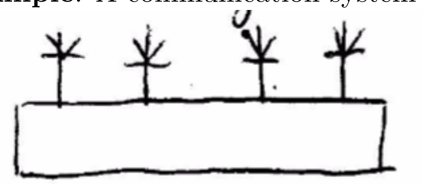
1 Combinational Analysis

1.1 Introduction

Example: A communication system consists of 4 antennas



Assume that this system will be functional if no two consecutive antennas are defective.

Q: If there are exactly 2 antennas defective, what is the probability that the resulting system will be functional?

Solution:

0 1 1 0	0 1 0 1	1 0 1 0	0 0 1 1	1 0 0 1	1 1 0 0
$\surd$	$\surd$	$\surd$	$\times$	$\times$	$\times$

If all the cases are equally likely, the probability is $\frac{3}{6}=\frac{1}{2}$.

1.2 The Basic Principle of Counting

Suppose that two experiments are to be performed. If experiment 1 can result in any one of m possible outcomes, and if for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are $m \times n$ possible outcomes of the two experiments.

$$\begin{array}{cccc} (1,1) & (1,2) & \dots & (1,n) \\ (2,1) & (2,2) & \dots & (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & \dots & (n,m) \end{array}$$

Example: A small community, 10 women, each has 3 children. If one women and one of her children are to be selected as mother and child of the year. How many possibilities?
Solution: $10 \times 3 = 30$

Generalize:
Experiment 1 with n_1, possible outcomes, for each of them. There are n_2

outcomes of experiment 2, for each of them. There are n_3 possible outcomes of experiment 3...... then there are totally,

$$n_1 \cdot n_2 \cdot n_3 \ldots n_r$$

possible outfcomes of the r experiments.

Example: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution:

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$$

Q: What if repetition among letters or numbers are prohibited?

Solution:

$$26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$$

1.3 Permutations

Example: How many different ordered arrangements of a, b, c?

Solution:

$$abc; acb; bac; bca; cab; cba$$

Each is known as a permutation. By the basic principle of counting:

$$3 \cdot 2 \cdot 1 = 6$$

Generally, for n objects, there are:

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$$

different permutations.

Example: To put 10 books on the bookshelf. Of these, 4 are math, 3 are chemistry, 2 are history, 1 is language. The books of the same subject should be put together. How many different arrangements?

Solution:

$4! \cdot 3! \cdot 2! \cdot 1! \cdot 4! = 6912$

(10! without the restriction that the books of the same subject should be put together)

Example: A class consists of 6 men and 4 women. An exam is given an no two students obtain the same score.

(a) How many different rankings are possible?

$$10! = 3,6628,800$$

(b) If men and women are ranked separately, how many?

$$4! \cdot 6! = 17,280$$

Example: How many different letter arrangements can be formed using the letters $PEPPER$?

Solution:

$$\frac{6!}{3! \cdot 2!} = 60, \quad \left.\begin{array}{r} P_1EP_2P_3ER \\ P_2EP_1P_3ER \\ P_3EP_2P_1ER \\ \vdots \\ \vdots \\ P_1EP_3P_2ER \end{array}\right\} 3! \text{ arrangements}$$

In general, there are

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \ldots n_r!}, (n = n_1 + n_2 + n_3 + \cdots + n_r)$$

diffierent permutations of n objects, of which n_1 are alike, n_2, n_3 are alike, $\ldots$, n_r are alike.

Example: How many different signals, by hanging 9 flags in a line, 4 white, 3 red, 2 blue?

Solution:

$$\frac{9!}{4! \cdot 3! \cdot 2!} = 1260$$

1.4 Combinations

First, a *question*: How many different groups of 3 can be selected from the 5 items A, B, C, D, E?

$$\frac{5 \times 4 \times 3}{3!} = 10$$

(Hint:$\{ABC\}, \{ACB\}, \dots \{BAC\}$are the same group.)

In general, selecting r items from n items, then the number of different groups is

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$
$$= \frac{n!}{(n-r)! \cdot r!}$$
$$= \binom{n}{r}$$

$$\binom{n}{n} = 1, \qquad \binom{n}{1} = n, \qquad \binom{n}{0} = 1 = \frac{n!}{n! \cdot 0!}, (0! = 1 \text{ by convention})$$

Example: From 5 women and 7 men, how many different committees of 2 women and 3 men can be formed?

Solution:

$$\binom{5}{2} \cdot \binom{7}{3} = \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$$

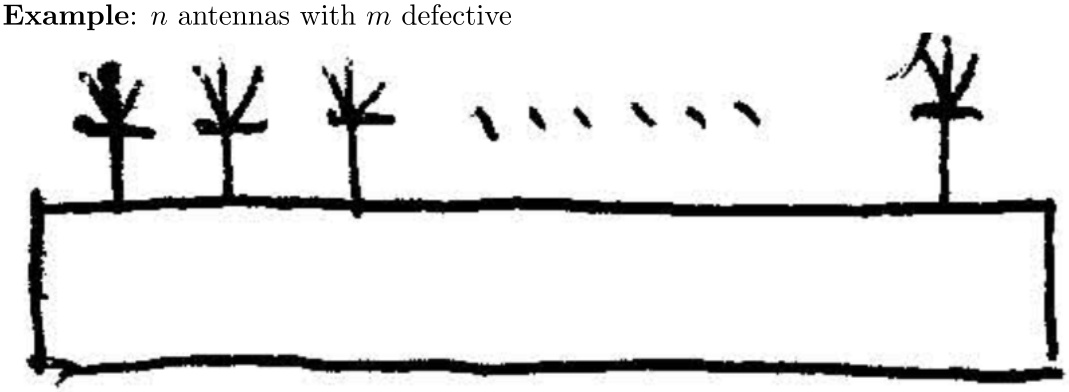
Q: What if 2 of the men refuse to serve together? Suppose the 2 men are A and B, then there are 4 situations:

1. both A and B are in $\binom{5}{2} \cdot \binom{5}{1}$
2. A in, B out $\binom{5}{2} \cdot \binom{5}{2}$
3. A out, B in $\binom{5}{2} \cdot \binom{5}{2}$
4. neither is in $\binom{5}{2} \cdot \binom{5}{3}$

So the answer is $350 - \binom{5}{2}\binom{5}{1}$

(Alternative answer is $\binom{5}{2}\binom{5}{2} \cdot 2 + \binom{5}{2}\binom{5}{3}$ or $\frac{\binom{5}{2}\binom{6}{3}}{2} + \frac{\binom{5}{2}\binom{6}{3}}{2} \times 2$

Example: n antennas with m defective



The system will be function if no two consecutive ones are defective. What is the probability of being funtional?

Hint: There are two steps to consider the problem:

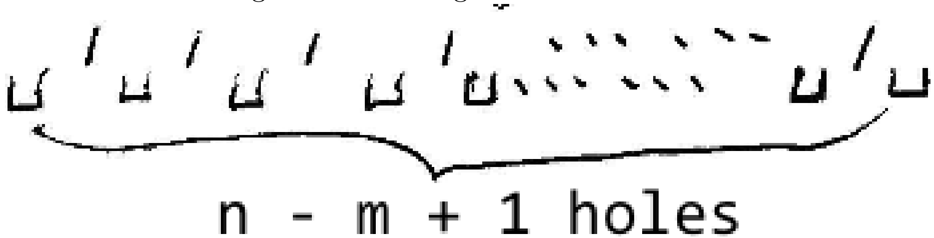
1. Totally, how many possibilities of finding m defective in n? (M)
2. Among them, how many where no two consecutive is defective? (N)

The probability should be $\frac{M}{N}$.

<u>Solution:</u>

1. The total number of possibilities: $\binom{n}{m}$
2. The total number of cases where no two consecutive ones: $\binom{n-m+1}{m}$

Hint: $n - m$ working ones in a line give rise to $n - m + 1$ holes.



Each hole can contain at most one defective antenna, so it is required to choose m holes from the $n-m+1$ holes to place defective antennas.

So, the probability of being functional is $\frac{\binom{n-m+1}{m}}{\binom{n}{m}}$. where of $n-m+1=m$, the probability is 0.

A useful combinational identity is $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Proof:

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{(n-1)!r}{(n-r)!r!} + \frac{(n-1)!(n-r)}{r!(n-r)!} \\
&= \frac{(n-1)!(n-r+r)}{(n-r)!r!} \\
&= \frac{n!}{(n-r)!r!} = \binom{n}{r}
\end{aligned}$$

Combinational Argument: Suppose there are n items, then $\binom{n}{r}$ is the number of different ways to choose r items from the n items. Besides, we con consider the first item, which can be chosen or not. if the 1st is chosen, then we only have to choose $r-1$ from the $n-1$ items, which gives rise to the number $-\binom{n-1}{r-1}$. If we don't choose the 1st, then there are $\binom{n-1}{r}$ ways to choose r from the $n-1$ items. Combining the above two cases, it follows:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

The binomial thorem:

$$
\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k} \qquad (\binom{n}{k} \text{ terms of } x^k y^{n-k}) \\
&= \sum_{k=0}^{n} \binom{n}{n-k} \cdot x^k \cdot y^{n-k} \quad (\binom{n}{n-k} \text{ terms of } x^k y^{n-k})
\end{aligned}
$$

$$
\text{Obviously, } \binom{n}{k} = \binom{n}{n-k}, \text{ since } \binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}.
$$

For mathematical proof of the theorem, please see the textbook.

Example: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Example: How many subsets are there of a set consisting of n elements?

$$\{1, 2, 3, \ldots n\}$$

Solution:

$$
\begin{array}{l}
\varnothing, \{1\}, \{2\}, \{3\} \ldots \{n\}; \\
\{1, 2\}, \{1, 3\} \ldots \{n, n-1\}; \\
\vdots \\
\{1, 2, 3 \ldots n\}
\end{array}
$$

$$\Downarrow$$

$$
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} =? \qquad (2^n)
$$

Applying the binomial theorem and letting x and y being 1s, then

$$
\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1+1)^n = 2^n
$$

Besides, we can consider this problem in another way: Forming a subset from the set consisting of n elements can be regarded as choosing some elements from that set. So, for each element, it has two choices, whether to be chosen to form the subset or not.

$$\{1, 2,3, 4 \ldots n\}$$
$$\downarrow$$
each of them could be chosen for subset or not.

Thus, by the basic principle of counting, there are 2^n subsets.