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Discrete Mathematics

Let Us Count

Examples of Catalan numbers

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We have discussed of what are Catalan numbers. Let us now see some examples of Catalan numbers.

Example 1: Parenthesis.

We will be given n pair of parenthesis and we would like to form a valid grouping of them. What do I mean by valid? Valid grouping means for every open parenthesis there must be a closed parenthesis.

Now the question is how many valid groupings are there for each value of n? Let us check.

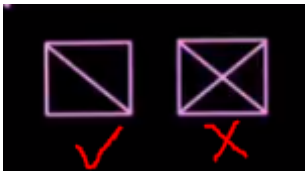
If n = 0, that is we do not have any parenthesis, we say that it's one way of having no parenthesis. When n = 1, we have 1 open and 1 closed parenthesis. Hence we have one way. When n = 2, we have 2 closed and 2 open parenthesis. So the 2 possibilities are there. When n = 3, we have 3 closed and 3 open parenthesis. We have 5 ways of grouping 3 pairs of parenthesis. When n = 4, you have 14 ways of doing this. You might want to stop here and do it all by yourself.

n=0	–	1 way
n = 1	( )	1 way
n = 2	( )( ) (( ))	2 ways
n = 3	( )( )( ) ((( ))) ( ) (( )) (( )( )) (( ))( )	5 ways

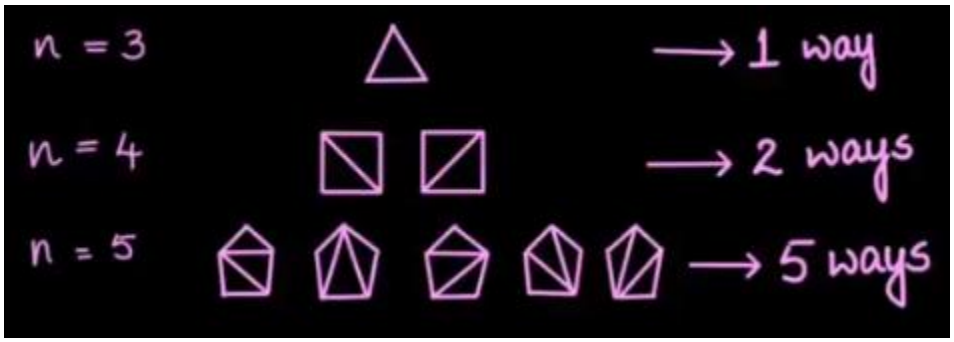
So we see that the sequence is 1, 1, 2, 5, 14 and so on. So here we see that Catalan numbers hold true in the case of grouping of parenthesis.

Example 2: Polygon Triangulation.

By triangulation we mean constructing triangles inside the polygon in a particular way. Let me tell you how. I mean constructing triangles by adding diagonals so that two diagonal do not cross over each other.



So let's start with n = 3.

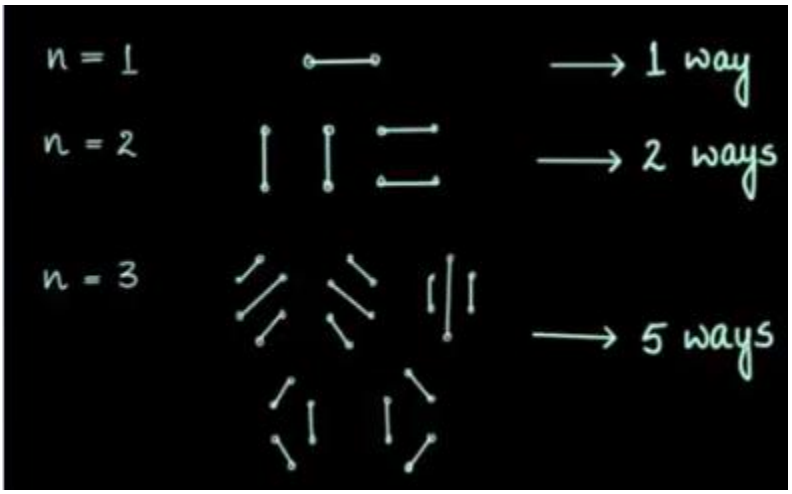


It can be done in 14 ways for n = 6. So we get the sequence 1, 1, 2, 5, 14 and so on. So this gives the Catalan numbers.

Example 3: Handshakes across a table.

This is interesting. If 2n people are seated around a round table, in how many ways can they all simultaneously shake hands with another person at the table such that nobody crosses arms across each other?

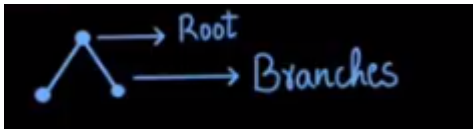
So we don't want to cross handshake. So we have the following cases:



when n = 1, there are two people and so 1 handshake. So this can be done only in one way. When n = 2, there are 4 people and they can shake their hands in 2 ways. Now when n = 3, there are 6 people and in 5 ways 6 people can shake their hands without crossing. Now when n = 4, there are 8 people it becomes slightly complicated and there are 14 ways of doing this and so on. As you keep increasing you see that the number increases. So we see that 1, 1, 2, 5, 14. So this again here when n = 0 we have 0 people and 0 people can shake their hands only in one way and hence we see the Catalan numbers here.

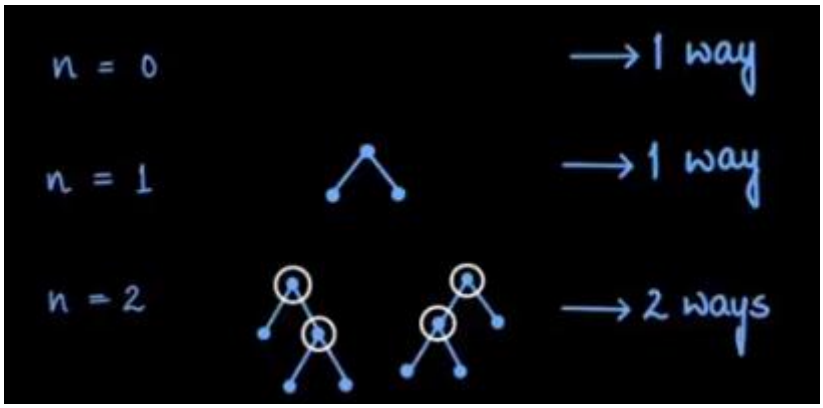
Example 4: Binary trees.

Do not get confused as I say binary trees. By tree I mean a dot as shown in the figure along with are some lines coming out from the dot connected to some other dots. So this I will call it as a tree for now. The starting dot is called a root and it is branching out into two branches, or let me say two leaves.



In binary tree a dot has two branches. So when n = 0, that means that there are no branches. There is only one node and this can be done in one way. When n = 1,

there is only 1 node which is having 2 branches and this also can be done in 1 way. When n = 2, there are 2 nodes having 2 branches each. So I can draw it in 2 ways as shown in the figure.



Now when n = 3, there are 3 roots which have 2 branches each. So this can be drawn in 5 ways.

Please note why did we use the word binary because we are here concerned only about two branches coming out. So as we increase n we see that the sequence becomes 1, 1, 2, 5, 14 etc.

So these were some of the examples of Catalan numbers. Well there are several many. I hope it was interesting.

We have now come to the end of the first chapter. We learned several elementary techniques on counting. These are going to be the builder blocks for more advanced techniques on counting.

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