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Discrete Mathematics

Let Us Count

Catalan Numbers - Part 3

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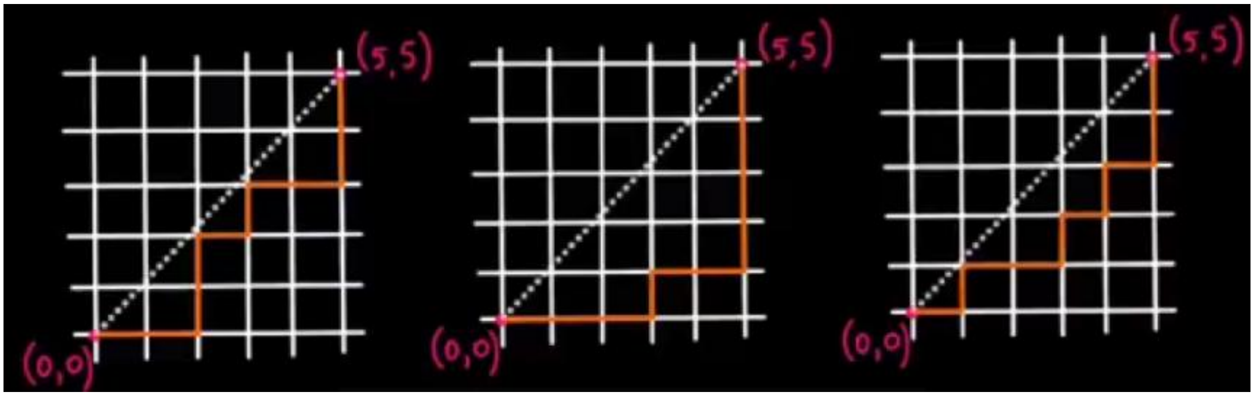
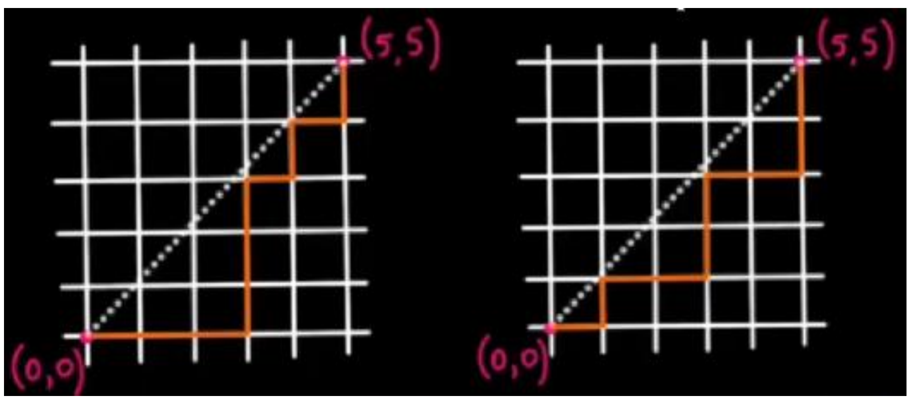
So I have tried all the possibilities manually for the total number of paths without crossing the diagonal from (0,0) to (5,5). Let me show you.

These are some of the possibilities:





It might surprise you that there are 42 such possibilities to go from (0,0) to (5,5) without crossing the diagonal.

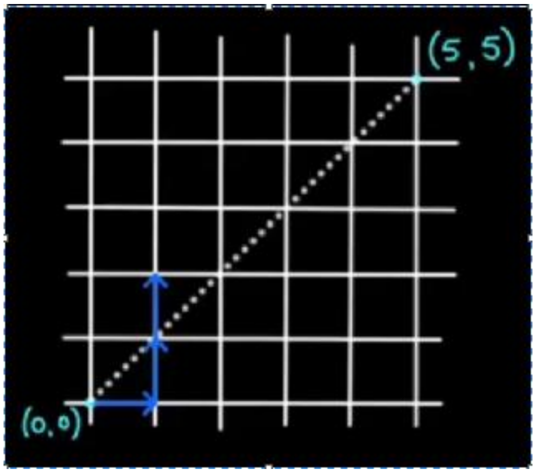
We saw that the answer is not necessarily $\frac{\binom{10}{5}}{2}$, which is all possibilities divided by 2. It was something else. So a straightforward intuition of seeing that crossing and not crossing would be of the same numbers and hence it's going to be $\frac{\binom{10}{5}}{2}$ may not work. Something to note here.

Let's see what works here. Before going any further I will try recollecting a small tip that I have been giving you when it comes to counting. When you want to count the number of elements in a set what you do is you try creating a one-to-one correspondence to another set where counting is very easy and then show that this set has the same number of elements as the other set. And since counting in that set is very easy, number of elements in the original set is now known. I'm going to use this technique right now in a very subtle way.

The concept is slightly deep. You may want to watch this video multiple times to understand what we are saying. So now let's get started. What was the question? What are the total number of ways in which you can reach (5,5) from (0,0) without trespassing the diagonal?

Let me now observe an instance where I actually trespassed the diagonal and I will count all those possibilities which crosses the diagonal. And then I will subtract that from my all possibilities which is $\binom{10}{5}$ and I am done. So all that remains right now is to count the number of ways in which you can trespass the fence.

I will start with just one such instance. Look at this instance RUURURRURU, where R stands for right and U stands for up. When you take a first step that is R you are well below the fence and next when you take U, you touch the fence and another U you trespass the fence. And then you take a few more steps.

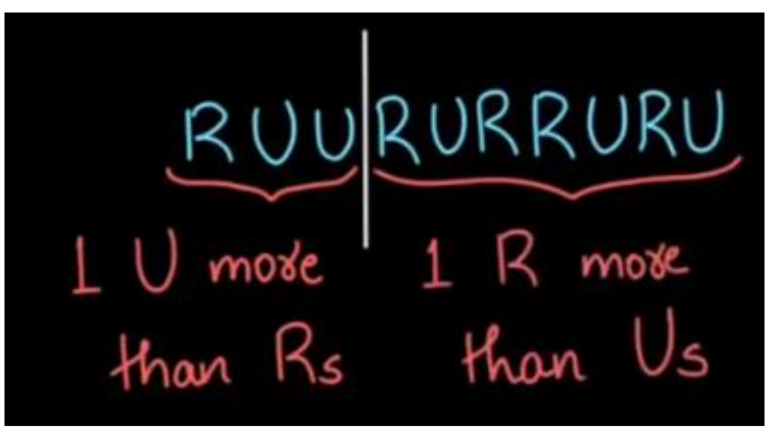


Let me put a vertical line immediately after that state when you trespass the fence.

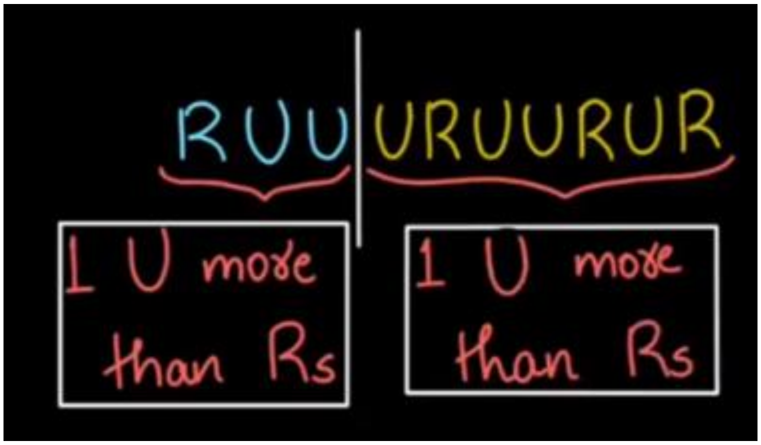
And I do is a small trick here, I take the right side of this vertical line and take the compliment of it. So an R becomes U and a U becomes R. I will take a compliment of it like this. You will get to know why I am doing this in some time but as of now you don't worry much about why exactly is I am doing this compliment business.

On the left side of the vertical line you have one U more than the R that is when you put a vertical line. The rule is not just you put a vertical line after two Us, it is whenever you trespass. Trespassing is when you have one U more than the number of Rs. There are more Us than R, that is when you trespass the diagonal. When you are on the diagonal the number of Rs and Us are the same. When you trespass there is one U more than the number of Rs.

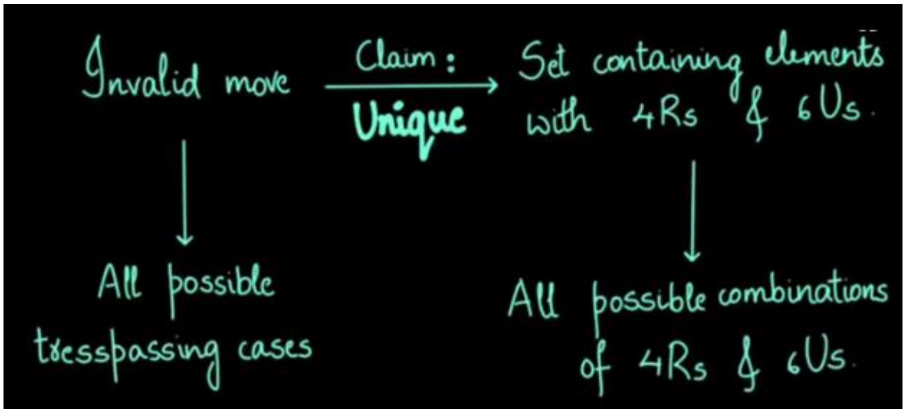
When you have one U more than R on the left side of the vertical line, obviously there should be one R more than the number of Us in the right side, as total number of Rs and Us are equal.



On the left side you always have one U more than the number of Rs and on the right side you always have one R more than the number of Us whenever you put a vertical line. Now I take the compliment of the right side which had one R more than a number of Us.



In the compliment you will get one U more than the number of Rs. On the left side you already had one U more than the number of Rs. So net total right now will be 2 Us more than the number of Rs. Which means you now have 4 Rs and 6 Us. What on earth does this denote? This doesn't denote a way from (0,0) to (5,5). This simply denotes a combination of 4 Rs and 6 Us.



Now what exactly happened so far? I took an invalid move and I created a unique mapping to a set containing elements with 4 Rs and 6 Us. And my claim right now is this mapping is unique. Unique in what sense? Every element on left side, denotes all possible trespassing cases and on the right side all possible 4 Rs and 6 Us and its combinations.

How many such things are there? There are $\binom{10}{6}$ such possibilities. And now what do we observe, for any 4 Rs and 6 Us possibilities you can actually put a vertical line after one U more than the R possibility and then take the complement. For an element to the left side you have a unique element on the right side. For every element in the right side you have a unique element on the left side.

total number of possibilities that are trespassing = $\binom{10}{6}$

total number of possibilities = $\binom{10}{5}$

So all those possibilities that doesn't trespass the line = $\binom{10}{5} - \binom{10}{6}$

We can even generalize this.

All possibilities to go from (0,0) to (n,n) without trespassing the diagonal

$$= \binom{2n}{n} - \binom{2n}{n-1}.$$

I believe this lecture was quick. It was intended to be that way. I request you all to please pause here and there and watch the video multiple times so that you understand every piece of what is being told. There are actually several pieces here. There's a beautiful one-to-one correspondence idea that we use to count the total number of trespassing instances and you use that to subtract from all possibilities and hence finally you will get those possibilities where you do not trespass.

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