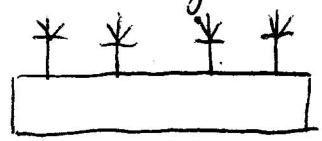
Chapter 1. Combinatorial Analysis

1.1 Introduction

Example: A communication System consists of 4 antenas.



Assume that this system will be functional if no two consecutive antenas are defective.

Q: If there are exactly 2 antenas defective, what is the probability that the resulting system will be functional?

Solution: 0110, 0101, 1010, 0011, 1001, 1100

$$\begin{array}{cccccc} 0110 & 0101 & 1010 & 0011 & 1001 & 1100 \\ \checkmark & \checkmark & \checkmark & \times & \times & \times \end{array}$$

If all the cases are equally likely, the probability is $\frac{3}{6} = \frac{1}{2}$.

1.2 The Basic Principle of Counting

Suppose that two experiments are to be performed. If experiment 1 can result in any one of m possible outcomes, and if for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are $m \cdot n$ possible outcomes of the two experiments.

$$\begin{array}{cccc} (1,1) & (1,2) & \cdots\cdots & (1,n) \\ (2,1) & (2,2) & \cdots\cdots & (2,n) \\ \vdots & & & \vdots \\ (m,1) & (m,2) & \cdots\cdots & (m,n) \end{array} \qquad \underline{m \cdot n \text{ possible outcomes}}$$

Example: A small community, 10 women, each has 3 children. If one women and one of her children are to be selected as mother and child of the year, How many possibilities?

Solution: $10 \times 3 = 30$.

Generalize:

Experiment 1 with n_1 possible outcomes, for each of them, there are n_2 outcomes of experiment 2, for each of them, there are n_3 possible outcomes of ex. 3, ……, then there are totally

$$n_1 \cdot n_2 \cdot n_3 \cdots n_r$$

possible outcomes of the r experiments.

Example: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution:

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$$

Q: What if repetition among letters or numbers are prohibited?

$$26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$$

1.3 Permutations

Example: How many different ordered arrangements of a, b, c?

Solution: abc ; acb ; bac ; bca ; cab ; cba

Each is known as a permutation.

By the basic principle of counting: $3 \cdot 2 \cdot 1 = 6$

Generally, for n objects, there are:

$$n \cdot (n-1) \cdot (n-2) \cdots\cdots 2 \cdot 1 = n!$$

different permutations.

Example: To put 10 books on the bookshelf. Of these, 4 are math, 3 are chemistry, 2 are history, 1 is language. The books of the same subject should be put together. How many different arrangements?

Solution: $4! \cdot 3! \cdot 2! \cdot 1! \cdot 4! = 6912$

($10!$ without the restriction that the books of the same subject should be put together)

Example: A class consists of 6 men and 4 women. An exam is given and no two students obtain the same score.

(a). How many different rankings are possible?

$$10! = 3,628,800$$

(b). If men and women are ranked seperately, how many?

$$4! \cdot 6! = 17,280$$

Example: How many different letter arrangements can be formed using the letters PEPPER?

Solution: $\dfrac{6!}{3!\,2!} = 60$

$$\left.\begin{array}{l} P_1 E P_2 P_3 E R \\ P_2 E P_1 P_3 E R \\ P_3 E P_2 P_1 E R \\ \vdots \\ P_1 E P_3 P_2 E R \end{array}\right\} 3! \text{ arrangements}$$

In general, there are

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, n_3 are alike, ⋯, n_r are alike.

$$n = n_1 + n_2 + n_3 + \cdots n_r$$

Example: How many different signals, by hanging 9 flags in a line, 4 white, 3 red, 2 blue?

Solution: $$\frac{9!}{4! \cdot 3! \cdot 2!} = 1260$$

1.4 Combinations

First, a question:

How many different groups of 3 can be selected from the 5 items A, B, C, D, E?

$$\frac{5 \times 4 \times 3}{3!} = 10$$

(Hint: $\{ABC\}$, $\{ACB\}$ ⋯⋯ $\{BAC\}$ are the same group.)

In general, selecting r items from n items, then the number of different groups is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$= \frac{n!}{(n-r)! \cdot r!} = \binom{n}{r}$$

$$\binom{n}{n} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{0} = 1 = \frac{n!}{n!\ 0!} \quad (0! = 1 \text{ by convention})$$

Example: From 5 women and 7 men, how many different committees of 2 women and 3 men can be formed?

Solution: $\binom{5}{2}\binom{7}{3} = \frac{5\cdot 4}{2\cdot 1} \cdot \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 350$

Q: What if 2 of the men refuse to serve together?

Suppose the 2 men are A and B, then there are 4 situations:

① both A and B are in $\binom{5}{2}\binom{5}{1}$

② A in, B out $\binom{5}{2}\binom{5}{2}$

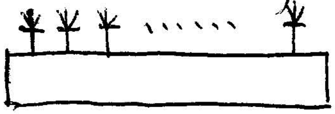
③ A out, B in $\binom{5}{2}\binom{5}{2}$

④ neither is in $\binom{5}{2}\binom{5}{3}$

So ther answer is $350 - \binom{5}{2}\binom{5}{1}$

(Alternative answer is $\binom{5}{2}\binom{5}{2}\cdot 2 + \binom{5}{2}\binom{5}{3}$ or $\frac{\binom{5}{2}\binom{6}{3}}{2} + \frac{\binom{5}{2}\binom{6}{3}}{2} \times 2$.)

Example: n antenas with m defective.



The system will be functional if no two consecutive ones are defective. What is the probability of being functional?

Hint: There are two steps to consider the problem:

① Totally, how many possibilities of finding m defective in n? (M)

② Among them, how many where no two consecutive is defective? (N)

The probability should be $\frac{N}{M}$

Solution: ① The total number of possibilities: $\binom{n}{m}$

② The number of cases where no two consecutive ones? $\binom{n-m+1}{m}$

(Hint: $n-m$ working ones in a line give rise to $n-m+1$ holes.

$$\underbrace{\sqcup \,/\, \sqcup \,/\, \sqcup \,/\, \sqcup \,/\, \sqcup \cdots\cdots\cdots \sqcup \,/\, \sqcup}_{n-m+1 \text{ holes}}$$

← each hole can contain at most one defective antena, so it is required to choose m holes from the $n-m+1$ holes to place defective antenas.)

So, the probability of being functional is $\dfrac{\binom{n-m+1}{m}}{\binom{n}{m}}$.

where if $n-m+1<m$, the probability is 0.

A useful combinatorial identity is $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Proof:
$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-1-r)!} = \frac{(n-1)!\,r}{(n-r)!\,r!} + \frac{(n-1)!\,(n-r)}{(n-r)!\,r!}$$
$$= \frac{(n-1)!\,(n-r+r)}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}.$$

Combinatorial Argument: Suppose there are n items, then $\binom{n}{r}$ is the number of different ways to choose r items from the n items. Besides, we can consider the first item, which can be chosen or not. If the 1st is chosen, then we only have to choose $r-1$ from the $n-1$ items, which gives rise to the number $\binom{n-1}{r-1}$. If we don't choose the 1st, then there are $\binom{n-1}{r}$ ways to choose r from the $n-1$ items. Combining the above two cases, it follows:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

The binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k} \quad \left(\binom{n}{k} \text{ terms of } x^k y^{n-k} \right)$$

$$= \sum_{k=0}^{n} \binom{n}{n-k} x^k \cdot y^{n-k} \quad \left(\binom{n}{n-k} \text{ terms of } x^k \cdot y^{n-k} \right)$$

Obviously, $\binom{n}{k} = \binom{n}{n-k}$, since $\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$.

For mathematical proof of the theorem, please see the textbook.

Example: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Example: How many subsets are there of a set consisting of n elements?
$\{1, 2, 3, \cdots n\}$.

Solution: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\cdots$ $\{n\}$; $\{1,2\}$, $\{1,3\}$, $\cdots$ $\{n, n-1\}$; $\cdots\cdots$ $\{1, 2, 3 \cdots n\}$.

$\Downarrow$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots\cdots \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = ? \quad (2^n)$$

Applying the binomial theorem and letting x and y being 1's,

then $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$

Besides, we can consider this problem in another way: Forming a subset from the set consisting of n elements can be regarded as choosing some elements from that set. So, for each element, it has two choices, whether to be chosen to form the subset or not.

$$\{1, 2, \underline{3}, 4, \cdots n\}.$$

$\downarrow$ each of them could be chosen for subset or not.

Thus, by the basic principle of counting, there are 2^n subsets.

1.5 Multinomial Coefficients:

Q: n distinct items are to be divided into r distinct groups of respective sizes $n_1, n_2, \cdots n_r$ ($n_1+n_2+n_3+\cdots+n_r = n$). How many different divisions are possible?

To fill the r groups step by step, we note that there are $\binom{n}{n_1}$ possible choices for the first group, $\binom{n-n_1}{n_2}$ possible choices for the second group, $\cdots$ and $\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$ choices for the rth group. Thus, it follows from the generalized version of basic principle of counting that there are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

$$= \frac{n!}{\cancel{(n-n_1)!}\,n_1!}\cdot\frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\,n_2!}\cdot\frac{\cancel{(n-n_1-n_2)!}}{(n-\cancel{n_1-n_2}-n_3)!\,n_3!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})}{0!\;n_r!}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_r!}$$ possible divisions.

(Can the answer be r^n? No! Consider the specific situation where $n_1 = 1$, for the first item there are r choices, but for the second item, the number of possible choices will depend and may not be r any longer).

Alternative reasoning: we can change the problem to how to throw n items into n holes as the following figure:

$$\{1,\ 2,\ 3,\ 4,\ \cdots\cdots\ n\}$$

$\Downarrow$ throw them into n holes.

$$\underbrace{\sqcup\sqcup\sqcup\cdots\sqcup}_{n_1}\quad \underbrace{\sqcup\cdots\sqcup}_{n_2}\quad \cdots\quad \underbrace{\sqcup\cdots\sqcup}_{n_r}\qquad (n_1+n_2+\cdots n_r=n).$$

Each permutation can be regarded as a way to group them, but if we take $n!$ as the total number of possible ways to group them, we have over counted, because it doesn't matter how we permute the items in the same group among themselves.

Thus, there are $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$ possible divisions.

Still, we can take this problem as how to assign n labels to n items. Of the n labels, n_1 are alike, n_2 are alike, $\cdots$ n_r are alike. If one item receives the kth label, then it's contained by the kth group.

$$\underbrace{\textcircled{1}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1}\cdots\textcircled{1}}_{n_1 \text{ labels}}\quad \underbrace{\textcircled{2}\ \textcircled{2}\cdots\textcircled{2}}_{n_2 \text{ labels}}\quad \cdots\quad \underbrace{\textcircled{r}\ \textcircled{r}\cdots\textcircled{r}}_{n_r \text{ labels}}.$$

Notation: $\dfrac{n!}{n_1!n_2!\cdots n_r!} = \dbinom{n}{n_1, n_2, \cdots n_r}$.

Example: A class of 68 students divided into 3 sections if sizes are 21, 22, 25. How many different ways of division?

$$\frac{68!}{21!\,22!\,25!} = 2.78\times 10^{30}$$

Example: 10 children are to be divided into 2 basketball teams, each of 5 to play against each other, how many different divisions?

$$\frac{\dfrac{10!}{5!\,5!}}{2}$$ (There is a 2 in the denominator because the order of the two teams is irrelevant.)

Recall the binomial theorem, which is $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$

Accordingly, the multinomial theorem is:

$$(x_1+x_2+\cdots x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \cdots n_r} x_1^{n_1} \cdot x_2^{n_2} \cdots x_r^{n_r}$$