Setup: Consider an undirected and unweighted network graph $G(V, E)$, with order $N_v := |V|$, size $N_e := |E|$, and adjacency matrix $\mathbf{A}$. Let $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_{N_v})$ be the degree matrix and $\mathbf{L} := \mathbf{D} - \mathbf{A}$ the Laplacian of G.

Part 1: $\mathbf{1}$ is an eigenvector of $\mathbf{L}$ with eigenvalue 0

Claim: $\mathbf{1} := [1, \ldots, 1]^\top \in \mathbb{R}^{N_v}$ satisfies $\mathbf{L}\mathbf{1} = \mathbf{0}$.

Proof:
For each row i of $\mathbf{L} = \mathbf{D} - \mathbf{A}$:

- The i-th row of $\mathbf{D}$ has entry d_i in column i and 0 elsewhere, so $(\mathbf{D}\mathbf{1})_i = d_i$.
- The i-th row of $\mathbf{A}$ has a 1 in each column j such that $\{i, j\} \in E$, and 0 elsewhere, so $(\mathbf{A}\mathbf{1})_i = \sum_{j:\{i,j\}\in E} 1 = d_i$.

Therefore

$$(\mathbf{L}\mathbf{1})_i = (\mathbf{D}\mathbf{1})_i - (\mathbf{A}\mathbf{1})_i = d_i - d_i = 0$$

for all i. Hence $\mathbf{L}\mathbf{1} = \mathbf{0}$, i.e. $\mathbf{1}$ is an eigenvector of $\mathbf{L}$ with eigenvalue 0. ∎

Part 2: Laplacian factorization $\mathbf{L} = \tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top$

Setup: Assign an arbitrary orientation to each edge (head/tail). The *signed incidence matrix* $\tilde{\mathbf{B}} \in \{-1, 0, 1\}^{N_v \times N_e}$ has

$$\tilde{\mathbf{B}}_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is the tail of edge } j, \\ -1, & \text{if vertex } i \text{ is the head of edge } j, \\ 0, & \text{otherwise.} \end{cases}$$

Claim: $\mathbf{L} = \tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top$.

Proof:
Compute $(\tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top)_{ik} = \sum_{j=1}^{N_e} \tilde{\mathbf{B}}_{ij}\tilde{\mathbf{B}}_{kj}$.

- **Case $i = k$:** For each edge j, vertex i is either tail (+1) or head (-1) or not incident (0). So $\tilde{\mathbf{B}}_{ij}^2 \in \{0, 1\}$ and $\sum_j \tilde{\mathbf{B}}_{ij}^2$ counts edges incident to i, i.e. $(\tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top)_{ii} = d_i = (\mathbf{L})_{ii}$.
- **Case $i \neq k$:** $\tilde{\mathbf{B}}_{ij}\tilde{\mathbf{B}}_{kj} \neq 0$ only when edge j is incident to both i and k. For such an edge, one of i, k is tail (+1) and the other head (-1), so $\tilde{\mathbf{B}}_{ij}\tilde{\mathbf{B}}_{kj} = -1$. Thus

$$(\tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top)_{ik} = -\#\{\text{edges between } i \text{ and } k\} = -A_{ik} = (\mathbf{L})_{ik}.$$

So $\tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top = \mathbf{D} - \mathbf{A} = \mathbf{L}$. ∎

Part 3: Quadratic form $\mathbf{x}^\top\mathbf{L}\mathbf{x}$ and positive semi-definiteness

Claim: For any $\mathbf{x} = [x_1, \ldots, x_{N_v}]^\top \in \mathbb{R}^{N_v}$,

$$\mathbf{x}^\top\mathbf{L}\mathbf{x} = \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Hence $\mathbf{L}$ is symmetric positive semi-definite.

Proof:
Using $\mathbf{L} = \tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top$,

$$\mathbf{x}^\top\mathbf{L}\mathbf{x} = \mathbf{x}^\top\tilde{\mathbf{B}}\tilde{\mathbf{B}}^\top\mathbf{x} = \|\tilde{\mathbf{B}}^\top\mathbf{x}\|^2.$$

The j-th entry of $\tilde{\mathbf{B}}^\top\mathbf{x}$ corresponds to edge j. If edge j connects vertices i (tail) and k (head), then $(\tilde{\mathbf{B}}^\top\mathbf{x})_j = x_i - x_k$. So

$$(\tilde{\mathbf{B}}^\top\mathbf{x})_j^2 = (x_i - x_k)^2.$$

Summing over edges (each undirected edge (i, j) appears once in the sum, with one orientation),

$$\mathbf{x}^\top\mathbf{L}\mathbf{x} = \sum_{j=1}^{N_e}(\tilde{\mathbf{B}}^\top\mathbf{x})_j^2 = \sum_{(i,j)\in E}(x_i - x_j)^2.$$

- **Symmetric:** $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and both $\mathbf{D}$ and $\mathbf{A}$ are symmetric, so $\mathbf{L}^\top = \mathbf{L}$.
- **Positive semi-definite:** $\mathbf{x}^\top\mathbf{L}\mathbf{x} = \sum_{(i,j)\in E}(x_i - x_j)^2 \geq 0$ for all $\mathbf{x}$, so $\mathbf{L} \succeq 0$. ∎

Part 4: Disconnected graph and second smallest eigenvalue

Claim: If G is disconnected, then $\mathbf{L}$ is block diagonal (with blocks corresponding to connected components), and the second smallest eigenvalue of $\mathbf{L}$ is zero.

Proof:

1. **Block structure:** Label vertices so that vertices in the same connected component have consecutive indices. Then there is no edge between different components, so $\mathbf{A}$ (and hence $\mathbf{D}$ and $\mathbf{L}$) has no off-diagonal blocks between components. Thus $\mathbf{L} = \mathrm{blkdiag}(\mathbf{L}_1, \ldots, \mathbf{L}_C)$, where $C \geq 2$ is the number of components and $\mathbf{L}_c$ is the Laplacian of the c-th component.
2. **Two independent zero eigenvectors:** For each component c, the indicator vector $\mathbf{v}^{(c)} \in \mathbb{R}^{N_v}$ with $v_i^{(c)} = 1$ if vertex i belongs to component c and $v_i^{(c)} = 0$ otherwise satisfies $\mathbf{L}\mathbf{v}^{(c)} = \mathbf{0}$ (within that block, $\mathbf{L}_c\mathbf{1}_c = \mathbf{0}$; outside the block, zeros). So each $\mathbf{v}^{(c)}$ is an eigenvector of $\mathbf{L}$ with eigenvalue 0. The vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(C)}$ are linearly independent (their supports are disjoint and non-empty). So the eigenvalue 0 has geometric multiplicity at least $C \geq 2$.

3. **Second smallest eigenvalue:** The eigenvalues of $\mathbf{L}$ are the union of the eigenvalues of the $\mathbf{L}_c$. Each $\mathbf{L}_c$ has eigenvalue 0 with eigenvector $\mathbf{1}_c$. So $\mathbf{L}$ has at least two linearly independent eigenvectors for eigenvalue 0. Ordering the eigenvalues as $0 = \lambda_1 \leq \lambda_2 \leq \cdots$, we get $\lambda_1 = \lambda_2 = 0$; hence the second smallest eigenvalue of $\mathbf{L}$ is zero. ∎