Endsem | Answers hints/sketched

Sunday, April 27, 2025 8:59 pm

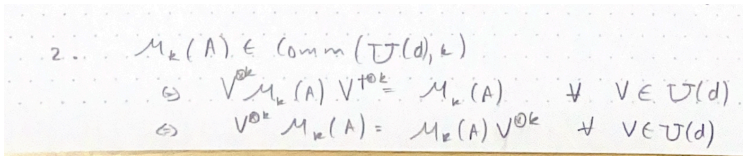
Exercise 1.

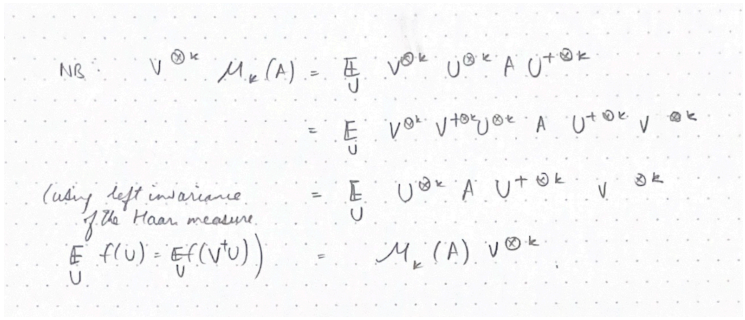
1.1 Haar Measure

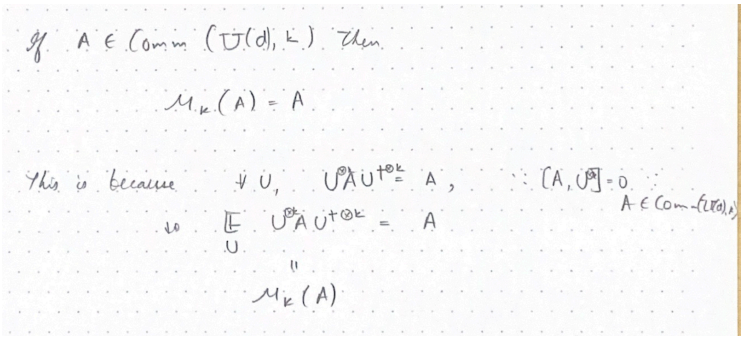
Reminder:

- The moment operator is defined as $\mathcal{M}^{(k)}(\cdot) := \mathbb{E}_{U \leftarrow \mu_H}[U^{\otimes k}(\cdot)U^{\dagger \otimes k}]$ where μ_H is the Haar measure.
- The commutant
$$\mathrm{Comm}(\mathrm{U}(d), k) := \{A \in \mathcal{L}((\mathbb{C}^d)^{\otimes k}) : [A, U^{\otimes k}] = 0 \; \forall \; U \in \mathrm{U}(d)\}$$
is the set of linear operators A that commute with all$U^{\otimes k}$. Also, it holds that

Exercise 1 (1 points; Haar). Show that $\mathcal{M}^{(k)} \in \mathrm{Comm}(\mathrm{U}(d), k)$ and that if $A \in \mathrm{Comm}(\mathrm{U}(d), k)$, then $\mathcal{M}^{(k)}(A) = A$.







Question 2

Reminder:

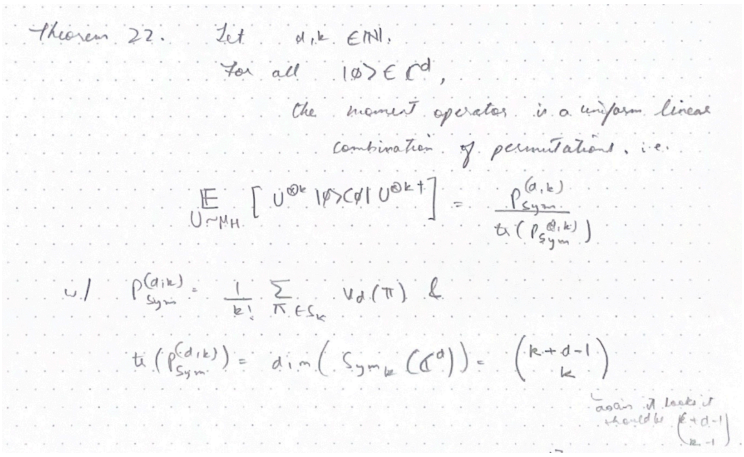
- $P_{\mathrm{sym}} := \frac{1}{k!} \sum_{\pi \in S_k} V_d(\pi)$ where S_k is the symmetric group (group of permutations over k indices), and $V_d(\pi) = \sum_{i_1, \ldots i_k \in [d]} |i_{\pi^{-1}(1)} \ldots i_{\pi^{-1}(k)}\rangle \langle i_1 \ldots i_k|$ affects this permutation on k many d-dimensional quantum systems.
- Schur-Weyl duality says that $\mathcal{M}^k(O) = \mathbb{E}_{U \leftarrow \mu_H}[U^{\otimes k} O U^{\otimes k}]$ can be expressed as a linear combination of $V_d(\pi)$, where O is a linear operator on $(\mathbb{C}^d)^{\otimes k}$.

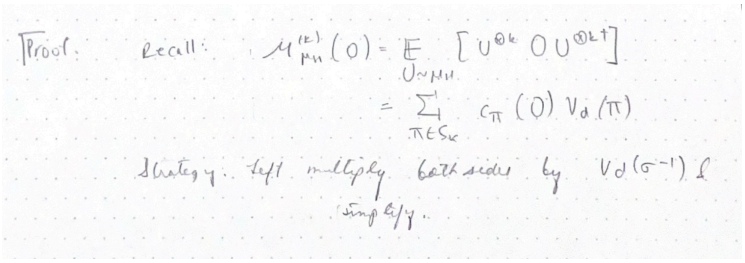
Question 2 (3 points). *Prove that*

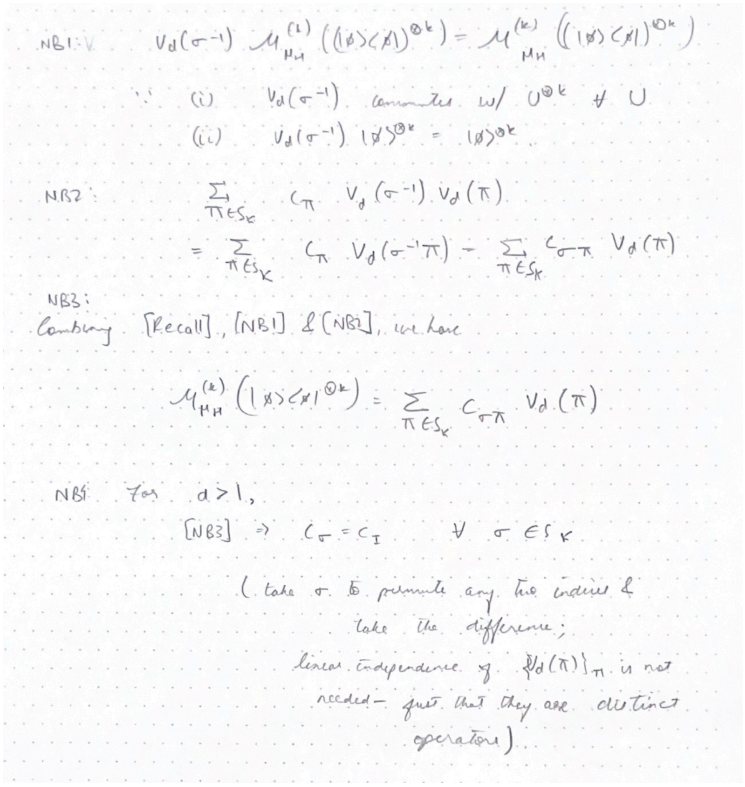
$$\mathcal{M}^k(|\phi\rangle \langle\phi|^{\otimes k}) = \mathbb{E}_{U \leftarrow \mu_H}[U^{\otimes k} |\phi\rangle \langle\phi|^{\otimes k} U^{\dagger \otimes k}] = \frac{P_{\mathrm{sym}}}{\mathrm{tr}(P_{\mathrm{sym}})}$$

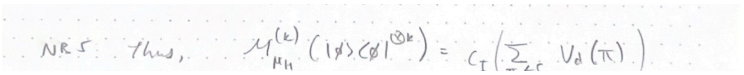
where $|\phi\rangle \in \mathbb{C}^d$. You could either do this directly yourself, or do it by establishing the following sub-steps.

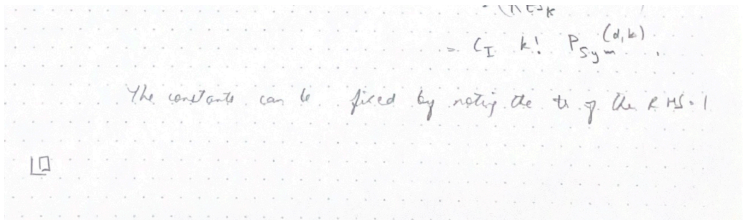
1. *Show that $V_d(\sigma^{-1})\mathcal{M}(\phi) = \mathcal{M}(\phi)$ where we use $\mathcal{M}$ to denote $\mathcal{M}^k$ and ϕ to denote $|\phi\rangle\langle\phi|$.*
2. *Show that $\sum_{\pi\in S_k} c_\pi V_d(\sigma^{-1})V_d(\pi) = \sum_{\pi\in S_k} c_{\sigma\pi} V_d(\pi)$.*
3. *Can you use these two observations to conclude that $\mathcal{M}(\phi) = \sum_{k\in S_k} c_{\sigma\pi} V_d(\pi)$.*
4. *Using your result above, reason that $c_\sigma = c_I$ for all $\sigma \in \pi$ (here I is the identity permutation)?*
5. *Now, show that $\mathcal{M}^k(\phi) \propto P_{\text{sym}}$*
6. *Finally, establish the normalisation to complete the proof.*







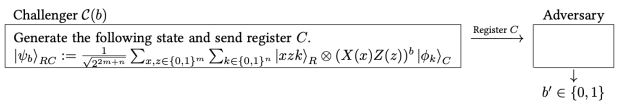




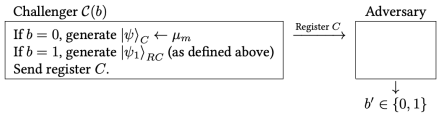
Exercise 2

Reminder: Recall that for proving computational hiding of their commitment scheme, [4] consider the following hybrids. Below, $|\phi_k\rangle$ is an m-qubit state produced by the state generator function of PRSG (with security parameter λ) and μ_m is the Haar measure over m-qubit states.

- $\mathrm{Hyb}_0(b)$



- $\mathrm{Hyb}_1(b)$



Exercise 2 (1 point; PRSG $\Longrightarrow$ Commitments). Show that $|\Pr[1 \leftarrow \mathrm{Hyb}_0(b)] - \Pr[1 \leftarrow \mathrm{Hyb}_1(b)| \leq \mathrm{negl}$ for $b = 1$ (note, $b = 0$ is not asked, only $b = 1$).

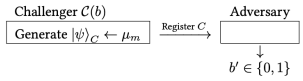
The two cases are in fact identical.

Question 3

4 PRS $\Longrightarrow$ Commitments

Reminder (PRSG $\Longrightarrow$ Commitments). In addition to $\mathrm{Hyb}_0, \mathrm{Hyb}_1$ as defined above 2, [4] also define Hyb_2 as follows.

- $\mathrm{Hyb}_2(b)$

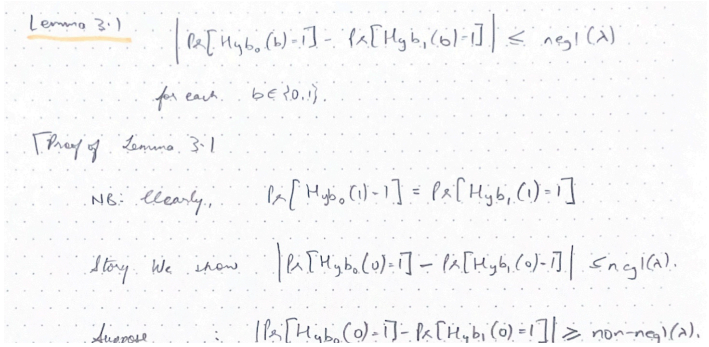


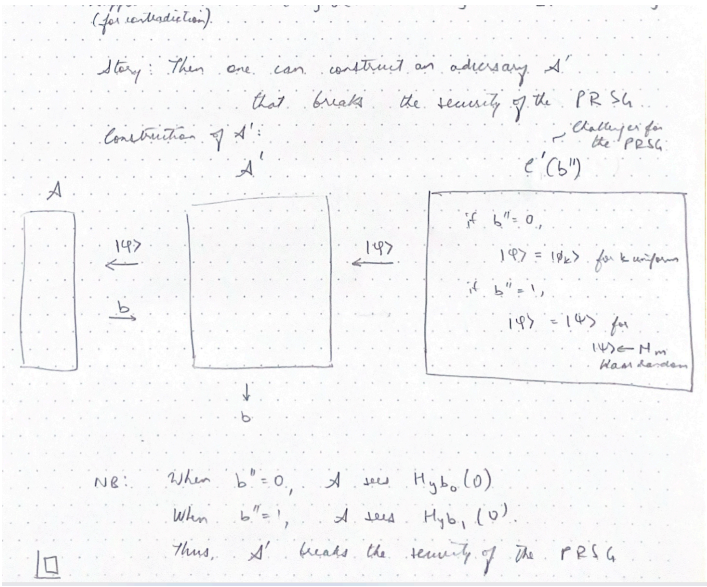
Question 3 (2 points). *We want to establish computational hiding of [4]'s commitment scheme by showing*

$$|\Pr[1 \leftarrow \mathrm{Hyb}_0(0)] - \Pr[1 \leftarrow \mathrm{Hyb}_0(1)]| \leq \mathrm{negl}(\lambda). \tag{2}$$

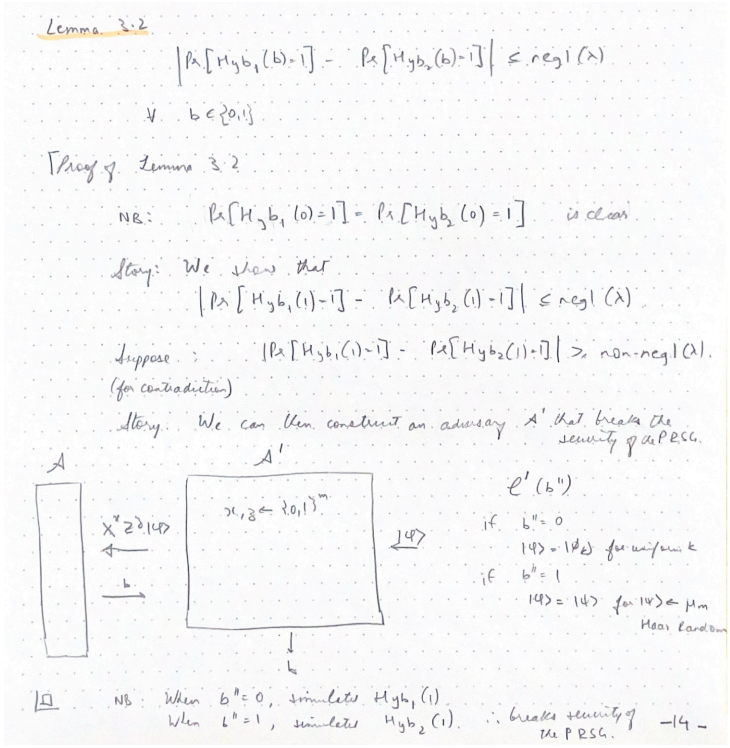
To this end, prove the following sub-claims (where we use $a \approx b$ to denote $|a - b| \leq \mathrm{negl}$).

1. $\Pr[1 \leftarrow \mathrm{Hyb}_0(b)] \approx \Pr[1 \leftarrow \mathrm{Hyb}_1(b)]$ *for $b = 0$ as well. Show that if this were not the case, then one can construct an adversary $\mathcal{A}'$ that breaks the security of the underlying PRSG (pseudorandom state generator) by distinguishing a pseudorandom state from a Haar random state.*
2. $\Pr[1 \leftarrow \mathrm{Hyb}_1(b)] \approx \Pr[1 \leftarrow \mathrm{Hyb}_2(b)]$ *for $b = 0$ should be immediate (write one line justifying it). How will you modify $\mathcal{A}'$ above to establish the $b = 1$ case? Using these observations, establish (2).*





For the second part, it suffices to show that the reduction $\mathcal{A}'$
behaves only slightly differently, i.e. sends $X^x Z^z|\psi\rangle$ instead of sending $|\psi\rangle$



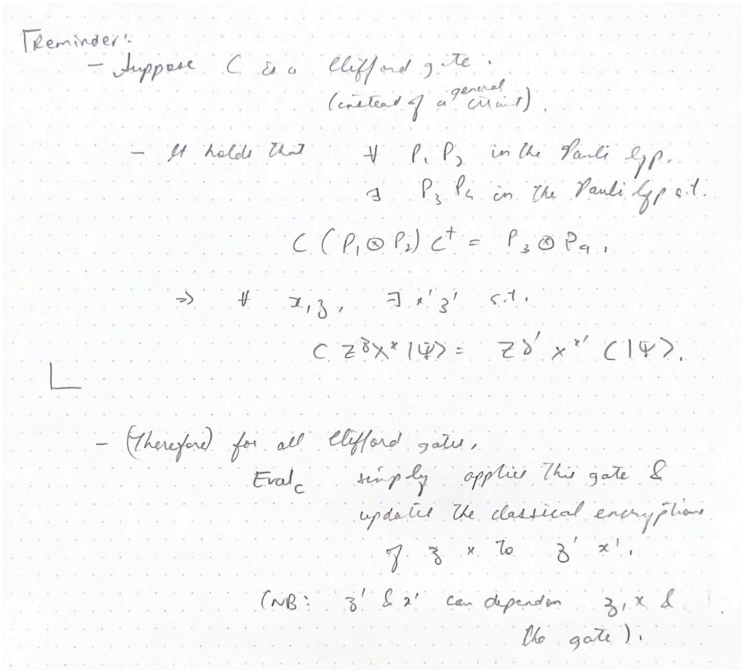
Exercise 4

1.4 QFHE

Reminder:

- Recall that Mahadev's QFHE scheme encrypts a quantum state $|\psi\rangle$ as $\left(X^x Z^z |\psi\rangle, (\hat{x}, \hat{z})\right)$ where x, z are sampled uniformly and $\hat{x}$ and $\hat{z}$ denote classical FHE encryptions of x and z respectively.

Exercise 4 (1 point; QFHE). Explain how the Eval function is implemented for Clifford gates, in Mahadev's construction.



Question 6

5.2 QFHE

Reminder: We look at some of the steps that go into applying the encrypted CNOT operation in Mahadev's QFHE scheme. The question will ask you to fill in a detail or two. Let $|\psi\rangle = \sum_{a,b\in\{0,1\}} \alpha_{ab} |ab\rangle$ be an arbitrary pure two qubit state. The goal is to apply CNOT^s where $\hat{s}$ is given (recall $\hat{s}$ is a classical FHE encryption of s).
[BEGIN SNIPPET]

1. Clasically, compute a description of the claw-free pair (f_0, f_1) corresponding to $\hat{s}$ (a reminder about the claw-free pair is appended, just in case).
2. Prepare the state
$$|\psi_1\rangle = \sum_{\substack{a,b,\mu\in\{0,1\}\\ r\in\mathcal{R}}} \alpha_{ab} |ab\rangle |\mu r\rangle |f_a(\mu, r)\rangle$$
and measure the last register to obtain a y. Denote by $|\psi_{1'}\rangle$ the resulting state.
3. XOR μ_a into the second register. This should result in
$$|\psi_3\rangle := \sum_{ab\in\{0,1\}} \alpha_{ab} (\mathbb{I} \otimes X^{\mu_0}) \mathsf{CNOT}^s_{12} |a, b\rangle |\mu_a r_a\rangle$$
where $f_0(\mu_0, r_0) = f_1(\mu_1, r_1) = y$.
4. The last register is measured in Hadamard basis...

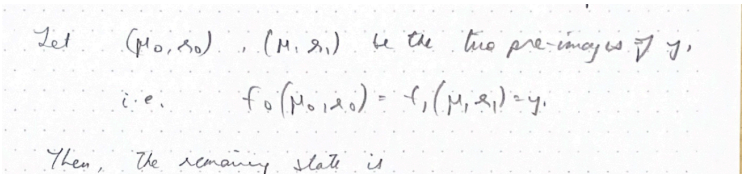
[END SNIPPET]

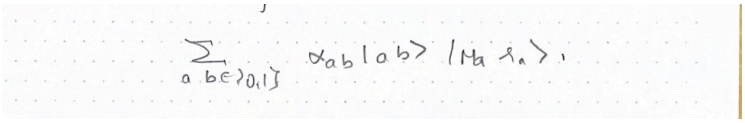
Question 6 (1 point; QFHE). *Write down $|\psi_{1'}\rangle$ and prove that following step 3, one indeed gets the state $|\psi_3\rangle$ (up to a change in variable names, depending on the choice you make in specifying $|\psi_{1'}\rangle$).*
(Hint: it may help to observe that $|b \oplus \mu_a\rangle = X^{\mu_0} |b + a \cdot s\rangle$; if you use this, prove that it is true first)

Reminder: Additional properties we need from (N)TCFs (Noisy Trapdoor Claw-free Functions), given encryption $\hat{s}$ of a bit s.

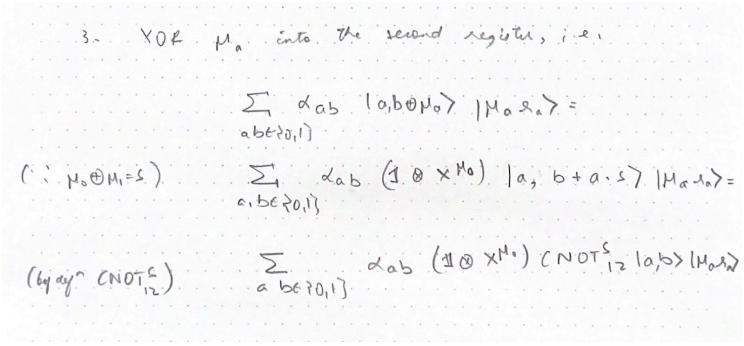
1. One can efficiently compute trapdoor claw-free functions $f_0, f_1 : \{0,1\} \times \mathcal{R} \to \mathcal{Y}$ s.t. $\forall (\mu_0, r_0), (\mu_1, r_1) \in \{0,1\} \times \mathcal{R}$ which is a claw (i.e. $f_0(\mu_0, r_0) = f_1(\mu_1, r_1)$), it holds that $\mu_0 \oplus \mu_1 = s$.
2. One can also compute a classical FHE encryption of the trapdoor (which allows one to invert the function) corresponding to the pair (f_0, f_1).

$|\psi_1'\rangle$ should look something like this





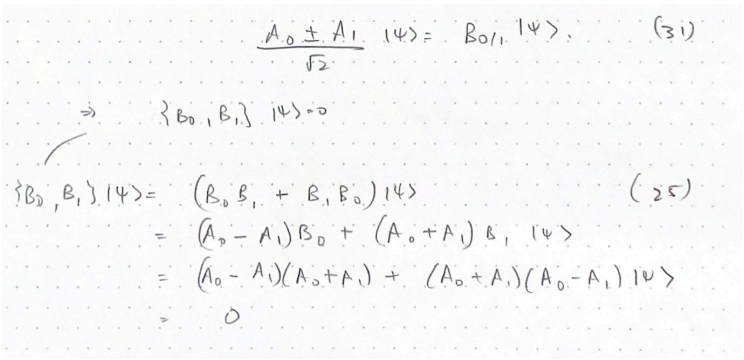
and $|\psi_3\rangle$ should be justified along the following lines



Exercise 5

Reminder: In class, we only considered self-testing of the two-qubit maximally entangled state $|\phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We considered the CHSH game where Alice and Bob are asked questions $x, y \in \{0, 1\}$ and respond with answers $a, b \in \{\pm 1\}$ respectively. To describe the most general quantum strategy, denote the quantum state shared by Alice and Bob by $|\psi\rangle_{AB}$ where Alice holds register A and Bob register B. Denote the observable used by Alice (resp. Bob) to answer question x, by A_x (resp. B_y). We observed that A_x and B_y are Hermitian and unitary.

Exercise 5 (1 point; Self-testing). Suppose $\frac{A_0 \pm A_1}{\sqrt{2}} |\psi\rangle = B_{0/1} |\psi\rangle$ where $\{A_x\}_{x\in\{0,1\}}$ and $\{B_y\}_{y\in\{0,1\}}$ denote Alice's and Bob's operators in a general strategy they follow to play the CHSH game. Prove that B_0 and B_1 anti-commute on $|\psi\rangle$, i.e. $\{B_0, B_1\} |\psi\rangle = 0$.



Question 5

5.1 Self-testing

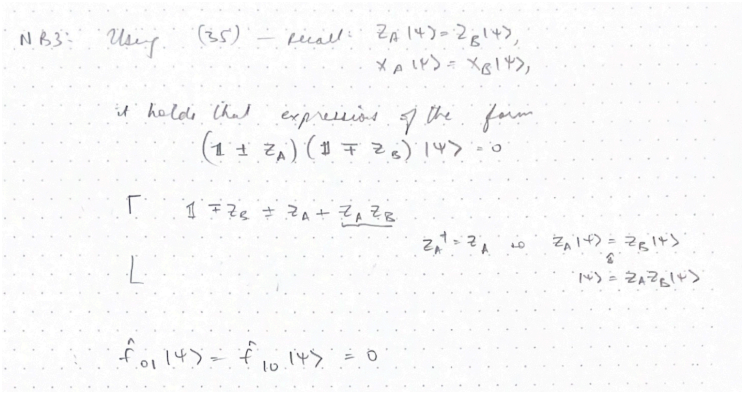
Reminder: Continuing with the discussion preceding Exercise 5, recall that we explicitly defined the local isometry Φ as shown in Figure 1. For $Z_A = \frac{1}{\sqrt{2}}(A_0 + A_1), Z_B = B_0, X_A = \frac{1}{\sqrt{2}}(A_0 - A_1), X_B = B_1$, we proved in class that

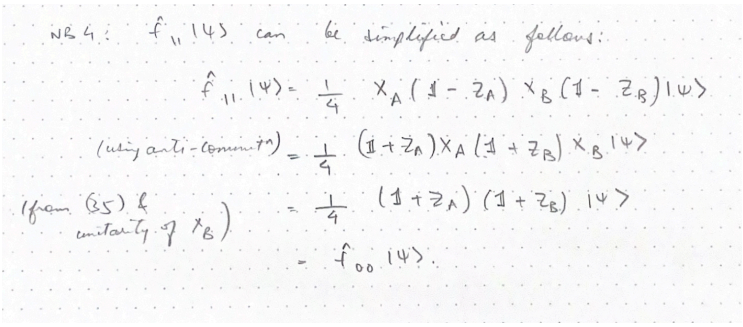
$$\Phi[|\psi\rangle] = \sum_{i,j\in\{0,1\}} |ij\rangle_{A'B'} \otimes \underbrace{\left(\frac{1}{4} X_A^i (\mathbb{I} + (-1)^i Z_A) X_B^j (\mathbb{I} - (-1)^j Z_B)\right)}_{=:\hat{f}_{ij}} |\psi\rangle_{AB}.$$

It may help to also recall that we proved the following:

- $\{Z_A, X_A\} = 0, \{Z_B, X_B\} |\psi\rangle = 0$
- $Z_A |\psi\rangle = Z_B |\psi\rangle$ and $X_A |\psi\rangle = X_B |\psi\rangle$

Question 5 (1 point; self-testing). *Using the aforementioned, show that that* $\hat{f}_{01}|\psi\rangle = \hat{f}_{10}|\psi\rangle = 0$ *and* $\hat{f}_{11}|\psi\rangle = \hat{f}_{00}|\psi\rangle$.





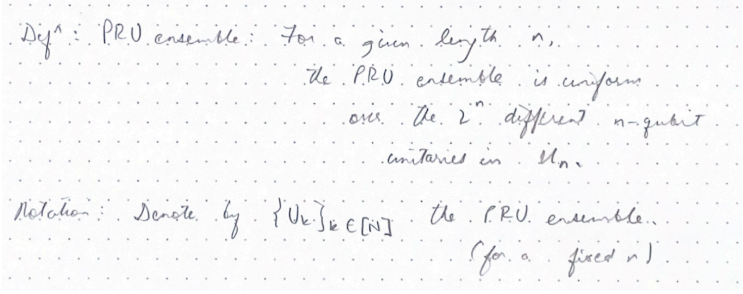
Exercise 3

1.3 BQP=QMA yet PRUs exist

Reminder:

- Semi-formally, a pseduorandom unitary is a keyed family of unitaries $\{U_k\}_{k\in\{0,1\}^\kappa}$ acting on $n(\kappa)$-qubits, where κ is a security parameter with the following two properties:
  - It can be applied efficiently (i.e. given k and a quantum state $|\psi\rangle$, one can efficiently compute $U_k|\psi\rangle$) and
  - No efficient quantum algorithm $\mathcal{A}$ can distinguish whether it is given oracle access to a Haar random unitary, i.e. $U \leftarrow \mu_{2^n}$, or a pseudorandom unitary, i.e. U_k where $k \leftarrow \{0,1\}^\kappa$.
- Kretschmer [3] uses $\mathcal{U}$ (quantum oracle) and $\mathcal{C}$ (a classical oracle) which is independent of $\mathcal{U}$. The quantum oracle $\mathcal{U}$ is defined as the set $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ where for each $n, \mathcal{U}_n$ denotes a direct sum 2^n independent Haar random unitaries, acting on n qubits, i.e. $\mathcal{U}_n \leftarrow \mu_{2^n}^{2^n}$.

Exercise 3 (1 point; BQP=QMA yet PRUs exist). State the construction for the PRU ensemble used by Kretschmer.



So, basically, for a fixed n, $\mathcal{U}_n = (U_1 \oplus U_2 \ldots \oplus U_{2^n})$
and the ensemble is given by
$\{U_k\}_{k\in\{1,\ldots,2^n\}}$.

Question 4

5 QMA=BQP and yet PRUs exist

Question 4 (3 points). *The answers to the following, should be enough for you to prove that the relative to $\mathcal{U}$ (as defined above Exercise 3), PRUs exist.*

1. *Fix the system size (on which the unitaries act) to be n qubits. Denote the PRU ensemble you defined in Exercise 3 as $\tilde{U} := \{\tilde{U}_k\}_{k\in\{0,1\}^n}$. To establish that $\tilde{U}$ is a PRU, does it suffice to show that for all efficient quantum oracle-algorithms $\mathcal{A}$,*

$$\mathsf{adv}(\mathcal{A}) := \Pr_{k\leftarrow\{0,1\}^n}[1 \leftarrow \mathcal{A}^{\tilde{U}_k}] - \Pr_{U\leftarrow\mu_{2^n}}[1 \leftarrow \mathcal{A}^{U}] \leq \mathrm{negl}(n)? \qquad [*]$$

If so, justify your answer (Hint: Do you need an $\mathcal{U}$ to efficiently generate $\tilde{U}$? If so, should that appear somewhere in your definition of a secure PRU, relative to $\mathcal{U}$).
If not, how will you modify adv*? (Hint: Note that $\mathcal{C}$ is uncorrelated to $\mathcal{U}$)*
(NB. You can neglect non-uniformity considerations i.e. it suffices, in the definition of PRUs to be secure against uniform adversaries—no need to consider advice, for the purposes of this exam).

2. *Suppose $\mathcal{A}^{O,U}$ is an oracle algorithm that makes queries to $U = (U_1, \ldots U_N) \in \mathbb{U}(D)^N$. Prove that*

$$\mathbb{E}_{U\leftarrow\mu_D^N}\left[\Pr_{k\leftarrow[N]}[1 \leftarrow \mathcal{A}^{U_k,U}]\right] = \Pr_{k\leftarrow[N]}[1 \leftarrow \mathcal{B}^{e_k}] \tag{3}$$

where $[N] = \{1, \ldots, N\}$, $e_k = (0, 0 \ldots, 0, 1, 0, \ldots 0)$ with 1 at the kth position, and $\mathcal{B}$ is defined as follows:

- *$\mathcal{B}$ samples $(V_0, V_1 \ldots V_N) \leftarrow \mu_D^{N+1}$ (which are unitaries in $\mathbb{U}(D)^{N+1}$).*
- *$\mathcal{B}$ runs $\mathcal{A}$, replacing queries to O by queries to V_0 and queries to $U_k \in U$ by V_0 if $x_k = 1$, and by V_k if $x_k = 0$.*

Prove also that

$$\mathbb{E}_{U\leftarrow\mu_D^N}\left[\Pr_{O\leftarrow\mu_D}[1 \leftarrow \mathcal{A}^{O,U}]\right] = \Pr[1 \leftarrow \mathcal{B}^{0^N}]. \tag{4}$$

(NB. Using a bound on unstructured search, one can now argue that the difference between Eq. (3) and (4) must be at most cT^2/N, where T is the number of queries made by $\mathcal{A}$ and c is a universal constant.).

Q4.1

[This question expects $\widetilde{U}$ was defined as in Exercise 3.]

No, to establish that $\widetilde{U}$ is a PRU,
one must, at the very least, show that [*] holds
even when access to $\mathcal{U}$ is given (take n to be fixed for the moment).

More concretely, one must show, at the very least that
the difference between the LHS of (3) and LHS of (4) is
negligible.

"Very least" because one could also consider classical advice
but we didn't cover this in class.

Q4.2

Here's the crux of the argument:
(I am taking $N = 2^n$ but this doesn't change anything)

$\mathcal{A}^{U_k,U}$ basically sees $2^n + 1$ oracles
$U_k, U_1, \ldots U_k, \ldots U_{2^n}$
where the first oracle matches one of the remaining 2^n oracles
(in fact the kth of the remaining 2^n oracles).

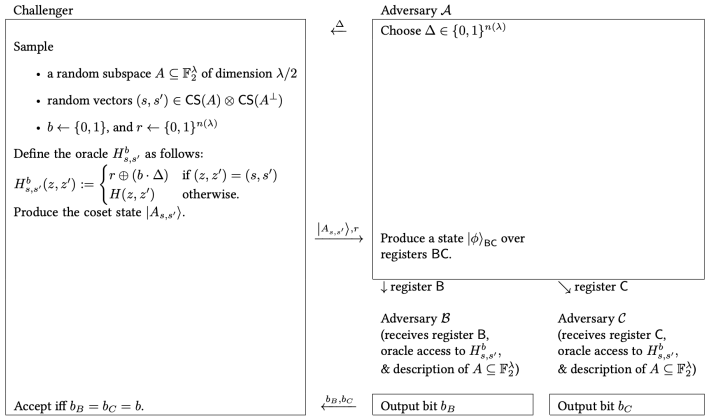
Now $\mathcal{B}^{e_k}$ also runs $\mathcal{A}$ with $2^n + 1$ oracles,
$V_0, V_1, \ldots, V_{k-1}, V_0, V_{k+1}, \ldots V_{2^n}$
where the first oracle matches one of the remaining 2^n oracles
(in fact the kth of the remaining 2^n oracles, as above).

And the distribution over $(U_1 \ldots U_{2^n})$ is identical to that over $(V_0, \ldots V_{k-1}, V_0, V_{k+1}, \ldots V_{2^n})$

because they are both sampled from μ_D^N.

Question 1

Reminder: Let us recall Hybrid 2 in the security proof of the uncloneable encryption scheme introduced by [1].



where

- H is a random oracle $H : \mathbb{F}_2^\lambda \times \mathbb{F}_2^\lambda \to \{0,1\}^{n(\lambda)}$,
- $\mathcal{A}, \mathcal{B}, \mathcal{C}$ all get access to P_{A+s} and $P_{A^\perp + s'}$ after the first message is sent by the challenger, and

- $\mathsf{CS}(A)$ is the set of canonical representatives for A, i.e. $\mathsf{CS}(A) = \{\mathsf{Can}_A(s) : s \in \mathbb{F}_2^n\}$.
  - $\mathsf{Can}_A(s)$ is the (lexicographically) smallest vector in $A + s := \{a + s : a \in A\}$.
- $|A_{s,s'}\rangle := \frac{1}{\sqrt{|A|}} \sum_{a \in A} (-1)^{s' \cdot a} |a + s\rangle$

For the proof, we will need to define projectors Π_0^B, Π_1^B and Π_0^C, Π_1^C. For a fixed A, r, Δ, s, s' they are defined as follows:

- Π_0^B: runs $\mathcal{B}$ with oracle access to $H^0_{s,s'}$ (B and A are the same), projects on outcome 0, and undoes the computation.
- Π_1^B : runs $\mathcal{B}$ with oracle access to $H^1_{s,s'}$ (B and A are the same), projects on outcome 1, and undoes the computation.
- Π_0^C and Π_1^C are defined similarly, with $\mathcal{B}, \mathsf{B}$ replaced with $\mathcal{C}, \mathsf{C}$.

We will need some more notation.

- Denote by $\{(|\phi_i\rangle, \lambda_i)\}_i$ the (eigenvectors, eigenvalues) of $(\Pi_0^B + \Pi_1^B)/2$. Similarly, let $\{(|\psi_j\rangle, \mu_j)\}_j$ denote the corresponding quantities for $(\Pi_0^C + \Pi_1^C)/2$.
- NB: $|\phi\rangle_{BC} = \sum_{ij} \alpha_{ij} |\phi_i\rangle_B \otimes |\psi_j\rangle_C$ without loss of generality.

Finally, we will assume the following holds about the state $|\phi\rangle_{BC}$ produced by $\mathcal{A}$: for every polynomial p, we have

$$\sum_{\substack{i:|\lambda_i - 1/2| > 1/p \\ j:|\mu_j - 1/2| > 1/p}} |\alpha_{ij}|^2 \leq \mathrm{negl}(n).$$

Question 1 (3 points). *Denote by p_2 the probability that Hybrid 2 above outputs accept. We will show some of the main steps in establishing that $p_2 \leq \frac{1}{2} + \mathrm{negl}(\lambda)$.*

1. *Show that the state $|\phi\rangle_{BC}$ is negligibly close to*

$$|\phi'\rangle_{BC} := \sum_{i:|\lambda_i - 1/2| \leq 1/p} \alpha_{ij} |\phi_i\rangle_B \otimes |\psi_j\rangle_C + \sum_{\substack{i:|\lambda_i - 1/2| > 1/p \\ j:|\mu_j - 1/2| \leq 1/p}} \alpha_{ij} |\phi_i\rangle_B \otimes |\psi_j\rangle_C .$$

2. *Show that*

$$\frac{\left(\left| (\Pi_0^B \otimes \Pi_0^C) |\phi'\rangle_{BC} \right|^2 + \left| (\Pi_1^B \otimes \Pi_1^C) |\phi'\rangle_{BC} \right|^2 \right)}{2} \tag{1}$$

   is negligibly close to the probability that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is accepted by the Challenger.

3. *Show that the expression above (Eq 1) can be upper bounded by*

$$\frac{1}{2} \cdot \left(\underbrace{\langle \phi'_B | (\Pi_0^B \otimes I) |\phi'_B\rangle + \langle \phi'_B | (\Pi_1^B \otimes I) |\phi'_B\rangle}_{\mathrm{I}} + \underbrace{\langle \phi'_C | (I \otimes \Pi_0^C) |\phi'_C\rangle + \langle \phi'_C | (I \otimes \Pi_1^C) |\phi'_C\rangle}_{\mathrm{II}} \right)$$
$$+ \Re \left(\underbrace{\langle \phi'_C | (\Pi_0^B \otimes \Pi_0^C) |\phi'_C\rangle}_{\mathrm{III}} + \underbrace{\langle \phi'_B | (\Pi_1^B \otimes \Pi_1^C) |\phi'_C\rangle}_{\mathrm{IV}} \right).$$

4. *Establish that* I *(resp.* II*) is at most* $\left(\frac{1}{2} + \frac{1}{p}\right) \||\phi'_B\rangle\|^2$ *(resp.* $\left(\frac{1}{2} + \frac{1}{p}\right) \||\phi'_C\rangle\|^2$*).*

5. *Suppose it holds that for indices i, i' satisfying $\lambda_i + \lambda_{i'} \neq 1$, we have $\langle \phi_i | \Pi_0^B |\phi_{i'}\rangle = 0$ (we proved this in class). Using this, show that both* III *and* IV *are zero.*

6. *Quick conceptual questions:*

(a) In your answers above, where did you (most directly) use properties of the coset state $|A_{s,s'}\rangle$?

(b) Assertion: $\Pi_0^B + \Pi_1^B = \mathbb{I}$. Either prove the assertion or justify why you don't expect the projectors to sum to identity.

(NB. Combining these, you should be able to deduce that $p_2 \leq \left(\frac{1}{2} + \frac{1}{p}\right)\left(||\phi'_B\rangle|^2 + ||\phi'_C\rangle|^2\right) + \text{negl} \leq \frac{1}{2} + \frac{1}{p} + \text{negl}$ and since this is true for all polynomials p, we have proved $p_2 \leq \frac{1}{2} + \text{negl}$.)

Question 1.1, 1.2, 1.3, 1.4 and 1.5 are basically straight from the paper (see below).

Question 1.6

(a) The answer for me is step (2) because this is where we use the fact that $|\phi\rangle_{BC}$ has the form we assumed it does and also corresponds to the success probability of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

(The coset states are basically used to prove that $|\phi\rangle_{BC}$ has the form it does.)

Other answers may also be fine, as long as they are justified appropriately. Judge for yourselves.

(b) We don't expect the projectors to sum to 1 because they don't correspond to different outcomes of the same measurement, i.e. Π_0^B uses oracle $H^0_{s,s'}$, while Π_1^B uses oracle $\Pi^1_{s,s'}$.
(if they were using the same oracle, then one would expect them to sum to 1).

Let $|\phi_{\mathsf{BC}}\rangle$ be the state prepared by $\mathcal{A}$. Without loss of generality, we can assume the state is pure. We write the state under the basis $\{|\phi_i\rangle\}_i$ and $\{|\psi_j\rangle\}_j$:

$$|\phi_{\mathsf{BC}}\rangle = \sum_{i,j} \alpha_{i,j} |\phi_i\rangle_{\mathsf{B}} \otimes |\psi_j\rangle_{\mathsf{C}}.$$

Lemma 4.12. *Taken the randomness of A, s, s' and $H_{-s,s'}$, for every polynomial $p(\cdot)$, there exists a negligible function* negl *such that with overwhelming probability the following weight is bounded:*

$$\sum_{\substack{i:\,|\lambda_i - 1/2| > 1/p \\ j:\,|\mu_j - 1/2| > 1/p}} |\alpha_{i,j}|^2 \leq \mathsf{negl}(n).$$

The proof for this lemma is given at the end of this section.

With the above lemma, we can claim that over the randomness of A, s, s' and $H_{-s,s'}$, for every polynomial $p(\cdot)$, $|\phi_{\mathsf{BC}}\rangle$ is negligibly close to the following state $|\phi'_{\mathsf{BC}}\rangle$:

$$\sum_{i:|\lambda_i - 1/2| \leq 1/p} \alpha_{i,j} |\phi_i\rangle_{\mathsf{B}} \otimes |\psi_j\rangle_{\mathsf{C}} + \sum_{\substack{i:|\lambda_i - 1/2| > 1/p \\ j:|\mu_j - 1/2| \leq 1/p}} \alpha_{i,j} |\phi_i\rangle_{\mathsf{B}} \otimes |\psi_j\rangle_{\mathsf{C}}.$$

For convenience, we name the left part as $|\phi'_{\mathcal{B}}\rangle$ (indicating $\mathcal{B}$ can not win) and the right part as $|\phi'_{\mathcal{C}}\rangle$ (indicating $\mathcal{C}$ can not win). Thus, for every polynomial $p(\cdot)$, there exists a negligible function $\mathsf{negl}(\cdot)$, $||\phi_{\mathsf{BC}}\rangle - (|\phi'_{\mathcal{B}}\rangle + |\phi'_{\mathcal{C}}\rangle)|_1$ is at most $\mathsf{negl}(\cdot)$ (in expectation, taken the randomness of A, s, s', r and $H_{-s,s'}$).

The probability that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ wins is at most:

$$\left(\left|(\Pi_0^B \otimes \Pi_0^C)|\phi'_{\mathsf{BC}}\rangle\right|^2 + \left|(\Pi_1^B \otimes \Pi_1^C)|\phi'_{\mathsf{BC}}\rangle\right|^2\right)/2.$$

$\Pi_0^B \otimes \Pi_0^C$ is the case that they both get access to H_0 and $\Pi_1^B \otimes \Pi_1^C$ for H_1.

The probability is at most

$$\begin{aligned}
&\left(\left|(\Pi_0^B \otimes \Pi_0^C)(|\phi'_{\mathcal{B}}\rangle + |\phi'_{\mathcal{C}}\rangle)\right|^2 + \left|(\Pi_1^B \otimes \Pi_1^C)(|\phi'_{\mathcal{B}}\rangle + |\phi'_{\mathcal{C}}\rangle)\right|^2\right)/2 \\
=&\frac{1}{2} \cdot \left(\langle\phi'_{\mathcal{B}}|(\Pi_0^B \otimes \Pi_0^C)|\phi'_{\mathcal{B}}\rangle + \langle\phi'_{\mathcal{B}}|(\Pi_1^B \otimes \Pi_1^C)|\phi'_{\mathcal{B}}\rangle + \langle\phi'_{\mathcal{C}}|(\Pi_0^B \otimes \Pi_0^C)|\phi'_{\mathcal{C}}\rangle + \langle\phi'_{\mathcal{C}}|(\Pi_1^B \otimes \Pi_1^C)|\phi'_{\mathcal{C}}\rangle\right) \\
&+\mathsf{Re}\left(\langle\phi'_{\mathcal{B}}|(\Pi_0^B \otimes \Pi_0^C)|\phi'_{\mathcal{C}}\rangle + \langle\phi'_{\mathcal{B}}|(\Pi_1^B \otimes \Pi_1^C)|\phi'_{\mathcal{C}}\rangle\right) \\
\leq&\frac{1}{2} \cdot \left(\langle\phi'_{\mathcal{B}}|(\Pi_0^B \otimes I)|\phi'_{\mathcal{B}}\rangle + \langle\phi'_{\mathcal{B}}|(\Pi_1^B \otimes I)|\phi'_{\mathcal{B}}\rangle + \langle\phi'_{\mathcal{C}}|(I \otimes \Pi_0^C)|\phi'_{\mathcal{C}}\rangle + \langle\phi'_{\mathcal{C}}|(I \otimes \Pi_1^C)|\phi'_{\mathcal{C}}\rangle\right) \\
&+\mathsf{Re}\left(\langle\phi'_{\mathcal{B}}|(\Pi_0^B \otimes \Pi_0^C)|\phi'_{\mathcal{C}}\rangle + \langle\phi'_{\mathcal{B}}|(\Pi_1^B \otimes \Pi_1^C)|\phi'_{\mathcal{C}}\rangle\right).
\end{aligned}$$

We bound each term separately.

- $\frac{1}{2}\left(\langle\phi'_{\mathcal{B}}|(\Pi_0^B \otimes I)|\phi'_{\mathcal{B}}\rangle + \langle\phi'_{\mathcal{B}}|(\Pi_1^B \otimes I)|\phi'_{\mathcal{B}}\rangle\right)$. It is equal to $\langle\phi'_{\mathcal{B}}|(\Pi_0^B + \Pi_1^B)/2 \otimes I|\phi'_{\mathcal{B}}\rangle$; by the definition of $|\phi'_{\mathcal{B}}\rangle$, it will be at most $(\frac{1}{2} + \frac{1}{p})||\phi'_{\mathcal{B}}\rangle|^2$.
- $\frac{1}{2}\left(\langle\phi'_{\mathcal{C}}|(I \otimes \Pi_0^C)|\phi'_{\mathcal{C}}\rangle + \langle\phi'_{\mathcal{C}}|(I \otimes \Pi_1^C)|\phi'_{\mathcal{C}}\rangle\right)$. Similar to the above case, it is at most $(\frac{1}{2}+\frac{1}{p})||\phi'_{\mathcal{C}}\rangle|^2$.
- $\mathsf{Re}\left(\langle\phi'_{\mathcal{B}}|(\Pi_0^B \otimes \Pi_0^C)|\phi'_{\mathcal{C}}\rangle\right)$. By Corollary 2.4, the inner product will be 0:

$$\langle\phi'_{\mathcal{B}}|(\Pi_0^B \otimes \Pi_0^C)|\phi'_{\mathcal{C}}\rangle = \sum_{i:|\lambda_i - 1/2| \leq 1/p} \sum_{\substack{i':|\lambda_{i'} - 1/2| > 1/p \\ j':|\mu_{j'} - 1/2| \leq 1/p}} \alpha^\dagger_{i,j}\alpha_{i',j'}\langle\phi_i|\Pi_0^B|\phi_{i'}\rangle\langle\psi_j|\Pi_0^C|\psi_{j'}\rangle;$$

  since every possible i, i' satisfy $\lambda_i + \lambda_{i'} \neq 1$, we have $\langle\phi_i|\Pi_0^B|\phi_{i'}\rangle = 0$.
- $\mathsf{Re}\left(\langle\phi'_{\mathcal{B}}|(\Pi_1^B \otimes \Pi_1^C)|\phi'_{\mathcal{C}}\rangle\right)$. By Corollary 2.4, the inner product will be 0 as well.

Therefore, the total probability will be at most $\left(\frac{1}{2} + \frac{1}{p}\right)\left(||\phi'_{\mathcal{B}}\rangle|^2 + ||\phi'_{\mathcal{C}}\rangle|^2\right) + \mathsf{negl}(n) \leq \frac{1}{2} + \frac{1}{p} +$

$\mathsf{negl}(n)$.

Since the above statement holds for every polynomial $p(\cdot)$, it finishes the proof for Theorem 4.8. □

Corollary 2.4. *For any two projectors Π_0, Π_1, let $|\phi_0\rangle$ and $|\phi_1\rangle$ be two eigenvectors of $w\Pi_0 + (1-w)\Pi_1$ with eigenvalues λ_0, λ_1. If $\lambda_0 + \lambda_1 \neq 1$ and $\lambda_0 \neq \lambda_1$, then*

$$\langle\phi_0|\Pi_0|\phi_1\rangle = \langle\phi_0|\Pi_1|\phi_1\rangle = 0.$$

Proof. If $\lambda_0 + \lambda_1 \neq 1$, by Lemma 2.3, $|\phi_0\rangle$ and $|\phi_1\rangle$ cannot be in the same Jordan block. Because $|\phi_0\rangle$ still belongs to the corresponding subspace $\mathcal{S}_0$ of its Jordan block after the action of Π_0, $\Pi_0|\phi_0\rangle$ is orthogonal to $|\phi_1\rangle$. Similarly, $\Pi_1|\phi_0\rangle$ is orthogonal to $|\phi_1\rangle$. □

Question 7

Reminder:

- An XZ Local Hamiltonian acting on n qubits is of the form $H = \sum_{\ell\in[m]} \gamma_\ell H_\ell$ where H_ℓ consists of tensor products of σ_x and σ_z (and $\mathbb{I}$) matrices, acting non-trivially (i.e. non $\mathbb{I}$) on at most k qubits, i.e. for each ℓ, $H_\ell = \otimes_{j\in n}\sigma_{W_{j,\ell}} \in \{\sigma_x, \sigma_z, \sigma_I\}^{\otimes n}$ with $\left|\{j | j \in [n] \wedge \sigma_{W_{j,\ell}} \neq \sigma_I\}\right| \leq k$.
- The Hamiltonian Test $G(H)$ for such Hamiltonians, as constructed by [2], is as shown in Figure 2 (t below is taken to be something like $n \log n$).
- In class, we proved the following (as Lemma 7). Let $\omega_h(H) := 1 - p\left(\frac{1}{2m}\sum_{\ell\in[m]}|\gamma_\ell| - \frac{1}{2}\lambda_0(H)\right)$ where $\lambda_0(H)$ denotes the smallest eigenvalue of H. If the provers use the honest strategy in the Pauli-Braiding test, then the maximum acceptance probability in $G(H)$ is $\omega_h(H)$. We now write down some of the main steps that go into this proof and you will be asked to fill in some of the gaps in the argument.

Question 7 (2 points, Verification). *Prove the following assertions.*

1. *Assertion: The acceptance probability in $G(H)$ depends uniquely on the strategy of the first prover in the energy test (in particular, it does not depend on the strategy of the second prover in the energy test).*

2. *Assertion: For a fixed H_ℓ, the verifier rejects with probability*

$$\frac{|\gamma_\ell| + \gamma_\ell \mathbb{E}[\Pi_{i\in[n]} d_i]}{2}. \tag{5}$$

Denote by τ the reduced state held by the second prover on qubits $\mathcal{T} := (\mathcal{T}_1, \ldots \mathcal{T}_n)$ of his EPR halves (after 'teleportation').

3. *Assertion: Given the answers (a, b) of the first prover, the following behaviours of the second prover are equivalent:*
   (a) *Measure τ using H_ℓ and obtain outcome $\Pi_{i\in\mathcal{T}} c_i$.*
   (b) *Measure $\rho = Z^b X^a \tau X^a Z^b$ using H_ℓ and obtain outcome $\Pi_{i\in\mathcal{T}} d_i$.*

   Hint: Measuring a qubit $|\phi\rangle$ in the Z-basis w/ outcome $f \in \{\pm 1\}$ is equivalent to getting outcome $(-1)^g f$ when measuring $Z^h X^g |\phi\rangle$ in the Z-basis. Similarly, measuring $|\phi\rangle$ in the X-basis w/ outcome $f \in \{\pm 1\}$ is equivalent to getting outcome $(-1)^h f$ when measuring $Z^h X^g |\phi\rangle$ in the X-basis. (Equivalence as in the output distribution is identical).

4. *Assertion: The rejection probability in Eq (5), averaged over ℓ, is*

$$\frac{1}{2m}\sum_{\ell\in[m]}|\gamma_\ell| + \frac{1}{2}\mathrm{tr}(\rho H).$$

   The acceptance *probability in $G(H)$ is at most*

$$1 - p\left(\frac{1}{2m}\sum_{\ell\in[m]}|\gamma_\ell| - \frac{1}{2}\lambda_0(H)\right) = w_h(H).$$

(a) Pauli-Braiding Test: (We don't need the details here, except that the view of Prover 2 is exactly the same as that in the Energy Test).

(b) Energy Test

Prover 1 (receives the ground state of H as a witness)	Verifier	Prover 2

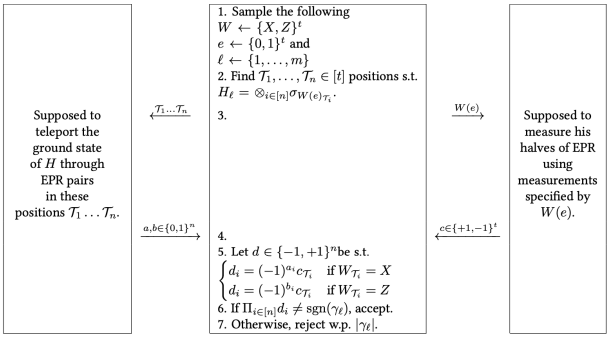


Figure 2: The verifier performs the Pauli-Braiding Test with probability $1 - p$ and the Energy Test with probability p. The provers are assumed to share t EPR pairs.

Question 7.1

NB1.

The view of the second prover is exactly the same
whether the Pauli-braiding test is run
or
the Energy test is run.

NB2:

Both provers
are assumed to be honest
in the Pauli-braiding test.

Conclusion.

The second prover
must behave identically
to how it behaves
in the energy test.

And therefore,
the acceptance probability in $G(H)$
depends uniquely on the strategy
of the first prover in the
energy test.

(these are hints, the actual answers must explain these in detail)

Question 7.2

One has to work out the probability that
the signs match and after some algebra,
one obtains the right expression.

Question 7.3

This essentially follows from the definitions of c_i of d_i
and the property about measuring
$X^g Z^h |\phi\rangle$ in Z-basis and getting outcome
$(-1)^g f$.
See that it is explained clearly.

Question 7.4

from the previous step
you get $\text{tr}(H_\ell \rho) = \mathbb{E}[\Pi_{i \in [n]} d_i]$
and then the weighted average
gives you $\text{tr}(H\rho)$.