{1 \dots n\}} y_i \alpha_i$ and α is as above, then

$$\alpha + \beta = \sum_i (x_i + y_i)\alpha_i$$

i.e. the ith coordinate of $(\alpha + \beta)$ in this ordered basis
is $(x_i + y_i)$.

NB2. Similarly,

the ith coordinate of $(c\alpha)$ is cx_i.

NB3. Every n-tuple $(x_1 \dots x_n)$ in F^n
is the n-tuple of coordinates of some vector in V
i.e.
$\sum_i x_i \alpha_i$.

Story:

- To summarise
  each ordered basis for V
  determines a one-to-one correspondence
  $\alpha \to (x_1 \dots x_n)$
  b/w
  the set of all vectors in V
  and
  the set of n-tuples in F^n.

  This correspondence has the property that
  - the correspondent of $(\alpha + \beta)$
    is the sum in F^n of the correspondents of
    α and β
  - the correspondent of $(c\alpha)$
    is the product in F^n
    of the scalar c and the correspondent of α.

[Me: I don't know why the book is going on and on about something
so trivial—maybe this becomes important later?]

Story:

- Why don't we simply select some ordered basis for V
  and describe each vector using
  the corresponding n-tuple of coordinates?
  That way,
  one only needs to operate with n-tuples.

- This defeats our purpose here
  - We are attempting to learn to reason with
    vector spaces as abstract algebraic systems

(so even things that don't a priori look like vectors
could be analysed as such)

- Even in those situations where we do use coordinates
it helps to be able to change coordinate systems
(i.e. change the ordered basis).

- It will be convenient to use the "coordinate matrix"
relative to the ordered basis $\mathcal{B}$.

Notation.

The coordinate matrix of α (given by coordinates $(x_1 \ldots x_n)$)
relative to the ordered basis $\mathcal{B}$ is defined as

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

To indicate